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Pradip Debnath
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Metric Fixed Point Theory

Applications in Science, Engineering and Behavioural Sciences

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Preface

Fixed point theory emerged as an indispensable tool over the last few decades in nonlinear sciences and engineering including behavioral science, mathematical economics, physics, etc. To be precise, while formulating an experiment mathematically, we often have to investigate the solvability of a functional equation in terms of differential, integral, fractional differential, or matrix equations. Such a solution is often achieved by finding fixed point of a particular mapping. The three major approaches in fixed point theory are metric approach, topological approach, and discrete approach. In this book, we mainly focus on the theory and applications of metric fixed point theory.

This book is meant for researchers, graduate students, and teachers interested in the theory of fixed points. Mathematicians, engineers, and behavioral scientists will also find the book useful. The readers of this book will require minimum pre-requisites of undergraduate studies in functional analysis and topology. This book has a collection of chapters authored by several renowned contemporary researchers across the world in fixed point theory. Here, readers will find several useful tools and techniques to develop their skills and expertise in fixed point theory. The book contains sufficient theory and applications of fixed points in several areas. The book presents a survey of the existing knowledge and also the current state-of-the-art development through original new contributions from the famous researchers all over the world.

This book consists of total 15 chapters. Chapter 1 provides a detailed review of the most important basic fixed point theorems in metric spaces, which are essential for the sequel. In Chap. 2, fixed point theorems related to the infinite system of integral equations have been studied. Chapter 3 presents the study of common fixed points in a generalized metric space. Fixed point results and their applications in various modular metric spaces have been discussed in Chaps. 4–6. Chapter 7 provides a new insight into parametric metric spaces, whereas variational in equalities and variational control problems have been studied in Chaps. 8–10. Some optimization techniques in terms of best proximity points and coincidence best proximity results have been presented in Chaps. 11 and 12, respectively. Application of fixed points to the mathematics of fractals has been presented in Chap. 13. A survey on nonexpansive mappings and their extensions in Banach spaces is provided in Chap. 14. Finally, in Chap. 15, we explore the applications of fixed point theory in behavioral sciences.

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Basic Fixed Point Theorems in Metric Spaces

Binayak S. Choudhury and Nikhilesh Metiya

Abstract This chapter is a review work on the development of metric fixed point theory. It begins with the description of Banach's Contraction Mapping Principle and finally contains results established in the recent years as well. The proofs are presented for every theorem discussed here. Several illustrations are given. The development is presented separately for functions with and without continuity property. Only results on metric spaces without any additional structures are considered.

1 Introduction

It is widely held that metric fixed point theory originated in the year 1922 through the work of S. Banach when he established the famous Contraction Mapping Principle [2] which has come to be known by his name. It is a versatile domain of mathematics having implications in several other branches of science, technology and economics [1, 31, 43]. At present even after a century of its initiation, the subject area remains vibrant with research activities.

Admittedly, putting together all basic theorems in metric fixed point theory in a single chapter is an impossible task. One has to be selective on this issue. We do not mean to undermine those results which are left out of our selection. They can even be more important than those which are included in this chapter. For instance,

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Generalized Banach Contraction Conjecture (GBCC) result of Merryfield et al. [26] is not included in this chapter. There are many important fixed point results which are deduced in metric spaces having additional structures like partial order, graph, etc. But here we consider only those results which are relevant to metric spaces without any additional structures. The only additional property which we consider here is the completeness property of the metric space. Further, we describe theorems for mappings with or without continuity assumption.

Definition 1 (*Fixed point*) Let M be a nonempty set and $S : M \to M$ be a mapping. A fixed point of S is a point $\xi \in M$ such that $S\xi = \xi$, that is, a fixed point of S is a solution of the functional equation $Sz = z$, $z \in M$.

A self-mapping may have no fixed point, a unique fixed point and more than one fixed point. This is illustrated in the following examples.

Example 1 Take R the set of all real numbers equipped with usual metric.

(i) The mapping $S : R \to R$, $Sz = z^3$, $z \in R$ has three fixed points $z = 0$, $z = 1$ and $z = -1$.
(ii) The mapping $S : R \to R$ defined by $Sz = -z^3$, $z \in R$ has only fixed point $z = 0$.
(iii) The mapping $S : R \to R$ where $Sz = z + \sin z$, $z \in R$ has fixed points $z = n\pi,\ n = 0, \pm 1, \pm 2, ...$
(iv) The mapping $S : R \to R$ defined as $Sz = z + 1$, $z \in R$ has no fixed point.

2 Banach's Contraction Mapping Principle

The first result we describe is the famous Contraction Mapping Principle.

Definition 2 (*Contraction mapping*) A mapping $S : M \to M$, where (M, ρ) is a metric space, is called a Lipschitz mapping if there exists a real number $k > 0$ such that $\rho(Su, Sv) \leq k\, \rho(u, v)$ holds for all $u, v \in M$. The smallest positive real number k for which the Lipschitz condition is valid is called the Lipschitz constant of S.

If the Lipschitz constant k lies between 0 and 1, that is, if $0 < k < 1$, then the Lipschitz mapping S is called a contraction mapping.

Obviously, a contraction mapping is continuous.

Example 2 (i) The mapping $S : [0, 1) \to [0, 1)$ defined by $Sz = \frac{z}{5}$ is a contraction mapping.
(ii) The mapping $S : R \to R$ defined by $Sz = \frac{5z+3}{2}$ is not a contraction mapping.

In 1922, Banach established a fixed point result for a self-map S of a complete metric space using a contractive condition, which is known as Banach's contraction mapping principle.

Theorem 1 (Banach's contraction mapping principle [2]) *A self-mapping S of a complete metric space $(M,\ \rho)$ admits a unique fixed point if for all $u,\ v \in M$,*

$$\rho(Su,\ Sv) \leq k\ \rho(u,\ v),\ \text{where } 0 < k < 1. \tag{1}$$

Proof Suppose $\zeta, \eta \in M$ with $\zeta \neq \eta$ are two fixed points of S. From (1), we have $\rho(\zeta, \eta) = \rho(S\zeta, S\eta) \leq k\ \rho(\zeta, \eta)$, which is a contradiction. Hence the fixed point of S, if it exists is unique.

Choose any point $z_0 \in M$. We construct a sequence $\{z_n\}$ in M such that

$$z_n = Sz_{n-1} = S^n z_0 \ \text{ for all } \ n \geq 1. \tag{2}$$

For each positive integer n, we have

$$\begin{aligned} \rho(z_n,\ z_{n+1}) &= \rho(Sz_{n-1}, Sz_n) \\ &\leq k\ \rho(z_{n-1}, z_n) \\ &\leq k^2\ \rho(z_{n-2}, z_{n-1}) \\ &\quad ... \\ &\leq k^n\ \rho(z_0, z_1). \end{aligned}$$

By triangular inequality, we have for $n > m$,

$$\begin{aligned} \rho(z_m, z_n) &\leq \rho(z_m, z_{m+1}) + \rho(z_{m+1}, z_{m+2}) + ... + \rho(z_{n-1}, z_n) \\ &\leq k^m\ \rho(z_0, z_1) + k^{m+1}\ \rho(z_0, z_1) + ... + k^{n-1}\ \rho(z_0, z_1) \\ &\leq k^m[1 + k + k^2 + ... + k^{n-m-1}]\ \rho(z_0, z_1) \\ &< k^m[1 + k + k^2 + ...]\ \rho(z_0, z_1) \\ &= \frac{k^m}{1-k}\ \rho(z_0, z_1) \to 0,\ \text{ as } n \to +\infty \text{ [since } \alpha < 1], \end{aligned}$$

which implies that $\{z_n\}$ is a Cauchy sequence in M. By the completeness of M, there exists $\xi \in M$ such that $z_n \to \xi$, as $n \to +\infty$.

Being a contraction mapping, S is continuous. Therefore, we have $S\xi = \lim_{n\to+\infty} Sz_n = \lim_{n\to+\infty} z_{n+1} = \xi$. Hence, ξ is a fixed point S. By what we have already proved, ξ is the unique fixed point of S.

Example 3 Take the complete metric space R equipped with usual metric and the contraction mapping $S : R \to R$ defined as $Sz = 2(1 - \frac{z}{5})$. We see that $z = \frac{10}{7}$ is the unique fixed point of S.

3 Generalizations of Contraction Mapping Principle

In 1969, Boyd and Wong [4] made a very interesting generalization of the Banach's contraction mapping principle in complete metric spaces. They replaced the constant k in (1) of Theorem 1 by a function $\varphi : [0, +\infty) \to [0, +\infty)$ which is upper semicontinuous from the right (that is, $t_n \downarrow t \geq 0 \Rightarrow \limsup \varphi(t_n) \leq \varphi(t)$).

The following result is due to Boyd and Wong [4].

Theorem 2 A self-mapping S of a complete metric space (M, ρ) admits a unique fixed point if there exists a function $\varphi : [0, +\infty) \to [0, +\infty)$ which is upper semicontinuous from the right with $0 \leq \varphi(t) < t$ for $t > 0$ and the following inequality holds:

$$\rho(Su, Sv) \leq \varphi(\rho(u, v)), \text{ for all } u,\ v \in M. \tag{3}$$

Proof Let $z_0 \in M$ be any arbitrary element. We define a sequence $\{z_n\}$ in M such that $z_n = Sz_{n-1} = S^n z_0$, for all $n \geq 1$. If $z_l = z_{l+1}$ for some positive integer l, then z_l is a fixed point of S. So we assume that $z_n \neq z_{n+1}$, for all $n \geq 0$.

Applying (3) and using the property of φ, we have

$$\rho(z_{n+1}, z_{n+2}) = \rho(Sz_n, Sz_{n+1}) \leq \varphi(\rho(z_n, z_{n+1})) < \rho(z_n, z_{n+1}), \text{ for all } n \geq 0. \tag{4}$$

Therefore, $\{\rho(z_n,\ z_{n+1})\}$ is a monotonic decreasing sequence which is bounded below by 0 and hence there exists an $\delta \geq 0$ for which

$$\lim_{n\to+\infty} \rho(z_n, z_{n+1}) = \delta. \tag{5}$$

From (4), we have

$$\rho(z_{n+1}, z_{n+2}) \leq \varphi(\rho(z_n, z_{n+1})), \text{ for all } n \geq 0.$$

Taking limit supremum as $n \to +\infty$ on both sides and using (5) and the properties of φ, we have $\delta \leq \varphi(\delta) < \delta$. It is a contradiction unless $\delta = 0$. Hence

$$\lim_{n\to+\infty} \rho(z_n, z_{n+1}) = 0. \tag{6}$$

We prove that $\{z_n\}$ is a Cauchy sequence by method of contradiction. If possible, suppose that $\{z_n\}$ is not a Cauchy sequence. Then we have an $\epsilon > 0$ for which there exist two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that

$$n(k) > m(k) > k, \ \ \rho(z_{m(k)},\ z_{n(k)}) \geq \epsilon \ \text{ and } \ \rho(z_{m(k)},\ z_{n(k)-1}) < \epsilon.$$

Now,

$$\begin{aligned}\epsilon \le \rho(z_{m(k)}, z_{n(k)}) &\le \rho(z_{m(k)}, z_{n(k)-1}) + \rho(z_{n(k)-1},\ z_{n(k)}) \\ &< \epsilon + \rho(z_{n(k)-1},\ z_{n(k)}).\end{aligned}$$

Using (6), we have

$$\lim_{k\to+\infty} \rho(z_{m(k)},\ z_{n(k)}) = \epsilon. \tag{7}$$

Again,

$$\begin{aligned}\rho(z_{m(k)}, z_{n(k)}) &\le \rho(z_{m(k)}, z_{m(k)+1}) + \rho(z_{m(k)+1},\ z_{n(k)+1}) + \rho(z_{n(k)}, z_{n(k)+1}) \\ &\le \rho(z_{m(k)}, z_{m(k)+1}) + \varphi(\rho(z_{m(k)}, z_{n(k)})) + \rho(z_{n(k)},\ z_{n(k)+1}).\end{aligned}$$

Taking limit supremum as $n \to +\infty$ on both sides of the inequality and using (6), (7) and the properties of φ, we have $\epsilon \le \varphi(\epsilon) < \epsilon$. This is a contradiction. Hence $\{z_n\}$ is a Cauchy sequence. As (M, ρ) is complete, there exists $\xi \in M$ such that $z_n \to \xi$, as $n \to +\infty$.

We now show that ξ is a fixed point of S. It follows by the contraction condition that S is continuous. Therefore, $S\xi = \lim_{n\to+\infty} Sz_n = \lim_{n\to+\infty} z_{n+1} = \xi$. Hence ξ is a fixed point S.

Let z be a fixed point of S other than ξ. Then $\rho(z, \xi) > 0$. From (3), we have $\rho(z, \xi) = \rho(Sz, S\xi) \le \varphi(\rho(z, \xi)) < \rho(z, \xi)$, which is a contradiction. Hence, ξ is the unique fixed point of S.

Example 4 Take the metric space $M = [0, 1]$ equipped with usual metric. Define $S : M \to M$ as $Sz = z - \frac{z^2}{2}$, for $z \in M$. Let $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ be defined by

$$\varphi(t) = \begin{cases} t - \frac{t^2}{2}, & \text{if } 0 \le t \le 1, \\ \frac{t}{2}, & \text{otherwise.} \end{cases}$$

Boyd and Wong fixed point theorem is applicable and $z = 0$ is the unique fixed point of S.

In 1969, Meir and Keeler [25] established that the conclusion of Banach's theorem holds more generally from the following condition of weakly uniformly strict contraction:

Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\epsilon \le \rho(x, y) < \epsilon + \delta \ \text{ implies } \ \rho(Sx, Sy) < \epsilon. \tag{8}$$

The following result is due to Meir and Keeler [25].

Theorem 3 A self-mapping S of a complete metric space (M, ρ) admits a unique fixed point if (8) holds.

Proof We first observe that (8) implies that

$$\rho(Sx, Sy) < \rho(x, y) \ \text{ whenever } \ x \ne y. \tag{9}$$

Suppose that ζ and η are two distinct fixed points of S. Then from (9), we have $\rho(\zeta, \eta) = \rho(S\zeta, S\eta) < \rho(\zeta, \eta)$, which is a contradiction. Hence S may have at most one fixed point.

Let $z_0 \in M$ be any arbitrary element. Take the same sequence $\{z_n\}$ in M as in the proof of Theorem 2. We take $z_n \neq z_{n+1}$, for all $n \geq 0$. This is because in the case $z_l = z_{l+1}$, for some positive integer l, z_l is a fixed point of S.

Let $c_n = \rho(z_n, z_{n+1})$. From (9), we can show that $\{c_n = \rho(z_n, \ z_{n+1})\}$ is a monotonic decreasing sequence of nonnegative real numbers. Then there exists an $\epsilon \geq 0$ such that $c_n \to \epsilon$, as $n \to +\infty$. If possible, suppose that $\epsilon > 0$. As $\{c_n\}$ is decreasing and $c_n \to \epsilon$, as $n \to +\infty$, for $\delta > 0$ there exists m such that $\epsilon \leq c_n < \epsilon + \delta$ for all $n \geq m$. Therefore, $\epsilon \leq c_m < \epsilon + \delta$. Then from (8) it follows that $c_{m+1} = \rho(z_{m+1}, z_{m+2}) = \rho(Sz_m, Sz_{m+1}) < \epsilon$, which is a contradiction. Hence $\epsilon = 0$. Therefore,

$$\lim_{n \to +\infty} \rho(z_n, z_{n+1}) = 0. \tag{10}$$

We suppose that $\{z_n\}$ is not a Cauchy sequence. Then there exists $2\epsilon > 0$ such that $\limsup \rho(z_m, z_n) > 2\epsilon$. By the hypothesis, there exists a $\delta > 0$ such that

$$\epsilon \leq \rho(x, y) < \epsilon + \delta \text{ implies } \rho(Sx, Sy) < \epsilon. \tag{11}$$

Formula (11) remains true if we replace δ by $\delta^{'} = \min\{\delta, \epsilon\}$. By (10), there exists a positive integer P for which $c_P < \frac{\delta^{'}}{3}$. Choose $m, n > P$ so that $\rho(z_m, z_n) > 2\epsilon$. Now for any $j \in [m, n]$, we have

$$| \ \rho(z_m, z_j) - \rho(z_m, z_{j+1}) \ | \leq c_j < \frac{\delta^{'}}{3}.$$

This implies, since $\rho(z_m, z_{m+1}) < \epsilon$ and $\rho(z_m, z_n) > \epsilon + \delta^{'}$, that there exists $j \in [m, n]$ with

$$\epsilon + \frac{2\delta^{'}}{3} < \rho(z_m, z_j) < \epsilon + \delta^{'}. \tag{12}$$

However, for all m and j,

$$\rho(z_m, z_j) \leq \rho(z_m, z_{m+1}) + \rho(z_{m+1}, z_{j+1}) + \rho(z_{j+1}, z_j).$$

From (11) and (12), we have

$$\rho(z_m, z_j) \leq c_m + \epsilon + c_j < \frac{\delta^{'}}{3} + \epsilon + \frac{\delta^{'}}{3},$$

which contradicts (12). Therefore, $\{z_n\}$ is a Cauchy sequence.

Now (9) implies that S is continuous. As discussed in the proof of Theorem 1, we conclude that S has a unique fixed point.

Example 5 ([25]) Let $M = [0, 1] \cup \{3, 4, 6, 7, ..., 3n, 3n + 1, ...\}$ be equipped with Euclidean metric and $S : M \to M$ be defined by

$$S(u) = \begin{cases} \frac{u}{2}, & \text{if } 0 \le u \le 1, \\ 0, & \text{if } u = 3n, \\ 1 - \frac{1}{n+2}, & \text{if } u = 3n + 1. \end{cases}$$

Here, Theorem 3 is applicable and the unique fixed point of S is $u = 0$.

It is observed that in Banach's contraction mapping principle, the contraction condition is global, that is, the operators satisfy the contraction condition for every pair of points taken from the metric space. A natural question arises whether the conclusion of Banach's theorem is true if the contraction condition is satisfied locally, that is, for sufficiently close points only. The answer was given in the affirmative in a paper by Michael Edelstein [14] in 1961.

Definition 3 *(Local Contraction* [14]) A self-mapping $S : M \to M$, where (M, ρ) is a metric space, is locally contractive if for every $x \in M$ there exist $\epsilon > 0$ and $\lambda \in [0, 1)$, which may depend on x, such that

$$p, q \in S(x, \epsilon) = \{y : \rho(x, y) < \epsilon\} \text{ implies } \rho(Sp,\ Sq) < \lambda\, \rho(p,\ q). \tag{13}$$

Definition 4 (*Uniform Local Contraction* [14]) A uniformly locally contractive mapping on a metric space (M, ρ) is a locally contractive mapping $S : M \to M$ where both ϵ and λ do not depend on x.

Definition 5 ([14]) Let (M, ρ) be a metric space such that for every $a, b \in M$ there exists an η-chain, that is, a finite set of points $a = x_0, x_1, ..., x_n = b$ (n may depend on both a and b) satisfying $\rho(x_{j-1}, x_j) < \eta$ $(j = 1, 2, ..., n)$. Then (M, ρ) is η-chainable.

Theorem 4 (Edelstein [14]) *An (ϵ, λ)—uniformly locally contractive mapping $S : M \to M$ on a ϵ-chainable complete metric space (M, ρ) has a unique fixed point.*

Proof Choose any point $z \in M$. Take the ϵ-chain : $z = z_0,\ z_1,\ ...,\ z_n = Sz$. By the triangular property, we have

$$\rho(z, Sz) \le \sum_{1}^{n} \rho(z_{i-1}, z_i) < n\epsilon. \tag{14}$$

For pairs of consecutive points of the ϵ-chain, condition (13) is satisfied. Hence, denoting $S(S^m z) = S^{m+1} z$ $(m = 1, 2, ...)$, we have

$$\rho(Sz_{i-1}, Sz_i) < \lambda\, \rho(z_{i-1}, z_i) < \lambda\, \epsilon;$$

and, by repeated application of the above inequality, we have

$$\rho(S^m z_{i-1}, S^m z_i) < \lambda\, \rho(S^{m-1} z_{i-1}, S^{m-1} z_i) < \lambda^m\, \epsilon. \tag{15}$$

Using (14) and (15), we have

$$\rho(S^m z, S^{m+1} z) \le \sum_{i=1}^{n} \rho(S^m z_{i-1}, S^m z_i) < \lambda^m\, n\, \epsilon. \tag{16}$$

Now, for any two positive integers $j, k (j < k)$, we have

$$\begin{aligned}\rho(S^j z, S^k z) &\le \sum_{i=j}^{k-1} \rho(S^i z, S^{i+1} z) < n\epsilon\, [\lambda^j + \lambda^{j+1} + ... + \lambda^{k-1}] \\ &< \frac{\lambda^j}{1-\lambda} n\epsilon \to 0, \text{ as } j \to +\infty.\end{aligned}$$

It follows that $\{S^i z\}$ is a Cauchy sequence in M. Now, M being complete, there exists a point $\xi \in M$ such that $S^i z \to \xi$, as $i \to +\infty$.

Now (13) implies that S is continuous. Therefore, we have $S\xi = \lim_{i\to+\infty} S(S^i z) = \lim_{i\to+\infty} S^{i+1} z = \xi$. Hence ξ is a fixed point S.

If possible, let ζ $(\zeta \neq \xi)$ be another fixed point of S. Now $\rho(\xi, \zeta) > 0$. Let $\xi = z_0, z_1, ..., z_k = \zeta$ be an ϵ-chain. Using (15), we have

$$\begin{aligned}\rho(\xi, \zeta) = \rho(S\xi, S\zeta) &\le \rho(S^l \xi, S^l \zeta) \\ &\le \sum_{i=1}^{k} \rho(S^l z_{i-1}, S^l z_i) < \lambda^l\, k\epsilon \to 0 \text{ as } l \to +\infty,\end{aligned}$$

which is a contradiction. Hence, $\xi = \zeta$ and our proof is completed.

Example 6 Let $M = \{(u, v) : u = \cos\theta,\ \ v = \sin\theta,\ \ 0 \le \theta \le \frac{3}{2}\pi\}$ be equipped with Euclidean metric. Define $S : M \to M$ as $Sp = (\frac{u}{2}, \frac{v}{2})$, for $p = (u, v) \in M$. Theorem 4 is applicable here and $p = (0, 0)$ is the unique fixed point of S.

In 2012, Samet et al. [37] introduced the new concept of $\alpha - \psi$-contractive type mapping and established a fixed point theorem for such mappings in complete metric spaces. The presented theorem therein extends, generalizes and improves the famous Banach's contraction mapping principle. We describe here the notions of $\alpha - \psi$-contractive and α-admissible mappings.

Let Ψ denote the family of nondecreasing functions $\psi : [0,\ +\infty) \to [0,\ +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is nth iterate of ψ.

Lemma 1 ([37]) *If* $\psi : [0,\ +\infty) \to [0,\ +\infty)$ *be a nondecreasing function satisfying* $\lim_{n\to+\infty} \psi^n(t) = 0$ *for each* $t > 0$*, then* $\psi(t) < t$ *for each* $t > 0$.

Definition 6 ([37]) Let $S : M \to M$ and $\alpha : M \times M \to [0,\ +\infty)$ be two mappings. The mapping T is α-admissible if $\alpha(u,\ v) \ge 1 \Longrightarrow \alpha(Tu,\ Tv) \ge 1$, for $u, v \in M$.

Example 7 Let $M = [0,\ 1]$. Let $S : M \to M$ and $\alpha : M \times M \to [0,\ +\infty)$ be respectively defined as follows:

$$Sz = \frac{\sin^2 z}{16},\ \text{for } z \in M \text{ and } \alpha(u,\ v) = \begin{cases} e^{u+v}, \text{ if } 0 \le u \le 1,\ 0 \le v \le \frac{1}{8}, \\ 0, \text{ otherwise.} \end{cases}$$

Here S is α-admissible.

Definition 7 ([37]) A mapping $T : M \to M$, where (M, d) is a metric space, is called an $\alpha - \psi$-contractive mapping if there exist two functions $\alpha : M \times M \to [0,\ +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(u, v)\ \rho(Tu, Tv) \le \psi(\rho(u, v)),\ \text{for all } u, v \in M.$$

Remark 1 If $\alpha(u, v) = 1$ for all $u, v \in M$ and $\psi(t) = kt$ for all $t \ge 0$ and some $k \in [0, 1)$, the $\alpha - \psi$-contractive mapping reduces to Banach's contraction mapping.

Theorem 5 (Samet et al. [37]) *Let (M, ρ) be a complete metric space, $S : M \to M$ and $\alpha : M \times M \to [0,\ +\infty)$. Suppose that (i) S is α-admissible, (ii) there exists $z_0 \in M$ such that $\alpha(z_0,\ Sz_0) \ge 1$, (iii) S is continuous and (iv) there exists $\psi \in \Psi$ such that S is an $\alpha - \psi$-contractive mapping. Then S admits a fixed point.*

Proof Let $z_0 \in M$ such that $\alpha(z_0,\ Sz_0) \ge 1$. We construct a sequence $\{z_n\}$ in M such that

$$z_{n+1} = Sz_n,\ \text{for all } n \ge 0. \tag{17}$$

Then $\alpha(z_0,\ z_1) \ge 1$. As S is α-admissible, we have $\alpha(Sz_0, Sz_1) = \alpha(z_1, z_2) \ge 1$. Again, applying the admissibility assumption, we have $\alpha(Sz_1, Sz_2) = \alpha(z_2, z_3) \ge 1$. Continuing this process, we have

$$\alpha(z_n, z_{n+1}) \ge 1,\ \text{for all } n \ge 0. \tag{18}$$

Like in the proof of Theorem 2, we show that the possibility of $z_l = z_{l+1}$ occurring, for some positive integer l, ensures that z_l is a fixed point of S. So we consider the case $z_n \ne z_{n+1}$, for all $n \ge 0$.

Applying (iv) with $z = z_{n-1}$ and $y = z_n$, where $n \ge 1$, and using (17) and (18), we obtain

$$\rho(z_n, z_{n+1}) = \rho(Sz_{n-1}, Sz_n) \le \alpha(z_{n-1}, z_n)\ \rho(Sz_{n-1}, Sz_n) \le \psi(\rho(z_{n-1}, z_n)).$$

By repeated the application of the above inequality and a property of ψ, we have

$$\rho(z_n, z_{n+1}) \le \psi^n(\rho(z_0, z_1)),\ \text{for all } n \ge 1.$$

With the help of the above inequality, we have

$$\sum_{n=1}^{+\infty} \rho(z_n,\ z_{n+1}) \leq \sum_{n=1}^{+\infty} \psi^n(\rho(z_0, z_1)) < +\infty,$$

which implies that $\{z_n\}$ is a Cauchy sequence in M. As M is complete, we get $\xi \in M$ such that $\lim_{n\to+\infty} z_n = \xi$. From the continuity of S, it follows that $S\xi = \lim_{n\to+\infty} Sz_n = \lim_{n\to+\infty} z_{n+1} = \xi$. Hence ξ is a fixed point S.

Example 8 ([37]) Take $M = R$ the set of all real numbers endowed with the usual metric ρ. Let $S : M \to M$ be defined as follows:

$$Sz = \begin{cases} 2z - \frac{3}{2}, & \text{if } z > 1, \\ \frac{z}{2}, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{if } z < 0. \end{cases}$$

As $\rho(S1, S2) = 2 > 1 = \rho(2, 1)$, the Banach's contraction mapping principle cannot be applied in this case.

Define $\psi : [0,\ +\infty) \to [0,\ +\infty)$ and $\alpha : M \times M \to [0,\ +\infty)$ as follows:

$$\psi(t) = \frac{t}{2} \text{ and } \alpha(u,\ v) = \begin{cases} 1, & \text{if } \ u, v \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Here Theorem 5 is applicable and $z = 0$ is a fixed point of S.

In 1973, Geraghty [17] introduced a class of functions to generalize the Banach's contraction mapping principle. Let S be the class of all functions $\beta : [0, +\infty) \to [0, 1)$ satisfying the property: $\beta(t_n) \to 1, \ \text{ as } \ t_n \to 0.$

An example of a function in S may be given by $\beta(t) = e^{-2t}$ for $t > 0$ and $\beta(0) \in [0, 1)$.

Theorem 6 (Geraghty [17]) *A self-mapping S of a complete metric space (M, ρ) admits a unique fixed point if there exists a function $\beta \in S$ such that*

$$\rho(Su, Sv) \leq \beta(\rho(u, v))\ \rho(u, v), \quad \text{for all } u, v \in M. \tag{19}$$

Proof Suppose that S has two fixed points ζ and η with $\zeta \neq \eta$. From (19), we have $\rho(\zeta, \eta) = \rho(S\zeta, S\eta) \leq \beta(\rho(\zeta, \eta))\,\rho(\zeta, \eta) < \rho(\zeta, \eta)$, which is a contradiction. Hence the fixed point of S, if it exists, is unique.

Let $z_0 \in M$ be any arbitrary element. Take the same sequence $\{z_n\}$ in M as in the proof of Theorem 2. Like in the proof of Theorem 2, we show that the possibility of $z_l = z_{l+1}$ occurring, for some positive integer l, implies the existence of a fixed point of S. So we assume that $z_n \neq z_{n+1}$, for all $n \geq 0$.

First we prove $\lim_{n\to+\infty} \rho(z_n, z_{n+1}) = 0$. Applying (19) and using the property of β, we have for all $n \geq 0$,

$$\rho(z_{n+1}, z_{n+2}) = \rho(Sz_n, Sz_{n+1}) \leq \beta(\rho(z_n, z_{n+1}))\ \rho(z_n, z_{n+1}) < \rho(z_n, z_{n+1}). \tag{20}$$

Therefore, $\{\rho(z_n,\ z_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. We get an $\delta \geq 0$ such that $\lim_{n\to+\infty} \rho(z_n, z_{n+1}) = \delta$.

Suppose that $\delta > 0$. From (20), we have

$$\frac{\rho(z_{n+1}, z_{n+2})}{\rho(z_n, z_{n+1})} \leq \beta(\rho(z_n, z_{n+1})) < 1, \text{ for all } n \geq 0.$$

Then

$$1 \leq \lim_{n\to+\infty} \beta(\rho(z_n, z_{n+1})) < 1,$$

which implies that

$$\lim_{n\to+\infty} \beta(\rho(z_n, z_{n+1})) = 1. \tag{21}$$

It follows by the property of β that $\lim_{n\to+\infty} \rho(z_n, z_{n+1}) = 0$, which contradicts our assumption. Hence $\delta = 0$, that is, $\lim_{n\to+\infty} \rho(z_n, z_{n+1}) = 0$.

Next we show that $\{z_n\}$ is a Cauchy sequence. If $\{z_n\}$ is not a Cauchy sequence then arguing similarly as in the proof of Theorem 2, we get an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that $\lim_{k\to+\infty} \rho(z_{m(k)}, z_{n(k)}) = \epsilon$.

Now,

$$\rho(z_{m(k)+1},\ z_{n(k)+1}) \leq \rho(z_{m(k)+1},\ z_{m(k)}) + \rho(z_{m(k)},\ z_{n(k)}) + \rho(z_{n(k)},\ z_{n(k)+1}).$$

Again,

$$\rho(z_{m(k)},\ z_{n(k)}) \leq \rho(z_{m(k)},\ z_{m(k)+1}) + \rho(z_{m(k)+1},\ z_{n(k)+1}) + \rho(z_{n(k)+1},\ z_{n(k)})$$

that is,

$$\rho(z_{m(k)},\ z_{n(k)}) - \rho(z_{m(k)},\ z_{m(k)+1}) - \rho(z_{n(k)+1},\ z_{n(k)}) \leq \rho(z_{m(k)+1},\ z_{n(k)+1}).$$

From the above inequalities we have that

$$\begin{aligned} \rho(z_{m(k)},\ z_{n(k)}) - \rho(z_{m(k)},\ z_{m(k)+1}) - \rho(z_{n(k)+1},\ z_{n(k)}) &\leq \rho(z_{m(k)+1},\ z_{n(k)+1}) \\ &\leq \rho(z_{m(k)+1},\ z_{m(k)}) + \rho(z_{m(k)},\ z_{n(k)}) + \rho(z_{n(k)},\ z_{n(k)+1}). \end{aligned}$$

Taking limit as $k \to +\infty$ in the above inequality and using the fact $\lim_{n\to+\infty} \rho(z_n, z_{n+1}) = 0$ and $\lim_{k\to+\infty} \rho(z_{m(k)},\ z_{n(k)}) = \epsilon$, we have

$$\lim_{k\to+\infty} \rho(z_{m(k)+1},\ z_{n(k)+1}) = \epsilon. \tag{22}$$

Applying (19), we have

$$\rho(z_{m(k)+1},\ z_{n(k)+1}) = \rho(Sz_{m(k)}, Sz_{n(k)}) \leq \beta(\rho(z_{m(k)},\ z_{n(k)}))\ \rho(z_{m(k)},\ z_{n(k)}) < \rho(z_{m(k)},\ z_{n(k)}),$$

that is,

$$\frac{\rho(z_{m(k)+1},\ z_{n(k)+1})}{\rho(z_{m(k)},\ z_{n(k)})} \leq \beta(\rho(z_{m(k)},\ z_{n(k)})) < 1.$$

Then

$$1 \leq \lim_{k\to+\infty} \beta(\rho(z_{m(k)},\ z_{n(k)})) < 1,$$

which implies that

$$\lim_{k\to+\infty} \beta(\rho(z_{m(k)},\ z_{n(k)})) = 1. \tag{23}$$

It follows by the property of β that $\lim_{k\to+\infty} \rho(z_{m(k)},\ z_{n(k)}) = 0$, that is, $\epsilon = 0$, which is a contradiction. Hence $\{z_n\}$ is a Cauchy sequence. As (M, ρ) is complete, there exists an $\xi \in M$ such that $z_n \to \xi$ as $n \to +\infty$. Now applying (19), we have

$$\rho(z_{n+1},\ S\xi) = \rho(Sz_n, S\xi) \leq \beta(\rho(z_n, \xi))\ \rho(z_n, \xi) < \rho(z_n, \xi).$$

Taking limit as $n \to +\infty$ in the above inequality, we have $\rho(\xi,\ S\xi) = 0$, that is, $\xi = S\xi$, that is, ξ is a fixed point of S. From what we have already proved, ξ is the unique fixed point of S.

Example 9 Take the metric space $M = [0, +\infty)$ equipped with usual metric. Let $\beta(t) = \frac{1}{1+t}$, for all $t \geq 0$. Then $\beta \in S$. Define $S : M \to M$ as

$$Su = \begin{cases} \frac{u}{3}, & \text{if } 0 \leq u \leq 1, \\ \frac{1}{3}, & \text{if } u > 1. \end{cases}$$

Theorem 6 is applicable and here $u = 0$ is the unique fixed point of S.

The next theorem is a generalized weak contraction mapping theorem due to Choudhury et al. [9] which was proved in 2013. It is the culmination of a series of papers generalizing and weakening Banach's result in a specific way. In metric spaces, this line of research was originated by Rhoades [34] and was further contributed through works like [7, 13, 44]. Prior to the work of Rhoades [34], such contractions were considered in different settings and under different conditions, a description of which can be found in [18, 19]. Although most of these results including [9] are worked out in partially ordered metric spaces, we present the theorem here in a complete metric space without order.

We denote by Ψ the set of all functions $\psi : [0,\ +\infty) \to [0,\ +\infty)$ satisfying
(i_ψ) ψ is continuous and nondecreasing,
(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$;

and by Θ we denote the set of all functions $\alpha\colon [0,\ +\infty) \to [0,\ +\infty)$ such that
(i_α) α is bounded on any bounded interval in $[0,\ +\infty)$,
(ii_α) α is continuous at 0 and $\alpha(0) = 0$.

Theorem 7 Let S be a self-mapping of a complete metric space (M, ρ). Suppose that there exist $\psi \in \Psi$ and $\varphi,\ \theta \in \Theta$ such that

$$\psi(x) \leq \varphi(y) \ \Rightarrow\ x \leq y, \tag{24}$$

for any sequence $\{x_n\}$ in $[0,\ +\infty)$ with $x_n \to t > 0$,

$$\psi(t) - \overline{\lim}\, \varphi(x_n) + \underline{\lim}\, \theta(x_n) > 0, \tag{25}$$

and

$$\psi(\rho(Su, Sv)) \leq \varphi(\rho(u, v)) - \theta(\rho(u, v)),\ \text{for all } u, v \in M. \tag{26}$$

Then S has a unique fixed point in M.

Proof Choose an arbitrary element $z_0 \in M$ and define a sequence $\{z_n\}$ in M such that

$$z_{n+1} = Sz_n, \quad \text{for all } n \geq 0. \tag{27}$$

Let $R_n = \rho(z_{n+1}, z_n)$, for all $n \geq 0$.
Applying (26), we have

$$\psi(\rho(z_{n+2}, z_{n+1})) = \psi(\rho(Sz_{n+1},\ Sz_n)) \leq \varphi(\rho(z_{n+1}, z_n)) - \theta(\rho(z_{n+1}, z_n)),$$

that is,

$$\psi(R_{n+1}) \leq \varphi(R_n) - \theta(R_n), \tag{28}$$

which, in view of the fact that $\theta \geq 0$, yields $\psi(R_{n+1}) \leq \varphi(R_n)$, which by (24) implies that $R_{n+1} \leq R_n$, for all positive integers n, that is, the sequence $\{R_n\}$ is monotonic decreasing. Then we get an $r \geq 0$ such that

$$R_n = \rho(z_{n+1}, z_n) \to r \text{ as } n \to +\infty. \tag{29}$$

Taking limit supremum on both sides of (28), using (29), the property (i_α) of φ and θ, and the continuity of ψ, we obtain

$$\psi(r) \leq \overline{\lim}\, \varphi(R_n) + \overline{\lim}\, (-\,\theta(R_n)).$$

Since $\overline{\lim}\, (-\,\theta(R_n)) = -\,\underline{\lim}\, \theta(R_n)$, we obtain

$$\psi(r) \leq \overline{\lim}\, \varphi(R_n) - \underline{\lim}\, \theta(R_n),$$

that is,

$$\psi(r) - \overline{\lim}\, \varphi(R_n) + \underline{\lim}\, \theta(R_n) \leq 0,$$

which by (25) is a contradiction unless $r = 0$. Therefore,

$$R_n = \rho(z_{n+1},\ z_n) \to 0, \quad \text{as } n \to +\infty. \tag{30}$$

Next we prove that $\{z_n\}$ is a Cauchy sequence. On the contrary, there exists an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad \rho(z_{m(k)}, z_{n(k)}) \geq \epsilon \ \text{ and } \ \rho(z_{m(k)}, z_{n(k)-1}) < \epsilon.$$

Arguing similarly as in the proof of Theorem 5, we prove that

$$\lim_{k\to+\infty} \rho(z_{m(k)}, z_{n(k)}) = \epsilon \ \text{ and } \ \lim_{k\to+\infty} \rho(z_{m(k)+1}, z_{n(k)+1}) = \epsilon. \tag{31}$$

Applying from (26) and (27), we have

$$\begin{aligned}\psi(\rho(z_{n(k)+1}, z_{m(k)+1})) &= \psi(\rho(Sz_{n(k)}, Sz_{m(k)})) \\ &\leq \varphi(\rho(z_{n(k)}, z_{m(k)})) - \theta(\rho(z_{n(k)}, z_{m(k)})).\end{aligned}$$

Using (31), the property (i_α) of φ and θ, and the continuity of ψ, we obtain

$$\psi(\epsilon) \leq \overline{\lim}\, \varphi(\rho(z_{n(k)}, z_{m(k)})) + \overline{\lim}\, (-\, \theta(\rho(z_{n(k)}, z_{m(k)}))).$$

As $\overline{\lim}\, (-\, \theta(\rho(z_{n(k)}, z_{m(k)}))) = -\, \underline{\lim}\, \theta(\rho(z_{n(k)}, z_{m(k)}))$, we get

$$\psi(\epsilon) \leq \overline{\lim}\, \varphi(\rho(z_{n(k)},\ z_{m(k)})) - \underline{\lim}\, \theta(\rho(z_{n(k)},\ z_{m(k)})),$$

that is,

$$\psi(\epsilon) - \overline{\lim}\, \varphi(\rho(z_{n(k)}, z_{m(k)})) + \underline{\lim}\, \theta(\rho(z_{n(k)}, z_{m(k)})) \leq 0,$$

which is a contradiction by (25). Therefore, $\{z_n\}$ is a Cauchy sequence in M and hence there exists $\xi \in M$ such that

$$\lim_{n\to+\infty} z_{n+1} = \lim_{n\to+\infty} Sz_n = \lim_{n\to+\infty} = \xi. \tag{32}$$

Now, applying (26), we have

$$\psi(\rho(z_{n+1}, S\xi)) = \psi(\rho(Sz_n, S\xi)) \leq \varphi(\rho(z_n, \xi)) - \theta(\rho(z_n, \xi)).$$

Taking limit as $n \to +\infty$ and using (32), the properties of ψ, φ and θ, we obtain $\psi(\rho(\xi, S\xi)) = 0$, which implies that $\rho(\xi, S\xi) = 0$, that is, $\xi = S\xi$, that is, ξ is a fixed point of S.

Suppose that $\zeta \in M$ $(\zeta \neq \xi)$ be another fixed point of S. Then $\rho(\xi, \zeta) > 0$. Now, we consider a sequence $\{y_n\}$ in M such that $y_n \to \zeta$ as $n \to +\infty$. Therefore,

$$\rho(\xi, y_n) \to \rho(\xi, \zeta) > 0, \text{ as } n \to +\infty. \tag{33}$$

By (26), we have

$$\psi(\rho(\xi, Sy_n)) = \psi(\rho(S\xi, Sy_n)) \leq \varphi(\rho(\xi, y_n)) - \theta(\rho(\xi, y_n)).$$

Using (33), the property (i_α) of φ and θ, and the continuity of ψ, we obtain

$$\psi(\rho(\xi, \zeta)) \leq \overline{\lim}\, \varphi(\rho(\xi, y_n)) + \overline{\lim}\, (-\,\theta(\rho(\xi, y_n))),$$

that is,

$$\psi(\rho(\xi, \zeta)) - \overline{\lim}\, \varphi(\rho(\xi, y_n)) + \underline{\lim}\, \theta(\rho(\xi, y_n)) \leq 0,$$

which is a contradiction by (25). Therefore, $\rho(\xi, \zeta) = 0$, that is, $\xi = \zeta$. Hence, T has a unique fixed point.

Example 10 Let $M = [0,\ 1]$ and $\rho(x,\ y) = |x - y|$, for $x,\ y \in M$. Let $S\colon M \to M$ be defined by $Sx = x - \frac{x^2}{2}$, for all $x \in M$. Let $\theta,\ \varphi,\ \psi\colon [0,\ +\infty) \to [0,\ +\infty)$ be given, respectively, by the formulas

$$\theta(t) = \frac{t^2}{2}, \quad \varphi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ 0, & \text{otherwise,} \end{cases} \quad \psi(t) = \begin{cases} t, & \text{if } 0 \leq t \leq 1, \\ t^2, & \text{otherwise.} \end{cases}$$

Applying Theorem 7, we see that the unique fixed point of S is $x = 0$.

Remark 2 Considering ψ and φ to be the identity mappings and $\theta(t) = (1 - k)t$, where $0 \leq k < 1$, in Theorem 7 we have Theorem 1.

Pata-type contractions are introduced in a recent paper due to Pata [29] in 2011 in which a fixed point theorem for such contractions was proved by using a new approach. The result due to Pata [29] appeared to be stronger than Banach's Contraction Mapping Principle, even stronger than the well-known Boyd-Wong fixed point theorem.

We use the following class of functions for the following result. Let Ψ denote the family of all functions $\psi : [0,\ 1] \to [0,\ +\infty)$ such that ψ is increasing and continuous at zero with $\psi(0) = 0$.

Theorem 8 (Pata [29]) *Let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0,\ \alpha]$ be some constants and $\psi \in \Psi$. Let (M, d) be a complete metric space and $S : M \to M$ be such that for every $\varepsilon \in [0,\ 1]$ and all $x,\ y \in M$,*

$$\rho(Sx,\ Sy) \leq (1-\varepsilon)\rho(x,\ y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\Big[1 + \|x\| + \|y\|\Big]^{\beta}, \tag{34}$$

where $||x|| = \rho(x, u)$ *and* $||y|| = \rho(y, u)$ *for an arbitrary but fixed* $u \in M$. *Then* S *has a unique fixed point in* M.

Proof Suppose that S has two fixed points ζ and η with $\zeta \neq \eta$. Then $\rho(\zeta, \eta) > 0$. Applying (34) with $0 < \varepsilon \leq 1$, we have

$$\rho(\zeta, \eta) = \rho(S\zeta, S\eta) \leq (1-\varepsilon)\ \rho(\zeta, \eta) + \Lambda\ \varepsilon^{\alpha}\psi(\varepsilon)\Big[1 + ||\zeta|| + ||\eta||\Big]^{\beta},$$

that is,

$$\varepsilon\ \rho(\zeta, \eta) \leq \Lambda\ \varepsilon^{\alpha}\ \psi(\varepsilon)\Big[1 + ||\zeta|| + ||\eta||\Big]^{\beta},$$

that is,

$$\rho(\zeta, \eta) \leq \Lambda\ \varepsilon^{\alpha-1}\ \psi(\varepsilon)\Big[1 + ||\zeta|| + ||\eta||\Big]^{\beta}.$$

Taking $\varepsilon \to 0$ and using the property of ψ, we have $\rho(\zeta, \eta) \leq 0$, which is a contradiction. Hence S may have at most one fixed point.

Choosing an arbitrary element $z_0 \in M$, we construct a sequence $\{z_n\}$ in M such that

$$z_{n+1} \in Sz_n \ \text{ for all } \ n \geq 0. \tag{35}$$

Let

$$c_n = ||z_n|| = \rho(z_n,\ z_0), \quad \text{for all } \ n \geq 0. \tag{36}$$

Applying (34) with $0 < \varepsilon \leq 1$, we get

$$\begin{aligned}\rho(z_{n+2},\ z_{n+1}) &\leq \rho(Sz_{n+1},\ Sz_n)\\ &\leq (1-\varepsilon)\ \rho(z_{n+1}, z_n) + \Lambda\ \varepsilon^{\alpha}\psi(\varepsilon)\Big[1 + ||z_{n+1}|| + ||z_n||\Big]^{\beta}.\end{aligned}$$

Since $\alpha \geq 1$, taking $\varepsilon \to 0$ and using the property of ψ, we have

$$\rho(z_{n+2},\ z_{n+1}) \leq \rho(z_{n+1},\ z_n) \ \text{ for all } n \geq 0, \tag{37}$$

that is, the sequence $\{\rho(z_{n+1},\ z_n)\}$ is a decreasing. So

$$\rho(z_{n+1},\ z_n) \leq \rho(z_1,\ z_0) = c_1 = ||z_1||, \quad \text{for all } n \geq 0, \tag{38}$$

and also there exists a real number $l \geq 0$ such that

$$\rho(z_{n+1},\ z_n) \to l \text{ as } n \to +\infty. \tag{39}$$

We claim that $\{c_n\}$ is bounded.
Applying (34) of the theorem, (35), (36), (37) and (38), we have

$$\begin{aligned}
c_n = \rho(z_n,\ z_0) &\leq \rho(z_n,\ z_{n+1}) + \rho(z_{n+1},\ z_1) + \rho(z_1,\ z_0)\\
&= \rho(z_{n+1},\ z_n) + \rho(z_{n+1},\ z_1) + c_1\\
&\leq \rho(z_1,\ z_0) + \rho(z_{n+1},\ z_1) + c_1 = c_1 + \rho(z_{n+1},\ z_1) + c_1\\
&\leq \rho(Sz_n,\ Sz_0) + 2c_1\\
&\leq (1-\varepsilon)\ \rho(z_n,\ z_0) + \Lambda\ \varepsilon^\alpha\ \psi(\varepsilon)\ \Big[1 + ||z_n|| + ||z_0||\Big]^\beta + 2c_1\\
&\leq (1-\varepsilon)\ c_n + \Lambda\ \varepsilon^\alpha\ \psi(\varepsilon)\Big[1 + ||z_n||\Big]^\beta + 2c_1\\
&\leq (1-\varepsilon)\ c_n + \Lambda\ \varepsilon^\alpha\ \psi(\varepsilon)\ \Big[1 + c_n\Big]^\alpha + 2c_1,\ \ \Big(\text{since } \beta \leq \alpha\Big).\\
&\leq (1-\varepsilon)\ c_n + \Lambda\ \varepsilon^\alpha\ \psi(\varepsilon)\ \Big[1 + c_n + c_1\Big]^\alpha + 2c_1,\ \ \Big(\text{since } \beta \leq \alpha\Big).
\end{aligned}$$

So, we have

$$c_n \leq (1-\varepsilon)\ c_n + \Lambda\ \varepsilon^\alpha\ \varphi(\varepsilon)\ \Big[1 + c_n + c_1\Big]^\alpha + 2c_1. \tag{40}$$

Now

$$\Big(1 + c_n + c_1\Big)^\alpha = (1+c_n)^\alpha\Big(1 + \frac{c_1}{1+c_n}\Big)^\alpha \leq (1+c_n)^\alpha(1+c_1)^\alpha. \tag{41}$$

If possible, suppose that the sequence $\{c_n\}$ is unbounded. Then we have a subsequence $\{c_{n_k}\}$ with $c_{n_k} \to +\infty$ as $k \to +\infty$. Then there exist a natural number N^* such that

$$c_{n_k} \geq 1 + 2c_1 \text{ for all } k \geq N^*. \tag{42}$$

Now, for all $k \geq N^*$ from (40) and using (41), we have

$$\Big(1 + c_{n_k} + c_1\Big)^\alpha = (1+c_{n_k})^\alpha(1+c_1)^\alpha \leq c_{n_k}^\alpha(1 + \frac{1}{c_{n_k}})^\alpha\ (1+c_1)^\alpha,$$

which implies

$$\Big(1 + c_{n_k} + c_1\Big)^\alpha \leq c_{n_k}^\alpha(1+1)^\alpha\ (1+c_1)^\alpha = 2^\alpha c_{n_k}^\alpha\ (1+c_1)^\alpha. \tag{43}$$

Then for all $k \geq N^*$, we have from (40) and (43) that

$$c_{n_k} \leq (1-\varepsilon)c_{n_k} + \Lambda\ \varepsilon^\alpha\ \psi(\varepsilon)\ 2^\alpha\ c_{n_k}^\alpha\ (1+c_1)^\alpha + 2c_1,$$

that is,

$$\begin{aligned}\varepsilon\, c_{n_k} &\leq \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, 2^{\alpha}\, c_{n_k}^{\alpha}\, (1+c_1)^{\alpha} + 2c_1 \\ &= \Big[\Lambda\, 2^{\alpha}\, (1+c_1)^{\alpha}\Big]\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, c_{n_k}^{\alpha} + 2c_1.\end{aligned}$$

Let $a = \Lambda\, 2^{\alpha}\, (1+c_1)^{\alpha}$ and $b = 2c_1$. Here a and b are fixed positive real numbers. So, we have

$$\varepsilon\, c_{n_k} \leq a\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, c_{n_k}^{\alpha} + b.$$

Choose $\varepsilon = \varepsilon_k = \frac{1+b}{c_{n_k}} = \frac{1+2c_1}{c_{n_k}}$, where $k \geq N^*$. Then by (42), $0 < \varepsilon \leq 1$. Now we have

$$1 \leq a\, (1+b)^{\alpha}\, \psi(\varepsilon_k) \to 0 \text{ as } k \to +\infty,$$

which is a contradiction. Hence $\{c_n\}$ is bounded.

Applying (34) with $\varepsilon \in (0,\, 1]$, we have

$$\begin{aligned}\rho(z_{n+2},\, z_{n+1}) &\leq \rho(Sz_{n+1},\, Sz_n) \\ &\leq (1-\varepsilon)\, \rho(z_{n+1},\, z_n) + \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, \Big[1 + \|z_{n+1}\| + \|z_n\|\Big]^{\beta}.\end{aligned}$$

Since $\{c_n\}$ is bounded, there exists a real number $H > 0$ such that $c_n = \|z_n\| \leq H$ for all $n \geq 0$. Then

$$\begin{aligned}\rho(z_{n+2},\, z_{n+1}) &\leq (1-\varepsilon)\, \rho(z_{n+1},\, z_n) + \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, \Big[1 + \|z_{n+1}\| + \|z_n\|\Big]^{\beta} \\ &\leq (1-\varepsilon)\, \rho(z_{n+1},\, z_n) + \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, \Big[1 + 2H\Big]^{\beta}.\end{aligned}$$

Taking $n \to +\infty$ and using (39), we have

$$l \leq (1-\varepsilon)\, l + \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, \Big[1 + 2H\Big]^{\beta},$$

which implies that

$$\varepsilon\, l \leq \Lambda\, \varepsilon^{\alpha}\, \psi(\varepsilon)\, \Big[1 + 2H\Big]^{\beta},$$

that is,

$$l \leq \Lambda\, \varepsilon^{\alpha-1}\, \varphi(\varepsilon)\, \Big[1 + 2H\Big]^{\beta}.$$

Taking $\varepsilon \to 0$ and using the property of ψ, we have $l \leq 0$, which implies that $l = 0$. So, we get

$$\lim_{n\to+\infty} \rho(z_{n+1},\ z_n) = 0. \tag{44}$$

Next we prove that the sequence $\{z_n\}$ is Cauchy. On the contrary, there exists a $\xi > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k,\ \ \rho(z_{m(k)},\ z_{n(k)}) \geq \xi \ \text{ and } \ \rho(z_{m(k)},\ z_{n(k)-1}) < \xi.$$

Now,

$$\xi \leq \rho(z_{m(k)},\ z_{n(k)}) \leq \rho(z_{m(k)},\ z_{n(k)-1}) + \rho(z_{n(k)-1},\ z_{n(k)}),$$

that is,

$$\xi \leq \rho(z_{m(k)},\ z_{n(k)}) < \xi + \rho(z_{n(k)-1},\ z_{n(k)}).$$

Using (44), we have

$$\lim_{k\to+\infty} \rho(z_{m(k)},\ z_{n(k)}) = \xi. \tag{45}$$

Again,

$$\rho(z_{m(k)},\ z_{n(k)}) \leq \rho(z_{m(k)},\ z_{m(k)+1}) + \rho(z_{m(k)+1},\ z_{n(k)+1}) + \rho(z_{n(k)+1},\ z_{n(k)})$$

and

$$\rho(z_{m(k)+1},\ z_{n(k)+1}) \leq \rho(z_{m(k)+1},\ z_{m(k)}) + \rho(z_{m(k)},\ z_{n(k)}) + \rho(z_{n(k)},\ z_{n(k)+1}).$$

Using (44) and (45), we have

$$\lim_{k\to+\infty} \rho(z_{m(k)+1},\ z_{n(k)+1}) = \xi. \tag{46}$$

Applying (34) with $\varepsilon \in (0,\ 1]$, we have

$$\begin{aligned}\rho(z_{m(k)+1},\ z_{n(k)+1}) &\leq \rho(Sz_{m(k)},\ Sz_{n(k)})\\ &\leq (1-\varepsilon)\ \rho(z_{m(k)},\ z_{n(k)}) + \Lambda\ \varepsilon^{\alpha}\ \psi(\varepsilon)\Big[1 + \|z_{m(k)}\| + \|z_{n(k)}\|\Big]^{\beta}.\end{aligned}$$

Since $c_n = \|z_n\| \leq H$ for all $n \geq 0$,

$$\rho(z_{m(k)+1},\ z_{n(k)+1}) \leq (1-\varepsilon)\ \rho(z_{m(k)},\ z_{n(k)}) + \Lambda\ \varepsilon^{\alpha}\ \psi(\varepsilon)\Big[1 + 2H\Big]^{\beta}.$$

Taking limit as $k \to +\infty$ and using (45), (46) and the property of ψ, we have

$$\xi \leq (1-\varepsilon)\,\xi + \Lambda\,\varepsilon^{\alpha}\ \psi(\varepsilon)\Big[1+2H\Big]^{\beta},$$

which implies that

$$\varepsilon\,\xi \leq \Lambda\,\varepsilon^{\alpha}\ \psi(\varepsilon)\Big[1+2H\Big]^{\beta},$$

that is,

$$\xi \leq \Lambda\,\varepsilon^{\alpha-1}\ \varphi(\varepsilon)\Big[1+2H\Big]^{\beta}.$$

Taking limit as $\varepsilon \to 0$ and using the property of ψ, we have $\xi \leq 0$, which is a contradiction. Therefore, $\{z_n\}$ is a Cauchy sequence in M and hence there exists $y \in M$ such that

$$z_n \to y \quad \text{as} \quad n \to +\infty. \tag{47}$$

Applying (34) with $\varepsilon \in (0,\ 1]$, we have

$$\begin{aligned}\rho(z_{n+1},\ Sy) &\leq \rho(Sz_n,\ Sy)\\ &\leq (1-\varepsilon)\ \rho(z_n,\ y) + \Lambda\,\varepsilon^{\alpha}\psi(\varepsilon)\Big[1+||z_n||+||y||\Big]^{\beta}.\end{aligned}$$

Since $c_n = ||z_n|| \leq H$ for all $n \geq 0$. Then

$$\rho(z_{n+1},\ Sy) \leq (1-\varepsilon)\ \rho(z_n,\ y) + \Lambda\,\varepsilon^{\alpha}\psi(\varepsilon)\Big[1+H+||y||\Big]^{\beta}.$$

Taking $n \to +\infty$ and using (44), (47), we get

$$\rho(y,\ Sy) \leq \Lambda\,\varepsilon^{\alpha}\psi(\varepsilon)\Big[1+H+||y||\Big]^{\beta}.$$

Taking limit as $\varepsilon \to 0$ and using the property of ψ, we have $\rho(y,\ Sy) = 0$, that is, $y = Sy$, that is, y is a fixed point of S. From what we have already proved, y is the unique fixed point of S.

Example 11 ([29]) Let $M = [1, +\infty)$ and let $S : M \to M$ be defined by

$$Sz = -2 + z - 2\sqrt{z} + 4\sqrt[4]{z}.$$

It has a unique fixed point $z = 1$. For any given $r > 0$ and $z \geq 1$, if

$$Q(z,r) = 2[\sqrt{z+r} - \sqrt{z}] - 4[\sqrt[4]{z+r} - \sqrt[4]{z}],$$

then

$$|S(z+r) - S(z)| = r - Q(z,r)$$

holds for all r and z. On the other hand, for every $\varepsilon \in [0, 1]$, one can prove that

$$-\varepsilon r + \varepsilon^2 (2z+r)^{3/2} + Q(z,r) \geq Q(z,r) - \frac{r^2}{4(r+2z)^{3/2}} \geq 0.$$

It follows that

$$|S(z+r) - S(z)| = r - Q(z,r) \leq (1-\varepsilon) r + \varepsilon^2 (2z+r)^{3/2},$$

and the conditions of Theorem 8 are fulfilled.

4 Metric Fixed Point Without Continuity

In 1976, Caristi [5] proved an elegant fixed point theorem on complete metric spaces, which is a generalization of the Banach's contraction mapping principle and is equivalent to the Ekeland variational principle [15].

Definition 8 A function $\varphi : X \to R$ is said to be lower semicontinuous at x if for any sequence $\{x_n\} \subset X$, we have

$$x_n \to x \in X \quad \Rightarrow \quad \varphi(x) \leq \liminf_{n \to +\infty} \varphi(x_n).$$

Definition 9 Let (M, ρ) be a metric space. A mapping $S : M \to M$ is called a Caristi mapping if there exists a lower semicontinuous function $\varphi : M \to R^+$ such that

$$\rho(u, Su) \leq \varphi(u) - \varphi(Su), \quad \text{for all } u \in M.$$

Theorem 9 ([24]) *Let (M, ρ) be a complete metric space. A mapping $S\colon M \to M$ admits a fixed point in M if there exists a lower semicontinuous function $\varphi : M \to R^+$ such that*

$$\rho(u, Su) \leq \varphi(u) - \varphi(Su), \quad \text{for all } u \in M. \tag{48}$$

Proof From (48) it follows immediately that

$$\varphi(Su) \leq \varphi(u), \quad \text{for every } u \in M. \tag{49}$$

For $u \in M$, define

$$Q(u) = \{y \in M : \rho(u, y) \leq \varphi(u) - \varphi(y)\}.$$

$Q(u)$ is nonempty because $u \in Q(u)$ and $Su \in Q(u)$. Let $y \in Q(u)$. Now, we have

$$\rho(u, Sy) \leq \rho(u, y) + \rho(y, Sy) \leq \varphi(u) - \varphi(y) + \varphi(y) - \varphi(Sy),$$

that is,

$$\rho(u, Sy) \leq \varphi(u) - \varphi(Sy). \tag{50}$$

It follows that $Sy \in Q(u)$. Hence, we have that if $y \in Q(u)$ then $Sy \in Q(u)$.

Define

$$q(u) = \inf \{\varphi(y) : y \in Q(u)\}.$$

As $Q(u)$ is nonempty for each $u \in M$ and the function φ is nonnegative, the function $q(u)$ is well-defined. Then, we have that for any $u \in M$,

$$0 \leq q(u) \leq \varphi(Su) \leq \varphi(u). \tag{51}$$

Let $u_1 \in M$ be arbitrary. By the definition of $q(u_1)$, there exists $u_2 \in Q(u_1)$ such that $\varphi(u_2) < q(u_1) + 1$. Again, by the definition of $q(u_2)$, there exists $u_3 \in Q(u_2)$ such that $\varphi(u_3) < q(u_2) + \frac{1}{2}$. In this way, we define a sequence $\{u_n\}$ in M such that $u_{n+1} \in Q(u_n)$ with

$$\varphi(u_{n+1}) < q(u_n) + \frac{1}{n}, \quad \text{for} \ \ n \geq 1. \tag{52}$$

Since $u_{n+1} \in Q(u_n)$, we have

$$0 \leq \rho(u_n, u_{n+1}) \leq \varphi(u_n) - \varphi(u_{n+1}), \tag{53}$$

that is,

$$\varphi(u_{n+1}) \leq \varphi(u_n), \quad \text{for} \ \ n \geq 1. \tag{54}$$

Hence $\{\varphi(u_n)\}$ is a nonincreasing sequence of nonnegative numbers and therefore there exists $r \geq 0$ such that

$$\lim_{n \to +\infty} \varphi(u_n) = r. \tag{55}$$

Therefore, $\{\varphi(u_n)\}$ is a Cauchy sequence. Hence, for every $k \in N$ (set of all natural number), there exists $N_k \in N$ such that for every pair of natural numbers m, n with $m \geq n \geq N_k$, we have

$$0 \leq \varphi(u_n) - \varphi(u_m) < \frac{1}{k}. \tag{56}$$

From (51) and (52), we have

$$\varphi(u_{n+1}) < q(u_n) + \frac{1}{n} \leq \varphi(u_n) + \frac{1}{n}.$$

Taking limit as $n \to +\infty$ and using (55), we have

$$\lim_{n\to+\infty} q(u_n) = r. \tag{57}$$

We claim that for $m \geq n \geq N_k$,

$$\rho(u_n, u_m) \leq \varphi(u_n) - \varphi(u_m) < \frac{1}{k}. \tag{58}$$

(58) is trivially valid for $n = m$. Therefore, it is sufficient to show that (58) is true for $m > n$. Using triangular inequality, (53) and (56), we have for $m > n$ that

$$\begin{aligned}\rho(u_n, u_m) &\leq \rho(u_n, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \cdots + \rho(u_{m-1}, u_m)\\ &\leq \varphi(u_n) - \varphi(u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_{n+2}) + \cdots + \varphi(u_{m-1}) - \varphi(u_m).\end{aligned}$$

It follows that

$$\rho(u_n, u_m) \leq \varphi(u_n) - \varphi(u_m) < \frac{1}{k}. \tag{59}$$

Therefore, (58) is true for $m \geq n \geq N_k$. From (58), it follows that $\{u_n\}$ is a Cauchy sequence and hence by completeness of M, there exists $z \in M$ such that

$$\lim_{n\to+\infty} \rho(u_n, z) = 0. \tag{60}$$

Hence, for every $n \in N$,

$$\lim_{m\to+\infty} \rho(u_n, u_m) = \rho(u_n, z).$$

Using this, (59) and the lower semicontinuity of φ,

$$\begin{aligned}\rho(u_n, z) = \lim_{m\to+\infty} \rho(u_n, u_m) &\leq \limsup_{m\to+\infty} [\varphi(u_n) - \varphi(u_m)]\\ &\leq \varphi(u_n) - \liminf_{m\to+\infty} \varphi(u_m)\\ &\leq \varphi(u_n) - \varphi(z).\end{aligned}$$

Therefore,

$$\rho(u_n, z) \leq \varphi(u_n) - \varphi(z), \tag{61}$$

which implies that $z \in Q(u_n)$ for every $n \in N$. Then we have

$$q(u_n) \leq \varphi(z) \leq \varphi(u_n) - \rho(u_n, z), \quad \text{for every } \ n \in N. \tag{62}$$

Taking limit as $n \to +\infty$ in the above inequality and using (55) and (57), we have

$$\varphi(z) = r. \tag{63}$$

Since, as proved above, $z \in Q(u_n)$ for every $n \in N$, (50) implies that $Sz \in Q(u_n)$ for every $n \in N$. Therefore, by (49), we conclude from (63) that

$$q(u_n) \leq \varphi(Sz) \leq \varphi(z) = r. \tag{64}$$

Letting $n \to +\infty$ and using (57), we obtain $\varphi(Sz) = \varphi(z)$. By (48) again,

$$0 \leq \rho(z, Sz) \leq \varphi(z) - \varphi(Sz) = 0.$$

Hence $\rho(z, Sz) = 0$, that is, $Sz = z$. Therefore, S has a fixed point.

Example 12 Take $M = [0, 1]$ endowed with the usual metric ρ. Define $S : M \to M$ as

$$Su = \begin{cases} \frac{u}{2}, \text{ if } u \neq 1, \\ 1, \text{ if } u = 1. \end{cases}$$

The conditions of Theorem 9 are satisfied and S has fixed points 0 and 1.

It is easy to see that Caristi's fixed point theorem is a generalization of the Banach's contraction mapping principle by defining $\varphi(u) = \frac{1}{1-k}\ \rho(u, Su)$, where $0 < k < 1$ is the Lipschitz constant associated with the contraction S from Banach's principle. It has been shown by Kirk in [22] that the validity of Caristi's fixed point theorem implies that the corresponding metric space is complete while the Banach's contraction mapping principle does not characterize completeness. The above example shows that Caristi's contraction can also be discontinuous.

Suzuki [42] in the year 2008 established a new fixed point theorem which is a generalization of Theorem 1 and characterizes the metric completeness. Though there are many generalizations of Theorem 1, the direction of Suzuki is new and very simple. Suzuki-type contractions form an important class of contractions in the domain of fixed point theory.

Define a function $\theta : [0,\ 1) \to (\frac{1}{2},\ 1]$ as

$$\theta(r) = \begin{cases} 1, & \text{if } \ 0 \leq r \leq \frac{\sqrt{5}-1}{2}; \\ \frac{1-r}{r^2}, & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}; \\ \frac{1}{1+r}, & \text{if } \ \frac{1}{\sqrt{2}} \leq r < 1. \end{cases}$$

Theorem 10 (Suzuki [42]) *A self-mapping S of a complete metric space (M, ρ) admits a unique fixed point if there exists a real number $r \in [0,\ 1)$ such that for all $x,\ y \in M$,*

$$\theta(r)\ \rho(x,\ Sx) \leq \rho(x,\ y) \text{ implies } \rho(Sx,\ Sy) \leq r\ \rho(x,\ y). \tag{65}$$

Proof Since $\theta(r) \leq 1,\ \theta(r)\ \rho(x,\ Sx) \leq \rho(x,\ Sx)$ holds for every $x \in M$. By (65), we have

$$\rho(Sx,\ S^2x) \leq r\ \rho(x,\ Sx), \text{ for all } x \in M. \tag{66}$$

Choose any point $u \in M$ and construct a sequence $\{u_n\}$ in M such that

$$u_n = S^n u \text{ for all } n \geq 1. \tag{67}$$

It follows from (66) that $\rho(u_n, u_{n+1}) \leq r^n\ \rho(u, Su)$. Then $\sum_1^+ \infty \rho(u_n, u_{n+1}) < +\infty$, which implies that $\{u_n\}$ is a Cauchy sequence. As M is complete, $\{u_n\}$ converges to some point $z \in M$. Next, we show

$$\rho(Sx, z) \leq r\rho(x, z), \text{ for all } x \in M \setminus \{z\}. \tag{68}$$

For $x \in M \setminus \{z\}$, there exists a positive integer m such that $\rho(u_n, z) \leq \frac{\rho(x,z)}{3}$, for all $n \geq m$. Then we have for all $n \geq m$ that

$$\begin{aligned}
\theta(r)\rho(u_n, Su_n) &\leq \rho(u_n, Su_n) = \rho(u_n, u_{n+1}) \\
&\leq \rho(u_n, z) + \rho(u_{n+1}, z) \\
&\leq \frac{2}{3}\ \rho(x, z) = \rho(x, z) - \frac{1}{3}\ \rho(x, z) \\
&\leq \rho(x, z) - \rho(u_n, z) \leq \rho(u_n, x).
\end{aligned}$$

Then it follows by (65) that $\rho(u_{n+1}, Sx) \leq r\ \rho(u_n, x)$, for all $n \geq m$. Taking $n \to +\infty$, we get $\rho(Sx, z) \leq r\rho(x, z)$. Hence (68) is true. Assume that $S^n z \neq z$ for all $n \in N$. By (68), we have

$$\rho(S^{n+1}z, z) \leq r^n \rho(Sz, z), \text{ for all } n \in N. \tag{69}$$

We consider the following three cases:

- $0 \leq r \leq \frac{\sqrt{5}-1}{2}$;
- $\frac{\sqrt{5}-1}{2} < r < \frac{1}{\sqrt{2}}$;
- $\frac{1}{\sqrt{2}} \leq r < 1$.

