

Nonlinear Models of the Fluid Flow in Porous Media and Their Methods of Study



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Abstract We survey mathematical models of the fluid flow in porous media based on quasilinear parabolic partial differential equations. We focus on singular and/or degenerate parabolic equations, which are suitable for modeling of turbulent filtration such as groundwater flow through gravel and/or fractured crystalline rocks and turbulent polytropic filtration of natural gas through rocks in standard deposits, on one hand, and isothermic nanoporous (slow) filtration of natural gas in shale formations, on the other hand. Since in the case of singular and/or degenerate parabolic equations, it is almost impossible to find explicit solutions, we survey some existence and regularity theory together with maximum and comparison principles. We apply this theory on some selected examples from practice.

Keywords Ground water · Drought · Flow in porous medium · Turbulence · Nonlinear Darcy law · Leibenson's equations · Natural gas · p -Laplacian · Doubly nonlinear equation · Comparison principles

1 Introduction

Climate change and shortage of natural freshwater resources are becoming very serious issues nowadays. There is a need for better management of existing resources, while looking for unconventional resources of this vital substance. Our aim is to contribute to these important issues by surveying several nonlinear mathematical

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models of the fluid flow in porous media and their methods of study. We hope that people from practice may find them useful.

Long-lasting droughts become serious problem not only in traditionally arid and/or semi-arid areas, but newly also in countries with moderate climate such as countries in Central Europe. Indeed, several regions of Europe including those in Central and Northern Europe experienced severe drought conditions during June and July 2019, resulting from a combination of the 2018 drought, the heatwaves of 2019 and below-average precipitations in spring 2019, according to JRC European Drought Observatory report [39]. Moreover, below-average precipitations in 2018–2019 lead to lowering of the groundwater level which caused drying of wells in many places in the Czech Republic as it can be seen from the weekly observations of water table in shallow boreholes (ca. 2–15 m deep) conducted by the Czech Hydrometeorological Institute [13], where most of the observations are significantly below long-term average values (collected data since 1950s). The drought events of 2015–2019 also contributed to bark beetle calamity, see, e.g., [31, 32, 34, 55, 69], peaking in Central Europe in 2019. Of course, the problems of drought were not limited to Europe in 2019, significant problems were experienced also in many more areas worldwide, e.g., in Southeast Australia [40], Southern Africa [42, 43], and India [41] in 2019. According to [53], two-thirds of the global population live under conditions of severe water scarcity for at least 1 month of the year and half a billion people face severe water scarcity all year round. It was already in 2008, when Goldman Sachs [28] estimated that the annual consumption of freshwater approximately doubles every 20 years, claimed that water will be oil of the forthcoming century, and recommended to private investors to invest into infrastructure related with freshwater supply.

Most of the mathematical models of the groundwater flow used in practice are based on the linear Darcy (constitutive) law relating groundwater flux with piezometric head loss per length:

$$q = \text{const.} \frac{\Delta h}{\Delta L}, \quad (1.1)$$

where $h = \frac{P}{\rho g} + z$ is the piezometric head, P is hydrostatic pressure, ρ is density, g is acceleration due to gravity and z is vertical coordinate measured from arbitrary (but fixed) horizontal level, Δh stands for the piezometric head loss (difference of h), ΔL is distance, and q is flux. This law was established empirically by Henry Darcy [15] already in 1856 and it is sufficiently accurate in the case that the flow is laminar, that is, when the Reynolds number related to flux is not “too high” (to be clarified in Sect. 4). If, however, the Reynolds number of the flux is “too high” (see Sect. 4), the turbulence occurs and the linear Darcy law should be replaced by a nonlinear one such as the Smreker–Izbash–Missbach law

$$\frac{\Delta h}{\Delta L} = \text{const.} q^m \quad \text{or, equivalently,} \quad q = \text{const.} \left(\frac{\Delta h}{\Delta L} \right)^{\frac{1}{m}}, \quad (1.2)$$

or the Forchheimer law

$$\frac{\Delta h}{\Delta L} = aq + bq^2, \quad (1.3)$$

where the positive multiplicative constants and the exponent $m \in (1, 2]$ are to be determined empirically. Note that the turbulence often occurs for reasonable and realistic fluxes in practice in the case of coarse porous materials such as gravel or fractured impermeable media with sufficiently wide fractures. A thorough historical survey of development constitutive laws and their history is presented in [8].

With increasing demand on water supply, crystalline rock (or hard rock) aquifers are gaining attention in the last decades [29, 60]. By crystalline rock (or hard rock), we mean impermeable rocks of igneous or metamorphic origin (of negligible permeability) such as, e.g., basalts, granites, or gneisses, where the groundwater flow occurs only in a system of cracks and fractures. Since the water is stored and flows only in cracks and fractures, wells and boreholes in the crystalline rock aquifers have significantly smaller yield as compared to those in porous sedimentary rocks or alluvial aquifers. Nevertheless, crystalline rocks of the Precambrian continental shields occupy approx. 20% of the land surface [29]. Hence, crystalline rock aquifers may become important source of freshwater in rural areas. More importantly, crystalline rocks are commonly found in semi-arid areas where they may represent important source of scarce freshwater. Indeed, continental shields occupy approx. 40% of the semi-arid areas of the sub-Saharan Africa [50, 77]. It is estimated that 40% of groundwater in Australia is stored in the crystalline aquifers [27]. Crystalline aquifers are intensively exploited by farming communities as a source of freshwater mostly used for irrigation in semi-arid southern India [57]. Thus, good understanding of the flow in crystalline aquifers can improve quality of life in these areas. The crystalline rocks are commonly found in continental shields and massifs also in areas which do not have lack of precipitations such as Brazil, Canada, and Scandinavia. On one hand, the crystalline rock aquifers are used for water supply to rural communities in these areas. On the other hand, there are also large underground construction projects such as tunnels, mines, nuclear-waste disposal sites, and similar, see [29]. Thus, understanding groundwater flow in hard rock aquifers is important not only from the point of view of water extraction, but also from the point of view of dewatering of these construction projects.

Hand in hand with climate change, global water cycle intensifies and hydrological extremes including floods may occur more frequently, see, e.g., [30, 33, 71, 76]. Thus, further research and development of effective drainage systems is needed. It appears that coarse porous media such as gravel or geosynthetic materials are suitable for this task, but it turns out that movement of water in these materials is again governed by the nonlinear Smreker–Izbash–Missbach or Forchheimer law [10, 23].

Recent serious drought events are closely related to ongoing climate changes, see, e.g., [14, 74, 75] and references therein. Although it may be the case that the CO₂ emissions are not the main reason of global warming, see, e.g., *pro et contra* arguments in [11, 46, 47, 61, 62, 70], preference for fossil fuels with lower CO₂ emissions will most likely not make the situation worse. Natural gas is a hydro-

carbon gas mixture consisting primarily of methane (CH_4), and thus has the most favorable ratio between carbon and hydrogen in terms of emission reduction of all fossil fuels. For comparison, the amount of CO_2 produced by burning natural gas to get a unit of energy is a half that of black coal (117 lb CO_2 per 1 million Btu versus 205–228.6 lb CO_2 per 1 million Btu) according to the U.S. Energy Information Administration (see [72]). With geographically narrowly localized conventional gas fields, unconventional deposits (e.g., shale gas deposits) are now increasingly being opened worldwide to meet increasing demand. In order to better exploit valuable natural resources, one needs good mathematical models. Natural gas flow in the rock is a very complicated process which involves heat exchange with collector rock and may involve turbulence. One of the first to develop satisfactory mathematical models of non-stationary flow of natural gas in a collector rock of a conventional gas field was Leibenson [48].

It turns out that the archetypal parabolic partial differential equation

$$\frac{\partial v}{\partial t} - \operatorname{div}(|v|^l |\nabla v|^{p-2} \nabla v) = f(x, t) \quad (1.4)$$

is a suitable model for all above situations of the fluid flow in porous medium. Note that (1.4) becomes Leibenson's equation of filtration of a polytropic gas in a porous strata for $3/2 < p \leq 2$, $l > 0$, see Sect. 3, and equation for the water table in an unconfined aquifer for $3/2 \leq p \leq 2$, $l = 1$, see Sect. 2. Note that the case $p = 2$ corresponds to *laminar flow* in both Leibenson's equation and the water table equation, while the case $p = 3/2$ corresponds to a *flow with fully developed turbulence*. Most importantly, for practical considerations, the intermediate case $3/2 < p < 2$ corresponds to a *flow with some effects of turbulence*. Moreover, (1.4) with $l = 1$ and $2 < p < 10$ is also model of fluid flow in nanoporous media (see [54]). Note that such type of gas filtration occurs in the shale deposits, whose importance in natural gas extraction has recently increased significantly.

2 Basic Terminology in Hydrology

2.1 Porous Medium

The attempt to formulate an exact definition of *porous medium* brings many pitfalls, see Bear [6, Sects. 1.2 and 1.3]. We adopt the conceptual model presented in [6, Sects. 1.3 and 4.5.2]. Moreover, we restrict ourselves to the case where a portion of space (domain from mathematical point of view) is occupied by two homogeneous kinds of matter. Solid phase (say rock) forms a rigid container for fluid phase. The space occupied by solid phase is called *solid matrix* and the space filled by fluid phase is called *pore space*. Porous medium contains solid matrix and pore space in any sufficiently large subdomain (but still much smaller than the whole domain). In

fact, the pore space includes many relatively narrow channels or tubes of various length, cross-section, and orientation. We call a *junction* the part of void space where at least three channels meet each other. The channels and the junctions have more or less uniform spatial distribution.

In the case of fluid flow, we can assume that any two points in pore space may be connected by a curve that lies completely within it since there is no flow in isolated *pores* (subsets of pore space). Consequently, the isolated pores are considered as the part of solid matrix, see [6, Sect. 1.2]. The remaining pore space (interconnected by channels) is usually called effective pore space. We will assume that the pore space includes only effective pore space for simplicity and hence we will omit the term “effective”.

