Basic Fixed Point Theorems in Metric Spaces

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Abstract This chapter is a review work on the development of metric fixed point theory. It begins with the description of Banach's Contraction Mapping Principle and finally contains results established in the recent years as well. The proofs are presented for every theorem discussed here. Several illustrations are given. The development is presented separately for functions with and without continuity property. Only results on metric spaces without any additional structures are considered.

1 Introduction

It is widely held that metric fixed point theory originated in the year 1922 through the work of S. Banach when he established the famous Contraction Mapping Principle [2] which has come to be known by his name. It is a versatile domain of mathematics having implications in several other branches of science, technology and economics [1, 31, 43]. At present even after a century of its initiation, the subject area remains vibrant with research activities.

Admittedly, putting together all basic theorems in metric fixed point theory in a single chapter is an impossible task. One has to be selective on this issue. We do not mean to undermine those results which are left out of our selection. They can even be more important than those which are included in this chapter. For instance,

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Generalized Banach Contraction Conjecture (GBCC) result of Merryfield et al. [26] is not included in this chapter. There are many important fixed point results which are deduced in metric spaces having additional structures like partial order, graph, etc. But here we consider only those results which are relevant to metric spaces without any additional structures. The only additional property which we consider here is the completeness property of the metric space. Further, we describe theorems for mappings with or without continuity assumption.

Definition 1 (*Fixed point*) Let M be a nonempty set and $S : M \to M$ be a mapping. A fixed point of S is a point $\xi \in M$ such that $S\xi = \xi$, that is, a fixed point of S is a solution of the functional equation $Sz = z$, $z \in M$.

A self-mapping may have no fixed point, a unique fixed point and more than one fixed point. This is illustrated in the following examples.

Example 1 Take R the set of all real numbers equipped with usual metric.

- (i) The mapping $S : R \to R$, $Sz = z^3$, $z \in R$ has three fixed points $z = 0$, $z = 1$ and $z = -1$.
- (ii) The mapping $S : R \to R$ defined by $Sz = -z^3$, $z \in R$ has only fixed point $z = 0$.
- (iii) The mapping $S : R \to R$ where $Sz = z + \sin z$, $z \in R$ has fixed points $z = n\pi,\ n = 0, \pm 1, \pm 2, ...$
- (iv) The mapping $S : R \to R$ defined as $Sz = z + 1$, $z \in R$ has no fixed point.

2 Banach's Contraction Mapping Principle

The first result we describe is the famous Contraction Mapping Principle.

Definition 2 (*Contraction mapping*) A mapping $S : M \to M$, where (M, ρ) is a metric space, is called a Lipschitz mapping if there exists a real number $k > 0$ such that $\rho(Su, Sv) \leq k\, \rho(u, v)$ holds for all $u, v \in M$. The smallest positive real number k for which the Lipschitz condition is valid is called the Lipschitz constant of S.

If the Lipschitz constant k lies between 0 and 1, that is, if $0 < k < 1$, then the Lipschitz mapping S is called a contraction mapping.

Obviously, a contraction mapping is continuous.

Example 2 (i) The mapping $S : [0, 1) \to [0, 1)$ defined by $Sz = \frac{z}{5}$ is a contraction mapping.

(ii) The mapping $S : R \to R$ defined by $Sz = \frac{5z+3}{2}$ is not a contraction mapping.

In 1922, Banach established a fixed point result for a self-map S of a complete metric space using a contractive condition, which is known as Banach's contraction mapping principle.

Theorem 1 (Banach's contraction mapping principle [2]) *A self-mapping S of a complete metric space $(M,\ \rho)$ admits a unique fixed point if for all $u,\ v \in M$,*

$$\rho(Su,\ Sv) \leq k\ \rho(u,\ v),\ \text{where } 0 < k < 1. \tag{1}$$

Proof Suppose $\zeta, \eta \in M$ with $\zeta \neq \eta$ are two fixed points of S. From (1), we have $\rho(\zeta, \eta) = \rho(S\zeta, S\eta) \leq k\ \rho(\zeta, \eta)$, which is a contradiction. Hence the fixed point of S, if it exists is unique.

Choose any point $z_0 \in M$. We construct a sequence $\{z_n\}$ in M such that

$$z_n = Sz_{n-1} = S^n z_0 \ \text{ for all }\ n \geq 1. \tag{2}$$

For each positive integer n, we have

$$\begin{aligned}
\rho(z_n,\ z_{n+1}) &= \rho(Sz_{n-1}, Sz_n) \\
&\leq k\ \rho(z_{n-1}, z_n) \\
&\leq k^2\ \rho(z_{n-2}, z_{n-1}) \\
&\quad ... \\
&\leq k^n\ \rho(z_0, z_1).
\end{aligned}$$

By triangular inequality, we have for $n > m$,

$$\begin{aligned}
\rho(z_m, z_n) &\leq \rho(z_m, z_{m+1}) + \rho(z_{m+1}, z_{m+2}) + ... + \rho(z_{n-1}, z_n) \\
&\leq k^m\ \rho(z_0, z_1) + k^{m+1}\ \rho(z_0, z_1) + ... + k^{n-1}\ \rho(z_0, z_1) \\
&\leq k^m[1 + k + k^2 + ... + k^{n-m-1}]\ \rho(z_0, z_1) \\
&< k^m[1 + k + k^2 + ...]\ \rho(z_0, z_1) \\
&= \frac{k^m}{1-k}\ \rho(z_0, z_1) \to 0,\ \text{ as } n \to +\infty \text{ [since } \alpha < 1],
\end{aligned}$$

which implies that $\{z_n\}$ is a Cauchy sequence in M. By the completeness of M, there exists $\xi \in M$ such that $z_n \to \xi$, as $n \to +\infty$.

Being a contraction mapping, S is continuous. Therefore, we have $S\xi = \lim_{n\to+\infty} Sz_n = \lim_{n\to+\infty} z_{n+1} = \xi$. Hence, ξ is a fixed point S. By what we have already proved, ξ is the unique fixed point of S.

Example 3 Take the complete metric space R equipped with usual metric and the contraction mapping $S : R \to R$ defined as $Sz = 2(1 - \frac{z}{5})$. We see that $z = \frac{10}{7}$ is the unique fixed point of S.

3 Generalizations of Contraction Mapping Principle

In 1969, Boyd and Wong [4] made a very interesting generalization of the Banach's contraction mapping principle in complete metric spaces. They replaced the constant k in (1) of Theorem 1 by a function $\varphi : [0, +\infty) \to [0, +\infty)$ which is upper semicontinuous from the right (that is, $t_n \downarrow t \geq 0 \Rightarrow \limsup \varphi(t_n) \leq \varphi(t)$).

The following result is due to Boyd and Wong [4].

Theorem 2 A self-mapping S of a complete metric space (M, ρ) admits a unique fixed point if there exists a function $\varphi : [0, +\infty) \to [0, +\infty)$ which is upper semicontinuous from the right with $0 \leq \varphi(t) < t$ for $t > 0$ and the following inequality holds:

$$\rho(Su, Sv) \leq \varphi(\rho(u, v)), \text{ for all } u,\ v \in M. \tag{3}$$

Proof Let $z_0 \in M$ be any arbitrary element. We define a sequence $\{z_n\}$ in M such that $z_n = Sz_{n-1} = S^n z_0$, for all $n \geq 1$. If $z_l = z_{l+1}$ for some positive integer l, then z_l is a fixed point of S. So we assume that $z_n \neq z_{n+1}$, for all $n \geq 0$.

Applying (3) and using the property of φ, we have

$$\rho(z_{n+1}, z_{n+2}) = \rho(Sz_n, Sz_{n+1}) \leq \varphi(\rho(z_n, z_{n+1})) < \rho(z_n, z_{n+1}), \text{ for all } n \geq 0. \tag{4}$$

Therefore, $\{\rho(z_n,\ z_{n+1})\}$ is a monotonic decreasing sequence which is bounded below by 0 and hence there exists an $\delta \geq 0$ for which

$$\lim_{n\to+\infty} \rho(z_n, z_{n+1}) = \delta. \tag{5}$$

From (4), we have

$$\rho(z_{n+1}, z_{n+2}) \leq \varphi(\rho(z_n, z_{n+1})), \text{ for all } n \geq 0.$$

Taking limit supremum as $n \to +\infty$ on both sides and using (5) and the properties of φ, we have $\delta \leq \varphi(\delta) < \delta$. It is a contradiction unless $\delta = 0$. Hence

$$\lim_{n\to+\infty} \rho(z_n, z_{n+1}) = 0. \tag{6}$$

We prove that $\{z_n\}$ is a Cauchy sequence by method of contradiction. If possible, suppose that $\{z_n\}$ is not a Cauchy sequence. Then we have an $\epsilon > 0$ for which there exist two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that

$$n(k) > m(k) > k,\ \ \rho(z_{m(k)},\ z_{n(k)}) \geq \epsilon \ \text{ and } \ \rho(z_{m(k)},\ z_{n(k)-1}) < \epsilon.$$

Now,

$$\begin{aligned}\epsilon \le \rho(z_{m(k)}, z_{n(k)}) &\le \rho(z_{m(k)}, z_{n(k)-1}) + \rho(z_{n(k)-1},\ z_{n(k)})\\ &< \epsilon + \rho(z_{n(k)-1},\ z_{n(k)}).\end{aligned}$$

Using (6), we have

$$\lim_{k\to+\infty} \rho(z_{m(k)},\ z_{n(k)}) = \epsilon. \tag{7}$$

Again,

$$\begin{aligned}\rho(z_{m(k)}, z_{n(k)}) &\le \rho(z_{m(k)}, z_{m(k)+1}) + \rho(z_{m(k)+1},\ z_{n(k)+1}) + \rho(z_{n(k)}, z_{n(k)+1})\\ &\le \rho(z_{m(k)}, z_{m(k)+1}) + \varphi(\rho(z_{m(k)}, z_{n(k)})) + \rho(z_{n(k)},\ z_{n(k)+1}).\end{aligned}$$

Taking limit supremum as $n \to +\infty$ on both sides of the inequality and using (6), (7) and the properties of φ, we have $\epsilon \le \varphi(\epsilon) < \epsilon$. This is a contradiction. Hence $\{z_n\}$ is a Cauchy sequence. As (M, ρ) is complete, there exists $\xi \in M$ such that $z_n \to \xi$, as $n \to +\infty$.

We now show that ξ is a fixed point of S. It follows by the contraction condition that S is continuous. Therefore, $S\xi = \lim_{n\to+\infty} Sz_n = \lim_{n\to+\infty} z_{n+1} = \xi$. Hence ξ is a fixed point S.

Let z be a fixed point of S other than ξ. Then $\rho(z, \xi) > 0$. From (3), we have $\rho(z, \xi) = \rho(Sz, S\xi) \le \varphi(\rho(z, \xi)) < \rho(z, \xi)$, which is a contradiction. Hence, ξ is the unique fixed point of S.

Example 4 Take the metric space $M = [0, 1]$ equipped with usual metric. Define $S : M \to M$ as $Sz = z - \frac{z^2}{2}$, for $z \in M$. Let $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ be defined by

$$\varphi(t) = \begin{cases} t - \frac{t^2}{2}, & \text{if } 0 \le t \le 1,\\ \frac{t}{2}, & \text{otherwise.}\end{cases}$$

Boyd and Wong fixed point theorem is applicable and $z = 0$ is the unique fixed point of S.

In 1969, Meir and Keeler [25] established that the conclusion of Banach's theorem holds more generally from the following condition of weakly uniformly strict contraction:

Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \le \rho(x, y) < \epsilon + \delta \ \text{ implies } \ \rho(Sx, Sy) < \epsilon. \tag{8}$$

The following result is due to Meir and Keeler [25].

Theorem 3 A self-mapping S of a complete metric space (M, ρ) admits a unique fixed point if (8) holds.

Proof We first observe that (8) implies that

$$\rho(Sx, Sy) < \rho(x, y) \ \text{ whenever } \ x \ne y. \tag{9}$$

Suppose that ζ and η are two distinct fixed points of S. Then from (9), we have $\rho(\zeta, \eta) = \rho(S\zeta, S\eta) < \rho(\zeta, \eta)$, which is a contradiction. Hence S may have at most one fixed point.