If $0 \leq r \leq \frac{\sqrt{5}-1}{2}$, then $r^2 + r - 1 \leq 0$ and $2r^2 < 1$. If we assume $\rho(S^2z, z) < \rho(S^2z, S^3z)$, then we have

$$\begin{aligned}\rho(z, Sz) &\le \rho(z, S^2z) + \rho(Sz, S^2z)\\ &< \rho(S^2z, S^3z) + \rho(Sz, S^2z)\\ &\le r^2\rho(z, Sz) + r\rho(z, Sz)\\ &\le \rho(z, Sz),\end{aligned}$$

which is a contradiction. So we have $\rho(S^2z, z) \ge \rho(S^2z, S^3z) \ge \theta(r)\rho(S^2z, SS^2z)$. By hypothesis and (69), we have

$$\begin{aligned}\rho(z, Sz) &\le \rho(z, S^3z) + \rho(S^3z, Sz)\\ &\le r^2\rho(z, Sz) + r\rho(S^2z, z)\\ &\le r^2\rho(z, Sz) + r^2\rho(Sz, z) = 2r^2\rho(z, Sz)\\ &< \rho(z, Sz).\end{aligned}$$

It is a contradiction. If $\frac{\sqrt{5}-1}{2} < r < \frac{1}{\sqrt{2}}$, then $2r^2 < 1$. If we assume $\rho(S^2z, z) < \theta(r)\ \rho(S^2z, S^3z)$, then we have in view of (66)

$$\begin{aligned}\rho(z, Sz) &\le \rho(z, S^2z) + \rho(Sz, S^2z)\\ &< \theta(r)\ \rho(S^2z, S^3z) + \rho(Sz, S^2z)\\ &\le \theta(r)\ r^2\rho(z, Sz) + r\rho(z, Sz) = \rho(z, Sz),\end{aligned}$$

which is a contradiction. Hence $\rho(S^2z, z) \ge \theta(r)\ \rho(S^2z, SS^2z)$. As in the previous case, we can prove

$$\rho(z, Sz) \le 2r^2\rho(z, Sz) < \rho(z, Sz).$$

This is a contradiction. Take the case $\frac{1}{\sqrt{2}} \le r < 1$. We note that for $x, y \in M$, either

$$\theta(r)\ \rho(x, Sx) \le \rho(x, y) \ \text{ or } \ \theta(r)\ \rho(Sx, S^2x) \le \rho(Sx, y)$$

holds. Indeed, if

$$\theta(r)\ \rho(x, Sx) > \rho(x, y) \ \text{ and } \ \theta(r)\ \rho(Sx, S^2x) > \rho(Sx, y),$$

then we have

$$\begin{aligned}\rho(x, Sx) &\le \rho(x, y) + \rho(Sx, y)\\ &< \theta(r)\ (\rho(x, Sx) + \rho(Sx, S^2x))\\ &\le \theta(r)\ (\rho(x, Sx) + r\rho(x, Sx))\\ &= \rho(x, Sx).\end{aligned}$$

This is a contradiction. Since either

$$\theta(r)\ \rho(u_{2n}, u_{2n+1}) \leq \rho(u_{2n}, z) \ \text{ or } \ \theta(r)\rho(u_{2n+1}, u_{2n+2}) \leq \rho(u_{2n+1}, z)$$

holds for every $n \in N$, either

$$\rho(u_{2n+1}, Sz) \leq r\ \rho(u_{2n}, z) \ \text{ or } \ \rho(u_{2n+2}, Sz) \leq r\ \rho(u_{2n+1}, z)$$

holds for every $n \in N$. Since $\{u_n\}$ converges to z, the above inequalities imply there exists a subsequence of $\{u_n\}$ which converges to Sz. This implies $Sz = z$. This is a contradiction. Therefore, there exists $n \in N$ such that $S^n z = z$. Since $\{S^n z\}$ is a Cauchy sequence, we obtain $Sz = z$, that is, z is a fixed point of S. The uniqueness of a fixed point follows easily from (68).

Example 13 ([42]) Take the metric space $M = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\}$ equipped with metric ρ defined as $\rho((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Let $S : M \to M$ be defined by

$$S(x_1, x_2) = \begin{cases} (x_1, 0), & \text{if } x_1 \leq x_2, \\ (0, x_2), & \text{if } x_1 > x_2. \end{cases}$$

Here, Theorem 10 is applicable and the unique fixed point of S is (0, 0).

All the results described above are generalizations of Banach's result. In the next theorem, we deal with a contraction condition which is of a different category and does not generalize Banach's contraction. The contraction condition is also satisfied by discontinuous functions. The result is due to Kannan [20, 21] which was established in the year 1968.

Definition 10 (*Kannan-type mapping* [20, 21]) A mapping $S : M \to M$, where (M, ρ) is a metric space, is called a Kannan-type mapping if there exists $0 < k < \frac{1}{2}$ such that

$$\rho(Sx,\ Sy) \leq k\ [\rho(x,\ Sx) + \rho(y,\ Sy)], \quad \text{for all } \ x,\ y \in M. \tag{70}$$

Theorem 11 (Kannan [20, 21]) *Let $(M,\ \rho)$ be a complete metric space and $S : M \to M$ be a Kannan type mapping. Then T admits a unique fixed point.*

Proof Let $z_0 \in M$ be any arbitrary element. We take the same sequence $\{z_n\}$ in M as in the proof of Theorem 1. Applying (70), we have

$$\begin{aligned} \rho(z_{n+1}, z_{n+2}) &= \rho(Sz_n, Sz_{n+1}) \leq k\ [\rho(z_n, Sz_n) + \rho(z_{n+1}, Sz_{n+1})] \\ &= k\ [\rho(z_n, z_{n+1}) + \rho(z_{n+1}, z_{n+2})], \ \text{for all } n \geq 0, \end{aligned}$$

which implies that

$$\rho(z_{n+1}, z_{n+2}) \leq \frac{k}{1-k}\ \rho(z_n, z_{n+1}), \quad \text{for all } \ n \geq 0. \tag{71}$$

Now $0 < k < \frac{1}{2}$ implies that $0 < 2k < 1$, that is, $0 < k < 1 - k$. Hence $0 < \frac{k}{1-k} < 1$. Let $\alpha = \frac{k}{1-k}$. Then we have from (71) that

$$\rho(z_{n+1}, z_{n+2}) \leq \alpha \ \rho(z_n, z_{n+1}), \quad \text{for all } \ n \geq 0.$$

Applying similar arguments as in the proof of Theorem 1, we prove that $\{z_n\}$ is a Cauchy sequence and there exists $\xi \in M$ such that $z_n \to \xi$, as $n \to +\infty$.

Now applying (70), we have

$$\begin{aligned} \rho(z_{n+1}, S\xi) &= \rho(Sz_n, S\xi) \leq k \ [\rho(z_n, Sz_n) + \rho(\xi, S\xi)] \\ &= k \ [\rho(z_n, z_{n+1}) + \rho(\xi, S\xi)], \ \text{for all } n \geq 0. \end{aligned}$$

Taking the limit as $n \to +\infty$, we have

$$\rho(\xi, S\xi) \leq k \ \rho(\xi, S\xi), \quad \text{that is,} \ \ (1 - k) \ \rho(\xi, S\xi) \leq 0.$$

As $(1 - k) > 0$, it follows that $\rho(\xi, S\xi) = 0$, that is, $\xi = S\xi$, that is, ξ is a fixed point of S.

If possible, suppose that ζ be another fixed point of S. Applying (70), we have

$$\rho(\zeta, \xi) = \rho(S\zeta, S\xi) \leq k \ [\rho(\zeta, \ S\zeta) + \rho(\xi, \ S\xi)] = 0,$$

which implies that $\rho(\zeta, \xi) = 0$, that is, $\zeta = \eta$, which is a contradiction. Hence the fixed point of S is unique.

Example 14 ([32], p. 262) Take $M = [0, 1]$ endowed with the usual metric. Define $S : M \to M$ as

$$Sz = \begin{cases} \frac{z}{3}, \text{ if } 0 \leq z < 1, \\ \frac{1}{6}, \quad \text{if } z = 1. \end{cases}$$

Theorem 11 is applicable and $z = 0$ is the unique fixed point of S. It is observed that S is not continuous on M.

Following the appearance of the results in [20, 21], many persons created contractive conditions not requiring continuity of the mapping and established fixed point and common fixed point results for them; see, for example, [6, 35, 36].

There is another reason for which the Kannan-type mappings are considered to be important. The Banach's contraction mapping principle does not characterize completeness. In fact, there are examples of noncomplete spaces where every contraction has a fixed point [11]. It has been shown in [38, 40] that the necessary existence of fixed points for Kannan-type mappings implies that the corresponding metric space is complete. The above are some reasons for which the Kannan-type mappings are considered important in mathematical analysis. There are several extensions and generalizations of Kannan-type mappings in various spaces as, for instance, those in the works noted in [8, 12, 16].

Fixed point theorem due to Chatterjea [6] which was established in the year 1972 and which is actually a sort of dual of the Kannan fixed point theorem is based on a condition similar to (70).

Definition 11 (*C-contraction* [6]) A mapping $S : M \to M$, where $(M,\ \rho)$ is a metric space, is called a C-contraction if there exists $0 < k < \frac{1}{2}$ such that

$$\rho(Sx,\ Sy) \leq k\,[\rho(x,\ Sy) + \rho(y,\ Sx)], \quad \text{for all } \ x,\ y \in X. \tag{72}$$

Theorem 12 (Chatterjea [6]) *Let $(M,\ \rho)$ be a complete metric space and $S : M \to M$ be a C-contraction. Then T admits a unique fixed point.*

Proof The proof follows by the same method as in Theorem 11. The details are omitted.

Example 15 Take $M = [0, 1]$ equipped with usual metric ρ. Define $S : M \to M$ as

$$Sz = \begin{cases} 0, \ \text{if } 0 \leq z < 1, \\ \frac{1}{6}, \quad \text{if } z = 1. \end{cases}$$

The conditions of Theorem 12 are satisfied and here $z = 0$ is the unique fixed point of S. It is observed that S is not continuous on M.

One of the most general contractive conditions was given by Ćirić [10] in 1974 which is known as quasi-contraction.

Definition 12 (*Quasi-contraction* [10]) A mapping $S : M \to M$, where $(M,\ d)$ is a metric space, is called a quasi-contraction if there exists $0 \leq k < 1$ such that, for all $u,\ v \in M$,

$$d(Su,\ Sv) \leq k \max\{d(u,\ v),\ d(u,\ Su),\ d(v,\ Sv),\ d(u,\ Sv),\ d(v,\ Su)\}. \tag{73}$$

Let S be a self-mapping of a metric space M. For $A \subset M$ let $\delta(A) = \sup\{d(a, b) : a, b \in A\}$ and for each $u \in M$, let

$$O(u, n) = \{u, Su, S^2u, ..., S^n u\}, \quad n = 1, 2, ...$$
$$O(u, \infty) = \{u, Su, S^2u, ...\}.$$

A space M is said to be S-orbitally complete if and only if every Cauchy sequence which is contained in $O(u, \infty)$ for some $u \in M$ converges in M.

Lemma 2 (Ćirić [10]) *Let (M, d) be a metric space, $S : M \to M$ be a quasi-contraction and n be any positive integer. Then for each $z \in M$ and for all positive integers i and j, $i,\ j \in \{1, 2, ..., n\}$ implies $d(S^i z, S^j z) \leq k\delta[O(z, n)]$.*

Proof Let $z \in M$ be arbitrary. Let n be any positive integer and let i and j satisfy the condition of lemma 2. Then $S^{i-1}z,\ S^i z,\ S^{j-1}z,\ S^j z \in O(z, n)$ (where $S^0 z = z$) and since S is a quasi-contraction, we have

$$\begin{aligned} d(S^i z, S^j z) &= d(SS^{i-1}z, SS^{j-1}z) \\ &\leq k \max\{d(S^{i-1}z, S^{j-1}z), d(S^{i-1}z, S^i z), d(S^{j-1}z, S^j z), \\ &\qquad d(S^{i-1}z, S^j z), d(S^{j-1}z, S^i z)\} \\ &\leq k\, \delta[O(z, n)], \end{aligned}$$

which proves the lemma.

Remark 3 From this lemma, it follows that if S is quasi-contraction and $z \in M$, then for every positive integer n there exists a positive integer $k \leq n$, such that $d(z, S^k z) = \delta[O(z, n)]$.

Lemma 3 (Ćirić [10]) *Let* (M, d) *be a metric space and* $S : M \to M$ *be a quasi-contraction. Then*

$$\delta[O(z, \infty)] \leq \frac{1}{1-k}\, d(z, Sz)$$

holds for all $z \in M$.

Proof Let $z \in M$ be arbitrary. Since $\delta[O(z, 1)] \leq \delta[O(z, 2)] \leq \ldots$, we have that $\delta[O(z, \infty)] = \sup\{\delta[O(z, n)] : n \in N\}$. Now it is sufficient to prove that $\delta[O(z, n)] \leq \frac{1}{1-k}\, d(z, Sz)$, for all $n \in N$.

Let n be any positive integer. From the remark of the previous lemma, there exists $S^k z \in O(z, n)]$ $(1 \leq k \leq n)$ such that $d(z, S^k z) = \delta[O(z, n)]$. By a triangular inequality and Lemma 2, we have

$$\begin{aligned} d(z, S^k z) &= d(z, Sz) + d(Sz, S^k z) \leq d(z, Sz) + k\delta[O(z, n)] \\ &\leq d(z, Sz) + kd(z, S^k z). \end{aligned}$$

Therefore, $\delta[O(z, n)] = d(z, S^k z) \leq \frac{1}{1-k}\, d(z, Sz)$. Since n is arbitrary, the proof is completed.

Now we state the main result.

Theorem 13 (Ćirić [10]) *Let* $S : M \to M$, *where* $(M,\ d)$ *is a metric space, be a quasi-contraction. If* M *is* S*-orbitally complete, then* S *has a unique fixed point in* M.

Proof Let $z \in M$ be arbitrary. First, we prove that the sequence $\{S^n z\}$ is a Cauchy sequence. Let n and m be two positive integers with $n < m$. By Lemma 2, we have

$$d(S^n z, S^m z) = d(SS^{n-1}z, S^{m-n+1}S^{n-1}z) \leq k\, \delta[O(S^{n-1}z, m - n + 1)].$$

Following Remark 3, we get an integer l with $1 \leq l \leq m - n + 1$ such that

$$\delta[O(S^{n-1}z, m - n + 1)] = d(S^{n-1}z, S^l S^{n-1}z).$$

By Lemma 2, we have

$$\begin{aligned} d(S^{n-1}z, S^l S^{n-1}z) &= d(SS^{n-2}z, S^{l+1}S^{n-2}z) \\ &\leq k\ \delta[O(S^{n-2}z, l+1)] \\ &\leq k\ \delta[O(S^{n-2}z, m-n+2)]. \end{aligned}$$

Therefore, we have

$$d(S^n z, S^m z) \leq k\ \delta[O(S^{n-1}z, m-n+1)] \leq k^2\ \delta[O(S^{n-2}z, m-n+2)].$$

Continuing this process, we obtain

$$d(S^n z, S^m z) \leq k\ \delta[O(S^{n-1}z, m-n+1)] \leq k^2\ \delta[O(S^{n-2}z, m-n+2)] \leq \cdots \leq k^n \delta[O(z, m)].$$

Now it follows from Lemma 3 that

$$d(S^n z, S^m z) \leq \frac{k^n}{1-k}\, d(z, Sz) \to 0 \ \text{ as } \ n \to +\infty, \tag{74}$$

which implies that $\{S^n z\}$ is a Cauchy sequence. As M is S-orbitally complete, there exists $\xi \in M$ such that $S^n z \to \xi$ as $n \to +\infty$. Now

$$\begin{aligned} d(S\xi, S^{n+1}z) &= d(S\xi, SS^n z) \\ &\leq k \max\{d(\xi, S^n z), d(\xi, S\xi), d(S^n z, S^{n+1}z), d(\xi, S^{n+1}z), d(S^n z, S\xi)\}. \end{aligned}$$

Taking the limit as $n \to +\infty$, we have

$$d(\xi, S\xi) \leq k\, d(\xi, S\xi), \ \text{ that is, } \ (1-k)\, d(\xi, S\xi) \leq 0.$$

As $(1-k) > 0$, it follows that $d(\xi, S\xi) = 0$, that is, $\xi = S\xi$, that is, ξ is a fixed point of S.

Suppose that $\zeta \in M$ $(\zeta \neq \xi)$ be another fixed point of S. As S is a quasi-contraction, we have

$$\begin{aligned} d(\xi, \zeta) &= d(S\xi, S\zeta) \\ &\leq k \max\{d(\xi, \zeta), d(\xi, S\xi), d(\zeta, S\zeta), d(\xi, S\zeta), d(\zeta, S\xi)\} \\ &\leq k \max\{d(\xi, \zeta), 0, 0, d(\xi, \zeta), d(\zeta, \xi)\} \\ &\leq k\, d(\xi, \zeta) \end{aligned}$$

which is a contradiction. Therefore, $d(\xi, \zeta) = 0$, that is, $\xi = \zeta$. Hence fixed point of S is unique.

Example 16 Take the metric space $M = [0, 1]$ equipped with usual metric. Define $S : M \to M$ as

$$Sz = \begin{cases} 0, & \text{if } 0 \le z < 1, \\ \frac{1}{2}, & \text{if } z = 1. \end{cases}$$

Then Theorem 13 is applicable and $z = 0$ is the unique fixed point of S. It is observed that S is not continuous on M.

In 1988, Rhoades [33] examined that there exists a large number of discontinuous contractive mappings which produce a fixed point but do not require the map to be continuous at the fixed point. Rhoades [33] raised an open question whether there exists a contractive definition which produces a fixed point but which does not require the map to be continuous at the fixed point. In 1999, Pant [27] answered the open question in the affirmative. In 2017, Bisht et al. [3] gave one more solution to the open question of the existence of contractive definitions which ensure the existence of a fixed point where the fixed point is not a point of continuity [33].

In the following theorem, the notation $Q(u, v)$ stands for

$$Q(u, v) = \max\{\rho(u, v), \rho(u, Tu), \rho(v, Tv), \frac{\rho(u, Tv) + \rho(v, Tu)}{2}\}.$$

Theorem 14 (Bisht et al. [3]) *Let (M, ρ) be a complete metric space and S be a self-mapping on M such that S^2 is continuous. Suppose that (i) $\rho(Su, Sv) \le \phi(Q(u, v))$, where $\phi : R_+ \to R_+$ is such that $\phi(t) < t$ for each $t > 0$; (ii) for a given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $\epsilon < Q(u, v) < \epsilon + \delta$ implies $\rho(Su, Sv) \le \epsilon$. Then there exists unique $z \in M$ such that $Sz = z$. Moreover, S is discontinuous at z if and only if $\lim_{u \to z} Q(u, z) \ne 0$.*

Proof Let $z_0 \in M$ be any arbitrary element. We define a sequence $\{z_n\}$ in M such that $z_n = Sz_{n-1} = S^n z_0$ for all $n \ge 1$. If $z_l = z_{l+1}$ for some positive integer l, then z_l is a fixed point of S. So we assume $z_n \ne z_{n+1}$, for all $n \ge 0$. Let $c_n = \rho(z_n, z_{n+1})$, for $n \ge 0$. By assumption (i)

$$\begin{aligned} c_{n+1} &= \rho(z_{n+1}, z_{n+2}) = \rho(Sz_n, Sz_{n+1}) \\ &\le \phi(\max\{\rho(z_n, z_{n+1}), \rho(z_n, Sz_n), \rho(z_{n+1}, Sz_{n+1}), \\ &\qquad \frac{\rho(z_n, Sz_{n+1}) + \rho(z_{n+1}, Sz_n)}{2}\}) \\ &\le \phi(\max\{\rho(z_n, z_{n+1}), \rho(z_n, z_{n+1}), \rho(z_{n+1}, z_{n+2}), \\ &\qquad \frac{\rho(z_n, z_{n+2}) + \rho(z_{n+1}, z_{n+1})}{2}\}) \\ &\le \phi(\max\{\rho(z_n, z_{n+1}), \rho(z_n, z_{n+1}), \rho(z_{n+1}, z_{n+2}), \\ &\qquad \frac{\rho(z_n, z_{n+1}) + \rho(z_{n+1}, z_{n+2})}{2}\}) \\ &\le \phi(\max\{\rho(z_n, z_{n+1}), \rho(z_{n+1}, z_{n+2})\}) \\ &= \phi(\max\{c_n, c_{n+1}\}) < \max\{c_n, c_{n+1}\}. \end{aligned}$$

Suppose that $c_n \leq c_{n+1}$. Then we have from the above inequality that $c_{n+1} < c_{n+1}$, which is a contradiction. Hence $c_{n+1} < c_n$, for all n. Then $\{c_n\}$ tends to a limit $c \geq 0$.

If possible, suppose $c > 0$. Then we have a positive integer k such that $n \geq k$ implies

$$c < c_n < c + \delta(c). \tag{75}$$

It follows from assumption (ii) and $c_{n+1} < c_n$ that $c_{n+1} \leq c$, for $n \geq k$, which contradicts the above inequality. Thus we have c = 0.

Let us fix $\epsilon > 0$. Without loss of generality, we may assume that $\delta(\epsilon) < \epsilon$. Since $c_n \to 0$ as $n \to +\infty$, there exists a positive integer k such that $c_n < \frac{\delta}{2}$, for all $n \geq k$. We shall use induction to show that for any $n \in N$,

$$\rho(z_k, z_{k+n}) < \epsilon + \frac{\delta}{2}. \tag{76}$$

The inequality (76) is true for $n = 1$. Assuming (76) is true for some n, we shall prove it for $n + 1$. Now

$$\rho(z_k, z_{k+n+1}) \leq \rho(z_k, z_{k+1}) + \rho(z_{k+1}, z_{k+n+1}). \tag{77}$$

It sufficient to show that

$$\rho(z_{k+1}, z_{k+n+1}) \leq \epsilon. \tag{78}$$

By assumption (i),

$$\rho(z_{k+1}, z_{k+n+1}) = \rho(Sz_k, Sz_{k+n}) \leq \phi(Q(z_k, z_{k+n})) < Q(z_k, z_{k+n}), \tag{79}$$

where

$$\begin{aligned} Q(z_k, z_{k+n}) &= \max\{\rho(z_k, z_{k+n}), \rho(z_k, Sz_k), \rho(z_{k+n}, Sz_{k+n}), \\ &\qquad \frac{\rho(z_k, Sz_{k+n}) + \rho(z_{k+n}, Sz_k)}{2}\} \\ &= \max\{\rho(z_k, z_{k+n}), \rho(z_k, z_{k+1}), \rho(z_{k+n}, z_{k+n+1}), \\ &\qquad \frac{\rho(z_k, z_{k+n+1}) + \rho(z_{k+n}, z_{k+1})}{2}\}. \end{aligned}$$

Now, $\rho(z_k, z_{k+n}) < \epsilon + \frac{\delta}{2}$, $\rho(z_k, z_{k+1}) < \frac{\delta}{2}$, $\rho(z_{k+n}, z_{k+n+1}) < \frac{\delta}{2}$, $\frac{\rho(z_k, z_{k+n+1}) + \rho(z_{k+n}, z_{k+1})}{2} \leq \frac{\rho(z_k, z_{k+n}) + \rho(z_{k+n}, z_{k+n+1}) + \rho(z_{k+n}, z_k) + \rho(z_k, z_{k+1})}{2} < \epsilon + \delta$. Hence $Q(z_k, z_{k+n}) < \epsilon + \delta$. If $0 \leq Q(z_k, z_{k+n}) \leq \epsilon$, then by (79), it follows that $\rho(z_{k+1}, z_{k+n+1}) \leq \epsilon$, that is, (78) is true. Again, if $\epsilon < Q(z_k, z_{k+n}) < \epsilon + \delta$, then by assumption (ii) and (79) we have that $\rho(z_{k+1}, z_{k+n+1}) \leq \epsilon$, that is, (78) is true. Therefore, $\rho(z_{k+1}, z_{k+n+1}) \leq \epsilon$, that is, (78) is true. Then from (77), we have that $\rho(z_k, z_{k+n+1}) < \epsilon + \frac{\delta}{2}$. Then by the induction method, (76) is true for any $n \in N$.

This implies that $\{z_n\}$ is a Cauchy sequence. Since M is complete, there exists a point $y \in M$ such that $z_n \to y$ as $n \to +\infty$. Also $Sz_n \to y$ and $S^2 z_n \to y$. By continuity of S^2, we have $S^2 z_n \to S^2 y$. This implies $S^2 y = y$.

We claim that $Sy = y$.

If possible, suppose that $y \neq Sy$. Then by (i), we get

$$\begin{aligned}\rho(y, Sy) &= \rho(S^2 y, Sy) \leq \phi(Q(Sy, y)) < Q(Sy, y)\\ &= \max\{\rho(Sy, y), \rho(Sy, S^2 y), \rho(y, Sy), \frac{\rho(Sy, Sy) + \rho(y, S^2 y)}{2}\} = \rho(y, Sy),\end{aligned}$$

which is a contradiction. Thus $y = Sy$, that is, y is a fixed point of S.

Suppose that $\zeta \in M$ ($\zeta \neq y$) is another fixed point of S. Then $\rho(y, \zeta) > 0$. By (i), we have

$$\begin{aligned}\rho(y, \zeta) &= \rho(Sy, S\zeta) \leq \phi(Q(y, \zeta)) < Q(y, \zeta)\\ &= \max\{\rho(y, \zeta), \rho(y, Sy), \rho(\zeta, S\zeta), \frac{\rho(y, S\zeta) + \rho(\zeta, Sy)}{2}\} = \rho(y, \zeta),\end{aligned}$$

which is a contradiction. Therefore, $\rho(y, \zeta) = 0$, that is, $y = \zeta$. Hence, S has a unique fixed point.

Example 17 ([3]) Take the metric space $M = [0, 2]$ with the metric. Define $S : M \to M$ as

$$Su = \begin{cases} 1, \text{ if } u \leq 1,\\ 0, \text{ if } u > 1.\end{cases}$$

The mapping S satisfies assumption (i) with $\phi(t) = 1$ for $t > 1$ and $\phi(t) = \frac{t}{2}$ for $t \leq 1$. Also, S satisfies assumption (ii) with $\delta(\epsilon) = 1$ for $\epsilon \geq 1$ and $\delta(\epsilon) = 1 - \epsilon$ for $\epsilon < 1$. Hence S satisfies all the assumptions of Theorem 14 and has a unique fixed point $u = 1$. Here, $\lim\limits_{u \to 1} Q(u, 1) \neq 0$ and S is discontinuous at the fixed point $u = 1$.

5 Remark

We have already mentioned that the present chapter is not sufficient for a comprehensive description of the topic under consideration. Among important results which form integral parts of the theory but are not covered here are the following. Asymptotic contractions in fixed point theory were introduced by Kirk [23]. Further generalizations of Kirk's result were done in works like [39, 41]. A very generalized fixed point theorem unifying many important results was introduced by Pant [28] which is significantly important. In 2006, Proinov [30] introduced a generalization of Banach's contraction mapping principle in a new direction which was subsequently shown to be even more general than Ćirić's quasi-contraction [10]. The review paper of Rhoades [32] is important for comprehending comparisons between several contractive conditions used in fixed point theory. Although not discussed in their technical details, the reader is strongly advised to consult these works.

Many of the results described above have initiated new lines of research in fixed point theory. For instance, the result of Caristi [5] is the origin of a study in fixed point theory and variational principles which by its vastness and importance is itself a chapter of mathematics. We do not dwell on these matters within the limited scope of this chapter. But we must say that without these considerations, the appreciation of the results presented here is bound to be partial.

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Study of Fixed Point Theorem and Infinite Systems of Integral Equations

Anupam Das and Bipan Hazarika

Abstract In this chapter, we propose some generalizations of the Meir-Keeler fixed point theorem involving a measure of noncompactness and prove the existence of a solution for an infinite system of functional integral equations by using this new type of fixed point theorem in Banach space. With the help of suitable examples, we illustrate our results.

1 Introduction

In 1930, Kuratowski [17] initiated the concept of measure of noncompactness in metric space. Measure of noncompactness (MNC) and fixed point theory can be applied to study various types of integral equations which we come across in different real-life situations. For more details on MNC, one can see [7, 9] and references therein. Darbo [11] introduced the measure of noncompactness to generalized the Banach fixed theorem. In the recent past, many researchers solved the different types of differential and non-linear integral equations in Banach spaces using the Darbo fixed point theorem. For example, we refer Aghajani et al. [2] and the references therein. Aghajani et al. [3] generalized the Meir-Keeler fixed point theorem (see [18]) with the help of MNC and used it for solving non-linear integral equations in Banach spaces. Mursaleen and Rizvi [20] solved the systems of second-order differential equations in Banach sequence spaces using the Meir-Keeler condensing

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operator. One can see [6, 12–15, 19, 21, 22] for the application of Darbo fixed point theorem and Meir-Keeler fixed point theorem in terms of MNC.

These ligatures motivated us to introduce the concept of an operator-type contraction and MNC, to study the solvability of infinite systems of functional integral equations in Banach sequence spaces. Further, we verified our results with the help of some suitable examples.

1.1 Axiomatic Approach to the Concept of an MNC

Assume a real Banach space $(E, \| \,.\, \|)$ and denote a closed ball in E centered at x_0 and with radius r by $B(x_0, r)$. If $X \subset E$ is nonempty, then $\bar{X}$ and $\text{Conv}(X)$ represent the closure and convex closure of X. Also, assume $\mathcal{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathcal{N}_E$ its subfamily consisting of all relatively compact sets.

The concept of an MNC is introduced by Banaś and Lecko [8].

Definition 1 A function $\mu : \mathcal{M}_E \to [0, \infty)$ is called an MNC if the following assumptions are satisfied:

(i) the family $\ker \mu = \{\Delta \in \mathcal{M}_E : \mu(\Delta) = 0\} \neq \phi$ and $\ker \mu \subset \mathcal{N}_E$.
(ii) $\Delta_1 \subset \Delta_2 \Rightarrow \mu(\Delta_1) \leq \mu(\Delta_2)$.
(iii) $\mu\left(\bar{\Delta}\right) = \mu(\Delta)$.
(iv) $\mu(\text{Conv}\Delta) = \mu(\Delta)$.
(v) $\mu(\gamma\Delta_1 + (1-\gamma)\Delta_2) \leq \gamma\mu(\Delta_1) + (1-\gamma)\mu(\Delta_2)$ for $\gamma \in [0, 1]$.
(vi) if $\Delta_j \in \mathcal{M}_E,\ \Delta_j = \bar{\Delta}_j,\ \Delta_{j+1} \subset \Delta_j$ for $j = 1, 2, 3, \ldots$ and $\lim_{j\to\infty} \mu\left(\Delta_j\right) = 0$ then $\bigcap_{j=1}^{\infty} \Delta_j \neq \emptyset$.

1.2 Hausdorff Measure of Noncompactness (HMNC)

Definition 2 [8] Suppose (Δ, ϱ) is a metric space, $\Gamma \subset \Delta$ is bounded and $B(w, d) = \{z \in \Delta : d(z, w) < d\}$. Then the HMNC $\chi(\Gamma)$ of Γ is defined by

$$\chi(\Gamma)0 := \inf\left\{\delta > 0 : \Gamma \subset \bigcup_{j=1}^{n} B(w_j, d_j), x_j \in \Delta, d_j < \delta \quad (j = 1, 2, \ldots, k), k \in \mathbf{N}\right\}.$$

It can equivalently be stated as follows:

$$\chi(\Gamma) = \inf\{\delta > 0 : \Gamma \text{ has a finite } \delta - \text{net in } \Delta\}.$$

Some Banach spaces are given as follows:

$$c_0 = \left\{ a = (a_i)_{i=1}^{\infty} \in \omega : \lim_{i \to \infty} a_i = 0, \ \| a \|_{\infty} = \sup_i |a_i| \right\}$$

the space of all sequences converging to zero
and

$$\ell_p = \left\{ a = (a_i)_{i=1}^{\infty} \in \omega : \sum_{i=1}^{\infty} |a_k|^p < \infty \ (1 \le p < \infty), \ \| a \|_p = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p} \right\}$$

the space of all absolutely $p-$summable series.

In the Banach space $\left(c_0, \| \, . \, \|_{c_0}\right)$, the Hausdorff MNC χ is defined by (see [8])

$$\chi_{c_0}(\mathcal{D}) = \lim_{n \to \infty} \sup_{a \in \mathcal{D}} \left[\max_{k \ge n} |a_k| \right],$$

where $a = (a_i)_{i=1}^{\infty} \in c_0$ and $\mathcal{D} \in \mathcal{M}_{c_0}$.

Banaś and Mursaleen [8] defined the Hausdorff MNC χ (in Theorem 5.18 (a)) on $\left(\ell_p, \| \, . \, \|_{\ell_p}\right)$, $(1 \le p < \infty)$ as follows:

$$\chi_{\ell_p}(\mathcal{D}) = \lim_{n \to \infty} \left[\sup_{a \in \mathcal{D}} \left(\sum_{k=n}^{\infty} | \ a_k \ |^p \right)^{1/p} \right],$$

where $a = (a_i)_{i=1}^{\infty} \in \ell_p$ and $\mathcal{D} \in \mathcal{M}_{\ell_p}$.

We recall the following important theorems.

Theorem 1 *[1, Shauder] Suppose E is a Banach space and $\Delta \subset E$ is nonempty, closed and convex. Then every compact, continuous map $T : \Delta \to \Delta$ has at least one fixed point.*

Theorem 2 *[11, Darbo] Suppose E is a Banach space and $\Delta \subset E$ is nonempty, bounded, closed and convex. Let $T : \Delta \to \Delta$ be a continuous mapping. Suppose that there is a constant $\gamma \in [0, 1)$ such that*

$$\mu(T\Lambda) \le \gamma \mu(\Lambda), \ \ \Lambda \subseteq \Delta.$$

Then T has a fixed point.

In 1969, Meir and Keeler [18] introduced a fixed point theorem in a metric space $(\Delta, \hat{d})$ for operator satisfying the following condition that for each $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\epsilon \le \hat{d}(\alpha, \beta) < \epsilon + \delta(\epsilon) \Rightarrow \hat{d}(T\alpha, T\beta) < \epsilon,$$

for all $x, y \in X$. This condition is called the Meir-Keeler (MK) contractive-type condition.

Definition 3 A mapping $\xi : \mathbf{R}_+ \to \mathbf{R}_+$ is called an MK mapping if $\xi(0) = 0$ and for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $w \in \mathbf{R}_+$,

$$\epsilon \leq w < \epsilon + \delta \Rightarrow \xi(w) < \epsilon.$$

Remark 1 It can be observed that if ξ is an MK mapping, then $\xi(w) < w$ for all $w > 0$.

We now introduce the notion of weaker Meir-Keeler function as follows:

Definition 4 [10] $\xi : \mathbf{R}_+ \to \mathbf{R}_+$ is a weaker MK mapping if for each $\epsilon > 0$, there exists $\delta > 0$ such that for any $w \geq 0$ with $\epsilon \leq w < \epsilon + \delta$, there exists $k_0 \in \mathbf{N}$ such that $\xi^{k_0}(w) < \epsilon$.

Definition 5 [16] A function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ is said to be a Jachymski function (JF) if $\psi(0) = 0$ and for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $w \in \mathbf{R}_+$,

$$\epsilon < w < \epsilon + \delta \Rightarrow \psi(w) \leq \epsilon.$$

Remark 2 [4] We observe that each MK mapping is a JF. However, the converse does not follow.

To establish our results, the following related concepts will be needed.

Definition 6 [5] Let $\hat{F}([0,\infty))$ denote the class of all functions $\hat{f} : [0,\infty) \to [0,\infty)$ and let Θ be the class of all operators

$$O(.;.) : \hat{F}([0,\infty)) \to \hat{F}([0,\infty)),\ \ \hat{f} \to O(\hat{f};.)$$

satisfying the following conditions:

1. $O(\hat{f};\hat{t}) > 0$ for $\hat{t} > 0$ and $O(\hat{f};0) = 0$;
2. $O(\hat{f};\hat{t}) \leq O(\hat{f};\hat{s})$ for $\hat{t} \leq \hat{s}$;
3. $\lim_{n\to\infty} O(\hat{f};\hat{t}_n) = O(\hat{f};\lim_{n\to\infty}\hat{t}_n)$;
4. $O(\hat{f};\max\left\{\hat{t},\hat{s}\right\}) = \max\left\{O(\hat{f};\hat{t}), O(\hat{f};\hat{s})\right\}$ for some $\hat{f} \in \hat{F}([0,\infty))$.

Example 1 If $\hat{f} : \mathbf{R}_+ \to \mathbf{R}_+$ is a Lebesgue integrable function with finite integral on each compact subset of $\mathbf{R}_+$, such that for each $w > 0$, $\int_0^w \hat{f}(s)ds > 0$, then

$$O(\hat{f};w) = \int_0^w \hat{f}(s)ds$$

satisfies all the assumptions.

Example 2 If $\hat{f} : \mathbf{R}_+ \to \mathbf{R}_+$ is a nondecreasing, continuous mapping satisfying $\hat{f}(0) = 0$ and $\hat{f}(w) > 0$ for $w > 0$, then

$$O(\hat{f}; w) = \frac{\hat{f}(w)}{1 + \hat{f}(w)}$$

satisfies all the assumptions.

Example 3 If $\hat{f} : \mathbf{R}_+ \to \mathbf{R}_+$ is any mapping, then

$$O(\hat{f}; w) = w$$

satisfies all the assumptions.

Definition 7 Let $\mathbf{F}$ be the class of all mappings $F : \mathbf{R}_+^3 \to \mathbf{R}_+$ satisfying

1. $\max\{p, q, r\} \leq F(p, q, r)$ for $p, q, r \geq 0$.
2. F is continuous and nondecreasing.
3. $F(p, 0, 0) = p$.

For example, $F(p, q, r) = p + q + r$.

2 Fixed Point Theorems

Definition 8 Let E be a Banach space and $D \subseteq E$ be nonempty and μ be an arbitrary MNC on E. Suppose that the operator $T : D \to D$ is a generalized MK-type function if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ satisfying for any $\Delta \subseteq D$,

$$\begin{aligned} &\epsilon \leq O\left(f; F\left(\mu(\Delta), \phi_1(\mu(\Delta)), \phi_2(\mu(\Delta))\right)\right) < \epsilon + \delta \\ \Rightarrow\ &O\left(f; F\left(\mu(T\Delta), \phi_1(\mu(T\Delta)), \phi_2(\mu(T\Delta))\right)\right) < \epsilon, \end{aligned}$$

where $\phi_1,\ \phi_2 : \mathbf{R}_+ \to \mathbf{R}_+$ are continuous mappings, $O(.;.) \in \Theta,\ F \in \mathbf{F}$ and $f \in \hat{F}([0, \infty)$.

Theorem 3 *Suppose that $D \subseteq E$ is a nonempty, bounded, closed and convex of a Banach space E and μ is an arbitrary MNC on E. Also $T : D \to D$ is a continuous and generalized MK condensing operator, then T has at least one fixed point on D.*

Proof Consider a sequence (D_n) satisfying $D_0 = D$ and $D_{n+1} = \text{Conv}(TD_n)$ for $n \geq 0$. We observe that $TD_0 = TD \subseteq D = D_0,\ D_1 = \text{Conv}(TD_0) \subseteq D = D_0$, therefore by continuing this process, we have $D_0 \supseteq D_1 \supseteq D_2 \supseteq \ldots \supseteq D_n \supseteq D_{n+1} \supseteq \ldots$.

If a natural number N can be found satisfying $\mu(D_N) = 0$ then D_N is compact. By Schauder's theorem, it can be concluded that T has a fixed point.

If $\mu(D_n) > 0$ for some $n \geq 0$, and also

$$F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right) > 0$$

and

$$O\left(f; F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right) > 0,$$

define $\epsilon_n = O\left(f; F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right)$ and $\delta_n = \delta(\epsilon_n)$.

As $\epsilon_n < \epsilon_n + \delta_n$, we obtain

$$\begin{aligned}\epsilon_{n+1} &= O\left(f; F\left(\mu(D_{n+1}), \phi_1(\mu(D_{n+1})), \phi_2(\mu(D_{n+1}))\right)\right)\\ &= O\left(f; F\left(\mu(\text{Conv}\,(TD_n)), \phi_1(\mu(\text{Conv}\,(TD_n))), \phi_2(\mu(\text{Conv}\,(TD_n)))\right)\right)\\ &= O\left(f; F\left(\mu(TD_n), \phi_1(\mu(TD_n)), \phi_2(\mu(TD_n))\right)\right)\\ &\leq O\left(f; F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right)\\ &= \epsilon_n.\end{aligned}$$

Therefore, $\{\epsilon_n\}$ is a positive decreasing sequence of real numbers, and there exists $\gamma \geq 0$ satisfying $\epsilon_n \to \gamma$ as $n \to \infty$.

We have to prove that $\gamma = 0$.

If possible, assume that $\gamma > 0$ then there exists a natural number n_0 satisfying $n \geq n_0$ that gives $\gamma \leq \epsilon_n < \gamma + \delta(\gamma)$, and so by applying generalized MK condensing operator we get $\epsilon_{n+1} < \gamma$ which is a contradiction. Therefore, $\gamma = 0$, i.e.$\epsilon_n \to 0$ as $n \to \infty$. Therefore,

$$O\left(f; \lim_{n\to\infty} F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right) = 0$$

which gives

$$\lim_{n\to\infty} F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right) = 0.$$

Using the property of F, we get

$$\lim_{n\to\infty} \mu(D_n) = \lim_{n\to\infty} \phi_1(\mu(D_n)) = \lim_{n\to\infty} \phi_2(\mu(D_n)) = 0.$$

Since $D_n \supseteq D_{n+1}$ for all $n \in \mathbf{N}$, therefore applying Definition 1, we obtain that $D_\infty = \bigcap_{n=1}^{\infty} D_n$ is a nonempty, closed and convex subset of D and D_∞ is invariant under T. Thus, by applying Schauder's theorem it can be said that T has a fixed point in $D_\infty \subseteq D$. This completes the proof.

Theorem 4 *Suppose E is a Banach space and $D \subseteq E$ is nonempty, bounded, closed and convex, and μ is an arbitrary MNC on E. Also $T : D \to D$ is a continuous mapping, and for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ satisfying for any $X \subseteq D$,*

$$\begin{aligned}&\epsilon \leq O\left(f; \mu(X) + \phi_1(\mu(X)) + \phi_2(\mu(X))\right) < \epsilon + \delta\\ &\Rightarrow O\left(f; \mu(TX) + \phi_1(\mu(TX)) + \phi_2(\mu(TX))\right) < \epsilon,\end{aligned}$$

where $\phi_1,\ \phi_2 : \mathbf{R}_+ \to \mathbf{R}_+$ are continuous mappings, $O(.;.) \in \Theta$ and $f \in F([0,\infty)$. Then T has at least one fixed point.

Proof Taking $F(a, b, c) = a + b + c$ in Theorem 3, we obtain the results.

Theorem 5 *Assume that $D \subseteq E$ is nonempty, bounded, closed and convex of a Banach space E, and μ is an arbitrary MNC on E. Assume that $T : D \to D$ is a continuous mapping, and for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ satisfying for any $X \subseteq D$,*

$$\epsilon \le \mu(X) + \phi_1(\mu(X)) + \phi_2(\mu(X)) < \epsilon + \delta$$
$$\Rightarrow \mu(TX) + \phi_1(\mu(TX)) + \phi_2(\mu(TX)) < \epsilon,$$

where $\phi_1,\ \phi_2 : \mathbf{R}_+ \to \mathbf{R}_+$ are continuous functions. Then T has at least one fixed point.

Proof Taking $O(f; w) = w$ in Theorem 4, the result can be obtained.

Theorem 6 *Assume D is a nonempty, bounded, closed and convex subset of a Banach space E, and μ is an arbitrary MNC on E. Also $T : D \to D$ is a continuous mapping, and for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ satisfying for any $X \subseteq D$,*

$$\epsilon \le \mu(X) + \phi_1(\mu(X)) < \epsilon + \delta$$
$$\Rightarrow \mu(TX) + \phi_1(\mu(TX)) < \epsilon,$$

where $\phi_1 : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous function. Then T has at least one fixed point.

Proof Taking $\phi_2 \equiv 0$ in Theorem 5, the result can be obtained.

Theorem 7 *Suppose a Banach space E and $D \subseteq E$ is nonempty, bounded, closed and convex with μ is an arbitrary MNC on E. Assume $T : D \to D$ is a continuous mapping. Suppose that there exists a JF $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying $\psi(t) < t$ for all $t > 0$ and for any $X \subseteq D$,*

$$O\left(f; F\left(\mu(TX), \phi_1(\mu(TX)), \phi_2(\mu(TX))\right)\right) \le \psi\left(O\left(f; F\left(\mu(X), \phi_1(\mu(X)), \phi_2(\mu(X))\right)\right)\right)$$

where $\phi_1,\ \phi_2, \psi : \mathbf{R}_+ \to \mathbf{R}_+$ are continuous mappings, $O(.;.) \in \Theta,\ F \in \mathbf{F}$ and $f \in F([0, \infty)$. Then T has at least one fixed point.

Proof Consider a sequence (D_n) satisfying $D_0 = D$ and $D_{n+1} = \text{Conv}(TD_n)$ for $n \ge 0$. We observe that $TD_0 = TD \subseteq D = D_0,\ D_1 = \text{Conv}(TD_0) \subseteq D = D_0$, therefore by continuing this process, we have $D_0 \supseteq D_1 \supseteq D_2 \supseteq \ldots \supseteq D_n \supseteq D_{n+1} \supseteq \ldots$.

If a natural number N can be found satisfying $\mu(D_N) = 0$, then D_N is compact. By Schauder's theorem, it can be concluded that T has a fixed point.

If $\mu(D_n) > 0$ for some $n \ge 0$, and also

$$F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right) > 0$$

and

$$O\left(f;\, F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right) > 0,$$

then we have

$$\begin{aligned} & O\left(f;\, F\left(\mu(D_{n+1}), \phi_1(\mu(D_{n+1})), \phi_2(\mu(D_{n+1}))\right)\right) \\ = {} & O\left(f;\, F\left(\mu(\mathrm{Conv}\,(TD_n)), \phi_1(\mu(\mathrm{Conv}\,(TD_n))), \phi_2(\mu(\mathrm{Conv}\,(TD_n)))\right)\right) \\ = {} & O\left(f;\, F\left(\mu(TD_n), \phi_1(\mu(TD_n)), \phi_2(\mu(TD_n))\right)\right) \\ \le {} & \psi\left(O\left(f;\, F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right)\right) \\ < {} & O\left(f;\, F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right). \end{aligned}$$

Then $\{O\left(f;\, F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right)\}_{n\in\mathbf{N}}$ is a nonincreasing sequence and thus it converges to some point $\epsilon \ge 0$ satisfying

$$\epsilon < O\left(f;\, F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right)$$

for all $n \in \mathbf{N}$.

If $\epsilon > 0$, then there exists $\delta = \delta(\epsilon)$ satisfying

$$\epsilon < t < \epsilon + \delta \Rightarrow \psi(t) \le \epsilon.$$

Take $n_\delta \in \mathbf{N}$ satisfying

$$O\left(f;\, F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right) < \epsilon + \delta$$

for all $n \ge n_\delta$. Therefore,

$$\psi\left(O\left(f;\, F\left(\mu(D_n), \phi_1(\mu(D_n)), \phi_2(\mu(D_n))\right)\right)\right) \le \epsilon$$

and so

$$O\left(f;\, F\left(\mu(D_{n+1}), \phi_1(\mu(D_{n+1})), \phi_2(\mu(D_{n+1}))\right)\right) \le \epsilon$$

for all $n \in \mathbf{N}$, which is a contradiction so, $\epsilon = 0$ and $\lim_{n\to\infty} \mu(D_n) = 0$.

Since $D_n \supseteq D_{n+1}$ for all $n \in \mathbf{N}$, therefore, applying Definition 1, we obtain that $D_\infty = \bigcap_{n=1}^{\infty} D_n$ is a nonempty, closed and convex subset of D and D_∞ is invariant under T. Thus, applying Schauder's theorem it can be said that T has a fixed point in $D_\infty \subseteq D$. This completes the proof.

Theorem 8 *Assume $D \subseteq E$ is a nonempty bounded closed convex of a Banach space E and μ is an arbitrary MNC on E. Also $T : D \to D$ is a continuous mapping. Suppose that there exists a JF $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ satisfying $\psi(t) < t$ for all $t > 0$ and for any $X \subseteq D$,*

$$O\left(f;\, \mu(TX) + \phi_1(\mu(TX)) + \phi_2(\mu(TX))\right) \le \psi\left(O\left(f;\, \mu(X) + \phi_1(\mu(X)) + \phi_2(\mu(X))\right)\right),$$

where $\phi_1,\ \phi_2, \psi : \mathbf{R}_+ \to \mathbf{R}_+$ *are continuous mappings,* $O(.;.) \in \Theta$ *and* $f \in F([0,\infty))$. *Then* T *has at least one fixed point.*

Proof Taking $F(p,q,r) = p+q+r$ in Theorem 7, the results can be obtained.

Theorem 9 *Assume* $D \subseteq E$ *is nonempty, bounded, closed and convex of a Banach space* E*, and* μ *is an arbitrary MNC on* E. *Assume* $T : D \to D$ *is a continuous mapping. Suppose that there exists a JF* $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ *satisfying* $\psi(t) < t$ *for all* $t > 0$ *and for any* $X \subseteq D$,

$$O\left(f; \mu(TX)\right) \le \psi\left(O\left(f; \mu(X)\right)\right),$$

where $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ *are continuous mappings,* $O(.;.) \in \Theta$ *and* $f \in F([0,\infty))$. *Then* T *has at least one fixed point.*

Proof Taking $\phi_1 \equiv 0$ and $\phi_2 \equiv 0$ in Theorem 8, the results can be obtained.

3 Application of Fixed Point on Integral Equations

In this part, the following infinite system of functional integral equations shall be studied:

$$z_n(t) = F_n\left(t, \int_0^{a(t)} G_n(t,v,z(v))dv, z(t)\right) \tag{1}$$

where $z(t) = (z_i(t))_{i=1}^{\infty},\ t \in I = [0,1]$.

3.1 Solvability of Infinite System of Functional Integral Equations in c_0

Assumptions

(1) $a : I \to \mathbf{R}_+$ is continuous.
(2) $F_n : I \times \mathbf{R} \times c_0 \to \mathbf{R}$ $(n \in \mathbf{N})$ are continuous functions with

$$\hat{F}_n = \sup\left\{\left|F_n(t,0,z^0(t))\right| : t \in I\right\},$$

where $z^0(t) = \left(z_n^0(t)\right)_{n=1}^{\infty}$ and $z_n^0(t) = 0$ for all $n \in \mathbf{N},\ t \in I$.
Also $u_n,\ m_n : I \to \mathbf{R}_+$ $(n \in \mathbf{N})$ are continuous functions satisfying

$$|F_n(t,p,z(t)) - F_n(t,q,\bar{z}(t))| \le u_n(t)\max_{i \ge n}|z_i(t) - \bar{z}_i(t)| + m_n(t)\,|p-q|,$$

where $\bar{z}(t) = (\bar{z}_i(t))_{i=1}^{\infty} \in c_0$.

(3) $G_n : I \times I \times c_0 \to \mathbf{R}$ $(n \in \mathbf{N})$ are continuous. Also,

$$\hat{G}_n = \sup \left\{ m_n(t) \left| \int_0^{a(t)} G_n(t, v, z(v)) dv \right| : t \in I \right\}.$$

(4) Define an operator Z on $I \times c_0$ to c_0 as follows:

$$(t, z(t)) \to (Zz)(t) = \left(F_n \left(t, \int_0^{a(t)} G_n(t, v, z(v)) dv, z(t) \right) \right)_{n=1}^{\infty}.$$

(5) As $n \to \infty$, then $\hat{F}_n \to 0$, $\hat{G}_n \to 0$. Also

$$\sup_{n \in \mathbf{N}} \hat{F}_n = \hat{F}, \quad \sup_{n \in \mathbf{N}} \hat{G}_n = \hat{G},$$

and $\sup\{u_n(t) : t \in I,\ n \in \mathbf{N}\} = \hat{U}$ such that $0 < \hat{U} < 1$.

Theorem 10 *If assumptions (1)–(5) hold, the system of equations (1) has at least one solution in $z(t) \in c_0,\ t \in I$.*

Proof For all $t \in I$,

$$\begin{aligned}
\| z(t) \|_{c_0} &= \max_{n \geq 1} \left| F_n \left(t, \int_0^{a(t)} G_n(t, v, z(v)) dv, z(t) \right) \right| \\
&\leq \max_{n \geq 1} \left| F_n \left(t, \int_0^{a(t)} G_n(t, v, z(v)) dv, z(t) \right) - F_n \left(t, 0, z^0(t) \right) \right| \\
&\quad + \max_{n \geq 1} \left| F_n \left(t, 0, z^0(t) \right) \right| \\
&\leq \max_{n \geq 1} \left\{ u_n(t) \max_{i \geq n} |z_i(t)| + m_n(t) \left| \int_0^{a(t)} G_n(t, v, z(v)) dv \right| \right\} \\
&\leq \hat{U} \| z(t) \|_{c_0} + \hat{G} + \hat{F},
\end{aligned}$$

i.e.

$$\| z(t) \|_{c_0} \leq \frac{\hat{G} + \hat{F}}{1 - \hat{U}} = r \ (say).$$

Let $\hat{B} = \hat{B}(z^0(t), r)$ be a closed ball with center at $z^0(t)$ and radius r, thus $\hat{B}$ is an NBCC subset of c_0. By assumption (4), for all $t \in I$,

$$(Zz)(t) = \{(Z_n z)(t)\}_{n=1}^{\infty} = \left\{ F_n \left(t, \int_0^{a(t)} G_n(t, v, z(v)) dv, z(t) \right) \right\}_{n=1}^{\infty},$$

where $z(t) \in \hat{B}$. Also,

$$\lim_{n\to\infty}(Z_n z)(t) = \lim_{n\to\infty} F_n\left(t, \int_0^{a(t)} G_n(t, v, z(v))dv, z(t)\right) = 0.$$

Hence $(Zz)(t) \in c_0$. Since $\| (Zz)(t) - z^0(t) \|_{c_0} \leq r$, therefore, Z maps $\hat{B}$ to $\hat{B}$.

Now, we claim that Z is continuous on $\hat{B}$.

Let $\epsilon > 0$ and $x(t) = (x_n(t))_{n=1}^{\infty},\ y(t) = (y_n(t))_{n=1}^{\infty} \in \hat{B}$ satisfying $\| x - y \|_{c_0} < \frac{\epsilon}{2\hat{U}} = \delta$.

For all $t \in I$,

$$\begin{aligned}
&|(Z_n x)(t) - (Z_n y)(t)| \\
&= \left| F_n\left(t, \int_0^{a(t)} G_n(t, v, x(v))dv, x(t)\right) - F_n\left(t, \int_0^{a(t)} G_n(t, v, y(v))dv, y(t)\right)\right| \\
&\leq \hat{U} \max_{i\geq n} |x_i(t) - y_i(t)| + m_n(t)\left|\int_0^{a(t)} G_n(t, v, x(v))dv - \int_0^{a(t)} G_n(t, v, y(v))dv\right| \\
&< \frac{\epsilon}{2} + m_n(t)\int_0^{a(t)} |G_n(t, v, x(v)) - G_n(t, v, y(v))|\, dv.
\end{aligned}$$

Let

$$A = \sup\{a(t) : t \in I\} \text{ and } M = \sup\{m_n(t) : t \in I,\ n \in \mathbf{N}\}.$$

As G_n is a continuous function, for $\| x - y \|_{c_0} < \delta$ we get

$$|G_n(t, v, x(v)) - G_n(t, v, y(v))| < \frac{\epsilon}{2(M+1)(A+1)}, \forall n \in \mathbf{N}.$$

Therefore,

$$\begin{aligned}
&m_n(t)\int_0^{a(t)} |G_n(t, v, x(v)) - G_n(t, v, y(v))|\, dv \\
&\leq M\int_0^{A} |G_n(t, v, x(v)) - G_n(t, v, y(v))|\, dv \\
&< \frac{MA\epsilon}{2(M+1)(A+1)} \\
&< \frac{\epsilon}{2}.
\end{aligned}$$

Thus, $|(Z_n x)(t) - (Z_n y)(t)| < \epsilon$ for $\| x - y \|_{c_0} < \delta$.

For $t \in I$,

$$\| Zx - Zy \|_{c_0} < \epsilon \text{ whenever } \| x - y \|_{c_0} < \delta.$$

Hence, Z is continuous on $\hat{B}$. Now,

$$\begin{aligned}\chi(Z(\hat{B})) &= \lim_{n\to\infty}\sup_{z(t)\in\hat{B}}\max_{k\geq n}\left|F_k\left(t,\int_0^{a(t)}G_n(t,v,z(v))dv,z(t)\right)\right|\\&\leq \lim_{n\to\infty}\sup_{z(t)\in\hat{B}}\max_{k\geq n}\left\{u_k(t)\max_{i\geq n}|z_i(t)|+m_k(t)\left|\int_0^{a(t)}G_n(t,v,z(v))dv\right|+F_k\right\}\\&\leq \hat{U}\chi(\hat{B}).\end{aligned}$$

Observe that $\chi(Z(\hat{B}))\leq\hat{U}\chi(\hat{B})<\epsilon$ gives $\chi(\hat{B})<\frac{\epsilon}{\hat{U}}$. Taking $\delta=\frac{\epsilon(1-\hat{U})}{\hat{U}}$, we get $\epsilon\leq\chi(\hat{B})<\epsilon+\delta$.

Applying Theorem 5 for $\phi_1\equiv\phi_2\equiv 0$, we imply that Z has at least one fixed point on $\hat{B}\subset c_0$, i.e. Eq. (1) has at least one solution in c_0. This completes the proof.

Example 4 Consider the following infinite system:

$$z_n(t)=\frac{1}{t+n^2}+\sum_{i\geq n}\frac{|z_i(t)|}{2i^2}+\frac{1}{n^3e^t}\int_0^t\frac{\sin(z_n(v))+\cos(v)\sin\left(\sum_{i=1}^n z_i(v)\right)}{2+\sin\left(\sum_{i=1}^n z_i(v)\right)}dv \tag{2}$$

for $t\in[0,1]=I,\ n\in\mathbf{N}$.

For this problem $a(t)=t$,

$$\begin{aligned}&F_n\left(t,\int_0^{a(t)}G_n(t,v,z(v))dv,z(t)\right)\\&=\frac{1}{t+n^2}+\sum_{i\geq n}\frac{|z_i(t)|}{2i^2}+\frac{1}{n^3e^t}\int_0^t\frac{\sin(z_n(v))+\cos(v)\sin\left(\sum_{i=1}^n z_i(v)\right)}{2+\sin\left(\sum_{i=1}^n z_i(v)\right)}dv\end{aligned}$$

and

$$G_n(t,v,z(v))=\frac{\sin(z_n(v))+\cos(v)\sin\left(\sum_{i=1}^n z_i(v)\right)}{2+\sin\left(\sum_{i=1}^n z_i(v)\right)}.$$

If $z(t)\in c_0$, then $F_n\left(t,\int_0^{a(t)}G_n(t,v,z(v))dv,z(t)\right)\in c_0$.

Now, if $x(t)=(x_i(t))_{i=1}^{\infty}$, $y(t)=(y_i(t))_{i=1}^{\infty}\in c_0$, then

$$\begin{aligned}&\left|F_n\left(t,\int_0^{a(t)}G_n(t,v,x(v))dv,x(t)\right)-F_n\left(t,\int_0^{a(t)}G_n(t,v,y(v))dv,y(t)\right)\right|\\&\leq\sum_{i\geq n}\frac{1}{2i^2}|x_i(t)-y_i(t)|+\frac{1}{n^3e^t}\int_0^{a(t)}|G_n(t,v,x(v))-G_n(t,v,y(v))|dv\\&\leq\left(\sum_{i\geq n}\frac{1}{2i^2}\right)\max_{i\geq n}|x_i(t)-y_i(t)|+\frac{1}{n^3e^t}\int_0^{a(t)}|G_n(t,v,x(v))-G_n(t,v,y(v))|dv\\&\leq\frac{\pi^2}{12}|x_i(t)-y_i(t)|+\frac{1}{n^3e^t}\int_0^{a(t)}|G_n(t,v,x(v))-G_n(t,v,y(v))|dv.\end{aligned}$$

Here, $u_t = \frac{\pi^2}{12}$, $m_n(t) = \frac{1}{n^3 e^t}$. Also, $0 < \hat{U} < 1$, ; $\hat{F}_n \leq 1$ and $\hat{F}_n \to 0$ as $n \to \infty$. Again,

$$\hat{G}_n = \sup\left\{\frac{1}{n^3 e^t}\left|\int_0^t \frac{\sin(z_n(v)) + \cos(v)\sin\left(\sum_{i=1}^n z_i(v)\right)}{2 + \sin\left(\sum_{i=1}^n z_i(v)\right)} dv\right| : t, v \in I\right\} \leq \frac{2}{n^3},$$

i.e. as $n \to \infty$ then $\hat{G}_n \to 0$.

The functions F_n and G_n are continuous for all $n \in \mathbf{N}$. As assumptions from (1)–(5) are satisfied, by applying Theorem 10 it can be obtained that Eq. (2) has at least one solution in c_0.

3.2 Solvability of Infinite System of Functional Integral Equations in ℓ_1

Assumptions

(1) $a : I \to \mathbf{R}_+$ is continuous.

(2) $F_n : I \times \mathbf{R} \times \ell_1 \to \mathbf{R}$ $(n \in \mathbf{N})$ are continuous functions with

$$\sum_{n \geq 1} \left|F_n(t, 0, z^0(t))\right|$$

converging to zero where $z^0(t) = \left(z_n^0(t)\right)_{n=1}^{\infty}$ and $z_n^0(t) = 0$ for all $n \in \mathbf{N}$, $t \in I$. Also $\alpha_n, \beta_n : I \to \mathbf{R}_+$ $(n \in \mathbf{N})$ are continuous functions satisfying

$$|F_n(t, p, z(t)) - F_n(t, q, \bar{z}(t))| \leq \alpha_n(t)\,|z_n(t) - \bar{z}_n(t)| + \beta_n(t)\,|p - q|\,,$$

where $\bar{z}(t) = (\bar{z}_i(t))_{i=1}^{\infty} \in \ell_1$.

(3) $G_n : I \times I \times \ell_1 \to \mathbf{R}$ $(n \in \mathbf{N})$ are continuous. Also, there exists Q_k satisfying

$$Q_k = \sup\left\{\sum_{n \geq k}\left[\beta_n(t)\left|\int_0^{a(t)} G_n(t, v, z(v))dv\right|\right] : t \in I\right\}.$$

(4) Define an operator Z on $I \times \ell_1$ to ℓ_1 as follows:

$$(t, z(t)) \to (Zz)(t) = \left(F_n\left(t, \int_0^{a(t)} G_n(t, v, z(v))dv, z(t)\right)\right)_{n=1}^{\infty}.$$

(5) As $n \to \infty$ then $Q_n \to 0$. Also

$$\sup_{n \in \mathbf{N}} Q_n = \hat{Q}, \quad \sup\{\alpha_n(t) : t \in I,\ n \in \mathbf{N}\} = \hat{\alpha}$$

such that $0 < \hat{\alpha} < 1$ and for all $t \in I$,

$$\hat{\beta} = \sup \left\{ \sum_{n} \beta_n(t) : n \in \mathbf{N},\ t \in I \right\}.$$

Theorem 11 *If assumptions (1)–(5) hold, the system of equations (1) has at least one solution in $z(t) \in \ell_1,\ t \in I$.*

Proof For all $t \in I$,

$$\begin{aligned}
\| z(t) \|_{\ell_1} &= \sum_{n\geq 1} \left| F_n\left(t, \int_0^{a(t)} G_n(t, v, z(v))dv, z(t)\right)\right| \\
&\leq \sum_{n\geq 1} \left| F_n\left(t, \int_0^{a(t)} G_n(t, v, z(v))dv, z(t)\right) - F_n\left(t, 0, z^0(t)\right)\right| \\
&\quad + \sum_{n\geq 1} \left| F_n\left(t, 0, z^0(t)\right)\right| \\
&\leq \sum_{n\geq 1} \left\{ \alpha_n(t)\, |z_n(t)| + \beta_n(t) \left| \int_0^{a(t)} G_n(t, v, z(v))dv \right| \right\} \\
&\leq \hat{\alpha} \| z(t) \|_{\ell_1} + \hat{Q},
\end{aligned}$$

i.e.

$$\| z(t) \|_{\ell_1} \leq \frac{\hat{Q}}{1-\hat{\alpha}} = \hat{r}\ (say).$$

Let $\hat{D} = \hat{D}(z^0(t), r)$ be a closed ball with center at $z^0(t)$ and radius $\hat{r}$, thus $\hat{D}$ is an NBCC subset of ℓ_1. By assumption (4), for all $t \in I$,

$$(Zz)(t) = \{(Z_n z)(t)\}_{n=1}^{\infty} = \left\{ F_n\left(t, \int_0^{a(t)} G_n(t, v, z(v))dv, z(t)\right)\right\}_{n=1}^{\infty},$$

where $z(t) \in \hat{D}$. Also,

$$\sum_{n\geq 1} |(Z_n z)(t)| = \sum_{n\geq 1} \left| F_n\left(t, \int_0^{a(t)} G_n(t, v, z(v))dv, z(t)\right)\right| < \infty.$$

Hence, $(Zz)(t) \in \ell_1$. Since $\| (Zz)(t) - z^0(t) \|_{\ell_1} \leq \hat{r}$, therefore, Z maps $\hat{D}$ to $\hat{D}$.

Now, we claim that Z is continuous on $\hat{D}$.

Let $\epsilon > 0$ and $x(t) = (x_n(t))_{n=1}^{\infty},\ y(t) = (y_n(t))_{n=1}^{\infty} \in \hat{D}$ satisfying $\| x - y \|_{\ell_1} < \frac{\epsilon}{2\hat{\alpha}} = \delta$.

For all $t \in I$,

$$\begin{aligned}
&|(Z_n x)(t) - (Z_n y)(t)| \\
&= \left| F_n\left(t, \int_0^{a(t)} G_n(t, v, x(v))dv, x(t)\right) - F_n\left(t, \int_0^{a(t)} G_n(t, v, y(v))dv, y(t)\right) \right| \\
&\leq \hat{\alpha}\, |x_n(t) - y_n(t)| + \beta_n(t) \left| \int_0^{a(t)} G_n(t, v, x(v))dv - \int_0^{a(t)} G_n(t, v, y(v))dv \right| \\
&\leq \hat{\alpha}\, |x_n(t) - y_n(t)| + \beta_n(t) \int_0^{a(t)} |G_n(t, v, x(v)) - G_n(t, v, y(v))|\, dv.
\end{aligned}$$

Let $A = \sup\{a(t) : t \in I\}$.

As G_n is a continuous function, for $\| x - y \|_{\ell_1} < \delta$ we get

$$|G_n(t, v, x(v)) - G_n(t, v, y(v))| < \frac{\epsilon}{2(\hat{\beta} + 1)(A + 1)}$$

for all $n \in \mathbf{N}$. Therefore,

$$\begin{aligned}
&\beta_n(t) \int_0^{a(t)} |G_n(t, v, x(v)) - G_n(t, v, y(v))|\, dv \\
&\leq \beta_n(t) \int_0^{A} |G_n(t, v, x(v)) - G_n(t, v, y(v))|\, dv \\
&< \frac{A\epsilon\beta_n(t)}{2(\hat{\beta} + 1)(A + 1)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{n\geq 1} |(Z_n x)(t) - (Z_n y)(t)| \\
&\leq \hat{\alpha} \sum_{n\geq 1} |x_n(t) - y_n(t)| + \frac{A\epsilon}{2(\hat{\beta} + 1)(A + 1)} \sum_{n\geq 1} \beta_n(t) \\
&< \frac{\epsilon}{2} + \frac{A\epsilon\hat{\beta}}{2(\hat{\beta} + 1)(A + 1)} = \epsilon.
\end{aligned}$$

Therefore,

$$\| Zx - Zy \|_{\ell_1} < \epsilon \quad \text{whenever } \| x - y \|_{\ell_1} < \delta.$$

Hence Z is continuous on $\hat{D}$.

Now,

$$\begin{aligned}\chi(Z(\hat{D})) &= \lim_{n\to\infty} \sup_{z(t)\in\hat{D}} \sum_{k\geq n} \left| F_k\left(t, \int_0^{a(t)} G_n(t,v,z(v))dv, z(t)\right)\right| \\ &\leq \lim_{n\to\infty} \sup_{z(t)\in\hat{D}} \sum_{k\geq n} \left\{ \alpha_k(t)\,|z_k(t)| + \beta_k(t) \left| \int_0^{a(t)} G_n(t,v,z(v))dv \right| \right\} \\ &\leq \hat{\alpha}\chi(\hat{B}).\end{aligned}$$

Observe that $\chi(Z(\hat{D})) \leq \hat{\alpha}\chi(\hat{D}) < \epsilon$ gives $\chi(\hat{D}) < \frac{\epsilon}{\hat{\alpha}}$. Taking $\delta = \frac{\epsilon(1-\hat{\alpha})}{\hat{\alpha}}$, we get $\epsilon \leq \chi(\hat{D}) < \epsilon + \delta$.

By applying Theorem 5 for $\phi_1 \equiv \phi_2 \equiv 0$, we imply that Z has at least one fixed point on $\hat{D} \subset \ell_1$, i.e. Eq. (1) has at least one solution in ℓ_1. This completes the proof.

Example 5 Consider the following infinite system:

$$z_n(t) = \sum_{i\geq n} \frac{z_i(t)\sin(t)}{2i^2} + \frac{1}{n^2 e^t} \int_0^t \frac{\cos\left(\sum_{i=1}^n z_i(v)\right)}{2+\sin\left(z_n(v)\right)} dv \tag{3}$$

for $t \in [0,1] = I,\ n \in \mathbf{N}$.