2.2 *Groundwater*

Typical porous media considered in hydrogeology are soils, sands, gravels, porous rocks such as sandstones, and fractured crystalline rocks such as basalts, granites, and gneisses. In general, the pore space of these porous media can be filled by air, vapor, and liquid phase of water. Part of the porous medium where all pores are filled by the water in liquid phase is called *saturated zone* and the part where the pores contain gaseous phase (air and vapor) and also liquid phase (of water) for at least part of the time is called *aeration zone*. For the purposes of this paper, the term *groundwater* is limited to the water present beneath Earth’s surface in the saturated zone. Mathematical models presented in this paper are restricted to the motion of water in saturated zone.

2.3 *Aquifer*

Note that the saturated zone can be either overlain by an impermeable layer (of rock or clay) or it can have a free upper surface, which is called water table. The water table is characterized as a surface where the pressure is equal to the atmospheric pressure. An *aquifer* is such saturated zone which allows groundwater flow. The aquifer with the free upper surface is called *unconfined aquifer* while the aquifer enclosed between two impermeable layers is called *confined aquifer*. In the presented paper, we are interested in groundwater flow through unconfined aquifer.

2.4 *Velocities and Flux*

The real velocity of the groundwater in the porous medium is highly and unpredictably fluctuating in space and time due to irregularity of the channels and their

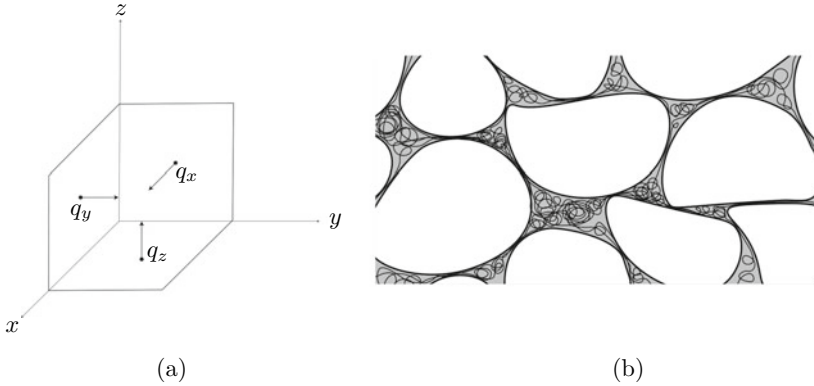


Fig. 1 Flow through porous medium. (a) specific discharge (or Darcy velocity) versus (b) streamlines of the real velocity field of the fluid

joints (and due to turbulence for high values of the Reynolds number). Thus, the real velocity is useless for the practical purposes. Instead, *average velocity* (which can be measured in practice) is used. Let us choose Cartesian coordinate system xyz , with z being the vertical axis. Now, let us consider cross-sectional area A_x perpendicular to x -axis. Let Q_x be the volume of water that passes through A_x per unit of time. The sign of Q_x is positive, if the water (in bulk) passes through A_x in the direction of axis x and negative otherwise. Then

$$q_x \stackrel{\text{def}}{=} \frac{Q_x}{A_x}.$$

In analogous way, we define q_y and q_z . Then, $\vec{q} = (q_x, q_y, q_z)$ and $q = \sqrt{q_x^2 + q_y^2 + q_z^2}$. The quantity \vec{q} is called *specific discharge* or *Darcy velocity*. We also define *average velocity* $\vec{v} = \vec{q}/n$, where n is porosity. Similarly, $v \stackrel{\text{def}}{=} |\vec{v}| = q/n$. This approach works for any incompressible fluid (Fig. 1).

2.5 Groundwater Energy and Piezometric Head

The total mechanical energy of a unit volume of groundwater (or any other incompressible fluid) is the sum of gravitational potential energy, pressure energy, and kinetic energy

$$E_T = z\rho g + P + \frac{1}{2}\rho v^2,$$

see, e.g., [59]. Here, v stands for the magnitude of average velocity of the flow, see above. Groundwater is losing total energy while flowing due to friction with porous

medium. Thus, its total energy decreases in the direction of the flow. The *total head* h_T is the height of the fictive column of groundwater with the gravitational potential energy equal to E_T , i.e.,

$$h_T = z + \frac{P}{\rho g} + \frac{1}{2g} v^2.$$

Since the average velocity of the groundwater flow in real situations is maximally of the order of a meter per day (that is, of the order 0.00001 m/s), the term corresponding to kinetic energy is negligible and can be dropped. In this way, we obtain *piezometric head*

$$h = z + \frac{P}{\rho g},$$

which is the state variable in the mathematical models of underground movement. Constitutive relations between specific discharge and piezometric head were observed by in-field observations [67] as well as experimentally established in laboratory conditions [15]. In general (for isotropic medium), these relations can be written as

$$q = \Phi \left(\frac{\Delta h}{\Delta L} \right),$$

where Φ is some nondecreasing function such that $\Phi(0) = 0$. Note that the constitutive laws are inferred from experiments for one-dimensional flow. However, groundwater flow in the real world is three dimensional. The properties of the isotropic porous medium are the same in all directions. Thus, in this case, the three-dimensional constitutive law can be inferred from the one-dimensional one in a straightforward manner, taking into account that the specific discharge takes the opposite direction of the gradient of the piezometric head and no flow occurs if the gradient of the piezometric head is zero, i.e.,

$$\vec{q} = \begin{cases} \vec{0} & \text{for } \nabla h = \vec{0}, \\ -\Phi(|\nabla h|) \frac{\nabla h}{|\nabla h|} & \text{for } \nabla h \neq \vec{0}. \end{cases} \quad (2.1)$$

In particular, we obtain the linear Darcy law (1.1) for

$$\Phi(r) = k r, \quad r > 0, \quad (2.2)$$

the Smreker–Izbash–Missbach power law (1.2) for

$$\Phi(r) = c r^{\frac{1}{m}}, \quad r > 0, \quad (2.3)$$

$1 < m < 2$, inverse Forchheimer law (inverse formula to (1.3)) for

$$\Phi(r) = \frac{\sqrt{a^2 + 4br} - a}{2b} = \frac{2r}{\sqrt{a^2 + 4br} + a}, \quad r > 0. \quad (2.4)$$

It has been observed by King [45] that the flow of water in low-permeable clays obeys (2.3) with $0 < m < 1$. The work [68, p. 239] contains an overview of values of m for various materials where m ranges from 0.27 to 0.89 (note that $1/m = n$, exponent n taken from [68, Table (A), Appendix I, p. 239]). Recently, it has been found that very slow filtration (i.e., $0 < m < 1$) occurs in petroleum and gas extraction from tight shales reservoirs. For laboratory experiments with real fluids and media, see [25, 63].

2.6 Problem of the Free Surface, the Dupuit–Forchheimer Assumption, and Simplified Problem

In the case of unconfined aquifers, the free surface of the groundwater is the upper boundary of the aquifer. Thus, we need to solve a partial differential equation for both an unknown $h = h(x, y, z, t)$ and an unknown bounded domain $\Theta \equiv \Theta(t)$ in \mathbb{R}^3 that represents the aquifer.

In 1863, Dupuit [21] simplified the problem of unknown boundary by observing that the maximal piezometric head loss per length $\Delta h/\Delta L$ is between 0.001 and 0.01 in typical unconfined aquifers and unconfined aquifer is bounded from below by horizontal impermeable layer. Based on these observations, he formulated the following assumptions on the flow:

- (DF1) groundwater flows horizontally (and thus piezometric head is constant in vertical direction z) and
- (DF2) the Darcy law (1.1) applies to this flow (Dupuit assumed that the groundwater flow is slow enough at these values of piezometric head loss per length so that the nonlinear effects can be neglected).

We will derive a simplified model of groundwater flow in unconfined aquifer using the assumption (DF1). We also assume that the lower boundary of the aquifer formed by impermeable layer is the xy -plane. We choose Ω a bounded domain in \mathbb{R}^2 such that orthogonal projection to $\Theta(t)$ to the xy -plain is contained in Ω for every $t \in [0, T]$. We remind that for a fixed x, y, t , the point (x, y, z) belongs to the water table if and only if $h(x, y, z, t) = z$. In case there is no water above $(x, y) \in \Omega$ at t , we extend the definition of the water table to contain the point $(x, y, 0)$. We assume that the water table is the graph of a function of x, y , and t , that is, there exists nonnegative and sufficiently smooth function $H : \mathbb{R}^2 \times [0, T] \rightarrow \mathbb{R}$ such that $h(x, y, H(x, y, t), t) = H(x, y, t)$. Then the mass of the water column stacked above arbitrary two-dimensional disk $A \subset \Omega$ at time t is

$$m_A(t) = \int_A n \varrho_{\text{water}} H(x, y, t) dx dy.$$

Hence, the integral form of mass conservation law for water (or any other incompressible fluid) has the following form:

$$m_A(t_2) - m_A(t_1) + \int_{t_1}^{t_2} \int_{\partial A} H(x, y, t) \vec{j}(x, y, t) \cdot \vec{n} \, ds = \int_{t_1}^{t_2} \int_A f(x, y, t) \, dx dy dt, \tag{2.5}$$

where $\vec{j} = n \varrho_{\text{water}} \vec{v}$ is the mass flow, \vec{n} is normal vector of ∂A , and f quantifies the sources or absorption in column over the point (x, y) at time t . Let us recall that $\vec{j}(x, y, z, t) = \vec{j}(x, y, t)$ by the assumption (DF1).

Using $m_A(t_2) - m_A(t_1) = \int_{t_1}^{t_2} m'_A(t) \, dt = \int_A \int_{t_1}^{t_2} n \varrho_{\text{water}} \frac{\partial H}{\partial t}(x, y, t) \, dt dx dy$, and the divergence theorem on the second term in (2.5), we arrive at

$$\int_A \int_{t_1}^{t_2} n \varrho_{\text{water}} \frac{\partial H}{\partial t}(x, y, t) \, dt dx dy - \int_{t_1}^{t_2} \int_A \text{div}(H(x, y, t) \varrho_{\text{water}} \vec{q}) \, dx dy dt = \int_{t_1}^{t_2} \int_A f(x, y, t) \, dx dy dt.$$

Since the integral identity is valid for any test disk $A \subset \Omega$ and any interval $[t_1, t_2] \subset [0, T]$, we infer the local form of the mass conservation law

$$n \frac{\partial H}{\partial t}(x, y, t) - \text{div}(H(x, y, t) \vec{q}) = \frac{f(x, y, t)}{\varrho_{\text{water}}}$$

a.e. in $\Omega \times [0, T]$. Since $h(x, y, \cdot, t) \equiv \text{const.}$ by the assumption (DF1) and $h(x, y, H(x, y, t), t) = H(x, y, t)$ on the water table, $h(x, y, z, t) \equiv h(x, y, t) = H(x, y, t)$ and

$$n \frac{\partial h}{\partial t}(x, y, t) - \text{div}(h(x, y, t) \vec{q}) = \frac{f(x, y, t)}{\varrho_{\text{water}}}. \tag{2.6}$$

By (DF2), we apply the Darcy law, i.e., (2.1) with (2.2) to conclude

$$\frac{\partial h}{\partial t} - \frac{k}{n} \text{div}(|h| \nabla h) = f(x, y, t) \tag{2.7}$$

with a little bit of abuse of notation (“hiding” multiplicative constants into f). Note that we can assume that $k/n = 1$ since we can get rid of this multiplicative constant by a linear substitution in the time variable.