Let $z_0 \in M$ be any arbitrary element. Take the same sequence $\{z_n\}$ in M as in the proof of Theorem 2. We take $z_n \neq z_{n+1}$, for all $n \geq 0$. This is because in the case $z_l = z_{l+1}$, for some positive integer l, z_l is a fixed point of S.

Let $c_n = \rho(z_n, z_{n+1})$. From (9), we can show that $\{c_n = \rho(z_n,\ z_{n+1})\}$ is a monotonic decreasing sequence of nonnegative real numbers. Then there exists an $\epsilon \geq 0$ such that $c_n \to \epsilon$, as $n \to +\infty$. If possible, suppose that $\epsilon > 0$. As $\{c_n\}$ is decreasing and $c_n \to \epsilon$, as $n \to +\infty$, for $\delta > 0$ there exists m such that $\epsilon \leq c_n < \epsilon + \delta$ for all $n \geq m$. Therefore, $\epsilon \leq c_m < \epsilon + \delta$. Then from (8) it follows that $c_{m+1} = \rho(z_{m+1}, z_{m+2}) = \rho(Sz_m, Sz_{m+1}) < \epsilon$, which is a contradiction. Hence $\epsilon = 0$. Therefore,

$$\lim_{n \to +\infty} \rho(z_n, z_{n+1}) = 0. \tag{10}$$

We suppose that $\{z_n\}$ is not a Cauchy sequence. Then there exists $2\epsilon > 0$ such that $\limsup \rho(z_m, z_n) > 2\epsilon$. By the hypothesis, there exists a $\delta > 0$ such that

$$\epsilon \leq \rho(x, y) < \epsilon + \delta \ \text{ implies } \ \rho(Sx, Sy) < \epsilon. \tag{11}$$

Formula (11) remains true if we replace δ by $\delta^{'} = \min\{\delta, \epsilon\}$. By (10), there exists a positive integer P for which $c_P < \frac{\delta^{'}}{3}$. Choose $m, n > P$ so that $\rho(z_m, z_n) > 2\epsilon$. Now for any $j \in [m, n]$, we have

$$\mid \rho(z_m, z_j) - \rho(z_m, z_{j+1}) \mid \leq c_j < \frac{\delta^{'}}{3}.$$

This implies, since $\rho(z_m, z_{m+1}) < \epsilon$ and $\rho(z_m, z_n) > \epsilon + \delta^{'}$, that there exists $j \in [m, n]$ with

$$\epsilon + \frac{2\delta^{'}}{3} < \rho(z_m, z_j) < \epsilon + \delta^{'}. \tag{12}$$

However, for all m and j,

$$\rho(z_m, z_j) \leq \rho(z_m, z_{m+1}) + \rho(z_{m+1}, z_{j+1}) + \rho(z_{j+1}, z_j).$$

From (11) and (12), we have

$$\rho(z_m, z_j) \leq c_m + \epsilon + c_j < \frac{\delta^{'}}{3} + \epsilon + \frac{\delta^{'}}{3},$$

which contradicts (12). Therefore, $\{z_n\}$ is a Cauchy sequence.

Now (9) implies that S is continuous. As discussed in the proof of Theorem 1, we conclude that S has a unique fixed point.

Example 5 ([25]) Let $M = [0, 1] \cup \{3, 4, 6, 7, ..., 3n, 3n + 1, ...\}$ be equipped with Euclidean metric and $S : M \to M$ be defined by

$$S(u) = \begin{cases} \frac{u}{2}, & \text{if } 0 \le u \le 1, \\ 0, & \text{if } u = 3n, \\ 1 - \frac{1}{n+2}, & \text{if } u = 3n + 1. \end{cases}$$

Here, Theorem 3 is applicable and the unique fixed point of S is $u = 0$.

It is observed that in Banach's contraction mapping principle, the contraction condition is global, that is, the operators satisfy the contraction condition for every pair of points taken from the metric space. A natural question arises whether the conclusion of Banach's theorem is true if the contraction condition is satisfied locally, that is, for sufficiently close points only. The answer was given in the affirmative in a paper by Michael Edelstein [14] in 1961.

Definition 3 *(Local Contraction* [14]) A self-mapping $S : M \to M$, where (M, ρ) is a metric space, is locally contractive if for every $x \in M$ there exist $\epsilon > 0$ and $\lambda \in [0, 1)$, which may depend on x, such that

$$p, q \in S(x, \epsilon) = \{y : \rho(x, y) < \epsilon\} \text{ implies } \rho(Sp, Sq) < \lambda\, \rho(p, q). \tag{13}$$

Definition 4 (*Uniform Local Contraction* [14]) A uniformly locally contractive mapping on a metric space (M, ρ) is a locally contractive mapping $S : M \to M$ where both ϵ and λ do not depend on x.

Definition 5 ([14]) Let (M, ρ) be a metric space such that for every $a, b \in M$ there exists an η-chain, that is, a finite set of points $a = x_0, x_1, ..., x_n = b$ (n may depend on both a and b) satisfying $\rho(x_{j-1}, x_j) < \eta$ ($j = 1, 2, ..., n$). Then (M, ρ) is η-chainable.

Theorem 4 (Edelstein [14]) *An (ϵ, λ)—uniformly locally contractive mapping $S : M \to M$ on a ϵ-chainable complete metric space (M, ρ) has a unique fixed point.*

Proof Choose any point $z \in M$. Take the ϵ-chain : $z = z_0,\ z_1,\ ...,\ z_n = Sz$. By the triangular property, we have

$$\rho(z, Sz) \le \sum_{1}^{n} \rho(z_{i-1}, z_i) < n\epsilon. \tag{14}$$

For pairs of consecutive points of the ϵ-chain, condition (13) is satisfied. Hence, denoting $S(S^m z) = S^{m+1} z$ ($m = 1, 2, ...$), we have

$$\rho(Sz_{i-1}, Sz_i) < \lambda\, \rho(z_{i-1}, z_i) < \lambda\, \epsilon;$$

and, by repeated application of the above inequality, we have

$$\rho(S^m z_{i-1}, S^m z_i) < \lambda\, \rho(S^{m-1} z_{i-1}, S^{m-1} z_i) < \lambda^m\, \epsilon. \tag{15}$$

Using (14) and (15), we have

$$\rho(S^m z, S^{m+1} z) \le \sum_{i=1}^{n} \rho(S^m z_{i-1}, S^m z_i) < \lambda^m\, n\, \epsilon. \tag{16}$$

Now, for any two positive integers $j, k (j < k)$, we have

$$\begin{aligned}\rho(S^j z, S^k z) &\le \sum_{i=j}^{k-1} \rho(S^i z, S^{i+1} z) < n\epsilon\, [\lambda^j + \lambda^{j+1} + ... + \lambda^{k-1}] \\ &< \frac{\lambda^j}{1-\lambda} n\epsilon \to 0, \text{ as } j \to +\infty.\end{aligned}$$

It follows that $\{S^i z\}$ is a Cauchy sequence in M. Now, M being complete, there exists a point $\xi \in M$ such that $S^i z \to \xi$, as $i \to +\infty$.

Now (13) implies that S is continuous. Therefore, we have $S\xi = \lim_{i\to+\infty} S(S^i z) = \lim_{i\to+\infty} S^{i+1} z = \xi$. Hence ξ is a fixed point S.

If possible, let ζ $(\zeta \ne \xi)$ be another fixed point of S. Now $\rho(\xi, \zeta) > 0$. Let $\xi = z_0, z_1, ..., z_k = \zeta$ be an ϵ-chain. Using (15), we have

$$\begin{aligned}\rho(\xi, \zeta) = \rho(S\xi, S\zeta) &\le \rho(S^l \xi, S^l \zeta) \\ &\le \sum_{i=1}^{k} \rho(S^l z_{i-1}, S^l z_i) < \lambda^l\, k\epsilon \to 0 \text{ as } l \to +\infty,\end{aligned}$$

which is a contradiction. Hence, $\xi = \zeta$ and our proof is completed.

Example 6 Let $M = \{(u, v) : u = \cos\theta,\ \ v = \sin\theta,\ \ 0 \le \theta \le \frac{3}{2}\pi\}$ be equipped with Euclidean metric. Define $S : M \to M$ as $Sp = (\frac{u}{2}, \frac{v}{2})$, for $p = (u, v) \in M$. Theorem 4 is applicable here and $p = (0, 0)$ is the unique fixed point of S.

In 2012, Samet et al. [37] introduced the new concept of $\alpha - \psi$-contractive type mapping and established a fixed point theorem for such mappings in complete metric spaces. The presented theorem therein extends, generalizes and improves the famous Banach's contraction mapping principle. We describe here the notions of $\alpha - \psi$-contractive and α-admissible mappings.

Let Ψ denote the family of nondecreasing functions $\psi : [0,\ +\infty) \to [0,\ +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is nth iterate of ψ.

Lemma 1 ([37]) *If $\psi : [0,\ +\infty) \to [0,\ +\infty)$ be a nondecreasing function satisfying $\lim_{n\to+\infty} \psi^n(t) = 0$ for each $t > 0$, then $\psi(t) < t$ for each $t > 0$.*

Definition 6 ([37]) Let $S : M \to M$ and $\alpha : M \times M \to [0,\ +\infty)$ be two mappings. The mapping T is α-admissible if $\alpha(u,\ v) \ge 1 \Longrightarrow \alpha(Tu,\ Tv) \ge 1$, for $u, v \in M$.

Example 7 Let $M = [0,\ 1]$. Let $S : M \to M$ and $\alpha : M \times M \to [0,\ +\infty)$ be respectively defined as follows:

$$Sz = \frac{\sin^2 z}{16},\ \text{for } z \in M \ \text{ and } \ \alpha(u,\ v) = \begin{cases} e^{u+v}, \ \text{if} \ \ 0 \le u \le 1,\ 0 \le v \le \frac{1}{8}, \\ 0, \ \ \text{otherwise.} \end{cases}$$

Here S is α-admissible.

Definition 7 ([37]) A mapping $T : M \to M$, where (M, d) is a metric space, is called an $\alpha - \psi$-contractive mapping if there exist two functions $\alpha : M \times M \to [0,\ +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(u, v)\ \rho(Tu, Tv) \le \psi(\rho(u, v)),\ \text{for all } u, v \in M.$$

Remark 1 If $\alpha(u, v) = 1$ for all $u, v \in M$ and $\psi(t) = kt$ for all $t \ge 0$ and some $k \in [0, 1)$, the $\alpha - \psi$-contractive mapping reduces to Banach's contraction mapping.

Theorem 5 (Samet et al. [37]) *Let (M, ρ) be a complete metric space, $S : M \to M$ and $\alpha : M \times M \to [0,\ +\infty)$. Suppose that (i) S is α-admissible, (ii) there exists $z_0 \in M$ such that $\alpha(z_0,\ Sz_0) \ge 1$, (iii) S is continuous and (iv) there exists $\psi \in \Psi$ such that S is an $\alpha - \psi$-contractive mapping. Then S admits a fixed point.*

Proof Let $z_0 \in M$ such that $\alpha(z_0,\ Sz_0) \ge 1$. We construct a sequence $\{z_n\}$ in M such that

$$z_{n+1} = Sz_n,\ \text{for all } n \ge 0. \tag{17}$$

Then $\alpha(z_0,\ z_1) \ge 1$. As S is α-admissible, we have $\alpha(Sz_0, Sz_1) = \alpha(z_1, z_2) \ge 1$. Again, applying the admissibility assumption, we have $\alpha(Sz_1, Sz_2) = \alpha(z_2, z_3) \ge 1$. Continuing this process, we have

$$\alpha(z_n, z_{n+1}) \ge 1,\ \text{for all } n \ge 0. \tag{18}$$

Like in the proof of Theorem 2, we show that the possibility of $z_l = z_{l+1}$ occurring, for some positive integer l, ensures that z_l is a fixed point of S. So we consider the case $z_n \ne z_{n+1}$, for all $n \ge 0$.