For this problem $a(t) = t$,

$$F_n\left(t, \int_0^{a(t)} G_n(t,v,z(v))dv, z(t)\right) = \sum_{i\geq n} \frac{z_i(t)\sin(t)}{2i^2} + \frac{1}{n^2 e^t} \int_0^t \frac{\cos\left(\sum_{i=1}^n z_i(v)\right)}{2+\sin\left(z_n(v)\right)} dv$$

and

$$G_n(t,v,z(v)) = \frac{\cos\left(\sum_{i=1}^n z_i(v)\right)}{2+\sin\left(z_n(v)\right)}.$$

If $z(t) \in \ell_1$, then

$$\begin{aligned}&\sum_{n=1}^{\infty} \left| F_n\left(t, \int_0^{a(t)} G_n(t,v,z(v))dv, z(t)\right)\right| \\ &\leq \sum_{n=1}^{\infty}\sum_{i\geq n} \left| \frac{z_i(t)\sin(t)}{2i^2} \right| + \frac{t}{e^t}\sum_{n\geq 1}\frac{1}{n^2} \\ &\leq \ \| z(t) \|_{\ell_1} + \frac{\pi^2}{6e} < \infty,\end{aligned}$$

i.e. $F_n\left(t, \int_0^{a(t)} G_n(t,v,z(v))dv, z(t)\right) \in \ell_1$.

Now, if $x(t) = (x_i(t))_{i=1}^{\infty},\ \ y(t) = (y_i(t))_{i=1}^{\infty} \in \ell_1$, then

$$\left| F_n\left(t, \int_0^{a(t)} G_n(t, v, x(v))dv, x(t)\right) - F_n\left(t, \int_0^{a(t)} G_n(t, v, y(v))dv, y(t)\right)\right|$$
$$\leq \frac{\pi^2}{12}\left|x_n(t) - y_n(t)\right| + \frac{1}{n^2 e^t}\int_0^{a(t)} \left|G_n(t, v, x(v)) - G_n(t, v, y(v))\right| dv.$$

Here, $\alpha_n(t) = \frac{\pi^2}{12}$, $\beta_n(t) = \frac{1}{n^2 e^t}$. Also, $0 < \hat{\alpha} < 1$ and $\sum_{n\geq 1}\left|F_n(t, 0, z^0(t))\right|$ converges to zero for all $t \in I$.

Again,

$$\sum_{n\geq 1}\beta_n(t)\left|\int_0^{a(t)} G_n(t, v, z(v))dv\right| \leq \frac{t}{e^t}\sum_{n\geq k}\frac{1}{n^2}$$

and

$$Q_k \leq \sup\left\{\frac{t}{e^t}\sum_{n\geq k}\frac{1}{n^2} : t \in I\right\}.$$

As $k \to \infty$ then $Q_k \to 0$ and $\hat{Q} = \frac{\pi^2}{6e}$, we also have $\hat{\beta} = \frac{\pi^2}{6}$.

The functions F_n and G_n are continuous for all $n \in \mathbf{N}$. As assumptions (1)–(5) are satisfied, by applying Theorem 11 we imply that Eq. (3) has at least one solution in ℓ_1.

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Common Fixed Point Theorems and Applications in Intuitionistic Fuzzy Cone Metric Spaces

Nabanita Konwar

Abstract The aim of this chapter is to establish some common fixed point theorems in intuitionistic fuzzy cone metric space (in short, IFCMS) for compatible and weakly compatible mappings. The existence and uniqueness of the common fixed point have been studied. In order to substantiate the non-triviality of results some examples are provided.

1 Introduction

Sometimes the mathematical modeling of many real-life experiments becomes infeasible because of the inadequate measure of distance between two elements or points. For such type of situations, fuzzy logic or fuzzy set contribute a consequential platform to construct and improve the modeling and designing mathematical systems. The generalized form of fuzzy set, i.e., the intuitionistic fuzzy set (in short, IFS) can control more complex situations efficiently and also reduce the complexity of modeling systems for higher order sets. The flexible nature of such type of models helps to improve the applications of science and mathematics and also motivates our current study.

Fuzzy set theory was established by Zadeh [23] in 1965. A generalized concept called IFS was initiated by Atanassaov [3] in 1986. With the help of this newly generalized set one can deal with the degree of membership as well as non-membership properties of an elements of a set. Kaleva and Seikkala [15] put forward the notion of fuzzy metric space (in short, FMS). Consequently the idea of FMS was modified by several mathematicians like Kramosil and Michalek [16], George and Veeramani [7], etc. Jungck and Rhoads [14] established the concept of weakly compatible maps on metric spaces. Further, Huang and Zhang [10] established the idea of cone metric space, whereas Abbas and Jungck [1] introduced some results on non-commuting mapping in cone metric spaces.

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Initially, Heilpern [9] introduced the conception of fixed point theory and established an extended version of Banach's contraction principle in the setting of FMS. The idea of fuzzy cone metric space (in short, FCMS) was initiated by Oner et al. [18]. This famous work has been further generalized and extended by many mathematicians in the settings of fuzzy set [2, 4–6, 8, 17, 19–22].

In this chapter, we investigate the existence and uniqueness of common fixed point for a pair of self-mappings in an IFCMS.

2 Preliminaries

Below we discuss a few preliminary definitions.

Definition 1 Consider a binary operation $b_* : [0, 1] \times [0, 1] \to [0, 1]$. Then b_* is known as a continuous $t - norm$ if:

(i) b_* satisfy the associativity and commutativity property,
(ii) b_* satisfy the continuous property,
(iii) $\alpha b_* 1 = \alpha, \forall\, \alpha \in [0, 1]$,
(iv) $\alpha b_* b \leq \beta b_* d$ whenever $\alpha \leq \beta$ and $b \leq d$ and $\alpha, \beta, c, d \in [0, 1]$.

Definition 2 Consider a binary operation $b_\circ : [0, 1] \times [0, 1] \to [0, 1]$. Then $b_\circ$ is called a continuous $t - co - norm$ if

(i) $b_\circ$ satisfy the associativity and commutativity property,
(ii) $b_\circ$ satisfy the continuous property,
(iii) $\alpha b_\circ 0 = \alpha, \forall\, \alpha \in [0, 1]$,
(iv) $\alpha b_\circ b \leq \beta b_\circ d$ whenever $\alpha \leq \beta$ and $b \leq d$ and $\alpha, \beta, c, d \in [0, 1]$.

Definition 3 ([11]) Suppose S is a subset of a real Banach space X then S is called a cone if it satisfies the following conditions:

(i) S is closed, non-empty and $S \neq 0$.
(ii) if $\alpha, \beta \in [0, \infty)$ and $x, y \in S$, then $\alpha x + \beta y \in S$
(iii) if both $x \in S$ and $-x \in S$, then $x = 0$.

For $S \subset X$, a partial ordering $\leq$ on X is defined by $x \leq y$ if and only if $y - x \in S$. And $x << y$ stands for $y - x \in int(S)$. All cones have non-empty interior.

Definition 4 ([3]) Suppose Y is a non-empty set and I is a subset of Y. Define the mappings $\hbar_I : Y \to [0, 1]$ and $\wp_I : Y \to [0, 1]$. If I is defined as $I = \{(x, \hbar_I(x), \wp_I(x)) : x \in X, 0 \leq \hbar_I + \wp_I \leq 1\}$, where $\hbar_I$ is the degree of membership and $\wp_I$ is non-membership function of the element $x \in Y$, then I is known as IFS.

Definition 5 ([12]) Consider a non-empty set X. Suppose $(X, \mu_d, \nu_d, *, \circ)$ is a five-tuple, where $*, \circ$ is a continuous t-norm and t-co-norm, Y is a closed cone and μ_d, ν_d are fuzzy set on $X^2 \times int(Y)$ where $int(Y)$ denotes interior of the set Y. Then $(X, \mu_d, \nu_d, *, \circ)$ is known as intuitionistic fuzzy cone metric space (in short, IFCMS) if $\forall\, \alpha, \xi, z \in X$ and $s, t \in int(Y)$ following conditions are holds

(i) $\mu_d(\alpha, \xi, s) + \nu_d(\alpha, \xi, s) \le 1$,
(ii) $\mu_d(\alpha, \xi, s) > 0$,
(iii) $\mu_d(\alpha, \xi, s) = 1$ iff $\alpha = \xi$,
(iv) $\mu_d(\alpha, \xi, s) = \mu_d(\xi, \alpha, s)$,
(v) $\mu_d(\alpha, z, s+t) \ge (\mu_d(\alpha, \xi, \frac{s}{b})) * (\mu_d(\xi, z, \frac{t}{b}))$,
(vi) $\mu_d(\alpha, \xi, \cdot) : int(Y) \to [0, 1]$ is continuous,
(vii) $\nu_d(\alpha, \xi, s) < 1$,
(viii) $\nu_d(\alpha, \xi, s) = 0$ iff $\alpha = \xi$,
(ix) $\nu_d(\alpha, \xi, s) = \nu_d(\xi, \alpha, s)$,
(x) $\nu_d(\alpha, z, s+t) \le (\nu_d(\alpha, \xi, \frac{s}{b})) * (\nu_d(\xi, z, \frac{t}{b}))$,
(xi) $\nu_d(\alpha, \xi, \cdot) : int(Y) \to [0, 1]$ is continuous.

Definition 6 ([13]) Suppose $(X, \mu_d, \nu_d, *, \circ)$ is a IFCMS. Then the pair (μ_d, ν_d) is said to be triangular, if $\forall\, \alpha, \xi, z \in X$ and $s, t \in int(Y)$,
$\left(\frac{1}{\mu_d(\upsilon,\xi,s)} - 1\right) \le \left(\frac{1}{\mu_d(\upsilon,z,s)} - 1\right) + \left(\frac{1}{\mu_d(z,\xi,s)} - 1\right)$,
and $\nu_d(\upsilon, \xi, s) \le \nu_d(\upsilon, z, s) + \nu_d(z, \xi, s)$.

Definition 7 ([1]) Consider a pair of self-mapping (S, f) of a IFCMS $(X, \mu_d, \nu_d, *, \circ)$. Then (S, f) is said to be compatible, if for some sequence (x_i) in X, $\lim_{i\to\infty} \mu_d (fS(x_i), Sf(x_i), s) = 1$ and $\lim_{i\to\infty} \nu_d(fS(x_i), Sf(x_i), s) = 0$, for $s \in int(y)$ such that $\lim_{i\to\infty} f(x_i) = \lim_{i\to\infty} S(x_i) = x$, for some $x \in X$.

Definition 8 ([1]) Suppose S and f are two self-maps of a set X. Then a point $\upsilon \in X$ is said to be a coincidence point of S and f if we have $\upsilon = S(\upsilon) = f(\upsilon)$. And S and f are known as weakly compatible if they commutes at their coincidence point, i.e., for some $\upsilon \in X$ $S(\upsilon) = f(\upsilon)$ we have $Sf(\upsilon) = fS(\upsilon)$.

Proposition 1 ([1]) *Suppose that S and f are two weakly compatible self-maps of a set X. If S and f have a unique point of coincidence $\upsilon = S(\upsilon) = f(\upsilon)$, then υ is the unique common fixed point of S and f.*

Next we categorize the main result of the chapter.

3 Some Common Fixed Point Theorems in IFCMS

In this section we present a common fixed point theorem in IFCMS for compatible and weakly compatible mappings. We also deduce some consequences of this main result.

Theorem 1 *Suppose $(X, \mu_d, \nu_d, *, \circ)$ is a complete IFCMS. Consider four self-mappings $S, T, f, g : X \to X$ having the properties that $S(X) \subseteq g(X)$, $T(X) \subseteq f(X)$ and f is continuous, (f, S) is compatible and (g, T) is weakly compatible. And $\forall\, x, y \in X$, $s \in int(P)$ μ_d, ν_d satisfies the following conditions:*

$$\frac{1}{\mu_d(S(x),T(y),s)}-1 \le \alpha_1(\frac{1}{\mu_d(f(x),g(y),s)}-1)+\alpha_2(\frac{1}{\mu_d(f(x),S(x),s)}-1)+$$
$$\alpha_3(\frac{1}{\mu_d(g(y),T(y),s)}-1)+\alpha_4(\frac{1}{\mu_d(f(x),T(y),s)}-1)+$$
$$\alpha_5(\frac{1}{\mu_d(g(y),S(x),s)}-1)$$

and

$$\begin{aligned}\nu_d(S(x),T(y),s) \le \alpha_1(\nu_d(f(x),g(y),s))+\alpha_2(\nu_d(f(x),S(x),s))\\ +\alpha_3(\nu_d(g(y),T(y),s))+\alpha_4(\nu_d(f(x),T(y),s))\\ +\alpha_5(\nu_d(g(y),S(x),s)) \qquad (1)\end{aligned}$$

where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5 \in [0,\infty)$ *with* $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5<1$ *and* $\alpha_2=\alpha_3$ *or* $\alpha_4=\alpha_5$.

Then S, T, f *and* g *have a unique common fixed point in* X.

Proof First we consider the given condition $S(X)\subseteq g(X)$, $T(X)\subseteq f(X)$.

For some fixed $x_0\in X$, considering a sequence $(x_i)\in X$ such that $\forall\, i\ge 0$

$$y_{2i+1}=g(x_{2i+1})=S(x_{2i}) \text{ and } y_{2i+2}=f(x_{2i+2})=T(x_{2i+1})$$

Now for $s\in int(P)$,

$$\begin{aligned}
&\frac{1}{\mu_d(g(x_{2i+1}),f(x_{2i+2}),s)}-1=\frac{1}{\mu_d(S(x_{2i}),T(x_{2i+1}),s)}-1\\
&\le \alpha_1\left(\frac{1}{\mu_d(f(x_{2i}),g(x_{2i+1}),s)}-1\right)+\alpha_2\left(\frac{1}{\mu_d(f(x_{2i}),S(x_{2i}),s)}-1\right)+\\
&\alpha_3\left(\frac{1}{\mu_d(g(x_{2i+1}),T(x_{2i+1}),s)}-1\right)+\alpha_4\left(\frac{1}{\mu_d(f(x_{2i}),T(x_{2i+1}),s)}-1\right)\\
&\quad+\alpha_5\left(\frac{1}{\mu_d(g(x_{2i+1}),S(x_{2i}),s)}-1\right)\\
&\le \alpha_1\left(\frac{1}{\mu_d(f(x_{2i}),g(x_{2i+1}),s)}-1\right)+\alpha_2\left(\frac{1}{\mu_d(f(x_{2i}),g(x_{2i+1}),s)}-1\right)+\\
&\alpha_3\left(\frac{1}{\mu_d(g(x_{2i+1}),f(x_{2i+2}),s)}-1\right)+\alpha_4\left(\frac{1}{\mu_d(f(x_{2i}),f(x_{2i+2}),s)}-1\right)\\
&\quad+\alpha_5\left(\frac{1}{\mu_d(g(x_{2i+1}),g(x_{2i+1}),s)}-1\right)\\
&\le \alpha_1\left(\frac{1}{\mu_d(f(x_{2i}),g(x_{2i+1}),s)}-1\right)+\alpha_2\left(\frac{1}{\mu_d(f(x_{2i}),g(x_{2i+1}),s)}-1\right)+\\
&\alpha_3\left(\frac{1}{\mu_d(g(x_{2i+1}),f(x_{2i+2}),s)}-1\right)\\
&\quad+\alpha_4\left(\frac{1}{\mu_d(f(x_{2i}),g(x_{2i+1}),s)}-1+\frac{1}{\mu_d(g(x_{2i+1}),f(x_{2i+2}),s)}-1\right)\\
&=(\alpha_1+\alpha_2+\alpha_4)\left(\frac{1}{\mu_d(f(x_{2i}),g(x_{2i+1}),s)}-1\right)\\
&\quad+(\alpha_3+\alpha_4)\left(\frac{1}{\mu_d(g(x_{2i+1}),f(x_{2i+2}),s)}-1\right)
\end{aligned}$$

and

$$
\begin{aligned}
\nu_d(g(x_{2i+1}), f(x_{2i+2}), s) = \nu_d(S(x_{2i}), T(x_{2i+1}), s) \\
&\le \alpha_1 \nu_d(f(x_{2i}), g(x_{2i+1}), s) + \alpha_2 \nu_d(f(x_{2i}), S(x_{2i}), s) + \\
&\quad \alpha_3 \nu_d(g(x_{2i+1}), T(x_{2i+1}), s) + \alpha_4 \nu_d(f(x_{2i}), T(x_{2i+1}), s) + \\
&\quad \alpha_5 \nu_d(g(x_{2i+1}), S(x_{2i}), s) \\
&\le \alpha_1 \nu_d(f(x_{2i}), g(x_{2i+1}), s) + \alpha_2 \nu_d(f(x_{2i}), g(x_{2i+1}), s) + \\
&\quad \alpha_3 \nu_d(g(x_{2i+1}), f(x_{2i+2}), s) + \alpha_4 \nu_d(f(x_{2i}), f(x_{2i+2}), s) + \\
&\quad \alpha_5 \nu_d(g(x_{2i+1}), g(x_{2i+1}), s) \\
&\le \alpha_1 \nu_d(f(x_{2i}), g(x_{2i+1}), s) + \alpha_2 \nu_d(f(x_{2i}), g(x_{2i+1}), s) + \\
&\quad \alpha_3 \nu_d(g(x_{2i+1}), f(x_{2i+2}), s) + \\
&\quad \alpha_4 (\nu_d(f(x_{2i}), g(x_{2i+1}), s) + \nu_d(g(x_{2i+1}), f(x_{2i+2}), s)) \\
&= (\alpha_1 + \alpha_2 + \alpha_4) \nu_d(f(x_{2i}), g(x_{2i+1}), s) + (\alpha_3 + \\
&\quad \alpha_4) \nu_d(g(x_{2i+1}), f(x_{2i+2}), s).
\end{aligned}
$$

Therefore we have,

$$
\frac{1}{\mu_d(g(x_{2i+1}), f(x_{2i+2}), s)} - 1 \le \alpha \left(\frac{1}{\mu_d(f(x_{2i}), g(x_{2i+1}), s)} - 1 \right) \text{ and}
$$
$$
\nu_d(g(x_{2i+1}), f(x_{2i+2}), s) \le \alpha(\nu_d(f(x_{2i}), g(x_{2i+1}), s)), \text{ where } \alpha = \frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - (\alpha_3 + \alpha_4)}. \tag{2}
$$

Again we have

$$
\begin{aligned}
&\frac{1}{\mu_d(f(x_{2i+2}), g(x_{2i+3}), s)} - 1 = \frac{1}{\mu_d(T(x_{2i+1}), S(x_{2i+2}), s)} - 1 \\
&\le \alpha_1 \left(\frac{1}{\mu_d(f(x_{2i+1}), g(x_{2i+2}), s)} - 1 \right) + \alpha_2 \left(\frac{1}{\mu_d(f(x_{2i+2}), S(x_{2i+2}), s)} - 1 \right) \\
&\quad + \alpha_3 \left(\frac{1}{\mu_d(g(x_{2i+1}), T(x_{2i+1}), s)} - 1 \right) + \alpha_4 \left(\frac{1}{\mu_d(f(x_{2i+2}), T(x_{2i+1}), s)} - 1 \right) \\
&\quad + \alpha_5 \left(\frac{1}{\mu_d(g(x_{2i+1}), S(x_{2i+2}), s)} - 1 \right) \\
&\le \alpha_1 \left(\frac{1}{\mu_d(f(x_{2i+1}), g(x_{2i+2}), s)} - 1 \right) + \alpha_2 \left(\frac{1}{\mu_d(f(x_{2i+2}), g(x_{2i+3}), s)} - 1 \right) \\
&\quad + \alpha_3 \left(\frac{1}{\mu_d(g(x_{2i+1}), f(x_{2i+2}), s)} - 1 \right) + \alpha_4 \left(\frac{1}{\mu_d(f(x_{2i+2}), f(x_{2i+2}), s)} - 1 \right) \\
&\quad + \alpha_5 \left(\frac{1}{\mu_d(g(x_{2i+1}), g(x_{2i+3}), s)} - 1 \right)
\end{aligned}
$$

$$\leq \alpha_1\left(\frac{1}{\mu_d(f(x_{2i+1}), g(x_{2i+2}), s)} - 1\right) + \alpha_2\left(\frac{1}{\mu_d(f(x_{2i+2}), g(x_{2i+3}), s)} - 1\right)$$
$$+ \alpha_3\left(\frac{1}{\mu_d(g(x_{2i+1}), f(x_{2i+2}), s)} - 1\right)$$
$$+ \alpha_5\left(\frac{1}{\mu_d(g(x_{2i+1}), f(x_{2i+2}), s)} - 1 + \frac{1}{\mu_d(f(x_{2i+2}), g(x_{2i+3}), s)} - 1\right)$$
$$= (\alpha_1 + \alpha_3 + \alpha_5)\left(\frac{1}{\mu_d(g(x_{2i+1}), f(x_{2i+2}), s)} - 1\right)$$
$$+ (\alpha_2 + \alpha_5)\left(\frac{1}{\mu_d(f(x_{2i+2}), g(x_{2i+3}), s)} - 1\right)$$

and

$$\begin{aligned}
\nu_d(f(x_{2i+2}), g(x_{2i+3}), s) &= \nu_d(T(x_{2i+1}), S(x_{2i+2}), s)\\
&\leq \alpha_1\nu_d(f(x_{2i+1}), g(x_{2i+2}), s) + \alpha_2\nu_d(f(x_{2i+2}), S(x_{2i+2}), s) +\\
&\quad \alpha_3\nu_d(g(x_{2i+1}), T(x_{2i+1}), s) + \alpha_4\nu_d(f(x_{2i+2}), T(x_{2i+1}), s) +\\
&\quad \alpha_5\nu_d(g(x_{2i+1}), S(x_{2i+2}), s)\\
&\leq \alpha_1\nu_d(f(x_{2i+1}), g(x_{2i+2}), s) + \alpha_2\nu_d(f(x_{2i+2}), g(x_{2i+3}), s) +\\
&\quad \alpha_3\nu_d(g(x_{2i+1}), f(x_{2i+2}), s) + \alpha_4\nu_d(f(x_{2i+2}), f(x_{2i+2}), s) +\\
&\quad \alpha_5\nu_d(g(x_{2i+1}), g(x_{2i+3}), s)\\
&\leq \alpha_1\nu_d(f(x_{2i+1}), g(x_{2i+2}), s) + \alpha_2\nu_d(f(x_{2i+2}), g(x_{2i+3}), s) +\\
&\quad \alpha_3\nu_d(g(x_{2i+1}), f(x_{2i+2}), s) +\\
&\quad \alpha_5(\nu_d(g(x_{2i+1}), f(x_{2i+2}), s) + \nu_d(f(x_{2i+2}), g(x_{2i+3}), s))\\
&= (\alpha_1 + \alpha_3 + \alpha_5)(\nu_d(g(x_{2i+1}), f(x_{2i+2}), s))\\
&\quad + (\alpha_2 + \alpha_5)(\nu_d(f(x_{2i+2}), g(x_{2i+3}), s)).
\end{aligned}$$

Hence we have

$$\frac{1}{\mu_d(f(x_{2i+2}), g(x_{2i+3}), s)} - 1 \leq \beta\left(\frac{1}{\mu_d(g(x_{2i+1}), f(x_{2i+2}), s)} - 1\right) \text{ and}$$
$$\nu_d(f(x_{2i+2}), g(x_{2i+3}), s) \leq \beta(\nu_d(g(x_{2i+1}), f(x_{2i+2}), s)), \text{ where } \beta = \frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - (\alpha_2 + \alpha_5)}. \tag{3}$$

Now form Eqs. 2 and 3, we have

$$\begin{aligned}
\frac{1}{\mu_d(g(x_{2i+1}), f(x_{2i+2}), s)} - 1 &\leq \alpha\left(\frac{1}{\mu_d(f(x_{2i}), g(x_{2i+1}), s)} - 1\right)\\
&\leq \alpha\beta\left(\frac{1}{\mu_d(g(x_{2i-1}), f(x_{2i}), s)} - 1\right)
\end{aligned}$$

$$\le \alpha\beta\alpha\left(\frac{1}{\mu_d(f(x_{2i-2}),g(x_{2i-1}),s)}-1\right)$$
$$\le \ldots \le \alpha(\beta\alpha)^i\left(\frac{1}{\mu_d(f(x_0),g(x_1),s)}-1\right) \quad (4)$$

and

$$\begin{aligned}\nu_d(g(x_{2i+1}),f(x_{2i+2}),s) &\le \alpha(\nu_d(f(x_{2i}),g(x_{2i+1}),s)),\\ &\le \alpha\beta(\nu_d(g(x_{2i-1}),f(x_{2i}),s))\\ &\le \alpha\beta\alpha(\nu_d(f(x_{2i-2}),g(x_{2i-1}),s))\\ &\le \ldots \le \alpha(\beta\alpha)^i(\nu_d(f(x_0),g(x_1),s)) \quad (5)\end{aligned}$$

And

$$\begin{aligned}\frac{1}{\mu_d(f(x_{2i+2}),g(x_{2i+3}),s)}-1 &\le \beta\left(\frac{1}{\mu_d(g(x_{2i+1}),f(x_{2i+2}),s)}-1\right)\\ &\le \beta\alpha\left(\frac{1}{\mu_d(f(x_{2i}),g(x_{2i+1}),s)}-1\right)\\ &\le \beta\alpha\beta\left(\frac{1}{\mu_d(g(x_{2i-1}),f(x_{2i}),s)}-1\right)\\ &\le \ldots \le (\beta\alpha)^{i+1}(\frac{1}{\mu_d(f(x_0),g(x_1),s)}-1) \quad (6)\end{aligned}$$

and

$$\begin{aligned}\nu_d(f(x_{2i+2}),g(x_{2i+3}),s) &\le \beta(\nu_d(g(x_{2i+1}),f(x_{2i+2}),s)),\\ &\le \beta\alpha(\nu_d(f(x_{2i}),g(x_{2i+1}),s))\\ &\le \beta\alpha\beta(\nu_d(g(x_{2i-1}),f(x_{2i}),s))\\ &\le \ldots \le (\beta\alpha)^{i+1}(\nu_d(f(x_0),g(x_1),s)) \quad (7)\end{aligned}$$

Now we consider the condition $a_2 = a_3$

$$\begin{aligned}\alpha\beta &= \frac{\alpha_1+\alpha_2+\alpha_4}{1-(\alpha_3+\alpha_4)}\cdot\frac{\alpha_1+\alpha_3+\alpha_5}{1-(\alpha_2+\alpha_5)}\\ &= \frac{\alpha_1+\alpha_2+\alpha_4}{1-(\alpha_2+\alpha_4)}\cdot\frac{\alpha_1+\alpha_3+\alpha_5}{1-(\alpha_3+\alpha_5)} < 1\cdot 1 = 1 \quad (8)\end{aligned}$$

and $a_4 = a_5$ we have

$$\alpha\beta = \frac{\alpha_1+\alpha_2+\alpha_4}{1-(\alpha_3+\alpha_4)}\cdot\frac{\alpha_1+\alpha_3+\alpha_5}{1-(\alpha_2+\alpha_5)}$$

$$= \frac{\alpha_1 + \alpha_2 + \alpha_4}{1 - (\alpha_3 + \alpha_5)} \cdot \frac{\alpha_1 + \alpha_3 + \alpha_5}{1 - (\alpha_2 + \alpha_4)} < 1 \cdot 1 = 1. \tag{9}$$

As μ_d and ν_d are triangular, therefore for $j > i \geq i_0$ we have

$$\begin{aligned}
\frac{1}{\mu_d(y_{2i+1}, y_{2j+1}), s)} - 1 &\leq \left(\frac{1}{\mu_d(y_{2i+1}, y_{2j+2}, s)} - 1\right) + \ldots + \left(\frac{1}{\mu_d(y_{2m}, y_{2m+1}, s)} - 1\right) \\
&\leq (\alpha \sum_{k=i}^{j-1} (\alpha\beta)^k + \sum_{k=i+1}^{j} (\alpha\beta)^k) \left(\frac{1}{\mu_d(y_0, y_1, s)} - 1\right) \\
&\leq (\frac{\alpha(\alpha\beta)^i}{1-\alpha\beta} + \frac{(\alpha\beta)^{i+1}}{1-\alpha\beta}) \left(\frac{1}{\mu_d(y_0, y_1), s)} - 1\right) \\
&= (1+\beta)\frac{\alpha(\alpha\beta)^i}{1-\alpha\beta} \left(\frac{1}{\mu_d(y_0, y_1), s)} - 1\right)
\end{aligned}$$

and

$$\begin{aligned}
\nu_d(y_{2i+1}, y_{2j+1}), s) &\leq (\nu_d(y_{2i+1}, y_{2j+2}, s)) + \ldots + (\nu_d(y_{2i+1}, y_{2j+1}, s)) \\
&\leq (\alpha \sum_{k=i}^{j-1} (\alpha\beta)^k + \sum_{k=i+1}^{j} (\alpha\beta)^k) \nu_d(y_0, y_1, s) \\
&\leq (\frac{\alpha(\alpha\beta)^i}{1-\alpha\beta} + \frac{(\alpha\beta)^{i+1}}{1-\alpha\beta}) \nu_d(y_0, y_1, s) \\
&= (1+\beta)\frac{\alpha(\alpha\beta)^i}{1-\alpha\beta} \nu_d(y_0, y_1, s).
\end{aligned}$$

Continuing in this way, we have

$$\frac{1}{\mu_d(y_{2i}, y_{2j+1}), s)} - 1 \leq (1+\alpha)\frac{(\alpha\beta)^i}{1-\alpha\beta} \left(\frac{1}{\mu_d(y_0, y_1), s)} - 1\right)$$
$$\text{and } \nu_d(y_{2i}, y_{2j+1}), s) \leq (1+\alpha)\frac{(\alpha\beta)^i}{1-\alpha\beta} \nu_d(y_0, y_1, s)$$

$$\frac{1}{\mu_d(y_{2i}, y_{2j}), s)} - 1 \leq (1+\alpha)\frac{(\alpha\beta)^i}{1-\alpha\beta} \left(\frac{1}{\mu_d(y_0, y_1), s)} - 1\right)$$
$$\text{and } \nu_d(y_{2i}, y_{2j}), s) \leq (1+\alpha)\frac{(\alpha\beta)^i}{1-\alpha\beta} \nu_d(y_0, y_1, s)$$

$$\frac{1}{\mu_d(y_{2i+1}, y_{2j}), s)} - 1 \leq (1+\beta)\frac{\alpha(\alpha\beta)^i}{1-\alpha\beta} \left(\frac{1}{\mu_d(y_0, y_1), s)} - 1\right)$$

$$\text{and } \nu_d(y_{2i+1}, y_{2j}), s) \leq (1+\alpha)\frac{\alpha(\alpha\beta)^i}{1-\alpha\beta}\nu_d(y_0, y_1, s).$$

For $j > i$ we have

$$\frac{1}{\mu_d(y_{2i+1}, y_{2j+1}), s)} - 1 \leq \max\left\{(1+\alpha)\frac{(\alpha\beta)^i}{1-\alpha\beta}, (1+\beta)\frac{\alpha(\alpha\beta)^i}{1-\alpha\beta}\right\}\left(\frac{1}{\mu_d(y_0, y_1), s)} - 1\right)$$
$$\longrightarrow 0, \text{ as } i \to \infty.$$

and

$$\nu_d(y_{2i+1}, y_{2j+1}), s) \leq \max\left\{(1+\alpha)\frac{(\alpha\beta)^i}{1-\alpha\beta}, (1+\beta)\frac{\alpha(\alpha\beta)^i}{1-\alpha\beta}\right\}\nu_d(y_0, y_1, s)$$
$$\longrightarrow 0, \text{ as } i \to \infty.$$

this implies that (y_i) for $i \geq 0$ is a Cauchy sequence.

By the completeness properties of X, $\exists\, v \in X$ such that $y_i \to v$ as $i \to \infty$, then we obtain

$$g(x_{2i+1}) \to v,\ f(x_{2i+2}) \to v,\ S(x_{2i}) \to v \text{ and } T(x_{2i+1}) \to v. \tag{10}$$

As f is a continuous self-mapping on X and satisfies

$$f(g(x_{2i+1})) \to f(v),\ f(f(x_{2i+2})) \to f(v),\ f(S(x_{2i})) \to f(v) \text{ and}$$
$$f(T(x_{2i+1})) \to f(v).$$

Therefore $f(S(x_{2i})) \to f(v)$ and (S, f) is compatible. Hence we have

$$\lim_{i\to\infty} \mu_d(S(f(x_{2i})), f(S(x_{2i})), s) = \lim_{i\to\infty} \mu_d(S(f(x_{2i})), f(v), s) = 1,$$
$$\lim_{i\to\infty} \mu_d(f(S(x_{2i})), f(v), s) = 1, \text{ for } s \in int(p). \tag{11}$$

and

$$\lim_{i\to\infty} \nu_d(S(f(x_{2i})), f(S(x_{2i})), s) = \lim_{i\to\infty} \nu_d(S(f(x_{2i})), f(v), s) = 0,$$
$$\lim_{i\to\infty} \mu_d(f(S(x_{2i})), f(v), s) = 0, \text{ for } s \in int(p). \tag{12}$$

Next we have to show that $f(v) = v$.

From the Definition 5,

$\mu_d(f(v), v, 2s) \geq \mu_d(f(v), S(f(x_{2i})), s) * \mu_d(S(f(x_{2i})), v, s)$

and $\nu_d(f(v), v, 2s) \leq \nu_d(f(v), S(f(x_{2i})), s) \circ \mu_d(S(f(x_{2i})), v, s)$.

Since (S, f) is compatible therefore,

$$\mu_d(f(v), v, 2s) \geq \lim_{i\to\infty} (\mu_d(f(v), S(f(x_{2i})), s) * \mu_d(S(f(x_{2i})), v, s)) = 1 * 1 = 1$$

and

$$\nu_d(f(v), v, 2s) \leq \lim_{i\to\infty} (\nu_d(f(v), S(f(x_{2i})), s) \circ \nu_d(S(f(x_{2i})), v, s)) = 0 \circ 0 = 0.$$

Hence
$\mu_d(f(v), v, 2s) = 1$ and $\nu_d(f(v), v, 2s) = 0$ which implies $f(v) = v$.
Next we have to show that $S(v) = v$.
Again from the Definition 5,
$\mu_d(S(v), v, 2s) \geq \mu_d(S(v), f(S(x_{2i})), s) * \mu_d(f(S(x_{2i})), v, s)$
and $\nu_d(S(v), v, 2s) \leq \nu_d(S(v), f(S(x_{2i})), s) \circ \mu_d(f(S(x_{2i})), v, s)$.
From the compatible property we have

$$\mu_d(S(v), v, 2s) \geq \lim_{i\to\infty} (\mu_d(S(v), f(S(x_{2i})), s) * \mu_d(f(S(x_{2i})), v, s) = 1 * 1 = 1$$

and

$$\nu_d(S(v), v, 2s) \leq \lim_{i\to\infty} (\nu_d(S(v), f(S(x_{2i})), s) \circ \nu_d(f(S(x_{2i})), v, s) = 0 \circ 0 = 0.$$

Then $\mu_d(S(v), v, 2s) = 1$ and $\nu_d(S(v), v, 2s) = 0$ which implies $S(v) = v$.
Hence $v = f(v) = S(v)$.
Next we have to show that $T(v) = g(v)$.
Since $S(X) \subseteq g(X)$ therefore $\exists\, u \in X$ such that $v = S(v) = g(u)$ we have

$$\begin{aligned}
&\frac{1}{\mu_d(T(u), g(u), s)} - 1 = \frac{1}{\mu_d(S(u), T(u), s)} - 1 \\
&\leq \alpha_1\left(\frac{1}{\mu_d(f(v), g(u), s)} - 1\right) + \alpha_2\left(\frac{1}{\mu_d(f(v), S(v), s)} - 1\right) + \\
&\quad \alpha_3\left(\frac{1}{\mu_d(g(u), T(u), s)} - 1\right) + \alpha_4\left(\frac{1}{\mu_d(f(v), T(u), s)} - 1\right) \\
&\quad + \alpha_5\left(\frac{1}{\mu_d(g(u), S(v)} - 1\right) \\
&= \alpha_1\left(\frac{1}{\mu_d(f(v), v, s)} - 1\right) + \alpha_2\left(\frac{1}{\mu_d(v, f(v), s)} - 1\right) + \\
&\quad \alpha_3\left(\frac{1}{\mu_d(g(u), T(u), s)} - 1\right) + \alpha_4\left(\frac{1}{\mu_d(g(u), T(u), s)} - 1\right) \\
&\quad + \alpha_5\left(\frac{1}{\mu_d(g(u), g(u), s)} - 1\right) \\
&= (\alpha_3 + \alpha_4)\left(\frac{1}{\mu_d(g(u), T(u), s)} - 1\right)
\end{aligned}$$

and

$$\nu_d(T(u), g(u), s) = \nu_d(S(u), T(u), s)$$

$$\leq \alpha_1 \nu_d(f(v), g(u), s) + \alpha_2 \nu_d(f(v), S(v), s)+$$
$$\alpha_3 \nu_d(g(u), T(u), s) + \alpha_4 \nu_d(f(v), T(u), s)$$
$$+ \alpha_5 \nu_d(g(u), S(v)$$
$$= \alpha_1 \nu_d(f(v), v, s) + \alpha_2 \nu_d(v, f(v), s)+$$
$$\alpha_3 \nu_d(g(u), T(u), s) + \alpha_4 \nu_d(g(u), T(u), s)$$
$$+ \alpha_5 \mu_d(g(u), g(u), s)$$
$$= (\alpha_3 + \alpha_4)\nu_d(g(u), T(u), s).$$

Since $(\alpha_3 + \alpha_4) < 1$ therefore
$\mu_d(g(u), T(u), s) = 1$ and $\nu_d(g(u), T(u), s) = 0$.

This implies that $T(u) = g(u) = v$ and from the weak compatibility of T and g we have

$$g(v) = g(T(u)) = T(g(u)) = T(v).$$

Finally we have to show that $T(v) = v$.

From the Definition 5,

$$\frac{1}{\mu_d(T(v), v, s)} - 1 = \frac{1}{\mu_d(T(v), S(v), s)} - 1$$
$$\leq \alpha_1 \left(\frac{1}{\mu_d(f(v), g(v), s)} - 1\right) + \alpha_2 \left(\frac{1}{\mu_d(f(v), S(v), s)} - 1\right)+$$
$$\alpha_3 \left(\frac{1}{\mu_d(g(v), T(v), s)} - 1\right) + \alpha_4 \left(\frac{1}{\mu_d(f(v), T(v), s)} - 1\right)$$
$$+ \alpha_5 \left(\frac{1}{\mu_d(g(v), S(v), s)} - 1\right)$$
$$= \alpha_1 \left(\frac{1}{\mu_d(v, T(v), s)} - 1\right) + \alpha_2 \left(\frac{1}{\mu_d(f(v), f(v), s)} - 1\right)+$$
$$\alpha_3 \left(\frac{1}{\mu_d(g(v), g(v), s)} - 1\right) + \alpha_4 \left(\frac{1}{\mu_d(v, T(v), s)} - 1\right)$$
$$+ \alpha_5 \left(\frac{1}{\mu_d(T(v), v, s)} - 1\right)$$
$$= (\alpha_1 + \alpha_4 + \alpha_5) \left(\frac{1}{\mu_d(v, T(v), s)} - 1\right)$$

and

$$\nu_d(T(v), v, s) = \nu_d(T(v), S(v), s))$$
$$\leq \alpha_1 \nu_d(f(v), g(v), s) + \alpha_2 \nu_d(f(v), S(v), s)+$$
$$\alpha_3 \nu_d(g(v), T(v), s) + \alpha_4 \nu_d(f(v), T(v), s)$$

$$
\begin{aligned}
&+ \alpha_5 \nu_d(g(v), S(v), s)\\
&= \alpha_1 \nu_d(v, T(v), s) + \alpha_2 \nu_d(f(v), f(v), s) +\\
&\alpha_3 \nu_d(g(v), g(v), s) + \alpha_4 \nu_d(v, T(v), s)\\
&+ \alpha_5 \mu_d(T(v), v, s)\\
&= (\alpha_1 + \alpha_4 + \alpha_5) \nu_d(v, T(v), s).
\end{aligned}
$$

Since $(\alpha_1 + \alpha_4 + \alpha_5) < 1$ therefore $\mu_d(v, T(v), s) = 1$ and $\nu_d(v, T(v), s) = 0$.

This implies $v = T(v)$ implies $g(v) = v$.

Hence, $f(v) = g(v) = S(v) = T(v) = v$, this implies v is a common fixed point of the four self-mappings f, g, S and T in X.

Next we have to show the uniqueness of v.

Suppose $v^{'} \in X$ is another point satisfies $f(v^{'}) = g(v^{'}) = S(v^{'}) = T(v^{'}) = v^{'}$ then from Eq. 1

$$
\begin{aligned}
\frac{1}{\mu_d(v^{'}, v, s)} - 1 &= \frac{1}{\mu_d(S(v^{'}), T(v), s)} - 1\\
&\leq \alpha_1 \left(\frac{1}{\mu_d(f(v^{'}), g(v), s)} - 1 \right) + \alpha_2 \left(\frac{1}{\mu_d(f(v^{'}), S(v^{'}), s)} - 1 \right)\\
&+ \alpha_3 \left(\frac{1}{\mu_d(g(v), T(v), s)} - 1 \right) + \alpha_4 \left(\frac{1}{\mu_d(f(v^{'}), T(v), s)} - 1 \right)\\
&+ \alpha_5 \left(\frac{1}{\mu_d(g(v), S(v^{'}), s)} - 1 \right)\\
&= (\alpha_1 + \alpha_4 + \alpha_5) \left(\frac{1}{\mu_d(v^{'}, v, s)} - 1 \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\nu_d(v^{'}, v, s) &= \nu_d(S(v^{'}), T(v), s))\\
&\leq \alpha_1 \nu_d(v^{'}), g(v), s) + \alpha_2 \nu_d(f(v^{'}), S(v^{'}), s) +\\
&\alpha_3 \nu_d(g(v), T(v), s) + \alpha_4 \nu_d(f(v^{'}), T(v), s)\\
&+ \alpha_5 \nu_d(g(v), S(v), s)\\
&= \alpha_1 \nu_d(v, T(v), s) + \alpha_2 \nu_d(f(v), f(v), s) +\\
&\alpha_3 \nu_d(g(v), g(v), s) + \alpha_4 \nu_d(v, T(v), s)\\
&+ \alpha_5 \mu_d(g(v), S(v^{'}), s)\\
&= (\alpha_1 + \alpha_4 + \alpha_5) \nu_d(v^{'}, v, s).
\end{aligned}
$$

Since $(\alpha_1 + \alpha_4 + \alpha_5) < 1$ therefore $\mu_d(v^{'}, v, s) = 1$ and $\nu_d(v^{'}, v, s) = 0$. this implies that $v^{'} = v$.

Hence the common fixed point of f, g, S and T in X is unique.

Next we deduct some corollary from the above theorem.

Corollary 1 *Consider a complete IFCMS* $(X, \mu_d, \nu_d, *, \circ)$ *where* μ_d, ν_d *are triangular. Consider four self-mappings* $S, T, f, g : X \to X$ *having the properties that* $S(X) \subseteq g(X)$, $T(X) \subseteq f(X)$ *and* f *is continuous,* (f, S) *is compatible and* (g, T) *is weakly compatible. And* $\forall\, x, y \in X$ *satisfying the condition:*

$$\frac{1}{\mu_d(S(x), T(y), s)} - 1 \le \alpha_1 \left(\frac{1}{\mu_d(f(x), g(y), s)} - 1 \right) + \alpha_2 \left(\frac{1}{\mu_d(f(x), S(x), s)} - 1 \right) + \alpha_3 \left(\frac{1}{\mu_d(g(y), T(y), s)} - 1 \right) \tag{13}$$

and

$$\nu_d(S(x), T(y), s) \le \alpha_1 \nu_d(f(x), g(y), s) + \alpha_2 \nu_d(f(x), S(x), s) + \alpha_3 \nu_d(g(y), T(y), s) \tag{14}$$

where $a_1, a_2, a_3 \in [0, \infty)$ *with* $a_1 + a_2 + a_3 < 1$.

Then S, T, f *and* g *have a unique common fixed point in* X.

Corollary 2 *Consider a complete IFCMS* $(X, \mu_d, \nu_d, *, \circ)$ *where* μ_d, ν_d *are triangular. Consider four self-mappings* $S, T, f, g : X \to X$ *having the properties that* $S(X) \subseteq g(X)$, $T(X) \subseteq f(X)$ *and* f *is continuous,* (f, S) *is compatible and* (g, T) *is weakly compatible. And* $\forall\, x, y \in X$ *satisfying the condition:*

$$\frac{1}{\mu_d(S(x), T(y), s)} - 1 \le \alpha_1 \left(\frac{1}{\mu_d(f(x), g(y), s)} - 1 \right) + \alpha_2 \left(\frac{1}{\mu_d(f(x), T(y), s)} - 1 \right) + \alpha_3 \left(\frac{1}{\mu_d(g(y), S(x), s)} - 1 \right) \tag{15}$$

and

$$\nu_d(S(x), T(y), s) \le \alpha_1 \nu_d(f(x), g(y), s) + \alpha_2 \nu_d(f(x), T(y), s) + \alpha_3 \nu_d(g(y), S(x), s) \tag{16}$$

where $a_1, a_2, a_3 \in [0, \infty)$ *with* $a_1 + a_2 + a_3 < 1$.

Then S, T, f *and* g *have a unique common fixed point in* X.

Corollary 3 *Consider a complete IFCMS* $(X, \mu_d, \nu_d, *, \circ)$ *where* μ_d, ν_d *are triangular. Consider four self-mappings* $S, T, f, g : X \to X$ *having the properties that* $S(X) \subseteq g(X)$, $T(X) \subseteq f(X)$ *and* f *is continuous,* (f, S) *is compatible and* (g, T) *is weakly compatible. And* $\forall\, x, y \in X$ *satisfying the condition:*

$$\frac{1}{\mu_d(S(x), T(y), s)} - 1 \leq \alpha \left(\frac{1}{\mu_d(f(x), g(y), s)} - 1 \right) \textit{ and}$$
$$\nu_d(S(x), T(y), s) \leq \alpha \nu_d(f(x), g(y), s) \tag{17}$$

where $a \in [0, \infty)$.

Then S, T, f *and* g *have a unique common fixed point in* X.

Corollary 4 *Consider a complete IFCMS* $(X, \mu_d, \nu_d, *, \circ)$ *where* μ_d, ν_d *are triangular. Consider two self-mappings* $T, g : X \to X$ *having the properties that* $T(X) \subseteq f(X)$ *and* f *is continuous,* (T, f) *is weakly compatible. And* $\forall\, x, y \in X$ *satisfying the condition:*

$$\frac{1}{\mu_d(T(x), T(y), s)} - 1 \leq \alpha_1 \left(\frac{1}{\mu_d(f(x), f(y), s)} - 1 \right) + \alpha_2 \left(\frac{1}{\mu_d(f(x), T(x), s)} - 1 \right) + \alpha_3 \left(\frac{1}{\mu_d(f(y), T(y), s)} - 1 \right) + \alpha_4 \left(\frac{1}{\mu_d(f(x), T(y), s)} - 1 \right) + \alpha_5 \left(\frac{1}{\mu_d(f(y), T(x), s)} - 1 \right) \tag{18}$$

and

$$\nu_d(T(x), T(y), s) \leq \alpha_1 \nu_d(f(x), f(y), s) + \alpha_2 \nu_d(f(x), T(x), s) + \alpha_3 \nu_d(f(y), T(y), s) + \alpha_4 \nu_d(f(x), T(y), s) + \alpha_5 \nu_d(f(y), T(x), s) \tag{19}$$

where $a_1, a_2, a_3, a_4, a_5 \in [0, \infty)$ *with* $a_1 + a_2 + a_3 + a_4 + a_5 < 1$*, and* $a_2 = a_3$ *or* $a_4 = a_5$.

Then T *and* f *have a unique common fixed point in* X.

Example 1 Consider a IFCMS $(X, \mu_d, \nu_d, *, \circ)$ where $X = [0, 1]$ and $\mu_d, \nu_d : X^2 \times (0, \infty) \to [0, 1]$ are triangular and $\forall\, x, y \in X$ and $s > 0$ μ_d, ν_d defined as:

$$\mu_d(x, y, s) = \frac{s}{s + |x - y|} \text{ and } \nu_d(x, y, s) = \frac{|x - y|}{s + |x - y|}.$$

Then $(X, \mu_d, \nu_d, *, \circ)$ is a complete IFCMS.

Next we consider four self-mapping $S, T, f, g : X \to X$ defined as

$$S(x) = T(x) = \begin{cases} \frac{1}{2}\left(\frac{2x}{3} + \frac{1}{4}\right), & x \neq 0 \\ 0, & x = 0, \end{cases}$$

and

$$g(x) = f(x) = \begin{cases} \frac{2x}{3} + \frac{1}{4}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

As $S(x) = T(X)$ and $g(X) = f(X)$ we have $S(x) \subseteq g(X)$ or $T(x) \subseteq f(X)$. Then from the Eq. 1, we have

$$\begin{aligned}\frac{1}{\mu_d(S(x), T(y), s) - 1} &= \frac{|S(x) - T(y)|}{s} = \frac{|x - y|}{3s}\\ &\leq \alpha_1\left(\frac{1}{\mu_d(f(x), g(y), s)} - 1\right) + \alpha_2\left(\frac{1}{\mu_d(f(x), S(x), s)} - 1\right)\\ &+ \alpha_3\left(\frac{1}{\mu_d(g(y), T(y), s)} - 1\right) + \alpha_4\left(\frac{1}{\mu_d(f(x), T(y), s)} - 1\right)\\ &+ \alpha_5\left(\frac{1}{\mu_d(g(y), S(x), s)} - 1\right)\end{aligned}$$

and

$$\begin{aligned}\nu_d(S(x), T(y), s) &= \frac{|S(x) - T(y)|}{s + |S(x) - T(y)|} = \frac{|x - y|}{3s + |x - y|}\\ &\leq \alpha_1\nu_d(f(x), g(y), s) + \alpha_2\nu_d(f(x), S(x), s)\\ &+ \alpha_3\nu_d(g(y), T(y), s) + \alpha_4\nu_d(f(x), T(y), s)\\ &+ \alpha_5\nu_d(g(y), S(x), s).\end{aligned}$$

Then we can verify that all the condition of Eq. 1 is satisfied with $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \alpha_3 = \frac{1}{6}$ and $\alpha_4 = \alpha_5 = 0$.

Hence 0 is the unique common fixed point of S, T, f and g in X.

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Modular Spaces and Fixed Points of Generalized Contractions

Tayebe Laal Shateri and Ozgur Ege

Abstract In the present paper, some common fixed point theorems for a couple of self-maps in modular spaces are proved. We have found sufficient conditions for the existence and uniqueness of common fixed points for a couple of self-maps in some classes of modular spaces, where the modular is satisfying the Δ_2-condition. In fact, we generalize the kind of nonlinear contraction for self-maps that is the result in [4].

1 Introduction

The modular spaces were investigated by Nakano [15] and then generalized by some researchers [10, 13, 24]. The detailed information on Orlicz spaces can be found in [9]. The references [11, 14] contain more reviews on Orlicz and modular spaces.

Fixed point theorems are used to show the existence of solution concept in such different fields such as engineering, medicine, statistics, chemistry, and economics. Banach contraction principle is an essential tool in fixed point theory, which has been used and extended in many different directions. In modular spaces, various fixed point theorems have been studied by many researchers, see [1–3, 5–7, 16, 17, 21, 22].

In addition, Razani et al. [18] proved some fixed point theorems of asymptotic and nonlinear contractions in modular spaces. Also, Khamsi [8] introduced quasi-contraction mappings without Δ_2-condition in modular spaces. For the existence results on asymptotic pointwise contractions in modular spaces, see [12]. Cyclic (α, β)-admissible mappings in modular spaces were investigated in [19, 20].

Definition 1 Let $\mathcal{X}$ be a vector space over $\mathbb{F}$ $(= \mathbb{C}$ or $\mathbb{R})$. A functional $\sigma : \mathcal{X} \to [0, \infty]$ is said to be modular if for all $x, y \in \mathcal{X}$,

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(i) $\sigma(x) = 0 \Leftrightarrow x = 0$,
(ii) $\sigma(\alpha x) = \sigma(x)$ for every $\alpha \in \mathbb{F}$ with $|\alpha| = 1$,
(iii) $\sigma(\alpha x + \beta y) \leq \sigma(x) + \sigma(y)$ if $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

Definition 2 If (iii) in Definition 1 is replaced by

$$\sigma(\alpha x + \beta y) \leq \alpha^s \sigma(x) + \beta^s \sigma(y),$$

for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ with an $s \in (0, 1]$, then σ is called an s-convex modular, and if $s = 1$, σ is said to be a convex modular.

A modular σ defines a corresponding modular space, i.e., the vector space $\mathcal{X}_\sigma$ stated as

$$\mathcal{X}_\sigma = \{x \in \mathcal{X} : \sigma(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

σ is said to satisfy the Δ_2-type condition if there exists $\kappa > 0$ such that $\sigma(2x) \leq \kappa\sigma(x)$ for all $x \in \mathcal{X}_\sigma$.

Definition 3 Let $\mathcal{X}_\sigma$ be a modular space and let $\{x_n\}$ and x be in $\mathcal{X}_\sigma$. Then
(i) $\{x_n\}$ is said to be σ-convergent to x and write $x_n \xrightarrow{\sigma} x$ if $\sigma(x_n - x) \to 0$ as $n \to \infty$.
(ii) We say that $\{x_n\}$ is σ-Cauchy if $\sigma(x_n - x_m) \to 0$ as $n, m \to \infty$.
(iii) A subset $\mathcal{B}$ of $\mathcal{X}_\sigma$ is said to be σ-complete if any σ-Cauchy sequence is σ-convergent to an element of $\mathcal{B}$.
(iv) A subset S of $\mathcal{X}_\sigma$ is called σ-closed if for any sequence $\{x_n\} \subseteq S$ with $x_n \xrightarrow{\sigma} x$, we have $x \in S$.
(v) A subset S of $\mathcal{X}_\sigma$ is called σ-bounded if

$$\delta_\sigma(S) = \sup_{x,y \in S} \sigma(x - y) < \infty,$$

where $\delta_\sigma(S)$ is the σ-diameter of S.

Remark 1 The function $\sigma(x)$ is increasing for any $x \in \mathcal{X}$. If we assume $0 < a < b$, then the condition (iii) in Definition 1 with $y = 0$ indicates that

$$\sigma(ax) = \sigma\left(\frac{a}{b} bx\right) \leq \sigma(bx)$$

for all $x \in \mathcal{X}$. Moreover, if σ is a convex modular on $\mathcal{X}$ and $|\alpha| \leq 1$, then $\sigma(\alpha x) \leq \alpha\sigma(x)$ and $\sigma(x) \leq \frac{1}{2}\sigma(2x)$ for all $x \in \mathcal{X}$.

Example 1 Let ξ be a nondecreasing, continuous, and convex function defined on $[0, \infty)$ such that $\xi(0) = 0$, $\xi(\alpha) > 0$ for $\alpha > 0$ and $\xi(\alpha) \to \infty$ as $\alpha \to \infty$. The function ξ is called an Orlicz function. ξ satisfies the Δ_2-condition if there exists $\kappa > 0$ such that $\xi(2\alpha) \leq \kappa\xi(\alpha)$ for all $\alpha > 0$. Let $(\Gamma, \mathfrak{M}, \mu)$ be a measure space.

Assume that $L^0(\mu)$ is the space of all measurable real- or complex-valued functions on Γ. Define for every $f \in L^0(\mu)$ the Orlicz modular $\sigma_\varphi(f)$ as

$$\sigma_\xi(f) = \int_\Gamma \xi(|f|)d\mu.$$

An Orlicz space is the associated modular function space with regard to this modular and it will be denoted by $L^\xi(\Gamma, \mu)$ or L^ξ which can be alternatively stated as

$$L^\xi = \{f \in L^0(\mu) : \ \sigma_\xi(\lambda f) \to 0 \text{ as } \lambda \to 0\}$$

or

$$L^\xi = \{f \in L^0(\mu) : \ \sigma_\xi(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

The Orlicz space L^ξ is σ_ξ-complete and $(L^\xi, \|.\|_{\sigma_\xi})$ is a Banach space with

$$\|f\|_{\sigma_\xi} = \inf\left\{\lambda > 0 : \ \int_\Gamma \xi\left(\frac{|f|}{\lambda}\right) d\mu \le 1\right\}.$$

2 Main Results

Throughout this study, it will be assumed that the modular σ satisfies the Δ_2-type condition with $\kappa \ge 1$. Also we assume that Φ is the family of all increasing and upper semicontinuous functions $\varphi : \mathbb{R}^+ \to [0, \infty)$ satisfying

$$\varphi(t) < t \quad (t > 0) \ \ and \ \ \varphi(0) = 0. \tag{1}$$

In [23], it is proved that if $t > 0$, then $\varphi(t) < t$ if and only if $\lim_{n\to\infty} \varphi^n(t) = 0$.

In this section, by using some techniques in [4], we obtain some common fixed point theorems for self-maps in modular spaces as follows.

Theorem 1 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space. Suppose that $\varphi : \mathbb{R}^+ \to [0, \infty)$ satisfies in (1). Let Φ be a σ-closed subset of $\mathcal{X}_\sigma$ and $H, K : \Phi \to \Phi$ be mappings such that there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$, and*

$$\sigma\Big(\alpha(Hx - Ky)\Big) \le \varphi\Big[\max\Big\{\sigma\big(\beta(x - Hx)\big), \sigma\big(\beta(y - Ky)\big)\Big\}\Big], \tag{2}$$

for all $x, y \in \Phi$. Then H and K have a unique common fixed point in Φ.

Proof First we prove that any fixed point of H is also a fixed point of K, and conversely. Suppose $Hx = x$, but $Kx \neq x$, then (2) implies that

$$\sigma\Big(\alpha(x-Kx)\Big) \le \varphi\Big(\sigma(\beta(x-Kx))\Big) < \varphi\Big(\sigma(\alpha(x-Kx))\Big), \tag{3}$$

which contradicts with (1), so $Kx = x$. Similarly, if $Kx = x$, then $Hx = x$.

Now, we prove that if H and K have a common fixed point, then the fixed point is unique. Let $Hx = Kx = x$ and $Hy = Ky = y$. If $x \neq y$, then (2) implies that

$$\sigma\Big(\beta(x-y)\Big) < \sigma\Big(\alpha(x-y)\Big) = \sigma\Big(\alpha(Hx-Ky)\Big) \le \varphi\Big(\sigma(\beta(x-y))\Big) \tag{4}$$

which is a contradiction. Hence $x = y$.

Let $x_0 \in \Phi$ and put $x_{2n+1} = Hx_{2n},\ x_{2n+2} = Kx_{2n+1}$ for all $n = 0, 1, 2, \ldots$. We may assume that for any n, $x_{n+1} \neq x_n$, otherwise H or K has a fixed point and the proof is complete. Now, we have

$$\begin{aligned}
\sigma\Big(\alpha(x_{2n+1}-x_{2n})\Big) &= \sigma\Big(\alpha(Hx_{2n}-Kx_{2n-1})\Big)\\
&\le \varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n}-Hx_{2n})\big), \sigma\big(\beta(x_{2n-1}-Kx_{2n-1})\big)\}\Big],\\
&= \varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n}-x_{2n+1})\big), \sigma\big(\beta(x_{2n-1}-x_{2n})\big)\Big\}\Big],\\
&= \varphi\Big(\sigma\Big(\beta(x_{2n}-x_{2n-1})\Big)\Big),
\end{aligned} \tag{5}$$

otherwise from (5) we get

$$\begin{aligned}
\sigma\Big(\alpha(x_{2n+1}-x_{2n})\Big) &\le \varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n}-x_{2n-1})\big), \sigma\big(\beta(x_{2n}-x_{2n+1})\big)\Big\}\Big]\\
&= \varphi\Big(\sigma\Big(\beta(x_{2n}-x_{2n+1})\Big)\Big) < \sigma\Big(\alpha(x_{2n+1}-x_{2n})\Big),
\end{aligned} \tag{6}$$

and this is impossible. Therefore, from (5) and (6), we have

$$\begin{aligned}
\sigma\Big(\alpha(x_{2n+1}-x_{2n})\Big) &= \sigma\Big(\alpha(Hx_{2n}-Kx_{2n-1})\Big) \le \varphi\Big(\sigma\big(\beta(x_{2n}-x_{2n-1})\big)\Big)\\
&< \sigma\big(\beta(x_{2n}-x_{2n-1})\big) < \sigma\Big(\alpha(x_{2n}-x_{2n-1})\Big).
\end{aligned} \tag{7}$$

Similarly,

$$\begin{aligned}
\sigma\Big(\alpha(x_{2n+2}-x_{2n+1})\Big) &= \sigma\Big(\alpha(Kx_{2n+1}-Hx_{2n})\Big) \le \varphi\Big(\sigma\big(\beta(x_{2n+1}-x_{2n})\big)\Big)\\
&< \sigma\Big(\beta(x_{2n+1}-x_{2n})\Big) < \sigma\Big(\alpha(x_{2n+1}-x_{2n})\Big).
\end{aligned} \tag{8}$$

By (7) and (8), therefore, we have

$$\sigma\Big(\alpha(x_{n+1}-x_n)\Big) \le \varphi\Big(\sigma\big(\beta(x_n-x_{n-1})\big)\Big) < \sigma\Big(\beta(x_n-x_{n-1})\Big)\ (n \ge 1). \tag{9}$$

Consequently, $\{\sigma\Big(\alpha(x_{n+1}-x_n)\Big)\}$ is decreasing and bounded from below. Hence $\{\sigma\Big(\alpha(x_{n+1}-x_n)\Big)\}$ converges to z. Now, if $z\neq 0$,

$$\begin{aligned} z=\lim_{n\to\infty}\sigma\Big(\alpha(x_{n+1}-x_n)\Big) &\le \lim_{n\to\infty}\varphi\Big(\sigma\Big(\beta(x_n-x_{n-1})\Big)\Big)\\ &< \lim_{n\to\infty}\varphi\Big(\sigma\Big(\alpha(x_n-x_{n-1})\Big)\Big), \end{aligned}$$

then $z\le\varphi(z)$, which is a contradiction, hence $z=0$.

Now, we show that $\{x_n\}$ is a σ-Cauchy sequence in $\mathcal{X}_\sigma$. If $\{\beta x_n\}$ is not a σ-Cauchy sequence, then there exists $\varepsilon>0$ and sequences $\{m_k\}$, $\{n_k\}$ of integers with $m_k>n_k\ge k$ such that

$$\sigma\Big(\beta(x_{m_k}-x_{n_k})\Big)\ge\varepsilon\quad(k\in\mathbb{N}). \tag{10}$$

Moreover, corresponding to odd numbers n_k, we can choose even numbers m_k in such a way that it is the smallest integer with $m_k>n_k$ such that

$$\sigma\Big(\beta(x_{m_{k-2}}-x_{n_k})\Big)<\varepsilon. \tag{11}$$

In fact, let m_k be the smallest even number exceeding n_k for which (10) holds, and

$$N_k=\Big\{m\in\mathbb{N}_e \mid \exists\, n_k\in\mathbb{N}_o;\ \sigma\Big(\beta(x_m-x_{n_k})\Big)\ge\varepsilon,\ m>n_k\ge k\Big\}.$$

It is clear that $N_k\neq\emptyset$ and by well-ordering principle, the minimum element of N_k exists and is denoted by m_k, and clearly (11) holds.

Now, let $\alpha_0\in\mathbb{R}^+$ be such that $\frac{\beta}{\alpha}+\frac{1}{\alpha_0}=1$. Also assume that r is the smallest integer number such that $\alpha_0<2^r$, then from (11) and Δ_2-type condition we have

$$\begin{aligned} \sigma\Big(\beta(x_{m_k}-x_{n_k})\Big) &= \sigma\Big(\frac{\beta}{\alpha}\Big(\alpha(x_{m_k}-x_{n_{k+2}})\Big)+\frac{1}{\alpha_0}\Big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\Big)\Big)\\ &\le \sigma\Big(\alpha(x_{m_k}-x_{n_{k+2}})\Big)+\sigma\Big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\Big)\\ &\le \varphi\Big(\sigma\Big(\beta(x_{m_{k-1}}-x_{n_{k+1}})\Big)\Big)+\sigma\Big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\Big)\\ &< \varepsilon+\kappa^r\sigma\Big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\Big). \end{aligned}$$

Since $\lim_{k\to\infty}\sigma\Big(\beta(x_{n_{k+2}}-x_{n_k})\Big)=0$, hence $\lim_{k\to\infty}\sigma\Big(\beta(x_{m_k}-x_{n_k})\Big)=\varepsilon$. Therefore,

$$\begin{aligned}\sigma\Big[\beta(x_{m_k}-x_{n_k})\Big] =&\sigma\Big(\frac{\beta}{\alpha}\Big(\alpha(x_{m_{k+1}}-x_{n_{k+1}})\Big)\\ &+\frac{1}{2\alpha_0}\Big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}}+x_{n_{k+1}}-x_{n_k})\Big)\Big]\\ \leq&\sigma\Big(\alpha(x_{m_{k+1}}-x_{n_{k+1}})\Big)+\sigma\Big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}})\Big)\\ &+\sigma\Big(2\alpha_0\beta(x_{n_{k+1}}-x_{n_k})\Big)\\ \leq&\varphi\Big(\sigma\Big(\beta(x_{m_k}-x_{n_k})\Big)\Big)+\sigma\Big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}})\Big)\\ &+\sigma\Big(2\alpha_0\beta(x_{n_{k+1}}-x_{n_k})\Big).\end{aligned}$$

Therefore, as $k\to\infty$, we get $\varepsilon\leq\varphi(\varepsilon)$, which is a contradiction. Hence $\{\beta x_n\}$ is a σ-Cauchy sequence, and by Δ_2-type condition, $\{x_n\}$ is a σ-Cauchy sequence. Since $\mathcal{X}_\sigma$ is complete, there is a $w\in\Phi$ such that $\sigma(x_n-w)\to 0$, as $n\to\infty$. Now, it will be shown that w is the common fixed point of H and K. Put $x=x_{2n}$ and $y=w$ in (12), we have

$$\begin{aligned}\sigma\Big(\alpha(w-Kw)\Big)&=\lim_{n\to\infty}\sigma\Big(\alpha(x_{2n+1}-Kw)\Big)=\sigma\Big(\alpha(Hx_{2n}-Kw)\Big)\\ &\leq\varphi\Big[\max\{\sigma(\beta(x_{2n}-w)),\sigma(\beta(x_{2n}-Hx_{2n})),\sigma(\beta(w-Kw))\}\Big]\\ &=\varphi\Big[\max\{\sigma(\beta(x_{2n}-x_{2n+1})),\sigma(\beta(w-Kw))\}\Big]\\ &\to\varphi\Big(\sigma(\beta(w-Kw))\Big)<\varphi\Big(\sigma(\alpha(w-Kw))\Big)\end{aligned}$$

therefore $\sigma\Big(\alpha(w-Kw)\Big)=0$, and so $w=Kw$ which is the required result.

Putting $H=K$ in Theorem 1, we have the following.

Corollary 1 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space. Suppose that $\varphi:\mathbb{R}^+\to[0,\infty)$ satisfies (1). Let Φ be a σ-closed subset of $\mathcal{X}_\sigma$ and let $H:\Phi\to\Phi$ be a mapping such that there exist $\alpha,\beta\in\mathbb{R}^+$ with $\alpha>\beta$, and*

$$\sigma\Big(\alpha(Hx-Hy)\Big)\leq\varphi\Big[\max\Big\{\sigma(\beta(x-Hx)),\sigma(\beta(y-Hy))\Big\}\Big],$$

for all $x,y\in\Phi$. Then H has a unique fixed point in Φ.

Setting $\varphi(t)=\eta t$ for $\eta\in(0,1)$ in Theorem 1, we obtain the next result.

Corollary 2 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space. Let Φ be a σ-closed subset of $\mathcal{X}_\sigma$ and let $H,K:\Phi\to\Phi$ be mappings such that there exist $\alpha,\beta,\eta\in\mathbb{R}^+$ with $\alpha>\beta$ and $\eta\in(0,1)$, and*

$$\sigma\Big(\alpha(Hx-Ky)\Big)\leq\ \eta\Big[\max\{\sigma(\beta(x-Hx)),\sigma(\beta(y-Ky))\}\Big]$$

for all $x, y \in \Phi$. *Then* H *and* K *have a unique common fixed point in* Φ.

Also if we set $\varphi(t) = \eta t$ for $\eta \in (0, 1)$ and $H = K$ in Theorem 1, we get the following.

Corollary 3 *Let* $\mathcal{X}_\sigma$ *be a* σ*-complete modular space. Let* Φ *be a* σ*-closed subset of* $\mathcal{X}_\sigma$ *and let* $H : \Phi \to \Phi$ *be a mapping such that there exist* $\alpha, \beta, \eta \in \mathbb{R}^+$ *with* $\alpha > \beta$ *and* $\eta \in (0, 1)$*, and*

$$\sigma\Big(\alpha(Hx - Hy)\Big) \leq \eta\Big[\max\{\sigma(\beta(x - Hx)), \sigma(\beta(y - Hy))\}\Big]$$

for all $x, y \in \Phi$. *Then* H *and* K *have a unique fixed point in* Φ.

The following example gives a modular space $\mathcal{X}$ and two self-maps on $\mathcal{X}$, which satisfied the requirements of Theorem 1.

Example 2 Let $\mathcal{X}_\sigma = [0.1, \frac{1}{4}]$, and $\sigma(x) = |x|$ for all $x \in \mathcal{X}_\sigma$. Define H and K on $\mathcal{X}_\sigma$ as $Hx = \frac{\sqrt{x}}{2}$, $Kx = \frac{1}{4}$. Suppose $\varphi : \mathbb{R} \to [0, \infty)$ defined as $\varphi(t) = \frac{t}{2}$. Then the hypothesis (12) is satisfied with H and K for $\alpha = 4$ and $\beta = 3$. In fact

$$\sigma\Big(\alpha(Hx - Ky)\Big) = |2\sqrt{x} - 1|$$

and

$$\varphi\Big[\max\{\sigma(\beta(x - Hx)), \sigma(\beta(y - Ky))\}\Big] = \frac{3}{2}\max\left\{\left|x - \frac{\sqrt{x}}{2}\right|, \left|y - \frac{1}{4}\right|\right\}.$$

Since $x, y \in [0.1, \frac{1}{4}]$, hence we have

$$|2\sqrt{x} - 1| \leq \frac{3}{2}|x - \frac{\sqrt{x}}{2}|,$$

and so

$$\sigma\Big(\alpha(Hx - Ky)\Big) \leq \varphi\Big[\max\{\sigma(\beta(x - Hx)), \sigma(\beta(y - Ky))\}\Big].$$

Therefore, Theorem 1 implies that H and K have a unique common fixed point, that is, $\frac{1}{4}$.