Based on numerous experiments and in-field observations summarized in [26], Ph. Forchheimer [26, see p. 1782 and “Anhang,” pp. 1787–1788] pointed out that the assumption (DF2) (i.e., the Darcy law (1.1)) is not accurate enough for piezometric head loss per length greater than 0.0005 for certain porous media (sands) and thus (1.3) has to be used instead while the assumption (DF1) is still applicable. Following Forchheimer, we apply Forchheimer law, i.e., (2.1) with (2.4) to conclude

$$\frac{\partial h}{\partial t} - \frac{1}{n} \text{div} \left(\frac{2|h| \nabla h}{\sqrt{a^2 + 4b|\nabla h|} + a} \right) = f(x, y, t). \tag{2.8}$$

Or alternatively, we apply Smreker–Izbash–Missbach law, i.e., (2.1) with (2.3) to conclude

$$\frac{\partial h}{\partial t} - \frac{c}{n} \operatorname{div} (|h| |\nabla h|^{p-2} \nabla h) = f(x, y, t), \quad (2.9)$$

where $p = 1 + 1/m$. It turns out that the equation (2.9) is easier to handle both theoretically and computationally and thus it is preferred in the literature.

3 Leibenson's Equation and Flow of the Natural Gas

Following Leibenson [49], we assume that the porous medium is nondeformable, isotropic, and homogeneous at macroscopic scale with constant porosity n and the gas is a homogeneous mixture. The condition on the gas ensures that its density depends on the pressure only. We also suppose that the examined thermodynamic process is *polytropic*, i.e., it obeys the following relation:

$$\frac{P}{\varrho^\gamma} = \beta^\gamma. \quad (3.1)$$

Here, $x \in \mathbb{R}^3$, $\varrho = \varrho(t, x)$ is the density, $P = P(t, x)$ is the pressure, $\gamma > 1$ is the *polytropic index* of the process, and $\beta > 0$ is a constant. The flow of the gas (as of any fluid) in the porous medium is governed by continuity equation in the form

$$n \frac{\partial \varrho}{\partial t} + \operatorname{div} (\varrho \vec{q}) = 0 \quad (3.2)$$

and an appropriate constitutive law which relates specific discharge $\vec{q} = n \vec{v}$ and pressure gradient ∇P . Specific discharge is volumetric flux per unit area and the term $\varrho \vec{q}$ represents mass flux per unit area. We refer to [6, Sect. 6.2] for derivation of (3.2) for homogeneous mixture.

For compressible fluid, the specific discharge \vec{q} does not provide relevant information and mass flux must be used instead. In this way, a similar power law for compressible gas subjected to polytropic process,

$$\varrho \vec{q} = -C |\nabla P_1|^{s-1} \nabla P_1, \quad \frac{1}{2} < s < 1, \quad (3.3)$$

was experimentally established, where $P_1 = P^{(\gamma+1)/\gamma}$ (see Leibenson [49]).

Plugging (3.3) into (3.2), we obtain

$$n \frac{\partial}{\partial t} \left(\frac{P_1^{\frac{1}{\gamma+1}}}{\beta} \right) - C \operatorname{div} (|\nabla P_1|^{s-1} \nabla P_1) = 0$$

by (3.1). This equation is often called the equation of turbulent polytropic filtration of gas in porous medium and it has attracted attention of many researches, see, e.g., [4, 17, 19, 20, 24, 36].

4 Turbulence in Porous Medium and Real-World Observations

The turbulence in porous medium was probably first *conjectured* from the experimentally established deviations from the Darcy law by Pavlovskii [56], who proposed to use the Reynolds number for the distinction of the validity range of the linear Darcy law from the validity range of nonlinear laws. He also observed that formula for the Reynolds number in the porous medium must be different than the one for a pipe. He proposed a definition suitable for grained porous media (e.g., sand or gravel) formed of grains of approximately the same diameter. His formula reads

$$\text{Re} = \frac{6.5 q d \rho}{\mu (0.75 n + 0.23)},$$

where d is effective diameter of the grain, ρ is density of the incompressible fluid (water), and μ is its dynamic viscosity. For this definition of Reynolds number, it follows from the experiments that the Darcy law (2.2) is valid if the value of Re is approximately below 50 to 60 (the boundary between the two cases is somewhat blurred) and, for higher values of Re , the Smreker–Izbash–Missbach law (2.3) with $1 < m < 2$ or the Forchheimer law (2.4) must be used instead. According to V. I. Aravin and S. N. Numerov [1, p. 4 and p. 33], this was the first time in [56] when such specification of ranges of the Reynolds number appears in the literature. As pointed out in [1, p. 33], the value of the Reynolds number when the Darcy law becomes inaccurate does not have to be the same as the critical value of the Reynolds number when the turbulence in the flow occurs. Nowadays, it is known that there are at least three ranges of Reynolds number with three different laws:

- pre-Darcy law (2.3) with $0 < m < 1$ for very low values of the Reynolds number;
- Darcy law (2.2) for moderate values of the Reynolds number;
- post-Darcy law (2.3) with $1 < m < 2$ or (2.4) for high values of the Reynolds number.

To get the picture complete, experimental study of flow through porous media over the complete flow regime is presented in, e.g., [2, 3, 66, 68].

Since the constitutive law can become nonlinear even in the laminar regime (as pointed out by [1, p. 33]), we are often asked at conferences if the turbulence in the flow through the porous medium was indeed observed in the laboratory. The modern laboratory techniques can indeed capture the structure of turbulent vortexes, see the recent paper [78].

5 Functional Framework

In this section we survey relevant existence, uniqueness and regularity results concerning generalized solutions of doubly nonlinear parabolic equations. There are several approaches to generalized solutions of (singular/degenerate) doubly nonlinear parabolic equations, see, e.g., [16, 18, 22, 36, 52]. For our purposes, we chose the least technical approach presented in the survey paper by Ivanov [36] (for the complete proofs of results surveyed in [36], see [35, 37] for $p > 1$ and [38] for $p > 2$).

Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 1$, and $T > 0$. We assume that Ω has $C^{0,1}$ -boundary (i.e., Lipschitz boundary) $\partial\Omega$, see [58]. Then $Q_T \stackrel{\text{def}}{=} \Omega \times (0, T]$, $S_T \stackrel{\text{def}}{=} \partial\Omega \times [0, T]$ and $\Gamma_T \stackrel{\text{def}}{=} S_T \cup (\Omega \times \{0\})$. We will use standard function spaces for parabolic problems and, for the convenience of the reader, their traditional notation, which is often different from the notation used in [36]. By $C([0, T] \rightarrow L^q(\Omega))$, for $q \geq 1$, we denote the space of all measurable functions v on Q_T such that the mapping $t \mapsto v(\cdot, t)$ acting from $[0, T]$ to $L^q(\Omega)$ is continuous, i.e.,

$$\lim_{n \rightarrow +\infty} |t_n - t| = 0 \Rightarrow \lim_{n \rightarrow +\infty} \|v(\cdot, t_n) - v(\cdot, t)\|_{L^q(\Omega)} = 0$$

for every sequence $\{t_n\}_{n=1}^{+\infty}$, $t_n \in [0, T]$ and $t \in [0, T]$.

By $L^p([0, T] \rightarrow W^{1,p}(\Omega))$, we mean a space of all measurable functions on Q_T such that $v(\cdot, t) \in W^{1,p}(\Omega)$ for a.e. $t \in [0, T]$ and

$$\|v\|_{L^p([0, T] \rightarrow W^{1,p}(\Omega))} \stackrel{\text{def}}{=} \left(\int_0^T \|v(\cdot, t)\|_{W^{1,p}(\Omega)}^p dt \right)^{1/p} < +\infty.$$

Note that if $v \in L^p([0, T] \rightarrow W^{1,p}(\Omega))$ then the trace of $v(\cdot, t)$ on $\partial\Omega$ is defined for a.e. $t \in [0, T]$.

Finally, by $C^{\lambda, \lambda/p}(\overline{Q_T})$ we mean a space of all continuous functions v on $\overline{Q_T}$ such that

$$\|v\|_{C^{\lambda, \lambda/p}(\overline{Q_T})} = \max_{(x,t) \in \overline{Q_T}} |v(x, t)| + \sup_{\substack{(x,t), (y,s) \in \overline{Q_T} \\ (x,t) \neq (y,s)}} \frac{|u(x, t) - u(y, s)|}{|x - y|^\lambda + |t - s|^{\lambda/p}} < +\infty.$$

We consider the prototype initial-boundary-value problem

$$\begin{cases} \frac{\partial v}{\partial t} - \operatorname{div} \vec{a}(x, t, v, \nabla v) = f(x, t) & \text{in } Q_T; \\ v = \psi & \text{on } \Gamma_T, \end{cases} \quad (5.1)$$

where $f \in L^\infty(Q_T)$, and $\psi \in C^{\lambda, \lambda/p}(\overline{Q_T})$ such that $\hat{\psi} \stackrel{\text{def}}{=} \psi^{1/(p-1)+1} \in L^p([0, T] \rightarrow W^{1,p}(\Omega))$, are given functions.