Applying (iv) with $z = z_{n-1}$ and $y = z_n$, where $n \ge 1$, and using (17) and (18), we obtain

$$\rho(z_n, z_{n+1}) = \rho(Sz_{n-1}, Sz_n) \le \alpha(z_{n-1}, z_n)\ \rho(Sz_{n-1}, Sz_n) \le \psi(\rho(z_{n-1}, z_n)).$$

By repeated the application of the above inequality and a property of ψ, we have

$$\rho(z_n, z_{n+1}) \le \psi^n(\rho(z_0, z_1)),\ \text{for all } n \ge 1.$$

With the help of the above inequality, we have

$$\sum_{n=1}^{+\infty} \rho(z_n,\ z_{n+1}) \leq \sum_{n=1}^{+\infty} \psi^n(\rho(z_0, z_1)) < +\infty,$$

which implies that $\{z_n\}$ is a Cauchy sequence in M. As M is complete, we get $\xi \in M$ such that $\lim_{n\to+\infty} z_n = \xi$. From the continuity of S, it follows that $S\xi = \lim_{n\to+\infty} Sz_n = \lim_{n\to+\infty} z_{n+1} = \xi$. Hence ξ is a fixed point S.

Example 8 ([37]) Take $M = R$ the set of all real numbers endowed with the usual metric ρ. Let $S : M \to M$ be defined as follows:

$$Sz = \begin{cases} 2z - \frac{3}{2}, & \text{if } z > 1, \\ \frac{z}{2}, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{if } z < 0. \end{cases}$$

As $\rho(S1, S2) = 2 > 1 = \rho(2, 1)$, the Banach's contraction mapping principle cannot be applied in this case.

Define $\psi : [0,\ +\infty) \to [0,\ +\infty)$ and $\alpha : M \times M \to [0,\ +\infty)$ as follows:

$$\psi(t) = \frac{t}{2} \text{ and } \alpha(u,\ v) = \begin{cases} 1, & \text{if } u, v \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Here Theorem 5 is applicable and $z = 0$ is a fixed point of S.

In 1973, Geraghty [17] introduced a class of functions to generalize the Banach's contraction mapping principle. Let S be the class of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the property: $\beta(t_n) \to 1, \text{ as } t_n \to 0$.

An example of a function in S may be given by $\beta(t) = e^{-2t}$ for $t > 0$ and $\beta(0) \in [0, 1)$.

Theorem 6 (Geraghty [17]) *A self-mapping S of a complete metric space (M, ρ) admits a unique fixed point if there exists a function $\beta \in S$ such that*

$$\rho(Su, Sv) \leq \beta(\rho(u, v))\ \rho(u, v), \quad \text{for all } u, v \in M. \tag{19}$$

Proof Suppose that S has two fixed points ζ and η with $\zeta \neq \eta$. From (19), we have $\rho(\zeta, \eta) = \rho(S\zeta, S\eta) \leq \beta(\rho(\zeta, \eta))\, \rho(\zeta, \eta) < \rho(\zeta, \eta)$, which is a contradiction. Hence the fixed point of S, if it exists, is unique.

Let $z_0 \in M$ be any arbitrary element. Take the same sequence $\{z_n\}$ in M as in the proof of Theorem 2. Like in the proof of Theorem 2, we show that the possibility of $z_l = z_{l+1}$ occurring, for some positive integer l, implies the existence of a fixed point of S. So we assume that $z_n \neq z_{n+1}$, for all $n \geq 0$.

First we prove $\lim_{n\to+\infty} \rho(z_n, z_{n+1}) = 0$. Applying (19) and using the property of β, we have for all $n \geq 0$,

$$\rho(z_{n+1}, z_{n+2}) = \rho(Sz_n, Sz_{n+1}) \leq \beta(\rho(z_n, z_{n+1}))\ \rho(z_n, z_{n+1}) < \rho(z_n, z_{n+1}). \tag{20}$$

Therefore, $\{\rho(z_n,\ z_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. We get an $\delta \geq 0$ such that $\lim_{n\to+\infty}\ \rho(z_n, z_{n+1}) = \delta$.

Suppose that $\delta > 0$. From (20), we have

$$\frac{\rho(z_{n+1}, z_{n+2})}{\rho(z_n, z_{n+1})} \leq \beta(\rho(z_n, z_{n+1})) < 1, \text{ for all } n \geq 0.$$

Then

$$1 \leq \lim_{n\to+\infty} \beta(\rho(z_n, z_{n+1})) < 1,$$

which implies that

$$\lim_{n\to+\infty} \beta(\rho(z_n, z_{n+1})) = 1. \tag{21}$$

It follows by the property of β that $\lim_{n\to+\infty}\ \rho(z_n, z_{n+1}) = 0$, which contradicts our assumption. Hence $\delta = 0$, that is, $\lim_{n\to+\infty}\ \rho(z_n, z_{n+1}) = 0$.

Next we show that $\{z_n\}$ is a Cauchy sequence. If $\{z_n\}$ is not a Cauchy sequence then arguing similarly as in the proof of Theorem 2, we get an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that $\lim_{k\to+\infty}\ \rho(z_{m(k)}, z_{n(k)}) = \epsilon$.

Now,

$$\rho(z_{m(k)+1},\ z_{n(k)+1}) \leq \rho(z_{m(k)+1},\ z_{m(k)}) + \rho(z_{m(k)},\ z_{n(k)}) + \rho(z_{n(k)},\ z_{n(k)+1}).$$

Again,

$$\rho(z_{m(k)},\ z_{n(k)}) \leq \rho(z_{m(k)},\ z_{m(k)+1}) + \rho(z_{m(k)+1},\ z_{n(k)+1}) + \rho(z_{n(k)+1},\ z_{n(k)})$$

that is,

$$\rho(z_{m(k)},\ z_{n(k)}) - \rho(z_{m(k)},\ z_{m(k)+1}) - \rho(z_{n(k)+1},\ z_{n(k)}) \leq \rho(z_{m(k)+1},\ z_{n(k)+1}).$$

From the above inequalities we have that

$$\begin{aligned}\rho(z_{m(k)},\ z_{n(k)}) - \rho(z_{m(k)},\ z_{m(k)+1}) - \rho(z_{n(k)+1},\ z_{n(k)}) &\leq \rho(z_{m(k)+1},\ z_{n(k)+1}) \\ &\leq \rho(z_{m(k)+1},\ z_{m(k)}) + \rho(z_{m(k)},\ z_{n(k)}) + \rho(z_{n(k)},\ z_{n(k)+1}).\end{aligned}$$

Taking limit as $k \to +\infty$ in the above inequality and using the fact $\lim_{n\to+\infty}\ \rho(z_n, z_{n+1}) = 0$ and $\lim_{k\to+\infty}\ \rho(z_{m(k)},\ z_{n(k)}) = \epsilon$, we have

$$\lim_{k\to+\infty} \rho(z_{m(k)+1},\ z_{n(k)+1}) = \epsilon. \tag{22}$$

Applying (19), we have

$$\rho(z_{m(k)+1},\ z_{n(k)+1}) = \rho(Sz_{m(k)}, Sz_{n(k)}) \le \beta(\rho(z_{m(k)},\ z_{n(k)}))\ \rho(z_{m(k)},\ z_{n(k)}) < \rho(z_{m(k)},\ z_{n(k)}),$$

that is,

$$\frac{\rho(z_{m(k)+1},\ z_{n(k)+1})}{\rho(z_{m(k)},\ z_{n(k)})} \le \beta(\rho(z_{m(k)},\ z_{n(k)})) < 1.$$

Then

$$1 \le \lim_{k\to+\infty} \beta(\rho(z_{m(k)},\ z_{n(k)})) < 1,$$

which implies that

$$\lim_{k\to+\infty} \beta(\rho(z_{m(k)},\ z_{n(k)})) = 1. \tag{23}$$

It follows by the property of β that $\lim_{k\to+\infty} \rho(z_{m(k)},\ z_{n(k)}) = 0$, that is, $\epsilon = 0$, which is a contradiction. Hence $\{z_n\}$ is a Cauchy sequence. As (M, ρ) is complete, there exists an $\xi \in M$ such that $z_n \to \xi$ as $n \to +\infty$. Now applying (19), we have

$$\rho(z_{n+1},\ S\xi) = \rho(Sz_n, S\xi) \le \beta(\rho(z_n, \xi))\ \rho(z_n, \xi) < \rho(z_n, \xi).$$

Taking limit as $n \to +\infty$ in the above inequality, we have $\rho(\xi,\ S\xi) = 0$, that is, $\xi = S\xi$, that is, ξ is a fixed point of S. From what we have already proved, ξ is the unique fixed point of S.

Example 9 Take the metric space $M = [0, +\infty)$ equipped with usual metric. Let $\beta(t) = \frac{1}{1+t}$, for all $t \ge 0$. Then $\beta \in S$. Define $S : M \to M$ as

$$Su = \begin{cases} \frac{u}{3}, & \text{if } 0 \le u \le 1, \\ \frac{1}{3}, & \text{if } u > 1. \end{cases}$$

Theorem 6 is applicable and here $u = 0$ is the unique fixed point of S.

The next theorem is a generalized weak contraction mapping theorem due to Choudhury et al. [9] which was proved in 2013. It is the culmination of a series of papers generalizing and weakening Banach's result in a specific way. In metric spaces, this line of research was originated by Rhoades [34] and was further contributed through works like [7, 13, 44]. Prior to the work of Rhoades [34], such contractions were considered in different settings and under different conditions, a description of which can be found in [18, 19]. Although most of these results including [9] are worked out in partially ordered metric spaces, we present the theorem here in a complete metric space without order.

We denote by Ψ the set of all functions $\psi : [0,\ +\infty) \to [0,\ +\infty)$ satisfying

(i_ψ) ψ is continuous and nondecreasing,

(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$;

and by Θ we denote the set of all functions $\alpha : [0, +\infty) \to [0, +\infty)$ such that
(i_α) α is bounded on any bounded interval in $[0, +\infty)$,
(ii_α) α is continuous at 0 and $\alpha(0) = 0$.

Theorem 7 Let S be a self-mapping of a complete metric space (M, ρ). Suppose that there exist $\psi \in \Psi$ and $\varphi, \theta \in \Theta$ such that

$$\psi(x) \leq \varphi(y) \Rightarrow x \leq y, \tag{24}$$

for any sequence $\{x_n\}$ in $[0, +\infty)$ with $x_n \to t > 0$,

$$\psi(t) - \overline{\lim}\, \varphi(x_n) + \underline{\lim}\, \theta(x_n) > 0, \tag{25}$$

and

$$\psi(\rho(Su, Sv)) \leq \varphi(\rho(u, v)) - \theta(\rho(u, v)), \text{ for all } u, v \in M. \tag{26}$$

Then S has a unique fixed point in M.

Proof Choose an arbitrary element $z_0 \in M$ and define a sequence $\{z_n\}$ in M such that

$$z_{n+1} = Sz_n, \quad \text{for all } n \geq 0. \tag{27}$$

Let $R_n = \rho(z_{n+1}, z_n)$, for all $n \geq 0$.