Theorem 2 *Let* $\mathcal{X}_\sigma$ *be a* σ*-complete modular space. Suppose that* $\varphi : \mathbb{R}^+ \to [0, \infty)$ *satisfies in (1). Let* Φ *be a* σ*-closed subset of* $\mathcal{X}_\sigma$ *and let* $H, K : \Phi \to \Phi$ *be mappings such that there exist* $\alpha, \beta \in \mathbb{R}^+$ *with* $\alpha > \beta$*, and*

$$\sigma\Big(\alpha(Hx - Ky)\Big) \leq \varphi\Big[\max\Big\{\sigma\big(\beta(x - y)\big), \sigma\big(\beta(x - Hx)\big), \sigma\big(\beta(y - Ky)\big)\Big\}\Big], \tag{12}$$

for all $x, y \in \Phi$. *Then* H *and* K *have a unique common fixed point in* Φ.

Proof As the proof of Theorem 1, we can prove that any fixed point of H is also a fixed point of S, and, conversely, also H and K have a unique common fixed point.

Let $x_0 \in \Phi$ and put $x_{2n+1} = Hx_{2n},\ x_{2n+2} = Kx_{2n+1}$ for all $n = 0, 1, 2, \ldots$. We may assume that for any n, $x_{n+1} \neq x_n$, otherwise H or K has a fixed point and the proof is complete. Now, we have

$$\begin{aligned}\sigma\Big(\alpha(x_{2n+1} - x_{2n})\Big) =&\sigma\Big(\alpha(Hx_{2n} - Kx_{2n-1})\Big)\\ \leq&\varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n} - x_{2n-1})\big), \sigma\big(\beta(x_{2n} - Hx_{2n})\big), \qquad (13)\\ &\sigma\big(\beta(x_{2n-1} - Kx_{2n-1})\big)\}\Big],\\ =&\varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n} - x_{2n-1})\big), \sigma\big(\beta(x_{2n} - x_{2n+1})\big), \qquad (14)\\ &\sigma\big(\beta(x_{2n-1} - x_{2n})\big)\Big\}\Big],\\ =&\varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n} - x_{2n-1})\big), \sigma\big(\beta(x_{2n} - x_{2n+1})\big)\Big\}\Big] \qquad (15)\\ =&\varphi\Big(\sigma\Big(\beta(x_{2n} - x_{2n-1})\Big)\Big), \qquad (16)\end{aligned}$$

otherwise from (13) we get

$$\begin{aligned}\sigma\Big(\alpha(x_{2n+1} - x_{2n})\Big) &\leq \varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n} - x_{2n-1})\big), \sigma\big(\beta(x_{2n} - x_{2n+1}\big)\Big\}\Big]\\ &= \varphi\Big(\sigma\Big(\beta(x_{2n} - x_{2n+1})\Big)\Big) < \sigma\Big(\alpha(x_{2n+1} - x_{2n})\Big), \qquad (17)\end{aligned}$$

and this is impossible. Therefore, from (13) and (17), we have

$$\begin{aligned}\sigma\Big(\alpha(x_{2n+1} - x_{2n})\Big) &= \sigma\Big(\alpha(Hx_{2n} - Kx_{2n-1})\Big) \leq \varphi\Big(\sigma\Big(\beta(x_{2n} - x_{2n-1})\Big)\Big)\\ &< \sigma\Big(\beta(x_{2n} - x_{2n-1})\Big) < \sigma\Big(\alpha(x_{2n} - x_{2n-1})\Big). \qquad (18)\end{aligned}$$

Similarly

$$\begin{aligned}\sigma\Big(\alpha(x_{2n+2} - x_{2n+1})\Big) &= \sigma\Big(\alpha(Kx_{2n+1} - Hx_{2n})\Big) \leq \varphi\Big(\sigma\Big(\beta(x_{2n+1} - x_{2n})\Big)\Big)\\ &< \sigma\Big(\beta(x_{2n+1} - x_{2n})\Big) < \sigma\Big(\alpha(x_{2n+1} - x_{2n})\Big). \qquad (19)\end{aligned}$$

By (18) and (19), therefore, we have

$$\sigma\Big(\alpha(x_{n+1} - x_n)\Big) \leq \varphi\Big(\sigma\Big(\beta(x_n - x_{n-1})\Big)\Big) < \sigma\Big(\beta(x_n - x_{n-1})\Big)\ (n \geq 1). \qquad (20)$$

Consequently, $\{\sigma\Big(\alpha(x_{n+1}-x_n)\Big)\}$ is decreasing and bounded from below. Hence $\{\sigma\Big(\alpha(x_{n+1}-x_n)\Big)\}$ converges to z. Now, if $z \neq 0$,

$$
\begin{aligned}
z = \lim_{n\to\infty} \sigma\Big(\alpha(x_{n+1}-x_n)\Big) &\leq \lim_{n\to\infty} \varphi\Big(\sigma\Big(\beta(x_n - x_{n-1})\Big)\Big) \\
&< \lim_{n\to\infty} \varphi\Big(\sigma\Big(\alpha(x_n - x_{n-1})\Big)\Big),
\end{aligned}
$$

then $z \leq \varphi(z)$, which is a contradiction, hence $z = 0$.

Now, we show that $\{x_n\}$ is a σ-Cauchy sequence in $\mathcal{X}_\sigma$. If $\{\beta x_n\}$ is not a σ-Cauchy sequence, then there exists $\varepsilon > 0$ and sequences $\{m_k\}$, $\{n_k\}$ of integers with $m_k > n_k \geq k$ such that

$$
\sigma\Big(\beta(x_{m_k} - x_{n_k})\Big) \geq \varepsilon \quad (k \in \mathbb{N}). \tag{21}
$$

Moreover, corresponding to odd numbers n_k, we can choose even numbers m_k in such a way that it is the smallest integer with $m_k > n_k$ such that

$$
\sigma\Big(\beta(x_{m_{k-2}} - x_{n_k})\Big) < \varepsilon. \tag{22}
$$

In fact, let m_k be the smallest even number exceeding n_k for which (21) holds, and

$$
N_k = \Big\{m \in \mathbb{N}_e \mid \exists\, n_k \in \mathbb{N}_o;\ \sigma\Big(\beta(x_m - x_{n_k})\Big) \geq \varepsilon,\ m > n_k \geq k\Big\}.
$$

It is clear that $N_k \neq \emptyset$ and by well-ordering principle, the minimum element of N_k exists and is denoted by m_k, and (22) holds.

Now, let $\alpha_0 \in \mathbb{R}^+$ be such that $\frac{\beta}{\alpha} + \frac{1}{\alpha_0} = 1$. Also assume that r is the smallest integer number such that $\alpha_0 < 2^r$, then from (22) and Δ_2-type condition we have

$$
\begin{aligned}
\sigma\Big(\beta(x_{m_k} - x_{n_k})\Big) &= \sigma\Big(\frac{\beta}{\alpha}\Big(\alpha(x_{m_k} - x_{n_{k+2}})\Big) + \frac{1}{\alpha_0}\Big(\alpha_0\beta(x_{n_{k+2}} - x_{n_k})\Big)\Big) \\
&\leq \sigma\Big(\alpha(x_{m_k} - x_{n_{k+2}})\Big) + \sigma\Big(\alpha_0\beta(x_{n_{k+2}} - x_{n_k})\Big) \\
&\leq \varphi\Big(\sigma\Big(\beta(x_{m_{k-1}} - x_{n_{k+1}})\Big)\Big) + \sigma\Big(\alpha_0\beta(x_{n_{k+2}} - x_{n_k})\Big) \\
&< \varepsilon + \kappa^r \sigma\Big(\alpha_0\beta(x_{n_{k+2}} - x_{n_k})\Big).
\end{aligned}
$$

Since $\lim_{k\to\infty} \sigma\Big(\beta(x_{n_{k+2}} - x_{n_k})\Big) = 0$, hence $\lim_{k\to\infty} \sigma\Big(\beta(x_{m_k} - x_{n_k})\Big) = \varepsilon$. Therefore,

$$\begin{aligned}\sigma\Big[\beta(x_{m_k}-x_{n_k})\Big) &= \sigma\Big(\frac{\beta}{\alpha}\Big(\alpha(x_{m_{k+1}}-x_{n_{k+1}})\Big)\\ &\quad+\frac{1}{2\alpha_0}\Big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}}+x_{n_{k+1}}-x_{n_k})\Big)\Big]\\ &\le \sigma\Big(\alpha(x_{m_{k+1}}-x_{n_{k+1}})\Big)+\sigma\Big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}})\Big)\\ &\quad+\sigma\Big(2\alpha_0\beta(x_{n_{k+1}}-x_{n_k})\Big)\\ &\le \varphi\Big(\sigma\Big(\beta(x_{m_k}-x_{n_k})\Big)\Big)+\sigma\Big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}})\Big)\\ &\quad+\sigma\Big(2\alpha_0\beta(x_{n_{k+1}}-x_{n_k})\Big).\end{aligned}$$

Therefore, as $k \to \infty$, we get $\varepsilon \le \varphi(\varepsilon)$, which is a contradiction. Hence $\{\beta x_n\}$ is a σ-Cauchy sequence, and by Δ_2-type condition, $\{x_n\}$ is a σ-Cauchy sequence. Since $\mathcal{X}_\sigma$ is complete, there is a $w \in \Phi$ such that $\sigma(x_n - w) \to 0$, as $n \to \infty$. Now, we show that w is the common fixed point of H and K. Put $x = x_{2n}$ and $y = w$ in (12), we have

$$\begin{aligned}\sigma\Big(\alpha(w-Kw)\Big) &= \lim_{n\to\infty}\sigma\Big(\alpha(x_{2n+1}-Kw)\Big)=\sigma\Big(\alpha(Hx_{2n}-Kw)\Big)\\ &\le \varphi\Big[\max\{\sigma(\beta(x_{2n}-w)),\sigma(\beta(x_{2n}-Hx_{2n})),\sigma(\beta(w-Kw))\}\Big]\\ &= \varphi\Big[\max\{\sigma(\beta(x_{2n}-w)),\sigma(\beta(x_{2n}-x_{2n+1})),\sigma(\beta(w-Kw))\}\Big]\\ &\to \varphi\Big(\sigma(\beta(w-Kw))\Big)<\varphi\Big(\sigma(\alpha(w-Kw))\Big)\end{aligned}$$

therefore $\sigma\Big(\alpha(w-Kw)\Big)=0$, and so $w = Kw$. This completes the proof.

Putting $S = T$ in Theorem 2, we deduce the next result.

Corollary 4 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space. Suppose that $\varphi : \mathbb{R}^+ \to [0, \infty)$ satisfies (1). Let Φ be a σ-closed subset of $\mathcal{X}_\sigma$ and let $T : \Phi \to \Phi$ be a mapping such that there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$, and*

$$\sigma\Big(\alpha(Hx-Hy)\Big)\le\varphi\Big[\max\Big\{\sigma\big(\beta(x-y)\big),\sigma\big(\beta(x-Hx)\big),\big(\beta(y-Hy)\big)\Big\}\Big],$$

for all $x, y \in \Phi$. Then H has a unique fixed point in Φ.

If we set $\varphi(t) = \eta t$ for $\eta \in (0, 1)$ in Theorem 2, we deduce the following.

Corollary 5 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space. Let Φ be a σ-closed subset of $\mathcal{X}_\sigma$ and let $H, K : \Phi \to \Phi$ be mappings such that there exist $\alpha, \beta, \eta \in \mathbb{R}^+$ with $\alpha > \beta$ and $\eta \in (0, 1)$, and*

$$\sigma\Big(\alpha(Hx - Ky)\Big) \leq \eta\Big[\max\big\{\sigma\big(\beta(x-y)\big), \sigma\big(\beta(x - Hx)\big), \sigma\big(\beta(y - Ky)\big)\big\}\Big]$$

for all $x, y \in \Phi$. Then H and K have a unique common fixed point in Φ.

Also if we set $\varphi(t) = \eta t$ for $\eta \in (0, 1)$ and $H = K$ in Theorem 2, we have the other result.

Corollary 6 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space. Let Φ be a σ-closed subset of $\mathcal{X}_\sigma$ and let $H : \Phi \to \Phi$ be a mapping such that there exist $\alpha, \beta, \eta \in \mathbb{R}^+$ with $\alpha > \beta$ and $\eta \in (0, 1)$, and*

$$\sigma\Big(\alpha(Hx - Hy)\Big) \leq \eta\Big[\max\big\{\sigma\big(\beta(x-y)\big), \sigma\big(\beta(x - Hx)\big), \sigma\big(\beta(y - Hy)\big)\big\}\Big]$$

for all $x, y \in \Phi$. Then H and K have a unique fixed point in Φ.

In the following, we give an example for Theorem 2.

Example 3 Let $\mathcal{X}_\sigma = [0, \infty)$, and $\sigma(x) = |x|$ for all $x \in \mathcal{X}_\sigma$. Define T and S on $\mathcal{X}_\sigma$ as $Hx = \frac{\sqrt{x}}{4}$, $Kx = \frac{x}{4}$. Suppose $\varphi : \mathbb{R} \to [0, \infty)$ defined as $\varphi(t) = \frac{t}{2}$. Then the hypothesis (12) is satisfied with H and K for $\alpha = 4$ and $\beta = 3$. Therefore, Theorem 2 implies that H and K have a unique common fixed point.

Theorem 3 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space. Suppose that $\varphi : \mathbb{R}^+ \to [0, \infty)$ satisfies (1). Let Φ be a σ-closed subset of $\mathcal{X}_\sigma$ and let $H, K : \Phi \to \Phi$ be mappings such that there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$, and*

$$\begin{aligned}\sigma\Big(\alpha(Hx - Ky)\Big) \leq \varphi\Big[\max\Big\{&\sigma\big(\beta(x-y)\big), \sigma\big(\beta(x - Hx)\big), \big(\beta(y - Ky)\big), \\ &\frac{1}{2\kappa}\Big(\sigma\big(\beta(x - Hy)\big) + \sigma\big(\beta(y - Kx)\big)\Big)\Big\}\Big],\end{aligned} \tag{23}$$

for all $x, y \in \Phi$. Then H and K have a unique common fixed point in Φ.

Proof As the proof of Theorem 1, one can see that any fixed point of H is also a fixed point of K, and conversely. Also, the common fixed point is unique.

Suppose $x_0 \in \Phi$ and put $x_{2n+1} = Hx_{2n}$, $x_{2n+2} = Kx_{2n+1}$ for all $n = 0, 1, 2, \ldots$. We may assume that for any n, $x_{n+1} \neq x_n$, otherwise H or K has a fixed point and the proof is complete. Now, we have

$$
\begin{aligned}
\sigma\Big(\alpha(x_{2n+1}-x_{2n})\Big) &= \sigma\Big(\alpha(Hx_{2n}-Kx_{2n-1})\Big) \\
&\leq \varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n}-x_{2n-1})\big), \sigma\big(\beta(x_{2n}-Hx_{2n})\big), \\
&\quad \sigma\big(\beta(x_{2n-1}-Kx_{2n-1})\big), \frac{1}{2\kappa}\Big(\sigma\big(\beta(x_{2n}-Kx_{2n-1})\big) \\
&\quad + \sigma\big(\beta(x_{2n-1}-Hx_{2n})\big)\Big)\Big\}\Big], \\
&= \varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n}-x_{2n-1})\big), \sigma\big(\beta(x_{2n}-x_{2n+1})\big), \\
&\quad \sigma\big(\beta(x_{2n-1}-x_{2n})\big), \frac{1}{2\kappa}\Big(\sigma\big(\beta(x_{2n}-x_{2n})\big) \\
&\quad + \sigma\big(\beta(x_{2n-1}-x_{2n+1})\big)\Big)\Big\}\Big], \\
&= \varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n}-x_{2n-1})\big), \sigma\big(\beta(x_{2n}-x_{2n+1})\big), \\
&\quad \frac{1}{2\kappa}\sigma\big(\beta(x_{2n-1}-x_{2n+1})\big)\Big\}\Big]. \qquad (24)
\end{aligned}
$$

But

$$
\begin{aligned}
\sigma\Big(\beta(x_{2n-1}-x_{2n+1})\Big) &= \sigma\Big(\frac{1}{2}\Big[2\beta(x_{2n-1}-x_{2n})\Big] + \frac{1}{2}\Big[2\beta(x_{2n}-x_{2n+1})\Big]\Big) \\
&\leq \sigma\Big(2\beta(x_{2n-1}-x_{2n})\Big) + \sigma\Big(2\beta(x_{2n}-x_{2n+1})\Big) \\
&\leq k\Big[\sigma\Big(\beta(x_{2n-1}-x_{2n})\Big) + \sigma\Big(\beta(x_{2n}-x_{2n+1})\Big)\Big], \qquad (25)
\end{aligned}
$$

hence

$$
\begin{aligned}
&\varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n}-x_{2n-1})\big), \sigma\big(\beta(x_{2n}-x_{2n+1})\big), \frac{1}{2\kappa}\Big(\sigma(\beta(x_{2n}-x_{2n})) \\
&\qquad + \sigma\big(\beta(x_{2n-1}-x_{2n+1})\big)\Big)\Big\}\Big] = \varphi\Big(\sigma\Big(\beta(x_{2n}-x_{2n-1})\Big)\Big), \qquad (26)
\end{aligned}
$$

in otherwise from (24), we get

$$
\begin{aligned}
\sigma\Big(\alpha(x_{2n+1}-x_{2n})\Big) &\leq \varphi\Big[\max\Big\{\sigma\big(\beta(x_{2n}-x_{2n-1})\big), \sigma\big(\beta(x_{2n}-x_{2n+1})\big), \\
&\quad \frac{1}{2\kappa}\Big(\sigma\big(\beta(x_{2n}-x_{2n})\big) + \sigma\big(\beta(x_{2n-1}-x_{2n+1})\big)\Big)\Big\}\Big] \\
&= \varphi\Big(\sigma\Big(\beta(x_{2n}-x_{2n+1})\Big)\Big) < \sigma\Big(\alpha(x_{2n+1}-x_{2n})\Big),
\end{aligned}
$$

and this is impossible. Therefore, from (24) and (25) we have

$$\sigma\Big(\alpha(x_{2n+1}-x_{2n})\Big)=\sigma\Big(\alpha(Hx_{2n}-Kx_{2n-1})\Big)\le\varphi\Big(\sigma\Big(\beta(x_{2n}-x_{2n-1})\Big)\Big)$$
$$<\sigma\Big(\beta(x_{2n}-x_{2n-1})\Big)<\sigma\Big(\alpha(x_{2n}-x_{2n-1})\Big). \tag{27}$$

similarly

$$\sigma\Big(\alpha(x_{2n+2}-x_{2n+1})\Big)=\sigma\Big(\alpha(Kx_{2n+1}-Hx_{2n})\Big)\le\varphi\Big(\sigma\Big(\beta(x_{2n+1}-x_{2n})\Big)\Big)$$
$$<\sigma\Big(\beta(x_{2n+1}-x_{2n})\Big)<\sigma\Big(\alpha(x_{2n+1}-x_{2n})\Big). \tag{28}$$

By (27) and (28), therefore, we have

$$\sigma\Big(\alpha(x_{n+1}-x_n)\Big)\le\varphi\Big(\sigma\Big(\beta(x_n-x_{n-1})\Big)\Big)<\sigma\Big(\beta(x_n-x_{n-1})\Big)\ (n\ge 1). \tag{29}$$

Consequently, $\Big\{\sigma\Big(\alpha(x_{n+1}-x_n)\Big)\Big\}$ is decreasing and bounded from below. Hence $\Big\{\sigma\Big(\alpha(x_{n+1}-x_n)\Big)\Big\}$ converges to z. Now, if $z\neq 0$,

$$z=\lim_{n\to\infty}\sigma\Big(\alpha(x_{n+1}-x_n)\Big)\le\lim_{n\to\infty}\varphi\Big(\sigma\Big(\beta(x_n-x_{n-1})\Big)\Big)$$
$$<\lim_{n\to\infty}\varphi\Big(\sigma\Big(\alpha(x_n-x_{n-1})\Big)\Big),$$

then $z\le\varphi(z)$, which is a contradiction, hence $z=0$.

Now, we show that $\{x_n\}$ is a σ-Cauchy sequence in $\mathcal{X}_\sigma$. If $\{\beta x_n\}$ is not a σ-Cauchy sequence, then there exists $\varepsilon>0$ and sequences $\{m_k\}$, $\{n_k\}$ of integers with $m_k>n_k\ge k$ such that

$$\sigma\Big(\beta(x_{m_k}-x_{n_k})\Big)\ge\varepsilon\quad(k\in\mathbb{N}). \tag{30}$$

Moreover, corresponding to odd numbers n_k, we can choose even numbers m_k in such a way that it is the smallest integer with $m_k>n_k$ such that

$$\sigma\Big(\beta(x_{m_{k-2}}-x_{n_k})\Big)<\varepsilon. \tag{31}$$

In fact, let m_k be the smallest even number exceeding n_k for which (30) holds, and

$$N_k=\Big\{m\in\mathbb{N}_e:\ \exists\, n_k\in\mathbb{N}_o;\ \sigma\Big(\beta(x_m-x_{n_k})\Big)\ge\varepsilon,\ m>n_k\ge k\Big\}.$$

It is clear that $N_k\neq\emptyset$ and by well-ordering principle, the minimum element of N_k exists and is denoted by m_k, and (31) holds.

Now, let $\alpha_0\in\mathbb{R}^+$ be such that $\frac{\beta}{\alpha}+\frac{1}{\alpha_0}=1$. Also assume that r is the smallest integer number such that $\alpha_0<2^r$, then from (31) and Δ_2-type condition we have

$$\begin{aligned}\sigma\Big(\beta(x_{m_k}-x_{n_k})\Big) &= \sigma\Big(\frac{\beta}{\alpha}\Big(\alpha(x_{m_k}-x_{n_{k+2}})\Big)+\frac{1}{\alpha_0}\Big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\Big)\Big)\\ &\leq \sigma\Big(\alpha(x_{m_k}-x_{n_{k+2}})\Big)+\sigma\Big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\Big)\\ &\leq \varphi\Big(\sigma\Big(\beta(x_{m_{k-1}}-x_{n_{k+1}})\Big)\Big)+\sigma\Big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\Big)\\ &< \varepsilon+\kappa^r\sigma\Big(\alpha_0\beta(x_{n_{k+2}}-x_{n_k})\Big).\end{aligned}$$

Since $\lim_{k\to\infty}\sigma\Big(\beta(x_{n_{k+2}}-x_{n_k})\Big)=0$, hence $\lim_{k\to\infty}\sigma\Big(\beta(x_{m_k}-x_{n_k})\Big)=\varepsilon$. Therefore,

$$\begin{aligned}\sigma\Big[\beta(x_{m_k}-x_{n_k})\Big) =&\sigma\Big(\frac{\beta}{\alpha}\Big(\alpha(x_{m_{k+1}}-x_{n_{k+1}})\Big)\\ &+\frac{1}{2\alpha_0}\Big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}}+x_{n_{k+1}}-x_{n_k})\Big)\Big]\\ &\leq \sigma\Big(\alpha(x_{m_{k+1}}-x_{n_{k+1}})\Big)+\sigma\Big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}})\Big)\\ &+\sigma\Big(2\alpha_0\beta(x_{n_{k+1}}-x_{n_k})\Big)\\ &\leq \varphi\Big(\sigma\Big(\beta(x_{m_k}-x_{n_k})\Big)\Big)+\sigma\Big(2\alpha_0\beta(x_{m_k}-x_{m_{k+1}})\Big)\\ &+\sigma\Big(2\alpha_0\beta(x_{n_{k+1}}-x_{n_k})\Big).\end{aligned}$$

Therefore, as $k\to\infty$, we get $\varepsilon\leq\varphi(\varepsilon)$, which is a contradiction. Thus $\{\beta x_n\}$ is a σ-Cauchy sequence, and by Δ_2-type condition, $\{x_n\}$ is a σ-Cauchy sequence. Since $\mathcal{X}_\sigma$ is complete, there is a $w\in\Phi$ such that $\sigma(x_n-w)\to 0$, as $n\to\infty$. Now, we show that w is the common fixed point of H and K. Put $x=x_{2n}$ and $y=w$ in (23), we obtain

$$\begin{aligned}\sigma\Big(\alpha(w-Kw)\Big) &= \lim_{n\to\infty}\sigma\Big(\alpha(x_{2n+1}-Kw)\Big) = \sigma\Big(\alpha(Hx_{2n}-Kw)\Big)\\ &\leq\varphi\Big[\max\{\sigma(\beta(x_{2n}-w)),\sigma(\beta(x_{2n}-Hx_{2n})),\sigma(\beta(w-Kw)),\\ &\quad\frac{1}{2\kappa}[\sigma(\beta(x_{2n}-Kw))+\sigma(\beta(w-Hx_{2n}))]\}\Big]\\ &=\varphi\Big[\max\{\sigma(\beta(x_{2n}-w)),\sigma(\beta(x_{2n}-x_{2n+1})),\sigma(\beta(w-Kw)),\\ &\quad\frac{1}{2\kappa}[\sigma(\beta(x_{2n}-Kw))+\sigma(\beta(w-x_{2n+1}))]\}\Big]\\ &\to\varphi\Big(\sigma(\beta(w-Kw))\Big)<\varphi\Big(\sigma(\alpha(w-Kw))\Big)\end{aligned}$$

therefore $\sigma\Big(\alpha(w-Kw)\Big)=0$, and so $w=Kw$. This completes the proof.

On putting $K=H$ in Theorem 3 reduces to a result of [4].

Corollary 7 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space. Assume that $\varphi : \mathbb{R}^+ \to [0, \infty)$ satisfies in (1). Let Φ be a σ-closed subset of $\mathcal{X}_\sigma$ and let $H : \Phi \to \Phi$ be a mapping such that there exist $\alpha, \beta \in \mathbb{R}^+$ with $\alpha > \beta$, and*

$$\sigma\Big(\alpha(Hx - Hy)\Big) \le \varphi\Big[\max\{\sigma(\beta(x-y)), \sigma(\beta(x-Hx)), \sigma(\beta(y-Hy)), \frac{1}{2\kappa}[\sigma(\beta(x-Hy)) + \sigma(\beta(y-Hx))]\}\Big],$$

for all $x, y \in \Phi$. Then H has a unique fixed point in Φ.

The following corollary is an immediate consequence of Theorem 3, if we consider $\varphi(t) = \eta t$ for $\eta \in (0, 1)$.

Corollary 8 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space. Let Φ be a σ-closed subset of $\mathcal{X}_\sigma$ and let $H, K : \Phi \to \Phi$ be mappings such that there exist $\alpha, \beta, \eta \in \mathbb{R}^+$ with $\alpha > \beta$ and $\eta \in (0, 1)$, and*

$$\sigma\Big(\alpha(Hx - Ky)\Big) \le \eta\Big[\max\{\sigma(\beta(x-y)), \sigma(\beta(x-Hx)), \sigma(\beta(y-Ky)), \frac{1}{2\kappa}[\sigma(\beta(x-Ky)) + \sigma(\beta(y-Hx))]\}\Big]$$

for all $x, y \in \Phi$. Then H and K have a unique common fixed point in Φ.

If we put $\varphi(t) = \eta t$ for $\eta \in (0, 1)$ and $H = K$ in Theorem 3, then we get the following corollary.

Corollary 9 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space and Φ be a σ-closed subset of $\mathcal{X}_\sigma$. Let $H : \Phi \to \Phi$ be a mapping such that there exist $\alpha, \beta, \eta \in \mathbb{R}^+$ with $\alpha > \beta$ and $\eta \in (0, 1)$, and*

$$\sigma\Big(\alpha(Hx - Hy)\Big) \le \eta\Big[\max\{\sigma(\beta(x-y)), \sigma(\beta(x-Hx)), \sigma(\beta(y-Hy)), \frac{1}{2\kappa}[\sigma(\beta(x-Ky)) + \sigma(\beta(y-Hx))]\}\Big]$$

for all $x, y \in \Phi$. Then H has a unique fixed point in Φ.

The final result holds if σ is s-convex and $\varphi(t) = \eta^s t$.

Corollary 10 *Let $\mathcal{X}_\sigma$ be a σ-complete modular space, where σ is s-convex and satisfies the Δ_2-type condition. Let Φ be a σ-closed subset of $\mathcal{X}_\sigma$ and let $H, K : \Phi \to \Phi$ be mappings such that there exist $\alpha, \beta, \eta \in \mathbb{R}^+$ with $\alpha > \max\{\beta, \eta\beta\}$ and*

$$\sigma\Big(\alpha(Hx - Ky)\Big) \le \eta^s\Big[\max\{\sigma(\beta(x-y)), \sigma(\beta(x-Hx)), \sigma(\beta(y-Ky)), \frac{1}{2\kappa}[\sigma(\beta(x-Ky)) + \sigma(\beta(y-Hx))]\}\Big], \tag{32}$$

for all $x, y \in \Phi$. *Then* H *and* K *have a unique common fixed point in* Φ.

Proof Let β_0 be a constant such that $\alpha > \beta_0 > \max\{\beta, \eta\beta\}$. Then we have

$$\begin{aligned}\sigma\Big(\alpha(Hx-Ky)\Big) &\le \eta^s\Big[\max\{\sigma(\beta(x-y)), \sigma(\beta(x-Hx)), \sigma(\beta(y-Ky)),\\ &\quad \frac{1}{2\kappa}[\sigma(\beta(x-Ky))+\sigma(\beta(y-Hx))]\}\Big]\\ &= \eta^s\Big[\max\{\sigma(\frac{\beta}{\beta_0}\beta_0(x-y)), \sigma(\frac{\beta}{\beta_0}\beta_0(x-Hx)), \sigma(\frac{\beta}{\beta_0}\beta_0(y-Ky)),\\ &\quad \frac{1}{2\kappa}[\sigma(\frac{\beta}{\beta_0}\beta_0(x-Ky))+\sigma(\frac{\beta}{\beta_0}\beta_0(y-Hx))]\}\Big]\\ &\le \Big(\frac{\beta\eta}{\beta_0}\Big)^s\eta^s\Big[\max\{\sigma(\beta_0(x-y)), \sigma(\beta_0(x-Hx)), \sigma(\beta_0(y-Ky)),\\ &\quad \frac{1}{2\kappa}[\sigma(\beta_0(x-Ky))+\sigma(\beta_0(y-Hx))]\}\Big],\end{aligned}$$

where $\left(\frac{\beta\eta}{\beta_0}\right)^s < 1$. Hence, by using Corollary 9, the result follows.

Remark 2 Note that Example 3 is also satisfied in the requirements of Theorem 3.

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Fixed-Point Theorems in Generalized Modular Metric Spaces

N. Manav

Abstract Modular function spaces are one of the unique conditions of modular vector spaces which were defined by Nakano [37] in 1950. Later on, Khamsi, Kozlowski, and Reich [28] introduced the fixed-point principle in modular function spaces in 1990. Chistyakov introduced concept of a modular metric space in 2011 [14]. Abdou and Khamsi introduced fixed-point theory into the modular metric spaces using different techniques from the viewpoint of Chistyakov [14, 15], the similar approach continues in this part as they used in [1]. In this chapter, the Banach Contraction Principle and Ćirić Quasi-contraction are proven in Generalized Modular Metric Spaces (briefly GMMS). The usual topology is defined on these spaces, and then, using Nadler [36] and Edelstein's results in [1], two fixed-point theorems are given for a multivalued contractive-type map in the construction of modular metric spaces. They are Caristi and Feng-Liu types in GMMS with their applications as in [42] and [43].

1 Metrics and Modulars

The story of modulars, after giving the definition of metrics and modulars might help better understanding what is their relation. There is a growing interest over the last few years in studying the fixed-point properties in modular metric spaces after Khamsi et al. [28] introduced the fixed-point principle in modular function spaces in 1990. Modular function spaces are singular situation of modular vector spaces which Nakano [37] defined them in 1950. Chistyakov [14] introduced modular metric definition in 2011 after his early works [13]. Vyacheslav V. Chistyakov was the first who suggest such a generalization which had a physical understanding of

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the generalization of the modular function spaces, while a metric defined on a set describes nonnegative finite distances between random two points of the set, the modulars allow various perceptions relying on the context; for example, the quantity $\omega_\lambda(a, b)$ can be considered as a mean velocity from point a to b in time $\lambda > 0$ (absolute value of nonlinear), then a modular creates distance function between randomly selected two points from X. For a nonlinear approximation to modular function spaces, readers can check these sources [1, 2, 42].

Metric spaces are defined as a nonlinear variant of a norm-endowed vector spaces and a nonlinear version of the modular function spaces pursuing certain path as a vector space equipped with a modular function. Hence, considering it is logical [29]. To understand the idea of modularity on a set, we start by reminding the concept of metric. Maurice Fréchet [23] in his 1906 doctoral thesis recognized a useful distance functional between any elements of a set X, then d is shown to be a metric on X if it fulfills the conditions (axioms) for all points.

Metric space: Let take X be an abstract set. While $d : X \times X \to \mathcal{R}$ is a function on a set X is said to be a metric on X, if it provides the following axioms for all $a, b, z \in X$:

(M_1) $d(a, b) = 0$ if and only if $a = b$,
(M_2) $d(a, b) = d(b, a)$,
(M_3) $d(a, b) \leq d(a, z) + d(b, z)$.

The pair (X, d) is said to be a metric space and these axioms are commonly known as nondegeneracy, symmetry, and triangle inequality in order. In some sources, the definition of a metric is given with two axioms due to the criteria (M1)–(M3) are equivalent to (M1) and (M3) which is expressed in the format as $d(a, b) \leq d(a, z) + d(b, z)$ while $z = a$ and replacing b with a, then we impose (M2).

In 1950 [37], Nakano introduced the theory of a modular on a vector space, and in 1959, it was refined by Musielak and Orlicz [33], then in 1983 by Musielak [34].

Modular vector spaces [37]: Let X be a linear vector space over the field $\mathcal{R}$. While $\rho : X \to [0, \infty]$ is a function which is called a regular modular if the following satisfied for any $a, b \in X$ such:

(1) $\rho(a) = 0 \iff a = 0$,
(2) $\rho(\alpha a) = \rho(a) \Rightarrow |\alpha| = 1$,
(3) $\rho(\alpha a + (1 - \alpha)b) \leq \rho(a) + \rho(b), \quad \forall \alpha \in [0, 1]$,

Let ρ be a regular modular which is settled on a vector space X. The set

$$X_\rho = \{a \in X; \lim_{\alpha \to 0} \rho(\alpha\, a) = 0\}$$

is called a modular vector space. Let $\{a_n\}_{n \in \mathcal{N}} \subseteq X_\rho$ is a sequence and $a \in X_\rho$. If $\lim_{n\to\infty} \rho(a_n - a) = 0$, then $\{a_n\}_{n \in \mathcal{N}}$ is said to ρ-converge to a. ρ is revealed to satisfy the Δ_2-condition if there exists $C \neq 0$ such that $\rho(2a) \leq C\rho(a)$, for any $a \in X_\rho$. Furthermore, ρ is said to fulfill the Fatou property if $\rho(a - b) \leq \liminf_{n\to\infty} \rho(a_n - b)$ whenever $\{a_n\}$ ρ-converges to a, for any $a, b, a_n \in X_\rho$.

These definitions are given in an article by Chistyakov [14].

Definition 1 Let take $w : (0, \infty) \times X \times X \to [0, \infty]$ is a function on X which is revealed to be a metric modular(or simply a modular) as $\omega(\lambda, a, b) = \omega_\lambda(a, b)$ for $\lambda > 0$. If it assures the following three axioms:

(i) For any given $a, b \in X$, $w_\lambda(a, b) = 0, \quad \forall\lambda > 0 \iff a = b$;
(ii) $w_\lambda(a, b) = w_\lambda(b, a), \quad \forall\lambda > 0$ and $a, b \in X$;
(iii) $w_{\lambda+\mu}(a, b) \le w_\lambda(a, z) + w_\mu(b, z), \quad \forall\lambda, \mu > 0$ and $a, b, z \in X$.

If we replace (i) with the condition following the new condition of modular, which is given as

(i') $\quad w_\lambda(a, a) = 0, \quad \forall\lambda > 0$ and $a \in X$,

then w is said to be *a (metric)pseudo modular* on X.

A modular metric w on X is said to be *regular* while the next version of (i) is fulfilled such as
(i-r) $\quad a = b \iff w_\lambda(a, b) = 0$ for some $\lambda > 0$.

Let w be a (pseudo) modular on a set X. The binary relation $\overset{w}{\sim}$ on X defined for $a, b \in X$ by $a \overset{w}{\sim} b$ if and only if $\lim_{\lambda\to\infty} w_\lambda(a, b) = 0$, is, as a result of axioms (i), (ii), and (iii), an equivalence relation.

Denote by $X/_{\underset{\sim}{w}}$ the quotient-set of X with respect to $\overset{w}{\sim}$ and by $X_w^o(a) = \{b \in X : b \overset{w}{\sim} a\}$ the equivalence class of the element $a \in X$ in the quotient set $X/_{\underset{\sim}{w}}$. $\tilde{d}$: $(X/_{\underset{\sim}{w}}) \times (X/_{\underset{\sim}{w}}) \to [0, \infty]$ given by $\tilde{d}\ (X_w^o(a), X_w^o(b)) = \lim_{\lambda\to\infty} w_\lambda(a, b), \quad a, b \in X$, is well defined and satisfies the axioms of a metric.

Modular Set [16] Now, we fix an element $a_0 \in X$ arbitrarily and set $X_w = X_w^o(a_0)$. The set X_w is said *a modular set*. And if we take as a w is a metric (pseudo)modular on X , then the modular set X_w is a *(pseudo)metric space* with (pseudo)metric such that $d_w^o(a, b) = \inf\{\lambda > 0 : w_\lambda(a, b) \le \lambda\}, \quad a, b \in X_w$.

Let w be a (pseudo)modular on a set X. Then we take a sequence $\{a_n\}_{n=1}^\infty \subset X_w$ and $a \in X_w$, now we have

$$d_w^0(a_n, a) \to 0 \text{ as } n \to \infty \iff w_\lambda(a_n, a) \to 0 \text{ as } n \to \infty \text{ for all } \lambda > 0.$$

An analogous acceptance holds for Cauchy sequences.

To show the set of a modular space, let hold an $a_0 \in X$. We show that the two sets are given as

$$X_\omega = X_\omega(a_0) = \{a \in X : \omega_\lambda(a, a_0) \to 0 \ \text{ as } \ \lambda \to \infty\}$$
$$X_\omega^* = X_\omega^*(a_0) = \{a \in X : \exists\lambda = \lambda(a) > 0 \text{ such that } \ \omega_\lambda(a, a_0) < \infty\}$$

and they are called modular spaces (around a_0). It is clear that $X_\omega \subset X_\omega^*$ on the other hand, this subsumption be allowed legitimate in general.

Standard metric space generalizations are important because they allow for a strong knowledge of the fundamental conclusions drawn into the setting of classical metric spaces. When we proposed a new generalization on metric spaces, we must always be careful. For instance, if we challenge the triangle inequality, our results won't meet with some established principles in metric spaces. It is the point with the Jleli and Samet presented generalized metric distance in [24]. This generalization has been shown by the authors including metric spaces, dislocated metric spaces, b-metric spaces, and modular vector spaces. But the case of modular metric spaces didn't generalized [26]. When we scrutinize a modular metric space and a generalized metric space wisdom of Jleli and Samet [24], authors introduce a contemporary perception of generalized modular metric space in [42]. We highly recommend for curious readers the important readings on metric fixed-point theory by Khamsi and Kirk [27] and for more [21, 30].

First, we give the definition of generalized modular metric spaces.

Definition 2 [42] Let take X as a concrete set and a function $D : (0, \infty) \times X \times X \to [0, \infty]$ is called to be a generalized modular metric on X (GMMS for short), if it provides the next three axioms:

(GMM_1) If $D_\lambda(a, b) = 0$, for some $\lambda > 0$, then $a = b$, $\forall a, b \in X$;
(GMM_2) $D_\lambda(a, b) = D_\lambda(b, a)$, $\forall \lambda > 0$ and $a, b \in X$;
(GMM_3) there exists $C > 0$ such that, if $(a, b) \in X \times X$, $\{a_n\} \subset X$

when $\lim_{n\to\infty} D_\lambda(a_n, a) = 0$, for some $\lambda > 0$, then

$$D_\lambda(a, b) \leq C \limsup_{n\to\infty} D_\lambda(a_n, b).$$

The pair (X, D) is called to be a *generalized modular metric space*.

When we investigate this situation whether there exist $a, b \in X$ such that there exists $\{a_n\} \subset X$ with $\lim_{n\to\infty} D_\lambda(a_n, a) = 0$, for some $\lambda > 0$, and $D_\lambda(a, b) < \infty$, then we realize $C \geq 1$ effortlessly. During this work, we literally accept $C \geq 1$.

Now we take D as a generalized modular metric on X and $a_0 \in X$ is fixed. The defined sets below;

$$\begin{cases} X_D = X_D(a_0) = \{a \in X : D_\lambda(a, a_0) \to 0 \text{ } as \text{ } \lambda \to \infty\} \\ X_D^\star = \{a \in X : \exists\lambda = \lambda(a) > 0 \text{ } such \text{ } that \text{ } D_\lambda(a, a_0) < \infty\} \end{cases}$$

are called *generalized modular sets*. Next, we give some examples that motivated our definition of a generalized modular metric space.

Example 1 (*Modular vector spaces* [37]) Let X be a vector space over the field $\mathcal{R}$. We shall show that a modular vector space may be built-in with a generalized modular metric format. Now, let (X, ρ) be a modular vector space. Define $D : (0, +\infty) \times X \times X \to [0, +\infty]$ by

$$D_\lambda(a, b) = \rho\left(\frac{a - b}{\lambda}\right).$$

Then the following are satisfied:

(i) If $D_\lambda(a,b)=0$, for some $\lambda>0$ and $\forall a,b\in X \Rightarrow a=b$;
(ii) $D_\lambda(a,b)=D_\lambda(b,a)$, $\forall\lambda>0$ and $a,b\in X$;
(iii) if ρ satisfies the Fatou property, then $\forall\lambda>0$ and $\{a_n\}$ such that $\{a_n/\lambda\}$ ρ-converges to a/λ, we impose

$$\rho\left(\frac{a-y}{\lambda}\right)\le \liminf_{n\to\infty}\rho\left(\frac{a_n-b}{\lambda}\right)\le \limsup_{n\to\infty}\rho\left(\frac{a_n-b}{\lambda}\right),$$

which shows

$$D_\lambda(a,b)\le \liminf_{n\to\infty} D_\lambda(a_n,b)\le \limsup_{n\to\infty} D_\lambda(a_n,b),$$

for any $a, y, a_n \in X_\rho$.

Accordingly, (X, D) provides all the requirements of Definition 2 as asserted. Considering the constant C which is showed in the feature (GMM_3) is equal to 1 determines the Fatou property is satisfied by ρ.

The case of modular metric spaces is mentioned in the subsequent example.

Example 2 (*Modular metric spaces* [14, 15]) Let X be an abstract set and a defined function $\omega:(0,+\infty)\times X\times X\to[0,\infty]$, we have

$$\omega(\lambda,a,b)=\omega_\lambda(a,b).$$

Then, we have X_ω be a modular metric space. If $\lim_{n\to\infty}\omega_\lambda(a_n,a)=0$, for some $\lambda>0$, then we could not take $\lim_{n\to\infty}\omega_\lambda(a_n,a)=0$, for all $\lambda>0$. For this reason, as it is carried out in modular vector spaces, we will observe that ω fulfills the Δ_2-condition whether this is the condition, i.e., $\lim_{n\to\infty}\omega_\lambda(a_n,a)=0$, for some $\lambda>0$ refers $\lim_{n\to\infty}\omega_\lambda(a_n,a)=0$, for all $\lambda>0$. By extension, the sequence $\{a_n\}_{n\in\mathcal{N}}$ in X_ω is ω-convergent to $a\in X_\omega$ if $\lim_{n\to\infty}\omega_\lambda(a_n,a)=0$, for some $\lambda>0$. The modular function ω is said to provides the Fatou property if $\{a_n\}$ is such that $\lim_{n\to\infty}\omega_\lambda(a_n,a)=0$, for some $\lambda>0$, we impose

$$\omega_\lambda(a,b)\le \liminf_{n\to\infty}\omega_\lambda(a_n,b),$$

for any $b\in X_\omega$. While ω is a regular modular, let X_ω be a modular metric space and let take a function $D:(0,+\infty)\times X_\omega\times X_\omega\to[0,+\infty]$ by

$$D_\lambda(a,b)=\omega_\lambda(a,b).$$

Then the next properties hold:

(i) if $D_\lambda(a,b)=0$, for some $\lambda>0$ and $a,b\in X_\omega$, then $a=b$;
(ii) $D_\lambda(a,b)=D_\lambda(b,a)$, for any $\lambda>0$ and $a,b\in X_\omega$;
(iii) if ω provides the Fatou property, then for any $a\in X_\omega$ and $\{a_n\}\subset X_\omega$ such that $\lim_{n\to\infty} D_\lambda(a_n,a)=0$, for some $\lambda>0$, we capture

$$\omega_\lambda(a,b)\le \liminf_{n\to\infty}\omega_\lambda(a_n,b)\le \limsup_{n\to\infty}\omega_\lambda(a_n,b),$$

for any $y\in X_\omega$, which refers

$$D_\lambda(a,b)\le \liminf_{n\to\infty} D_\lambda(a_n,b)\le \limsup_{n\to\infty} D_\lambda(a_n,b).$$

In other words, (X_ω, D) is a generalized modular metric space.

Example 3 (*Generalized metric spaces* [24]) Throughout X is an abstract set and $\mathcal{D}: X\times X\to[0,\infty]$ is a function and $a\in X$, then we demonstrate the set

$$\mathcal{C}(\mathcal{D},X,a)=\{\{a_n\}\subset X;\ \lim_{n\to\infty}\mathcal{D}(a_n,a)=0\}.$$

With respect to [24], the function $\mathcal{D}: X\times X\to[0,\infty]$ is assumed to define a generalized metric on X if it fulfills the next axioms:

($\mathcal{D}_1$) $\forall(a,b)\in X\times X$, we have $\mathcal{D}(a,b)=0\Rightarrow a=b$;
($\mathcal{D}_2$) $\forall(a,b)\in X\times X$, we have $\mathcal{D}(a,b)=\mathcal{D}(b,a)$,
($\mathcal{D}_3$) there exists $C>0$ such that, if $(a,b)\in X\times X$, $\{a_n\}\in\mathcal{C}(\mathcal{D},X,a)$, we have

$$\mathcal{D}(a,b)\le C\ \limsup_{n\to\infty}\mathcal{D}(a_n,b).$$

The couple $(X,\mathcal{D})$ is then called a generalized metric space. Now, display that such formation may be seen as a generalized modular metric space. In fact, let $(X,\mathcal{D})$ be a generalized metric space. Define $D:(0,+\infty)\times X\times X\to[0,+\infty]$ by

$$D_\lambda(a,b)=\frac{\mathcal{D}(a,b)}{\lambda}.$$

Obviously, if $\{a_n\}\in\mathcal{C}(\mathcal{D},X,a)$, for some $a\in X$, then we impose

$$\lim_{n\to\infty} D_\lambda(a_n,a)=0,$$

for any $\lambda>0$. Then the next hold:

(i) if $D_\lambda(a,b)=0$, for some $\lambda>0$ and $a,b\in X\Rightarrow a=b$;
(ii) $D_\lambda(a,b)=D_\lambda(b,a)$, for any $\lambda>0$ and $a,b\in X$;
(iii) there exists $C>0$ such that, if $(a,b)\in X\times X$, $\{a_n\}\in\mathcal{C}(D_\lambda,X,a)$, for some $\lambda>0$, we have

$$D_\lambda(a, b) \le C \limsup_{n\to\infty} D_\lambda(a_n, b).$$

These properties show that (X, D) is a generalized modular metric space.

2 Fixed-Point Theorems in Generalized Modular Metric Spaces

Abdou and Khamsi conveyed Fixed-point theory in modular metric spaces [1]. Their way was distinct from Chistyakov's perspective [14, 15]. In here, we maintain the same touch as they used in [1]. So, the definitions, propositions, and their proofs from a work [42] below resemble the metric properties and are subsequently beneficial to prove Banach Contraction Principle and Ćirić quasicontraction. After giving Banach Contraction Principle part Ćirić quasicontraction part is helping more generalised versons

Definition 3 Let (X_D, D) be a generalized modular metric space.

(1) The sequence $\{a_n\}_{n\in\mathcal{N}}$ in X_D is said to be D-convergent to $a \in X_D$ if and only if $D_\lambda(a_n, a) \to 0$, as $n \to \infty$, for some $\lambda > 0$.
(2) The sequence $\{a_n\}_{n\in\mathcal{N}}$ in X_D is said to be D-Cauchy if $D_\lambda(a_m, a_n) \to 0$, as $m, n \to \infty$, for some $\lambda > 0$.
(3) A subset C of X_D is said to be D-closed if for any $\{a_n\}$ from C which D-converges to a, we have $a \in C$.
(4) A subset C of X_D is said to be D-complete if for any $\{a_n\}$ D-Cauchy sequence in C such that $\lim_{n,m\to\infty} D_\lambda(a_n, a_m) = 0$ for some λ, there exists a point $a \in C$ such that $\lim_{n,m\to\infty} D_\lambda(a_n, a) = 0$.
(5) A subset C of X_D is said to be D-bounded if for some $\lambda > 0$, we have

$$\delta_{D,\lambda}(C) = \sup\{D_\lambda(a, b); a, b \in C\} < \infty.$$

In fact, if $\lim_{n\to\infty} D_\lambda(a_n, a) = 0$, for some $\lambda > 0$, then it cannot be $\lim_{n\to\infty} D_\lambda(a_n, a) = 0$, for all $\lambda > 0$. For that reason, as this axiom is fulfilled in modular function spaces, it is possible to say that D compensates Δ_2-condition if and only if $\lim_{n\to\infty} D_\lambda(a_n, a) = 0$, for some $\lambda > 0$ refer $\lim_{n\to\infty} D_\lambda(a_n, a) = 0$, for all $\lambda > 0$.

Now we can give an answer to a question that arises in the concept of D-limit and D-limit's uniqueness.

Proposition 1 *Let (X_D, D) be a generalized modular metric space. Let $\{a_n\}$ be a sequence in X_D. Let $(a, b) \in X_D \times X_D$ such that $D_\lambda(a_n, a) \to 0$ and $D_\lambda(a_n, b) \to 0$ as $n \to \infty$, for some $\lambda > 0$, then $a = b$.*

Proof Using the property (GMM_3), we impose

$$D_\lambda(a, b) \leq C \limsup_{n\to\infty} D_\lambda(a_n, b) = 0.$$

which signifies from the property (GMM_1) that $a = b$.

2.1 The Banach Contraction Principle in Generalized Modular Metric Spaces

In 1922, Banach gave the Banach Contraction Principle in his thesis [9]. Banach's theory deals with their presence and uniqueness fixed points of certain metric space with self-mappings. Many researchers have studied contractive form mappings in different metric spaces for several fixed-point theorems. Modular metric spaces are one of them, more important and practical one. The representation to the definition of modular metric spaces and especially nonlinear version of the classical modular spaces have been developed by many authors [1, 5, 10, 14–16, 24, 29–31, 39]. Now in this work, we demonstrate an extension of the Banach Contraction Principle to the formerly implemented context of generalized modular metric spaces.

Definition 4 Let (X_D, D) be a generalized modular metric space and $f : X_D \to X_D$ be a mapping. f is called a D-contraction mapping, if there exists $k \in (0, 1)$ such that

$$D_1(f(a), f(b)) \leq k\ D_1(a, b), \quad \forall (a, b) \in X_D \times X_D.$$

a is said to be a fixed point of f if $f(a) = a$.

Proposition 2 *Let (X_D, D) be a generalized modular metric space. Let $f : X_D \to X_D$ be a D-contraction mapping. If ω_1 and ω_2 are fixed points of f and $D_1(\omega_1, \omega_2) < \infty$, then $\omega_1 = \omega_2$.*

Proof Let $\omega_1, \omega_2 \in X_D$ be two fixed points of f such that $D_1(\omega_1, \omega_2) < \infty$. Since f is a D-contraction, there exists $k \in (0, 1)$ such that

$$D_1(\omega_1, \omega_2) = D_1(f(\omega_1), f(\omega_2)) \leq k\ D_1(\omega_1, \omega_2).$$

Since $D_1(\omega_1, \omega_2) < \infty$, we finalize that $D_1(\omega_1, \omega_2) = 0$, which implies $\omega_1 = \omega_2$ from (GMM_1).

Now let us take (X_D, D) be a generalized modular metric space and $f : X_D \to X_D$ be a mapping, for any $a \in M$, define the *orbit* of a by

$$\mathcal{O}(a) = \{a, f(a), f^2(a), ...\}.$$

Set $\delta_{D,\lambda}(a) = \sup\{D_\lambda(f^n(a), f^m(a)); n, m \in \mathcal{N}\}$, where $\lambda > 0$. An extension of the Banach Contraction Principle is given in generalized modular metric spaces.

Theorem 1 *Let (X_D, D) be a generalized modular metric space as X_D is D-complete. Let $f : X_D \to X_D$ be a D-contraction mapping. Suppose that $\delta_{D,1}(a_0)$ is finite, for some $a_0 \in X_D$. Then $\{f^n(a_0)\}$ D-converges to a fixed point ω of f. Furthermore, if $D_1(a, \omega) < \infty$, for $a \in X_D$, then $\{f^n(a)\}$ D-converges to ω.*

Proof Let $a_0 \in X_D$ be such that the $\delta_{D,1}(a_0) < \infty$. Then

$$D_1(f^{n+p}(a_0), f^n(a_0)) \le k^n\ D_1(f^p(a_0), a_0) \le k^n\ \delta_{D,1}(a_0),$$

for any $n, p \in \mathcal{N}$. Since $k < 1$, $\{f^n(a_0)\}$ is D-Cauchy. Since X_D is D-complete, then there exists $\omega \in X_D$ such that $\lim_{n\to\infty} D_1(f^n(a_0), \omega) = 0$. Since

$$D_1(f^n(a_0), f(\omega)) \le k\ D_1(f^{n-1}(a_0), \omega); \quad n = 1, 2, ...,$$

we have $\lim_{n\to\infty} D_1(f^n(a_0), f(\omega)) = 0$. Proposition 1 shows that $f(\omega) = \omega$, i.e., ω is a fixed point of f. Let $a \in X_D$ be such that $D_1(a, \omega) < \infty$. Then

$$D_1(f^n(a), \omega) = D_1(f^n(a), f^n(\omega)) \le k^n D_1(a, \omega),$$

for any $n \ge 1$. Since $k < 1$, we have $\lim_{n\to\infty} D_1(f^n(a), \omega) = 0$, i.e., $\{f^n(a)\}$ D-converges to x.

If we have $D_1(a, b) < \infty$, for any $a, b \in X_D$, then we have f, which has only a fixed point. Furthermore, if X_D is D-complete and $\delta_{D,1}(a) < \infty$ for any $a \in X_D$, then all orbits D-converge to the given unique fixed point of f. In metric spaces, $d(a, b)$ is always finite. Due to this reason, any contraction have at most one fixed point. In addition, the orbits of the contraction are all bounded. Indeed, where M is a metric space presented with a metric distance d, let $f : M \to M$ be a contraction. We show that

$$d(f^{n+1}(a), f^n(a)) \le k^n\ d(f(a), a),$$

for any $n \in \mathcal{N}$ and $a \in M$, which indicates by adopting the triangle inequality

$$\begin{aligned} d(f^{n+p}(a), f^n(a)) &\le \sum_{k=0}^{p-1} d(f^{n+k+1}(a), f^{n+k}(a)) \\ &\le \sum_{k=0}^{p-1} k^{n+k} d(f(a), a) \\ &\le \frac{1}{1-k} d(f(a), a), \end{aligned}$$

considering $k < 1$. So

$$\sup\{d(f^n(a), f^m(a)); n, m \in \mathcal{N}\} \leq \frac{1}{1-k} d(f(a), a) < \infty,$$

for any $a \in M$.

Now, the extension of the Ćirić's fixed-point theorem [17] is examined for quasi-contraction type mappings in generalized modular metric spaces and an accurate version of Theorem 4.3 in [24] seeing that its proof is needed to be modified [26].

2.2 Ćirić Quasi-contraction in Generalized Modular Metric Spaces

Let us describe the view of quasi-contraction mappings initially in the processing of generalized modular metric spaces.

Definition 5 Let (X_D, D) be a generalized modular metric space. The mapping $f : X_D \to X_D$ is said to be a D-quasi-contraction, if there exists $k \in (0, 1)$ such that

$$D_1(f(a), f(b)) \leq k \max\big\{D_1(a, b), D_1(a, f(a)), D_1(b, f(b)), D_1(a, f(b)), D_1(b, f(a))\big\},$$

for any $(a, b) \in X_D \times X_D$.

Proposition 3 *Let (X_D, D) be a generalized modular metric space. Let $f : X_D \to X_D$ be a D-quasi-contraction mapping. If x is a fixed point of f such that $D_1(x, x) < \infty$, then we impose $D_1(x, x) = 0$. Furthermore, if x_1 and x_2 are two fixed points of f such that $D_1(x_1, x_2) < \infty$, $D_1(x_1, x_1) < \infty$ and $D_1(x_2, x_2) < \infty$, then we have $x_1 = x_2$.*

Proof Let x be the fixed point of f, then

$$\begin{aligned} D_1(x, x) = D_1(f(x), f(x)) &\leq k \max\big\{D_1(x, x), D_1(x, f(x)), D_1(x, f(x)), D_1(x, f(x)), D_1(x, f(x))\big\} \\ &= k\, D_1(x, x). \end{aligned}$$

Since $k < 1$ and $D_1(x, x) < \infty$, then $D_1(x, x) = 0$. Let $x_1, x_2 \in X_D$ be two fixed points of f, such that $D_1(x_1, x_2) < \infty$, $D_1(x_1, x_1) < \infty$ and $D_1(x_2, x_2) < \infty$. Since f is a D-quasi-contraction, there exists $k < 1$ such that

$$\begin{aligned} D_1(x_1, x_2) = D_1(f(x_1), f(x_2)) &\leq k \max\big\{D_1(x_1, x_2), D_1(x_1, f(x_1)), D_1(x_2, f(x_2)), D_1(x_1, f(x_2)), D_1(x_2, f(x_1))\big\}. \\ &= k \max\big\{D_1(x_1, x_2), D_1(x_1, x_1), D_1(x_2, x_2)\big\}. \end{aligned}$$

Since $D_1(x_1, x_1) < \infty$ and $D_1(x_2, x_2) < \infty$, then $D_1(x_1, x_1) = D_1(x_2, x_2) = 0$. Now we impose

$$D_1(x_1, x_2) \leq k\ D_1(x_1, x_2).$$

Since $D_1(x_1, x_2) < \infty$ and $k < 1$, then $D_1(x_1, x_2) = 0$.

The next proof may be seen for quasi-contraction type mappings as an extension of the Ćirić's fixed-point theorem [17] in generalized modular metric spaces.

Theorem 2 *Let (X_D, D) be a D-complete generalized modular metric space. Let $f : X_D \to X_D$ be a D-quasi-contraction mapping. Suppose that $k < \frac{1}{C}$, where C is the constant from (GMM_3), and there exists $a_0 \in X_D$ such that $\delta_{D,1}(a_0) < \infty$. Then $\{f^n(a_0)\}$ D-converges to some $x \in X_D$. If $D_1(a_0, f(x)) < \infty$ and $D_1(x, f(x)) < \infty$, then x is a fixed point of f.*

Proof Let f is a D-quasi-contraction, then there exists $k \in (0, 1)$ such that for all $p, r, n \in \mathcal{N}$ and $a \in X_D$, we impose

$$\begin{aligned} D_1(f^{n+p+1}(a), f^{n+r+1}(a)) \leq k\ \max\big\{&D_1(f^{n+p}(a), f^{n+r}(a)),\\ &D_1(f^{n+p}(a), f^{n+p+1}(a)), D_1(f^{n+r}(a), f^{n+r+1}(a)),\\ &D_1(f^{n+p}(a), f^{n+r+1}(a)), D_1(f^{n+r}(a), f^{n+p+1}(a))\big\}. \end{aligned}$$

$$\begin{aligned} D_1(f^{n+p+1}(a), f^{n+r+1}(a)) \leq k\ \max\big\{&D_1(f^{n+p}(a), f^{n+r}(a)),\\ &D_1(f^{n+p}(a), f^{n+p+1}(a)), D_1(f^{n+r}(a), f^{n+r+1}(a)),\\ &D_1(f^{n+p}(a), f^{n+r+1}(a)), D_1(f^{n+r}(a), f^{n+p+1}(a))\big\}. \end{aligned}$$

Hereinafter $\delta_{D,1}(f(a)) \leq k\ \delta_{D,1}(a)$, for any $a \in X_D$. As a result, we impose

$$1\delta_{D,1}(f^n(a_0)) \leq k^n\ \delta_{D,1}(a_0), \tag{1}$$

for any $n \geq 1$. Taking the inequality (1) just above

$$2D_1(f^n(a_0), f^{n+m}(a_0)) \leq \delta_{D,1}(f^n(a_0)) \leq k^n \delta_{D,1}(a_0), \tag{2}$$

for every $n, m \in \mathcal{N}$. Since $\delta_{D,1}(a_0) < \infty$ and $k < 1/C \leq 1$, we impose

$$\lim_{n,m\to\infty} D_1(f^n(a_0), f^{n+m}(a_0)) = 0,$$

which that $\{f^n(a_0)\}$ is a D-Cauchy sequence. When X_D is D-complete, there exists $x \in X_D$ such that $\lim_{n\to\infty} D_1(f^n(a_0), x) = 0$, i.e., $\{f^n(a_0)\}$ D-converges to x. Next, we suppose $D_1(a_0, f(x)) < \infty$ and $D_1(x, f(x)) < \infty$. Taking the inequality (2) and the property (GMM_3), we show that

$$3D_1(x, f^n(a_0)) \leq C\ \limsup_{m\to\infty} D_1(f^n(a_0), f^{n+m}(a_0)) \leq C\ k^n\ \delta_{D,1}(a_0) \tag{3}$$

for every $n, m \in \mathcal{N}$.
Hence

$$D_1(f(a_0), f(x)) \leq k \ \max\left\{D_1(a_0, x), D_1(a_0, f(a_0)), D_1(x, f(x)) \right. \\ \left. D_1(f(a_0), x), D_1(a_0, f(x))\right\}.$$

and, applying (1), (2), (3) and $k < 1/C \leq 1$, we have

$$D_1(f^2(a_0), f(x)) \leq \max\left\{k^2 \ C \ \delta_{D,1}(a_0), k \ D_1(x, f(x)), k^2 \ D_1(x, f(a_0))\right\}.$$

Consecutively, by induction, we can impose

$$D_1(f^n(a_0), f(x)) \leq \max\left\{k^n \ C \ \delta_{D,1}(a_0), k \ D_1(x, f(x)), k^n \ D_1(x, f(a_0))\right\},$$

for every $n \geq 1$. Furthermore, we get

$$\limsup_{n\to\infty} D_1(f^n(a_0), f(x)) \leq k \ D_1(x, f(x)),$$

when $D_1(a_0, f(x)) < \infty$ and $\delta_{D,1}(a_0) < \infty$. Again property (GMM_3) implies

$$D_1(x, f(x)) \leq C \ \limsup_{n\to\infty} D_1(f^n(a_0), f(x)) \leq k \ C \ D_1(x, f(x)).$$

Since $k \ C < 1$ and $D_1(x, f(x)) < \infty$, then $D_1(x, f(x)) = 0$, i.e., $f(x) = x$.

Example 4 Let $X = \{p, q, t\}$ and define $D : (0, \infty) \times X \times X \to [0, \infty]$ as $D_\lambda(p, p) = D_\lambda(q, q) = D_\lambda(t, t) = 0$, $D_\lambda(p, q) = D_\lambda(q, p) = 2$, $D_\lambda(p, t) = D_\lambda(t, p) = 6$, $D_\lambda(q, t) = D_\lambda(t, q) = 2$ for $\lambda > 0$. Then, GMM_1 and GMM_2 are clear. $D_{\lambda+\mu}(p, t) \leq D_\lambda(p, q) + D_\mu(q, t)$ is $6 \leq 4$, is false, so (X, D) is not a modular metric space. When we address the GMM_3, we get $\lim\limits_{n\to\infty} D_\lambda(a_n, a) = 0$, while $\lim\limits_{n\to\infty} a_n = a$. It is obvious that if we choose $a = y$ then GMM_3 is answered, if we choose $a \neq b$ it is easy to prove that $D_\lambda(a, b) \leq C \ \limsup_{n\to\infty} D_\lambda(a_n, b)$ is true for all elements from $X = \{p, q, t\}$.

2.3 Topology on Generalized Modular Metric Spaces

Now let us show how to describe topologies on generalized modular metric spaces by using results of [43] and take (X_D, D) be a GMMS, then $B \subset X_D$ $D-$sequentially open subset of X_D, when each sequence of X_D has $\lim\limits_{n\to\infty} D_\lambda(a_n, a) = 0$ for some λ. Then, there exists a point $a \in B_D$ all but a finite number of terms of the sequence contained in B_D. Let consider τ_{X_D} be a family of all sequentially open subsets of X_D. In a topological space (X_D, τ_{X_D}), any convergent sequence in X_D is convergent.

When we have $\mathcal{C}(X_D)$ as a family of all nonempty closed subsets of (X_D, τ_{X_D}) and we show $\mathcal{M}$ as a family of all nonempty subsets A of X_D, we impose the next characteristic. Then we are targeting those two subsets as they are similar. If $D_\lambda(a, A) = 0$, then we have $a \in A$ for all $a \in X_D$, while $D_\lambda(a, A) = \inf\{D_\lambda(a, y) : y \in A\}$. If our property is assured for any $B_D \subset \mathcal{C}(X_D)$ and $a \in X_D$, then there exists a sequence in B_D such that $\lim_{n\to\infty} D_\lambda(a_n, a) = 0$. We take $a \in B_D$ all but a finite number of terms of the sequence included in B_D, which means $B_D \cap A \neq \emptyset$, so $a \in A = \bar{A}$, in a topological space (X_D, τ_{X_D}). Consequently, $\mathcal{C}(X_D) \subset \mathcal{M}$. If we have $A \subset \mathcal{M}$, $a \in X_D - A$ and a sequence in X_D such that $\lim_{n\to\infty} D_\lambda(a_n, a) = 0$, then we have no subsequence in A provide $D_\lambda(a, A) = 0$ for any $a \in A$. Hence, $X_D - A \in \tau_{X_D}$ is found and realize that $A \in \mathcal{C}(X_D)$. The result shows us $\mathcal{C}(X_D) = \mathcal{M}$. And additionally, the definition of an open subset is showed by using open balls in GMMS in the next sentence. If A is a subset of X_D for any $a \in X_D$, there exists $\epsilon > 0$ such that $B_D(a, \epsilon) := \{b \in X_D : D_\lambda(a, b) < \epsilon\} \subseteq A$.

τ_{X_D} satisfies the usual properties of a topology in a work [42]. For instance, when we take modular vector spaces as in [4], the ρ-ball $B_\rho(a, r)$, where $a \in X_\rho$ and $r \geq 0$, is given by the definition $B_\rho(a, r) = \{b \in X_\rho; \rho(a - b) < r\}$. B_ρ is an open ball and then a subset of A in vector space X_ρ. For all ρ-open subsets of X_ρ, in the example of the topology (τ_ρ), is identical for the definition of open subsets of τ_{X_ρ} in a modular space X_ρ. Chistyakov [16] presented modular open balls and their topological properties such as: A nonempty set in X is said to ω-open if for every $a \in A$ and $\lambda > 0$ there exists $\mu > 0$ such that $B(a)_{\lambda,\mu} \subset A$ by using ω as a modular metric. It was given by $\tau(\omega)$ for all ω-open subsets of X_ω we take a ω-topology (modular topology) on X_ω, which is analogous to τ_{X_D} in a modular metric space. When we take a JS-metric space and the topology on JS-metric space, which is showed in [6], we realize that the normal topology on JS-metric space is equal to τ_{X_D} again.

3 Feng-Liu Theorem in GMMS

Multivalued mapping has numerous applications in pure and applied mathematics. Topology, nonlinear functional analysis, function theory of a real variable, game theory, and mathematical economics are very important examples for those fields.

In this part, new definitions in GMMS are analyzed, namely, multivalued Lipschitzian mapping and then D-multivalued contraction. We're also concentrating on the relation between those meanings as showing a generalization of the Banach contraction mapping theory and explaining a fixed-point property via a multivalued contraction mapping of the nonempty D-closed and bounded subsets in X_D. Caristi and Feng-Liu type approaches for the existence of a fixed point in GMMS are given by using the work of Nadler [36]. A non-homogeneous linear parabolic partial differential equation and then, an initial value problem in GMMS is given to show some applications of these results.

3.1 Multivalued Mappings in GMMS

Two fixed-point theorems are demonstrated by Nadler in [36] for multivalued contraction mapping. The first one is a generalization of Banach's contraction mapping principle, claims that it has a fixed point as a multivalued contraction mapping of a complete metric space within the nonempty closed and bounded subsets of the same metric space. The second one is a general statement of an Edelstein result, for compact set-valued local contractions is given for a fixed-point theorem. His works are applied through other metric spaces, for example, in [3, 8, 10, 11, 17–21, 25, 26, 28, 32, 35, 39, 40].

Feng and Liu [22], without using Pompei-Hausdorff distance, gave one of the most important generalization of Nadler's test. Then several studies based on those findings and applied them in various metric spaces, such as in [6]. Now it will be explored in GMMS.

Essentially, the generalized Hausdorff modular is explained here.