The following *structural hypotheses* on the Carathéodory function \vec{a} are assumed for a.e. $(x, t) \in Q_T$ and any $s \in \mathbb{R}$ and any $\vec{r} \in R^N$:

$$\vec{a}(x, t, s, \vec{r}) \cdot \vec{r} \geq \nu_0 |s|^l |\vec{r}|^p - \mu_0 (1 + |s|^\delta), \tag{5.2}$$

$$|\vec{a}(x, t, s, \vec{r})| \leq \nu_1 |s|^l |\vec{r}|^{p-1} + \mu_1 |s|^{l/p}. \tag{5.3}$$

Here $p > 1, l \geq 0, \nu_0, \nu_1 > 0$, and $\mu_0, \mu_1 \geq 0$ are certain given constants. Moreover, $0 \leq \delta < l + p$ is given constant for $l + p > 2$ and $\delta = 2$ for $1 < l + p \leq 2$. Note that these structural assumptions are satisfied in the particular case of the equation (1.4).

Note that Ω with Lipschitz boundary satisfies the following *structural hypothesis* from [36] (so-called *property of positive geometric density*) on the boundary $\partial\Omega$:

$$\exists \alpha_* \in (0, 1) \quad \exists \varrho_* > 0 \quad \forall x_0 \in \partial\Omega \quad \forall \rho \in (0, \varrho_*]: \text{meas}(\Omega \cap B_\rho(x_0)) \leq (1 - \alpha_*) \text{meas}(B_\rho(x_0)). \tag{5.4}$$

From [36, Def. 1.1 and Def. 2.1], we adapt the following notion of weak solution.

Definition 1 A nonnegative function $v \in L^\infty(Q_T)$ is a *weak solution* (*supersolution, subsolution*) if

- (a) $v \in C([0, T] \rightarrow L^1(\Omega)), \partial v^{\sigma+1} / \partial x_i \in L^p(Q_T)$ for $\sigma \stackrel{\text{def}}{=} l/(p-1), i = 1, \dots, N$, and $\hat{v} \stackrel{\text{def}}{=} v^{\sigma+1} \in L^p([0, T] \rightarrow W^{1,p}(\Omega))$.
- (b) for any $\phi \in C_0^1(Q_T)$ and any $t_1, t_2 \in [0, T]$,

$$\int_\Omega v \phi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega \left\{ -v \frac{\partial \phi}{\partial t} + \vec{a}(x, t, v, \vec{v}_x) \cdot \nabla \phi - f \phi \right\} dx dt = 0, \tag{5.5}$$

($\phi \geq 0$, for supersolution: ≥ 0 , for subsolution: ≤ 0), where $\vec{v}_x \stackrel{\text{def}}{=} (v_{x_1}, v_{x_2}, \dots, v_{x_N})$ and

$$v_{x_i} \stackrel{\text{def}}{=} \begin{cases} (1 + \sigma)^{-1} v^{-\sigma} \frac{\partial \hat{v}}{\partial x_i} & \text{in } \{(x, t) \in Q_T : v > 0\}, \\ 0 & \text{in } \{(x, t) \in Q_T : v = 0\}. \end{cases} \tag{5.6}$$

- (c) v coincide with ψ on Γ_T , that is,

$$v = \psi \quad \text{on } S_T \quad \text{in the sense of } v^{\sigma+1} \text{ trace}; \tag{5.7}$$

$$\lim_{t \rightarrow 0^+} \|v(\cdot, t) - \psi(\cdot, 0)\|_{L^1(\Omega)} = 0. \tag{5.8}$$

This definition makes sense, cf. [36, p. 24], in the general case $p > 1, l > 1 - p$. However, we limit ourselves to $p > 1$ and $l \geq 0$, which are values that appear in our models. Note that the conditions (5.7) and (5.8) do not appear in [36, Definition 2.1]

explicitly, however, they are mentioned in previous works by the same author, see, e.g., [38, Definition 1.2].

The following result is a basic weak comparison principle for the weak sub- and supersolutions of the doubly nonlinear equation from the initial-boundary-value problem (5.1).

Proposition 1 *cc(see [36, Prop. 4.1]) Let the assumptions (5.2), (5.3) be fulfilled. Assume that $v_1 \in L^p([0, T] \rightarrow W^{1,p}(\Omega))$ is a subsolution of the equation*

$$\frac{\partial v}{\partial t} - \operatorname{div} \vec{a}(x, t, v, \nabla v) = f_1(x, t) \quad \text{in } Q_T, \quad (5.9)$$

and $v_2 \in L^p([0, T] \rightarrow W^{1,p}(\Omega))$ is a supersolution of the equation

$$\frac{\partial v}{\partial t} - \operatorname{div} \vec{a}(x, t, v, \nabla v) = f_2(x, t) \quad \text{in } Q_T, \quad (5.10)$$

where $f_1, f_2 \in L^\infty(Q_T)$. If

$$v_1 \leq v_2 \quad \text{on } \Gamma_T \text{ (in the sense of traces)}, \quad \text{and } f_1 \leq f_2 \quad \text{in } Q_T \quad (5.11)$$

then, for any $\tau \in (0, T]$, we have

$$\int_{\Omega} (v_1(x, \tau) - v_2(x, \tau))^+ dx \leq \int_{\Omega} (v_1(x, 0) - v_2(x, 0))^+ dx. \quad (5.12)$$

From this proposition, we easily obtain uniqueness of weak solutions in the class $L^p([0, T] \rightarrow W^{1,p}(\Omega))$.

Proposition 2 (see [36, Prop. 4.2]) *Let assumptions (5.2), (5.3) be fulfilled. Then there is at most one weak solution of the initial-boundary-value problem (5.1) belonging to $L^p([0, T] \rightarrow W^{1,p}(\Omega))$.*

Note that in the case of the doubly nonlinear equation, a weak solution v is assumed to satisfy $v^{\sigma+1} \in L^p([0, T] \rightarrow W^{1,p}(\Omega))$ for $\sigma = l/(p-1)$, which reduces to $v \in L^p([0, T] \rightarrow W^{1,p}(\Omega))$ provided $l = 0$. For $l \neq 0$, weak solutions to (5.1) do not need to be of class $L^p([0, T] \rightarrow W^{1,p}(\Omega))$, in general. Note that if $\inf_{Q_T} v > 0$ then $v^{\sigma+1} \in L^p([0, T] \rightarrow W^{1,p}(\Omega))$ implies $v \in L^p([0, T] \rightarrow W^{1,p}(\Omega))$.

Proposition 3 (see [36, Theorems 5.3 and 6.1]) *Let $p > 1$ and either*

$$l \geq 0, \quad p + l \geq 2$$

or

$$1 < p + l \leq 2.$$

Moreover, assume

$$\Omega \subset \mathbb{R}^N \text{ is a bounded domain satisfying (20),} \tag{5.13}$$

$$f \in L^\infty(Q_T), \quad f \geq 0 \text{ a.e. in } Q_T, \tag{5.14}$$

$$\psi \in L^p\left([0, T] \rightarrow W_0^{1,p}(\Omega)\right) \cap C^{\beta,\beta/p}(\overline{Q_T}), \psi \geq 0, \text{ for some } \beta \in (0, 1). \tag{5.15}$$

Then there exists exactly one quasi-strong solution of the Cauchy–Dirichlet problem

$$\begin{cases} \frac{\partial v}{\partial t} - \operatorname{div}(|v|^l |\nabla v|^{p-2} \nabla v) = f(x, t) & \text{in } Q_T; \\ v = \psi & \text{on } \Gamma_T, \end{cases}$$

which is Hölder continuous on $\overline{Q_T}$.

Moreover, $\nabla(v^{\alpha+1}) \in L^p(Q_T)$, with $\alpha = l/p$, and

$$\sup_{(x,t),(x',t') \in Q_T} \frac{|v(x, t) - v(x', t')|}{|x - x'|^\lambda + |t - t'|^{\lambda/p}} \leq K$$

with some $\lambda \in (0, 1)$, $K > 0$ depending only on $N, p, l, \|f\|_{L^\infty(Q_T)}, \operatorname{meas} \Omega, T, \alpha^*$, and ρ^* (from condition (5.4)), $\|\psi\|_{L^p([0,T] \rightarrow W_0^{1,p}(\Omega))}$, $\|\psi\|_{C^{\beta,\beta/p}(\overline{Q_T})}$, and $\beta \in (0, 1)$.

Proposition 4 (see [36, Theorem 3.1]) *Let $1 < p < 2$, $p + l \geq 2$ and assume that the structural conditions (5.2) and (5.3) are satisfied. Moreover, suppose that*

- (a) *for a.e. $(x, t) \in Q_T$ and any $s \in \mathbb{R}$ there exist $\nu_1 > 0$ and $\vec{b} = \vec{b}(x, t, s) \in \mathbb{R}^N$, $|\vec{b}(x, t, s)| < +\infty$, such that for a.e. $(x, t) \in Q_T$ and all $s \in \mathbb{R}$ and $\vec{r}_1, \vec{r}_2 \in \mathbb{R}^N$*

$$[\vec{a}(x, t, s, \vec{r}_1) - \vec{a}(x, t, s, \vec{r}_2)] \cdot (\vec{r}_1 - \vec{r}_2) \geq \nu_1 |s|^l |\vec{r}_1 - \vec{r}_2|^2 \left\{ |\vec{r}_1 - \vec{b}|^p + |\vec{r}_2 - \vec{b}|^p \right\}^{1 - \frac{2}{p}}$$

holds.

- (b) *for a.e. $(x, t) \in Q_T$ and any $\vec{r} \in \mathbb{R}^N$, the functions $s^{-\alpha} a_i(x, t, s, \vec{r})$ and $s^{-\alpha} a_i(x, t, s, s^{-\alpha} \vec{r})$ are continuous on \mathbb{R} with respect to s . Here $\alpha = 1/p$.*
- (c) *$\psi(x, t)$ is nonnegative in Q_T , $\psi \in L^p\left([0, T] \rightarrow W_0^{1,p}(\Omega)\right) \cap L^\infty(Q_T)$, and we have the Hölder condition*

$$\sup_{(x,t),(x',t') \in \overline{Q_T}} \frac{|\psi(x, t) - \psi(x', t')|}{(|x - x'|^p + |t - t'|)^{\gamma_0/p}} \leq K_0$$

for some $K_0 > 0$ and $\gamma_0 \in (0, 1)$.