Applying (26), we have

$$\psi(\rho(z_{n+2}, z_{n+1})) = \psi(\rho(Sz_{n+1}, Sz_n)) \leq \varphi(\rho(z_{n+1}, z_n)) - \theta(\rho(z_{n+1}, z_n)),$$

that is,

$$\psi(R_{n+1}) \leq \varphi(R_n) - \theta(R_n), \tag{28}$$

which, in view of the fact that $\theta \geq 0$, yields $\psi(R_{n+1}) \leq \varphi(R_n)$, which by (24) implies that $R_{n+1} \leq R_n$, for all positive integers n, that is, the sequence $\{R_n\}$ is monotonic decreasing. Then we get an $r \geq 0$ such that

$$R_n = \rho(z_{n+1}, z_n) \to r \text{ as } n \to +\infty. \tag{29}$$

Taking limit supremum on both sides of (28), using (29), the property (i_α) of φ and θ, and the continuity of ψ, we obtain

$$\psi(r) \leq \overline{\lim}\, \varphi(R_n) + \overline{\lim}\, (-\theta(R_n)).$$

Since $\overline{\lim}\, (-\theta(R_n)) = -\underline{\lim}\, \theta(R_n)$, we obtain

$$\psi(r) \leq \overline{\lim}\, \varphi(R_n) - \underline{\lim}\, \theta(R_n),$$

that is,

$$\psi(r) - \overline{\lim}\, \varphi(R_n) + \underline{\lim}\, \theta(R_n) \le 0,$$

which by (25) is a contradiction unless $r = 0$. Therefore,

$$R_n = \rho(z_{n+1},\ z_n) \to 0, \quad \text{as } n \to +\infty. \tag{30}$$

Next we prove that $\{z_n\}$ is a Cauchy sequence. On the contrary, there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad \rho(z_{m(k)}, z_{n(k)}) \ge \epsilon \ \text{ and } \ \rho(z_{m(k)}, z_{n(k)-1}) < \epsilon.$$

Arguing similarly as in the proof of Theorem 5, we prove that

$$\lim_{k\to+\infty} \rho(z_{m(k)}, z_{n(k)}) = \epsilon \ \text{ and } \lim_{k\to+\infty} \rho(z_{m(k)+1}, z_{n(k)+1}) = \epsilon. \tag{31}$$

Applying from (26) and (27), we have

$$\begin{aligned}\psi(\rho(z_{n(k)+1}, z_{m(k)+1})) &= \psi(\rho(Sz_{n(k)}, Sz_{m(k)}))\\ &\le \varphi(\rho(z_{n(k)}, z_{m(k)})) - \theta(\rho(z_{n(k)}, z_{m(k)})).\end{aligned}$$

Using (31), the property (i_α) of φ and θ, and the continuity of ψ, we obtain

$$\psi(\epsilon) \le \overline{\lim}\, \varphi(\rho(z_{n(k)}, z_{m(k)})) + \overline{\lim}\, (-\, \theta(\rho(z_{n(k)}, z_{m(k)}))).$$

As $\overline{\lim}\, (-\, \theta(\rho(z_{n(k)}, z_{m(k)}))) = -\, \underline{\lim}\, \theta(\rho(z_{n(k)}, z_{m(k)}))$, we get

$$\psi(\epsilon) \le \overline{\lim}\, \varphi(\rho(z_{n(k)},\ z_{m(k)})) - \underline{\lim}\, \theta(\rho(z_{n(k)},\ z_{m(k)})),$$

that is,

$$\psi(\epsilon) - \overline{\lim}\, \varphi(\rho(z_{n(k)}, z_{m(k)})) + \underline{\lim}\, \theta(\rho(z_{n(k)}, z_{m(k)})) \le 0,$$

which is a contradiction by (25). Therefore, $\{z_n\}$ is a Cauchy sequence in M and hence there exists $\xi \in M$ such that

$$\lim_{n\to+\infty} z_{n+1} = \lim_{n\to+\infty} Sz_n = \lim_{n\to+\infty} = \xi. \tag{32}$$

Now, applying (26), we have

$$\psi(\rho(z_{n+1}, S\xi)) = \psi(\rho(Sz_n, S\xi)) \le \varphi(\rho(z_n, \xi)) - \theta(\rho(z_n, \xi)).$$

Taking limit as $n \to +\infty$ and using (32), the properties of ψ, φ and θ, we obtain $\psi(\rho(\xi, S\xi)) = 0$, which implies that $\rho(\xi, S\xi) = 0$, that is, $\xi = S\xi$, that is, ξ is a fixed point of S.

Suppose that $\zeta \in M$ $(\zeta \neq \xi)$ be another fixed point of S. Then $\rho(\xi, \zeta) > 0$. Now, we consider a sequence $\{y_n\}$ in M such that $y_n \to \zeta$ as $n \to +\infty$. Therefore,

$$\rho(\xi, y_n) \to \rho(\xi, \zeta) > 0, \text{ as } n \to +\infty. \tag{33}$$

By (26), we have

$$\psi(\rho(\xi, Sy_n)) = \psi(\rho(S\xi, Sy_n)) \leq \varphi(\rho(\xi, y_n)) - \theta(\rho(\xi, y_n)).$$

Using (33), the property (i_α) of φ and θ, and the continuity of ψ, we obtain

$$\psi(\rho(\xi, \zeta)) \leq \overline{\lim}\, \varphi(\rho(\xi, y_n)) + \overline{\lim}\, (-\,\theta(\rho(\xi, y_n))),$$

that is,

$$\psi(\rho(\xi, \zeta)) - \overline{\lim}\, \varphi(\rho(\xi, y_n)) + \underline{\lim}\, \theta(\rho(\xi, y_n)) \leq 0,$$

which is a contradiction by (25). Therefore, $\rho(\xi, \zeta) = 0$, that is, $\xi = \zeta$. Hence, T has a unique fixed point.

Example 10 Let $M = [0, 1]$ and $\rho(x, y) = |x - y|$, for $x, y \in M$. Let $S\colon M \to M$ be defined by $Sx = x - \frac{x^2}{2}$, for all $x \in M$. Let $\theta, \varphi, \psi\colon [0, +\infty) \to [0, +\infty)$ be given, respectively, by the formulas

$$\theta(t) = \frac{t^2}{2}, \quad \varphi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{otherwise,} \end{cases} \quad \psi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ t^2, & \text{otherwise.} \end{cases}$$

Applying Theorem 7, we see that the unique fixed point of S is $x = 0$.

Remark 2 Considering ψ and φ to be the identity mappings and $\theta(t) = (1 - k)t$, where $0 \leq k < 1$, in Theorem 7 we have Theorem 1.

Pata-type contractions are introduced in a recent paper due to Pata [29] in 2011 in which a fixed point theorem for such contractions was proved by using a new approach. The result due to Pata [29] appeared to be stronger than Banach's Contraction Mapping Principle, even stronger than the well-known Boyd-Wong fixed point theorem.

We use the following class of functions for the following result. Let Ψ denote the family of all functions $\psi : [0, 1] \to [0, +\infty)$ such that ψ is increasing and continuous at zero with $\psi(0) = 0$.

Theorem 8 (Pata [29]) *Let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be some constants and $\psi \in \Psi$. Let (M, d) be a complete metric space and $S : M \to M$ be such that for every $\varepsilon \in [0, 1]$ and all $x, y \in M$,*

$$\rho(Sx,\ Sy) \leq (1-\varepsilon)\rho(x,\ y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\Big[1 + \|x\| + \|y\|\Big]^{\beta}, \tag{34}$$

where $||x|| = \rho(x, u)$ and $||y|| = \rho(y, u)$ for an arbitrary but fixed $u \in M$. Then S has a unique fixed point in M.

Proof Suppose that S has two fixed points ζ and η with $\zeta \neq \eta$. Then $\rho(\zeta, \eta) > 0$. Applying (34) with $0 < \varepsilon \leq 1$, we have

$$\rho(\zeta, \eta) = \rho(S\zeta, S\eta) \leq (1-\varepsilon)\ \rho(\zeta, \eta) + \Lambda\ \varepsilon^{\alpha}\psi(\varepsilon)\Big[1 + ||\zeta|| + ||\eta||\Big]^{\beta},$$

that is,

$$\varepsilon\ \rho(\zeta, \eta) \leq \Lambda\ \varepsilon^{\alpha}\ \psi(\varepsilon)\Big[1 + ||\zeta|| + ||\eta||\Big]^{\beta},$$

that is,

$$\rho(\zeta, \eta) \leq \Lambda\ \varepsilon^{\alpha-1}\ \psi(\varepsilon)\Big[1 + ||\zeta|| + ||\eta||\Big]^{\beta}.$$

Taking $\varepsilon \to 0$ and using the property of ψ, we have $\rho(\zeta, \eta) \leq 0$, which is a contradiction. Hence S may have at most one fixed point.

Choosing an arbitrary element $z_0 \in M$, we construct a sequence $\{z_n\}$ in M such that

$$z_{n+1} \in Sz_n \ \text{ for all } \ n \geq 0. \tag{35}$$

Let

$$c_n = ||z_n|| = \rho(z_n,\ z_0), \quad \text{for all} \quad n \geq 0. \tag{36}$$

Applying (34) with $0 < \varepsilon \leq 1$, we get

$$\begin{aligned}\rho(z_{n+2},\ z_{n+1}) &\leq \rho(Sz_{n+1},\ Sz_n)\\ &\leq (1-\varepsilon)\ \rho(z_{n+1}, z_n) + \Lambda\ \varepsilon^{\alpha}\psi(\varepsilon)\Big[1 + ||z_{n+1}|| + ||z_n||\Big]^{\beta}.\end{aligned}$$

Since $\alpha \geq 1$, taking $\varepsilon \to 0$ and using the property of ψ, we have

$$\rho(z_{n+2},\ z_{n+1}) \leq \rho(z_{n+1},\ z_n) \ \text{ for all } n \geq 0, \tag{37}$$

that is, the sequence $\{\rho(z_{n+1},\ z_n)\}$ is a decreasing. So

$$\rho(z_{n+1},\ z_n) \leq \rho(z_1,\ z_0) = c_1 = ||z_1||, \quad \text{for all } n \geq 0, \tag{38}$$

and also there exists a real number $l \geq 0$ such that

$$\rho(z_{n+1},\ z_n) \to l \ \text{ as } \ n \to +\infty. \tag{39}$$

We claim that $\{c_n\}$ is bounded.
Applying (34) of the theorem, (35), (36), (37) and (38), we have

$$\begin{aligned}
c_n = \rho(z_n,\ z_0) &\leq \rho(z_n,\ z_{n+1}) + \rho(z_{n+1},\ z_1) + \rho(z_1,\ z_0)\\
&= \rho(z_{n+1},\ z_n) + \rho(z_{n+1},\ z_1) + c_1\\
&\leq \rho(z_1,\ z_0) + \rho(z_{n+1},\ z_1) + c_1 = c_1 + \rho(z_{n+1},\ z_1) + c_1\\
&\leq \rho(Sz_n,\ Sz_0) + 2c_1\\
&\leq (1-\varepsilon)\ \rho(z_n,\ z_0) + \Lambda\ \varepsilon^\alpha\ \psi(\varepsilon)\ \Big[1 + ||z_n|| + ||z_0||\Big]^\beta + 2c_1\\
&\leq (1-\varepsilon)\ c_n + \Lambda\ \varepsilon^\alpha\ \psi(\varepsilon)\Big[1 + ||z_n||\Big]^\beta + 2c_1\\
&\leq (1-\varepsilon)\ c_n + \Lambda\ \varepsilon^\alpha\ \psi(\varepsilon)\ \Big[1 + c_n\Big]^\alpha + 2c_1,\ \ \Big(\text{since } \beta \leq \alpha\Big).\\
&\leq (1-\varepsilon)\ c_n + \Lambda\ \varepsilon^\alpha\ \psi(\varepsilon)\ \Big[1 + c_n + c_1\Big]^\alpha + 2c_1,\ \ \Big(\text{since } \beta \leq \alpha\Big).
\end{aligned}$$

So, we have

$$c_n \leq (1-\varepsilon)\ c_n + \Lambda\ \varepsilon^\alpha\ \varphi(\varepsilon)\ \Big[1 + c_n + c_1\Big]^\alpha + 2c_1. \tag{40}$$

Now

$$\Big(1 + c_n + c_1\Big)^\alpha = (1 + c_n)^\alpha\Big(1 + \frac{c_1}{1 + c_n}\Big)^\alpha \leq (1 + c_n)^\alpha(1 + c_1)^\alpha. \tag{41}$$

If possible, suppose that the sequence $\{c_n\}$ is unbounded. Then we have a subsequence $\{c_{n_k}\}$ with $c_{n_k} \to +\infty$ as $k \to +\infty$. Then there exist a natural number N^* such that

$$c_{n_k} \geq 1 + 2c_1 \ \text{ for all } \ k \geq N^*. \tag{42}$$

Now, for all $k \geq N^*$ from (40) and using (41), we have

$$\Big(1 + c_{n_k} + c_1\Big)^\alpha = (1 + c_{n_k})^\alpha(1 + c_1)^\alpha \leq c_{n_k}^\alpha(1 + \frac{1}{c_{n_k}})^\alpha\ (1 + c_1)^\alpha,$$

which implies

$$\Big(1 + c_{n_k} + c_1\Big)^\alpha \leq c_{n_k}^\alpha(1 + 1)^\alpha\ (1 + c_1)^\alpha = 2^\alpha c_{n_k}^\alpha\ (1 + c_1)^\alpha. \tag{43}$$

Then for all $k \geq N^*$, we have from (40) and (43) that

$$c_{n_k} \leq (1-\varepsilon)c_{n_k} + \Lambda\ \varepsilon^\alpha\ \psi(\varepsilon)\ 2^\alpha\ c_{n_k}^\alpha\ (1 + c_1)^\alpha + 2c_1,$$

that is,

$$\begin{aligned}\varepsilon\, c_{n_k} &\le \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, 2^{\alpha}\, c_{n_k}^{\alpha}\, (1+c_1)^{\alpha} + 2c_1 \\ &= \Big[\Lambda\, 2^{\alpha}\, (1+c_1)^{\alpha}\Big]\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, c_{n_k}^{\alpha} + 2c_1.\end{aligned}$$

Let $a = \Lambda\, 2^{\alpha}\, (1+c_1)^{\alpha}$ and $b = 2c_1$. Here a and b are fixed positive real numbers. So, we have

$$\varepsilon\, c_{n_k} \le a\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, c_{n_k}^{\alpha} + b.$$

Choose $\varepsilon = \varepsilon_k = \frac{1+b}{c_{n_k}} = \frac{1+2c_1}{c_{n_k}}$, where $k \ge N^*$. Then by (42), $0 < \varepsilon \le 1$. Now we have

$$1 \le a\, (1+b)^{\alpha}\, \psi(\varepsilon_k) \to 0 \;\text{ as }\; k \to +\infty,$$

which is a contradiction. Hence $\{c_n\}$ is bounded.