Definition 6 Let (X_D, D) be a GMMS and for all nonempty $A, B \subset X_D$, the generalized Hausdorff modular has a definition such as:

$$H_D(\lambda, A, B) = \max\{\sup_{a\in A} D_\lambda(a, B), \sup_{b\in B} D_\lambda(b, A)\}$$

where $D_\lambda(a, B) = \inf_{b\in B} D_\lambda(a, b)$, on $\mathcal{C}(X_D)$-D-strongly complete version of X_D is defined in the next section.

If $\lambda = 1$, we get $H_D(A, B) = \max\{\sup_{a\in A} D_1(a, B), \sup_{b\in B} D_1(b, A)\}$ on $\mathcal{C}(X_D)$, where $D_1(a, B) = \inf_{b\in B} D_1(a, b)$.

Example 5 If the GMMS is taken in which is given in the first example, for $A = \{p, q\}$, $B = \{t\} \subset X$, we find that

$$H_D(\lambda, \{p, q\}, \{t\}) = \max\{\sup_{a\in\{p,q\}} D_\lambda(a, \{t\}), \sup_{b\in\{t\}} D_\lambda(b, \{p, q\})\},$$

where $D_\lambda(a, \{t\}) = \inf_{b \in \{t\}} D_\lambda(a, b)$ and $D_\lambda(\{p, q\}, b) = \inf_{a \in \{p,q\}} D_\lambda(a, b)$. All possible calculations could be made by readers.

3.1.1 Fixed Point for Multivalued Mappings

The existence of a fixed point for a multivalued contractive-type map in modular metric spaces and in their analysis, and then the existence of a unique fixed point of multivalued contractive mapping in these spaces by using Nadler [36] and Edelstein's results are reviewed by Abdou and Khamsi [1].

Definition 7 Let (X_D, D) be a GMMS and then a given mapping $f : X_D \to \mathcal{C}(X_D)$ is called a multivalued Lipschitzian mapping, if there exists a constant $k \geq 0$ such that for any $a, b \in X_D$, for every $x \in f(a)$ there exists $y \in f(b)$, while $D_1(x, y) \leq k\ D_1(a, b)$. A point $a \in X_D$ is named as a fixed point of f when $a \in f(a)$. $Fix(f)$ shows the set of fixed points of f. The mapping f is denoted as *D-multivalued contraction*, if the constant satisfies $k < 1$.

Example 6 Let us take the same example as we used before, and a mapping $f : X \to \mathcal{C}(X)$ such that $f(3) = f(6) = 3$ *and* $f(9) = 6$ for every $x \in f(a)$ there exists $y \in f(b)$, such the inequality $D_1(x, y) \leq k\ D_1(a, b)$ is proved in X.

We describe at this point that f has a fixed point of X_D, as D-multivalued contraction mapping f in particular space.

Theorem 3 *Let (X_D, D) be a $GMMS$. Suppose that X_D is D-strongly complete and D satisfy* 1*-Fatou property. If $f : X_D \to \mathcal{C}(X_D)$ is a D-multivalued contraction mapping. Suppose that $D_1(a_0, a)$ is finite for some $a_0 \in X_D$ and $a \in f(a_0)$. Then f has a fixed point.*

Proof Fix $a_0 \in X_D$ such that $D_1(a_0, f(a)) < \infty$ for some $a_1 \in f(a_0)$ then there exists $a_2 \in f(a_1)$ such that

$$D_1(a_1, a_2) \leq k\ D_1(a_0, a_1),$$

where $D_1(a_1, a_2) < \infty$.

$$D_1(a_2, a_3) \leq k^2\ D_1(a_0, a_1),$$

where $D_1(a_2, a_3) < \infty$. By induction, we choose elements of a sequence $\{a_n\}$ there is $a_1 \in f(a_{n+1})$, for every $a_0 \in f(a_n)$, then there exists $a_{n+1} \in f(a_n)$, when f is a D- multivalued contraction:

$$D_1(a_n, a_{n+1}) \leq k^n \, D_1(a_0, a_1),$$

where $D_1(a_n, a_{n+1}) < \infty$, for every $n \geq 0$. Since $k < 1$, $\sum_{n=1}^{\infty} D_1(a_n, a_{n+1})$ is convergent, i.e., $\{a_n\}$ is D-strongly Cauchy. Since X_D is D-strongly complete, then we say that there exists a point $a \in X_D$ such that $\lim_{n\to\infty} D_1(a_n, a) = 0$. Since there is $a_0 \in f(a)$, for every $a_1 \in f(a_n)$,

$$D_1(a_0, a_1) \leq k \; D_1(a_n, a),$$

and D_1 has 1-Fatou property,

$$D_1(a_0, a_1) \leq k \; D_1(a, a) \leq k \; \liminf_{n\to\infty} D_1(a_n, a),$$

we conclude that $\lim_{n\to\infty} D_1(a_0, a_1) = 0$, then a is fixed point of f.

3.1.2 Caristi-Type Fixed-Point Results for Multivalued Mappings

Caristi showed a general fixed-point theorem and gave us the application of it to achieve an important result in a complete metric space which is the generalization of the Contraction Mapping Principle, and the application along with the characterization of weakly inward mappings to show some important fixed-point theorems in Banach spaces [12]. Thereafter, several authors extended his approach via various metric spaces; for example, in [7, 41]. In this section, we discuss Caristi-type mappings and state results of the Feng-Liu-type in GMMS.

Theorem 4 *Let take X_D be a D-complete GMMS and $f : X_D \to CB(X_D)$ be a nonexpansive mapping such that for each $a \in X_D$ and $b \in f(a)$, we impose*

$$D_1(a, b) \leq \Theta_D(a, b) - \Theta_D(b, z)$$

for $z \in f(b)$, while $CB(X_D)$ is D-closed and bounded subsets of X_D and with its first variable the function $\Theta_D : X_D \times X_D \to [0, \infty]$ is lower semicontinuous. Then $D_1(a_n, a_{n+1}) < \infty$, so f has a fixed point.

Proof Let $a_0 \in X_D$ and $a_1 \in f(a_0)$. If $a_1 = a_0$, then proof is completed. Let $a_0 \neq a_1$. Using the above inequality of the theorem, then

$$D_1(a_0, a_1) \leq \Theta_D(a_0, a_1) - \Theta_D(a_1, a_2),$$

for $a_2 \in f(a_1)$. When we continue to use the technique, we have $a_n \in f(a_n)$ while $a_n \neq a_{n+1}$, then we take

$$0 < D_1(a_{n-1}, a_n) \leq \Theta_D(a_{n-1}, a_n) - \Theta_D(a_n, a_{n+1}),$$

for $a_{n+1} \in f(a_n)$. Suppose there exists $\Theta_D(a_{n-1}, a_n)_{n\in\mathcal{N}}$ nonincreasing sequence and converges to $x > 0$. If we impose limit from the latest inequality, we impose

$$\lim_{n\to\infty} D_1(a_{n-1}, a_n) \le \lim_{n\to\infty}\{\Theta_D(a_{n-1}, a_n) - \Theta_D(a_n, a_{n+1})\},$$
$$\lim_{n\to\infty} D_1(a_{n-1}, a_n) \le \lim_{n\to\infty}\Theta_D(a_{n-1}, a_n) - \lim_{n\to\infty}\Theta_D(a_n, a_{n+1}),$$
$$\lim_{n\to\infty} D_1(a_{n-1}, a_n) \le x - x = 0$$

for $n \in \mathcal{N}$. It is the same way to prove $\{a_n\}_{n\in\mathcal{N}}$ is D-Cauchy sequence. Then we suppose x is a fixed point of f:

$$\begin{aligned} D_1(x, f(x)) &\le D_{\frac{1}{2}}(x, a_{n+1}) - D_{\frac{1}{2}}(f(x), a_{n+1}), \\ &\le D_{\frac{1}{2}}(x, a_{n+1}) - H_D(f(x), f(a_{n+1})), \\ &\le D_{\frac{1}{2}}(x, a_{n+1}) - D_{\frac{1}{2}}(x, a_n), \end{aligned}$$

for the last equality in the process, when we pass the limit, and then we impose

$$\lim_{n\to\infty} D_1(x, f(x)) \le \lim_{n\to\infty} D_{\frac{1}{2}}(x, a_{n+1}) - \lim_{n\to\infty} D_{\frac{1}{2}}(x, a_n),$$
$$\lim_{n\to\infty} D_1(x, f(\omega)) \le D_{\frac{1}{2}}(x, x) - D_{\frac{1}{2}}(x, x) = 0.$$

Then x is a fixed point of f.

Now its more general version is given via υ function.

Theorem 5 *Let X_D be a D-complete GMMS and $f : X_D \to CB(X_D)$ be a multi-valued mapping*

$$H_D(f(a), f(b)) \le \upsilon(D_1(a, b))$$

for all $a, b \in X_D$ and $\upsilon : [0, \infty] \to [0, \infty]$ is a lower semicontinuous map such as defined: $\upsilon(t) < t$ for $t \in [0, \infty]$ and provides that $\frac{\upsilon(t)}{t}$ is nondecreasing. Then $D_1(a_n, a_{n+1}) < \infty$, so f has a fixed point.

The next theorem was given by Feng and Liu [22], rather than using Hausdorff distance and now we indicate their results for a multivalued mapping of f on XD, let and define

$$I^a_{D,\beta}(f) = \{x \in f(a); \beta\, D_\lambda(a, x) \le D_\lambda(a, f(a))\}.$$

The function f is named as D-lower semicontinuous, and for any sequence $\{a_n\} \in X_D$ is convergent to $a \in X_D$, if $D_1(a, f(a)) \le \liminf_{n\to\infty} D_1(a_n, f(a_n))$.

Example 7 Using the mapping $f : X \to \mathcal{C}(X)$ from Example 6; $f(a) = 9$, $\beta = \frac{1}{6}$, $a \in X$, we are able to take for any calculation of $\beta\ D_1(a, x) \leq D_1(a, f(a))$, where $a \in f(a)$, it is assured. Then a function f is called D-lower semicontinuous for any sequence $\{a_n\} \in X$ is convergent to $a \in X$, if $D_1(a, f(a)) \leq \liminf_{n\to\infty} D_1(a_n, f(a_n))$.

Theorem 6 *Let (X_D, D) be a complete GMMS and a function f be D-multivalued mapping on X_D. If there exists a constant $K > 0$ such that $\frac{K}{\beta} < 1$ for any $a \in X_D$ there is $b \in I^a_{D,\beta}(f)$ providing*

$$D_1(b, f(b)) \leq K\ D_1(a, b).$$

If there exists $a_0 \in X_D$ such that $D_1(a_0, f(a_0)) < \infty$. Suppose there exists a sequence $\{a_n\}$ in X_D such that $\beta\ D_1(a_{n+1}, a_{n+2}) \leq K\ D_1(a_n, a_{n+1})$ and $\beta\ D_1(a_{n+1}, f(a_{n+1})) \leq K\ D_1(a_n, f(a_n))$; while $a_{n+1} \in f(a_n)$ and $D_1(a_n, a_{n+1}) < \infty$ for any $n \in \mathcal{N}$.

The sequence we have is D-strongly Cauchy, and if we suppose $D_1(a, f(a))$ is D-lower semicontinuous, then we get f has a fixed point.

Proof Since $f(a) \in X_D$ for all $a \in X_D$, then $I^a_{D,b}(f)$ is nonempty. Let us take $a_0 \in X_D$ such as $D_1(a_0, f(a_0)) < \infty$. If $D_1(b, f(b)) \leq K\ D_1(a, b)$, there exists $a_1 \in I^{a_0}_{D,\beta}(f)$ such that,

$$D_1(a_1, f(a_1)) \leq K\ D_1(a_0, a_1).$$

When $a_1 \in I^{a_0}_{D,\beta}(f)$, then $a_1 \in f(a_0)$ and

$$\beta\ D_1(a_0, a_1) \leq D_1(a_0, f(a_0)) < \infty.$$

Taking $a_1 \in X_D$ such that $D_1(a_1, f(a_1)) < \infty$. From $D_1(b, f(b)) \leq K\ D_1(a, b)$, there exists $a_2 \in I^{a_1}_{D,\beta}(f)$ such that,

$$D_1(a_2, f(a_2)) \leq K\ D_1(a_1, a_2).$$

Since $a_2 \in I^{a_1}_{D,\beta}(f)$, then $a_2 \in f(a_1)$ and

$$b\ D_1(a_1, a_2) \leq D_1(a_1, f(a_1)) < \infty.$$

Next, going with the new take $a_{n+1} \in M$, we have $D_1(a_{n+1}, f(a_{n+1})) < \infty$. From $D_1(b, f(b)) \leq K\ D_1(a, b)$, there exists $a_{n+1} \in I^{a_n}_{D,\beta}(f)$ such that

$$D_1(a_n, f(a_n)) \leq K\ D_1(a_n, a_{n+1}).$$

While $a_{n+1} \in I^{a_n}_{D,\beta}(f)$, then $a_{n+1} \in f(a_n)$ and

$$\beta\ D_1(a_n, a_{n+1}) \leq D_1(a_n, f(a_n)) < \infty.$$

Then, we impose

$$\beta\ D_1(a_{n+1}, a_{n+2}) \leq D_1(a_n, f(a_{n+1})) \leq K\ D_1(a_n, a_{n+1}),$$

which is, while $a_{n+1} \in f(a_n)$,

$$\beta\ D_1(a_{n+1}, f(a_{n+1})) \leq D_1(a_n, f(a_{n+1})) \leq K\ D_1(a_n, f(a_n)).$$

Then, we show

$$D_1(a_{n+1}, f(a_{n+1})) \leq \frac{K}{\beta}\ D_1(a_n, f(a_n))$$

for $\frac{K}{\beta} < 1$ for any $a \in X_D$,

$$D_1(a_{n+1}, f(a_{n+1})) \leq \left(\frac{K}{\beta}\right)^n D_1(a_0, a_1),$$

while $\sum_{n=1}^{\infty} D_1(a_n, a_{n+1}) < \infty$ and $\{a_n\}$ is D-strongly Cauchy and X_D is D-strongly complete; then

$$0 = \lim_{n\to\infty} D_1(a_n, a_{n+1}) = \lim_{n\to\infty} D_1(a_n, f(a_n)).$$

$D_1(a, f(a))$ is D-lower semicontinuous,

$$0 \leq D_1(z, f(z)) \leq \liminf_{n\to\infty} D_1(a_n, f(a_n));$$

since $f(z) \in X_D$, then we impose $z \in f(z)$.

3.1.3 Application for Feng-Liu Theorem

One of the multivalued mapping applications for modular vector spaces is given by Alfuraidan et al. in [4]. They showed that a fixed-point theorem for uniformly Lipschitz mapping in modular vector spaces which in the modular sense has the uniform property of normal structure. They broadened their findings in the exponent variable space. Another important application of these is given in [38] by Padcharoen et al. They proved some fixed-point theorems in generalized metric spaces and applied the fixed-point theorems to demonstrate the presence and uniqueness of the solution to the ordinary differential equation (ODE), partial differential equation (PDEs), and fractional boundary value problems by using the generalized contraction.

Initial value problem is given in [11] for a non-homogeneous linear parabolic partial differential equation such as

$$f_t(a,t) = f_{aa}(a,t) + S(a,t,f(a,t),f_a(a,t)), -\infty < a < \infty, 0 < t \leq T,$$
$$f(a,0) = \phi(a) \geq 0$$

for same valued $a \in X_D$, where S is continuous and ϕ supposed to be continuously differentiable such that ϕ and ϕ' are bounded. A function $f = f(a, t)$ defined on $\mathcal{R} \times I = [0, T]$ by a solution of this problem, if we have I which fulfills the given next conditions:

(i) $f, f_t, f_a, f_{aa} \in C(\mathcal{R} \times I)$ while it denotes the space of all continuous real valued functions,
(ii) f, f_a are bounded $\in \mathcal{R} \times I$,
(iii) $f_t(a, t) = f_{aa}(a, t) + S(a, t, f(a, t), f_a(a, t)), (a, t) \in \mathcal{R} \times I$,
(iv) $f(a, 0) = \phi(a) \geq 0$ for all $a \in \mathcal{R}$,

The differential equation problem which is given below, is equivalent to the next integral equation problem such as:

$$f(a, t) = \int_{-\infty}^{\infty} K(a - \delta, t)\phi(\delta)d\delta + \int_0^t \int_{-\infty}^{\infty} K(a - \delta, t - u)S(\delta, u, f(\delta, u), f_a(\delta, u))d\delta du$$

for all $a \in \mathcal{R}$ and $0 < t \leq T$ where

$$K(a, t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-a^2}{4t}}.$$

This problem introduces a solution if and only if the equivalent problem just above has a solution. Let

$$B := \{f(a, t) : f, f_a \in C(\mathcal{R} \times I), ||f|| < \infty\}$$

where

$$||f|| := \sup_{a \in \mathcal{R}, t \in I} |f(a, t)| + \sup_{a \in \mathcal{R}, t \in I} |f_a(a, t)|.$$

Now, we have a function D_1 as

$$D_1(a, b) := \frac{1}{\lambda}\omega_1(a, b) = \frac{1}{\lambda^2}|a - b|$$

is a GMM on B. Clearly, the GMMS B_ω is a $D-$complete and separated from its generators.

Lower semicontinuity property is easy to show for Feng-Liu-type, while D_1 is a GMMS.

Theorem 7 *Let us take the problem*

$$f_t(a, t) = f_{aa}(a, t) + S(a, t, f(a, t), f_a(a, t)), -\infty < a < \infty, 0 < t \leq T,$$
$$f(a, 0) = \phi(a) \geq 0.$$

and suppose the following:

(i) *The function* $S(a,t,s,p)$ *is uniformly Hölder continuous in a and t for each compact subset of* $\mathcal{R} \times I$*, for* $c > 0$ *with* $|s| < c$ *and* $|p| < c$.

(ii) *There exists a constant* $c_S \leq T + 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}} \leq q$*, where* $q \in (0,1)$ *such that*

$$0 \leq \tfrac{1}{\lambda} S[(a,t,s_2,p_2) - S(a,t,s_1,p_1)]$$
$$c_S \leq [\tfrac{s_2 - s_1 + p_2 - p_1}{\lambda}]$$

for all $(s_1,p_1), (s_2,p_2) \in \mathcal{R} \times \mathcal{R}$ *with* $s_1 \leq s_2$ *and* $p_1 \leq p_2$*,*

(iii) *S is bounded for bounded s and p;*

Then it has a solution.

Proof Let us take $a \in B_\omega$ is a solving of the problem above, if and only if our answer is integral equivalent such as $a \in B_\omega$.

If we choose the graph G for $V(G) = B_\omega$ and the definition of $E(G) = \{(z,v) \in B_\omega \times B_\omega : z(a,t) \leq v(a,t) \text{ and } z_a(a,t) \leq v_a(a,t) \text{ for each } (a,t) \in \mathcal{R} \times I\}$ when $E(G)$ is partially ordered and $(B_\omega, E(G))$ provides property (A).

The mapping $\Omega : B_\omega \to B_\omega$ defined as

$$f(u(a,t)) := \int_{-\infty}^{\infty} K(a-\delta,t)\phi(\delta)d\delta + \int_0^t \int_{-\infty}^{\infty} K(a-\delta,t-u)S(\delta,u,f(\delta,u),f_a(\delta,u))d\delta du$$

for all $a \in \mathcal{R}$ and when the solution gives us the existence of fixed point of f of the problem.

Since $(z,v), (z_a,v_a), (f(z),f(v)), (f(z_a),f(v_a)) \in E(G)$ and from the definition of f and (ii)

$$\tfrac{1}{\lambda}|f(v(a,t)) - f(z(a,t))| \leq c_S D_1(z,v).$$

Then, we impose

$$\begin{aligned}\tfrac{1}{\lambda}|f_a(v(a,t)) - f_a(z(a,t))| &\leq c_S D_1(z,v) \textstyle\int_{-\infty}^{\infty} K(a-\delta,t)\phi(\delta)d\delta \\ &\leq 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}} c_S D_1(z,v).\end{aligned}$$

In the end, all the solutions gathered as

$$\begin{aligned} D_1(f(z),f(v)) &\leq (T + 2\pi^{-\frac{1}{2}} T^{\frac{1}{2}})\ c_S\ D_1(z,v) \\ D_1(f(z),f(v)) &\leq c\ D_1(z,v),\ c \in (0,1) \\ |f(z) - f(v)| &\leq \lambda^2\ |z-v|,\ \lambda \in (0,1). \end{aligned}$$

From Feng-Liu's viewpoint, we impose

$$\begin{aligned}
\lambda^2|v - f(v)| &\leq \lambda^2 H_d(f(z), f(v)) \leq \lambda^2\,|z - v| \\
d(v, f(v)) &\leq H_d(f(z), f(v)) \quad \leq d(z, v) \\
D_1(v, f(v)) &\leq H_D(f(z), f(v)) \quad \leq D_1(z, v)
\end{aligned}$$

then we have $b\, D_1(z, v) \leq D_1(z, f(z))$, while $b \in (0, 1)$. Then, we have results that there exists a $z^\star \in B_\omega$ so that $z^\star = f(z^\star)$. So this is the result of our problem.

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On Some Fixed Point Results in Various Types of Modular Metric Spaces

Mahpeyker Öztürk and Ekber Girgin

Abstract This study aims to introduce a new structure in the setting of non-Archimedean modular metric space structure called the generalized orthogonal $F_\varphi-$contraction in the sense of Suzuki and to prove some of the consequences obtained as a result of using this structure in fixed point theory. Also, graphical fixed point theorems are obtained as an application of these results. Since the orthogonal $F_\varphi-$contraction in the sense of Suzuki, which was put forward as a new idea in the first part of this study, is not applicable in non-Archimedean quasi modular metric spaces defined by the authors in [25], a new modification is made, and as a result, the new fixed point theorems to which the contraction is applicable are examined and various results are presented.

1 Introduction

The Banach contraction principle [1], which forms one of the main structures of the metric fixed point theory, which has emerged as a rich field of study, has been generalized and developed in various aspects in terms of its lucidity and simplicity in its application areas and is still among the structures that make progress today. Among these generalizations, one of the most productive ones is the study presented by Wardowski [2] in 2012. In his study, he obtained a new contraction using an increasing function and stated that this contraction generalized Banach's work, and also he proved some fixed point theorems. This study was later taken as a reference by many researchers and new studies have been made and are still being done.

Definition 1 [2] Let (M, d) be a metric space and $F : (0, \infty) \to (-\infty, +\infty)$ be a mapping satisfying

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- (F_1) F is strictly increasing,
- (F_2) For each sequence $\{\alpha_n\}_{n\in N} \subset (0,\infty)$, $\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\alpha_n) = -\infty$,
- (F_3) There exists $k \in (0,1)$ such that $\lim_{n\to\infty} \alpha^k F(\alpha) = 0$.

We say that $Z : M \to M$ is a $F-$contraction if there exists $\tau > 0$ such that for all $j, l \in M$,

$$d(Zj, Zl) > 0 \quad \text{implies} \quad \tau + F(d(Zj, Zl)) \le F(d(j,l)).$$

Besides, Wardowski [3] introduced the generalized F-contraction in his study presented in 2018 as noted below.

Definition 2 Let (M, d) be a metric space and $Z : M \to M$ be a self-mapping. For some functions $F : (0,\infty) \to (-\infty, +\infty)$ and $\varphi : (0,\infty) \to (0,\infty)$, we say that Z is $(\varphi, F)-$contraction if the followings hold:

- $(\hbar_1)$ F satisfies (F_1) and (F_2),
- $(\hbar_2)$ $\liminf_{s\to t^+} \varphi(s) > 0$ for all $t \ge 0$,
- $(\hbar_3)$ $\varphi(d(j,l)) + F(d(Zj, Zl)) \le F(d(j,l))$ for all $j, l \in M$ such that $Zj \ne Zl$.

After the fixed point results put forward by Wardowski, many researchers improved and generalized the $F-$contraction in abstract spaces [4–9].

Gordji et al. [10] added a new dimension to existing studies in the literature and defined the term called the orthogonal Banach contraction and presented new results.

Definition 3 [10] Let $\perp \subseteq M \times M$ be a binary relation defined on a non-void set M. The relation $\perp$ satisfying

$$\exists j_0 \ [(\forall l \in M,\ l \perp j_0,) \quad \text{or} \quad (\forall l \in M,\ j_0 \perp l)],$$

is named an orthogonal set and j_0 is an orthogonal element. The set M with the relation $\perp$ is an orthogonal set and we point out this orthogonal set by $(M_\perp)$.

Definition 4 [10] $\{j_p\}$ is named an orthogonal sequence if

$$(\forall p \in N,\ j_p \perp j_{p+1}) \quad \text{or} \quad (\forall p \in N,\ j_{p+1} \perp j_p).$$

Definition 5 [10] $Z : M_\perp \to M_\perp$ is called an orthogonal continuous if $j_p \to j$ as $p \to \infty$, we have $Zj_p \to Zj$ as $p \to \infty$.

Definition 6 [10] Let $(M_\perp)$ be an orthogonal set. $Z : M_\perp \to M_\perp$ is a $\perp-$preserving if $Zj \perp Zl$ whenever $j \perp l$.

Definition 7 [10] Let $(M_\perp)$ be an orthogonal set. $Z : M_\perp \to M_\perp$ is a $\perp-$transitive if $j \perp l$ and $l \perp k$ imply $l \perp k$.

Definition 8 [10] In an orthogonal set $(M_{\perp})$, the finite sequence $x_0, x_1, x_2, ..., x_r \subseteq M$ such that

$$x_0 = j, x_r = l, \quad x_p \perp x_{p+1} \quad or \quad x_{p+1} \perp x_p,$$

for all $p = 0, 1, 2, ..., r - 1$, is named as a path of length r in $\perp$ from j to l.

Orthogonality property is investigated and generalized in some particular types of spaces; see, e.g., [11–13].
Suzuki [14] proved the following fixed point theorems.

Theorem 1 *[14] Let Z be a self-mapping defined on a compact metric space (M, d). Assume that, for all elements j and l belong to M with $j \neq l$,*

$$\frac{1}{2} d(j, Zj) < d(j, l) \quad \Rightarrow \quad d(Zj, Zl) < d(j, l).$$

Then Z holds a unique fixed point in M.

2 Fixed Point Results in non-Archimedean Modular Metric Space

In 2010, Chistyakov [15, 16] established a modular metric space, in short MMS which is an extension of metric space and modular linear space. For a non-void set M, let $\kappa : (0, \infty) \times M \times M \to [0, \infty]$ be a function; for simplicity, we will write

$$\kappa_\lambda(j, l) = \kappa(\lambda, j, l)$$

for all $\lambda > 0$ and $j, l \in M$.

Definition 9 [15] Let $\kappa : (0, \infty) \times M \times M \to [0, \infty]$ be a function on a non-void set satisfying the following statements for all $\lambda,\ \mu > 0$ and $j, k, l \in M$:

- $(\kappa_1.)$ $\quad j = l \Leftrightarrow \kappa_\lambda(j, l) = 0;$
- $(\kappa_2.)$ $\quad \kappa_\lambda(j, l) = \kappa_\lambda(l, j);$
- $(\kappa_3.)$ $\quad \kappa_{\lambda+\mu}(j, l) \leq \kappa_\lambda(j, k) + \kappa_\mu(k, l).$

Therefore, κ is called modular metric in M and so M_κ is MMS.
κ is regular if the following holds:

- $(\kappa_4.)$ $\quad j = l \Leftrightarrow \kappa_\lambda(j, l) = 0$ for some $\lambda > 0$.

Moreover, κ is a convex function if for $j, l, k \in M$, the following expression holds:

- $(\kappa_5.)$ $\quad \kappa_{\lambda+\mu}(j, l) \leq \frac{\lambda}{\lambda+\mu}\kappa_\lambda(j, k) + \frac{\mu}{\lambda+\mu}\kappa_\mu(k, l), \quad \lambda,\ \mu > 0.$

For all $\lambda,\ \mu > 0$ and $j, l, k \in M_\kappa$, if we replace $(\kappa_3.)$ with the below

- $(\kappa_6.)$ $\quad \kappa_{\max\{\lambda,\mu\}}(j, l) \leq \kappa_\lambda(j, k) + \kappa_\mu(k, l),$

then we have more general property. Hence the pair (M, κ) that provides the property $(\kappa_6.)$ instead of $(\kappa_3.)$ is a non-Archimedean modular metric space, in short non-AMMS, and is represented by M_κ.

Definition 10 [15] Let $S \subseteq M_\kappa$ and $\left(j_p\right)_{p\in N}$ be a sequence. Then,

- $(i.)$ $\left(j_p\right)_{p\in N}$ is $\kappa-$convergent to $p \in M_\kappa$ if and only if $\kappa_\lambda\left(j_p, j\right) \to 0$ as $p \to \infty$ for all $\lambda > 0$.
- $(ii.)$ $\left(j_p\right)_{p\in N}$ is $\kappa-$Cauchy if $\kappa_\lambda\left(j_p, j_t\right) \to 0$, as $p, t \to \infty$ for all $\lambda > 0$.
- $(iii.)$ S is $\kappa-$complete if any $\kappa-$Cauchy sequence is $\kappa-$convergent.

Definition 11 [17] Let M_κ be a MMS. $Z : M_\kappa \to M_\kappa$ is a $\kappa-$continuous when if $\kappa_\lambda\left(j_k, j\right) \to 0$, then $\kappa_\lambda\left(Zj_k, Zj\right) \to 0$ as $k \to \infty$.

Now, we establish a new contraction and generate new fixed point results and applications. We assume that the function F satisfies only condition (F_1) whereas the function φ has a property $(\hbar_2)$.

Definition 12 Let $M_{\kappa_\perp}$ be an orthogonally non-AMMS. A mapping $Z : M_{\kappa_\perp} \to M_{\kappa_\perp}$ is called a generalized orthogonal Suzuki $F_\varphi-$contraction if there exist $\varphi \in \Phi$ and $F \in \Upsilon$ such that

$$\frac{1}{2}\kappa_\lambda\left(j, Zj\right) < \kappa_\lambda\left(j, l\right) \;\Rightarrow\; \varphi\left(\kappa_\lambda\left(j, l\right)\right) + F\left(\kappa_\lambda\left(Zj, Zl\right)\right) \le F\left(P\left(j, i\right)\right), \quad (1)$$

$$P\left(j, l\right) = \max\left\{\kappa_\lambda\left(j, l\right), \kappa_\lambda\left(j, Zj\right), \kappa_\lambda\left(l, Zl\right), \frac{\kappa_\lambda\left(j, Zl\right) + \kappa_\lambda\left(l, Zj\right)}{2}, \right.$$
$$\left. \frac{\kappa_\lambda\left(Z^2 j, j\right) + \kappa_\lambda\left(Z^2 j, Zl\right)}{2}, \kappa_\lambda\left(Z^2 j, Zj\right), \kappa_\lambda\left(Z^2 j, l\right), \kappa_\lambda\left(Z^2 j, Zl\right)\right\},$$

for all $j, l \in M_{\kappa_\perp}$.

Theorem 2 *Let $M_{\kappa_\perp}$ be a transitive orthogonally $\kappa-$complete non-AMMS and Z be a $\perp-$continuous, $\perp-$preserving and generalized orthogonal Suzuki $F_\varphi-$contraction. Then Z holds a fixed point in $M_{\kappa_\perp}$.*

Proof Since the set $M_{\kappa_\perp}$ has orthogonality property, there is a j_0 element in the set $M_{\kappa_\perp}$ that provides the following:

$$\left(\forall l \in M_{\kappa_\perp}, \quad j_0 \perp l\right) \quad \text{or} \quad \left(\forall l \in M_{\kappa_\perp}, \quad l \perp j_0\right).$$

It follows that $j_0 \perp Zj_0$ or $Zj_0 \perp j_0$. Let $j_{p+1} = Zj_p = Z^p j_0$ for all $p \in N$. For some $p^* \in N$, if $j_{p^*} = j_{p^*+1}$, then j_{p^*} is a fixed point of the mapping Z and this fact ends the proof. Accordingly, we presume that $j_p \neq j_{p+1}$ for all $p \in N$. In this way, we possess $\kappa_\lambda\left(j_p, j_{p+1}\right) > 0$ for some $p \in N$. Since Z is $\perp-$preserving, we have $\left(\forall p, \quad j_p \perp j_{p+1}\right)$ or $\left(\forall p, \quad j_{p+1} \perp j_p\right)$. Because of the generalized orthogonal Suzuki $F_\varphi-$contraction of Z, $\frac{1}{2}\kappa_\lambda\left(j_p, Zj_p\right) < \kappa_\lambda\left(j_p, Zj_p\right) = \kappa_\lambda\left(j_p, j_{p+1}\right)$ implies

$$F\left(\kappa_\lambda\left(j_p, j_{p+1}\right)\right) < \varphi\left(\kappa_\lambda\left(j_{p-1}, j_p\right)\right) + F\left(\kappa_\lambda\left(j_p, j_{p+1}\right)\right) \leq F\left(P\left(j_{p-1}, j_p\right)\right),$$

where

$$\begin{aligned} P\left(j_{p-1}, j_p\right) = \max\Big\{&\kappa_\lambda\left(j_{p-1}, j_p\right), \kappa_\lambda\left(j_{p-1}, Zj_{p-1}\right), \kappa_\lambda\left(j_p, Zj_p\right), \\ &\frac{\kappa_\lambda\left(j_{p-1}, Zj_p\right) + \kappa_\lambda\left(j_p, Zj_{p-1}\right)}{2}, \frac{\kappa_\lambda\left(Z^2 j_{p-1}, j_{p-1}\right) + \kappa_\lambda\left(Z^2 j_{p-1}, Zj_p\right)}{2}, \\ &\kappa_\lambda\left(Z^2 j_{p-1}, Zj_p\right), \kappa_\lambda\left(Z^2 j_{p-1}, Zj_p\right), \kappa_\lambda\left(Z^2 j_{p-1}, Zj_p\right)\Big\} \\ = \max\Big\{&\kappa_\lambda\left(j_{p-1}, j_p\right), \kappa_\lambda\left(j_{p-1}, j_p\right), \kappa_\lambda\left(j_p, j_{p+1}\right), \\ &\frac{\kappa_\lambda\left(j_{p-1}, j_{p+1}\right) + \kappa_\lambda\left(j_p, j_p\right)}{2}, \frac{\kappa_\lambda\left(j_{p+1}, j_{p-1}\right) + \kappa_\lambda\left(j_{p+1}, j_{p+1}\right)}{2}, \\ &\kappa_\lambda\left(j_{p+1}, j_{p+1}\right), \kappa_\lambda\left(j_{p+1}, j_{p+1}\right), \kappa_\lambda\left(j_{p+1}, j_{p+1}\right)\Big\} \\ = \max&\left\{\kappa_\lambda\left(j_{p-1}, j_p\right), \kappa_\lambda\left(j_p, j_{p+1}\right), \frac{\kappa_\lambda\left(j_{p-1}, j_{p+1}\right)}{2}\right\} \\ = \max&\left\{\kappa_\lambda\left(j_{p-1}, j_p\right), \kappa_\lambda\left(j_p, j_{p+1}\right), \frac{\kappa_{\max\{\lambda,\lambda\}}\left(j_{p-1}, j_{p+1}\right)}{2}\right\} \\ \leq \max&\left\{\kappa_\lambda\left(j_{p-1}, j_p\right), \kappa_\lambda\left(j_p, j_{p+1}\right), \frac{\kappa_\lambda\left(j_{p-1}, j_p\right) + \kappa_\lambda\left(j_p, j_{p+1}\right)}{2}\right\} \\ = \max&\left\{\kappa_\lambda\left(j_{p-1}, j_p\right), \kappa_\lambda\left(j_p, j_{p+1}\right)\right\}. \end{aligned}$$

We get

$$\begin{aligned} F\left(\kappa_\lambda\left(j_p, j_{p+1}\right)\right) &< \varphi\left(\kappa_\lambda\left(j_{p-1}, j_p\right)\right) + F\left(\kappa_\lambda\left(j_p, j_{p+1}\right)\right) \\ &\leq F\left(\max\left\{\kappa_\lambda\left(j_{p-1}, j_p\right), \kappa_\lambda\left(j_p, j_{p+1}\right)\right\}\right). \end{aligned}$$

It is clear that $\max\left\{\kappa_\lambda\left(j_{p-1}, j_p\right), \kappa_\lambda\left(j_p, j_{p+1}\right)\right\} = \kappa_\lambda\left(j_{p-1}, j_p\right)$. Otherwise, we have $\kappa_\lambda\left(j_p, j_{p+1}\right) < \kappa_\lambda\left(j_p, j_{p+1}\right)$, which is a contradiction. Hence, we obtain

$$F\left(\kappa_\lambda\left(j_p, j_{p+1}\right)\right) < \varphi\left(\kappa_\lambda\left(j_{p-1}, j_p\right)\right) + F\left(\kappa_\lambda\left(j_p, j_{p+1}\right)\right) \leq F\left(\kappa_\lambda\left(j_{p-1}, j_p\right)\right), \tag{2}$$

for all $p \in N$. Based on (F_1), the relation (2) means $\kappa_\lambda\left(j_p, j_{p+1}\right) < \kappa_\lambda\left(j_p, j_{p+1}\right)$, i.e., $\lim_{k\to\infty} \kappa_\lambda\left(j_p, j_{p+1}\right) = r \geq 0$. Assume that $r > 0$. Then by (H_2) follows that there exists $\tau > 0$ and $p_1 \in N$ such that for all $p \geq p_1$

$$\tau + F\left(\kappa_\lambda\left(j_p, j_{p+1}\right)\right) \leq \varphi\left(\kappa_\lambda\left(j_{p-1}, j_p\right)\right) + F\left(\kappa_\lambda\left(j_p, j_{p+1}\right)\right)$$

$$\leq F\left(\kappa_\lambda\left(j_{p-1}, j_p\right)\right),$$

i.e.,

$$\tau + F\left(\kappa_\lambda\left(j_p, j_{p+1}\right)\right) \leq F\left(\kappa_\lambda\left(j_{p-1}, j_p\right)\right).$$

From the last taking limit as $p \to \infty$, it follows

$$\tau + F\left(r + 0\right) \leq F\left(r + 0\right),$$

a contradiction. Therefore, we attain

$$\lim_{p\to\infty} \kappa_\lambda\left(j_p, j_{p+1}\right) = 0. \tag{3}$$

Now, we express that $\left\{j_p\right\}$ is a $\kappa-$Cauchy sequence by deeming contrary. Then for $\varepsilon > 0$ it is possible to find two subsequences $\{p_s\}$ and $\{t_s\}$ of positive integers with the property $k_s > t_s \geq s$ such the following inequalities hold:

$$\kappa_\lambda\left(j_{p_s}, j_{t_s}\right) \geq \varepsilon, \quad \text{and} \quad \kappa_\lambda\left(j_{p_s}, j_{t_s-1}\right) < \varepsilon. \tag{4}$$

From (4) and (κ_6), it follows that

$$\begin{aligned}\varepsilon \leq \kappa_\lambda\left(j_{p_s}, j_{t_s}\right) &= \kappa_{\max\{\lambda,\lambda\}}\left(j_{p_s}, j_{t_s}\right)\\ &\leq \kappa_\lambda\left(j_{p_s}, j_{t_s-1}\right) + \kappa_\lambda\left(j_{t_s-1}, j_{t_s}\right)\\ &< \varepsilon + \kappa_\lambda\left(j_{t_s-1}, j_{t_s}\right).\end{aligned} \tag{5}$$

As $s \to \infty$ in above relation, we attain that

$$\lim_{s\to\infty} \kappa_\lambda\left(j_{p_s}, j_{t_s}\right) = \varepsilon. \tag{6}$$

Also,

$$\begin{aligned}\kappa_\lambda\left(j_{p_s}, j_{t_s}\right) &= \kappa_{\max\{\lambda,\lambda\}}\left(j_{p_s}, j_{t_s}\right)\\ &\leq \kappa_\lambda\left(j_{p_s}, j_{p_s+1}\right) + \kappa_\lambda\left(j_{p_s+1}, j_{t_s}\right)\\ &= \kappa_\lambda\left(j_{p_s}, j_{p_s+1}\right) + \kappa_{\max\{\lambda,\lambda\}}\left(j_{p_s+1}, j_{t_s}\right)\\ &\leq \kappa_\lambda\left(j_{p_s}, j_{p_s+1}\right) + \kappa_\lambda\left(j_{p_s+1}, j_{t_s+1}\right) + \kappa_\lambda\left(j_{t_s+1}, j_{t_s}\right),\end{aligned} \tag{7}$$

and

$$\begin{aligned}\kappa_\lambda\left(j_{p_s+1}, j_{t_s+1}\right) &= \kappa_{\max\{\lambda,\lambda\}}\left(j_{p_s+1}, j_{t_s+1}\right)\\ &\leq \kappa_\lambda\left(j_{p_s+1}, j_{p_s}\right) + \kappa_\lambda\left(j_{p_s}, j_{t_s+1}\right)\\ &= \kappa_\lambda\left(j_{p_s+1}, j_{p_s}\right) + \kappa_{\max\{\lambda,\lambda\}}\left(j_{p_s}, j_{t_s+1}\right)\\ &\leq \kappa_\lambda\left(j_{p_s+1}, x_{p_s}\right) + \kappa_\lambda\left(j_{p_s}, j_{t_s}\right) + \kappa_\lambda\left(j_{t_s}, j_{t_s+1}\right). \end{aligned} \tag{8}$$

Using (4) and (6), by taking limit as $s \to \infty$ in (7) and (8), we deduce that

$$\lim_{s\to\infty} \kappa_\lambda\left(j_{p_s+1}, j_{t_s+1}\right) = \varepsilon. \tag{9}$$

Moreover, from (4) and (κ_6), it follows that

$$\begin{aligned}\kappa_\lambda\left(j_{p_s}, j_{t_s+1}\right) &= \kappa_{\max\{\lambda,\lambda\}}\left(j_{p_s}, j_{t_s+1}\right)\\ &\leq \kappa_\lambda\left(j_{p_s}, j_{t_s-1}\right) + \kappa_\lambda\left(j_{t_s-1}, j_{t_s+1}\right)\\ &= \kappa_\lambda\left(j_{p_s}, j_{t_s-1}\right) + \kappa_{\max\{\lambda,\lambda\}}\left(j_{t_s-1}, j_{t_s+1}\right)\\ &\leq \kappa_\lambda\left(j_{p_s}, j_{t_s-1}\right) + \kappa_\lambda\left(j_{t_s-1}, j_{t_s}\right) + \kappa_\lambda\left(j_{t_s}, j_{t_s+1}\right)\\ &< \kappa_\lambda\left(j_{t_s-1}, j_{t_s}\right) + \kappa_\lambda\left(j_{t_s}, j_{t_s+1}\right) + \varepsilon \end{aligned} \tag{10}$$

and

$$\begin{aligned}\kappa_\lambda\left(j_{t_s}, j_{p_s+1}\right) &= \kappa_{\max\{\lambda,\lambda\}}\left(j_{t_s}, j_{p_s+1}\right)\\ &\leq \kappa_\lambda\left(j_{t_s}, j_{p_s}\right) + \kappa_\lambda\left(j_{p_s}, j_{t_s+1}\right)\\ &= \kappa_\lambda\left(j_{p_s}, j_{p_s+1}\right) + \kappa_{\max\{\lambda,\lambda\}}\left(j_{t_s}, j_{p_s}\right)\\ &\leq \kappa_\lambda\left(j_{p_s}, j_{p_s+1}\right) + \kappa_\lambda\left(j_{t_s}, j_{t_s-1}\right) + \kappa_\lambda\left(j_{t_s-1}, j_{p_s}\right)\\ &< \varepsilon + \kappa_\lambda\left(j_{t_s}, j_{t_s-1}\right) + \kappa_\lambda\left(j_{p_s}, j_{p_s+1}\right). \end{aligned} \tag{11}$$

Next, we claim that

$$\frac{1}{2}\kappa_\lambda\left(j_{p_s}, Zj_{p_s}\right) \leq \kappa_\lambda\left(j_{p_s}, j_{t_s}\right). \tag{12}$$

If

$$\frac{1}{2}\kappa_\lambda\left(j_{p_s}, Zj_{p_s}\right) > \kappa_\lambda\left(j_{p_s}, j_{t_s}\right), \tag{13}$$

then taking limit $s \to \infty$ in (13), we have $0 > \varepsilon$ is a contradiction. Hence,

$$\frac{1}{2}\kappa_\lambda\left(j_{p_s}, Zj_{p_s}\right) \leq \kappa_\lambda\left(j_{p_s}, j_{t_s}\right).$$

Also as $M_{\kappa_\perp}$ is a transitive orthogonal set, we obtain

$$\left(\forall s, \quad j_{p_s} \perp j_{t_s}\right) \quad \text{or} \quad \left(\forall s, \quad j_{t_s} \perp j_{p_s}\right).$$

Since Z is generalized orthogonal Suzuki $F_{\varphi}-$contraction,

$$\begin{aligned} F\left(\kappa_{\lambda}\left(j_{p_s+1}, j_{t_s+1}\right)\right) &< \varphi\left(\kappa_{\lambda}\left(j_{p_s}, j_{t_s}\right)\right) + F\left(\kappa_{\lambda}\left(j_{p_s+1}, j_{t_s+1}\right)\right) \\ &= \varphi\left(\kappa_{\lambda}\left(j_{p_s}, j_{t_s}\right)\right) + F\left(\kappa_{\lambda}\left(Zj_{p_s}, Zj_{t_s}\right)\right) \\ &\leq F\left(P\left(j_{p_s}, j_{t_s}\right)\right), \end{aligned} \tag{14}$$

where

$$\begin{aligned} P\left(j_{p_s}, j_{t_s}\right) = \max\Big\{&\kappa_{\lambda}\left(j_{p_s}, j_{t_s}\right), \kappa_{\lambda}\left(j_{p_s}, j_{p_s+1}\right), \kappa_{\lambda}\left(j_{t_s}, j_{t_s+1}\right), \\ &\frac{\kappa_{\lambda}\left(j_{p_s}, j_{t_s+1}\right) + \kappa_{\lambda}\left(j_{t_s}, j_{p_s+1}\right)}{2}, \frac{\kappa_{\lambda}\left(j_{p_s+2}, j_{p_s}\right) + \kappa_{\lambda}\left(j_{p_s+2}, j_{t_s+1}\right)}{2}, \\ &\kappa_{\lambda}\left(j_{p_s+2}, j_{p_s+1}\right), \kappa_{\lambda}\left(j_{p_s+2}, j_{t_s}\right), \kappa_{\lambda}\left(j_{p_s+2}, j_{t_s+1}\right)\Big\} \\ \leq \max\Big\{&\kappa_{\lambda}\left(j_{p_s}, j_{t_s}\right), \kappa_{\lambda}\left(j_{p_s}, j_{p_s+1}\right), \kappa_{\lambda}\left(j_{t_s}, j_{t_s+1}\right), \\ &\frac{\kappa_{\lambda}\left(j_{p_s}, j_{t_s+1}\right) + \kappa_{\lambda}\left(j_{t_s}, j_{p_s+1}\right)}{2}, \\ &\frac{\kappa_{\lambda}\left(j_{p_s+2}, j_{p_s+1}\right) + \kappa_{\lambda}\left(j_{p_s+1}, j_{p_s}\right) + \kappa_{\lambda}\left(j_{p_s+2}, j_{p_s+1}\right) + \kappa_{\lambda}\left(j_{p_s+1}, j_{t_s+1}\right)}{2}, \\ &\kappa_{\lambda}\left(j_{p_s+2}, j_{p_s+1}\right), \kappa_{\lambda}\left(j_{p_s+2}, j_{p_s+1}\right) + \kappa_{\lambda}\left(j_{p_s+1}, j_{t_s}\right), \\ &\kappa_{\lambda}\left(j_{p_s+2}, j_{p_s+1}\right) + \kappa_{\lambda}\left(j_{p_s+1}, j_{t_s+1}\right)\Big\}. \end{aligned} \tag{15}$$

Taking limit as $s \to \infty$ in (10), (11), (13), (14) and (15), also from (3), (6) and (9), we get

$$F(\varepsilon) < \varphi(\varepsilon) + F(\varepsilon) \leq F(\varepsilon),$$

which yields a contradiction. Herewith, $\left\{j_p\right\}_{p\in N}$ is a $\kappa-$Cauchy sequence. Since $M_{\kappa_\perp}$ orthogonal $\kappa-$complete non-AMMS, there exists $u \in M_{\kappa_\perp}$ such that $\lim_{p\to\infty} \kappa_{\lambda}\left(j_p, u\right) = 0$. Also, because Z is $\perp-$continuous, we get $\lim_{p\to\infty} \kappa_{\lambda}\left(Zj_p, Zu\right) = 0$. Thus, $\lim_{p\to\infty} \kappa_{\lambda}\left(Zj_p, Zu\right) = \lim_{p\to\infty} \kappa_{\lambda}\left(j_{p+1}, Zu\right) = 0$. Due to the uniqueness of limit, we get $Zu = u$, i.e., u is a fixed point of Z. Now, we prove u is a unique fixed point. Let v is another fixed point of Z. From the choosing of j_0 in the first part of proof, we have

$$(j_0 \perp v) \quad \text{or} \quad (v \perp j_0).$$

Because Z is $\perp-$preserving and $M_{\kappa_\perp}$ is a transitive orthogonal set, we obtain

$$(u \perp v) \quad \text{or} \quad (v \perp u).$$

Also, we have

$$0 = \frac{1}{2}\kappa_{\lambda}(u, Zu) \leq \kappa_{\lambda}(u, v).$$

By the generalized orthogonal Suzuki $F_{\varphi}-$contraction of Z, we get

$$\begin{aligned}F(\kappa_{\lambda}(u,v)) &= F(\kappa_{\lambda}(Zu,Zv)) \\ &< \varphi(\kappa_{\lambda}(u,v)) + F(\kappa_{\lambda}(Zu,Zv)) \\ &\le F(P(u,v)),\end{aligned}$$

where

$$\begin{aligned}P(u,v) = \max\{&\kappa_{\lambda}(u,v), \kappa_{\lambda}(u,Zu), \kappa_{\lambda}(v,Zv), \\ &\frac{\kappa_{\lambda}(u,Zv)+\kappa_{\lambda}(v,Zu)}{2}, \frac{\kappa_{\lambda}\left(Z^{2}u,u\right)+\kappa_{\lambda}\left(Z^{2}u,Zv\right)}{2}, \\ &\kappa_{\lambda}\left(Z^{2}u,Zu\right), \kappa_{\lambda}\left(Z^{2}u,v\right), \kappa_{\lambda}\left(Z^{2}u,Zv\right)\} \\ &= \kappa_{\lambda}(u,v),\end{aligned}$$

that is a contradiction and so u is a unique fixed point of Z.

By replacing $\perp-$continuity of Z with continuity of F, we attain subsequent results.

Theorem 3 *Let $M_{\kappa_{\perp}}$ be a transitive orthogonally $\kappa-$complete non-AMMS and Z be a generalized orthogonal Suzuki $F_{\varphi}-$contraction. Suppose that following conditions hold:*

(i.) Z is $\perp-$preserving mapping,
(ii.) F is a continuous function.

Then Z holds a unique fixed point in $M_{\kappa_{\perp}}$.

Proof From the definition of orthogonality, there exists $j_0 \in M_{\kappa_{\perp}}$ such that

$$\left(\forall l \in M_{\kappa_{\perp}},\quad j_0 \perp l\right) \quad \text{or} \quad \left(\forall l \in M_{\kappa_{\perp}},\quad l \perp j_0\right).$$

Then from Theorem 2, we get $\{j_p\}$ is a $\kappa-$Cauchy sequence in $M_{\kappa_{\perp}}$. The $\kappa-$completeness of the space $M_{\kappa_{\perp}}$ indicates the existence of an element u that belongs to $M_{\kappa_{\perp}}$ provides $\lim_{p\to\infty} \kappa_{\lambda}\left(j_p,u\right) = 0$. Now, it should be shown that u is a fixed point of Z. Suppose that $\kappa_{\lambda}(Zu,u) > 0$. We assert that for each $p \in N$

$$\frac{1}{2}\kappa_{\lambda}\left(j_p,Zj_p\right) \le \kappa_{\lambda}\left(j_p,u\right).$$

On the contrary, suppose that

$$\frac{1}{2}\kappa_\lambda\left(j_p, Zj_p\right) > \kappa_\lambda\left(j_p, u\right)$$
$$= \frac{1}{2}\kappa_\lambda\left(j_p, j_{p+1}\right) > \kappa_\lambda\left(j_p, u\right).$$

Taking limit $p \to \infty$, we get a contradiction. Thus claim is true. Also, since $M_{\kappa_\perp}$ is transitive and Z is $\perp-$preserving, we have

$$j_p \perp u \quad \text{or} \quad u \perp j_p.$$

From the generalized orthogonal Suzuki $F_\varphi-$contraction, we get

$$\begin{aligned} F\left(\kappa_\lambda\left(j_{p+1}, Zu\right)\right) &< \varphi\left(\kappa_\lambda\left(j_p, u\right)\right) + F\left(\kappa_\lambda\left(Zj_p, Zu\right)\right) \\ &\leq F\left(P\left(j_p, u\right)\right), \end{aligned} \tag{16}$$

where

$$\begin{aligned} P\left(j_p, u\right) = \max\Big\{&\kappa_\lambda\left(j_p, u\right), \kappa_\lambda\left(j_p, Zj_p\right), \kappa_\lambda\left(u, Zu\right), \\ &\frac{\kappa_\lambda\left(j_p, Zu\right) + \kappa_\lambda\left(u, Zj_p\right)}{2}, \frac{\kappa_\lambda\left(Z^2 j_p, j_p\right) + \kappa_\lambda\left(Z^2 j_p, Zu\right)}{2}, \\ &\kappa_\lambda\left(Z^2 j_p, Zj_p\right), \kappa_\lambda\left(Z^2 j_p, u\right), \kappa_\lambda\left(Z^2 j_p, Zu\right)\Big\} \\ \leq \max\Big\{&\kappa_\lambda\left(j_p, u\right), \kappa_\lambda\left(j_p, j_{p+1}\right), \kappa_\lambda\left(u, Zu\right), \\ &\frac{\kappa_\lambda\left(j_p, Zu\right) + \kappa_\lambda\left(u, j_{p+1}\right)}{2}, \\ &\frac{\kappa_\lambda\left(j_{p+2}, j_{p+1}\right) + \kappa_\lambda\left(j_{p+1}, j_p\right) + \kappa_\lambda\left(j_{p+2}, j_{p+1}\right) + \kappa_\lambda\left(j_{p+1}, Zu\right)}{2}, \\ &\kappa_\lambda\left(j_{p+2}, j_{p+1}\right), \kappa_\lambda\left(j_{p+2}, j_{p+1}\right) + \kappa_\lambda\left(j_{p+1}, u\right), \\ &\kappa_\lambda\left(j_{p+2}, j_{p+1}\right) + \kappa_\lambda\left(j_{p+1}, Zu\right)\Big\}. \end{aligned} \tag{17}$$

Taking limit $k \to \infty$ in above, we get

$$\begin{aligned} F\left(\kappa_\lambda\left(u, Zu\right)\right) &< \liminf_{\kappa_\lambda(j_k, u)\to 0^+} \varphi\left(\kappa_\lambda\left(j_k, u\right)\right) + F\left(\kappa_\lambda\left(u, Zu\right)\right) \\ &\leq F\left(\kappa_\lambda\left(u, Zu\right)\right). \end{aligned}$$

The attained last inequality causes a contradiction. Thus, u is a fixed point of Z. Also, as in Theorem 2, u is a unique fixed point of Z in $M_{\kappa_\perp}$.

Next, by omitting Suzuki's condition, we achieve some consequences as noted below.

Corollary 1 *Let $M_{\kappa_\perp}$ be a transitive orthogonally $\kappa-$complete non-AMMS. $Z : M_{\kappa_\perp} \to M_{\kappa_\perp}$ satisfies the following conditions:*

- $(i.)$ *Z is $\perp-$preserving mapping,*
- $(ii.)$ *Z is $\perp-$continuous or F is a continuous function,*

- $(iii.)$ *there exist* $\varphi \in \Phi$ *and* $F \in \Upsilon$ *such that*

$$\varphi\left(\kappa_\lambda\left(j,l\right)\right)+F\left(\kappa_\lambda\left(Zj,Zl\right)\right)\leq F\left(P\left(j,i\right)\right), \tag{18}$$

$$P\left(j,l\right)=\max\left\{\kappa_\lambda\left(j,l\right),\kappa_\lambda\left(j,Zj\right),\kappa_\lambda\left(l,Zl\right),\frac{\kappa_\lambda\left(j,Zl\right)+\kappa_\lambda\left(l,Zj\right)}{2},\right.$$
$$\left.\frac{\kappa_\lambda\left(Z^2j,j\right)+\kappa_\lambda\left(Z^2j,Zl\right)}{2},\kappa_\lambda\left(Z^2j,Zj\right),\kappa_\lambda\left(Z^2j,l\right),\kappa_\lambda\left(Z^2j,Zl\right)\right\},$$

for all $j,l\in M_{\kappa_\perp}$.

Then Z *holds a unique fixed point in* $M_{\kappa_\perp}$.

Corollary 2 *Let* $M_{\kappa_\perp}$ *be a transitive orthogonally* $\kappa-$*complete non-AMMS and* $Z:M_{\kappa_\perp}\to M_{\kappa_\perp}$ *satisfies the following conditions:*

- $(i.)$ Z *is* $\perp-$*preserving mapping,*
- $(ii.)$ Z *is* $\perp-$*continuous or* F *is continuous,*
- $(iii.)$ *there exist* $\varphi \in \Phi$ *and* $F \in \Upsilon$ *such that*

$$\varphi\left(\kappa_\lambda\left(j,l\right)\right)+F\left(\kappa_\lambda\left(Zj,Zl\right)\right)\leq F\left(\kappa_\lambda\left(j,i\right)\right), \tag{19}$$

for all $j,l\in M_{\kappa_\perp}$.

Then Z *holds a unique fixed point in* $M_{\kappa_\perp}$.

Corollary 3 *Let* $M_{\kappa_\perp}$ *be a transitive orthogonally* $\kappa-$*complete non-AMMS and* $Z:M_{\kappa_\perp}\to M_{\kappa_\perp}$ *satisfies the following conditions:*

- $(i.)$ Z *is* $\perp-$*preserving mapping,*
- $(ii.)$ Z *is* $\perp-$*continuous or* F *is continuous,*
- $(iii.)$ *there exist* $\varphi \in \Phi$ *and* $F \in \Upsilon$ *such that*

$$\frac{1}{2}\kappa_\lambda\left(j,Zj\right)\leq\kappa_\lambda\left(j,l\right)\quad\Rightarrow\quad\varphi\left(\kappa_\lambda\left(j,l\right)\right)+F\left(\kappa_\lambda\left(Zj,Zl\right)\right)\leq F\left(\kappa_\lambda\left(j,i\right)\right), \tag{20}$$

for all $j,l\in M_{\kappa_\perp}$.

Then Z *holds a unique fixed point in* $M_{\kappa_\perp}$.

Corollary 4 *Let* $M_{\kappa_\perp}$ *be a transitive orthogonally* $\kappa-$*complete non-AMMS and* $Z:M_{\kappa_\perp}\to M_{\kappa_\perp}$ *satisfies the following conditions:*

- $(i.)$ Z *is* $\perp-$*preserving mapping,*
- $(ii.)$ Z *is* $\perp-$*continuous or* F *is continuous,*

- $(iii.)$ *there exist* $\tau > 0$ *and* $F \in \Upsilon$ *such that*

$$\frac{1}{2}\kappa_\lambda(j, Zj) \leq \kappa_\lambda(j,l) \quad \Rightarrow \quad \tau + F(\kappa_\lambda(Zj, Zl)) \leq F(\kappa_\lambda(j,i)), \tag{21}$$

for all $j, l \in M_{\kappa_\perp}$.

Then Z *holds a unique fixed point in* $M_{\kappa_\perp}$.

Corollary 5 *Let* $M_{\kappa_\perp}$ *be a transitive orthogonally* $\kappa-$*complete non-AMMS and* $Z : M_{\kappa_\perp} \to M_{\kappa_\perp}$ *satisfies the following conditions:*

- $(i.)$ Z *is* $\perp-$*preserving mapping,*
- $(ii.)$ Z *is* $\perp-$*continuous or* F *is continuous,*
- $(iii.)$ *there exist* $\tau > 0$ *and* $F \in \Upsilon$ *such that*

$$\tau + F(\kappa_\lambda(Zj, Zl)) \leq F(\kappa_\lambda(j,i)), \tag{22}$$

for all $j, l \in M_{\kappa_\perp}$.

Then Z *holds a unique fixed point in* $M_{\kappa_\perp}$.

Now, we establish new fixed point theorems involving a graph.

The use of graph in metric fixed point theory was established by Jachmyski [18]. You can find more detail about graph theory in [18]. Many mathematicians proved some fixed point results improving Jachymski's technique in [19–24].

Now, inspired by [8, 14, 25], we constitute a contractive condition and prove fixed point results via Jachymski's technique.

Definition 13 Let $M_{\kappa_\perp}$ be an orthogonally non-AMMS. $Z : M_{\kappa_\perp} \to M_{\kappa_\perp}$ is a generalized orthogonal Suzuki $(\varphi, F)_G-$contraction if the following conditions hold:

- $(G_1.)$ Z preserves edges of G,
- $(G_2.)$ Z is $\perp-$preserving,
- $(G_3.)$ there exist $\varphi \in \Phi$ and $F \in \Upsilon$ such that

$$\frac{1}{2}\kappa_\lambda(j, Zj) < \kappa_\lambda(j,l) \quad \Rightarrow \quad \varphi(\kappa_\lambda(j,l)) + F(\kappa_\lambda(Zj, Zl)) \leq F(P(j,i)), \tag{23}$$

$$P(j,l) = \max\left\{\kappa_\lambda(j,l), \kappa_\lambda(j,Zj), \kappa_\lambda(l,Zl), \frac{\kappa_\lambda(j,Zl) + \kappa_\lambda(l,Zj)}{2}, \right.$$
$$\left. \frac{\kappa_\lambda\left(Z^2j, j\right) + \kappa_\lambda\left(Z^2j, Zl\right)}{2}, \kappa_\lambda\left(Z^2j, Zj\right), \kappa_\lambda\left(Z^2j, l\right), \kappa_\lambda\left(Z^2j, Zl\right)\right\},$$

for all $(j,l) \in E(\tilde{G})$.

Definition 14 Let $M_{\kappa_\perp}$ be an orthogonal non-AMMS. $Z : M_{\kappa_\perp} \to M_{\kappa_\perp}$ is a $G_\perp-$ continuous if given $j \in M$ and sequence $\{j_p\}$,

- $(i.)$ $\lim_{p\to\infty} \kappa_\lambda \left(j_p, j\right) = 0,$
- $(ii.)$ $\left(j_p, j_{p+1}\right) \in E(G)$ and $\left(j_p \perp j_{p+1} \text{ or } j_{p+1} \perp j_p\right)$ for $p \in N$ imply

$Zj_p \to Zj.$

Theorem 4 *Let $M_{\kappa_\perp}$ be a transitive orthogonally $\kappa-$complete non-AMMS endowed with a graph G and Z be a generalized orthogonal Suzuki $(\varphi, F)_{\tilde{G}}-$contraction. Assume that following statements hold:*

- $(i.)$ *Z is $\perp-$preserving mapping,*
- $(ii.)$ *G is a weakly connected graph,*
- $(iii.)$ *Z is $G_\perp-$continuous mapping,*
- $(iv.)$ *there exists $j_0 \in M_{Z_\perp}$.*

Then Z holds a unique fixed point in $M_{\kappa_\perp}$.

Proof Let $\{j_p\}_{p\in N}$ be a sequence in $M_{\kappa_\perp}$ by

$$j_{p+1} = Zj_p,$$

for all $p \in N$. Let j_0 be a given point in $M_{Z_\perp}$. Because Z preserves edge of $\tilde{G}$ and $\perp-$preserving, we get $\left(j_p, j_{p+1}\right) \in E\left(\widetilde{G}\right)$ and $j_p \perp j_{p+1}$. Then by Theorem 2, we get $\{j_p\}$ is a $\kappa-$Cauchy sequence. By the orthogonal $\kappa-$completeness of $M_{\kappa_\perp}$, there exists $u \in M_{\kappa_\perp}$ such that

$$\lim_{p\to\infty} \kappa_\lambda \left(j_p, u\right) = 0.$$

Moreover, if Z is $G_\perp-$continuous mapping, we get $\lim_{p\to\infty} \kappa_\lambda \left(Zj_p, Zu\right) = \lim_{p\to\infty} \kappa_\lambda \left(j_{p+1}, Zu\right) = 0$. Due to the uniqueness of the limit, we have $u = Zu$, i.e., u is a fixed point of Z. Now, we show that u is a unique fixed point of Z. Conversely, we assume that w is another fixed point of Z, i.e., $Zw = w$ and $u \neq w$. Then as G is weakly connected and $M_{\kappa_\perp}$ is transitive, we get $(u, w) \in E\left(\tilde{G}\right)$ and $(u \perp w \text{ or } w \perp u)$. Furthermore,

$$0 = \frac{1}{2}\kappa_\lambda (u, Zu) < \kappa_\lambda (u, w).$$

From the generalized orthogonal Suzuki $(\varphi, F)_{\tilde{G}}-$contraction, we have

$$\begin{aligned} F(\kappa_\lambda (u, w)) &= F(\kappa_\lambda (Zu, Zw)) \\ &< \varphi (\kappa_\lambda (u, w)) + F(\kappa_\lambda (Zu, Zw)) \\ &\leq F(P(u, w)), \end{aligned}$$

where

$$\begin{aligned} P(u, w) = \max\{&\kappa_\lambda(u, w), \kappa_\lambda(u, Zu), \kappa_\lambda(w, Zw), \\ &\tfrac{\kappa_\lambda(u,Zw)+\kappa_\lambda(w,Zu)}{2}, \tfrac{\kappa_\lambda(Z^2u,u)+\kappa_\lambda(Z^2u,Zw)}{2}, \\ &\kappa_\lambda\left(Z^2u, Zu\right), \kappa_\lambda\left(Z^2u, w\right), \kappa_\lambda\left(Z^2u, Zw\right)\} \\ &= \kappa_\lambda(u, w). \end{aligned}$$

Thus, we get $F(\kappa_\lambda(u, w)) < F(\kappa_\lambda(u, w))$, which is a contradiction and so u is a unique fixed point of Z.

Corollary 6 *Let $M_{\kappa_\perp}$ be a transitive orthogonally $\kappa-$complete non-AMMS with a graph G. $Z : M_{\kappa_\perp} \to M_{\kappa_\perp}$ satisfies the following conditions:*

- $(i.)$ *Z is $\perp-$preserving mapping,*
- $(ii.)$ *G is a weakly connected graph,*
- $(iii.)$ *Z is $G_\perp-$continuous mapping,*
- $(iv.)$ *there exists $j_0 \in M_{Z_\perp}$,*
- $(v.)$ *there exist $\varphi \in \Phi$ and $F \in \Upsilon$ such that*

$$\frac{1}{2}\kappa_\lambda(j, Zj) < \kappa_\lambda(j, l) \quad \Rightarrow \quad \varphi(\kappa_\lambda(j, l)) + F(\kappa_\lambda(Zj, Zl)) \le F(\kappa_\lambda(j, i)), \tag{24}$$

for all $(j, l) \in E(\tilde{G})$.

Then Z holds a unique fixed point in $M_{\kappa_\perp}$.

Corollary 7 *Let $M_{\kappa_\perp}$ be a transitive orthogonally $\kappa-$complete non-AMMS with a graph G. $Z : M_{\kappa_\perp} \to M_{\kappa_\perp}$ satisfies the following conditions:*

- $(i.)$ *Z is $\perp-$preserving mapping,*
- $(ii.)$ *G is a weakly connected graph,*
- $(iii.)$ *Z is $G_\perp-$continuous mapping,*
- $(iv.)$ *there exists $j_0 \in M_{Z_\perp}$,*
- $(v.)$ *there exist $\tau > 0$ and $F \in \Upsilon$ such that*

$$\frac{1}{2}\kappa_\lambda(j, Zj) < \kappa_\lambda(j, l) \quad \Rightarrow \quad \tau + F(\kappa_\lambda(Zj, Zl)) \le F(\kappa_\lambda(j, i)), \tag{25}$$

for all $(j, l) \in E(\tilde{G})$.

Then Z holds a unique fixed point in $M_{\kappa_\perp}$.

Corollary 8 *Let $M_{\kappa_\perp}$ be a transitive orthogonally $\kappa-$complete non-AMMS endowed with a graph G and $Z : M_{\kappa_\perp} \to M_{\kappa_\perp}$ be a mapping. Suppose that following conditions hold:*

- $(i.)$ *Z is $\perp-$preserving mapping,*
- $(ii.)$ *G is a weakly connected graph,*

- *(iii.)* *Z is $G_\perp-$continuous mapping,*
- *(iv.)* *there exists $j_0 \in M_{Z_\perp}$,*
- *(v.)* *there exist $\varphi \in \Phi$ and $F \in \Upsilon$ such that*

$$\varphi\left(\kappa_\lambda\left(j,l\right)\right)+F\left(\kappa_\lambda\left(Zj,Zl\right)\right)\leq F\left(\kappa_\lambda\left(j,i\right)\right), \tag{26}$$

for all $(j,l) \in E(\tilde{G})$.

Then Z holds a unique fixed point in $M_{\kappa_\perp}$.

3 Fixed Point Results in Non-Archimedean Quasi Modular Metric Space

The authors in [25] introduced a quasi modular metric space, shortly indicated as QMMS, and proved some fixed point theorems for the mappings using rational expressions. Also in [26], the modified Suzuki-simulation type contractions have been identified and fixed point theorems have been examined in the context of non-AQMMS.

Definition 15 [25] A function $E:(0,\infty)\times M\times M\to[0,\infty]$ is named a quasi modular metric if the followings hold for all $m,\ n>0$ and $j,l,k\in M$:

- $(q_1.)$ $j=l \Leftrightarrow E_m\left(j,l\right)=0,$
- $(q_2.)$ $E_{m+n}\left(j,l\right)\leq E_m\left(j,k\right)+E_n\left(k,l\right).$

Then, M_E is a QMMS.