Then there exists a weak solution v of the Cauchy–Dirichlet problem (5.1) which is Hölder continuous in $\overline{Q_T}$. Moreover, $\nabla(v^{\alpha+1}) \in L^p(Q_T)$, $\alpha = 1/p$, and the estimate

$$\sup_{(x,t),(x',t') \in Q_T} \frac{|v(x, t) - v(x', t')|}{(|x - x'|^p + |t - t'|)^{\gamma/p}} \leq K$$

holds with constants $K > 0$ and $\gamma \in (0, 1)$ dependent only on the dimension N , the known parameters from (5.2), (5.3), a)–b), the constants α_* and ϱ_* , $|\Omega|$, T , $\|\psi\|_{W^{1,p}(Q_T)}$, $\sup_{Q_T}(\psi)$, γ_0 , and K_0 .

The following result stated in [38] guarantees the existence of a solution of (5.1) with time-dependent boundary conditions. Let us emphasize that the result is valid only for $p > 2$. As far as we know, a similar result has not been proved for $1 < p < 2$ yet. In Proposition 5, we use Einstein's summation convention as in [38].

Proposition 5 (see [38, Theorem 1.1]) *Let $p > 2$ and assume that the structural conditions (5.2) and (5.3) are satisfied. Moreover, suppose that*

(a) *for any $s \in \mathbb{R}$, $\vec{r}_1, \vec{r}_2 \in \mathbb{R}^N$ and a.e. $(x, t) \in Q_T$,*

$$(a_i(x, t, s, \vec{r}_1) - a_i(x, t, s, \vec{r}_2)) \cdot (r_{1,i} - r_{2,i}) \geq \nu_1 |s|^l |\vec{r}_1 - \vec{r}_2|^p$$

with $\nu_1 = \text{const.} > 0$.

(b) *for a.e. $(x, t) \in Q_T$ and all $\vec{r} \in \mathbb{R}^N$, the limit*

$$\lim_{s \rightarrow 0^+} s^{-\alpha} a_i(x, t, s, s^{-\alpha} \vec{r}), \quad \alpha = \frac{l}{p},$$

exists.

(c) *for any $\vec{r} \in \mathbb{R}^N$ and a.e. $(x, t) \in Q_T$,*

$$a_i(x, t, s, \vec{r}) r_i - f(x, t) s > -c_1 s^2, \quad c_1 = \text{const.} > 0,$$

for all $s < 0$.

(d) *concerning the function $\psi(x, t)$, $(x, t) \in \overline{Q_T}$, defining the boundary condition in (5.1), $\psi(x, t) \geq 0$, $\psi \in W^{1,p}(Q_T) \cap L^\infty(Q_T)$ and we have the Hölder condition*

$$\sup_{(x,t),(x',t') \in \overline{Q_T}} \frac{|\psi(x, t) - \psi(x', t')|}{(|x - x'|^p + |t - t'|)^{\gamma_0/p}} \leq K_0$$

for some $K_0 > 0$ and $\gamma_0 \in (0, 1)$.

Then problem (5.1) has at least one nonnegative weak solution $v(x, t)$ for which

$$\sup_{(x,t),(x',t') \in Q_T} \frac{|v(x, t) - v(x', t')|}{(|x - x'|^p + |t - t'|)^{\gamma/p}} \leq K$$

holds with $K > 0$ and $\gamma \in (0, 1)$.

6 Maximum and Comparison Principles

In case of singular and/or degenerate parabolic equations, it is impossible to find explicit solutions except for very rare cases, thus we heavily rely on qualitative methods of their study combined with numerical computations. *Maximum and comparison principles* play a prominent role among the qualitative methods. To remind what maximum and comparison principles are, let us start with the well-known *elliptic* Dirichlet Laplacian problem. Let $u_i \in W^{1,2}(\Omega)$, $i = 1, 2$, be the weak solutions of

$$-\Delta u_i = f_i(x) \quad \text{in } \Omega,$$

$f_i \in L^\infty(\Omega)$, in a bounded domain $\Omega \subset \mathbb{R}^N$. The *weak* comparison principle states that if $f_1 \leq f_2$ in Ω and $u_1 \leq u_2$ on $\partial\Omega$ (in the sense of traces) then $u_1 \leq u_2$ in Ω . The *strong* comparison principle states that if, moreover, $f_1 \not\equiv f_2$ in Ω or $u_1 \not\equiv u_2$ on $\partial\Omega$ then $u_1 < u_2$ in Ω . In particular, the strong comparison principle says that $f_1 < f_2$ in a small part of Ω of positive measure (and $f_1 \equiv f_2$ elsewhere) is sufficient to have $u_1 < u_2$ everywhere in Ω .

Similar principles hold for the *parabolic* Cauchy–Dirichlet Laplacian problem. Let $u_i \in L^2([0, T] \rightarrow W^{1,2}(\Omega))$, $i = 1, 2$, be the weak solutions of

$$\frac{\partial u_i}{\partial t} - \Delta u_i = f_i(x, t) \quad \text{in } Q_T,$$

$f_i \in L^\infty(Q_T)$. Notice that this equation is a special case of (5.1) with $\vec{a}(x, t, s, \vec{r}) = \vec{r}$ satisfying both (5.2) and (5.3) with $p = 2$. If $f_1 \leq f_2$ in Q_T and $u_1 \leq u_2$ on Γ_T (in the sense of traces) then $u_1 \leq u_2$ in Q_T (weak comparison principle, cf. Proposition 1). If, moreover, at least one of the following three conditions holds:

- $f_1 \not\equiv f_2$ in $\Omega \times (0, t_0)$ whenever $0 < t_0 \leq T$,
- $u_1 \not\equiv u_2$ on $\Omega \times \{0\}$ (in the sense of traces),
- $u_1 \not\equiv u_2$ on $\partial\Omega \times (0, t_0)$ (in the sense of traces) whenever $0 < t_0 \leq T$,

then $u_1 < u_2$ in Q_T (strong comparison principle).

For the linear case $p = 2$, it is usual to prove the *maximum* principles first since the comparison principles come forth as a consequence. Let $u \in L^2([0, T] \rightarrow W^{1,2}(\Omega))$ be the weak solution of

$$\frac{\partial u}{\partial t} - \Delta u = f(x, t) \quad \text{in } Q_T,$$

$f \in L^\infty(Q_T)$. The *weak* maximum principle states that if $f \geq 0$ in Q_T then $u \geq M \stackrel{\text{def}}{=} \text{ess inf}_{\Gamma_T} u$ (in the sense of traces) in Q_T . We note that although it would make more sense to call this statement a *minimum* principle and to call a *maximum principle* that $f \leq 0$ implies $u \leq \text{ess sup}_{\Gamma_T} u$, these two are equivalent (we get one from the other replacing u by $-u$) and thus we use only the term *maximum* principle. The

strong maximum principle states that if, moreover, at least one of the following three conditions holds:

- $f \neq 0$ in $\Omega \times (0, t_0)$ whenever $0 < t_0 \leq T$,
- $u \neq M$ on $\Omega \times \{0\}$ (in the sense of traces),
- $u \neq M$ on $\partial\Omega \times (0, t_0)$ (in the sense of traces) whenever $0 < t_0 \leq T$,

then $u > M$ in Q_T .

Once the maximum principle (weak or strong) is proved, the comparison principle (weak or strong, respectively) is easily obtained choosing $u = u_2 - u_1$ (thus $M \geq 0$) and $f = f_2 - f_1$. Notice that the linearity of the left-hand side of the equation is used. Conversely, if we have the comparison principle in our hands, the respective maximum principle can be derived choosing $u_1 \equiv M$, $f_1 \equiv 0$, $u_2 = u$ and $f_2 = f$ (no linearity is used here).

Let us now replace the Laplacian by the p -Laplace operator

$$\Delta_p u \stackrel{\text{def}}{=} \operatorname{div} (|\nabla u|^{p-2} \nabla u) = \operatorname{div} \vec{a}(x, t, u, \nabla u),$$

$p > 1$, where $\vec{a}(x, t, s, \vec{r}) = |\vec{r}|^{p-2} \vec{r}$ satisfies both (5.2) and (5.3). Similarly as above, the comparison principle implies the respective maximum principle. But since the operator is nonlinear, the maximum principle does not imply the comparison principle. In other words, the maximum principle is weaker because it is only a comparison with the constant solution. Moreover, the uniqueness of the weak solution is a consequence of the weak comparison principle (cf. Proposition 2) but not a consequence of the maximum principle.

As for the *elliptic* Dirichlet p -Laplacian problem for

$$-\Delta_p u = f(x) \quad \text{in } \Omega,$$

both the *weak* maximum and the *weak* comparison principle can be proved in a standard way choosing an appropriate test function. Basically, the *weak* comparison principle states that the p -Laplacian is a monotone operator. The *strong* maximum principle was proved by Vázquez in 1984 [73]. The *strong* comparison principle was proved by Cuesta and Takáč in 1998 [12] provided $0 \leq f_1 \leq f_2$, $f_1 \neq f_2$ and $u \equiv 0$ on $\partial\Omega$ (they focus on the influence of the right-hand side rather than the boundary data).

While the *weak* maximum and the *weak* comparison principle for the *parabolic* Cauchy–Dirichlet p -Laplacian problem for

$$\frac{\partial u}{\partial t} - \Delta_p u = f(x, t) \quad \text{in } Q_T \tag{6.1}$$

is still standard (see Proposition 1), the *strong* maximum and comparison principle is much more involved when $p \neq 2$. It follows from Barenblatt [4] that we cannot expect the strong maximum principle in the degenerate case $p > 2$ (weak diffusion) even locally in time. Indeed, an explicit radially symmetric solution $u(x, t) \equiv \varrho(|x|, t) =$

$\varrho(r, t)$, $r = |x|$, of (6.1) with $f \equiv 0$, is obtained from the well-known Barenblatt solution of [4, Eq. (1.3)]:

$$c \frac{\partial \varrho}{\partial t} = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left[r^{N-1} \left(\frac{\partial}{\partial r} \varrho^k \right) \left| \frac{\partial}{\partial r} \varrho^k \right|^{m-1} \right] \tag{6.2}$$

with $m = p - 1$, $k = 1$, and $c > 0$ a constant. Hence, the case $p > 2$ corresponds to $k > 1/m$. The support of such u (see [4, Fig. 1]) at each particular time is a compact ball with the radius starting from 0 at $t = 0$ (the initial condition is the Dirac distribution located at the origin) and increasing in time at finite speed. Consequently, if we choose Ω a ball in \mathbb{R}^N and an initial time in which the support of the solution is a smaller ball (replacing t by $t + \varepsilon$ with an $\varepsilon > 0$ small enough in [4]), then $u \not\equiv M = 0$ on $\Omega \times \{0\}$ and $u \not\equiv 0$ in Q_T since $u = 0$ in a part of Ω (spherical shell) for positive times until the support of the solution hits $\partial\Omega$. Another counterexample to the strong comparison principle in one spatial dimension where $u_1 \equiv u_2$ on Γ_T , $f_1 \leq f_2$, $f_1 \not\equiv f_2$ but $u_1 \not\leq u_2$ is presented in [9]. On the other hand, a certain stronger condition on the separation of f_1 and f_2 that guarantee the strong comparison principle is formulated in [9].