Applying (34) with $\varepsilon \in (0,\ 1]$, we have

$$\begin{aligned}\rho(z_{n+2},\ z_{n+1}) &\le \rho(Sz_{n+1},\ Sz_n) \\ &\le (1-\varepsilon)\, \rho(z_{n+1},\ z_n) + \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, \Big[1 + ||z_{n+1}|| + ||z_n||\Big]^{\beta}.\end{aligned}$$

Since $\{c_n\}$ is bounded, there exists a real number $H > 0$ such that $c_n = ||z_n|| \le H$ for all $n \ge 0$. Then

$$\begin{aligned}\rho(z_{n+2},\ z_{n+1}) &\le (1-\varepsilon)\, \rho(z_{n+1},\ z_n) + \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, \Big[1 + ||z_{n+1}|| + ||z_n||\Big]^{\beta} \\ &\le (1-\varepsilon)\, \rho(z_{n+1},\ z_n) + \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, \Big[1 + 2H\Big]^{\beta}.\end{aligned}$$

Taking $n \to +\infty$ and using (39), we have

$$l \le (1-\varepsilon)\, l + \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, \Big[1 + 2H\Big]^{\beta},$$

which implies that

$$\varepsilon\, l \le \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, \Big[1 + 2H\Big]^{\beta},$$

that is,

$$l \le \Lambda\, \varepsilon^{\alpha-1}\, \varphi(\varepsilon)\, \Big[1 + 2H\Big]^{\beta}.$$

Taking $\varepsilon \to 0$ and using the property of ψ, we have $l \le 0$, which implies that $l = 0$. So, we get

$$\lim_{n\to+\infty} \rho(z_{n+1},\ z_n) = 0. \tag{44}$$

Next we prove that the sequence $\{z_n\}$ is Cauchy. On the contrary, there exists a $\xi > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k,\ \ \rho(z_{m(k)},\ z_{n(k)}) \ge \xi \ \text{ and } \ \rho(z_{m(k)},\ z_{n(k)-1}) < \xi.$$

Now,

$$\xi \le \rho(z_{m(k)},\ z_{n(k)}) \le \rho(z_{m(k)},\ z_{n(k)-1}) + \rho(z_{n(k)-1},\ z_{n(k)}),$$

that is,

$$\xi \le \rho(z_{m(k)},\ z_{n(k)}) < \xi + \rho(z_{n(k)-1},\ z_{n(k)}).$$

Using (44), we have

$$\lim_{k\to+\infty} \rho(z_{m(k)},\ z_{n(k)}) = \xi. \tag{45}$$

Again,

$$\rho(z_{m(k)},\ z_{n(k)}) \le \rho(z_{m(k)},\ z_{m(k)+1}) + \rho(z_{m(k)+1},\ z_{n(k)+1}) + \rho(z_{n(k)+1},\ z_{n(k)})$$

and

$$\rho(z_{m(k)+1},\ z_{n(k)+1}) \le \rho(z_{m(k)+1},\ z_{m(k)}) + \rho(z_{m(k)},\ z_{n(k)}) + \rho(z_{n(k)},\ z_{n(k)+1}).$$

Using (44) and (45), we have

$$\lim_{k\to+\infty} \rho(z_{m(k)+1},\ z_{n(k)+1}) = \xi. \tag{46}$$

Applying (34) with $\varepsilon \in (0,\ 1]$, we have

$$\begin{aligned}\rho(z_{m(k)+1},\ z_{n(k)+1}) &\le \rho(Sz_{m(k)},\ Sz_{n(k)})\\ &\le (1-\varepsilon)\ \rho(z_{m(k)},\ z_{n(k)}) + \Lambda\ \varepsilon^{\alpha}\ \psi(\varepsilon)\Big[1 + ||z_{m(k)}|| + ||z_{n(k)}||\Big]^{\beta}.\end{aligned}$$

Since $c_n = ||z_n|| \le H$ for all $n \ge 0$,

$$\rho(z_{m(k)+1},\ z_{n(k)+1}) \le (1-\varepsilon)\ \rho(z_{m(k)},\ z_{n(k)}) + \Lambda\ \varepsilon^{\alpha}\ \psi(\varepsilon)\Big[1 + 2H\Big]^{\beta}.$$

Taking limit as $k \to +\infty$ and using (45), (46) and the property of ψ, we have

$$\xi \le (1-\varepsilon)\,\xi + \Lambda\,\varepsilon^{\alpha}\,\psi(\varepsilon)\Big[1+2H\Big]^{\beta},$$

which implies that

$$\varepsilon\,\xi \le \Lambda\,\varepsilon^{\alpha}\,\psi(\varepsilon)\Big[1+2H\Big]^{\beta},$$

that is,

$$\xi \le \Lambda\,\varepsilon^{\alpha-1}\,\varphi(\varepsilon)\Big[1+2H\Big]^{\beta}.$$

Taking limit as $\varepsilon \to 0$ and using the property of ψ, we have $\xi \le 0$, which is a contradiction. Therefore, $\{z_n\}$ is a Cauchy sequence in M and hence there exists $y \in M$ such that

$$z_n \to y \quad \text{as} \quad n \to +\infty. \tag{47}$$

Applying (34) with $\varepsilon \in (0,\ 1]$, we have

$$\begin{aligned}\rho(z_{n+1},\ Sy) &\le \rho(Sz_n,\ Sy)\\ &\le (1-\varepsilon)\ \rho(z_n,\ y) + \Lambda\,\varepsilon^{\alpha}\psi(\varepsilon)\Big[1+||z_n||+||y||\Big]^{\beta}.\end{aligned}$$

Since $c_n = ||z_n|| \le H$ for all $n \ge 0$. Then

$$\rho(z_{n+1},\ Sy) \le (1-\varepsilon)\ \rho(z_n,\ y) + \Lambda\,\varepsilon^{\alpha}\psi(\varepsilon)\Big[1+H+||y||\Big]^{\beta}.$$

Taking $n \to +\infty$ and using (44), (47), we get

$$\rho(y,\ Sy) \le \Lambda\,\varepsilon^{\alpha}\psi(\varepsilon)\Big[1+H+||y||\Big]^{\beta}.$$

Taking limit as $\varepsilon \to 0$ and using the property of ψ, we have $\rho(y,\ Sy) = 0$, that is, $y = Sy$, that is, y is a fixed point of S. From what we have already proved, y is the unique fixed point of S.

Example 11 ([29]) Let $M = [1, +\infty)$ and let $S : M \to M$ be defined by

$$Sz = -2 + z - 2\sqrt{z} + 4\sqrt[4]{z}.$$

It has a unique fixed point $z = 1$. For any given $r > 0$ and $z \ge 1$, if

$$Q(z,r) = 2[\sqrt{z+r} - \sqrt{z}] - 4[\sqrt[4]{z+r} - \sqrt[4]{z}],$$

then

$$|S(z+r) - S(z)| = r - Q(z,r)$$

holds for all r and z. On the other hand, for every $\varepsilon \in [0, 1]$, one can prove that

$$-\varepsilon r + \varepsilon^2 (2z+r)^{3/2} + Q(z,r) \geq Q(z,r) - \frac{r^2}{4(r+2z)^{3/2}} \geq 0.$$

It follows that

$$|S(z+r) - S(z)| = r - Q(z,r) \leq (1-\varepsilon)r + \varepsilon^2(2z+r)^{3/2},$$

and the conditions of Theorem 8 are fulfilled.

4 Metric Fixed Point Without Continuity

In 1976, Caristi [5] proved an elegant fixed point theorem on complete metric spaces, which is a generalization of the Banach's contraction mapping principle and is equivalent to the Ekeland variational principle [15].

Definition 8 A function $\varphi : X \to R$ is said to be lower semicontinuous at x if for any sequence $\{x_n\} \subset X$, we have

$$x_n \to x \in X \quad \Rightarrow \quad \varphi(x) \leq \liminf_{n \to +\infty} \varphi(x_n).$$

Definition 9 Let (M, ρ) be a metric space. A mapping $S : M \to M$ is called a Caristi mapping if there exists a lower semicontinuous function $\varphi : M \to R^+$ such that

$$\rho(u, Su) \leq \varphi(u) - \varphi(Su), \quad \text{for all } \; u \in M.$$

Theorem 9 ([24]) *Let (M, ρ) be a complete metric space. A mapping $S \colon M \to M$ admits a fixed point in M if there exists a lower semicontinuous function $\varphi : M \to R^+$ such that*

$$\rho(u, Su) \leq \varphi(u) - \varphi(Su), \quad \text{for all } \; u \in M. \tag{48}$$

Proof From (48) it follows immediately that

$$\varphi(Su) \leq \varphi(u), \quad \text{for every } \; u \in M. \tag{49}$$

For $u \in M$, define

$$Q(u) = \{y \in M : \rho(u, y) \leq \varphi(u) - \varphi(y)\}.$$

$Q(u)$ is nonempty because $u \in Q(u)$ and $Su \in Q(u)$. Let $y \in Q(u)$. Now, we have

$$\rho(u, Sy) \leq \rho(u, y) + \rho(y, Sy) \leq \varphi(u) - \varphi(y) + \varphi(y) - \varphi(Sy),$$

that is,

$$\rho(u, Sy) \leq \varphi(u) - \varphi(Sy). \tag{50}$$

It follows that $Sy \in Q(u)$. Hence, we have that if $y \in Q(u)$ then $Sy \in Q(u)$.