E is regular if the following holds:

- $(q_3.)$ $j=l \Leftrightarrow E_m\left(j,l\right)=0$ for some $m>0$.

Again, E is named convex if the inequality holds:

- $(q_4.)$ $E_{m+n}\left(j,l\right)\leq \frac{m}{m+n}E_m\left(j,k\right)+\frac{n}{m+n}E_n\left(k,l\right).$

Definition 16 [25] In Definition 15, if we exchange (q_2) by

- $(q_5.)$ $E_{\max\{m,n\}}\left(j,l\right)\leq E_m\left(j,k\right)+E_n\left(k,l\right).$

Then, M_Q is a non-Archimedean quasi modular metric space, in short non-AQMMS.

Note that every non-AQMMS is a QMMS.

Remark 1 [25] From the above, we conclude that

- $(i.)$ A conjugate quasi modular metric E^{-1} of E is introduced by $E_m^{-1}\left(j,l\right)=E_m\left(l,j\right).$
- $(ii.)$ A function E^H denoted by $E^H=E^{-1}\vee E$, that is, $E_m^H\left(j,l\right)=\max\left\{E_m\left(j,l\right),E_m\left(l,j\right)\right\}$, defines a modular metric.

Definition 17 [25]
A sequence $\{j_p\}_{p\in N}$ in M_E converges to $j \in M_E$ and is called

- $(i.)E-$convergent or left convergent if $j_p \to j \Leftrightarrow E_m(j, j_p) \to 0$, as $p \to \infty$.
- $(ii.)E^{-1}-$convergent or right convergent if $j_p \to j \Leftrightarrow E_m(j_p, j) \to 0$, as $p \to \infty$.
- $(iii.)E^H-$convergent if $E_m(j, j_p) \to 0$ and $E_m(j_p, j) \to 0$, as $p \to \infty$.

Definition 18 [25] A sequence $\{j_p\}_{p\in N}$ in M_E is named

- $(i.)$ left(right) E_K-Cauchy if for every $\varepsilon > 0$ there exists $p_\varepsilon \in N$ such that $E_m(j_r, j_p) < \varepsilon$ for all $p, r \in N$ with $p_\varepsilon \le r \le p$ $(p_\varepsilon \le p \le r)$ and for all $m > 0$.
- $(ii.)$ E^H-Cauchy if for every $\varepsilon > 0$ there exists $p_\varepsilon \in N$ such that $E_m(j_p, j_r) < \varepsilon$ for all $p, r \in N$ with $p, r \ge p_\varepsilon$.

Remark 2 [25] From the above, we conclude that $\{j_p\}_{p\in N}$

- $(i.)$ is left E_K-Cauchy to E if and only if it is right E_K-Cauchy to E^{-1}.
- $(ii.)$ is E^H-Cauchy if and only if it is left and right E_K-Cauchy.

Definition 19 [25] M_E is said to be a

- $(i.)$ left E_K-complete if every left E_K-Cauchy is $E-$convergent.
- $(ii.)$ $E-$Smyth-complete if every left E_K-Cauchy sequence is E^H- convergent.

Throughout this section, E is regular and convex. Also, we modify the generalized orthogonal Suzuki $F_\varphi-$contraction in non-AQMMS.

Definition 20 Let $M_{E_\perp}$ be a non-AQMMS. $Z : M_{E_\perp} \to M_{E_\perp}$ is the modified generalized orthogonal Suzuki $F_\varphi-$contraction if there exist

$$\frac{1}{2}E_m(j, Zj) < E_m(j, l) \;\Rightarrow\; \varphi(E_m(j, l)) + F(E_m(Zj, Zl)) \le F(S(j, i)),$$
$$S(j, l) = \max\{E_m(j, l), E_m(j, Zj), E_m(l, Zl)\} \tag{27}$$

such that $\varphi \in \Phi$ and $F \in \Upsilon$ for all $j, l \in M_{E_\perp}$.

Theorem 5 *Let $M_{E_\perp}$ be a transitive orthogonally $E-$Smyth-complete non-AQMMS and Z be a $\perp-$continuous, $\perp-$preserving and modified generalized orthogonal Suzuki $F_\varphi-$contraction. Then Z holds a unique fixed point in $M_{E_\perp}$.*

Proof Define a sequence $\{j_p\}_{p\in N}$ in $M_{E_\perp}$ by $j_{p+1} = Zj_p$, for all $p \in N$. There exists $j_0 \in M_{E_\perp}$ such that

$$(\forall l \in M_{E_\perp}, \;\; j_0 \perp l) \quad \text{or,} \quad (\forall l \in M_{E_\perp}, \;\; l \perp j_0).$$

It follows that $j_0 \perp Zj_0$ or $Zj_0 \perp j_0$. If $j_{p^*} = j_{p^*+1}$ for some $p^* \in N$, then j_{p^*} is a fixed point of Z and so the proof is completed. Consequently, we shall suppose that

$j_p \neq j_{p+1}$ for all $p \in N$. Therefore, we have $E_m\left(j_p, j_{p+1}\right) > 0$ for all $p \in N$. Since Z is $\perp-$preserving, we have $\left(\forall p, \quad j_p \perp j_{p+1}\right)$ or $\left(\forall p, \quad j_{p+1} \perp j_p\right)$. Because Z is a modified generalized orthogonal Suzuki $F_\varphi-$contraction,

$$\frac{1}{2} E_m\left(j_p, Z j_p\right) < E_m\left(j_p, Z j_p\right) = E_m\left(j_p, j_{p+1}\right)$$

implies

$$F\left(E_m\left(j_p, j_{p+1}\right)\right) < \varphi\left(E_m\left(j_{p-1}, j_p\right)\right) + F\left(E_m\left(j_p, j_{p+1}\right)\right) \leq F\left(S\left(j_{p-1}, j_p\right)\right),$$

where

$$\begin{aligned} S\left(j_{p-1}, j_p\right) &= \max\left\{E_m\left(j_{p-1}, j_p\right), E_m\left(j_{p-1}, Z j_{p-1}\right), E_m\left(j_p, Z j_p\right)\right\} \\ &= \max\left\{E_m\left(j_{p-1}, j_p\right), E_m\left(j_{p-1}, j_{p+1}\right)\right\}. \end{aligned}$$

We get

$$\begin{aligned} F\left(E_m\left(j_p, j_{p+1}\right)\right) &< \varphi\left(E_m\left(j_{p-1}, j_p\right)\right) + F\left(E_m\left(j_p, j_{p+1}\right)\right) \\ &\leq F\left(\max\left\{E_m\left(j_{p-1}, j_p\right), E_m\left(j_p, j_{p+1}\right)\right\}\right). \end{aligned}$$

It is clear that

$$\max\left\{E_m\left(j_{p-1}, j_p\right), E_m\left(j_p, j_{p+1}\right)\right\} = E_m\left(j_{p-1}, j_p\right).$$

Otherwise, we have $E_m\left(j_p, j_{p+1}\right) < E_m\left(j_p, j_{p+1}\right)$, which is a contradiction. Hence, we obtain

$$F\left(E_m\left(j_p, j_{p+1}\right)\right) < \varphi\left(E_m\left(j_{p-1}, j_p\right)\right) + F\left(E_m\left(j_p, j_{p+1}\right)\right) \leq F\left(E_m\left(j_{p-1}, j_p\right)\right), \tag{28}$$

for all $p \in N$. Based on (F_1), the relation (28) means

$$E_m\left(j_p, j_{p+1}\right) < E_m\left(j_p, j_{p+1}\right),$$

i.e., $\lim_{p \to \infty} E_m\left(j_p, j_{p+1}\right) = r \geq 0$. Assume that $r > 0$, then by $(\hbar_2)$, it follows that there exists $\tau > 0$ and $p_1 \in N$ such that for all $p \geq p_1$

$$\begin{aligned} \tau + F\left(E_m\left(j_p, j_{p+1}\right)\right) &\leq \varphi\left(E_m\left(j_{p-1}, j_p\right)\right) + F\left(E_m\left(j_p, j_{p+1}\right)\right) \\ &\leq F\left(E_m\left(j_{p-1}, j_p\right)\right), \end{aligned}$$

i.e.,

$$\tau + F\left(E_m\left(j_p, j_{p+1}\right)\right) \leq F\left(E_m\left(j_{p-1}, j_p\right)\right).$$

From the last taking limit as $r \to \infty$, it follows

$$\tau + F(r+0) \leq F(r+0),$$

which is a contradiction. Hence, we attain

$$\lim_{p \to \infty} E_m\left(j_p, j_{p+1}\right) = 0. \tag{29}$$

Now, we show that $\left\{j_p\right\}_{p \in N}$ is a left E_K-Cauchy sequence by supposing contrary. Then for $\varepsilon > 0$ there exist two subsequences of positive integers $\{p_s\}$ and $\{t_s\}$ satisfying $p_s > t_s \geq s$ such that the inequalities indicated below hold:

$$E_m\left(j_{p_s}, j_{t_s}\right) \geq \varepsilon, \quad \text{and} \quad E_m\left(j_{p_s}, j_{t_s-1}\right) < \varepsilon. \tag{30}$$

From (30) and (q_5), it follows that

$$\begin{aligned} \varepsilon \leq E_m\left(j_{p_s}, j_{t_s}\right) &= E_{\max\{m,m\}}\left(j_{p_s}, j_{t_s}\right) \\ &\leq E_m\left(j_{p_s}, j_{t_s-1}\right) + E_m\left(j_{t_s-1}, j_{t_s}\right) \\ &< \varepsilon + E_m\left(j_{t_s-1}, j_{t_s}\right). \end{aligned} \tag{31}$$

On taking limit as $s \to \infty$ in above relation, we obtain that

$$\lim_{s \to \infty} E_m\left(j_{p_s}, j_{t_s}\right) = \varepsilon. \tag{32}$$

Also,

$$\begin{aligned} E_m\left(j_{p_s}, j_{t_s}\right) &= E_{\max\{m,m\}}\left(j_{p_s}, j_{t_s}\right) \\ &\leq E_m\left(j_{p_s}, j_{p_s+1}\right) + E_m\left(j_{p_s+1}, j_{t_s}\right) \\ &= E_m\left(j_{p_s}, j_{p_s+1}\right) + E_{\max\{m,m\}}\left(j_{p_s+1}, j_{t_s}\right) \\ &\leq E_m\left(j_{p_s}, j_{p_s+1}\right) + E_m\left(j_{p_s+1}, j_{t_s+1}\right) + E_m\left(j_{t_s+1}, j_{t_s}\right) \end{aligned} \tag{33}$$

and

$$\begin{aligned} E_m\left(j_{p_s+1}, j_{t_s+1}\right) &= E_{\max\{m,m\}}\left(j_{p_s+1}, j_{t_s+1}\right) \\ &\leq E_m\left(j_{p_s+1}, j_{k_s}\right) + E_m\left(j_{p_s}, j_{t_s+1}\right) \\ &= E_m\left(j_{p_s+1}, j_{p_s}\right) + E_{\max\{m,m\}}\left(j_{p_s}, j_{t_s+1}\right) \\ &\leq E_m\left(j_{p_s+1}, j_{p_s}\right) + E_m\left(j_{p_s}, j_{t_s}\right) + E_m\left(j_{t_s}, j_{t_s+1}\right). \end{aligned} \tag{34}$$

Using (30) and (32), by taking limit as $s \to \infty$ in (33) and (34), we deduce that

$$\lim_{s \to \infty} E_m\left(j_{p_s+1}, j_{t_s+1}\right) = \varepsilon. \tag{35}$$

Next, we claim that

$$\frac{1}{2}E_m\left(j_{p_s}, Zj_{p_s}\right) \leq E_m\left(j_{p_s}, j_{t_s}\right). \tag{36}$$

If

$$\frac{1}{2}E_m\left(j_{p_s}, Zj_{p_s}\right) > E_m\left(j_{p_s}, j_{t_s}\right), \tag{37}$$

then taking limit $s \to \infty$ in (37), we have $0 > \varepsilon$ is a contradiction. Hence,

$$\frac{1}{2}E_m\left(j_{p_s}, Zj_{p_s}\right) \leq E_m\left(j_{p_s}, j_{t_s}\right).$$

Also as $M_{E_\perp}$ is a transitive orthogonal set, we obtain

$$\left(\forall s, \quad j_{p_s} \perp j_{t_s}\right) \text{ or } \left(\forall s, \quad j_{t_s} \perp j_{p_s}\right).$$

Since Z is the modified generalized orthogonal Suzuki $F_\varphi-$contraction,

$$\begin{aligned} F\left(E_m\left(j_{p_s+1}, j_{t_s+1}\right)\right) &< \varphi\left(E_m\left(j_{p_s}, j_{t_s}\right)\right) + F\left(E_m\left(j_{p_s+1}, j_{t_s+1}\right)\right) \\ &= \varphi\left(E_m\left(j_{p_s}, j_{t_s}\right)\right) + F\left(E_m\left(Zj_{p_s}, Zj_{t_s}\right)\right) \\ &\leq F\left(S\left(j_{p_s}, j_{t_s}\right)\right), \end{aligned} \tag{38}$$

where

$$\begin{aligned} S\left(j_{p_s}, j_{t_s}\right) &= \max\left\{E_m\left(j_{p_s}, j_{t_s}\right), E_m\left(j_{p_s}, Zj_{p_s}\right), E_m\left(j_{t_s}, Zj_{t_s}\right)\right\} \\ &= \max\left\{E_m\left(j_{p_s}, j_{t_s}\right), E_m\left(j_{p_s}, j_{p_s+1}\right), E_m\left(j_{t_s}, j_{t_s+1}\right)\right\}. \end{aligned} \tag{39}$$

Taking limit as $s \to \infty$ in (29), (32), (35), (38) and (39), then we get

$$F(\varepsilon) < \varphi(\varepsilon) + F(\varepsilon) \leq F(\varepsilon),$$

which gives a contradiction. Thus, $\left\{j_p\right\}_{p \in N}$ is a left E_K-Cauchy sequence. As $M_{E_\perp}$ orthogonal $E-$Smyth-complete non-AQMMS, there exists $u \in M_{E_\perp}$ such that

$$\lim_{p\to\infty} {E_m}^H\left(j_p, u\right) = 0.$$

Thus, we have

$$\lim_{p\to\infty} E_m\left(j_p, u\right) = 0 \quad \text{and} \quad \lim_{p\to\infty} E_m\left(u, j_p\right) = 0.$$

Now, we show that u is a fixed point of Z. Suppose that $E_m(Zu, u) > 0$. Since Z is $\perp-$continuous, we get $\lim_{p\to\infty} E_m\left(Zj_p, Zu\right) = 0$. Thus, $\lim_{p\to\infty} E_m\left(Zj_p, Zu\right) =$

$\lim_{p\to\infty} E_m\left(j_{p+1}, Zu\right) = 0$. Due to the uniqueness of limit, we get $Zu = u$, i.e., u is a fixed point of Z. Now, we prove u is a unique fixed point. Let w is another fixed point of Z. By our choice of j_0 in the first part of proof, we have

$$(j_0 \perp w) \quad \text{or} \quad (w \perp j_0).$$

As Z is $\perp-$preserving and $M_{E_\perp}$ is transitive orthogonal set, we obtain

$$(u \perp w) \quad \text{or} \quad (w \perp u).$$

Also, we have

$$0 = \frac{1}{2} E_m(u, Zu) \leq E_m(u, w).$$

Using the modified generalized orthogonal Suzuki $F_\varphi-$contraction of Z, we get

$$\begin{aligned} F(E_m(u, w)) = F(E_m(Zu, Zw)) \\ &< \varphi(E_m(u, w)) + F(E_m(Zu, Zw)) \\ &\leq F(S(u, w)), \end{aligned}$$

where

$$\begin{aligned} S(u, w) &= \max\{E_m(u, w), E_m(u, Zu), E_m(w, Zw)\} \\ &= \max\{E_m(u, w)\}. \end{aligned}$$

Thus, we have

$$F(E_m(u, w)) < F(E_m(u, w)),$$

which is a contradiction. Hence, u is a unique fixed point of Z.

Now, we give some consequences of our main results in this section.

Corollary 9 *Let $M_{E_\perp}$ be a transitive orthogonally $E-$Smyth-complete non-AQMMS and Z be a modified generalized orthogonal Suzuki $F_\varphi-$contraction. Suppose the statements noted below hold:*

- *(i.) Z is $\perp-$preserving mapping,*
- *(ii.) F is a continuous function.*

Then Z holds a unique fixed point $M_{E_\perp}$.

Corollary 10 *Let $M_{E_\perp}$ be a transitive orthogonally $E-$Smyth-complete non-AQMMS and $Z : M_{E_\perp} \to M_{E_\perp}$ satisfies the following:*

- *(i.) Z is $\perp-$preserving mapping,*
- *(ii.) Z is $\perp-$continuous mapping or F is a continuous function,*
- *(iii.) there exist $\varphi \in \Phi$ and $F \in \Upsilon$ such that*

$$\varphi\left(E_m\left(j,l\right)\right)+F\left(E_m\left(Zj,Zl\right)\right)\leq F\left(S\left(j,i\right)\right), \quad (40)$$

$$S\left(j,l\right)=\max\left\{E_m\left(j,l\right),E_m\left(j,Zj\right),E_m\left(l,Zl\right)\right\}$$

for all $j, l \in M_{E_\perp}$.

Then Z *holds a unique fixed point in* $M_{E_\perp}$.

Corollary 11 *Let* $M_{E_\perp}$ *be a transitive orthogonally* $E-$*Smyth-complete non-AQMMS and* $Z : M_{Q_\perp} \to M_{Q_\perp}$ *satisfies the following:*

- $(i.)$ Z *is* $\perp-$*preserving mapping,*
- $(ii.)$ Z *is* $\perp-$*continuous mapping or* F *is a continuous function,*
- $(iii.)$ *there exist* $\tau > 0$ *and* $F \in \Upsilon$ *such that*

$$\tau+F\left(E_m\left(Zj,Zl\right)\right)\leq F\left(S\left(j,i\right)\right), \quad (41)$$

$$S\left(j,l\right)=\max\left\{E_m\left(j,l\right),E_m\left(j,Zj\right),E_m\left(l,Zl\right)\right\}$$

for all $j, l \in M_{E_\perp}$.

Then Z *holds a unique fixed point in* $M_{E_\perp}$.

Corollary 12 *Let* $M_{E_\perp}$ *be a transitive orthogonally* $E-$*Smyth-complete non-AQMMS and* $Z : M_{E_\perp} \to M_{E_\perp}$ *satisfies the following:*

- $(i.)$ Z *is* $\perp-$*preserving mapping,*
- $(ii.)$ Z *is* $\perp-$*continuous mapping or* F *is a continuous function,*
- $(iii.)$ *there exist* $\tau > 0$ *and* $F \in \Upsilon$ *such that*

$$\tau+F\left(E_m\left(Zj,Zl\right)\right)\leq F\left(E_m\left(j,i\right)\right), \quad (42)$$

for all $j, l \in M_{E_\perp}$.

Then Z *holds a unique fixed point in* $M_{E_\perp}$.

4 Conclusions

In the first part of this study, we introduce the generalized orthogonal Suzuki $F_\varphi-$contraction, and using this notion, we prove fixed point results in orthogonal non-AMMS. Also, we attain new results in non-AMMS with a graph. Since our new contractive condition is not suitable to apply in the setting of orthogonal non-AQMMS, we need to modify this contractive condition. And then we achieve some fixed point results via modified contractive conditions which are filling the gap in the existing literature.

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On Parametric (b, θ)-Metric Space and Some Fixed Point Theorems

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Abstract The primary aim of this chapter is to introduce the notion of parametric (b, θ)-metric space as an extended form of parametric b-metric space and establish some theorems on the existence and uniqueness of fixed point for a class of admissible mapping, satisfying a certain contractive condition. Finally, the result obtained is applied to establish the existence of a solution of an integral equation.

1 Introduction

In the mathematical analysis, we are concerned inevitably with two basic concepts, namely, convergent of sequences and continuity of functions. Notice that these notions depend precisely on the distance between two points. The term metric (distance) between two abstract points plays a vital role in mathematical analysis and the related discipline of sciences, engineering and social sciences. The renowned French mathematician Maurice Fréchet (1878–1973) initiated the study of metric space in 1905. The notion of metric space has been extending with a continuous effort by many mathematicians in many directions. One such extension is b-metric space, which was introduced by Bakhtin [8] in 1989 (also see Czerwik [15], 1993). Recently, a new generalization of parametric metric space [18], namely, parametric b-metric space has been initiated by Hussain et al. [19] in 2015 and proved fixed point theorems of the almost weakly contractive condition and Geraghty type. On the other hand, Kamran et al. [20] introduced the concept of extended b-metric space as a generalization of b-metric space and proved fixed point theorems including an analog of Banach contraction principle. Nieto and Rodríguez-López ([24, 25]) generalized the results of Ran and Reurings [30] by weakening the conditions of continuity

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as well as monotonicity and applied to study periodic boundary value problems of first-order ordinary differential equations (for more details and recent development in fixed point theory, we refer to [1, 3, 4, 9, 14, 16, 18, 19, 24, 25, 30, 31], and references therein).

There are two aspects in this chapter: first, we discuss the notions of parametric (b, θ)-metric space and parametric α-admissibility of mapping and secondly using these notions, we establish theorems on the existence and uniqueness of fixed points with an application to an integral equation.

2 Preliminaries

Throughout the chapter, we shall denote $N =$ the set of natural numbers and $R =$ the set of real numbers.

Definition 1 [8, 15] The ordered paired (Ω, d_b) is called a b-metric space (in short, bMS) if $\Omega \neq \emptyset$ is an arbitrary set, $b \geq 1$ a real number and $d_b : \Omega \times \Omega \to [0, \infty)$ is a b-metric on Ω satisfying the following conditions: $\forall \eta, \rho, \sigma \in \Omega$,
$(d_b 1)$ $d_b(\eta, \rho) = 0$ iff $\eta = \rho$;
$(d_b 2)$ $d_b(\eta, \rho) = d_b(\rho, \eta)$;
$(d_b 3)$ $d_b(\eta, \rho) \leq b[d_b(\eta, \sigma) + d_b(\sigma, \rho)]$.

The condition $(d_b 3)$ reduces to triangular inequality if $b = 1$, but it does not true for $b > 1$. Hence, the class of b-metric spaces is an extended form of metric space in the broader sense.

Example 1 [2] Let (Ω, d) be a metric space, where $\Omega \neq \emptyset$ and $d_b(\eta, \rho) = [d(\eta, \rho)]^r, \forall \eta, \rho \in \Omega$. Then (Ω, d_b) is a bMS with $b = 2^{r-1}$, where $r > 1$ is a real number, but d_b is not a metric on Ω.

Note that the distance function d_b may not be continuous (see [13, 17]). For more results and examples on b-metric spaces, we refer to [2, 10, 13, 15, 17, 27] and references therein.

Definition 2 [20] The ordered paired (Ω, d_θ) is called an extended b-metric space (in short, $EbMS$) if $\Omega \neq \emptyset$ is an arbitrary set, $\theta : \Omega \times \Omega \to [1, \infty)$, and $d_b : \Omega \times \Omega \to [0, \infty)$ is an extended b-metric on Ω satisfying the following conditions: $\forall \eta, \rho, \sigma \in \Omega$,
$(d_\theta 1)$ $d_\theta(\eta, \rho) = 0$ iff $\eta = \rho$;
$(d_\theta 2)$ $d_\theta(\eta, \rho) = d_\theta(\rho, \eta)$;
$(d_\theta 3)$ $d_\theta(\eta, \rho) \leq \theta(\eta, \rho)[d_\theta(\eta, \sigma) + d_\theta(\sigma, \rho)]$.

Note that if $\theta(\eta, \rho) = b > 1, \forall \eta, \rho \in \Omega$, then $EbMS$ becomes a bMS. Therefore, every metric space is bMS and every bMS is $EbMS$, but the converse need not be true in general. For more results on $EbMS$ and references, we refer to [5–7, 20, 28].

Example 2 Consider $\Omega = R$ and define $d_\theta : \Omega \times \Omega \to [0, \infty)$ as: $d_\theta(\eta, \rho) = |\eta| + |\rho|,\ \eta \neq \rho$ and $d_\theta(\eta, \rho) = 0,\ \eta = \rho$, where $\theta(\eta, \rho) = 1 + |\eta| + |\rho|, \forall \eta, \zeta \in \Omega$. Then (Ω, d_θ) is an $EbMS$. However, for $\eta, \rho \in R \backslash \{0\}, \eta \neq \rho$, we have

$$\frac{d_\theta(\eta, \rho)}{d_\theta(\eta, 0) + d_\theta(0, \rho)} \leq 1 + |\eta| + |\rho| = \theta(\eta, \rho).$$

But $\sup_{\eta, \rho \in \Omega} \theta(\eta, \rho) = +\infty$, so it is impossible to find a finite $b = \theta(\eta, \rho) \geq 1$ satisfying $(d_\theta 3)$. Therefore, (Ω, d_θ) is not a bMS. Note that every finite $EbMS$ is obviously bMS.

Definition 3 [18] The ordered paired $(\Omega, \mathcal{P})$ is called a parametric metric space (in short, PMS) if $\Omega \neq \emptyset$ be a set, and $\mathcal{P} : \Omega^2 \times (0, \infty) \to [0, \infty)$ is a parametric metric satisfying the following conditions: $\forall\ \eta, \rho, \sigma \in \Omega$ and $\forall \tau > 0$,
$(\mathcal{P}1)$ $\mathcal{P}(\eta, \rho, \tau) = 0$, iff $\eta = \rho$;
$(\mathcal{P}2)$ $\mathcal{P}(\eta, \rho, \tau) = \mathcal{P}(\rho, \eta, \tau)$;
$(\mathcal{P}3)$ $\mathcal{P}(\eta, \rho, \tau) \leq \mathcal{P}(\eta, \sigma, \tau) + \mathcal{P}(\sigma, \rho, \tau)$.

Example 3 [18] Consider $\Omega \neq \emptyset$ is a set containing all continuous function $\eta : (0, \infty) \to R$ and define $\mathcal{P}(\eta, \rho, \tau) = |\eta(\tau) - \rho(\tau)|, \forall \tau > 0$. Then $(\Omega, \mathcal{P})$ is a PMS.

Example 4 [18] Consider $\Omega = [0, \infty)$ and define $\mathcal{P}(\eta, \rho, \tau) = \tau \max\{\eta, \rho\}, \eta \neq \rho$ and $\mathcal{P}(\eta, \rho, \tau) = 0, \eta = \rho, \forall \tau > 0$. Then $(\Omega, \mathcal{P})$ is a PMS.

Example 5 Consider (Ω, d) be a metric space with metric d, where $\Omega \neq \emptyset$ is a set. Define $\mathcal{P}(\eta, \rho, \tau) = \frac{d(\eta, \rho)}{\tau + d(\eta, \rho)}, \forall \tau > 0$. It is obvious that $\mathcal{P}$ satisfies $(\mathcal{P}1)$ and $(\mathcal{P}2)$. To verify $(\mathcal{P}3)$, let $\eta, \rho, \sigma \in \Omega$, we have

$$\begin{aligned}
\mathcal{P}(\eta, \sigma, \tau) + \mathcal{P}(\sigma, \rho, \tau) &= \frac{d(\eta, \sigma)}{\tau + d(\eta, \sigma)} + \frac{d(\sigma, \rho)}{\tau + d(\sigma, \rho)} \\
&\geq \frac{d(\eta, \sigma)}{\tau + d(\eta, \sigma) + d(\sigma, \rho)} + \frac{d(\sigma, \rho)}{\tau + d(\eta, \sigma) + d(\sigma, \rho)} \\
&= \frac{d(\eta, \sigma) + d(\sigma, \rho)}{\tau + d(\eta, \sigma) + d(\rho, \sigma)} = \frac{1}{\frac{\tau}{d(\eta, \sigma) + d(\sigma, \rho)} + 1} \\
&\geq \frac{1}{\frac{\tau}{d(\eta, \rho)} + 1} = \frac{d(\eta, \rho)}{\tau + d(\eta, \rho)} = \mathcal{P}(\eta, \rho, \tau).
\end{aligned}$$

It shows that $(\mathcal{P}3)$ is satisfied $\forall \tau > 0$. Thus $(\Omega, \mathcal{P})$ is a PMS.

Let $(\Omega, \mathcal{P})$ be a PMS and $c_0 \in \Omega$ and $\lambda > 0$, then $B(c_0, \lambda) = \{\eta \in \Omega : \mathcal{P}(c_0, \eta, \tau) < \lambda, \forall \tau > 0\}$ is called an open ball of radius λ centred at $c_0 \in \Omega$.

Remark 1 In parametric metric space $(\Omega, \mathcal{P})$, we say that $\mathcal{P}$ is continuous if it is continuous at all variables $\eta, \rho \in \Omega, \forall \tau > 0$. Note that $\mathcal{P}$ is a continuous function (Remark 9[18]).

Definition 4 [19] The ordered paired $(\Omega, \mathcal{P}_b)$ is called a parametric b-metric space (in short, $PbMS$) if $\Omega \neq \emptyset$ is a set, $b \geq 1$ is a real number and $\mathcal{P}_b : \Omega^2 \times (0, \infty) \to [0, \infty)$ is a parametric b-metric satisfying the following conditions: $\forall \eta, \rho, \sigma \in \Omega$ and $\forall \tau > 0$,
$(\mathcal{P}_b 1)$ $\mathcal{P}_b(\eta, \rho, \tau) = 0$, iff $\eta = \rho$;
$(\mathcal{P}_b 2)$ $\mathcal{P}_b(\eta, \rho, \tau) = \mathcal{P}_b(\rho, \eta, \tau)$;
$(\mathcal{P}_b 3)$ $\mathcal{P}_b(\eta, \rho, \tau) \leq b[\mathcal{P}_b(\eta, \sigma, \tau) + \mathcal{P}_b(\sigma, \rho, \tau)]$.

Clearly if $b = 1$, then $PbMS$ becomes a PMS and we remark that every PMS is $PbMS$, but the converse need not be true. Note that $\mathcal{P}_b$, for $b > 1$ may not be continuous (Example 1.7 [19]).

Example 6 [19] Consider $\Omega = [0, \infty)$ and define $\mathcal{P}_b(\eta, \rho, \tau) = \tau|\eta - \rho|^p$, $\forall \eta, \rho \in \Omega$ and $\forall \tau > 0$. Then $(\Omega, \mathcal{P}_b)$ is a $PbMS$ with constant $b = 2^p$, where $p \geq 1$.

Motivated by Kamran et al. [20], we define the notion of parametric (b, θ)-metric space.

Definition 5 The ordered paired $(\Omega, \mathcal{P}_\theta)$ is called a parametric (b, θ)-metric space (in short, $\mathcal{P}_\theta MS$) if $\Omega \neq \emptyset$ be a set, $\theta : \Omega^2 \times (0, \infty) \to [1, \infty)$ and $\mathcal{P}_\theta : \Omega^2 \times (0, \infty) \to [0, \infty)$ is a parametric (b, θ)-metric satisfying the following conditions: $\forall \eta, \rho, \sigma \in \Omega$ and $\forall \tau > 0$,
$(\mathcal{P}_\theta 1)$ $\mathcal{P}_\theta(\eta, \rho, \tau) = 0$, iff $\eta = \rho$;
$(\mathcal{P}_\theta 2)$ $\mathcal{P}_\theta(\eta, \rho, \tau) = \mathcal{P}_\theta(\rho, \eta, \tau)$;
$(\mathcal{P}_\theta 3)$ $\mathcal{P}_\theta(\eta, \rho, \tau) \leq \theta(\eta, \rho, \tau)[\mathcal{P}_\theta(\eta, \sigma, \tau) + \mathcal{P}_\theta(\sigma, \rho, \tau)]$.

If $\theta(\eta, \rho, \tau) = b \geq 1$, then $\mathcal{P}_\theta$ becomes $\mathcal{P}_b$. Note that every PMS is $PbMS$ and every $PbMS$ is $\mathcal{P}_\theta MS$. Recall that $\mathcal{P}_b$ with $b > 1$ is not a continuous function, so is $\mathcal{P}_\theta$. We discuss some examples on $\mathcal{P}_\theta MS$ as follows.

Example 7 Consider $\Omega = R$ and let $\mathcal{P}_\theta : \Omega^2 \times (0, \infty) \to [0, \infty)$ be defined by $\mathcal{P}_\theta(\eta, \rho, \tau) = \tau(\eta - \rho)^2$, where $\theta(\eta, \rho, \tau) = 2 + \tau(|\eta| + |\rho|)$, $\forall \eta, \rho \in \Omega$ and $\forall \tau > 0$. Then $(\Omega, \mathcal{P}_\theta)$ is a $\mathcal{P}_\theta MS$.

Example 8 Consider $\Omega = R$ and let $\theta : \Omega^2 \times (0, \infty) \to [1, \infty)$ be defined by $\theta(\eta, \rho, \tau) = 1 + \tau(|\eta| + |\rho|)$, $\forall \eta, \rho \in \Omega$ and $\forall \tau > 0$. Let $\mathcal{P}_\theta : \Omega^2 \times (0, \infty) \to [0, \infty)$ be given by $\mathcal{P}_\theta(\eta, \rho, \tau) = \tau(|\eta|^p + |\rho|^p)$, $\eta \neq \rho$ and $\mathcal{P}_\theta(\eta, \rho, \tau) = 0$, $\eta = \rho$ $\forall \tau > 0$, where $p \geq 1$. Then $(\Omega, \mathcal{P}_\theta)$ is a $\mathcal{P}_\theta MS$.

Example 9 Let $\theta : \Omega^2 \times (0, \infty) \to [1, \infty)$, where $\Omega = [0, 1]$ be a function defined by $\theta(\eta, \rho, \tau) = 2[\frac{1+\tau(\eta+\rho)}{\eta+\rho}]$, $\eta + \rho > 0$ and $\theta(0, 0, \tau) = 1, \forall \tau > 0$. Define $\mathcal{P}_\theta : \Omega^2 \times (0, \infty) \to [0, \infty)$ as

$$\mathcal{P}_\theta(\eta, \rho, \tau) = \frac{\tau}{\eta\rho}, \ \ \eta, \rho \in (0, 1], \eta \neq \rho;$$
$$\mathcal{P}_\theta(\eta, \rho, \tau) = 0, \ \ \eta = \rho;$$
$$\mathcal{P}_\theta(\eta, 0, \tau) = \mathcal{P}_\theta(0, \eta, \tau) = \frac{\tau}{\eta}, \ \ \eta \in (0, 1],$$

$\forall \tau > 0$. Clearly $(\mathcal{P}_\theta 1)$ and $(\mathcal{P}_\theta 2)$ are hold. For $(\mathcal{P}_\theta 3)$, we have the following cases:
(i) For $\eta, \rho, \sigma \in (0, 1]$, $\forall \tau > 0$, we have

$$\mathcal{P}_\theta(\eta, \rho, \tau) \leq \theta(\eta, \rho, \tau)[\mathcal{P}_\theta(\eta, \sigma, \tau) + \mathcal{P}_\theta(\sigma, \rho, \tau)]$$
$$\Longleftrightarrow \frac{\tau}{\eta\rho} \leq 2\frac{[1 + \tau(\eta + \rho)]}{(\eta + \rho)} \frac{\tau(\eta + \rho)}{\eta\rho\sigma} \Longleftrightarrow \sigma \leq 2[1 + \tau(\eta + \rho)].$$

(ii) For $\eta, \rho \in (0, 1]$ and $\sigma = 0$, $\forall \tau > 0$, we have

$$\mathcal{P}_\theta(\eta, \rho, \tau) \leq \theta(\eta, \rho, \tau)[\mathcal{P}_\theta(\eta, 0, \tau) + \mathcal{P}_\theta(0, \rho, \tau)]$$
$$\Longleftrightarrow \frac{\tau}{\eta\rho} \leq 2\frac{[1 + \tau(\eta + \rho)]}{\eta + \rho} \left(\frac{\tau}{\eta} + \frac{\tau}{\rho}\right) \Longleftrightarrow 1 \leq 2[1 + \tau(\eta + \rho)].$$

(iii) For $\eta, \sigma \in (0, 1]$ and $\rho = 0$, $\forall \tau > 0$, we have

$$\mathcal{P}_\theta(\eta, 0, \tau) \leq \theta(\eta, 0, \tau)[\mathcal{P}_\theta(\eta, \sigma, \tau) + \mathcal{P}_\theta(\sigma, 0, \tau)]$$
$$\Longleftrightarrow \frac{\tau}{\eta} \leq 2\frac{(1 + \tau\eta)}{\eta} \left(\frac{\tau}{\eta\sigma} + \frac{\tau}{\sigma}\right) \Longleftrightarrow \eta\sigma \leq 2(1 + \tau\eta)(1 + \eta).$$

It shows that $(\mathcal{P}_\theta 3)$ is satisfied. Thus $(\Omega, \mathcal{P}_\theta)$ is a $\mathcal{P}_\theta MS$.

Example 10 Consider $\Omega = l_p(R)$, $0 < p < 1$ where $l_p(R) = \{\{\rho_i\} \subseteq R : \sum_{i=1}^{\infty} |\rho_i|^p < \infty\}$. Define $\mathcal{P}_\theta(\rho, \sigma, \tau) = \left(\sum_{i=1}^{\infty} |\eta(\tau)\{\rho_i - \sigma_i\}|^p\right)^{\frac{1}{p}}$ with $\theta(\rho, \sigma, \tau) = 2^{\frac{1}{p}} + \tau(|\rho| + |\sigma|)$, where $0 < \eta(\tau) < \infty$, $\rho = \{\rho_i\}$, $\sigma = \{\sigma_i\} \in \Omega$ and $\forall \tau > 0$. Obviously, $(\mathcal{P}_\theta 1)$ and $(\mathcal{P}_\theta 2)$ are hold. For $(\mathcal{P}_\theta 3)$, let $\rho, \sigma, \omega \in \Omega$, where $\rho = \{\rho_i\}, \sigma = \{\sigma_i\}, \omega = \{\omega_i\} \in \Omega$, $\forall \tau > 0$, we obtain

$$\begin{aligned}
\left(\mathcal{P}_\theta(\rho, \sigma, \tau)\right)^p &= [\eta(\tau)]^p \sum_{i=1}^{\infty} |(\rho_i - \omega_i) + (\omega_i - \sigma_i)|^p \\
&\leq [\eta(\tau)]^p \sum_{i=1}^{\infty} \left(2 \max\left\{|\rho_i - \omega_i|, |\omega_i - \sigma_i|\right\}\right)^p \\
&\leq [2\eta(\tau)]^p \sum_{i=1}^{\infty} \left(|\rho_i - \omega_i|^p + |\omega_i - \sigma_i|^p\right) \\
&= 2^p \left(\sum_{i=1}^{\infty} |\eta(\tau)\{\rho_i - \omega_i\}|^p + \sum_{i=1}^{\infty} |\eta(\tau)\{\omega_i - \sigma_i\}|^p\right) \\
\Longrightarrow \mathcal{P}_\theta(\rho, \sigma, \tau) &\leq 2 \left(\sum_{i=1}^{\infty} u_i + \sum_{i=1}^{\infty} v_i\right)^{\frac{1}{p}},
\end{aligned}$$

setting with $u_i = |\eta(t)\{\rho_i - \omega_i\}|^p$ and $v_i = |\eta(t)\{\omega_i - \sigma_i\}|^p$. Since $0 < p < 1$ which is equivalent to $1 < \frac{1}{p} < \infty$. By convexity of $f(r) = r^{\frac{1}{p}}$, we have

$$
\begin{aligned}
& f\Big(\frac{r+s}{2}\Big) \leq \frac{1}{2}\Big(f(r)+f(s)\Big) \\
\Longrightarrow & \Big(\frac{r+s}{2}\Big)^{\frac{1}{p}} \leq \frac{1}{2}\Big(r^{\frac{1}{p}}+s^{\frac{1}{p}}\Big) \\
\Longrightarrow & (r+s)^{\frac{1}{p}} \leq 2^{\frac{1}{p}-1}\Big(r^{\frac{1}{p}}+s^{\frac{1}{p}}\Big).
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\mathcal{P}_\theta(\rho,\sigma,\tau) &\leq 2.2^{\frac{1}{p}-1}\left[\left(\sum_{i=1}^{\infty}u_i\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{\infty}v_i\right)^{\frac{1}{p}}\right] \\
&= 2^{\frac{1}{p}}\left[\left(\sum_{i=1}^{\infty}|\eta(\tau)\{\rho_i-\omega_i\}|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{\infty}|\eta(\tau)\{\omega_i-\sigma_i\}|^p\right)^{\frac{1}{p}}\right] \\
&\leq \theta(\rho,\sigma,\tau)\left[\mathcal{P}_\theta(\rho,\omega,\tau)+\mathcal{P}_\theta(\omega,\sigma,\tau)\right].
\end{aligned}
$$

Thus $(\Omega, \mathcal{P}_\theta)$ is a $\mathcal{P}_\theta MS$.

Example 11 Consider $\Omega = L_p[0,1]$, the set of all real function $f(u)$, $u \in [0,1]$ such that $\int_0^1 |f(u)|^p du < \infty$, where $0 < p < 1$. Define

$$
\mathcal{P}_\theta(f,g,\tau) = \left(\int_0^1 \left|\frac{f(u)-g(u)}{1+\tau}\right|^p du\right)^{\frac{1}{p}}
$$

and $\theta(f,g,\tau) = 2^{\frac{1}{p}} + \frac{1+\tau}{f+g}$, where $f+g>0$ and $\theta(0,0,\tau) = 2^{\frac{1}{p}}$, $\forall \tau > 0$. It is obvious that $(\mathcal{P}_\theta 1)$ and $(\mathcal{P}_\theta 2)$ are hold. For $(\mathcal{P}_\theta 3)$, let $f, g, h \in \Omega$, we have

$$
\begin{aligned}
[\mathcal{P}_\theta(f,g,\tau)]^p &= \frac{1}{(1+\tau)^p}\int_0^1 |f(u)-g(u)|^p du \\
&\leq \Big(\frac{2}{1+\tau}\Big)^p\left[\int_0^1 |f(u)-h(u)|^p du + \int_0^1 |h(u)-g(u)|^p du\right] \\
\Rightarrow \mathcal{P}_\theta(f,g,\tau) &\leq \frac{2}{1+\tau}\left[\int_0^1 |f(u)-h(u)|^p du + \int_0^1 |h(u)-g(u)|^p du\right]^{\frac{1}{p}}
\end{aligned}
$$

As in Example 10, we obtain

$$
\begin{aligned}
\mathcal{P}_\theta(f,g,\tau) &\leq 2^{\frac{1}{p}}\Big[\mathcal{P}_\theta(f,h,\tau)+\mathcal{P}_\theta(h,g,\tau)\Big] \\
&\leq \theta(f,g,\tau)\Big[\mathcal{P}_\theta(f,h,\tau)+\mathcal{P}_\theta(h,g,\tau)\Big],
\end{aligned}
$$

so $(\mathcal{P}_\theta 3)$ is satisfied. Thus $(\Omega, \mathcal{P}_\theta)$ is a $\mathcal{P}_\theta MS$.

Consider $\Omega \neq \emptyset$ be a set and let $(\Omega, \mathcal{P}_\theta)$ be a $\mathcal{P}_\theta MS$. Suppose $\mathcal{P}_\theta$ is a continuous function on Ω, $a \in \Omega$ and $r > 0$, we write

$$\mathcal{B}(a, r) = \{\eta \in \Omega : \mathcal{P}_\theta(a, \eta, \tau) < r, \forall \tau > 0\}.$$

Then we say that $\mathcal{B}(a, r)$ is an open ball of radius $r > 0$ centred at a.

Definition 6 Let $(\Omega, \mathcal{P}_\theta)$, where $\Omega \neq \emptyset$ be a $\mathcal{P}_\theta MS$ and $\{\eta_n\}$ be a sequence in Ω, then

(i) the sequence $\{\eta_n\}$ is said to be convergent to $\eta \in \Omega$ and symbolically, we write $\eta_n \to \eta$ as $n \to +\infty$ iff $\forall \tau > 0, \mathcal{P}_\theta(\eta_n, \eta, \tau) \to 0$ as $n \to +\infty$;

(ii) the sequence $\{\eta_n\}$ is said to be a Cauchy in Ω iff $\forall \tau > 0, \mathcal{P}_\theta(\eta_m, \eta_n, \tau) \to 0$ as $m, n \to +\infty$;

(iii) $(\Omega, \mathcal{P}_\theta)$ is said to be complete iff every Cauchy sequence $\{\eta_n\}$ in Ω is convergent.

Definition 7 Let $(\Omega, \mathcal{P}_\theta)$ be a $\mathcal{P}_\theta MS$ and $T : \Omega \to \Omega$ be a mapping. We say that T is continuous at $\eta \in \Omega$ if for any sequence $\{\eta_n\}$ in Ω such that $\eta_n \to \eta$ as $n \to +\infty$, we have $T\eta_n \to T\eta$ as $n \to +\infty$.

Example 12 Let $\Omega = [0, 1]$ and define $\mathcal{P}_\theta(\eta, \rho, \tau) = \tau(|\eta|^2 + |\rho|^p), \eta \neq \rho$ and $\mathcal{P}_\theta(\eta, \rho, \tau) = 0, \eta = \rho$, where $\theta(\eta, \rho, \tau) = 1 + \tau(|\eta| + |\rho|), \forall \eta, \rho \in \Omega$ and $\forall \tau >$. Suppose $T : \Omega \to \Omega$ be a mapping defined by $T\eta = \frac{\eta}{3}, \forall \eta \in \Omega$. For any $\eta_0 \in \Omega$, we define a sequence $\{\eta_n\}_{n \in N}$ in Ω such that $\eta_n = T^n \eta_0 = (\frac{1}{3})^n \eta_0$. Clearly, $\eta_n \to 0$ as $n \to +\infty$ and $T\eta_n = (\frac{1}{3})^{n+1}\eta_0 \to T0 = 0$ as $n \to +\infty$, i.e. $\lim_{n \to +\infty} \mathcal{P}_\theta(T\eta_n, T0, \tau) = 0$ whenever $\lim_{n \to +\infty} \mathcal{P}_\theta(\eta_n, 0, \tau) = 0$. Therefore, T is continuous at 0.

Throughout the following sections, we assume that $\mathcal{P}_\theta$ is a continuous function. Let $\Omega \neq \emptyset$ be a set. Suppose $T : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \to R$ are mappings. We denote $Fix(T) = \{\eta \in \Omega : T\eta = \eta\}$.

Definition 8 ([31]) A mapping $T : \Omega \to \Omega$ is said to be α-admissible if $\eta, \rho \in \Omega, \alpha(\eta, \rho) \geq 1 \Longrightarrow \alpha(T\eta, T\rho) \geq 1$.

Definition 9 [23] An α-admissible mapping T is said to be α^*-admissible if $\forall \eta, \eta^* \in Fix(T) \neq \emptyset, \alpha(\eta, \eta^*) \geq 1$.

Definition 10 [21] A mapping $T : \Omega \to \Omega$ is said to be triangular α-admissible if:
(T_1) T is an α-admissible;
(T_2) $\alpha(\eta, \rho) \geq 1$ and $\alpha(\rho, \sigma) \geq 1 \Longrightarrow \alpha(\eta, \sigma) \geq 1, \eta, \rho, \sigma \in \Omega$.

Definition 11 [4] A mapping $T : \Omega \to \Omega$ is said to be weak triangular α-admissible if:
(T_1) T is an α-admissible;
(T_3) $\alpha(\eta, T\eta) \geq 1 \Longrightarrow \alpha(\eta, T^2\eta) \geq 1$.

For the uniqueness of fixed point, Alsulami et al. [4] used the following hypothesis: Condition (B): For $\eta, \rho \in \Omega$, there exists $\sigma \in \Omega$ such that $\alpha(\eta, \sigma) \geq 1$ and $\alpha(\sigma, \rho) \geq 1$.

Definition 12 [29] A mapping $T : \Omega \to \Omega$ is said to be α-orbital-admissible if $\eta \in \Omega, \alpha(\eta, T\eta) \geq 1 \Longrightarrow \alpha(T\eta, T^2\eta) \geq 1$.

Definition 13 [29] A mapping $T : \Omega \to \Omega$ is said to be triangular α-orbital-admissible if:
(T_4) T is α-orbital-admissible;
(T_5) $\alpha(\eta, \rho) \geq 1$ and $\alpha(\rho, T\rho) \geq 1 \Longrightarrow \alpha(\eta, T\rho) \geq 1, \eta, \rho \in \Omega$.

For the uniqueness condition, many authors used the following hypothesis:

Definition 14 An α-orbital-admissible mapping T is said to be α^*-orbital-admissible if $\forall \eta, \eta^* \in Fix(T) \neq \emptyset, \alpha(\eta, \eta^*) \geq 1$.

Remark 2 In [29], Popescu remarked that every α-admissible mapping is α-orbital-admissible. However, it may be observed that every α-admissible mapping T is α-orbital-admissible if there exist $w \in \Omega$ and $r \in N \cup \{0\}$ such that $\alpha(\eta, \rho) = \alpha(T^r w, T^{r+1} w) \geq 1$. Moreover, in Definition 12, if $\rho = T\eta, \eta \in \Omega$, then $\alpha(\eta, \rho) \geq 1 \Longrightarrow \alpha(T\eta, T\rho) \geq 1$, that is, every α-orbital-admissible mapping T is α-admissible mapping.

Example 13 Let $\Omega = \{0, 1, 2\}$ with usual metric $d(\eta, \rho) = |\eta - \rho|$. Define $T : \Omega \to \Omega$ and $\alpha : \Omega \times \Omega \to R$ as: $T0 = 0, T1 = 2, T2 = 1$ and $\alpha(\eta, \rho) = 1$, if $(\eta, \rho) \in \{(0, 1), (0, 2)\}$ and $\alpha(\eta, \rho) = 0$ otherwise. Note that T is a α-admissible as $\alpha(0, 1) = \alpha(0, 2) = 1$, $\alpha(T0, T1) = \alpha(T0, T2) = 1$. But there does not exist $w \in \Omega$ such that $\alpha(\eta, \rho) = \alpha(T^r w, T^{r+1} w) = 1$, $r \in N \cup \{0\}$. So T is not an α-orbital-admissible mapping.

Example 14 Consider Ω and $T : \Omega \to \Omega$ are as in Example 13. Define $\alpha : \Omega \times \Omega \to R$ as $\alpha(\eta, \rho) = 1, (\eta, \rho) \in P$ and $\alpha(\eta, \rho) = 0$ otherwise, where $P = \{(0, 0), (1, 2), (2, 1)\}$. Since $\alpha(0, T0) = \alpha(1, T1) = \alpha(2, T2) = 1$, $\alpha(T0, T^2 0) = \alpha(T1, T^2 1) = \alpha(T2, T^2 2) = 1$, so T is an α-orbital-admissible. Also, we have $\alpha(0, 0) = \alpha(1, 2) = \alpha(2, 1) = 1, \alpha(T0, T0) = \alpha(T1, T2) = \alpha(T2, T1) = 1$. So T is α-admissible mapping. Notice that T is neither triangular α-admissible nor weak triangular α-admissible (respectively, triangular α-orbital-admissible).

Now we define the notion of parametric α-admissible mapping, an analog of α-admissible mapping ([31]). Let $\Omega \neq \emptyset$ be a set and $\alpha : \Omega^2 \times (0, \infty) \to R$ be a function.

Definition 15 A mapping $T : \Omega \to \Omega$ is said to be parametric α-admissible if $\eta, \rho \in \Omega, \alpha(\eta, \rho, \tau) \geq 1 \Longrightarrow \alpha(T\eta, T\rho, \tau) \geq 1, \forall \tau > 0$.
In addition, we say that T is a parametric α^*-admissible if $\forall \eta, \rho \in Fix(T) \neq \emptyset$, $\alpha(\eta, \rho, \tau) \geq 1, \forall \tau > 0$.

Example 15 Let Ω be a set of all continuous function $f : [0, \infty) \to [0, \infty)$. Define $\alpha : \Omega^2 \times (0, \infty) \to R$ and $T : \Omega \to \Omega$ as: $\alpha(f, g, \tau) = e^{2(f-g)}$, for $f(\tau) \geq g(\tau)$ and $\alpha(f, g, \tau) = 0$ otherwise, $\forall \tau > 0$ and $Tf = \ln(1 + f)$, $\forall f \in \Omega$. Then T is a parametric α-admissible.

Example 16 Let $\Omega = [0, \infty)$ and $T : \Omega \to \Omega$ be a mapping defined by $T\eta = \frac{\eta^2}{2}$, $\forall \eta \in \Omega$. Define $\alpha : \Omega^2 \times (0, \infty) \to R$ as $\alpha(\eta, \rho, \tau) = 1 + \tau(\eta + \rho)$, for $\eta, \rho \in [0, 2]$ and $\alpha(\eta, \rho, \tau) = 0$ otherwise, $\forall \tau > 0$. Note that $Fix(T) = \{0, 2\}$. Then T is a parametric α-admissible and parametric α^*-admissible as well.

Example 17 Consider $\Omega = [0, \infty)$ and let $T : \Omega \to \Omega$ be given by $T\eta = \frac{1+\eta}{2}$, $\eta \in [0, 1]$ and $T\eta = \eta$, $\eta > 1$. Define $\alpha(\eta, \rho, \tau) = 1$, $\eta, \rho \in [0, 2]$ and $\alpha(\eta, \rho, \tau) = 0$ otherwise, $\forall \tau > 0$. Then T is a parametric α-admissible but not a parametric α^*-admissible as $Fix(T) = \{1\} \cup \{c : c > 1\}$.

Definition 16 [26] A continuous function $\varphi : R^+ \to R^+$ is called an altering distance if it is non-decreasing and $\varphi(r) = 0$ iff $r = 0$ and Φ denotes the set of all altering distance function.

Example 18 Let $\varphi_i : R^+ \to R^+$, where $i = 1, 2$ be defined by

(i) $\varphi_1(r) = e^{ar} + br - 1$;

(ii) $\varphi_2(r) = ar^2 + \ln(br + 1)$, where $a, b > 0$.

Clearly, $\varphi_{i=1,2}$ is an altering distance function (for more examples on altering distance function, we refer to Sintunavarat [32]).

Lemma 1 [22] *Suppose $\eta : R^+ \to R^+$ is non-decreasing. Then, for every $r > 0$, $\lim_{n\to+\infty} \eta^n(r) = 0$ implies $\eta(r) < r$,where η^n denotes the n^{th}-iterate of η.*

Definition 17 [10, 11] A function $\psi : R^+ \to R^+$ is said to be a comparison function, if it is monotonically increasing and $\lim_{n\to+\infty} \psi^n(r) = 0$, for all $r > 0$. The symbol Ψ, the set of all comparison function.

Note that if ψ is comparison function, then by Lemma 1, $\psi(r) < r, \forall r > 0$ and $\psi(0) = 0$.

Example 19 [12] Let $\psi_{i=1,2,3} : R^+ \to R^+$, where $\psi_i \in \Psi$ be defined by
(ψ_1) $\psi_1(r) = \alpha r$, where $0 \leq \alpha < 1$;
(ψ_2) $\psi_2(r) = \frac{r}{1+r}$;
(ψ_3) $\psi_3(r) = \beta\gamma(r)$, where $\psi_3(r)$ is monotonically increasing, $0 \leq \beta < 1$ and $\gamma : R^+ \to R^+$ such that $\gamma^n(r) \to 0$ as $n \to +\infty$.

3 Main Results

In this section, we establish theorems on the existence and uniqueness of fixed point for a class of parametric α-admissible mapping in the setting of parametric (b, θ)-metric space and extend our result to parametric (b, θ)-metric space endowed with partial ordered.

Lemma 2 *Let $(\Omega, \mathcal{P}_\theta)$ be a parametric (b, θ)-metric space and $\{\eta_n\}$ be any sequence in Ω. If there exist two functions $\varphi \in \Phi$, $\psi \in \Psi$ with $\varphi(r) \geq r > \psi(r)$, for $r > 0$ such that*

$$0 < \varphi\Big(\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau)\Big) \leq \psi\Big(\mathcal{P}_\theta(\eta_{n-1}, \eta_n, \tau)\Big) \tag{1}$$

and

$$\lim_{n,m\to+\infty} \frac{\theta(\eta_n, \eta_m, \tau)\psi^n\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big)}{\psi^{n-1}\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big)} < 1 \tag{2}$$

for any $m > n \geq 1$ and $\forall\tau > 0$, then the sequence $\{\eta_n\}$ is a Cauchy in Ω.

Proof Assume that $\varphi(r) \geq r > \psi(r),\ r > 0$, then from (1), we obtain

$$\begin{aligned} 0 < \mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau) &\leq \varphi(\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau)) \\ &\leq \psi\Big(\mathcal{P}_\theta(\eta_{n-1}, \eta_n, \tau)\Big) \\ &\ \vdots \\ &\leq \psi^n\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big), \forall\tau > 0. \end{aligned} \tag{3}$$

Setting $\theta_i = \theta(\eta_i, \eta_{i+p}, \tau)\ \forall i \in N,\ p \geq 1$ and $\omega = \mathcal{P}_\theta(\eta_0, \eta_1, \tau)$, then by $(\mathcal{P}_\theta 3)$ with (3), we obtain

$$\begin{aligned}
\mathcal{P}_\theta(\eta_n, \eta_{n+p}, \tau) &\leq \theta(\eta_n, \eta_{n+p}, \tau)[\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau) + \mathcal{P}_\theta(\eta_{n+1}, \eta_{n+p}, \tau)] \\
&= \theta(\eta_n, \eta_{n+p}, \tau)\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau) \\
&\quad + \theta(\eta_n, \eta_{n+p}, \tau)\mathcal{P}_\theta(\eta_{n+1}, \eta_{n+p}, \tau) \\
&\leq \theta(\eta_n, \eta_{n+p}, \tau)\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau) \\
&\quad + \theta(\eta_n, \eta_{n+p}, \tau)\theta(\eta_{n+1}, \eta_{n+p})\mathcal{P}_\theta(\eta_{n+1}, \eta_{n+2}, \tau) + \dots \\
&\quad + \theta(\eta_n, \eta_{n+p}, \tau)\theta(\eta_{n+1}, \eta_{n+p}, \tau) \\
&\quad \dots \theta(\eta_{n+p-1}, \eta_{n+p}, \tau)\mathcal{P}_\theta(\eta_{n+p-1}, \eta_{n+p}, \tau) \\
&\leq \theta_n \psi^n\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big) + \theta_n\theta_{n+1}\psi^{n+1}\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big) + \dots \\
&\quad + \theta_n\theta_{n+1}\dots\theta_{n+p-1}\psi^{n+p-1}\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big) \\
&= \theta_n\psi^n(\omega) + \theta_n\theta_{n+1}\psi^{n+1}(\omega) + \dots + \theta_n\dots\theta_{n+p-1}\psi^{n+p-1}(\omega) \\
&= \sum_{i=n}^{n+p-1} \psi^i(\omega)\prod_{j=n}^{i}\theta_j.
\end{aligned}$$

Multiplying $\prod_{i=1}^{n-1}\theta_i$ on the right side of the above inequality, we obtain

$$\begin{aligned}
\mathcal{P}_\theta(\eta_n, \eta_{n+p}, \tau) &\leq \sum_{i=n}^{n+p-1} \psi^i(\omega)\prod_{j=1}^{i}\theta_j \\
&= \sum_{i=1}^{n+p-1} \psi^i(\omega)\prod_{j=1}^{i}\theta_j - \sum_{i=1}^{n-1}\psi^i(\omega)\prod_{j=1}^{i}\theta_j. \qquad (4)
\end{aligned}$$

Since from (2) for $i \geq 1$, we obtain

$$\lim_{i\to+\infty}\frac{\theta(\eta_i, \eta_{i+p}, \tau)\psi^i\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big)}{\psi^{i-1}\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big)} = \lim_{i\to+\infty}\frac{\theta_i\psi^i(\omega)}{\psi^{i-1}(\omega)} < 1.$$

Therefore, by ratio test the series $\sum_{i=1}^{\infty}\psi^i(\omega)\prod_{j=1}^{i}\theta_j$ converges. Let $S = \sum_{i=1}^{\infty}\psi^i(\omega)\prod_{j=1}^{i}\theta_j$ and $S_n = \sum_{i=1}^{n}\psi^i(\omega)\prod_{j=1}^{i}\theta_j$, the sequence of partial sum. Consequently, (4) becomes

$$\mathcal{P}_\theta(\eta_n, \eta_{n+p}, t) \leq \Big[S_{n+p-1} - S_{n-1}\Big]$$

for any $n \in N$ and $p \geq 1$. Letting limit as $n \to +\infty$, we obtain

$$\lim_{n\to+\infty}\mathcal{P}_\theta(\eta_n, \eta_{n+p}, \tau) = 0.$$

Hence, $\{\eta_n\}$ is Cauchy sequence in Ω.

Putting $\varphi(\xi) = \xi$ and $\psi(\xi) = k\xi$, where $\xi \in R^+$, $k \in [0, 1)$ in Lemma 2, then we obtain the following lemma.

Lemma 3 *Let $(\Omega, \mathcal{P}_\theta)$ be a parametric (b, θ)-metric space and $\{\eta_n\}$ be any sequence in Ω such that*

$$0 < \mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau) \leq k\mathcal{P}_\theta(\eta_{n-1}, \eta_n, \tau)$$

and

$$\lim_{n,m\to+\infty} \theta(\eta_n, \eta_m, \tau) < \frac{1}{k},$$

where $k \in [0, 1)$, for any $m > n \geq 1$ and $\forall \tau > 0$, then the sequence $\{\eta_n\}$ is a Cauchy in Ω.

Let $(\Omega, \mathcal{P}_\theta)$ be a parametric (b, θ)-metric space and $T : \Omega \to \Omega$ be a mapping. We denote

$$\Lambda(\eta, \rho, \tau) = \max\Big\{\mathcal{P}_\theta(\eta, \rho, \tau), \mathcal{P}_\theta(\eta, T\eta, \tau), \mathcal{P}_\theta(\rho, T\rho, \tau), \frac{\mathcal{P}_\theta(\eta, T\rho, \tau) + \mathcal{P}_\theta(\rho, T\eta, \tau)}{2\theta(\eta, \rho, \tau)}\Big\};$$

$$\mathcal{R}(\eta, \rho) = \max\Big\{d(\eta, \rho), d(\eta, T\eta), d(\rho, T\rho), \frac{d(\eta, T\rho) + d(\rho, T\eta)}{2}\Big\}$$

and

$$\mathcal{S}(\eta, \rho) = \max\Big\{d(\eta, \rho), \frac{d(\eta, T\eta) + d(\rho, T\rho)}{2}, \frac{d(\eta, T\rho) + d(\rho, T\eta)}{2}\Big\}.$$

Theorem 1 *Let $(\Omega, \mathcal{P}_\theta)$ be a complete parametric (b, θ)-metric space and $T : \Omega \to \Omega$ be a continuous mapping on Ω. Assume that there exist $\alpha : \Omega^2 \times (0, \infty) \to R$, $\varphi \in \Phi$ and $\psi \in \Psi$ such that $\varphi(r) \geq r > \psi(r)$, for $r > 0$ satisfying*

$$\alpha(\eta, \rho, \tau)\varphi\Big(\mathcal{P}_\theta(T\eta, T\rho, \tau)\Big) \leq \psi\Big(\Lambda(\eta, \rho, \tau)\Big)$$

$\forall \eta, \rho \in \Omega$ and $\forall \tau > 0$. If

(i) T is a parametric α-admissible;

(ii) there exists $\eta_0 \in \Omega$ such that $\alpha(\eta_0, T\eta_0, \tau) \geq 1, \forall \tau > 0$;

(iii) $\lim_{n,m\to+\infty} \frac{\theta(\eta_n, \eta_m, \tau)\psi^n\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big)}{\psi^{n-1}\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big)} < 1,$

where $\eta_n = T^n\eta_0$, $m > n \geq 1, \forall \tau > 0$. *Then there exists* $\zeta \in \Omega$ *such that* $T\zeta = \zeta$, *i.e.* $Fix(T) \neq \emptyset$.

Proof From condition (ii), there exists $\eta_0 \in \Omega$ such that $\alpha(\eta_0, T\eta_0, \tau) \geq 1, \forall \tau > 0$. Consider a sequence $\{\eta_n\}$ in Ω such that $\eta_n = T^n\eta_0, \forall n \in N$. If $\eta_{k-1} = \eta_k = T\eta_{k-1}$, for some $k \in N$, then we have $\eta_{k-1} = T\eta_{k-1}$ and hence η_{k-1} is a fixed point of T. Without lost of generality we assume that $\eta_{n-1} \neq \eta_n, \forall n \in N$, then $\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau) > 0, \forall \tau > 0$. Since by (i), T is a parametric α-admissible, $\alpha(\eta_0, \eta_1, \tau) = \alpha(\eta_0, T\eta_0, \tau) \geq 1$ implies $\alpha(\eta_1, \eta_2, \tau) = \alpha(T\tau_0, T^2\tau_0, \tau) \geq 1, \forall \tau > 0$. Similarly, $\alpha(\eta_1, \eta_2, \tau) = \alpha(T\eta_0, T^2\eta_0, \tau) \geq 1$ implies $\alpha(\eta_2, \eta_3, \tau) = \alpha(T^2\eta_0, T^3\eta_0, \tau) \geq 1, \forall \tau > 0$. Inductively we obtain that $\alpha(\eta_{n-1}, \eta_n, \tau) \geq 1$, where $n \in N$ and $\forall \tau > 0$. From the inequality, we obtain

$$\begin{aligned}\varphi\Big(\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau)\Big) &= \varphi\Big(\mathcal{P}_\theta(T\eta_{n-1}, T\eta_n, \tau)\Big)\\ &\leq \alpha(\eta_{n-1}, \eta_n, \tau)\varphi\Big(\mathcal{P}_\theta(T\eta_{n-1}, T\eta_n, \tau)\Big)\\ &\leq \psi\Big(\Lambda(\eta_{n-1}, \eta_n, \tau)\Big),\end{aligned}$$

where

$$\begin{aligned}\Lambda(\eta_{n-1}, \eta_n, \tau) &= \max\Big\{\mathcal{P}_\theta(\eta_{n-1}, \eta_n, \tau), \mathcal{P}_\theta(\eta_{n-1}, T\eta_{n-1}, \tau), \mathcal{P}_\theta(\eta_n, T\eta_n, \tau),\\ &\qquad \frac{\mathcal{P}_\theta(\eta_{n-1}, T\eta_n, \tau) + \mathcal{P}_\theta(\eta_n, T\eta_{n-1}, \tau)}{2\theta(\eta_{n-1}, \eta_n, \tau)}\Big\}\\ &= \max\Big\{\mathcal{P}_\theta(\eta_{n-1}, \eta_n, \tau), \mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau), \frac{\mathcal{P}_\theta(\eta_{n-1}, \eta_{n+1}, \tau)}{2\theta(\eta_{n-1}, \eta_n, \tau)}\Big\}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\varphi\Big(\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau)\Big) &\leq \psi\Big(\max\Big\{\mathcal{P}_\theta(\eta_{n-1}, \eta_n, \tau), \mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau),\\ &\qquad \frac{\mathcal{P}_\theta(\eta_{n-1}, \eta_{n+1}, \tau)}{2\theta(\eta_{n-1}, \eta_n, \tau)}\Big\}\Big)\\ &\leq \psi\Big(\max\Big\{\mathcal{P}_\theta(\eta_{n-1}, \eta_n, \tau), \mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau),\\ &\qquad \frac{\mathcal{P}_\theta(\eta_{n-1}, \eta_n, \tau) + \mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau)}{2\theta(\eta_{n-1}, \eta_n, \tau)}\Big\}\Big).\end{aligned} \tag{5}$$

If $\mathcal{P}_\theta(\eta_{n-1}, \eta_n, \tau) < \mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau)$, then from (5), we obtain

$$\varphi\Big(\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau)\Big) \leq \psi(\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau)) < \varphi(\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau)).$$

This is a contradiction and hence $\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau) \leq \mathcal{P}_\theta(\eta_{n-1}, \eta_n, \tau), \forall n \in N$. Therefore from (5) with $\varphi(r) \geq r > \psi(r)$, $r > 0$, we obtain

$$0 < \varphi\Big(\mathcal{P}_\theta(\eta_n, \eta_{n+1}, \tau)\Big) \leq \psi\Big(\mathcal{P}_\theta(\tau_{n-1}, \eta_n, \tau)\Big), \forall\tau > 0.$$

Also from (iii), we obtain

$$\lim_{n,m\to+\infty} \frac{\theta(x_n,\ x_m,\ t)\psi^n(\mathcal{P}_\theta(\eta_0,\ \eta_1,\ \tau))}{\psi^{n-1}(\mathcal{P}_\theta(\eta_0,\ \eta_1,\ \tau))} < 1,$$

$\forall\tau > 0$, where $m > n \geq 1$. By Lemma 2, we obtain that the sequence $\{\eta_n\}$ is a Cauchy in Ω. Since $(\Omega, \mathcal{P}_\theta)$ is complete, there is $\zeta \in \Omega$ such that $\eta_n \to \zeta$ as $n \to +\infty$, i.e. $\lim_{n\to+\infty} \mathcal{P}_\theta(\eta_n, \zeta, \tau) = 0, \forall\tau > 0$. Suppose that T is continuous on Ω, then $T\eta_n \to T\zeta$ as $n \to +\infty$, but $T\eta_n = \eta_{n+1} \to \zeta$ as $n \to +\infty$. Therefore $T\zeta = \zeta$.