Even in the singular case $1 < p < 2$ (strong diffusion) the strong maximum principle cannot hold for arbitrarily large T . It follows from the extinction in finite time (see DiBenedetto [19, Sect. 2 of Chap. VII.]) which implies that if $u > 0$ on $\Omega \times \{0\}$, $u \equiv 0$ on $\partial\Omega \times (0, T)$ and $f \equiv 0$ in Γ_T then $u(\cdot, t)$ vanishes in Ω for t large enough. Hence, the strong maximum principle $u > M = 0$ does not hold globally in time. A time-local version of the strong maximum principle was proved in [7] for even more general doubly nonlinear equation

$$\frac{\partial}{\partial t} b(u(x, t)) - \Delta_p u = f(x, t) \quad \text{in } Q_T, \tag{6.3}$$

where $b: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous function, $b(0) = 0$, and $b \in C^1(0, +\infty)$ with $b' > 0$ in $(0, +\infty)$. Notice that if $b(s) \equiv s$ then (6.3) reduces to (6.1).

Theorem 1 (see [7, Theorem 1.1]) *Let $1 < p < 2$ and*

$$\lim_{s \rightarrow 0^+} \frac{s^{2-p} b'(s)}{|\log s|^{p-1}} = 0. \tag{6.4}$$

Assume that $u: \overline{\Omega} \times [0, T) \rightarrow \mathbb{R}_+$ is a continuous, nonnegative, weak solution of (6.3). Then, for any fixed $t_0 \in (0, T)$, the solution $u(\cdot, t_0)$ is either positive everywhere on Ω or else identically zero on Ω .

In particular, if $u(\xi, 0) > 0$ for some $\xi \in \Omega$, then there exists $\tau \in (0, T]$ such that $u(x, t) > 0$ for all $(x, t) \in \Omega \times (0, \tau)$, i.e., the strong maximum principle is valid in the $(N + 1)$ -dimensional space-time cylinder $\Omega \times (0, \tau)$. The number $\tau \in (0, T)$ can be estimated as

$$\tau = \sup\{T' \in (0, T]: u(\xi, t) > 0 \text{ for all } t \in [0, T']\} > 0.$$

Notice that $u(x, t) \equiv \varrho^k(|x|, t)$ where ϱ is the Barenblatt solution of (6.2) is a solution of (6.3) where $b(s) = s^{1/k}$, $p = m + 1$, and $f \equiv 0$. If $k \leq 1/m$, i.e., $k \leq 1/(p - 1)$, then the Barenblatt solution is positive everywhere in \mathbb{R}^N for any positive time. In other words, the speed of propagation is infinite, and it is reasonable to expect at least the time-local strong maximum principle to hold in this case. Indeed, for $b(s) = s^{1/k}$ the condition (6.4) reads as

$$\lim_{s \rightarrow 0^+} \frac{s^{1-p+1/k}}{k |\log s|^{p-1}} = 0$$

which is satisfied if and only if $1 - p + 1/k \geq 0$, i.e., $k \leq 1/(p - 1)$. Obviously, condition (6.4) is natural and matches the Barenblatt result perfectly.

7 Basic Models

7.1 Parallel Ditches

Our first model is related to irrigation and drainage. Irrigation is especially important in agriculture while drainage is very important in building and construction. We assume that aquifer is homogeneous, isotropic, and resting on a horizontal impermeable layer. Bottom of all ditches reaches the impermeable layer and the water levels in all ditches are at equal elevation. In our first model, we will consider two infinite parallel ditches and we will study transient groundwater flow between them with the possible recharge due to rain. For the sketch of the problem, see the vertical cross-section perpendicular to the ditches in Fig. 2, where we place the axis x to be perpendicular to the ditches and $x = 0$ is set to be exactly in the middle between two ditches. Such problem has been intensively studied in [5, 51, 64, 65] (and others, see references therein). In the aforementioned works, Darcy law is used as constitutive law. Following Forchheimer's observations from [26], we use nonlinear Smreker–Izbash–Missbach law instead. Thus, the governing equation is (2.9), i.e.,

$$\frac{\partial h}{\partial t} - \operatorname{div}(|h| |\nabla h|^{p-2} \nabla h) = f(x, y, t) \geq 0. \quad (7.1)$$

We suppose that the problem is translation invariant with respect to y -axis, i.e., a possible recharge is described by $f(x, y, t) \equiv f(x, t) \geq 0$. Thus, $h(x, y, t) \equiv h(x, t)$ and Equation (2.9) reduces to

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x} \left(|h| \left| \frac{\partial h}{\partial x} \right|^{p-2} \frac{\partial h}{\partial x} \right) = f(x, t). \quad (7.2)$$

$$U(x, t) = \frac{1}{(t + \tau)^\lambda} \left[C - k \frac{|x - x_0|^{p'}}{(t + \tau)^{\frac{p'}{2p-1}}} \right]_+^{p'}$$

with

$$p' = \frac{p}{p-1}, \quad \lambda = \frac{p}{2p^2 - 3p + 1}, \quad k = \left(\frac{p-1}{p} \right)^2 \left(\frac{1}{2p-1} \right)^{\frac{p}{p-1}},$$

and some $\tau > 0$ and $C > 0$ such that $h_0(x) \leq U(x, 0)$ and

$$[x_0 - \delta, x_0 + \delta] \subset \text{supp } U(\cdot, 0) \subsetneq \left(-\frac{L}{2}, \frac{L}{2} \right).$$

Then

$$\text{supp } h(\cdot, t) \subset \left(x_0 - \left(\frac{C}{k} \right)^{\frac{1}{p'}} (t + \tau)^{\frac{1}{2p-1}}, x_0 + \left(\frac{C}{k} \right)^{\frac{1}{p'}} (t + \tau)^{\frac{1}{2p-1}} \right).$$

This means that the water from the localized sudden recharge $h_0(x)$ does not reach any of the shores immediately.

Case $H > 0$. In contrast, if both the water level in the ditches and the water table are at constant level $H > 0$, then the localized sudden recharge $h_0 - H \geq 0$ with $\text{supp}(h_0 - H) = [x_0 - \delta, x_0 + \delta] \subset (-L/2, L/2)$ and $h_0(x_0) > 0$ will cause immediate rise of the water table in the whole aquifer between the ditches. In order to apply theory from Sects. 5 and 6, we introduce a substitution $v(x, t) = h(x, t) - H$ and we arrive at

$$\left\{ \begin{array}{ll} \frac{\partial v}{\partial t} - \frac{\partial}{\partial x} \left(|v + H| \left| \frac{\partial v}{\partial x} \right|^{p-2} \frac{\partial v}{\partial x} \right) = f(x, t) & \text{for } (x, t) \in \left(-\frac{L}{2}, \frac{L}{2} \right) \times (0, T), \\ v \left(\pm \frac{L}{2}, t \right) = 0 & \text{for } t \in (0, T), \\ v(x, 0) = h_0(x) - H & \text{for } x \in \left(-\frac{L}{2}, \frac{L}{2} \right). \end{array} \right. \quad (7.3)$$

For any $(x, t) \in Q_T$, we set

$$a(x, t, s, r) \stackrel{\text{def}}{=} \begin{cases} H |r|^{p-2} r & \text{for } s < 0, \\ (s + H) |r|^{p-2} r & \text{for } s \in [0, M], \\ (M + H) |r|^{p-2} r & \text{for } s > M, \end{cases} \quad (7.4)$$

where $M = \|h_0 - H\|_{L^\infty(\Omega)} + T \|f\|_{L^\infty(Q_T)}$. Then $a(x, t, s, r)$ given by (7.4) satisfies the assumptions of Proposition 4 with $l = 0$ and $\frac{3}{2} < p < 2$. Then by Proposi-

tion 4, Cauchy–Dirichlet problem (5.1) possesses the unique quasi-strong solution $v \in L^p([0, T] \rightarrow W^{1,p}(-L/2, L/2))$ for $a(x, t, s, r)$ given by (7.4), $\psi \in L^p([0, T] \rightarrow W_0^{1,p}(-L/2, L/2))$ is an extension of $h_0 - H$. Since v is a subsolution of (5.9) with $f_1 = f$ and $\bar{v} = \|h_0 - H\|_{L^\infty(\Omega)} + t\|f\|_{L^\infty(Q_T)}$ is a supersolution of (5.10) with $f_2 = \|f\|_\infty$. Thus, by Proposition 1, $(0 \leq)v \leq \bar{v} \leq M$ on Q_T . It follows that v is also the weak solution of (7.3). Since

$$\frac{\partial v}{\partial x} = \frac{\partial(v + H)}{\partial x} \quad \text{and} \quad (v + H)^{\frac{1}{p-1}} \frac{\partial(v + H)}{\partial x} = \frac{p - 1}{p} \frac{\partial(v + H)^{\frac{p}{p-1}}}{\partial x}$$

we may rewrite PDE in (7.3) as

$$\frac{\partial v}{\partial t} - \left(\frac{p - 1}{p}\right)^{p-1} \frac{\partial}{\partial x} \left(\left| \frac{\partial(v + H)^{\frac{p}{p-1}}}{\partial x} \right|^{p-2} \frac{\partial(v + H)^{\frac{p}{p-1}}}{\partial x} \right) = f(x, t).$$

Introducing another substitution $u = (v + H)^{p/(p-1)} - H^{p/(p-1)}$, we arrive at

$$\frac{\partial \left(\left(u + H^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} - H \right)}{\partial t} - \left(\frac{p - 1}{p}\right)^{p-1} \frac{\partial}{\partial x} \left(\left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right) = f(x, t), \quad (7.5)$$

which is in fact (6.3) with $b(s) = \left(s + H^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} - H$. Note that $b(s)$ is a continuous function, $b(0) = 0$, $b \in C^1(0, +\infty)$ with $b' > 0$ in $(0, +\infty)$, and it satisfies (6.4) from Theorem 1. Since $v \in C^{\gamma, \gamma/p}(Q_T)$, $u = (v + H)^{p/(p-1)} - H^{p/(p-1)}$ is a continuous weak solution of (7.5) (and (6.3)), Theorem 1 guarantees the existence of $\tau \in (0, T)$ such that $u(x, t) > 0$ for all $(x, t) \in \Omega \times (0, \tau)$. In particular, for $f = 0$ this means that localized sudden recharge $h_0 - H$ causes the immediate water table rise in the whole aquifer between the ditches.