Define

$$q(u) = \inf \{\varphi(y) : y \in Q(u)\}.$$

As $Q(u)$ is nonempty for each $u \in M$ and the function φ is nonnegative, the function $q(u)$ is well-defined. Then, we have that for any $u \in M$,

$$0 \leq q(u) \leq \varphi(Su) \leq \varphi(u). \tag{51}$$

Let $u_1 \in M$ be arbitrary. By the definition of $q(u_1)$, there exists $u_2 \in Q(u_1)$ such that $\varphi(u_2) < q(u_1) + 1$. Again, by the definition of $q(u_2)$, there exists $u_3 \in Q(u_2)$ such that $\varphi(u_3) < q(u_2) + \frac{1}{2}$. In this way, we define a sequence $\{u_n\}$ in M such that $u_{n+1} \in Q(u_n)$ with

$$\varphi(u_{n+1}) < q(u_n) + \frac{1}{n}, \quad \text{for } \ n \geq 1. \tag{52}$$

Since $u_{n+1} \in Q(u_n)$, we have

$$0 \leq \rho(u_n, u_{n+1}) \leq \varphi(u_n) - \varphi(u_{n+1}), \tag{53}$$

that is,

$$\varphi(u_{n+1}) \leq \varphi(u_n), \quad \text{for } \ n \geq 1. \tag{54}$$

Hence $\{\varphi(u_n)\}$ is a nonincreasing sequence of nonnegative numbers and therefore there exists $r \geq 0$ such that

$$\lim_{n \to +\infty} \varphi(u_n) = r. \tag{55}$$

Therefore, $\{\varphi(u_n)\}$ is a Cauchy sequence. Hence, for every $k \in N$ (set of all natural number), there exists $N_k \in N$ such that for every pair of natural numbers m, n with $m \geq n \geq N_k$, we have

$$0 \leq \varphi(u_n) - \varphi(u_m) < \frac{1}{k}. \tag{56}$$

From (51) and (52), we have

$$\varphi(u_{n+1}) < q(u_n) + \frac{1}{n} \leq \varphi(u_n) + \frac{1}{n}.$$

Taking limit as $n \to +\infty$ and using (55), we have

$$\lim_{n \to +\infty} q(u_n) = r. \tag{57}$$

We claim that for $m \geq n \geq N_k$,

$$\rho(u_n, u_m) \leq \varphi(u_n) - \varphi(u_m) < \frac{1}{k}. \tag{58}$$

(58) is trivially valid for $n = m$. Therefore, it is sufficient to show that (58) is true for $m > n$. Using triangular inequality, (53) and (56), we have for $m > n$ that

$$\begin{aligned} \rho(u_n, u_m) &\leq \rho(u_n, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \cdots + \rho(u_{m-1}, u_m) \\ &\leq \varphi(u_n) - \varphi(u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_{n+2}) + \cdots + \varphi(u_{m-1}) - \varphi(u_m). \end{aligned}$$

It follows that

$$\rho(u_n, u_m) \leq \varphi(u_n) - \varphi(u_m) < \frac{1}{k}. \tag{59}$$

Therefore, (58) is true for $m \geq n \geq N_k$. From (58), it follows that $\{u_n\}$ is a Cauchy sequence and hence by completeness of M, there exists $z \in M$ such that

$$\lim_{n \to +\infty} \rho(u_n, z) = 0. \tag{60}$$

Hence, for every $n \in N$,

$$\lim_{m \to +\infty} \rho(u_n, u_m) = \rho(u_n, z).$$

Using this, (59) and the lower semicontinuity of φ,

$$\begin{aligned} \rho(u_n, z) = \lim_{m \to +\infty} \rho(u_n, u_m) &\leq \limsup_{m \to +\infty} [\varphi(u_n) - \varphi(u_m)] \\ &\leq \varphi(u_n) - \liminf_{m \to +\infty} \varphi(u_m) \\ &\leq \varphi(u_n) - \varphi(z). \end{aligned}$$

Therefore,

$$\rho(u_n, z) \leq \varphi(u_n) - \varphi(z), \tag{61}$$

which implies that $z \in Q(u_n)$ for every $n \in N$. Then we have

$$q(u_n) \le \varphi(z) \le \varphi(u_n) - \rho(u_n, z), \quad \text{for every } \; n \in N. \tag{62}$$

Taking limit as $n \to +\infty$ in the above inequality and using (55) and (57), we have

$$\varphi(z) = r. \tag{63}$$

Since, as proved above, $z \in Q(u_n)$ for every $n \in N$, (50) implies that $Sz \in Q(u_n)$ for every $n \in N$. Therefore, by (49), we conclude from (63) that

$$q(u_n) \le \varphi(Sz) \le \varphi(z) = r. \tag{64}$$

Letting $n \to +\infty$ and using (57), we obtain $\varphi(Sz) = \varphi(z)$. By (48) again,

$$0 \le \rho(z, Sz) \le \varphi(z) - \varphi(Sz) = 0.$$

Hence $\rho(z, Sz) = 0$, that is, $Sz = z$. Therefore, S has a fixed point.

Example 12 Take $M = [0, 1]$ endowed with the usual metric ρ. Define $S : M \to M$ as

$$Su = \begin{cases} \frac{u}{2}, & \text{if } u \ne 1, \\ 1, & \text{if } u = 1. \end{cases}$$

The conditions of Theorem 9 are satisfied and S has fixed points 0 and 1.

It is easy to see that Caristi's fixed point theorem is a generalization of the Banach's contraction mapping principle by defining $\varphi(u) = \frac{1}{1-k}\, \rho(u, Su)$, where $0 < k < 1$ is the Lipschitz constant associated with the contraction S from Banach's principle. It has been shown by Kirk in [22] that the validity of Caristi's fixed point theorem implies that the corresponding metric space is complete while the Banach's contraction mapping principle does not characterize completeness. The above example shows that Caristi's contraction can also be discontinuous.

Suzuki [42] in the year 2008 established a new fixed point theorem which is a generalization of Theorem 1 and characterizes the metric completeness. Though there are many generalizations of Theorem 1, the direction of Suzuki is new and very simple. Suzuki-type contractions form an important class of contractions in the domain of fixed point theory.

Define a function $\theta : [0,\ 1) \to (\frac{1}{2},\ 1]$ as

$$\theta(r) = \begin{cases} 1, & \text{if } \; 0 \le r \le \frac{\sqrt{5}-1}{2}; \\ \frac{1-r}{r^2}, & \text{if } \frac{\sqrt{5}-1}{2} \le r \le \frac{1}{\sqrt{2}}; \\ \frac{1}{1+r}, & \text{if } \; \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Theorem 10 (Suzuki [42]) *A self-mapping S of a complete metric space (M, ρ) admits a unique fixed point if there exists a real number $r \in [0,\ 1)$ such that for all $x,\ y \in M$,*

$$\theta(r)\ \rho(x,\ Sx) \leq \rho(x,\ y) \ \text{ implies } \rho(Sx,\ Sy) \leq r\ \rho(x,\ y). \tag{65}$$

Proof Since $\theta(r) \leq 1,\ \theta(r)\ \rho(x,\ Sx) \leq \rho(x,\ Sx)$ holds for every $x \in M$. By (65), we have

$$\rho(Sx,\ S^2x) \leq r\ \rho(x,\ Sx),\ \text{ for all } x \in M. \tag{66}$$

Choose any point $u \in M$ and construct a sequence $\{u_n\}$ in M such that

$$u_n = S^n u \ \text{ for all } \ n \geq 1. \tag{67}$$

It follows from (66) that $\rho(u_n, u_{n+1}) \leq r^n \ \rho(u, Su)$. Then $\sum_1^+ \infty \rho(u_n, u_{n+1}) < +\infty$, which implies that $\{u_n\}$ is a Cauchy sequence. As M is complete, $\{u_n\}$ converges to some point $z \in M$. Next, we show

$$\rho(Sx, z) \leq r\rho(x, z),\ \text{ for all } x \in M \setminus \{z\}. \tag{68}$$

For $x \in M \setminus \{z\}$, there exists a positive integer m such that $\rho(u_n, z) \leq \frac{\rho(x,z)}{3}$, for all $n \geq m$. Then we have for all $n \geq m$ that

$$\begin{aligned}
\theta(r)\rho(u_n, Su_n) &\leq \rho(u_n, Su_n) = \rho(u_n, u_{n+1}) \\
&\leq \rho(u_n, z) + \rho(u_{n+1}, z) \\
&\leq \frac{2}{3}\ \rho(x, z) = \rho(x, z) - \frac{1}{3}\ \rho(x, z) \\
&\leq \rho(x, z) - \rho(u_n, z) \leq \rho(u_n, x).
\end{aligned}$$

Then it follows by (65) that $\rho(u_{n+1}, Sx) \leq r\ \rho(u_n, x)$, for all $n \geq m$. Taking $n \to +\infty$, we get $\rho(Sx, z) \leq r\rho(x, z)$. Hence (68) is true. Assume that $S^n z \neq z$ for all $n \in N$. By (68), we have

$$\rho(S^{n+1}z, z) \leq r^n \rho(Sz, z),\ \text{ for all } n \in N. \tag{69}$$

We consider the following three cases:

- $0 \leq r \leq \frac{\sqrt{5}-1}{2}$;
- $\frac{\sqrt{5}-1}{2} < r < \frac{1}{\sqrt{2}}$;
- $\frac{1}{\sqrt{2}} \leq r < 1$.

If $0 \leq r \leq \frac{\sqrt{5}-1}{2}$, then $r^2 + r - 1 \leq 0$ and $2r^2 < 1$. If we assume $\rho(S^2z, z) < \rho(S^2z, S^3z)$, then we have

$$\begin{aligned}\rho(z, Sz) &\leq \rho(z, S^2z) + \rho(Sz, S^2z)\\ &< \rho(S^2z, S^3z) + \rho(Sz, S^2z)\\ &\leq r^2\rho(z, Sz) + r\rho(z, Sz)\\ &\leq \rho(z, Sz),\end{aligned}$$

which is a contradiction. So we have $\rho(S^2z, z) \geq \rho(S^2z, S^3z) \geq \theta(r)\rho(S^2z, SS^2z)$. By hypothesis and (69), we have

$$\begin{aligned}\rho(z, Sz) &\leq \rho(z, S^3z) + \rho(S^3z, Sz)\\ &\leq r^2\rho(z, Sz) + r\rho(S^2z, z)\\ &\leq r^2\rho(z, Sz) + r^2\rho(Sz, z) = 2r^2\rho(z, Sz)\\ &< \rho(z, Sz).\end{aligned}$$

It is a contradiction. If $\frac{\sqrt{5}-1}{2} < r < \frac{1}{\sqrt{2}}$, then $2r^2 < 1$. If we assume $\rho(S^2z, z) < \theta(r)\ \rho(S^2z, S^3z)$, then we have in view of (66)

$$\begin{aligned}\rho(z, Sz) &\leq \rho(z, S^2z) + \rho(Sz, S^2z)\\ &< \theta(r)\ \rho(S^2z, S^3z) + \rho(Sz, S^2z)\\ &\leq \theta(r)\ r^2\rho(z, Sz) + r\rho(z, Sz) = \rho(z, Sz),\end{aligned}$$

which is a contradiction. Hence $\rho(S^2z, z) \geq \theta(r)\ \rho(S^2z, SS^2z)$. As in the previous case, we can prove

$$\rho(z, Sz) \leq 2r^2\rho(z, Sz) < \rho(z, Sz).$$

This is a contradiction. Take the case $\frac{1}{\sqrt{2}} \leq r < 1$. We note that for $x, y \in M$, either

$$\theta(r)\ \rho(x, Sx) \leq \rho(x, y) \ \text{ or } \ \theta(r)\ \rho(Sx, S^2x) \leq \rho(Sx, y)$$

holds. Indeed, if

$$\theta(r)\ \rho(x, Sx) > \rho(x, y) \ \text{ and } \ \theta(r)\ \rho(Sx, S^2x) > \rho(Sx, y),$$

then we have

$$\begin{aligned}\rho(x, Sx) &\leq \rho(x, y) + \rho(Sx, y)\\ &< \theta(r)\ (\rho(x, Sx) + \rho(Sx, S^2x))\\ &\leq \theta(r)\ (\rho(x, Sx) + r\rho(x, Sx))\\ &= \rho(x, Sx).\end{aligned}$$

This is a contradiction. Since either

$$\theta(r)\ \rho(u_{2n}, u_{2n+1}) \leq \rho(u_{2n}, z) \ \text{ or } \ \theta(r)\rho(u_{2n+1}, u_{2n+2}) \leq \rho(u_{2n+1}, z)$$

holds for every $n \in N$, either

$$\rho(u_{2n+1}, Sz) \leq r\ \rho(u_{2n}, z) \ \text{ or } \ \rho(u_{2n+2}, Sz) \leq r\ \rho(u_{2n+1}, z)$$

holds for every $n \in N$. Since $\{u_n\}$ converges to z, the above inequalities imply there exists a subsequence of $\{u_n\}$ which converges to Sz. This implies $Sz = z$. This is a contradiction. Therefore, there exists $n \in N$ such that $S^n z = z$. Since $\{S^n z\}$ is a Cauchy sequence, we obtain $Sz = z$, that is, z is a fixed point of S. The uniqueness of a fixed point follows easily from (68).