Example 20 Let $\Omega = [0, \infty)$ and $\mathcal{P}_\theta : \Omega^2 \times (0, \infty) \to [0, \infty)$ be a parametric (b, θ)-metric equipped with $\mathcal{P}_\theta(\eta, \rho, \tau) = \tau|\eta - \rho|^2$, where $\theta(\eta, \rho, \tau) = 2 + \tau(\eta + \rho), \forall\eta, \rho \in \Omega$ and $\forall\tau > 0$. Consider $T : \Omega \to \Omega$ is a continuous mapping defined by $T\eta = \frac{3\eta}{5}, \eta \in [0, 1]$ and $T\eta = 2\eta - \frac{7}{5}, \eta > 1$. Define $\alpha : \Omega^2 \times (0, \infty) \to R$ as $\alpha(\eta, \rho, \tau) = 1,\ \eta, \rho \in [0, 1]$ and $\alpha(\eta, \rho, \tau) = 0$ otherwise, $\forall\tau > 0$. Note that $\alpha(\eta, \rho, \tau) = 1$ and $\alpha(T\eta, T\rho, \tau) = 1, \forall\eta, \rho \in [0, 1]$ and $\forall\tau > 0$. So T is a parametric α-admissible. Also setting $\psi(r) = kr$ and $\varphi(r) = r$, where $k = \frac{9}{25}$, then $\varphi(r) \geq r > \psi(r)$, for $r > 0$. In fact, $\forall\eta, \rho \in \Omega$ and $\forall\tau > 0$, we obtain

$$\begin{aligned}
\alpha(\eta, \rho, \tau)\varphi\Big(\mathcal{P}_\theta(T\eta, T\rho, \tau)\Big) &= \tau|T\eta - T\rho|^2 \\
&= \frac{9}{25}\tau|\eta - \rho|^2 = k\mathcal{P}_\theta(\eta, \rho, \tau) \\
&\leq \psi\Big(\Lambda(\eta, \rho, \tau)\Big).
\end{aligned}$$

Since T is parametric α-admissible, we construct a sequence $\{\eta_n\}$ in Ω such that $\alpha(\eta_n, \eta_{n+1}, \tau) = \alpha(T^n\eta_0, T^{n+1}\eta_0, \tau) \geq 1, \forall\tau > 0$. Since $\alpha(\eta_n, \eta_{n+1}, \tau) \geq 1, \forall n \in N \cup \{0\}$, so $\eta_n \in [0, 1], \forall n \in N \cup \{0\}$. In fact, $\eta_n = T^n\eta_0 = (\frac{3}{5})^n\eta_0 \to 0$ as $n \to +\infty$ and $\lim_{n,m\to+\infty} \theta(T^n\eta_0, T^m\eta_0, \tau) = 2 < \frac{1}{k}$. Thus, T satisfies all the conditions of Theorem 1 and hence $Fix(T) \neq \emptyset$. One can check that $Fix(T) = \{0, \frac{7}{5}\}$.

We omit the condition of continuity assumption in Theorem 1 as follows.

Theorem 2 *Let $(\Omega, \mathcal{P}_\theta)$ be a complete parametric (b, θ)-metric space and $T : \Omega \to \Omega$ be a mapping on Ω. Assume that there exist $\alpha : \Omega^2 \times (0, \infty) \to R$, $\varphi \in \Phi$ and $\psi \in \Psi$ such that $\varphi(r) \geq r > \psi(r)$, $r > 0$ satisfying*

$$\alpha(\eta, \rho, \tau)\varphi\Big(\mathcal{P}_\theta(T\eta, T\rho, \tau)\Big) \leq \psi\Big(\Lambda(\eta, \rho, \tau)\Big),$$

$\forall\eta, \rho \in \Omega$ and $\forall\tau > 0$. If

(i) T is a parametric α-admissible;

(ii) *there exists* $\eta_0 \in \Omega$ *such that* $\alpha(\eta_0, T\eta_0, \tau) \geq 1, \forall \tau > 0$*;*

(iii) $\lim_{n,m\to+\infty} \frac{\theta(\eta_n,\eta_m,\tau)\psi^n\left(\mathcal{P}_\theta(\eta_0,\eta_1,\tau)\right)}{\psi^{n-1}\left(\mathcal{P}_\theta(\eta_0,\eta_1,\tau)\right)} < 1$*, where* $\eta_n = T^n\eta_0$*,* $m > n \geq 1, \forall \tau > 0$*;*

(iv) $\{\eta_n\}$ *is a sequence in* Ω *such that* $\alpha(\eta_n, \eta_{n+1}, \tau) \geq 1$ *and* $\eta_n \to \zeta \in \Omega$ *as* $n \to +\infty$*, then there exists a subsequence* $\{\eta_{n_k}\}$ *of* $\{\eta_n\}$ *such that* $\alpha(\eta_{n_k}, \zeta, \tau) \geq 1, \forall \tau > 0$*, where* $n_k \geq n_0 \geq 1$*.*

Then there exists $\zeta \in \Omega$ *such that* $T\zeta = \zeta$*, i.e.* $Fix(T) \neq \emptyset$*.*

Proof As in Theorem 1, one can show that the sequence $\{\eta_n\}$ is a Cauchy in Ω. Since $(\Omega, \mathcal{P}_\theta)$ is complete, there exists $\zeta \in \Omega$ such that $\eta_n \to \zeta$ an $n \to +\infty$. From (iv) we obtain $\alpha(\eta_{n_k}, \zeta, \tau) \geq 1$, $n_k \geq n_0 \geq 1, \forall \tau > 0$. Taking $\eta = \eta_{n_k}$ and $\rho = \zeta, \forall \tau > 0$, we obtain

$$\begin{aligned}
\varphi\Big(\mathcal{P}_\theta(\eta_{n_k+1}, T\zeta, \tau)\Big) &= \varphi\Big(\mathcal{P}_\theta(T\eta_{n_k}, T\zeta, \tau)\Big) \\
&\leq \alpha(\eta_{n_k}, \zeta, \tau)\varphi\Big(\mathcal{P}_\theta(T\eta_{n_k}, T\zeta, \tau)\Big) \\
&\leq \psi\Big(\Lambda(\eta_{n_k}, \zeta, \tau)\Big) \\
&< \varphi\Big(\Lambda(\eta_{n_k}, \zeta, \tau)\Big),
\end{aligned}$$

where

$$\begin{aligned}
\Lambda(\eta_{n_k}, \zeta, \tau) &= \max\Big\{\mathcal{P}_\theta(\eta_{n_k}, \zeta, \tau), \mathcal{P}_\theta(\eta_{n_k}, T\eta_{n_k}, \tau), \mathcal{P}_\theta(\zeta, T\zeta, \tau), \\
&\qquad \frac{\mathcal{P}_\theta(\eta_{n_k}, T\zeta, \tau) + \mathcal{P}_\theta(\zeta, T\eta_{n_k}, \tau)}{2\theta(\eta_{n_k}, \zeta, \tau)}\Big\} \\
&= \max\Big\{\mathcal{P}_\theta(\eta_{n_k}, \zeta, \tau), \mathcal{P}_\theta(\eta_{n_k}, \eta_{n_k+1}, \tau), \mathcal{P}_\theta(\zeta, T\zeta, \tau), \\
&\qquad \frac{\mathcal{P}_\theta(\eta_{n_k}, T\zeta, \tau) + \mathcal{P}_\theta(\zeta, \eta_{n_k+1}, \tau)}{2\theta(\eta_{n_k}, \zeta, \tau)}\Big\}.
\end{aligned}$$

Letting $k \to +\infty$ and continuity of φ, we obtain

$$\varphi\Big(\mathcal{P}_\theta(\zeta, T\zeta, \tau)\Big) < \varphi\Big(\lim_{n_k\to+\infty} \Lambda(\eta_{n_k}, \zeta, \tau)\Big) = \varphi\Big(\mathcal{P}_\theta(\zeta, T\zeta, \tau)\Big)$$

which is a contradiction. Therefore, we conclude that $\mathcal{P}_\theta(\zeta, T\zeta, \tau) = 0$ and hence $T\zeta = \zeta$, i.e. $Fix(T) \neq \emptyset$.

Theorem 3 *Suppose parametric* α^**-admissibility of mapping* $T : \Omega \to \Omega$ *is subsuming to the hypothesis of Theorem 1 (resp. Theorem 2). Then there exists a unique* $\zeta \in \Omega$ *such that* $T\zeta = \zeta$ *i.e.* $Fix(T)$ *is a singleton.*

Proof By Theorem 1 (resp. Theorem 2), we obtain $Fix(T) \neq \emptyset$. Since T is a parametric α^*-admissible, then $\alpha(\zeta, \zeta^*, \tau) = \alpha(T\zeta, T\zeta^*, \tau) \geq 1, \forall \zeta, \zeta^* \in Fix(T)$ and $\forall \tau > 0$. Suppose that $\zeta \neq \zeta^*, \forall \tau > 0$, we obtain

$$\begin{aligned}\varphi\Big(\mathcal{P}_\theta(\zeta, \zeta^*, \tau)\Big) &= \varphi\Big(\mathcal{P}_\theta(T\zeta, T\zeta^*, \tau)\Big)\\ &\leq \alpha(\zeta, \zeta^*, t)\varphi\Big(\mathcal{P}_\theta(T\zeta, T\zeta^*, \tau)\Big)\\ &\leq \psi\Big(\Lambda(\zeta, \zeta^*, \tau)\Big) = \psi\Big(\mathcal{P}_\theta(\zeta, \zeta^*, \tau)\Big)\\ &< \varphi\Big(\mathcal{P}_\theta(\zeta, \zeta^*, \tau)\Big).\end{aligned}$$

This is a contradiction and hence T possesses a unique fixed point in Ω, i.e. $Fix(T)$ is a singleton.

Corollary 1 *Let $(\Omega, \mathcal{P}_\theta)$ be a complete parametric (b, θ)-metric space and $T : \Omega \to \Omega$ be a mapping such that*

$$\mathcal{P}_\theta(T\eta, T\rho, \tau) \leq k\Lambda(\eta, \rho, \tau)$$

$\forall \eta, \rho \in \Omega$ and $\forall \tau > 0$. Moreover, if for any $\eta_0 \in \Omega$,

$$\lim_{n,m\to+\infty} \theta(\eta_n, \eta_m, \tau) < \frac{1}{k},$$

where $\eta_n = T^n\eta_0$ and $0 \leq k < 1, \forall \tau > 0$. Then $Fix(T)$ is singleton.

Remark 3 (i) In Example 20, T is a parametric α-admissible and $Fix(T) = \{0, \frac{7}{5}\}$, but $\alpha(\frac{7}{5}, \frac{7}{5}, \tau) = \alpha(T\frac{7}{5}, T\frac{7}{5}, \tau) = 0, \forall \tau > 0$. This shows that T is not parametric α^*-admissible. In this case, Theorem 3 is not application in Example 20.

(ii) Moreover, in Example 20, taking $\eta = \frac{1}{2}$ and $\rho = 2$, then

$$\mathcal{P}_\theta(T\eta, T\rho, \tau) = \mathcal{P}_\theta(T\frac{1}{2}, T2, \tau) = \frac{64\tau}{25} > \frac{9\tau}{4} = \mathcal{P}_\theta(\frac{1}{2}, 2, \tau),$$

$\forall \tau > 0$. This shows that Corollary 1 is not application in Example 20.

We give the direct consequences of Theorems 1 and 2 (respectively, Theorem 3) as follows.

Theorem 4 *Let (Ω, d) be a complete metric space and $T : \Omega \to \Omega$ be a mapping on Ω. Assume that there exist $\alpha : \Omega \times \Omega \to R$, $\varphi \in \Phi$ and $\psi \in \Psi$ such that $\varphi(r) \geq r > \psi(r)$, for $r > 0$ satisfying*

$$\alpha(\eta, \rho)\varphi\Big(d(T\eta, T\rho)\Big) \leq \psi\Big(\mathcal{R}(\eta, \rho)\Big), \tag{6}$$

$\forall \eta, \rho \in \Omega$. If

(i) T is an α-admissible;
(ii) there exists $\eta_0 \in \Omega$ such that $\alpha(\eta_0, T\eta_0) \geq 1$;
(iii) $\lim_{n\to+\infty} \frac{\psi^n\left(d(\eta_0, \eta_1)\right)}{\psi^{n-1}\left(d(\eta_0, \eta_1)\right)} < 1$, where $\eta_n = T^n\eta_0$;
(iv) (a) T is continuous,
or (b) $\{\eta_n\}$ is a sequence in Ω such that $\alpha(\eta_n, \eta_{n+1}) \geq 1$ and $\eta_n \to \zeta \in \Omega$ as $n \to +\infty$, then there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\alpha(\eta_{n_k}, \zeta) \geq 1$, where $n_k \geq n_0 \geq 1$.

Then there exists $\zeta \in \Omega$ such that $T\zeta = \zeta$, i.e. $Fix(T) \neq \emptyset$.

Proof By (ii), there exists $\eta_0 \in \Omega$ such that $\alpha(\eta_0, T\eta_0) \geq 1$. Define a sequence in Ω such that $\eta_{n+1} = T^{n+1}\eta_0 = T\eta_n, \forall n \in N \cup \{0\}$. As in Theorem 1, we assume that $d(\eta_n, \eta_{n+1}) > 0$, then $\eta_{n+1} \neq \eta_n, \forall n \in N \cup \{0\}$. Since T is α-admissible, $\alpha(\eta_0, \eta_1) = \alpha(\eta_0, T\eta_0) \geq 1$ implies $\alpha(\eta_1, \eta_2) = \alpha(T\eta_0, T^2\eta_0) \geq 1$. Similarly, $\alpha(\eta_1, \eta_2) = \alpha(T\eta_0, T^2\eta_0) \geq 1$ implies $\alpha(\eta_2, \eta_3) = \alpha(T^2\eta_0, T^3\eta_0) \geq 1$. Repeating this process, we obtain inductively that $\alpha(\eta_n, \eta_{n+1}) \geq 1, \forall n \in N \cup \{0\}$. From (6), we obtain

$$\begin{aligned}\varphi\Big(d(\eta_{n+1}, \eta_{n+2})\Big) &\leq \alpha(\eta_n, \eta_{n+1})\varphi\Big(d(T\eta_n, T\eta_{n+1})\Big)\\ &\leq \psi\Big(\mathcal{R}(\eta_n, \eta_{n+1})\Big),\end{aligned}$$

where

$$\begin{aligned}\mathcal{R}(\eta_n, \eta_{n+1}) &= \max\Big\{d(\eta_n, \eta_{n+1}), d(\eta_n, T\eta_n), d(\eta_{n+1}, T\eta_{n+1}),\\ &\qquad \frac{d(\eta_n, T\eta_{n+1}) + d(\eta_{n+1}, T\eta_n)}{2}\Big\}\\ &= \max\Big\{d(\eta_n, \eta_{n+1}), d(\eta_{n+1}, \eta_{n+2}), \frac{d(\eta_n, \eta_{n+2})}{2}\Big\}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\varphi\Big(d(\eta_{n+1}, \eta_{n+2})\Big) &\leq \psi\left(\max\Big\{d(\eta_n, \eta_{n+1}), d(\eta_{n+1}, \eta_{n+2}), \frac{d(\eta_n, \eta_{n+2})}{2}\Big\}\right)\\ &\leq \psi\Big(\max\Big\{d(\eta_n, \eta_{n+1}), d(\eta_{n+1}, \eta_{n+2}),\\ &\qquad \frac{d(\eta_n, \eta_{n+1}) + d(\eta_{n+1}, \eta_{n+2})}{2}\Big\}\Big)\\ &\leq \psi\Big(\max\Big\{d(\eta_n, \eta_{n+1}), d(\eta_{n+1}, \eta_{n+2})\Big\}\Big). \qquad (7)\end{aligned}$$

If $d(\eta_n, \eta_{n+1}) < d(\eta_{n+1}, \eta_{n+2})$, then from (7), we obtain

$$\varphi\Big(d(\eta_{n+1},\eta_{n+2})\Big) \le \psi\Big(d(\eta_{n+1},\eta_{n+2})\Big) < \varphi\Big(d(\eta_{n+1},\eta_{n+2})\Big).$$

This is a contradiction and hence $d(\eta_{n+1},\eta_{n+2}) \le d(\eta_n,\eta_{n+1})$. From (7), it follows that

$$\varphi\Big(d(\eta_{n+1},\eta_{n+2})\Big) \le \psi\Big(d(\eta_n,\eta_{n+1})\Big). \tag{8}$$

Since $\varphi(r) \ge r > \psi(r)$ for $r > 0$, then from (8), we obtain

$$0 < d(\eta_n,\eta_{n+1}) \le \varphi\Big(d(\eta_n,\eta_{n+1})\Big) \le \psi\Big(d(\eta_{n-1},\eta_n)\Big). \tag{9}$$

Consequently from (9), we obtain

$$0 < d(\eta_n,\eta_{n+1}) \le \psi\Big(d(\eta_{n-1},\eta_n)\Big) \le \cdots \le \psi^n\Big(d(\eta_0,\eta_1)\Big). \tag{10}$$

Using triangular inequality and from (10) setting with $r = d(\eta_0,\eta_1)$, for $p \ge 1$ and $n \in N$, we obtain

$$\begin{aligned} d(\eta_n,\eta_{n+p}) &\le d(\eta_n,\eta_{n+1}) + d(\eta_{n+1},\eta_{n+2}) + \cdots + d(\eta_{n+p-1},\eta_{n+p}) \\ &\le \psi^n(d(\eta_0,\eta_1)) + \psi^{n+1}(d(\eta_0,\eta_1)) + \cdots + \psi^{n+p-1}(d(\eta_0,\eta_1)) \\ &= \sum_{i=n}^{n+p-1} \psi^i(r) \\ &= \sum_{i=1}^{n+p-1} \psi^i(r) - \sum_{i=1}^{n-1} \psi^i(r). \end{aligned} \tag{11}$$

Since from (iii), we obtain

$$\lim_{n\to+\infty} \frac{\psi^n\Big(d(\eta_0,\ \eta_1)\Big)}{\psi^{n-1}\Big(d(\eta_0,\ \eta_1)\Big)} = \lim_{n\to+\infty} \frac{\psi^n(r)}{\psi^{n-1}(r)} < 1.$$

Therefore, by Ration test, the series $\sum_{i=1}^{\infty} \psi^i(r)$ is convergent. Let $S = \sum_{i=1}^{\infty} \psi^i(r)$ and $S_n = \sum_{i=1}^{n} \psi^i(r)$, the sequence of partial sum. Consequently, (11) becomes

$$d(\eta_n,\eta_{n+p}) \le [S_{n+p-1} - S_{n-1}]$$

for $n \in N$ and $p \ge 1$. Letting $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} d(\eta_n,\eta_{n+p}) = 0.$$

Thus, the sequence $\{\eta_n\}$ is a Cauchy in Ω. Since (Ω, d) is complete, so the sequence $\{\eta_n\}$ converges to $\zeta \in \Omega$. Following the same steps as in Theorems 1 and 2, we obtain the required result.

Theorem 5 *Suppose α^*-admissibility of mapping $T : \Omega \to \Omega$ is subsuming to the hypothesis of Theorem 4. Then there exists a unique $\zeta \in \Omega$ such that $T\zeta = \zeta$. Moreover, the sequence $\{T^n \eta_0\}_{n \in N}$ converges to $\zeta \in \Omega$.*

Remark 4 (i) Theorem 4 improves Theorem 8 [4] (respectively, Theorems 9, 10 and 11 of [4]) as the continuity condition in the control function ψ has replaced by comparison function and weak triangular α-admissibility of mapping T by α-admissibility, and $\mathcal{R}(\eta, \rho) \leq \mathcal{S}(\eta, \rho), \forall \eta, \rho \in \Omega$.

(ii) The drawback to obtain a unique fixed point in Theorem 8 [4] (respectively, Theorem 9 [4]) is removed by using α^*-admissibility of mapping T.

Consider $(\Omega, \preceq)$ is a partial ordered set. We say $T : \Omega \to \Omega$ is monotone non-decreasing if $\eta, \rho \in \Omega$, $\eta \preceq \rho$ implies $T\eta \preceq T\rho$.

Theorem 6 *Let $(\Omega, \preceq)$ be a partial ordered set and suppose that there exists a parametric (b, θ)-metric $\mathcal{P}_\theta$ such that $(\Omega, \mathcal{P}_\theta)$ be a complete parametric (b, θ)-metric space. Let $T : \Omega \to \Omega$ be a monotone non-decreasing self-mapping w.r.t. $\preceq$ such that there exist $\varphi \in \Phi$ and $\psi \in \Psi$, $\varphi(r) \geq r > \psi(r)$, for $r > 0$ satisfying*

$$\varphi\Big(\mathcal{P}_\theta(T\eta, T\rho, \tau)\Big) \leq \psi\Big(\Lambda(\eta, \rho, \tau)\Big),$$

$\forall \eta, \rho \in \Omega$ with $\eta \preceq \rho$ and $\forall \tau > 0$. If

(i) there exists $\eta_0 \in \Omega$ such that $\eta_0 \preceq T\eta_0$;

(ii) $\lim_{n,m \to +\infty} \dfrac{\theta(\eta_n, \eta_m, \tau)\psi^n\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big)}{\psi^{n-1}\Big(\mathcal{P}_\theta(\eta_0, \eta_1, \tau)\Big)} < 1$, where $\eta_n = T^n \eta_0, \forall \tau > 0$;

(iii) (a) T is continuous, or (b) $\{\eta_n\}$ is a non-decreasing sequence in Ω such that $\eta_n \to \zeta$ as $n \to +\infty$, then there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\eta_{n_k} \preceq \zeta$, where $n_k \geq n_0$.

Then $Fix(T) \neq \emptyset$. Further, if every pair of elements $\zeta, \zeta^* \in Fix(T)$ is comparable, then $Fix(T)$ is a singleton.

Proof Define a mapping $\alpha : \Omega^2 \times (0, \infty) \to [0, \infty)$ as $\alpha(\eta, \rho, \tau) = 1$, $\eta \preceq \rho$ or, $\rho \preceq \eta$ and $\alpha(\eta, \rho, \tau) = 0$ otherwise, $\forall \tau > 0$. Then, we obtain

$$\alpha(\eta, \rho, \tau)\varphi\Big(\mathcal{P}_\theta(T\eta, T\rho, \tau)\Big) \leq \psi\Big(\Lambda(\eta, \rho, \tau)\Big),$$

$\forall \eta, \rho \in \Omega$ with $\eta \preceq \rho$ and $\forall \tau > 0$. Since T is monotone non-decreasing mapping w.r.t. $\preceq$, so T is a parametric α-admissible. Indeed if $\eta, \rho \in \Omega$ such that $\alpha(\zeta, \rho, \tau) \geq 1, \forall \tau > 0$, then $\eta \preceq \rho$, or $\rho \preceq \eta$. Since T is monotone non-decreasing mapping w.r.t $\preceq$, we have $T\eta \preceq T\rho$, or $T\rho \preceq T\eta$, which in turn gives $\alpha(T\eta, T\rho, \tau) \geq$

$1, \forall \tau > 0$. On the other hand, from (ii) there exists $\eta_0 \in \Omega$ such that $\eta_0 \preceq T\eta_0$, then $\alpha(\eta_0, T\eta_0, \tau) \geq 1, \forall \tau > 0$. From (iii)(a) if T is continuous, then all the hypothesis of Theorem 1 are satisfied. Again from (iii)(b) suppose that $\{\eta_n\}$ is a non-decreasing sequence in Ω such that $\eta_n \to \zeta$ as $n \to +\infty$, then there exists a subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$ such that $\eta_{n_k} \preceq \zeta, n_k \geq n_0$, which in turn gives $\alpha(\eta_{n_k}, \zeta, \tau) \geq 1, \forall \tau > 0$. Thus, all the hypothesis of Theorem 2 are satisfied. Consequently, T possesses a fixed point in Ω, i.e. $Fix(T) \neq \emptyset$. Further assume that every pair of elements $\zeta, \zeta^* \in Fix(T)$ are comparable, then $\zeta \preceq \zeta^*$, or $\zeta^* \preceq \zeta$ which in turn gives $\alpha(\zeta, \zeta^*, \tau) \geq 1, \forall \tau > 0$. Therefore, T is a parametric α^*-admissible. Thus, all the hypothesis of Theorem 3 are satisfied and hence $Fix(T)$ is a singleton.

4 Application

Let $\Omega = \mathcal{C}([0, l], R)$ be a set of all real-valued continuous functions on $[0, l]$ and define a parametric (b, θ)-metric $\mathcal{P}_\theta : \Omega \times \Omega \times (0, \infty) \to [0, \infty)$ as

$$\mathcal{P}_\theta(\eta, \rho, \tau) = \max_{\omega \in [0,l]} \left\{ \tau |\eta(\omega) - \rho(\omega)|^2 \right\}$$

with $\theta(\eta, \rho, \tau) = 2 + \tau(\eta + \rho), \forall \eta, \rho \in \Omega$ and $\forall \tau > 0$. Then $(\Omega, \mathcal{P}_\theta)$ is a complete parametric (b, θ)-metric space. Let $\preceq$ be a partial order on Ω defined by $\eta \preceq \rho$ if and only if $\eta(\omega) \preceq \rho(\omega), \forall \omega \in [0, l]$.

Consider an integral equation

$$\eta(r) = \sigma(r) + \int_0^l K(r, s) f\Big(s, \eta(s)\Big) ds \tag{12}$$

with the following assumption that:
(H_1) : $f : [0, l] \times R \to R$, $\sigma : [0, l] \to R$, and $K : [0, l] \times [0, l] \to [0, \infty)$ are continuous functions;
(H_2) : $\max_{r \in [0,l]} \Big(\int_0^l K^2(r, s) ds \Big)^{\frac{1}{2}} < \frac{\sqrt{k}}{l}$, where $k = \frac{1}{2^2}$;
(H_3) : $0 \leq \Big(f(s, \eta(s)) - f(s, \rho(s)) \Big) \leq \Big(\frac{|\eta(s) - T\rho(s)|^2 + |\rho(s) - T\eta(s)|^2}{2\theta(\eta(s), \rho(s), \tau)} \Big)^{\frac{1}{2}}$

$\forall \eta, \rho \in \Omega, \eta \preceq \rho$ and $\forall \tau > 0$, where $T\eta(s) = \eta(s), s \in [0, l]$;
(H_4) : there exists $\eta_0 \in \Omega$ such that

$$\eta_0(r) \leq \sigma(r) + \int_0^l K(r, s) f\Big(s, \eta_0(s)\Big) ds;$$

(H_5) : $\lim_{n,m\to\infty} \theta(\eta_n, \eta_m, \tau) < \frac{1}{k}$, where $\eta_n = T^n\eta_0, m > n \geq n_0 \in N$ and $\forall\tau > 0$.

We have the following theorem for the existence of solution of integral equation.

Theorem 7 *Suppose that* $(H_1) - (H_5)$ *are hold. Then the integral Eq. (12) has a solution in* Ω.

Proof Suppose $T : \Omega \to \Omega$ be a continuous mapping defined by

$$T\eta(r) = \sigma(r) + \int_0^l K(r,s) f\Big(s, \eta(s)\Big) ds, \tag{13}$$

$r \in [0, l]$ and $\forall\eta \in \Omega$. First we show that T is non-decreasing mapping with respect to $\preceq$. For this, let $\eta \preceq \rho$, then by (H_3), we have

$$0 \leq \Big(f(s, \eta(s)) - f(s, \rho(s))\Big),$$

$\forall s \in [0, l]$. Also we have

$$T\rho(r) - T\eta(r) = \int_0^l K(r,s)\Big[f\Big(s, \rho(s)\Big) - f\Big(s, \eta(s)\Big)\Big]ds \geq 0,$$

$\forall r \in [0, l]$. Then $T\eta \preceq T\rho$, i.e. T is monotone non-decreasing mapping with respect to $\preceq$. On the other hand by (H_2), (H_3) and $\forall\tau > 0$, we have

$$\begin{aligned}
\mathcal{P}_\theta(T\eta, T\rho, \tau) &= \max_{r\in[0,l]} \tau|T\eta(r) - T\rho(r)|^2 \\
&\leq \tau\left(\max_{r\in[0,l]} \int_0^l K(r,s)\Big[f\Big(s, \eta(s)\Big) - f\Big(s, \rho(s)\Big)\Big]ds\right)^2 \\
&\leq \tau \max_{r\in[0,l]}\left[\left(\int_0^l K^2(r,s)ds\right)^{\frac{1}{2}}\left(\int_0^l \Big[f\Big(s, \eta(s)\Big)\right.\right. \\
&\qquad \left.\left. - f\Big(s, \rho(s)\Big)\Big]^2 ds\right)^{\frac{1}{2}}\right]^2 \\
&\leq \frac{k\tau}{l^2}\left\{\frac{|\eta - T\rho|^2 + |\rho - T\eta|^2}{2\theta(\eta, \rho, \tau)}\right\}\left(\int_0^l ds\right)^2 \\
&\leq k\frac{\mathcal{P}_\theta(\eta, T\rho, \tau) + \mathcal{P}_\theta(\rho, T\eta, \tau)}{2\theta(\eta, \rho, \tau)} \\
&= k\Lambda(\eta, \rho, \tau).
\end{aligned}$$

From (H_4) there exists $\eta_0 \in \Omega$ such that $\eta_0 \preceq T\eta_0$. Setting $\varphi(r) = r$ and $\psi(r) = kr$, where $k \in (0, 1]$ and $r > 0$. Thus, the integral operator T defined by (13) satisfies all the conditions of Theorem 6 and hence $Fix(T) \neq \emptyset$, i.e. the integral equation (12) has a solution in Ω. Further if every pair of elements $\zeta, \zeta^* \in Fix(T) \subseteq \Omega$ is comparable, then the solution is unique.

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Some Extragradient Methods for Solving Variational Inequalities Using Bregman Projection and Fixed Point Techniques in Reflexive Banach Spaces

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Abstract In this chapter, we introduce some extragradient methods for solving variational inequalities using Bregman projections and a fixed point technique in reflexive Banach spaces. These algorithms are extensions of the prototypes which have been studied extensively in real Hilbert and 2-uniformly convex Banach spaces. We emphasize that there are some applicable examples (most especially in mechanics) which can be modelled as variational inequalities in reflexive Banach spaces outside Hilbert and 2-uniformly convex Banach spaces. Moreover, the usage of Bregman projections allows the consideration of more general structures of the feasible set. The convergence analysis of the algorithms are given using Bregman distance and fixed point techniques. More so, we present some computational examples to illustrate the effects of various type of convex functions on the proposed algorithm.

1 Introduction

Throughout this chapter, E is a real Banach space endowed with norm $\|\cdot\|$ and dual E^*. We denote the value of the functional $f^* \in E^*$ at $g \in E$ by $\langle f^*, g\rangle$. Let C be a nonempty, closed and convex subset of E and $A : C \to E^*$ be an operator. The Variational Inequalities (in short, VI) can be defined as finding a point $x^\dagger \in C$ such that

$$\langle Ax^\dagger, y - x^\dagger\rangle \geq 0 \quad \forall\, y \in C. \tag{1}$$

The theory of VI can be traced back to the work of Stampacchia [43] on Signorini problem [19, 20]. It has been developed and considered as a fruitful interaction between many fields of applied science and mathematical analysis, see for instance [17, 22, 32, 33]. The existence of solution of the VI was built on arguments of monotonicity of the operator A and convexity of the set C. It is well known that a point $x^\dagger$ solves the VI if and only if $x^\dagger$ solves the fixed point equation

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$$x^\dagger = P_C(x^\dagger - \beta A x^\dagger), \tag{2}$$

where $\beta > 0$ is a non-negative value and P_C is the projection onto C. Considerable efforts have been made by several researchers in proposing some fixed point methods which are based on projections for solving the VI in many directions, see, for instance [8, 9, 11, 34, 35]. One of the simplest methods for solving the VI is the gradient projection method which is a natural extension of gradient descent method for solving convex optimization problems [23]. The convergence of the gradient projection method is guaranteed under a strong condition known as strong monotonicity. However, if the operator A defining the VI does not satisfies the strong condition, the gradient projection method may fail to converge to a solution of the VI. Korpelevich [34] first introduced an *Extragradient Method (EM)* which does not require the strong monotonicity condition in finite dimensional spaces as follows:

$$\begin{cases} x_0 \in C \subset \mathbb{R}^n, \\ y_n = P_C(x_n - \beta A x_n), \\ x_{n+1} = P_C(x_n - \beta A y_n), \quad n \geq 0, \end{cases} \tag{3}$$

where $\beta \in (0, 1/L)$, $A : \mathbb{R}^n \to \mathbb{R}^n$ is a monotone and L-Lipschitz continuous operator. The *EM* has been extended to infinite-dimensional spaces by many authors, see [9, 10, 14, 18]. Though the *EM* yields a successful approximation of the solution of VI, it is considered to be too computational expensive since there is need to calculate two projections onto the whole feasible set C in each iteration. Other important modifications of the *EM* has been introduced by many researchers such as the *Subgradient Extragradient Method (SEM)* by Censor et al. [11, 12]. This was done by defining a constructible half-space whose projection can easily be calculated explicitly in real Hilbert spaces. The *SEM* is given as follows: Choose $x_0 \in C$ and compute

$$\begin{cases} y_n = P_C(x_n - \beta A x_n), \\ x_{n+1} = P_{T_n}(x_n - \beta A y_n), \\ \text{where} \quad T_n = \{x \in H : \langle x_n - \beta A x_n - y_n, x - y_n \rangle \leq 0\}, \quad n \geq 0. \end{cases} \tag{4}$$

The authors further proved that the sequence $\{x_n\}$ generated by the *SEM* converges weakly to a solution of the VI in real Hilbert spaces provided the condition $\beta \in (0, 1/L)$ is satisfied. Also, Tseng [49] introduced a single projection method for solving the VI as follows:

$$\begin{cases} x_0 \in C, \beta > 0, \\ y_n = P_C(x_n - \beta A x_n), \\ x_{n+1} = y_n - \beta(A y_n - A x_n), \quad n \geq 1. \end{cases} \tag{5}$$

It was proved that (5) converges weakly to a solution of the VI if the stepsize also satisfies $\beta \in \left(0, \frac{1}{L}\right)$. Note that the value of the operator A is evaluated at two points in the feasible set C when using the *SEM* and (5). This can be computational expensive if the operator A does not have simple structure. Example of such operators can be found in optimal control theory (see, [31, 50]). Hence, there is need to improve the *SEM* such that the evaluation of A will be minimal. An attempt in this direction was introduced by Popov [40] who introduced the following iteration with evaluation of A at a single point in the feasible set per each iteration: Given $x_0, y_0 \in C$, compute

$$\begin{cases} y_{n+1} = P_C(x_n - \beta A y_n), \\ x_{n+1} = P_C(x_{n+1} - \beta A y_n), \quad \forall n \geq 0, \end{cases} \tag{6}$$

where $\beta \in \left(0, \frac{1}{3L}\right)$. The author also proved a weak convergence of (6) to a solution of VI. Many other modifications of the above methods can be found in, for instance, see [1, 8–10, 15, 16, 27–31, 42, 45–48, 52].

Most results on iterative methods for solving the VI have been introduced in real Hilbert or 2-uniformly convex Banach spaces (see [13, 26, 36, 41] and references therein for examples of iterative methods for solving VI in 2-uniformly convex Banach spaces). We note that there are some interesting applicable models which can be formulated as VI in higher Banach spaces which are not Hilbert or 2-uniformly convex Banach spaces. Examples of such can be found in the mechanics and membrane problems (see, for instance, [2, Example 4.4.2]). Hence it becomes necessary to find iterative methods in Banach spaces which are more general than Hilbert and 2-uniformly convex Banach spaces.

In this chapter, we introduce some extragradient-type iterative methods for solving VI in real reflexive Banach spaces. The convergence of these methods are studied using Bregman distance technique and the fixed point Eq. (2) in reflexive Banach spaces. It should also be mention that another importance of using the Bregman projection is that a general structure of the feasible set can be considered for the VI. For example, we can consider the feasible set as a simplex and choose the Kullback-Leibler divergence which is a Bregman divergence on negative entropy as the distance and obtain a projection onto simplex which can easily be calculated explicitly. We further perform some computational experiments to illustrate the behaviour of the proposed methods in this chapter.

2 Preliminaries

In this section, we give some definitions and preliminary results which will we needed for our results. The strong and weak convergences of $\{x_n\}$ to $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$ respectively.

Definition 1 A Banach space E is called strictly convex if

$$\left\| \frac{x+y}{2} \right\| < 1,$$

whenever $x, y \in B_E = \{x \in E : \|x\| = 1\}$ and $x \neq y$. When the limit

$$\lim_{t\to\infty} \frac{\|x+ty\| - \|x\|}{t} \tag{7}$$

exists for all $x, y \in B_E$, we say E is Gâteaux differentiable. In this case, we called E a smooth Banach space. If for each $x \in B_E$ and the limit (7) exists for all $y \in B_E$, we say that the norm on E is Fréchet differentiable. Also, if the limit (7) is attained uniformly for any $x, y \in B_E$, then the norm on E is said to be uniformly Fréchet differentiable. In this case E is called uniformly smooth Banach space. It is well known that every uniformly smooth Banach space is smooth and reflexive.

Definition 2 Let $f : E \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. We denote the domain of f by $domf$ where $domf := \{u \in E : f(u) < +\infty\}$. When $domf \neq \emptyset$, then f is said to be proper. The Fenchel conjugate of f is the functional $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(\xi) = \sup\{\langle \xi, x\rangle - f(u) : u \in E\}, \quad \forall\, \xi \in E^*.$$

For any $u \in int(domf)$ and $y \in E$, we defined the directional derivative of f at u by

$$f'(u, v) = \lim_{t\to\infty} \frac{f(u+tv) - f(u)}{t}. \tag{8}$$

The gradient of f at u is the linear function $\nabla f(u)$ which is denoted by $\langle v, \nabla f(u)\rangle = f'(u, v)$ for all $v \in E$. If the limit (8) exists for every $u \in int(domf)$, we say that f is Gâteaux differentiable on E. When the subdifferential of f is single-valued, then $\nabla f = \partial f$, where ∂f is defined by

$$\partial f(u) = \{\xi \in E^* : f(v) - f(u) \geq \langle v - u, \xi\rangle, \quad \forall v \in E\},$$

which is the subdifferential of f at u.

Definition 3 Let E be a reflexive Banach space. We say that $f : E \to (-\infty, +\infty]$ is a Legendre function if it satisfies:

(L1) f is Gâteaux differentiable, $int(domf) \neq \emptyset$ and $dom\nabla f = int(domf)$,
(L2) f^* is Gâteaux differentiable, $int(domf^*) \neq \emptyset$ and $dom\nabla f^* = int(domf^*)$.

Since E is reflexive, then $\nabla f^* = (\nabla f)^{-1}$. If E is a smooth and strictly convex Banach space, then an example of the Legendre function is $f_p(u) = \frac{1}{p}\|u\|^p$ $(1 < p < \infty)$ with conjugate $f^*(u^*) = \frac{1}{q}\|u^*\|^q$ $(1 < q < \infty)$, where $\frac{1}{p} + \frac{1}{q} = 1$; see [3, Corollary 5.5, p. 634]. In this case, the gradient ∇f coincides with the generalized duality mapping $J_p : E \to 2^{E^*}$ which is defined by

$$J_p(u) = \{v^* \in E^* : \langle u, v^* \rangle = \|u\|^p, \|v^*\| = \|u\|^{p-1}\}, \quad \text{where} \quad u \in E.$$

If $J = J_2$, then J_2 is called the normalized duality mapping and when $E = H$, a real Hilbert space, then $J \equiv I$ is the duality mapping.

Definition 4 (*Bregman distance* [4]) Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : domf \times int(domf) \to [0, +\infty)$ defined by

$$D_f(u, v) := f(u) - f(v) - \langle u - v, \nabla f(v) \rangle \tag{9}$$

is called the Bregman distance with respect to f. Although, the Bregman distance fails to satisfies some properties of metric, it however, posses the following important properties (see [3]):

(i) For any $u, x \in domf$ and $v, w \in int(domf)$,

$$D_f(v, w) + D_f(w, u) - D_f(v, u) = \langle \nabla f(w) - \nabla f(u), w - v \rangle; \tag{10}$$

(ii) For

$$\begin{aligned} &D_f(w, x) + D_f(v, u) - D_f(w, u) - D_f(v, x) \\ &= \langle \nabla f(u) - \nabla f(x), w - v \rangle. \end{aligned} \tag{11}$$

Definition 5 The function $f : E \to (-\infty, +\infty]$ is said to be

(i) strongly coercive if [6]

$$\lim_{\|u\| \to \infty} \frac{f(u)}{\|u\|} = \infty.$$

(ii) strongly convex if there exists a constant $\sigma > 0$ such that [38]

$$f(v) \geq f(u) + \langle \nabla f(u), v - u \rangle + \frac{\sigma}{2}\|u - v\|^2.$$

Remark 1 If f is strongly coercive, then [6]

(a) $\nabla f : E \to E^*$ is bijective and norm-to-weak* continuous;
(b) $\{u \in E : D_f(u, v) \leq \alpha\}$ is bounded for all $v \in E$ and $\alpha > 0$.

In addition, if f is strongly convex with parameter $\sigma > 0$, then [51]

$$D_f(u, v) \geq \frac{\sigma}{2}\|u - v\|^2. \tag{12}$$

Definition 6 [7] Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a convex and G$\hat{a}$teaux differentiable function such that $C \subset \text{int}(\text{dom} f)$. The Bregman projection of $x \in \text{int}(\text{dom} f)$ onto C is defined as the necessarily unique vector $Proj_C^f(x) \in C$ such that

$$D_f(Proj_C^f(u), u) = \inf\{D_f(v, u) : v \in C\}.$$

Also, $p = \text{Proj}_C^f(u)$ if and only if

$$\langle \nabla f(u) - \nabla f(p), v - p \rangle \leq 0 \qquad \forall\, v \in C. \tag{13}$$

More so

$$D_f(v, Proj_C^f(u)) + D_f(Proj_C^f(u), u) \leq D_f(v, u) \qquad \forall\, u \in E,\ v \in C. \tag{14}$$

The following is an analogue of the celebrated Opial's lemma for Bregman distance in Banach space.

Lemma 1 [25] *Let $\{x_n\}$ be a sequence in E such that $x_n \rightharpoonup p$ for some $p \in E$. Then*

$$\limsup_{n\to\infty} D_f(p, x_n) < \limsup_{n\to\infty} D_f(q, x_n),$$

for all q in the interior of dom$f$ with $p \neq q$.

Next, we give some results which will be used to established the convergence of iterates to an element in the solution set of the VI (1).

Lemma 2 [5, 24, 44] *Let C be a nonempty, closed and convex subset or E. Let $A : C \to E^*$ be a continuous, monotone mapping and $z \in C$, then*

$$u^\dagger \in SOL(C, A) \quad \textit{if and only if} \quad \langle Aw, w - u^\dagger \rangle \geq 0, \quad \forall w \in C.$$

Moreover, $SOL(C, A)$ is closed and convex.

The following result is well known; see, e.g. [21, 39].

Lemma 3 *Let $\{a_n\}$ and $\{b_n\}$ be two non-negative real sequences such that*

$$a_{n+1} \leq a_n - b_n.$$

Then $\{a_n\}$ is bounded and $\sum_{n=0}^{\infty} b_n < \infty$.

3 Main Results

In this section, we present some extragradient methods which are based on Bregman projections and fixed point Eq. (2) for approximating solution of VI in reflexive Banach spaces E. Suppose C is a nonempty closed convex subset of E. For the rest of the paper, we assume that the following conditions hold.

Condition 1 The solution set of the VI (1), denoted by $SOL(C, A)$ is nonempty.

Condition 2 The mapping $A : C \to E^*$ is monotone, i.e.,

$$\langle Au - Av, u - v\rangle \geq 0 \quad \forall u, v \in C. \tag{15}$$

Condition 3 The mapping A is *Lipschitz continuous* on E with constant $L > 0$, i.e.,

$$\|Au - Av\| \leq L\|u - v\| \quad \forall u, v \in E. \tag{16}$$

Condition 4 The function $f : E \to (-\infty, +\infty]$ satisfies the following:

(B1) f is proper, convex and lower semicontinuous;
(B2) f is uniformlly Fréchet differentiable;
(B2) f is strongly convex on every E with strongly convexity constant $\sigma > 0$;
(B4) f is a strongly coercive and Legendre function which is bounded.

Next, we present a subgradient extragradient method for solving the VI.

Algorithm Subgradient extragradient method with Bregman projections

Step 0: Select a starting point $x_0 \in E$, $\beta > 0$ and set $n = 0$.
Step 1: Given the current iterate x_n, compute

$$y_n = Proj_C^f(\nabla f^*(\nabla f(x_n) - \beta Ax_n)), \tag{17}$$

construct the half-space T_n given by

$$T_n = \{u \in E : \langle\nabla f(x_n) - \beta Ax_n - \nabla f(y_n), u - y_n\rangle \leq 0\}. \tag{18}$$

Compute the next iterate via

$$x_{n+1} = Proj_{T_n}^f(\nabla f^*(\nabla f(x_n) - \beta Ay_n)). \tag{19}$$

Step 2: If $x_n = y_n$, then stop. Otherwise, set $n \leftarrow (n+1)$ and return to *Step 1*.

3.1 Convergence Analysis

The proof of the stopping criterion for Algorithm 5 follows from (2). However, we present the proof for completeness.

Lemma 4 *Suppose that* $x_n = y_n$ *for* $n \geq 1$, *then we are at a solution of the VI* (1).

Proof Since $x_n = y_n$, then from (13), we have

$$\langle\nabla f(x_n) - \beta Ax_n - \nabla f(x_n), u - x_n\rangle \leq 0 \quad \forall u \in C.$$

This implies that

$$\beta\langle Ax_n, u - x_n\rangle \geq 0 \quad \forall u \in C.$$

Since $\beta > 0$, then we get

$$\langle Ax_n, u - x_n\rangle \geq 0 \quad \forall\, u \in C.$$

Therefore $x_n \in SOL(C, A)$.

Theorem 1 *Assume that Condition 1–4 hold and $\beta \in \left(0, \frac{\sigma}{L}\right)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 5 converges weakly to a unique solution $\bar{x} \in SOL(C, A)$.*

Proof Let $z \in SOL(C, A)$, then from (14), we have

$$\begin{aligned} D_f(z, x_{n+1}) &= D_f(z, Proj^f_{T_n}(\nabla f^*(\nabla f(x_n) - \beta Ay_n))) \\ &\leq D_f(z, \nabla f^*(\nabla f(x_n) - \beta Ay_n)) - D_f(x_{n+1}, \nabla f(x_n) - \beta Ay_n) \\ &= f(z) - \langle z, \nabla f(x_n) - \beta Ay_n\rangle + f^*(\nabla f(x_n) - \beta Ay_n) \\ &\quad - f(x_{n+1}) + \langle x_{n+1}, \nabla f(x_n) - \beta Ay_n\rangle - f^*(\nabla f(x_n) - \beta Ay_n) \\ &= f(z) - \langle z, \nabla f(x_n)\rangle + f^*(x_n) - f(x_{n+1}) \\ &\quad + \langle x_{n+1}, \nabla f(x_n)\rangle - f^*(x_n) - \beta\langle z - x_{n+1}, Ay_n\rangle \\ &= D_f(z, x_n) - D_f(x_{n+1}, x_n) + \beta\langle z - x_{n+1}, Ay_n\rangle. \end{aligned} \tag{20}$$

Since A is monotone and $z \in SOL(C, A)$, then we have $\langle Ay_n, z - y_n\rangle \geq 0$. Consequently, we get

$$\beta\langle Ay_n, z - x_{n+1}\rangle \leq \beta\langle Ay_n, y_n - x_{n+1}\rangle.$$

Therefore from (20), we derive

$$D_f(z, x_{n+1}) \leq D_f(x, x_n) + \beta\langle Ay_n, y_n - x_{n+1}\rangle - D_f(x_{n+1}, x_n). \tag{21}$$

Also from (10), we get

$$D_f(x_{n+1}, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) = \langle \nabla f(y_n) - \nabla f(x_n), x_{n+1} - y_n\rangle. \tag{22}$$

Substituting (22) into (21), we obtain

$$\begin{aligned} D_f(z, x_{n+1}) &\leq D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\ &\quad - \langle \nabla f(y_n) - \nabla f(y_n), x_{n+1} - y_n\rangle + \beta\langle Ay_n, y_n - x_{n+1}\rangle \\ &= D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\ &\quad + \langle \nabla f(y_n) - \nabla f(x_n) + \beta Ay_n, y_n - x_{n+1}\rangle. \end{aligned} \tag{23}$$

Since $x_{n+1} \in T_n$ and from (18), we deduce that

$$\langle \nabla f(x_n) - \beta_n Ax_n - \nabla f(y_n), x_{n+1} - y_n\rangle \leq 0.$$

This implies that

$$\begin{aligned}\langle \nabla f(x_n) - \beta A y_n - \nabla f(y_n), x_{n+1} - y_n \rangle &\leq \langle \nabla f(x_n) - \beta A x_n - \nabla f(y_n), x_{n+1} - y_n \rangle \\ &\quad + \beta \langle A x_n - A y_n, x_{n+1} - y_n \rangle \\ &\leq \beta \langle A x_n - A y_n, x_{n+1} - y_n \rangle \\ &\leq \beta \| A x_n - A y_n \| \| x_{n+1} - y_n \| \\ &\leq \beta L \| x_n - y_n \| \cdot \| x_{n+1} - y_n \| \\ &\leq \frac{\beta L}{2} (\| x_n - y_n \|^2 + \| x_{n+1} - y_n \|^2).\end{aligned}$$

Therefore, (23) becomes

$$\begin{aligned}D_f(z, x_{n+1}) \leq & D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\ & + \frac{\beta L}{2} (\| y_n - x_n \|^2 + \| y_n - x_{n+1} \|^2).\end{aligned}$$

Using (12), we have

$$\begin{aligned}D_f(z, x_{n+1}) \leq & D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\ & + \frac{\beta L}{\sigma} (D_f(y_n, x_n) + D_f(x_{n+1}, y_n)) \\ = & D_f(z, x_n) - \left(1 - \frac{\beta L}{\sigma}\right) D_f(y_n, x_n) \\ & - \left(1 - \frac{\beta L}{\sigma}\right) D_f(x_{n+1}, y_n). \end{aligned} \tag{24}$$

Since $\beta \in (0, \sigma/L)$, then we have from (24)

$$D_f(z, x_{n+1}) \leq D_f(z, x_n).$$

This implies that $\{D_f(z, x_n)\}$ is non-increasing and hence the limit $\lim_{n\to\infty} D_f(z, x_n)$ exists. Consequently $\{D_f(z, x_n)\}$ is bounded. This means that $\{x_n\}$ is bounded. Also $\{y_n\}$ is bounded too.

Now from (24), we get

$$\left(1 - \frac{\beta L}{\sigma}\right) D_f(y_n, x_n) \leq D_f(z, x_n) - D_f(z, x_{n+1}) \to 0.$$

Thus

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0.$$

Then from (12)

$$\lim_{n\to\infty} \| x_n - y_n \| = 0. \tag{25}$$

Since f is uniformly Fréchet differentiable, then ∇f is uniformly continuous on bounded subsets of E^* and thus

$$\lim_{n\to\infty} \|\nabla f(y_n) - \nabla f(x_n)\|' = 0. \tag{26}$$

Since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x} \in C$. We now show that $\bar{x} \in SOL(C, A)$. For all $x \in C$ and since A is monotone, it follows from (13) that

$$\begin{aligned} 0 &\leq \langle \nabla f(y_{n_k}) - \nabla f(x_{n_k}) + \beta A x_{n_k}, x - y_{n_k}\rangle \\ &= \langle \nabla f(y_{n_k}) - \nabla f(x_{n_k}), x - y_{n_k}\rangle + \beta\langle A x_{n_k}, x_{n_k} - y_{n_k}\rangle \\ &\quad +\beta\langle A x_{n_k}, x - x_{n_k}\rangle \\ &\leq \langle \nabla f(y_{n_k}) - \nabla f(x_{n_k}), x - x_{n_k}\rangle + \beta\langle A x_{n_k}, x_{n_k} - y_{n_k}\rangle \\ &\quad +\beta\langle A x_{n_k}, x - x_{n_k}\rangle. \end{aligned}$$

Passing limit to the last inequality above, we get from (25) and (26) that

$$\langle Ax, x - \bar{x}\rangle \geq 0 \quad \forall x \in C.$$

Hence from Lemma 2, we derived that $\bar{x} \in SOL(C, A)$. Furthermore, using Bregman Opial property, we show that $\bar{x}$ is a unique solution of the VI (1). Suppose there exists another sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{y}$, such that $\hat{x} \neq \bar{x}$. Following similar argument has above, we get $\hat{x} \in SOL(C, A)$. It follows from the Bregman opial-like property of E (more precisely, Lemma 1) that

$$\begin{aligned} \lim_{n\to\infty} D_f(\hat{x}, x_n) &= \lim_{k\to\infty} D_f(\hat{x}, x_{n_k}) < \lim_{k\to\infty} D_f(\bar{x}, x_{n_k}) \\ &= \lim_{n\to\infty} D_f(\bar{x}, x_n) = \lim_{j\to\infty} D_f(\bar{x}, x_{n_j}) \\ &< \lim_{j\to\infty} D_f(\hat{x}, x_{n_j}) = \lim_{n\to\infty} D_f(\hat{x}, x_n), \end{aligned}$$

which is a contradiction. Thus, we have $\hat{x} = \bar{x}$ and the desired result follows. This completes the proof.

Next, we present a Popov's extragradient method for solving VI with its convergence analysis using Bregman distance technique in reflexive Banach spaces. We assume that *Condition* 1–4 are valid.

Algorithm Popov's extragradient method with Bregman projection

Step 0: Pick $x_0, y_0 \in H$ arbitrarily and let $\beta > 0$. Set $n = 0$.
Step 1: Compute

$$\begin{aligned} x_1 &= Proj_C^f(\nabla f^*(\nabla f(x_0) - \beta A y_0)), \\ y_1 &= Prof_C^f(\nabla f^*(\nabla f(x_1) - \beta A y_0)). \end{aligned}$$

Step 2: Given the current iterates x_n, y_n and y_{n-1}, calculate x_{n+1} and y_{n+1} as follows:

$$x_{n+1} = Proj_{T_n}^f(\nabla f^*(\nabla f(x_n) - \beta Ay_n)), \tag{27}$$
$$y_{n+1} = Proj_C^f(\nabla f^*(\nabla f(x_{n+1}) - \beta Ay_n)),$$

where

$$T_n := \{z \in E : \langle \nabla f(x_n) - \beta Ay_{n-1} - \nabla f(y_n), z - y_n\rangle \le 0\}. \tag{28}$$

Step 3: If $x_{n+1} = x_n$ and $y_n = x_n$ STOP. Otherwise, set $n \leftarrow n + 1$ and repeat Step 2.

Remark 2 Algorithm 7 is constructed based on the concepts of [37] and [40] using Bregman projections. Note that Algorithm 7 compute only one projection for y_{n+1} onto C and one evaluation of A at the current approximation y_n. Moreover, it is easy to see that $C \subset T_n$, for all $n \in \mathbb{N}$.

3.2 Convergence Analysis

First, we show that the stopping criterion of Algorithm 7 is valid.

Lemma 5 *Suppose $x_{n+1} = x_n = y_n$ in Algorithm 7. Then x_n is a solution of the VI* (1).

Proof Since $x_{n+1} = x_n$, we get from (13) that

$$\langle \nabla f(x_{n+1}) - \nabla f(x_n) + \beta Ay_n, z - x_{n+1}\rangle \ge 0 \quad \forall z \in T_n.$$

Thus

$$\beta\langle Ay_n, z - x_n\rangle \ge 0 \quad \forall z \in T_n.$$

Equivalently

$$\langle Ay_n, z - y_n\rangle \ge \langle Ay_n, x_n - y_n\rangle \quad \forall z \in T_n.$$

This implies that

$$\langle Ay_n, z - y_n\rangle \ge \langle Ay_n, x_n - y_n\rangle \quad \forall z \in C. \tag{29}$$

Moreover, it follows from the definition of T_n that

$$\langle \nabla f(x_n) - \beta Ay_{n-1} - \nabla f(y_n), u - y_n\rangle \le 0 \quad \forall u \in E.$$

Substituting $y_{n-1} = y_n$ and $z = x_n$ the above inequality, we get

$$\langle \nabla f(x_n) - \beta A y_n - \nabla f(y_n), x_n - y_n \rangle \leq 0.$$

Thus

$$\beta \langle A y_n, x_n - y_n \rangle \geq 0. \tag{30}$$

Combining (29) and (30), we have

$$\langle A y_n, z - y_n \rangle \geq 0 \quad \forall z \in C.$$

Using the fact that $x_n = y_n$, we get

$$\langle A x_n, z - x_n \rangle \geq 0 \quad \forall z \in C.$$

Hence, this implies that x_n is a solution of the VI (1).

We now prove the convergence of the iterates $\{x_n\}$, $\{y_n\}$ to a solution of the VI (1).

Theorem 2 *Assume that Condition 1–4 and let* $\beta \in \left(0, \frac{(\sqrt{2}-1)\sigma}{L}\right)$. *Then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by Algorithm 7 converges weakly to a unique solution* $\bar{x} \in SOL(C, A)$.

Proof Let $z \in SOL(C, A)$, then we have

$$\langle \nabla f(x_{n+1}) - (\nabla f(x_n) - \beta A y_n), z - x_{n+1} \rangle \geq 0.$$

This implies that

$$\langle \nabla f(x_{n+1}) - \nabla f(x_n) + \beta A y_n, z - x_{n+1} \rangle \geq 0.$$

Thus we have

$$\beta \langle A y_n, z - x_{n+1} \rangle \geq \langle \nabla f(x_{n+1}) - \nabla f(x_n), x_{n+1} - z \rangle. \tag{31}$$

Note that from (10), we get

$$D_f(z, x_n) = D_f(z, x_{n+1}) + D_f(x_{n+1}, x_n) + \langle \nabla f(x_{n+1}) - \nabla f(x_n), x_{n+1} - z \rangle.$$

Then (31) becomes

$$\beta \langle A y_n, z - x_{n+1} \rangle \geq D_f(z, x_n) - D_f(z, x_{n+1}) - D_f(x_{n+1}, x_n).$$

Hence

$$D_f(z, x_{n+1}) \leq D_f(z, x_n) + \beta \langle A y_n, z - x_{n+1} \rangle - D_f(x_{n+1}, x_n).$$

Since A is monotone and $z \in SOL(C, A)$, then we have $\langle Ay_n, z - y_n\rangle \geq 0$. Consequently, we get

$$\langle Ay_n, z - x_{n+1}\rangle \leq \langle Ay_n, y_n - x_{n+1}\rangle.$$

This implies that

$$\begin{aligned} D_f(z, x_{n+1}) &\leq D_f(z, x_n) - D_f(x_{n+1}, x_n) + \beta\langle Ay_n, y_n - x_{n+1}\rangle \\ &= D_f(z, x_n) - D_f(x_{n+1}, x_n) + \beta\langle Ay_{n-1}, y_n - x_{n+1}\rangle \\ &\quad +\beta\langle Ay_n - Ay_{n-1}, y_n - x_{n+1}\rangle. \end{aligned} \tag{32}$$

But

$$\begin{aligned} \beta\langle Ay_{n-1}, y_n - x_{n+1}\rangle &= \langle \nabla f(x_n) - \beta Ay_{n-1} - \nabla f(y_n), x_{n+1} - y_n\rangle \\ &\quad +\langle \nabla f(y_n) - \nabla f(x_n), x_{n+1} - y_n\rangle. \end{aligned}$$

Since $x_{n+1} \in T_n$ and by the definition of T_n, we get

$$\beta\langle Ay_{n-1}, y_n - x_{n+1}\rangle \leq \langle \nabla f(y_n) - \nabla f(x_n), x_{n+1} - y_n\rangle. \tag{33}$$

Using (10) in (33), we obtain

$$\beta\langle Ay_{n-1}, y_n - x_{n+1}\rangle \leq D_f(x_{n+1}, x_n) - D_f(x_{n+1}, y_n) - D_f(y_n, x_n). \tag{34}$$

Combining (32) and (34), we get

$$\begin{aligned} D_f(z, x_{n+1}) &\leq D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\ &\quad + \beta\langle Ay_n - Ay_{n-1}, y_n - x_{n+1}\rangle. \end{aligned} \tag{35}$$

Note that by the Cauchy-Schwartz inequality, we have

$$\begin{aligned} \beta\langle Ay_{n-1} - Ay_n, x_{n+1} - y_n\rangle &\leq \beta\|Ay_{n-1} - Ay_n\| \cdot \|x_{n+1} - y_n\| \\ &\leq \beta L\|y_{n-1} - y_n\| \cdot \|x_{n+1} - y_n\| \\ &\leq \beta L\left(\frac{1}{2\sqrt{2}}\|y_{n-1} - y_n\|^2 + \frac{1}{\sqrt{2}}\|x_{n+1} - y_n\|^2\right) \\ &\leq \frac{\beta L}{2\sqrt{2}}\left((2+\sqrt{2})\|x_n - y_n\|^2 + \sqrt{2}\|x_n - y_{n-1}\|^2\right) \\ &\quad +\frac{\beta L}{\sqrt{2}}\|x_{n+1} - y_n\|^2 \\ &\leq \frac{\beta L}{\sigma}(1+\sqrt{2})D_f(y_n, x_n) + \frac{\beta L}{\sigma}D_f(x_n, y_{n-1}) \\ &\quad +\frac{\beta L\sqrt{2}}{\sigma}D_f(x_{n+1}, y_n), \end{aligned} \tag{36}$$

where in (36), we have also used the following basic inequalities with (12):

$$ab \le \frac{\tau^2}{2}a^2 + \frac{1}{2\tau^2}b^2 \quad \text{and} \quad (a+b)^2 \le \sqrt{2}a^2 + (2+\sqrt{2})b^2.$$

It follows from (35) and (36) that

$$\begin{aligned} D_f(z, x_{n+1}) \le D_f(z, x_n) &- \left(1 - \frac{\beta L\sqrt{2}}{\sigma}\right) D_f(x_{n+1}, y_n) \\ &- \left(1 - \frac{\beta L}{\sigma}(1+\sqrt{2})\right) D_f(y_n, x_n) + \frac{\beta L}{\sigma} D_f(x_n, y_{n-1}). \end{aligned} \tag{37}$$

Now let

$$a_n = D_f(z, x_n) + \frac{\lambda L}{\sigma} D_f(x_n, y_{n-1})$$

and

$$b_n = \left(1 - \frac{\lambda L(1+\sqrt{2})}{\sigma}\right)\left(D_f(y_n, x_n) + D_f(x_{n+1}, y_n)\right).$$

Then we can rewrite (37) as

$$a_{n+1} \le a_n - b_n.$$

Then, it follows from Lemma 3 that $\{a_n\}$ is bounded and $\lim_{n\to\infty} D_f(y_n, x_n) = 0$. Consequently, $\{x_n\}$ is bounded and $\|x_n - y_n\| \to 0$, $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. This implies that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Furthermore

$$\begin{aligned} \|y_{n+1} - y_n\| \le \|x_{n+1} - x_n\| &+ \|y_{n+1} - x_{n+1}\| \\ &+ \|x_n - y_n\| \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{38}$$

Consequent

$$\lim_{n\to\infty} \|x_{n+1} - y_{n+1}\| \le \lim_{n\to\infty} (\|x_{n+1} - y_n\| + \|y_{n+1} - y_n\|) = 0.$$

Since f is uniformly Fréchet differentiable, then ∇f is uniformly continuous on bounded subsets of E^*, hence

$$\lim_{n\to\infty} \|\nabla f(y_{n+1}) - \nabla f(y_n)\|' = 0. \tag{39}$$

and

$$\lim_{n\to\infty} \|\nabla f(x_{n+1}) - \nabla f(y_{n+1})\|' = 0. \tag{40}$$

Since $\{x_n\}$ is bounded, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$. Consequently, $y_{n_k} \rightharpoonup \bar{x}$. Since $\{y_n\} \subset C$, thus $\bar{x} \in C$. Moreover, from the definition on y_{n+1} and (13), we get

$$\langle \nabla f(x_{n+1}) - \beta A y_n - \nabla f(y_{n+1}), x - y_{n+1} \rangle \leq 0, \quad \forall x \in C.$$

Equivalently

$$\langle \nabla f(x_{n+1}) - \nabla f(y_{n+1}), x - y_{n+1} \rangle \leq \langle \beta A y_n, x - y_{n+1} \rangle \quad \forall x \in C.$$

Thus

$$\left\langle \frac{\nabla f(x_{n+1}) - \nabla f(y_{n+1})}{\beta}, x - y_{n+1} \right\rangle + \langle A y_n, y_{n+1} - y_n \rangle \leq \langle A y_n, x - y_n \rangle \quad \forall x \in C.$$

This implies that

$$\left\langle \frac{\nabla f(x_{n_j+1}) - \nabla f(y_{n_j+1})}{\beta}, x - y_{n_j+1} \right\rangle + \langle A y_{n_j}, y_{n_j+1} - y_{n_j} \rangle \leq \langle A y_{n_j}, x - y_{n_j} \rangle \quad \forall x \in C.$$

Passing limits to the above inequality and using (38) and (40), we get

$$\langle A\bar{x}, x - \bar{x} \rangle \geq 0 \quad \forall x \in C.$$

Hence $\bar{x} \in SOL(C, A)$. Using the opial property as in the Proof of Theorem 1, we obtain that $\bar{x}$ is a unique solution of VI. This completes the proof.

Next, we present a Tseng extragradient method for solving the VI in reflexive Banach spaces.

Algorithm Tseng extragradient method using Bregman projection

Step 0: Pick $x_0 \in E$, $\beta > 0$ and set $n = 0$.
Step 1: Given the current iterate x_n, compute

$$y_n = Proj_C^f(\nabla f^*(\nabla f(x_n) - \beta A x_n)). \tag{41}$$

If $x_n = y_n$, STOP. Otherwise,
Step 2: Compute

$$x_{n+1} = \nabla f^*(\nabla f(y_n) - \beta(A y_n - A x_n)), \tag{42}$$

Set $n := n + 1$ and repeat Step 1.

Remark 3 We note that the stopping criterion is the same as that of Algorithm 5 and it is also valid for Algorithm 9.

3.3 Convergence Analysis

Theorem 3 *Assume that Condition 1–4 hold and $\beta \in (0, \sigma/L)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 9 converges weakly to a unique solution $\bar{x} \in SOL(C, A)$.*

Proof Let $z \in Sol(C, A)$, then

$$\begin{aligned} D_f(z, x_{n+1}) &= D_f(z, \nabla f^*(\nabla f(y_n) - \beta(Ay_n - Ax_n)) \\ &= f(z) - \langle z - x_{n+1}, \nabla f(y_n) - \beta(Ay_n - Ax_n)\rangle - f(x_{n+1}) \\ &= f(z) + \langle x_{n+1} - z, \nabla f(y_n)\rangle + \langle z - x_{n+1}, \beta(Ay_n - Ax_n)\rangle - f(x_{n+1}) \\ &= f(z) - \langle z - y_n, \nabla f(y_n)\rangle - f(y_n) + \langle z - y_n, \nabla f(y_n)\rangle + f(y_n) \\ &\quad +\langle x_{n+1} - z, \nabla f(y_n)\rangle + \langle z - x_{n+1}, \beta(Ay_n - Ax_n)\rangle - f(x_{n+1}) \\ &= D_f(z, y_n) + \langle x_{n+1} - y_n, \nabla f(y_n)\rangle + f(y_n) - f(x_{n+1}) \\ &\quad +\langle z - x_{n+1}, \beta(Ay_n - Ax_n)\rangle \\ &= D_f(z, y_n) - D_f(x_{n+1}, y_n) + \langle z - x_{n+1}, \beta(Ay_n - Ax_n)\rangle. \end{aligned} \tag{43}$$

Note that from (11), we have

$$\begin{aligned} D_f(z, y_n) - D_f(x_{n+1}, y_n) = D_f(z, x_n) - D_f(x_{n+1}, x_n) \\ +\langle \nabla f(x_n) - \nabla f(y_n), z - x_{n+1}\rangle. \end{aligned} \tag{44}$$

Then from (43) and (44), we have

$$\begin{aligned} D_f(z, x_{n+1}) = D_f(z, x_n) - D_f(x_{n+1}, x_n) + \langle z - x_{n+1}, \beta(Ay_n - Ax_n)\rangle \\ +\langle \nabla f(x_n) - \nabla f(y_n), z - x_{n+1}\rangle. \end{aligned} \tag{45}$$

Also, (10) yields

$$D_f(x_{n+1}, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) = \langle \nabla f(x_n) - \nabla f(y_n), y_n - x_{n+1}\rangle. \tag{46}$$

Substituting (46) into (45), we have

$$\begin{aligned} D_f(z, x_{n+1}) &= D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\ &\quad -\langle \nabla f(x_n) - \nabla f(y_n), y_n - x_{n+1}\rangle \\ &\quad +\langle z - x_{n+1}, \beta(Ay_n - Ax_n)\rangle + \langle \nabla f(x_n) - \nabla f(y_n), z - x_{n+1}\rangle \\ &= D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\ &\quad +\langle \nabla f(x_n) - \nabla f(y_n), z - y_n\rangle + \langle z - x_{n+1}, \beta(Ay_n - Ax_n)\rangle \\ &= D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\ &\quad +\langle \nabla f(x_n) - \nabla f(y_n), z - y_n\rangle - \langle x_{n+1} - y_n + y_n - z, \beta(Ay_n - Ax_n)\rangle \\ &= D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) \\ &\quad +\langle \nabla f(x_n) - \nabla f(y_n), z - y_n\rangle - \langle x_{n+1} - y_n, \beta(Ay_n - Ax_n)\rangle \\ &\quad -\langle y_n - z, \beta(Ay_n - Ax_n)\rangle \end{aligned}$$

$$= D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n) - \langle x_{n+1} - y_n, \beta(Ay_n - Ax_n)\rangle$$
$$-\langle y_n - z, \beta(Ay_n - Ax_n) - (\nabla f(y_n) - \nabla f(x_n))\rangle. \quad (47)$$

From the definition of y_n in (41), it follows from (13) that

$$\langle \nabla f(x_n) - \beta Ax_n - \nabla f(y_n), z - y_n\rangle \leq 0. \quad (48)$$

Since A is monotone and $z \in SOL(C, A)$, then we have $\langle Ay_n, y_n - z\rangle \leq 0$. Combining this with (48), we get

$$\langle \beta(Ay_n - Ax_n) - (\nabla f(y_n) - \nabla f(x_n)), y_n - z\rangle \geq 0.$$

Hence from (47), we obtain

$$D_f(z, x_{n+1}) \leq D_f(z, x_n) - D_f(y_n, x_n) - D_f(x_{n+1}, y_n)$$
$$-\langle x_{n+1} - y_n, \beta(Ay_n - Ax_n)\rangle. \quad (49)$$

More so, using the Cauchy-Schwartz inequality and (12), we have

$$\begin{aligned}\langle y_n - x_{n+1}, Ay_n - Ax_n\rangle &\leq \beta\|y_n - x_{n+1}\| \cdot \|Ay_n - Ax_n\|\\ &\leq \beta L\|y_n - x_{n+1}\| \cdot \|y_n - x_n\|\\ &\leq \frac{\beta L}{2}(\|y_n - x_{n+1}\|^2 + \|x_n - y_n\|^2)\\ &\leq \frac{\beta L}{\sigma}(D_f(x_{n+1}, y_n) + D_f(y_n, x_n)).\end{aligned}$$

Thus (49) becomes

$$D_f(z, x_{n+1}) \leq D_f(z, x_n) - \left(1 - \frac{\beta L}{\sigma}\right) D_f(x_{n+1}, y_n) - \left(1 - \frac{\beta L}{\sigma}\right) D_f(y_n, x_n). \quad (50)$$

Since $\beta \in \left(0, \frac{\sigma}{L}\right)$, then we have

$$D_f(z, x_{n+1}) \leq D_f(z, x_n).$$

Hence $\{D_f(z, x_n)\}$ is non-increasing, this implies that $\lim_{n\to\infty} D_f(z, x_n)$ exists and thus $\{D_f(z, x_n)\}$ is bounded. Consequently, $\{x_n\}$ and $\{y_n\}$ are bounded.
From (50), we have

$$\left(1 - \frac{\beta L}{\sigma}\right) D_f(y_n, x_n) \leq D_f(z, x_n) - D_f(z, x_{n+1}) \to 0.$$

This implies that

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0.$$

Since $\{x_n\}$ is bounded, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x} \in C$. Following similar argument as in the Proof of Theorem 1, we have that $\bar{x} \in SOL(C, A)$ and it is unique.