Conclusion. *In the case of dry ditches ($H = 0$), the water from a localized sudden recharge does not reach the shores of the ditches immediately and the boundaries of the water mound expand toward the ditches with finite speed. In contrast, for the flooded ditches ($H > 0$), the localized sudden recharge causes the immediate water table rise in whole aquifer between the ditches. In the real world, all movements take place at finite speeds. Thus, the above results should be interpreted as follows: for $H > 0$, the water mound expands toward the ditches much faster than for $H = 0$.*

7.2 Isothermic Nanoporous Filtration of Natural Gas

The shales are increasingly gaining importance in natural gas extraction due to their abundance in the world in comparison with classical gas reservoirs. The size of pores

and channels in shales is of order of several nanometers which leads to extremely low permeability and the standard mathematical models fail in this situation, see [54] for more details. Thus, in [54], the following mathematical model of isothermic nanoporous filtration of natural gas was proposed (we slightly change their notation in order not to interfere with ours)

$$\left\{ \begin{array}{ll} \frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \left(|P(x)| \left| \frac{\partial P}{\partial x} \right|^{p-2} \frac{\partial P}{\partial x} \right) & \text{for } (x, t) \in (0, +\infty) \times (0, T), \\ P(0, t) = P_0 & \text{for } t \in (0, T), \\ \lim_{x \rightarrow \infty} P(x, t) = \bar{P} & \text{for } t \in (0, T), \\ P(x, 0) = P_0 & \text{for } x \in (0, +\infty), \end{array} \right.$$

for $2 < p < 10$, $P_0 \geq 0$, $\bar{P} > 0$ being given constants. This model was analyzed using self-similarity of solutions in [54]. Using the methods of Sect. 5, we can analyze the problem in situations which are not self-similar including time-varying boundary conditions, but only on a bounded interval for x . Note that, e.g., if we assume $(x, t) \in Q_T \stackrel{\text{def}}{=} (0, L) \times (0, T]$, for some $L > 0$, $P(0, t) = P_0(t) \geq 0$, $P(L, t) = P_L(t) \geq 0$ for $t \in [0, T]$ and $P(x, 0) = P_{\text{init}}(x) \geq 0$ for $x \in [0, L]$ are Lipschitz functions such that $P_0(0) = P_{\text{init}}(0)$, $P_L(0) = P_{\text{init}}(L)$. Then Proposition 5 guarantees existence of at least one weak solution on $Q_T = (0, L) \times (0, T]$ together with a priori bounds on its Hölder norm.

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References

1. Aravin, V.I., Numerov, S.N.: *Teoriya dvizheniya zhidkosti i gazov v nedeformiruemoi poristo- toj srede*. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. English Transl. by A. Moscona: “Theory of Fluid Flow in Undeformable Porous Media”, Israel Program for Scientific Trans- lations, Jerusalem (1965)
2. Bağcı, Ö., Dukhan, N., Özdemir, M.: Flow regimes in packed beds of spheres from pre-darcy to turbulent. *Transp. Porous Media* **104**(3), 501–520 (2014)
3. Banerjee, A., Pasupuleti, S., Singh, M.K., Dutta, S.C., Kumar, G.N.P.: Modelling of flow through porous media over the complete flow regime. *Transp. Porous Media* **129**(1), 1–23 (2019)
4. Barenblatt, G.I.: On some unsteady motions of a liquid and gas in a porous medium. *Akad. Nauk SSSR. Prikl. Mat. Meh.* **16**, 67–78 (1952). In Russian
5. Basak, P.: An analytical solution for the transient ditch drainage problem. *J. Hydrol.* **41**(3), 377–382 (1979)
6. Bear, J.: *Dynamics of Fluids in Porous Media*. Dover Civil and Mechanical Engineering Series, Dover Publications Inc, New York (2014)

7. Benedikt, J., Girg, P., Kotrla, L., Takáč, P.: The strong maximum principle in parabolic problems with the p -Laplacian in a domain. *Appl. Math. Lett.* **63**, 95–101 (2017)
8. Benedikt, J., Girg, P., Kotrla, L., Takáč, P.: Origin of the p -Laplacian and A. Missbach. *Electron. J. Differ. Equ. Paper No. 16*, 17 pp (2018)
9. Benedikt, J., Girg, P., Kotrla, L., Takáč, P.: The strong comparison principle in parabolic problems with the p -Laplacian in a domain. *Appl. Math. Lett.* **98**, 365–373 (2019)
10. Bordier, C., Zimmer, D.: Drainage equations and non-darcian modelling in coarse porous media or geosynthetic materials. *J. Hydrol.* **228**(3), 174–187 (2000)
11. Bressler, S., Shaviv, G., Shaviv, N.: The sensitivity of the greenhouse effect to changes in the concentration of gases in planetary atmospheres. *Acta Polytech.* **53**(SUPPL. 1), 832–838 (2013)
12. Cuesta, M., Takáč, P.: A strong comparison principle for the Dirichlet p -Laplacian. In *Reaction diffusion systems. Lecture Notes in Pure and Applied Mathematics*, vol. 194. Dekker, Trieste, New York **1998**, pp. 79–87 (1995)
13. Czech Hydrometeorological Institute. Current Observations of Watertable in Shallow Boreholes [Data] (2020). [http://hydro.chmi.cz/hpps/hpps_pzv_list.php?&objtyp\[\]=p&objtyp\[\]=m&objtyp\[\]=h#](http://hydro.chmi.cz/hpps/hpps_pzv_list.php?&objtyp[]=p&objtyp[]=m&objtyp[]=h#) [Online; accessed 04-February-2020]. In *Czech*
14. Dai, A.: Drought under global warming: a review. *Wiley Interdiscip. Rev.: Clim. Change* **2**(1), 45–65 (2011)
15. Darcy, H.: *Les fontaines publiques de la ville de Dijon*. Victor Dalmont, Paris (1856)
16. Díaz, J.I.: Qualitative study of nonlinear parabolic equations: an introduction. *Extracta Math.* **16**(3), 303–341 (2001)
17. Díaz, J.I., de Thélin, F.: On a nonlinear parabolic problem arising in some models related to turbulent flows. *SIAM J. Math. Anal.* **25**(4), 1085–1111 (1994)
18. Díaz, J.I., Padial, J.F.: Uniqueness and existence of solutions in the $BV_r(Q)$ space to a doubly nonlinear parabolic problem. *Publ. Mat.* **40**(2), 527–560 (1996)
19. DiBenedetto, E.: *Degenerate Parabolic Equations*. Universitext. Springer, New York (1993)
20. DiBenedetto, E., Gianazza, U., Vespi, V.: *Harnack's Inequality for Degenerate and Singular Parabolic Equations*. Springer Monographs in Mathematics, Springer, New York (2012)
21. Dupuit, J.: *Études théoriques et pratiques sur le mouvement des eaux dans les canaux découverts et à travers les terrains perméables*. Dunod, Paris (1863)
22. Ebmeyer, C., Urbano, J.M.: The smoothing property for a class of doubly nonlinear parabolic equations. *Trans. Am. Math. Soc.* **357**(8), 3239–3253 (2005)
23. Eck, B., Barrett, M., Charbeneau, R.: Forchheimer flow in gently sloping layers: Application to drainage of porous asphalt. *Water Resour. Res.* **48**, 1 (2012)
24. Esteban, J.R., Vázquez, J.L.: On the equation of turbulent filtration in one-dimensional porous media. *Nonlinear Anal.* **10**(11), 1303–1325 (1986)
25. Farmani, Z., Azin, R., Fatehi, R., Escrochi, M.: Analysis of pre-darcy flow for different liquids and gases. *J. Petrol. Sci. Eng.* **168**, 17–31 (2018)
26. Forchheimer, P.: *Wasserbewegung durch boden*. *Zeit. Ver. Deutsch. Ing.* **45**, 1736–1741 and 1781–1788 (1901)
27. Geoscience Australia: (2014). <http://www.ga.gov.au/scientific-topics/water/groundwater/fractured-rocks>, Accessed 21-February-2020
28. Goldman Sachs: *The Essentials of Investing in the Water Sector*, version 2.0 (2008). <http://venturecenter.co.in/water/pdf/2008-goldman-sachs-water-primer.pdf>. [Online; accessed 04-February-2020]
29. Gustafson, G., Krásný, J.: Crystalline rock aquifers: their occurrence, use and importance. *Appl. Hydrogeol.* **2**(2), 64–75 (1994)
30. Hall, J., Arheimer, B., Borga, M., Brázdil, R., Claps, P., Kiss, A., Kjeldsen, T.R., Kriaučiūnienė, J., Kundzewicz, Z.W., Lang, M., Llasat, M.C., Macdonald, N., McIntyre, N., Mediero, L., Merz, B., Merz, R., Molnar, P., Montanari, A., Neuhold, C., Parajka, J., Perdigão, R.A.P., Plavcová, L., Rogger, M., Salinas, J.L., Sauquet, E., Schär, C., Szolgay, J., Viglione, A., Blöschl, G.: Understanding flood regime changes in Europe: a state-of-the-art assessment. *Hydrol. Earth Syst. Sci.* **18**(7), 2735–2772 (2014)