Example 13 ([42]) Take the metric space $M = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\}$ equipped with metric ρ defined as $\rho((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Let $S : M \to M$ be defined by

$$S(x_1, x_2) = \begin{cases} (x_1, 0), \ \text{if } x_1 \leq x_2, \\ (0, x_2), \ \text{if } x_1 > x_2. \end{cases}$$

Here, Theorem 10 is applicable and the unique fixed point of S is (0, 0).

All the results described above are generalizations of Banach's result. In the next theorem, we deal with a contraction condition which is of a different category and does not generalize Banach's contraction. The contraction condition is also satisfied by discontinuous functions. The result is due to Kannan [20, 21] which was established in the year 1968.

Definition 10 (*Kannan-type mapping* [20, 21]) A mapping $S : M \to M$, where (M, ρ) is a metric space, is called a Kannan-type mapping if there exists $0 < k < \frac{1}{2}$ such that

$$\rho(Sx,\ Sy) \leq k\ [\rho(x,\ Sx) + \rho(y,\ Sy)], \ \text{ for all } \ x,\ y \in M. \tag{70}$$

Theorem 11 (Kannan [20, 21]) *Let $(M,\ \rho)$ be a complete metric space and $S : M \to M$ be a Kannan type mapping. Then T admits a unique fixed point.*

Proof Let $z_0 \in M$ be any arbitrary element. We take the same sequence $\{z_n\}$ in M as in the proof of Theorem 1. Applying (70), we have

$$\begin{aligned} \rho(z_{n+1}, z_{n+2}) &= \rho(Sz_n, Sz_{n+1}) \leq k\ [\rho(z_n, Sz_n) + \rho(z_{n+1}, Sz_{n+1})] \\ &= k\ [\rho(z_n, z_{n+1}) + \rho(z_{n+1}, z_{n+2})], \ \text{ for all } n \geq 0, \end{aligned}$$

which implies that

$$\rho(z_{n+1}, z_{n+2}) \leq \frac{k}{1-k}\ \rho(z_n, z_{n+1}), \ \text{ for all } \ n \geq 0. \tag{71}$$

Now $0 < k < \frac{1}{2}$ implies that $0 < 2k < 1$, that is, $0 < k < 1 - k$. Hence $0 < \frac{k}{1-k} < 1$. Let $\alpha = \frac{k}{1-k}$. Then we have from (71) that

$$\rho(z_{n+1}, z_{n+2}) \leq \alpha \; \rho(z_n, z_{n+1}), \quad \text{for all} \;\; n \geq 0.$$

Applying similar arguments as in the proof of Theorem 1, we prove that $\{z_n\}$ is a Cauchy sequence and there exists $\xi \in M$ such that $z_n \to \xi$, as $n \to +\infty$.

Now applying (70), we have

$$\begin{aligned} \rho(z_{n+1}, S\xi) = \rho(Sz_n, S\xi) &\leq k\,[\rho(z_n, Sz_n) + \rho(\xi, S\xi)] \\ &= k\,[\rho(z_n, z_{n+1}) + \rho(\xi, S\xi)], \;\text{for all}\; n \geq 0. \end{aligned}$$

Taking the limit as $n \to +\infty$, we have

$$\rho(\xi, S\xi) \leq k \; \rho(\xi, S\xi), \quad \text{that is,} \;\; (1 - k) \; \rho(\xi, S\xi) \leq 0.$$

As $(1 - k) > 0$, it follows that $\rho(\xi, S\xi) = 0$, that is, $\xi = S\xi$, that is, ξ is a fixed point of S.

If possible, suppose that ζ be another fixed point of S. Applying (70), we have

$$\rho(\zeta, \xi) = \rho(S\zeta, S\xi) \leq k\,[\rho(\zeta,\; S\zeta) + \rho(\xi,\; S\xi)] = 0,$$

which implies that $\rho(\zeta, \xi) = 0$, that is, $\zeta = \eta$, which is a contradiction. Hence the fixed point of S is unique.

Example 14 ([32], p. 262) Take $M = [0, 1]$ endowed with the usual metric. Define $S : M \to M$ as

$$Sz = \begin{cases} \frac{z}{3}, \text{ if } 0 \leq z < 1, \\ \frac{1}{6}, \quad \text{if } z = 1. \end{cases}$$

Theorem 11 is applicable and $z = 0$ is the unique fixed point of S. It is observed that S is not continuous on M.

Following the appearance of the results in [20, 21], many persons created contractive conditions not requiring continuity of the mapping and established fixed point and common fixed point results for them; see, for example, [6, 35, 36].

There is another reason for which the Kannan-type mappings are considered to be important. The Banach's contraction mapping principle does not characterize completeness. In fact, there are examples of noncomplete spaces where every contraction has a fixed point [11]. It has been shown in [38, 40] that the necessary existence of fixed points for Kannan-type mappings implies that the corresponding metric space is complete. The above are some reasons for which the Kannan-type mappings are considered important in mathematical analysis. There are several extensions and generalizations of Kannan-type mappings in various spaces as, for instance, those in the works noted in [8, 12, 16].

Fixed point theorem due to Chatterjea [6] which was established in the year 1972 and which is actually a sort of dual of the Kannan fixed point theorem is based on a condition similar to (70).

Definition 11 (*C-contraction* [6]) A mapping $S : M \to M$, where $(M,\ \rho)$ is a metric space, is called a C-contraction if there exists $0 < k < \frac{1}{2}$ such that

$$\rho(Sx,\ Sy) \leq k\,[\rho(x,\ Sy) + \rho(y,\ Sx)], \quad \text{for all } x,\ y \in X. \tag{72}$$

Theorem 12 (Chatterjea [6]) *Let $(M,\ \rho)$ be a complete metric space and $S : M \to M$ be a C-contraction. Then T admits a unique fixed point.*

Proof The proof follows by the same method as in Theorem 11. The details are omitted.

Example 15 Take $M = [0, 1]$ equipped with usual metric ρ. Define $S : M \to M$ as

$$Sz = \begin{cases} 0, & \text{if } 0 \leq z < 1, \\ \frac{1}{6}, & \text{if } z = 1. \end{cases}$$

The conditions of Theorem 12 are satisfied and here $z = 0$ is the unique fixed point of S. It is observed that S is not continuous on M.

One of the most general contractive conditions was given by Ćirić [10] in 1974 which is known as quasi-contraction.

Definition 12 (*Quasi-contraction* [10]) A mapping $S : M \to M$, where $(M,\ d)$ is a metric space, is called a quasi-contraction if there exists $0 \leq k < 1$ such that, for all $u,\ v \in M$,

$$d(Su,\ Sv) \leq k \max\{d(u,\ v),\ d(u,\ Su),\ d(v,\ Sv),\ d(u,\ Sv),\ d(v,\ Su)\}. \tag{73}$$

Let S be a self-mapping of a metric space M. For $A \subset M$ let $\delta(A) = \sup\{d(a, b) : a, b \in A\}$ and for each $u \in M$, let

$$O(u, n) = \{u, Su, S^2u, \ldots, S^n u\}, \quad n = 1, 2, \ldots$$
$$O(u, \infty) = \{u, Su, S^2u, \ldots\}.$$

A space M is said to be S-orbitally complete if and only if every Cauchy sequence which is contained in $O(u, \infty)$ for some $u \in M$ converges in M.

Lemma 2 (Ćirić [10]) *Let (M, d) be a metric space, $S : M \to M$ be a quasi-contraction and n be any positive integer. Then for each $z \in M$ and for all positive integers i and j, $i, j \in \{1, 2, \ldots, n\}$ implies $d(S^i z, S^j z) \leq k\delta[O(z, n)]$.*

Proof Let $z \in M$ be arbitrary. Let n be any positive integer and let i and j satisfy the condition of lemma 2. Then $S^{i-1}z,\ S^i z,\ S^{j-1}z,\ S^j z \in O(z, n)$ (where $S^0 z = z$) and since S is a quasi-contraction, we have

$$\begin{aligned} d(S^i z, S^j z) &= d(SS^{i-1}z, SS^{j-1}z) \\ &\leq k \max\{d(S^{i-1}z, S^{j-1}z), d(S^{i-1}z, S^i z), d(S^{j-1}z, S^j z), \\ &\qquad d(S^{i-1}z, S^j z), d(S^{j-1}z, S^i z)\} \\ &\leq k\, \delta[O(z, n)], \end{aligned}$$

which proves the lemma.

Remark 3 From this lemma, it follows that if S is quasi-contraction and $z \in M$, then for every positive integer n there exists a positive integer $k \leq n$, such that $d(z, S^k z) = \delta[O(z, n)]$.

Lemma 3 (Ćirić [10]) *Let* (M, d) *be a metric space and* $S : M \to M$ *be a quasi-contraction. Then*

$$\delta[O(z, \infty)] \leq \frac{1}{1-k}\, d(z, Sz)$$

holds for all $z \in M$.

Proof Let $z \in M$ be arbitrary. Since $\delta[O(z, 1)] \leq \delta[O(z, 2)] \leq \ldots$, we have that $\delta[O(z, \infty)] = \sup\{\delta[O(z, n)] : n \in N\}$. Now it is sufficient to prove that $\delta[O(z, n)] \leq \frac{1}{1-k}\, d(z, Sz)$, for all $n \in N$.

Let n be any positive integer. From the remark of the previous lemma, there exists $S^k z \in O(z, n)]$ $(1 \leq k \leq n)$ such that $d(z, S^k z) = \delta[O(z, n)]$. By a triangular inequality and Lemma 2, we have

$$\begin{aligned} d(z, S^k z) &= d(z, Sz) + d(Sz, S^k z) \leq d(z, Sz) + k\delta[O(z, n)] \\ &\leq d(z, Sz) + kd(z, S^k z). \end{aligned}$$

Therefore, $\delta[O(z, n)] = d(z, S^k z) \leq \frac{1}{1-k}\, d(z, Sz)$. Since n is arbitrary, the proof is completed.

Now we state the main result.

Theorem 13 (Ćirić [10]) *Let* $S : M \to M$*, where* (M, d) *is a metric space, be a quasi-contraction. If* M *is* S*-orbitally complete, then* S *has a unique fixed point in* M.

Proof Let $z \in M$ be arbitrary. First, we prove that the sequence $\{S^n z\}$ is a Cauchy sequence. Let n and m be two positive integers with $n < m$. By Lemma 2, we have

$$d(S^n z, S^m z) = d(SS^{n-1}z, S^{m-n+1}S^{n-1}z) \leq k\, \delta[O(S^{n-1}z, m-n+1)].$$

Following Remark 3, we get an integer l with $1 \leq l \leq m - n + 1$ such that

$$\delta[O(S^{n-1}z, m-n+1)] = d(S^{n-1}z, S^l S^{n-1}z).$$

By Lemma 2, we have

$$\begin{aligned} d(S^{n-1}z, S^l S^{n-1}z) &= d(SS^{n-2}z, S^{l+1}S^{n-2}z) \\ &\leq k\ \delta[O(S^{n-2}z, l+1)] \\ &\leq k\ \delta[O(S^{n-2}z, m-n+2)]. \end{aligned}$$

Therefore, we have

$$d(S^n z, S^m z) \leq k\ \delta[O(S^{n-1}z, m-n+1)] \leq k^2\ \delta[O(S^{n-2}z, m-n+2)].$$

Continuing this process, we obtain

$$d(S^n z, S^m z) \leq k\ \delta[O(S^{n-1}z, m-n+1)] \leq k^2\ \delta[O(S^{n-2}z, m-n+2)] \leq \cdots \leq k^n \delta[O(z, m)].$$

Now it follows from Lemma 3 that

$$d(S^n z, S^m z) \leq \frac{k^n}{1-k}\, d(z, Sz) \to 0 \text{ as } n \to +\infty, \tag{74}$$

which implies that $\{S^n z\}$ is a Cauchy sequence. As M is S-orbitally complete, there exists $\xi \in M$ such that $S^n z \to \xi$ as $n \to +\infty$. Now

$$\begin{aligned} d(S\xi, S^{n+1}z) &= d(S\xi, SS^n z) \\ &\leq k \max\{d(\xi, S^n z), d(\xi, S\xi), d(S^n z, S^{n+1}z), d(\xi, S^{n+1}z), d(S^n z, S\xi)\}. \end{aligned}$$

Taking the limit as $n \to +\infty$, we have

$$d(\xi, S\xi) \leq k\ d(\xi, S\xi), \text{ that is, } (1-k)\, d(\xi, S\xi) \leq 0.$$

As $(1-k) > 0$, it follows that $d(\xi, S\xi) = 0$, that is, $\xi = S\xi$, that is, ξ is a fixed point of S.

Suppose that $\zeta \in M$ $(\zeta \neq \xi)$ be another fixed point of S. As S is a quasi-contraction, we have

$$\begin{aligned} d(\xi, \zeta) &= d(S\xi, S\zeta) \\ &\leq k \max\{d(\xi, \zeta), d(\xi, S\xi), d(\zeta, S\zeta), d(\xi, S\zeta), d(\zeta, S\xi)\} \\ &\leq k \max\{d(\xi, \zeta), 0, 0, d(\xi, \zeta), d(\zeta, \xi)\} \\ &\leq k\ d(\xi, \zeta) \end{aligned}$$

which is a contradiction. Therefore, $d(\xi, \zeta) = 0$, that is, $\xi = \zeta$. Hence fixed point of S is unique.

Example 16 Take the metric space $M = [0, 1]$ equipped with usual metric. Define $S : M \to M$ as

$$Sz = \begin{cases} 0, \text{ if } 0 \le z < 1, \\ \frac{1}{2}, \quad \text{if } z = 1. \end{cases}$$

Then Theorem 13 is applicable and $z = 0$ is the unique fixed point of S. It is observed that S is not continuous on M.

In 1988, Rhoades [33] examined that there exists a large number of discontinuous contractive mappings which produce a fixed point but do not require the map to be continuous at the fixed point. Rhoades [33] raised an open question whether there exists a contractive definition which produces a fixed point but which does not require the map to be continuous at the fixed point. In 1999, Pant [27] answered the open question in the affirmative. In 2017, Bisht et al. [3] gave one more solution to the open question of the existence of contractive definitions which ensure the existence of a fixed point where the fixed point is not a point of continuity [33].

In the following theorem, the notation $Q(u, v)$ stands for

$$Q(u, v) = \max\{\rho(u,\ v),\ \rho(u,\ Tu),\ \rho(v,\ Tv),\ \frac{\rho(u,\ Tv) + \rho(v,\ Tu)}{2}\}.$$

Theorem 14 (Bisht et al. [3]) *Let $(M,\ \rho)$ be a complete metric space and S be a self-mapping on M such that S^2 is continuous. Suppose that (i) $\rho(Su, Sv) \le \phi(Q(u, v))$, where $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$; (ii) for a given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $\epsilon < Q(u, v) < \epsilon + \delta$ implies $\rho(Su, Sv) \le \epsilon$. Then there exists unique $z \in M$ such that $Sz = z$. Moreover, S is discontinuous at z if and only if $\lim_{u \to z} Q(u, z) \ne 0$.*

Proof Let $z_0 \in M$ be any arbitrary element. We define a sequence $\{z_n\}$ in M such that $z_n = Sz_{n-1} = S^n z_0$ for all $n \ge 1$. If $z_l = z_{l+1}$ for some positive integer l, then z_l is a fixed point of S. So we assume $z_n \ne z_{n+1}$, for all $n \ge 0$. Let $c_n = \rho(z_n, z_{n+1})$, for $n \ge 0$. By assumption (i)

$$\begin{aligned} c_{n+1} &= \rho(z_{n+1}, z_{n+2}) = \rho(Sz_n, Sz_{n+1}) \\ &\le \phi(\max\{\rho(z_n,\ z_{n+1}),\ \rho(z_n,\ Sz_n),\ \rho(z_{n+1},\ Sz_{n+1}), \\ &\qquad \frac{\rho(z_n,\ Sz_{n+1}) + \rho(z_{n+1},\ Sz_n)}{2}\}) \\ &\le \phi(\max\{\rho(z_n,\ z_{n+1}),\ \rho(z_n,\ z_{n+1}),\ \rho(z_{n+1},\ z_{n+2}), \\ &\qquad \frac{\rho(z_n,\ z_{n+2}) + \rho(z_{n+1},\ z_{n+1})}{2}\}) \\ &\le \phi(\max\{\rho(z_n,\ z_{n+1}),\ \rho(z_n,\ z_{n+1}),\ \rho(z_{n+1},\ z_{n+2}), \\ &\qquad \frac{\rho(z_n,\ z_{n+1}) + \rho(z_{n+1},\ z_{n+2})}{2}\}) \\ &\le \phi(\max\{\rho(z_n,\ z_{n+1}),\ \rho(z_{n+1},\ z_{n+2})\}) \\ &= \phi(\max\{c_n,\ c_{n+1}\}) < \max\{c_n,\ c_{n+1}\}. \end{aligned}$$

Suppose that $c_n \leq c_{n+1}$. Then we have from the above inequality that $c_{n+1} < c_{n+1}$, which is a contradiction. Hence $c_{n+1} < c_n$, for all n. Then $\{c_n\}$ tends to a limit $c \geq 0$.

If possible, suppose $c > 0$. Then we have a positive integer k such that $n \geq k$ implies

$$c < c_n < c + \delta(c). \tag{75}$$

It follows from assumption (ii) and $c_{n+1} < c_n$ that $c_{n+1} \leq c$, for $n \geq k$, which contradicts the above inequality. Thus we have c = 0.

Let us fix $\epsilon > 0$. Without loss of generality, we may assume that $\delta(\epsilon) < \epsilon$. Since $c_n \to 0$ as $n \to +\infty$, there exists a positive integer k such that $c_n < \frac{\delta}{2}$, for all $n \geq k$. We shall use induction to show that for any $n \in N$,

$$\rho(z_k, z_{k+n}) < \epsilon + \frac{\delta}{2}. \tag{76}$$

The inequality (76) is true for $n = 1$. Assuming (76) is true for some n, we shall prove it for $n + 1$. Now

$$\rho(z_k, z_{k+n+1}) \leq \rho(z_k, z_{k+1}) + \rho(z_{k+1}, z_{k+n+1}). \tag{77}$$

It sufficient to show that

$$\rho(z_{k+1}, z_{k+n+1}) \leq \epsilon. \tag{78}$$

By assumption (i),

$$\rho(z_{k+1}, z_{k+n+1}) = \rho(Sz_k, Sz_{k+n}) \leq \phi(Q(z_k, z_{k+n})) < Q(z_k, z_{k+n}), \tag{79}$$

where

$$\begin{aligned} Q(z_k, z_{k+n}) &= \max\{\rho(z_k, z_{k+n}), \rho(z_k, Sz_k), \rho(z_{k+n}, Sz_{k+n}), \\ &\qquad \frac{\rho(z_k, Sz_{k+n}) + \rho(z_{k+n}, Sz_k)}{2}\} \\ &= \max\{\rho(z_k, z_{k+n}), \rho(z_k, z_{k+1}), \rho(z_{k+n}, z_{k+n+1}), \\ &\qquad \frac{\rho(z_k, z_{k+n+1}) + \rho(z_{k+n}, z_{k+1})}{2}\}. \end{aligned}$$

Now, $\rho(z_k, z_{k+n}) < \epsilon + \frac{\delta}{2}$, $\rho(z_k, z_{k+1}) < \frac{\delta}{2}$, $\rho(z_{k+n}, z_{k+n+1}) < \frac{\delta}{2}$, $\frac{\rho(z_k, z_{k+n+1}) + \rho(z_{k+n}, z_{k+1})}{2} \leq \frac{\rho(z_k, z_{k+n}) + \rho(z_{k+n}, z_{k+n+1}) + \rho(z_{k+n}, z_k) + \rho(z_k, z_{k+1})}{2} < \epsilon + \delta$. Hence $Q(z_k, z_{k+n}) < \epsilon + \delta$. If $0 \leq Q(z_k, z_{k+n}) \leq \epsilon$, then by (79), it follows that $\rho(z_{k+1}, z_{k+n+1}) \leq \epsilon$, that is, (78) is true. Again, if $\epsilon < Q(z_k, z_{k+n}) < \epsilon + \delta$, then by assumption (ii) and (79) we have that $\rho(z_{k+1}, z_{k+n+1}) \leq \epsilon$, that is, (78) is true. Therefore, $\rho(z_{k+1}, z_{k+n+1}) \leq \epsilon$, that is, (78) is true. Then from (77), we have that $\rho(z_k, z_{k+n+1}) < \epsilon + \frac{\delta}{2}$. Then by the induction method, (76) is true for any $n \in N$.

This implies that $\{z_n\}$ is a Cauchy sequence. Since M is complete, there exists a point $y \in M$ such that $z_n \to y$ as $n \to +\infty$. Also $Sz_n \to y$ and $S^2 z_n \to y$. By continuity of S^2, we have $S^2 z_n \to S^2 y$. This implies $S^2 y = y$.

We claim that $Sy = y$.

If possible, suppose that $y \neq Sy$. Then by (i), we get

$$\begin{aligned}\rho(y, Sy) &= \rho(S^2 y, Sy) \leq \phi(Q(Sy, y)) < Q(Sy, y)\\ &= \max\{\rho(Sy, y), \rho(Sy, S^2 y), \rho(y, Sy), \frac{\rho(Sy, Sy) + \rho(y, S^2 y)}{2}\} = \rho(y, Sy),\end{aligned}$$

which is a contradiction. Thus $y = Sy$, that is, y is a fixed point of S.

Suppose that $\zeta \in M$ ($\zeta \neq y$) is another fixed point of S. Then $\rho(y, \zeta) > 0$. By (i), we have

$$\begin{aligned}\rho(y, \zeta) &= \rho(Sy, S\zeta) \leq \phi(Q(y, \zeta)) < Q(y, \zeta)\\ &= \max\{\rho(y, \zeta), \rho(y, Sy), \rho(\zeta, S\zeta), \frac{\rho(y, S\zeta) + \rho(\zeta, Sy)}{2}\} = \rho(y, \zeta),\end{aligned}$$

which is a contradiction. Therefore, $\rho(y, \zeta) = 0$, that is, $y = \zeta$. Hence, S has a unique fixed point.

Example 17 ([3]) Take the metric space $M = [0, 2]$ with the metric. Define $S : M \to M$ as

$$Su = \begin{cases} 1, \text{ if } u \leq 1, \\ 0, \text{ if } u > 1. \end{cases}$$

The mapping S satisfies assumption (i) with $\phi(t) = 1$ for $t > 1$ and $\phi(t) = \frac{t}{2}$ for $t \leq 1$. Also, S satisfies assumption (ii) with $\delta(\epsilon) = 1$ for $\epsilon \geq 1$ and $\delta(\epsilon) = 1 - \epsilon$ for $\epsilon < 1$. Hence S satisfies all the assumptions of Theorem 14 and has a unique fixed point $u = 1$. Here, $\lim\limits_{u \to 1} Q(u, 1) \neq 0$ and S is discontinuous at the fixed point $u = 1$.

5 Remark

We have already mentioned that the present chapter is not sufficient for a comprehensive description of the topic under consideration. Among important results which form integral parts of the theory but are not covered here are the following. Asymptotic contractions in fixed point theory were introduced by Kirk [23]. Further generalizations of Kirk's result were done in works like [39, 41]. A very generalized fixed point theorem unifying many important results was introduced by Pant [28] which is significantly important. In 2006, Proinov [30] introduced a generalization of Banach's contraction mapping principle in a new direction which was subsequently shown to be even more general than Ćirić's quasi-contraction [10]. The review paper of Rhoades [32] is important for comprehending comparisons between several contractive conditions used in fixed point theory. Although not discussed in their technical details, the reader is strongly advised to consult these works.

Many of the results described above have initiated new lines of research in fixed point theory. For instance, the result of Caristi [5] is the origin of a study in fixed point theory and variational principles which by its vastness and importance is itself a chapter of mathematics. We do not dwell on these matters within the limited scope of this chapter. But we must say that without these considerations, the appreciation of the results presented here is bound to be partial.

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