31. Hinze, J., John, R.: Effects of heat on the dispersal performance of ips typographus. *J. Appl. Entomol.* **144**(1–2), 144–151 (2020)
32. Huang, J., Kautz, M., Trowbridge, A., Hammerbacher, A., Raffa, K., Adams, H., Goodson, D., Xu, C., Meddens, A., Kandasamy, D., Gershenson, J., Seidl, R., Hartmann, H.: Tree defence and bark beetles in a drying world: carbon partitioning, functioning and modelling. *New Phytol.* **225**(1), 26–36 (2020)
33. Huntington, T.G.: Evidence for intensification of the global water cycle: review and synthesis. *J. Hydrol.* **319**(1), 83–95 (2006)
34. Hussain, A., Classens, G., Guevara-Rozo, S., Cale, J.A., Rajabzadeh, R., Peters, B.R., Erbilgin, N.: Spatial variation in soil available water holding capacity alters carbon mobilization and allocation to chemical defenses along jack pine stems. *Environ. Exp. Bot.* **171**, 103902 (2020)
35. Ivanov, A.V.: The classes $B_{m,l}$ and Hölder estimates for quasilinear parabolic equations that admit double degeneration. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **197**, Kraev. Zadachi Mat. Fiz. Smezh. Voprosy Teor. Funktsii. **23**, 42–70, 179–180 (1992)
36. Ivanov, A.V.: Regularity for doubly nonlinear parabolic equations. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **209**, Voprosy Kvant. Teor. Polya i Statist. Fiz. **12**, 37–59, 261 (1994)
37. Ivanov, A.V.: Existence and uniqueness of a regular solution of the Cauchy-Dirichlet problem for doubly nonlinear parabolic equations. *Z. Anal. Anwendungen* **14**(4), 751–777 (1995)
38. Ivanov, A.V., Mkrtchyan, P.Z.: On the regularity up to the boundary of generalized solutions of the first initial-boundary value problem for quasilinear parabolic equations that admit double degeneration. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **196**, Modul. Funktsii Kvadrat. Formy. **2**, 83–98, 173–174 (1991). Translated in *J. Math. Sci.* **70**(6), 2112–2122 (1994), 35K65 (35D10)
39. JRC European Drought Observatory: Drought in Europe (2019). https://edo.jrc.ec.europa.eu/documents/news/EDODroughtNews201908_Europe.pdf. Accessed 04-February-2020
40. JRC Global Drought Observatory (GDO): Drought in New South Wales (Australia), (2019). https://edo.jrc.ec.europa.eu/documents/news/GDODroughtNews201910_South-East_Australia.pdf. Accessed 04-February-2020
41. JRC Global Drought Observatory (GDO) and ERCC Analytical Team : Drought in India (2019). https://edo.jrc.ec.europa.eu/documents/news/GDODroughtNews201906_India.pdf. Accessed 04-February-2020
42. JRC Global Drought Observatory (GDO) and ERCC Analytical Team : Drought in Southern Africa (2019). https://edo.jrc.ec.europa.eu/documents/news/GDODroughtNews201901_SouthernAfrica.pdf. Accessed 04-February-2020
43. JRC Global Drought Observatory (GDO) and ERCC Analytical Team: Drought in Southern Africa (2019). https://edo.jrc.ec.europa.eu/documents/news/GDODroughtNews201912_Southern_Africa.pdf. Accessed 04-February-2020
44. Kalašnikov, A.S.: On the concept of finite rate of propagation of perturbations. *Uspekhi Mat. Nauk* **34**, 2(206), 199–200 (1979)
45. King, F.: Principles and conditions of the movements of ground water. *Nineteenth Ann. Kept. U. S. Geol. Survey* pt. **2**, 9–12 (1898), 209–215
46. Lanci, L., Galeotti, S., Grimani, C., Huber, M.: Evidence against a long-term control on earth climate by galactic cosmic ray flux. *Global Planet. Change* **185** (2020)
47. Laut, P.: Solar activity and terrestrial climate: an analysis of some purported correlations. *J. Atmos. Solar Terr. Phys.* **65**(7), 801–812 (2003)
48. Leibenson, L.S.: Turbulent movement of gas in a porous medium. *Bull. Acad. Sci. USSR. Sér. Géophys. Géophys. [Izvestia Akad. Nauk SSSR]* **9**, 3–6 (1945). In Russian. Reprinted in Ref. [49], 499–502
49. Leibenson, L.S.: Sbranie trudov, Chast' II: Podzemnaya gidrodinamika [Collected Works, Vol. II: Underground Hydrodynamics]. Izdat'elstvo Akademii Nauk S.S.S.R., Moscow, U.S.S.R. (1953). In Russian
50. Macdonald, A., Davies, J., Calow, R.: African hydrogeology and rural water supply. *Appl. Groundwater Stud. Afr.*, 127–148 (2008)

51. Marino, M.: Rise and decline of the water table induced by vertical recharge. *J. Hydrol.* **23**(3–4), 289–298 (1974)
52. Matas, A., Merker, J.: Existence of weak solutions to doubly degenerate diffusion equations. *Appl. Math.* **57**(1), 43–69 (2012)
53. Mekonnen, M.M., Hoekstra, A.Y.: Four billion people facing severe water scarcity. *Sci. Adv.* **2**, 2 (2016)
54. Monteiro, P., Rycroft, C., Barenblatt, G.: A mathematical model of fluid and gas flow in nanoporous media. *Proc. Natl. Acad. Sci. U.S.A.* **109**(50), 20309–20313 (2012)
55. Netherer, S., Matthews, B., Katzensteiner, K., Blackwell, E., Henschke, P., Hietz, P., Pennerstorfer, J., Rosner, S., Kikuta, S., Schume, H., Schopf, A.: Do water-limiting conditions predispose norway spruce to bark beetle attack? *New Phytol.* **205**(3), 1128–1141 (2015)
56. Pavlovskii, N.N.: The theory of movement of ground water under hydraulic structures and its main applications. Scientific Amelioration Institute, St. Petersburg, lecture notes. Lithographic (1922). In Russian
57. Perrin, J., Ahmed, S., Hunkeler, D.: The effects of geological heterogeneities and piezometric fluctuations on groundwater flow and chemistry in a hard-rock aquifer, southern india. *Hydrogeol. J.* **19**(6), 1189–1201 (2011)
58. Pick, L., Kufner, A., John, O., Fučík, S.: Function spaces. Vol. 1, extended ed., *De Gruyter Series in Nonlinear Analysis and Applications*, vol. 14. Walter de Gruyter & Co., Berlin (2013)
59. Sen, Z.: Applied Hydrogeology for Scientists and Engineers. CRC Press, Boca Raton (1995)
60. Shapiro, A.M.: Fractured-rock aquifers understanding an increasingly important source of water (2002). <https://toxics.usgs.gov/pubs/FS-112-02/fs-112-02.pdf>. Accessed 21-February-2020
61. Shaviv, N., Veizer, J.: Celestial driver of phanerozoic climate? *GSA Today* **13**(7), 4–10 (2003)
62. Shaviv, N.J.: On climate response to changes in the cosmic ray flux and radiative budget. *J. Geophys. Res. Space Phys.* **110**, A8 (2005)
63. Siddiqui, F., Soliman, M., House, W., Ibragimov, A.: Pre-darcy flow revisited under experimental investigation. *J. Anal. Sci. Technol.* **7**, 1 (2016)
64. Singh, R., Rai, S.: On subsurface drainage of transient recharge. *J. Hydrol.* **48**(3–4), 303–311 (1980)
65. Singh, R., Rai, S.: A solution of the nonlinear boussinesq equation for phreatic flow using an integral balance approach. *J. Hydrol.* **109**(3–4), 319–323 (1989)
66. Sivanapillai, R., Steeb, H., Hartmaier, A.: Transition of effective hydraulic properties from low to high reynolds number flow in porous media. *Geophys. Res. Lett.* **41**(14), 4920–4928 (2014)
67. Smreker, O.: Entwicklung eines Gesetzes für den Widerstand bei der Bewegung des Grundwassers. *Zeitschr. des Vereines deutscher Ing.*, **22**, 117–128 and 193–204 4 and 5 (1878)
68. Soni, J., Islam, N., Basak, P.: An experimental evaluation of non-darcian flow in porous media. *J. Hydrol.* **38**(3–4), 231–241 (1978)
69. Stereńczak, K., Mielcarek, M., Kamińska, A., Kraszewski, B., Żaneta Piasecka, Miścicki, S., Heurich, M.: Influence of selected habitat and stand factors on bark beetle *Ips typographus* (L.) outbreak in the Białowięza forest. *Forest Ecol. Manag.* **459**, 117826 (2020)
70. Svensmark, H., Friis-Christensen, E.: Variation of cosmic ray flux and global cloud coverage - a missing link in solar-climate relationships. *J. Atmos. Solar Terr. Phys.* **59**(11), 1225–1232 (1997)
71. Trenberth, K.: Changes in precipitation with climate change. *Climate Res.* **47**(1–2), 123–138 (2011)
72. U.S.: Energy Information Administration. Carbon dioxide emissions coefficients (2016). https://www.eia.gov/environment/emissions/co2_vol_mass.php. Accessed 13-March-2020
73. Vázquez, J.L.: A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.* **12**(3), 191–202 (1984)
74. Vicente-Serrano, S., McVicar, T., Miralles, D., Yang, Y., Tomas-Burguera, M.: Unraveling the influence of atmospheric evaporative demand on drought and its response to climate change. *Climate Change, Wiley Interdisciplinary Reviews* (2019)

75. Vicente-Serrano, S., Quiring, S., Peña Gallardo, M., Yuan, S., Domínguez-Castro, F.: A review of environmental droughts: Increased risk under global warming? *Earth-Sci. Rev.* **201** (2020)
76. Westra, S., Fowler, H., Evans, J., Alexander, L., Berg, P., Johnson, F., Kendon, E., Lenderink, G., Roberts, N.: Future changes to the intensity and frequency of short-duration extreme rainfall. *Rev. Geophys.* **52**(3), 522–555 (2014)
77. Wright, E.: The hydrogeology of crystalline basement aquifers in Africa. *Geol. Soc. Spec. Pub.* **66**, 1–27 (1992)
78. Ziazi, R.M., Liburdy, J.A.: Vortical structure characteristics of transitional flow through porous media. In: Volume 1: Fluid Mechanics Fluids Engineering Division Summer Meeting. V001T01A070 (2019)