4 Numerical Experiments

In this section, we perform some experiments to illustrate the numerical behaviour of the algorithms in the previous section. We also consider the numerical experiments using some known convex functions with their corresponding Bregman distance given as follows:

(i) Squared Euclidean distance (SED) with $dom f = \mathbb{R}^n$, and

$$f(x) = \frac{1}{2}x^T x, \quad \nabla f(x) = x, \quad \nabla f^*(x) = x,$$
$$D_f(x, y) = \frac{1}{2}||x - y||^2.$$

(ii) Mahalanobis distance (MD) with $dom f = \mathbb{R}^n$, and

$$f(x) = \frac{1}{2}x^T Qx, \quad \nabla f(x) = Qx, \quad \nabla f^*(x) = Q^{-1}x,$$
$$D_f(x, y) = \frac{1}{2}(x - y)^T Q(x - y),$$

where Q is symmetric positive definite (in some applications, Q is positive semidefinite, but not positive definite).

(iii) Kullback-Leibler distance (KLD) with $dom f = \mathbb{R}^n_+$,

$$f(x) = \sum_{i=1}^{n} x_i \log x_i, \quad \nabla f(x) = \begin{bmatrix} \log x_1 + 1 \\ \vdots \\ \log x_n + 1 \end{bmatrix}, \quad \nabla f^*(x) = \begin{bmatrix} \exp(x_1 - 1) \\ \vdots \\ \exp(x_n - 1) \end{bmatrix},$$
$$D_f(x, y) = \sum_{i=1}^{n} \left(x_i \log \frac{x_i}{y_i} - x_i + y_i \right).$$

Here, the function f is called relative (Shannon) entropy.

(iv) Itakura-Saito divergence (ISD) with $dom f = \mathbb{R}^n_{++}$,

Table 1 Computation results showing No of iteration and Time taken by each algorithm using different convex functions

			Algorithm 5	Algorithm 7	Algorithm 9
SED	$m = 5$	Iter.	29	14	26
		Time (sec)	0.0062	0.0029	0.0054
	$m = 50$	Iter.	257	20	237
		Time (sec)	0.0191	0.0059	0.0133
	$m = 100$	Iter.	1605	6	1318
		Time (sec)	6.0354	0.0059	4.5291
KLD	$m = 5$	Iter.	965	53	709
		Time (sec)	3.0710	0.0218	1.4066
	$m = 50$	Iter.	775	14	410
		Time (sec)	1.3694	0.0250	0.4100
	$m = 100$	Iter.	416	8	119
		Time (sec)	0.4129	0.0045	0.0361
ISD	$m = 5$	Iter.	1126	735	927
		Time (sec)	2.5143	1.2777	1.9037
	$m = 50$	Iter.	1358	1057	1983
		Time (sec)	4.9860	3.2387	9.5451
	$m = 100$	Iter.	1563	1246	1596
		Time (sec)	6.6140	3.9076	7.0319
SMD	$m = 5$	Iter.	6	9	8
		Time (sec)	0.0026	0.0037	0.0029
	$m = 50$	Iter.	5	27	8
		Time (sec)	0.0031	0.0084	0.0037
	$m = 100$	Iter.	4	32	8
		Time (sec)	0.0027	0.0149	0.0067

$$f(x) = -\sum_{i=1}^{n} \log x_i, \quad \nabla f(x) = \begin{bmatrix} -\frac{1}{x_1} \\ \vdots \\ -\frac{1}{x_n} \end{bmatrix}, \quad \nabla f^*(x) = \begin{bmatrix} -\frac{1}{x_1} \\ \vdots \\ -\frac{1}{x_n} \end{bmatrix}$$

$$D_f(x, y) = \sum_{i=1}^{n} \left(\frac{x_i}{y_i} - \log \frac{x_i}{y_i} - 1 \right),$$

The function f is also called Burg entropy.

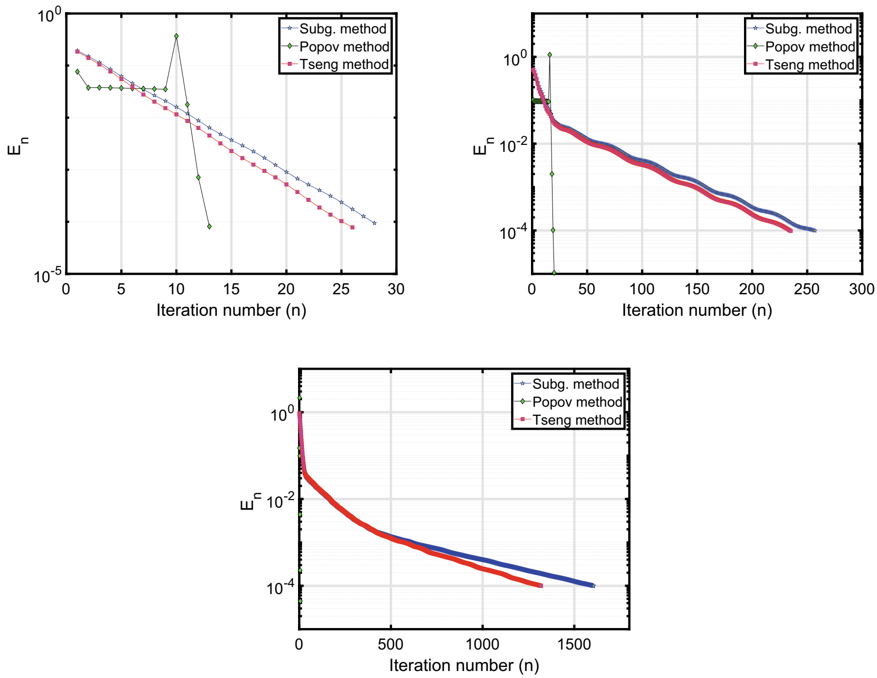

Fig. 1 SED; top left: $m = 5$; top right: $m = 50$, bottom: $m = 100$

All computation are carried out using Lenovo PC with the following specification: Intel(R)core i7-600, CPU 2.48GHz, RAM 8.0GB, MATLAB version 9.5 (R2019b). We consider the variational inequalities model with the operator $A : \mathbb{R}^m \to \mathbb{R}^m$ defined by $A(x) = Mx + q$ where

$$M = BB^T + S + D,$$

where B is a $m \times m$ matrix, S is a $m \times m$ skew symmetric matrix, D is a $m \times m$ diagonal matrix whose diagonal are non-negative (so M is positive definite) and q is a vector in $\mathbb{R}^m$. The feasible set C is defined by

$$C = \left\{ x = (x_1, \ldots, x_m)^T : \|x\| \leq 1 \quad \text{and} \quad x_i \geq a, \ i = 1, \ldots, m \right\}$$

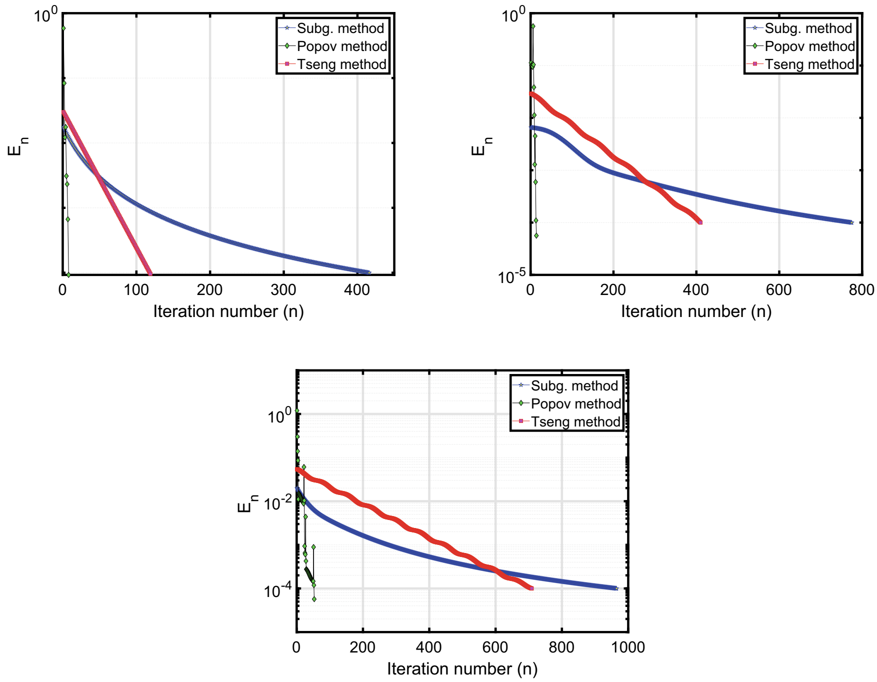

Fig. 2 KLD; top left: $m = 5$; top right: $m = 50$, bottom: $m = 100$

where $a < 1/\sqrt{m}$. It is clear that A is monotone and Lipschitz continuous with Lipschitz constant $L = \|M\|$. For $q = 0$, the unique solution of the corresponding VI is $\{0\}$. We compare the performance of Algorithm 5, 7 and 9 for each convex function given above and $m = 5, 50, 100$. The initial values are generated randomly in C with $a = 0.01$ and we take $\beta = \frac{1}{2L}$. Note that we are at a solution of the VI (1) if $x_n = y_n$ in Algorithm 5 and (9), and $x_{n+1} = x_n = y_n$ in Algorithm 7. Thus, we show the numerical behaviour of the sequence $E_n = \|y_n - x_n\|$ for Algorithm 5 and Algorithm 9, and $E_n = \|x_{n+1} - x_n\| + \|x_n - y_n\|$ for Algorithm 7. The computational results are shown in Table 1, Figs. 1, 2, 3 and 4.

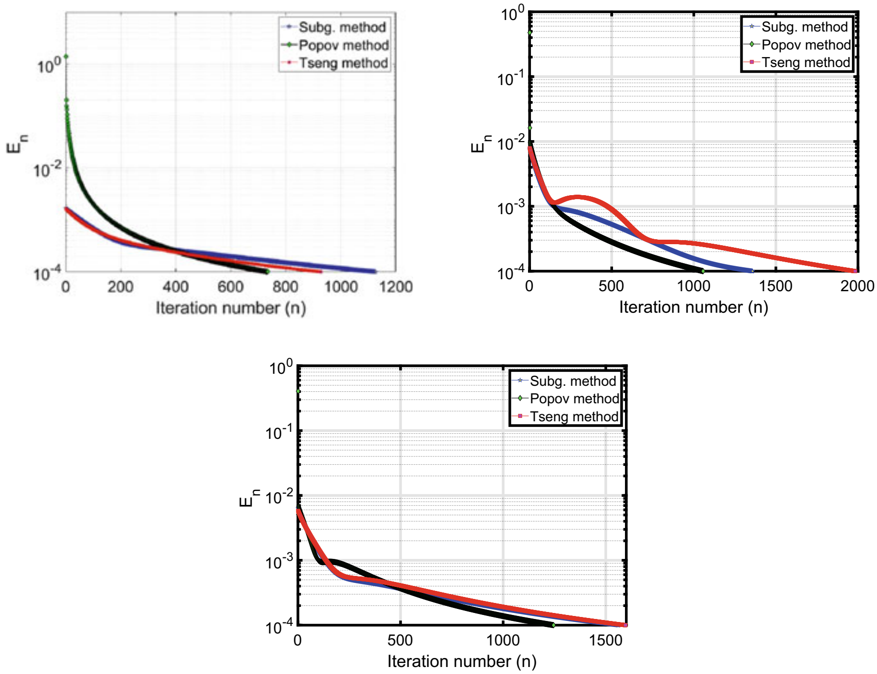

Fig. 3 ISD; top left: $m = 5$; top right: $m = 50$, bottom: $m = 100$

The numerical experiments show that the change in the convex function has significant effect on the behaviour of the sequence generated by each algorithm. It is important to also mention that the performance of the algorithm can be improved by using appropriate stepsize or using self-adaptive technique for selecting the stepsize.

5 Conclusion

In this chapter, we introduce some extragradient-type methods for solving variational inequalities in reflexive Banach spaces. The convergence of the algorithms are proved using the Bregman distance technique. Also, we provide some numerical illustration of the sequence generated by each algorithm using different type of convex functions. These results extend and improve several interesting results in the literature.

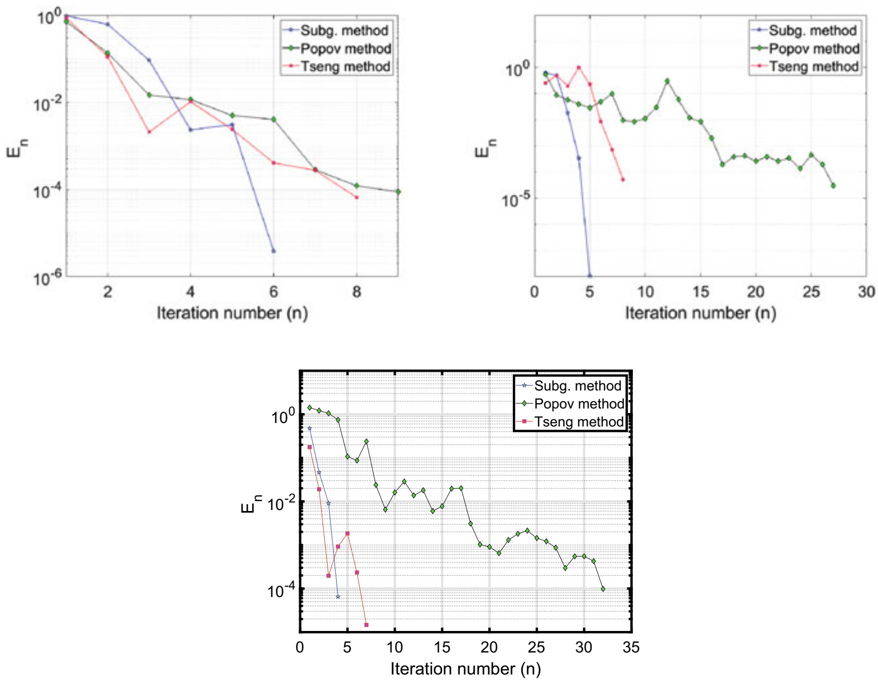

Fig. 4 SMD; top left: $m = 5$; top right: $m = 50$, bottom: $m = 100$

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Common Solutions to Variational Inequality Problem via Parallel and Cyclic Hybrid Inertial CQ-Subgradient Extragradient Algorithms in (HSs)

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Abstract The intent of this manuscript is to present new algorithms, so-called strongly convergent parallel and cyclic hybrid inertial CQ-subgradient extragradient (PCHICQ-SE) algorithms. Proposed algorithms are applied to find common solutions to the variational inequality problem (CSVIP) in the Hilbert spaces (HSs). Ultimately, numerical experiments are presented here to examine the efficiency of our algorithms.

1 Introduction

Assume that H is a (HS) with the induced norm $\|.\|$ and the inner product $\langle ., .\rangle$, and $C \neq \emptyset$ is a closed convex subset (CCS) of H.

In 1966, Hartman and Stampacchia [1] introduced the concept of variational inequality problem (VIP), for obtaining $y^* \in C$ so that

$$\langle Ay^*, y - y^*\rangle \geq 0, \ \forall y \in C, \tag{1}$$

where $A : H \to H$ is a nonlinear mapping (NM). The set of solutions of (VIP) (1) is denoted by $VI(A, C)$. Many algorithms that were based on projections onto (CCSs) have been proposed for solving (VIP). One of these, and the easiest, is the gradient method because one projection is calculated on the feasible set. Even so, studying convergence for this method requires strong assumptions on the involved operators.

Another (PM) for solving saddle point problems and generalizing (VIP) for the Lipschitz continuous and monotone mapping has been presented by Korpelevich [2]. He named it as the extragradient method (EM) which is built below:

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$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_C(x_n - \lambda A(y_n)), \end{cases}$$

where P_C is the metric projection onto C and λ is a suitable parameter. The projections can be found easily if C is a simple structure and the (EM) is computable and very useful. Otherwise, it is more complicated.

In order to obtain the weak convergence of a solution of the (VIP), Censor et al. [3] gave the subgradient extragradient (SE) method, for (VIP) in (HSs):

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ x_{n+1} = P_{T_n}(x_n - \lambda A(y_n)), \end{cases}$$

where a half-space T_n is described as

$$T_n = \{v \in H : \langle (x_n - \lambda A(x_n)) - y_n, v - y_n \rangle \leq 0\}.$$

To accelerate the convergence of this algorithm, authors [4, 5] presented the algorithm below, which combines the (SEM) with the hybrid method:

$$\begin{cases} y_n = P_C(x_n - \lambda A(x_n)), \\ z_n = \alpha_n x_n + (1 - \alpha_n) P_{T_n} x_n, \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

Here, the (CSVIP) is to find a point $y^* \in K = \cap_{i=1}^N K_i$ so that

$$\langle A_i(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in K_i, i = 1, \ldots, N, \tag{2}$$

where $A_i : H \to H$ is a NM and K_i is a finite collection of non-empty (CCSs) of H so that $\cap_{i=1}^N K_i \neq \emptyset$.

The study of (CSVIP) with $N > 1$ results from selecting $A_i = 0$, then the problem is reduced to finding a point $y^* = \cap_{i=1}^N K_i$ in the non-empty intersection of a finite family of closed and convex sets, and it is called the convex feasibility problem (CFP). For a family $T_i : H \to H$, if K_i are the fixed point sets, then the problem is called the common fixed point problem (CFPP). It is worth noting that these issues have been studied on a large scale in the past decades, theoretically and numerically, and this is what gives these issues their importance to many researchers; see [6–13] and the references therein.

For multi-valued mappings $A_i : H \to 2^H$, $i = 1, \ldots, N$, the authors [14] introduced a procedure to discuss the (CSVIP). This procedure is considered so that mappings A_i are single-valued as the following: Choose $x_1 \in H$ and calculate

$$\begin{cases} y_n^i = P_{K_i}(x_n - \lambda_n^i A_i(x_n)), \\ z_n^i = P_{K_i}(x_n - \lambda_n^i A_i(y_n)), \\ C_n^i = \{z \in H : \langle x_n - z_n^i, z - x_n - \gamma_n^i(z_n^i - x_n)\rangle\}, \\ C_n = \cap_{i=1}^N C_n^i, \\ W_n = \{z \in H : \langle x_1 - x_n, z - x_n\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap W_n} x_1. \end{cases} \tag{3}$$

In order to find the approximation x_{n+1} of the Algorithm (3), we have to construct $N+1$ subsets $C_n^1, C_n^2, \ldots, C_n^i, W_n$ and solve the following distance optimization problem:

$$\begin{cases} \min \|z - x_1\|^2, \\ \text{so that } z \in C_n^1 \cap \ldots \cap C_n^i \cap W_n. \end{cases} \tag{4}$$

When the number of mappings N is large, the task of solving (4) is very costly.

In the Banach spaces, for searching of the common solution to the (VIP), Anh and Hieu [15, 16] have presented an iterative method and this method is iterated in (HSs) as follows:

$$\begin{cases} x_0 \in C, \\ y_n^i = \alpha_n x_n + (1-\alpha_n) S_i x_n, \; i = 1, \ldots, N, \\ i_n = \arg\max\{\left\|y_n^i - x_n\right\| : i = 1, \ldots, N\}, \; y_n^- = y_n^{i_n}, \\ C_{n+1} = \{v \in C_n : \left\|v - y_n^-\right\| \le \|v - x_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0. \end{cases}$$

where $\alpha_n \in (0, 1)$, $\limsup_{n\to\infty} \alpha_n < 1$. According to this algorithm, the approximation x_{n+1} is defined as the projection of x_0 onto C_{n+1}. However, it seems difficult to find the explicit form of the sets C_n and perform numerical experiments. Continuing in this line, Hieu [17] introduced two (PCHSE) algorithms for (CSVIP) in (HSs) and analyzed the convergence by numerical results. Hasanen et al. [18–20] extended it by introducing advanced algorithms.

In this paper, we introduce (PCHICQ-SE) algorithms for solving (CSVIP) and generate sequences that converge strongly to the nearest point projection of the starting point onto the solution set of the (CSVIP). Also, we give some numerical experiments to support our results.

2 Crucial Lemmas

We begin this part with some important definitions and lemmas.

Definition 1 [21] A nonlinear operator A (for all $x, y \in H$) is called

(i) monotone if

$$\langle x - y, Ax - Ay\rangle \ge 0,$$

(ii) pseudo-monotone if $\langle Ax - Ay, x - y\rangle \ge 0$ implies that

$$\langle Ay - Ax, y - x\rangle \leq 0,$$

(iii) L-Lipschitz continuous if $\exists L > 0$ so that

$$\frac{||Ax - Ay||}{||x - y||} \leq L,$$

(iv) non-expansive if

$$\frac{||Ax - Ay||}{||x - y||} \leq 1.$$

The set of fixed points of a mapping A is defined by $F(A) = \{x \in H : Ax = x\}$.

Alber and Ryazantseva [22] illustrated the notion of maximal monotone (MM) mapping as follows: A monotone mapping $A : H \to H$ is maximal if and only if, for each $(x, y) \in H \times H$ so that

$$\langle x - u, y - v\rangle \geq 0, \ \forall (u, v) \in G(A),$$

then $y = T(x)$.

Lemma 1 *[23] Let H be a (HS). Then for each $(x, y) \in H \times H$ and $\alpha \in [0, 1]$,*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha \|x\|^2 + (1-\alpha) \|y\|^2 - \alpha(1-\alpha) \|x - y\|^2 .$$

For every $x \in H$, the projection $P_C x$ of x onto C is defined by $P_C x = \arg\min \{\|y - x\| : y \in C\}$. Since C is a non-empty (CCS) of H, $P_C x$ exists and is unique. The projection $P_C : H \to C$ has the following properties:

Lemma 2 *[21] Let $P_C : H \to C$. Then*

1. P_C is 1-inverse strongly monotone (ISM), i.e., $\forall x, y \in H$,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle.$$

2. $\forall y \in H, x \in C$,

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2 .$$

3. $z = P_C x$ iff

$$\langle x - z, z - y\rangle \geq 0, \ \forall y \in C.$$

For more details about the projection P_C, see [24]. The following lemma illustrate some topological properties of the $VI(A, C)$:

Lemma 3 *[23] Consider $C \neq \emptyset$ (CCS) of a (HS) H and A be a hemi-continuous and monotone mapping. Then,*

$$VI(A, C) = \{e \in C : \langle o - e, A(o)\rangle \geq 0, \ \forall o \in C\}.$$

A normal cone (NC) N_C to a set C at a point $\ell \in C$ is described as

$$N_C\ell = \{\ell^* \in H : \langle \ell - o, \ell^* \rangle \geq 0,\ \forall o \in C\},$$

we have the following:

Lemma 4 *[25] Assume that $C \neq \emptyset$ (CCS) of a (HS) H, A is a hemi-continuous and monotone mapping of C into H with $D(A) = C$ and B is a mapping described as*

$$B(\ell) = \begin{cases} A\ell + N_C\ell & \text{if } \ell \in C \\ \emptyset & \text{if } \ell \notin C \end{cases},$$

therefore B is a (MM) and $B^{-1}(0) = VI(A, C)$.

3 Main Theorems

This part is devoted to introduce and study the strongly convergent of two (PCHISE) algorithms.

Theorem 1 ***(PHICQ-SE algorithm)***
Let K_i, $i = 1, \ldots, N$ be (CCSs) of a real (HS) H so that $K = \cap_{i=1}^{N} K_i \neq \emptyset$. Assume $\{A_i\}_{i=1}^{N} : H \to H$ are finite family of monotone and $L-$Lipschitz continuous mappings. Further, let the solution set F is non-empty and $\{x_n\}$ be a sequence created by the following manner: Choose $x_\circ \in C_\circ^i = C = H$, for all $i = 1, \ldots, N$ and define w_n as follows:

$$w_\circ = x_\circ \text{ and for all } n \geq 1,\ w_n = x_n - \theta_n(x_n - x_{n-1}).$$

Moreover, choose $0 < \lambda_n < \frac{1}{L}$ and calculate

$$\begin{cases} y_n^i = P_{K_i}(w_n - \lambda_n A_i(w_n)), \\ z_n^i = P_{T_n^i}(w_n - \lambda_n A_i(y_n^i)), \\ C_{n+1}^i = \{v \in H : \left\| z_n^i - v \right\|^2 \leq \mu_n \|x_n - v\|^2 + (1 - \mu_n) \|x_{n-1} - v\|^2 - \epsilon_n^i\}, \\ Q_n = \{v \in H : \langle v - x_n, x_n - x_\circ \rangle \geq 0\}, \\ C_{n+1} = \cap_{i=1}^{N} C_{n+1}^i, \\ x_{n+1} = P_{C_{n+1} \cap Q_n}(x_\circ),\ n \geq 0, \end{cases} \tag{5}$$

where $y_n^i \in K_i$, $z_n^i \in T_n^i = \{v \in H : \langle (w_n - \lambda_n A_i(w_n)) - y_n^i, v - y_n^i \rangle \leq 0\}$,

$$\epsilon_n^i = \mu_n(1 - \mu_n)(1 - r) \|x_n - x_{n-1}\|^2 + r \left(\left\| z_n^i - y_n^i \right\|^2 + \mu_n \left\| y_n^i - x_n \right\|^2 + (1 - \mu_n) \left\| y_n^i - x_{n-1} \right\|^2 \right),$$

$r = 1 - \lambda_n L \geq 0$ and $\mu_n = 1 - \theta_n$, for all $i = 1, \ldots, N$. Assume that $\{\theta_n\}_{n \in N}$ is a real-valued sequence such that $\theta_n \in [0, 1]$. Then the sequence $\{x_n\}_{n \in N}$ converges strongly to $p = P_F(x_\circ)$, provided that the series $\sum_{i=1}^{n} \theta_n \|x_n - x_{n-1}\|^2$ is convergent.

Proof We discuss the steps below:
Step 1. Show that

$$\left\|z_n^i - x^*\right\|^2 \leq \left\|x_{n-1} - x^*\right\|^2 + \mu_n \left(\left\|x_n - x^*\right\|^2 - \left\|x_{n-1} - x^*\right\|^2\right) - \epsilon_n^i, \tag{6}$$

where $x^* \in F$.
Let $x^* \in F$, then by Lemma 1 , we can write

$$\begin{aligned}\left\|w_n - x^*\right\|^2 &= \left\|(1-\theta_n)(x_n - x^*) + \theta_n(x_{n-1} - x^*)\right\|^2 \\ &= (1-\theta_n)\left\|x_n - x^*\right\|^2 + \theta_n \left\|x_{n-1} - x^*\right\|^2 \\ &\quad - \theta_n(1-\theta_n)\left\|x_n - x_{n-1}\right\|^2.\end{aligned} \tag{7}$$

Similarly, we have

$$\left\|y_n^i - w_n\right\|^2 = (1-\theta_n)\left\|y_n^i - x_n\right\|^2 + \theta_n \left\|y_n^i - x_{n-1}\right\|^2 - \theta_n(1-\theta_n)\left\|x_n - x_{n-1}\right\|^2. \tag{8}$$

Since A_i is monotone on K_i and $y_n^i \in K_i$, we get

$$\langle y_n^i - x^*, A_i(y_n^i) - A_i(x^*)\rangle \geq 0, \text{ for all } x^* \in F,$$

yields together $x^* \in VI(A_i, K_i)$,

$$\langle y_n^i - x^*, A_i(y_n^i)\rangle \geq 0.$$

So

$$\langle A_i(y_n^i), z_n^i - x^*\rangle \geq \langle A_i(y_n^i), z_n^i - y_n^i\rangle. \tag{9}$$

From the property of the (MP) onto T_n^i, we have

$$\langle z_n^i - y_n^i, (w_n - \lambda_n A_i(w_n)) - y_n^i\rangle \leq 0. \tag{10}$$

Thus, by (10), we have

$$\begin{aligned}\langle z_n^i - y_n^i, (w_n - \lambda_n A_i(y_n^i)) - y_n^i\rangle &= \langle z_n^i - y_n^i, (w_n - \lambda_n A_i(w_n)) - y_n^i\rangle \\ &\quad + \lambda_n \langle z_n^i - y_n^i, A_i(w_n) - A_i(y_n^i)\rangle \\ &\leq \lambda_n \langle z_n^i - y_n^i, A_i(w_n) - A_i(y_n^i)\rangle.\end{aligned} \tag{11}$$

Put $s_n^i = w_n - \lambda_n A_i(y_n^i)$ and write again $z_n^i = P_{T_n^i}(s_n^i)$. Using Lemma 2 (ii) and (9), one sees that

$$\begin{aligned}\left\|z_n^i - x^*\right\|^2 &\le \left\|s_n^i - x^*\right\|^2 - \left\|P_{T_n^i}(s_n^i) - s_n^i\right\|^2 \\ &= \left\|w_n - \lambda_n A_i(y_n^i) - x^*\right\|^2 - \left\|z_n^i - (w_n - \lambda_n A_i(y_n^i))\right\|^2 \\ &= \left\|w_n - x^*\right\|^2 - \left\|z_n^i - w_n\right\|^2 + 2\lambda_n \langle x^* - z_n^i, A_i(y_n^i)\rangle \\ &\le \left\|w_n - x^*\right\|^2 - \left\|z_n^i - w_n\right\|^2 + 2\lambda_n \langle y_n^i - z_n^i, A_i(y_n^i)\rangle. \end{aligned} \tag{12}$$

From (11), we can write

$$\begin{aligned} &\left\|z_n^i - w_n\right\|^2 - 2\lambda_n \langle y_n^i - z_n^i, A_i(y_n^i)\rangle \\ &= \left\|z_n^i - y_n^i + y_n^i - w_n\right\|^2 - 2\lambda_n \langle y_n^i - z_n^i, A_i(y_n^i)\rangle \\ &= \left\|z_n^i - y_n^i\right\|^2 + \left\|y_n^i - w_n\right\|^2 - 2\langle z_n^i - y_n^i, (w_n - \lambda_n A_i(y_n^i) - y_n^i)\rangle \\ &= \left\|z_n^i - y_n^i\right\|^2 + \left\|y_n^i - w_n\right\|^2 - 2\lambda_n \langle z_n^i - y_n^i, A_i(w_n) - A_i(y_n^i)\rangle \\ &\ge \left\|z_n^i - y_n^i\right\|^2 + \left\|y_n^i - w_n\right\|^2 - 2\lambda_n \left\|z_n^i - y_n^i\right\| \left\|A_i(w_n) - A_i(y_n^i)\right\| \\ &\ge \left\|z_n^i - y_n^i\right\|^2 + \left\|y_n^i - w_n\right\|^2 - 2\lambda_n L \left\|z_n^i - y_n^i\right\| \left\|w_n - y_n^i\right\| \\ &\ge \left\|z_n^i - y_n^i\right\|^2 + \left\|y_n^i - w_n\right\|^2 - \lambda_n L \left(\left\|z_n^i - y_n^i\right\|^2 + \left\|w_n - y_n^i\right\|^2\right) \\ &\ge (1 - \lambda_n L)\left(\left\|z_n^i - y_n^i\right\|^2 + \left\|y_n^i - w_n\right\|^2\right). \end{aligned} \tag{13}$$

Applying (13) in (12) and applying (7) and (8), one gets

$$\begin{aligned}\left\|z_n^i - x^*\right\|^2 &\le \left\|w_n - x^*\right\| - (1 - \lambda_n L)\left(\left\|z_n^i - y_n^i\right\|^2 + \left\|y_n^i - w_n\right\|^2\right) \\ &= (1 - \theta_n)\left\|x_n - x^*\right\|^2 + \theta_n \left\|x_{n-1} - x^*\right\|^2 - \theta_n(1 - \theta_n)\left\|x_n - x_{n-1}\right\|^2 \\ &\quad - (1 - \lambda_n L)\begin{pmatrix} \left\|z_n^i - y_n^i\right\|^2 + (1 - \theta_n)\left\|y_n^i - x_n\right\|^2 \\ +\theta_n \left\|y_n^i - x_{n-1}\right\|^2 - \theta_n(1 - \theta_n)\left\|x_n - x_{n-1}\right\|^2 \end{pmatrix} \\ &= \mu_n \left\|x_n - x^*\right\|^2 + (1 - \mu_n)\left\|x_{n-1} - x^*\right\|^2 + \mu_n(\mu_n - 1)(1 - r)\left\|x_n - x_{n-1}\right\|^2 \\ &\quad - r\left(\left\|z_n^i - y_n^i\right\|^2 + \mu_n \left\|y_n^i - x_n\right\|^2 + (1 - \mu_n)\left\|y_n^i - x_{n-1}\right\|^2\right) \\ &\le \mu_n \left\|x_n - x^*\right\|^2 + (1 - \mu_n)\left\|x_{n-1} - x^*\right\|^2 - \mu_n(1 - \mu_n)(1 - r)\left\|x_n - x_{n-1}\right\|^2 \\ &\quad - r\left(\left\|z_n^i - y_n^i\right\|^2 + \mu_n \left\|y_n^i - x_n\right\|^2 + (1 - \mu_n)\left\|y_n^i - x_{n-1}\right\|^2\right) \\ &= \left\|x_{n-1} - x^*\right\|^2 + \mu_n\left(\left\|x_n - x^*\right\|^2 - \left\|x_{n-1} - x^*\right\|^2\right) - \epsilon_n^i, \end{aligned}$$

where

$$\begin{aligned}\epsilon_n^i &= \mu_n(1 - \mu_n)(1 - r)\left\|x_n - x_{n-1}\right\|^2 \\ &\quad + r\left(\left\|z_n^i - y_n^i\right\|^2 + \mu_n \left\|y_n^i - x_n\right\|^2 + (1 - \mu_n)\left\|y_n^i - x_{n-1}\right\|^2\right). \end{aligned}$$

Hence, we have the inequality (6).
Step 2. Prove that C^i_{n+1} satisfy the following inequality:

$$\begin{aligned}&\langle \mu_n(x_n - x_{n-1}) - (z^i_n - x_{n-1}), v\rangle\\&\leq \langle \mu_n\left(\frac{x_n + x_{n-1}}{2}\right), x_n - x_{n-1}\rangle - \langle \frac{z^i_n + x_{n-1}}{2}, z^i_n - x_{n-1}\rangle - \frac{\epsilon^i_n}{2},\end{aligned}$$

where $v \in F$.
Let $v \in F$. By (6), we have successively the following inequality:

$$\left\|z^i_n - v\right\|^2 \leq \|x_{n-1} - v\|^2 + \mu_n\left(\|x_n - v\|^2 - \|x_{n-1} - v\|^2\right) - \epsilon^i_n,$$

or

$$\left\|z^i_n - v\right\|^2 - \|x_{n-1} - v\|^2 \leq \mu_n\left(\|x_n - v\|^2 - \|x_{n-1} - v\|^2\right) - \epsilon^i_n.$$

By properties of the norm, we have

$$\begin{aligned}&\left((z^i_n - v) - (x_{n-1} - v)\right)\left((z^i_n - v) + (x_{n-1} - v)\right)\\&\leq \mu_n\left(\left[(x_n - v) - (x_{n-1} - v)\right]\left[(x_n - v) + (x_{n-1} - v)\right]\right) - \epsilon^i_n,\end{aligned}$$

and the above inequality can be written as

$$\langle \frac{z^i_n + x_{n-1}}{2}, z^i_n - x_{n-1}\rangle \leq \mu_n\langle\left(\frac{x_n + x_{n-1}}{2}\right), x_n - x_{n-1}\rangle - \frac{\epsilon^i_n}{2}.$$

Rearranging the terms, we get the desired.
Step 3. Illustrate that x_{n+1} is well-defined $\forall x_\circ \in H$ and $F \subset C_{n+1}$. Because A_i is Lipschitz continuous, A_i is continuous. Then, Lemma 3 ensures that $VI(A_i, K_i)$ is (CC) for all $i = 1, \ldots, N$. Hence, F is (CC). From the definition of C_{n+1} and Step 2, C_{n+1} is (CC) for each $n \geq 0$ as intersection of closed half-spaces.
Let $v \in F$, then it follows from Step 1 that

$$\left\|z^i_n - v\right\|^2 \leq \|x_{n-1} - v\|^2 + \mu_n\left(\|x_n - v\|^2 - \|x_{n-1} - v\|^2\right) - \epsilon^i_n.$$

So, we have $v \in C_{n+1}$. Thus $F \subset C_{n+1}$. Next, we will show that $F \subset C_{n+1} \cap Q_n$. By the induction, indeed $F \subset Q_\circ$ and so $F \subset C_1 \cap Q_\circ$. Suppose that x_l is given and $F \subset C_{l+1} \cap Q_l$ for some $l \geq 0$. There exists a unique element $x_{l+1} \in C_{l+1} \cap Q_l$ such that $x_{l+1} = P_{C_{l+1}\cap Q_l}(x_\circ)$. It follows that

$$\langle v - x_{l+1}, x_{l+1} - x_\circ\rangle \geq 0,$$

for each $v \in C_{l+1} \cap Q_l$. Since $F \subset C_{l+1} \cap Q_l$, we get $F \subset Q_{l+1}$. Therefore, we have $F \subset C_{l+2} \cap Q_{l+1}$ for all $l \geq 0$. Since $F \neq \emptyset$, so $x_{n+1} = P_{C_{n+1}\cap Q_n}(x_\circ)$ is well-defined.

Step 4. Prove that $\lim_{n\to\infty} \|x_n - x_\circ\| = 0$. From the algorithm of Theorem 1, we have

$$\langle v - x_n, x_n - x_\circ\rangle \geq 0 \text{ for all } v \in F.$$

This yields $x_n \in P_{Q_n}(x_\circ)$. Since $F \subset Q_n$, we get

$$\|x_n - x_\circ\| \leq \|x_\circ - v\|, \text{ for all } v \in F. \tag{14}$$

Also, since $x_{n+1} \in Q_n$, we get

$$\|x_n - x_\circ\| \leq \|x_{n+1} - x_\circ\|, \text{ for all } n \geq 0.$$

Using (14) and (15), the sequence $\{\|x_n - x_\circ\|\}$ is bounded and non-decreasing. Therefore, $\lim_{n\to\infty} \|x_n - x_\circ\| = 0$.
Furthermore, by Lemma 2 (ii) with $x_\circ \in H$ and $x_n \in Q_{n-1}$, we can write

$$\left\|x_n - P_{Q_{n-1}}(x_\circ)\right\|^2 + \left\|P_{Q_{n-1}}(x_\circ) - x_\circ\right\|^2 \leq \|x_n - x_\circ\|^2.$$

Since $x_{n-1} = P_{Q_{n-1}}(x_\circ)$, so the above inequality leads to

$$\|x_n - x_{n-1}\|^2 + \|x_{n-1} - x_\circ\|^2 \leq \|x_n - x_\circ\|^2.$$

Taking the limit as $n \to \infty$ in the above inequality, one can obtain

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{15}$$

Let $w_n = x_n - \theta_n(x_n - x_{n-1})$. Since $0 \leq \theta_n \leq 1$, we have

$$\lim_{n\to\infty} \|w_n - x_n\| = \lim_{n\to\infty} \|\theta_n(x_n - x_{n-1})\| \leq \lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{16}$$

Step 5. Prove that the following relations holds for all $i = 1, \ldots, N$.

$$\lim_{n\to\infty} \left\|z_n^i - x_n\right\| = 0, \ \lim_{n\to\infty} \left\|y_n^i - x_n\right\| = 0 = \lim_{n\to\infty} \left\|y_n^i - w_n\right\|.$$

By the definition of $C_{n+1}^i \subset C_n$, $x_{n+1} = C_{n+1}$, for all $i = 1, \ldots, N$, we get

$$\begin{aligned}\left\|z_n^i - x_{n+1}\right\|^2 &\leq \mu_n \|x_n - x_{n+1}\|^2 - \mu_n(1-r)(1-\mu_n)\|x_n - x_{n-1}\|^2 \\ &\quad + (1-\mu_n)\|x_{n-1} - x_{n+1}\|^2 \\ &\quad - r\left(\left\|z_n^i - y_n^i\right\|^2 + \mu_n\left\|y_n^i - x_n\right\|^2 + (1-\mu_n)\left\|y_n^i - x_{n-1}\right\|^2\right) \\ &\leq \mu_n \|x_n - x_{n+1}\|^2 - \mu_n(1-r)(1-\mu_n)\|x_n - x_{n-1}\|^2 \\ &\quad + (1-\mu_n)\|x_{n-1} - x_{n+1}\|^2.\end{aligned} \tag{17}$$

Passing the limit as $n \to \infty$ in (17), and since μ_n is a bounded real sequence, so we can write

$$\lim_{n\to\infty} \left\| z_n^i - x_{n+1} \right\| = 0. \tag{18}$$

By the triangle inequality and using (16) and (19), one sees that

$$\left\| z_n^i - x_n \right\| \leq \left\| z_n^i - x_{n+1} \right\| + \left\| x_{n+1} - x_n \right\| \to 0, \tag{19}$$

as $n \to \infty$ for all $i = 1, \ldots, N$. Further since $r \geq 0$, and using (17), we get

$$\begin{aligned} \left\| z_n^i - x_{n+1} \right\|^2 &\leq \mu_n \left\| x_n - x_{n+1} \right\|^2 - \mu_n(1-r)(1-\mu_n) \left\| x_n - x_{n-1} \right\|^2 \\ &\quad + (1-\mu_n) \left\| x_{n-1} - x_{n+1} \right\|^2 - r(1-\mu_n) \left\| y_n^i - x_{n-1} \right\|^2, \end{aligned}$$

or we get

$$\begin{aligned} r(1-\mu_n) \left\| y_n^i - x_{n-1} \right\|^2 &\leq \mu_n \left\| x_n - x_{n+1} \right\|^2 - \mu_n(1-r)(1-\mu_n) \left\| x_n - x_{n-1} \right\|^2 \\ &\quad + (1-\mu_n) \left\| x_{n-1} - x_{n+1} \right\|^2 - \left\| z_n^i - x_{n+1} \right\|^2. \end{aligned}$$

Taking the limit in the above inequity and using (16) and (19), we have

$$\lim_{n\to\infty} \left\| y_n^i - x_{n-1} \right\| = 0. \tag{20}$$

From the triangle inequality, (16) and (21), one can obtain

$$\left\| y_n^i - x_n \right\| \leq \left\| y_n^i - x_{n-1} \right\| + \left\| x_{n-1} - x_n \right\| \to 0, \tag{21}$$

as $n \to \infty$ for all $i = 1, \ldots, N$. Last, one can obtain

$$\left\| y_n^i - w_n \right\| = \left\| (y_n^i - x_n) + \theta_n (x_n - x_{n-1}) \right\| \leq \left\| y_n^i - x_n \right\| + \theta_n \left\| x_n - x_{n-1} \right\|.$$

Since θ_n is a bounded real sequence and using (22) and (17), we get

$$\left\| y_n^i - w_n \right\| \leq \left\| y_n^i - x_n \right\| + \left\| x_n - w_n \right\| \to 0, \tag{22}$$

as $n \to \infty$ for all $i = 1, \ldots, N$. The inequalities (20), (22) and (23) completes the proof of this step.

Step 6. Show that $\{x_n\}$, $\{y_n^i\}$ and $\{z_n^i\}$ generated by Algorithm (5) converges strongly to $p = P_F x_\circ$. Assume that p is a weak cluster point (WCP) of $\{x_n\}$ and there exists a subsequence of $\{x_n\}$ converging weakly to p, i.e., $x_n \rightharpoonup p$, from (21), $y_n^i \rightharpoonup p$.

Now, to prove $p \in \cap_{i=1}^N VI(A_i, K_i)$, we recall Lemma 4, which state the mappings

$$B_i(x) = \begin{cases} A_i x + N_{K_i}(x) & \text{if } x \in C \\ \emptyset & \text{if } x \notin C \end{cases}$$

are a (MM). $\forall(x, y) \in G(B_i)$, we get $y - A_i x \in N_{K_i}(x)$, where $G(B_i)$ is the graph of B_i. Based on $N_{K_i}(x)$, we see that

$$\langle y - A_i(x), x - z\rangle \geq 0,$$

$\forall z \in K_i$. Because $y_n^i \in K_i$,

$$\langle y - A_i(x), x - y_n^i\rangle \geq 0.$$

Then,

$$\langle y, x - y_n^i\rangle \geq \langle A_i(x), x - y_n^i\rangle. \tag{23}$$

Considering $y_n^i = P_{K_i}(w_n - \lambda_n A_i(w_n))$ and Lemma 2 (iii), we get, for all $x \in K_i$,

$$\langle x - y_n^i, y_n^i - w_n + \lambda_n A_i(w_n)\rangle \geq 0,$$

or

$$\langle x - y_n^i, A_i(w_n)\rangle \geq \langle x - y_n^i, \frac{w_n - y_n^i}{\lambda_n}\rangle. \tag{24}$$

Therefore, from (23) and (24) and because A_i is a (MM), we have

$$\begin{aligned}\langle y, x - y_n^i\rangle &\geq \langle A_i(x), x - y_n^i\rangle \\ &= \langle A_i(x) - A_i(y_n^i), x - y_n^i\rangle + \langle A_i(y_n^i) - A_i(w_n), x - y_n^i\rangle + \langle A_i(w_n), x - y_n^i\rangle \\ &\geq \langle A_i(y_n^i) - A_i(w_n), x - y_n^i\rangle + \langle \frac{w_n - y_n^i}{\lambda_n}, x - y_n^i\rangle.\end{aligned} \tag{25}$$

Applying (21) in (25) and A_i are $L-$Lipschitz continuous,

$$\lim_{n\to\infty} \left\| A_i(y_n^i) - A_i(w_n)\right\| = 0. \tag{26}$$

Taking the limit in (25) as $n \to \infty$ and using (26), $y_n^i \rightharpoonup p$, we have $\langle x - p, y\rangle \geq 0$ for all $(x, y) \in G(B_i)$. The MM of B_i implies that $p \in B_i^{-1}(0) = VI(A_i, K_i)$ for all $i = 1, \ldots, N$.

Ultimately, we illustrate that $x_n \to p = q = P_F x_\circ$. From (15) and $q \in F$, we get

$$\|x_n - x_\circ\| \leq \|q - x_\circ\|, \text{ for all } n \geq 0. \tag{27}$$

By (27) and lower weak semi-continuity of the norm, one sees that

$$\|p - x_\circ\| \leq \lim\inf_{n\to\infty} \|x_n - x_\circ\| \leq \lim\sup_{n\to\infty} \|x_n - x_\circ\| \leq \|q - x_\circ\|.$$

By the definition of q, $p = q$ and $\lim_{n\to\infty} \|x_n - x_\circ\| = \|q - x_\circ\|$. Hence, from $x_n - x_\circ \rightharpoonup q - x_\circ$ and the Kadec-Klee property of H, we get $x_n - x_\circ \to q - x_\circ$, and so $x_n \to p$. Also, Steps 3 and 5 ensure that the sequences $\{y_n^i\}$, $\{z_n^i\}$ converge strongly to $P_F x_\circ$. $\square$

Theorem 2 ***(CHICQ-SE algorithm)***
Let K_i, $i = 1, \ldots, N$ be (CCSs) of a real (HS) H so that $K = \cap_{i=1}^N K_i \neq \emptyset$. Assume $\{A_i\}_{i=1}^n : H \to H$ is a finite family of monotone and $L-$Lipschitz continuous mappings. In addition, F is non-empty. Suppose that $\{x_n\}$ is a sequence marked by $x_\circ \in C_\circ^i = C = H$, for all $i = 1, \ldots, N$ and define w_n as follows:

$$w_\circ = x_\circ \textit{ and for all } n \geq 1,\ w_n = x_n - \theta_n(x_n - x_{n-1}).$$

Then choose $0 < \lambda_n < \frac{1}{L}$. Set $n = 0$,

$$\begin{cases} y_n = P_{K_{[n]}}(w_n - \lambda_n A_{[n]}(w_n)), \\ z_n = P_{T_{[n]}}(w_n - \lambda_n A_{[n]}(y_n)), \\ C_{n+1} = \{v \in H : \|z_n - v\|^2 + \epsilon_n \leq \mu_n \|x_n - v\|^2 + (1-\mu_n)\|x_{n-1} - v\|^2\}, \\ Q_n = \{v \in H : \langle v - x_n, x_n - x_\circ\rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}\cap Q_n}(x_\circ),\ n \geq 0, \end{cases}$$

where $y_n \in K_{[n]}$, $z_n \in T_{[n]} = \{v \in H : \langle (w_n - \lambda_n A_{[n]}(w_n)) - y_n, v - y_n\rangle \leq 0\}$, $[n] = modulo(n, N) + 1$ with the $[n]$ function taking values in $\{1, 2, \ldots, N\}$,

$$\begin{aligned} \epsilon_n &= \mu_n(1-\mu_n)(1-r)\|x_n - x_{n-1}\|^2 \\ &\quad + r\left(\|z_n - y_n\|^2 + \mu_n\|y_n - x_n\|^2 + (1-\mu_n)\|y_n - x_{n-1}\|^2\right), \end{aligned}$$

$r = 1 - \lambda_n L \geq 0$ and $\mu_n = 1 - \theta_n$. Assume that $\{\theta_n\}_{n\in N}$ is a real-valued sequence such that $\theta_n \in [0, 1]$. Then the sequence $\{x_n\}_{n\in N}$ converges strongly to $p = P_F(x_\circ)$. Moreover, the series $\sum_{i=1}^n \theta_n \|x_n - x_{n-1}\|^2$ is convergent.

Proof The proof of the second theorem goes here. By arguing similarly as in the proof of Theorem 1, we obtain that F and C_{n+1} are (CC) and $F \subset C_{n+1}$ for all $n \geq 1$. Besides, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded and

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0,\ \lim_{n\to\infty}\|z_n - x_n\| = 0,\ \lim_{n\to\infty}\|y_n - x_n\| = 0 = \lim_{n\to\infty}\|y_n - w_n\|. \tag{28}$$

Suppose p is some (WCP) of $\{x_n\}$. Fixed an index $i \in \{1, 2, \ldots, N\}$ so that indexes i are finite. Then there is a subsequence $\{x_{n_k}\} \in \{x_n\}$ so that $x_{n_k} \rightharpoonup p$ and $[n_k] = i$, $\forall k$. From (28), we get $y_{n_k} \rightharpoonup p$ as $k \to \infty$. Similar to (23)–(26), one concludes that $p \in VI(A_i, K_i)$, and the rest of the proof follows immediately from the proof of Theorem 1. $\square$

Remark 1 (i) Since C_{n+1} and Q_n are either half-space or the whole space H, the projection $x_{n+1} = P_{C_{n+1}\cap Q_n}(x_\circ)$ computed explicitly as in Theorem 1. (ii) If the mapping A is α-(ISM), then A is $1/\alpha-$Lipschitz continuous. So, algorithms of Theorems 1 and 2 are successful for solving the (CSVIP) for the α-(ISMs) A_i, $i = 1, \dots, N$.

4 Numerical Results

In this section, we shall discuss the strong convergence of our algorithms numerically and graphically in $\mathbb{R}^3$ and L^2 spaces.

4.1 A Strong Convergence in $\mathbb{R}^3$

Let $H = \mathbb{R}^3$ be a finite dimensional space, $d(H)$ be the dimension of H and N be the number of functions A_i which is equal to the number of subsets K_i of H. Here, we shall choose $d = 3$ and $N = 3$ (in general d is not equal N). Consider the variable x defined by

$$x = \begin{pmatrix} x(1) \\ x(2) \\ x(3) \end{pmatrix},$$

and the operator $A_i(x)$ in given form

$$A_i(x) = M_i x + q_i, \; M_i = B_i B_i^T + C_i + D_i,$$

where B_i is an $(m \times m)-$matrix, B_i^T is a transpose of B_i, C_i is an $(m \times m)-$skew-symmetric matrix, D_i is an $(m \times m)-$diagonal matrix (here $m = 3$) and q_i is a vector in $\mathbb{R}^3$. Here, we consider the diagonal entries are nonnegative, so M_i is positive definite and each of the operators A_i is defined on the feasible set K_i for all $i = 1, \dots, N$.

It is clear that A_i is monotone and Lipschitz continuous with the Lipschitz constant

$$L = \max\{\|M_i\|, \; i = 1, \dots, N\}.$$

Suppose that B_i, C_i and D_i (i=1,2,3) are generated randomly matrices so that

$$B_1 = \begin{pmatrix} -2 & 1 & 3 \\ -1 & 4 & -1 \\ -2 & 1 & 1 \end{pmatrix}, C_1 = \begin{pmatrix} 5 & -1 & 2 \\ 1 & 7 & -1 \\ -2 & 1 & -3 \end{pmatrix}, D_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix},$$
$$B_2 = \begin{pmatrix} 8 & 1 & -3 \\ -1 & 2 & -1 \\ 2 & 1 & 5 \end{pmatrix}, C_2 = \begin{pmatrix} 3 & 0 & -5 \\ 0 & 9 & 1 \\ 5 & -1 & 1 \end{pmatrix}, D_2 = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$
$$B_3 = \begin{pmatrix} 9 & 1 & 3 \\ -3 & 7 & 1 \\ -5 & 1 & -7 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & 5 & -2 \\ -5 & 4 & 3 \\ 2 & -3 & 1 \end{pmatrix}, D_3 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

The feasible sets are given by

$$\begin{cases} K_1 = \{x \in H : x(1) \geq 0\} \\ K_2 = \{x \in H : x(2) \geq 0\} \\ K_3 = \{x \in H : x(1) = x(2)\} \end{cases}.$$

All the projections over K_i are computed explicitly.
Now consider the sets C^i_{n+1} and Q_n defined in the theorems as follows:

$$C^i_{n+1} = \left\{ \begin{array}{c} v \in H : \langle \mu_n(x_n - x_{n-1}) - (z^i_n - x_{n-1}), v \rangle \\ \leq \langle \mu_n \left(\frac{x_n + x_{n-1}}{2} \right), x_n - x_{n-1} \rangle - \langle \frac{z^i_n + x_{n-1}}{2}, z^i_n - x_{n-1} \rangle - \frac{\epsilon^i_n}{2} \end{array} \right\},$$
$$Q_n = \{v \in H : \langle v - x_n, x_n - x_\circ \rangle \geq 0\}.$$

These sets are as well half-spaces, but the projection $x_{n+1} = P_{C_n \cap Q_n}(x_\circ)$ (in both of the two proposed algorithms) will be computed numerically with a linear quadratic optimization technique. (The programs are written in Scilab and performed on a PC Desktop Intel(R)Core(TM) i5-7200U CPU @ 2.50GHz 2.70 GHz, RAM 8.00 GB.) It should be noted that the (CSVIP) in this case is $x^* = 0$. We take $\kappa_n = \|x_n - x^*\|_\infty$, $n = 0, 1, \ldots$, to check the convergence of the sequence $\{x_n\}_{n \in N}$ which is generated by Algorithm of Theorem 1 or Algorithm of Theorem 2, such that if $\{\kappa_n\}_{n \in N}$ converges to 0, it implies that $\{x_n\}_{n \in N}$ converges to the solution x^* of the problem (CSVIP). Define $L = \max\{\|M_1\|, \|M_2\|, \|M_3\|\}$.
Now, we shall discuss the behavior of the algorithms with $\theta_n = 0$ and $\theta_n = \left(\frac{1}{a}\right)^n$, respectively, where a is a natural number greater than 1.
Suppose that Let $x(1)$, $x(2)$ and $x(3)$ be the components of x for each iteration. We give below the details of the two iterations.

4.1.1 When the Sequence is Convergent

In this case, we access the behavior of the algorithms when $\theta_n = 0$ and $\theta_n = \left(\frac{1}{a}\right)^n$, respectively, where a is a natural number.

Table 1 The numerical results of iterations for $\lambda = \frac{0.5}{L}$

Niter	$x(1)$	$x(2)$	$x(3)$
2	2.6822788	4.1669622	−2.1926203
4	1.7428908	1.9905399	−1.1881078
8	0.8838627	1.063666	−0.147852
16	−0.114004	0.4020859	−0.2948171
32	0.1516955	0.1366903	−0.20404
64	0.0271699	0.0316335	−0.0050567
128	0.003	0.002955	−0.0016168
256	0.0006922	0.000994	−0.000963

Algorithm of Theorem 1 Choosing $w_\circ = x_\circ = \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$; $\theta_n = 0$ and $\mu_n = 1$.

The numerical results of iterations for $\lambda = \frac{0.5}{L}$ are presented as follows (Table 1):

One has a convergence for Niter=199, while, for $\lambda = \frac{0.1}{L}$, we obtain the convergence at Niter=332 (Note, the word Niter refers to the number of the iterations).

Tables 2 and 3 illustrate the numerical results of Algorithm of Theorem 1, when $\lambda = \frac{0.5}{L}$ and $\lambda = \frac{0.1}{L}$, respectively, for the various geometric sequence $\theta_n = \left(\frac{1}{a}\right)^n$, with $a \geq 1$.

Algorithm of Theorem 2 We use the same data of the Algorithm of Theorem 1. The function $[n]$ is linked to n by the following formula (which is available in Scilab):

$$[n] = modulo(n, N) + 1,$$

with the $[n]$ function taking the values in $\{1, 2, \dots, N\}$.
The numerical results of iterations for $\lambda = \frac{0.5}{L}$ are presented as follows (Table 4):

One has a convergence for Niter = 30279, while, for $\lambda = \frac{0.1}{L}$, we obtain the convergence at Niter = 93899.

Table 2 Niter for a minimum precision of 0.001 and $\lambda = \frac{0.5}{L}$ for various values of $a \geq 1$

a	2	3	4	5	6	7	8	9
Niter	239	251	245	282	256	265	203	218
Precision	0.000964	0.000916	0.000972	0.0009435	0.0008744	0.0008258	0.000894	0.0008877

Table 3 Niter for a minimum precision of 0.001 and $\lambda = \frac{0.1}{L}$ for various values of $a \geq 1$

a	2	3	4	5	6	7	8	9
Niter	468	425	441	412	436	473	366	398
Precision	0.0008626	0.0009609	0.0009059	0.0009781	0.0009738	0.0009586	0.0009793	0.0009965

Table 4 The numerical results of iterations for $\lambda = \frac{0.5}{L}$

Niter	$x(1)$	$x(2)$	$x(3)$
2	0.8591395	6.6280827	−2.68178
4	2.4721046	3.8131022	−2.3010322
8	1.3779545	2.6042651	−1.0132506
16	2.0633457	1.5218332	−1.004058
32	0.7153241	0.9836615	−0.3302871
64	0.4690249	0.41279	−0.2490068
128	0.1406275	0.2056841	−0.1215445
256	−0.0754801	0.2155831	0.1165085
512	0.0588911	0.0604353	−0.024352
1024	0.0162688	0.024721	−0.0504005
2048	0.0174299	0.0261066	0.0061292
4096	0.0191064	0.0105297	0.0005558
8192	0.0038712	0.0051106	−0.0027426
16384	0.0031451	0.0018751	−0.0013357
32768	0.0009576	0.000981	−0.0009486

Tables 5 and 6 illustrate the numerical results of Algorithm of Theorem 2, when $\lambda = \frac{0.5}{L}$ and $\lambda = \frac{0.1}{L}$, respectively, for the various geometric sequence $\theta_n = \left(\frac{1}{a}\right)^n$, with $a \geq 1$.

From Figs. 1 and 2, we see that the performance of Algorithm of Theorem 1 is better than that of Algorithm of Theorem 2. Also, as the value of λ increases, the algorithms converge faster.

Table 5 Niter for a minimum precision of 0.001 and $\lambda = \frac{0.5}{L}$ for various values of $a \geq 1$

a	2	3	4	5	6	7	8	9
Niter	31158	31916	23604	32695	33781	31640	32388	24753
Precision	0.0009737	0.0009972	0.0009887	0.0009999	0.0009858	0.000994	0.0009868	0.0009713

Table 6 Niter for a minimum precision of 0.001 and $\lambda = \frac{0.1}{L}$ for various values of $a \geq 1$

a	2	3	4	5	6	7	8	9
Niter	264693	276236	234029	257163	223990	236072	259420	237678
Precision	0.0009932	0.0009943	0.0009955	0.0009899	0.0009999	0.0009944	0.0009907	0.0009838

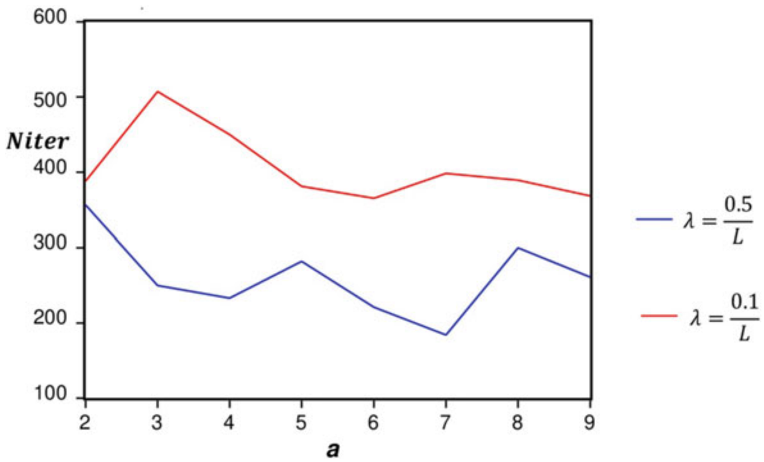

Fig. 1 Number of iterations for a parallel algorithm

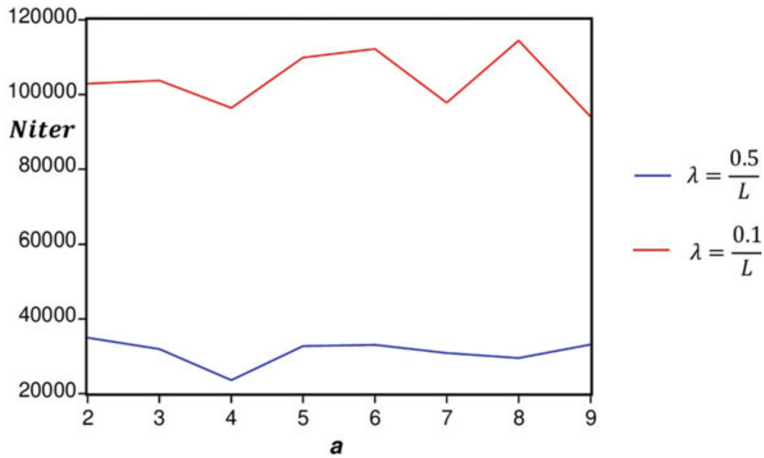

Fig. 2 Number of iterations for a parallel algorithm

4.1.2 When the Sequence is Bounded but not Convergent

Here, we define a sequence θ_n by

$$\theta_n = |\cos(n)| .$$

It is easy to see that θ_n verifies $0 \leq \theta_n < 1$ as required by our theorems.

Tables 7, 8 and 9 give a comparison between the number of iterations required to get the required precision ($\delta = 0.001$) for the two algorithms with two bounded and non-convergent sequences.

Table 7 Comparison between the two algorithms for a cosine sequence

λ	Parallel algorithm	Cyclic algorithm
$\frac{0.5}{L}$	218	28889
$\frac{0.1}{L}$	360	534555

Table 8 Comparison between the two algorithms for a uniformly random distributed sequence

λ	Parallel algorithm	Cyclic algorithm
$\frac{0.5}{L}$	171	34613
$\frac{0.1}{L}$	371	473818

Table 9 Comparison between the two algorithms for a constant sequence θ_n

λ	Parallel algorithm	Cyclic algorithm
$\frac{0.5}{L}$	182	68971
$\frac{0.1}{L}$	318	451432

Note that, conversely to the first case, θ_n is not a convergent sequence; however, the conclusion of the theorems remain verified (one has convergence) as illustrated by the above table. Moreover, the number of iterations to get the required precision will become much higher.

4.2 A Strong Convergence in L^2

Let $H = L^2(0, 1)$ be an infinite dimensional space, define A, K_1 and K_2 by $(Ax)(t) = \int_0^t x(s)ds$

$$\begin{cases} K_1 = \{x \in L^2(0, 1), \ \int_0^1 x(t)dt = b_1\} \\ K_2 = \left\{x \in L^2(0, 1), \ x(t) = b_2, \ 0 \le t \le 0.1\right\} \end{cases}, \tag{29}$$

such that $K = K_1 \cap K_2$.

For simplicity, we shall consider the subset of $L^2(0, 1)$ is a simple function. This choice is justified by the fact that it is easy to define integration for a simple function and also it is straightforward to approximate L^2 functions by sequences of simple functions.

Here, the issue is to find the mapping $t \mapsto x^*(t) \in K$ such that $\langle Ax, x - x^* \rangle \ge 0$ for all $x \in \cap K_i$.

Now, we shall show that the mapping $x \mapsto Ax$ is linear, $1-$Lipschitz and monotone.

Let us compute the L^2 norm of Ax, where $(Ax)(t) = \int_0^t x(s)ds$.

Decomposing x as the difference between its positive and negative parts, we get

$$\begin{aligned}\|Ax\|^2_{L^2(0,1)} &= \int_0^1 \left(\int_0^t x(s)ds \right)^2 dt = \int_0^1 \left(\int_0^t x_+(s)ds \right)^2 dt + \int_0^1 \left(\int_0^t x_-(s)ds \right)^2 dt \\ &- \int_0^1 \left(\int_0^t x_+(s)x_-(s)ds \right) dt \\ &\leq \int_0^1 \left(\int_0^t x_+(s)ds \right)^2 dt + \int_0^1 \left(\int_0^t x_-(s)ds \right)^2 dt \\ &\leq \left(\int_0^t x_+(t)dt \right)^2 dt + \left(\int_0^t x_-(t)dt \right)^2 dt = \|x\|^2_{L^2(0,1)}.\end{aligned}$$

Also, we have

$$\langle Ax, x\rangle_{L^2(0,1)} = \int_0^1 \left(\int_0^t x(s)ds \right)^2 x(t)dt = \int_0^1 \left(\int_0^t x(s)x(t)ds \right)^2 dt.$$

So it is clear that for the subspace of simple functions, the product $x(s)x(t) \geq 0$ for each $t \geq 0, 0 \leq s \leq t$ and partition $A_k \in [0, 1]$. Hence, we get $Ax, x\rangle_{L^2(0,1)} \geq 0$. So, we conclude that A is a linear, continuous Lipschitzian and monotone mapping.

Now, we will solve VIP (1) with the following data:

$$\begin{aligned}&x^{(0)} = 1 \;\; x^{(1)} = 1 \;\; \theta_n = 0 \;\; \lambda = 1, \\ &\mu_n = 1 \;\; A_i = A \;\; K_i = K \;\; L = 1.\end{aligned}$$

For the cases $b_1 = b_2 = 0$, the (CSVIP) here is $x^* = 0$. So we shall put $\|x_n - x^*\|_\infty$, $n = 0, 1, 2, \ldots$ and an error threshold $\delta = 10^{-3}$ to examine the convergence of $(x_n)_{n\in N}$ which is iterated by Algorithm 3.1 or Algorithm 3.2 as previously. For the other cases where $b_1 \neq 0$ or $b_2 \neq 0$, we will use the sequence $\|x_n - x_{n-1}\|_\infty$ $n = 0, 1, 2, \ldots$ and an error threshold $\delta = 10^{-3}$ to discuss the convergence of $(x_n)_{n\in N}$ because one do not, in advance, know the solution.

In the following cases, we shall consider the parameters b_1 and b_2 in (29) to study the behavior of the various solutions and we take

$$x_\circ(t) = 3,\; x_1(t) = 3,\; \theta_n = \left(\tfrac{1}{2}\right)^n,\; \lambda = \tfrac{0.01}{L},\; \delta = 10^{-3}.$$

Cyclic algorithm

Case 1. When $b_1 = 0$ and $b_2 = 0$, we have (Table 10)

Table 10 Numerical results of Case 1

$Niter$	x(1)	x(2)	x(3)	x(4)	x(5)	x(6)	x(7)	x(8)	x(9)
2	0.0148348	2.9970208	2.9940446	2.9910714	2.9881011	2.9851339	2.9821696	2.9792083	2.97625
4	0.0149541	0.0147809	0.0146125	0.0144487	0.0142897	0.0141354	0.0139857	0.0138408	0.0137005
8	0.0001981	0.0000718	−0.000016	−0.0000662	−0.0000799	−0.000058	−0.0000017	0.0000881	0.0002104

$Niter = 5, \int_0^1 x(t)dt = 0.0000711$ and $\|x\|_{L^2(0,1)} = 0.000186.$

Case 2. When $b_1 = 0$ and $b_2 = 1$, we get (Fig. 3; Table 11)

$Niter = 85057, \int_0^1 x(t)dt = 0.00006727$ and $\|x\|_{L^2(0,1)} = 0.4463904.$

Case 3. When $b_1 = 0.25$ and $b_2 = 1$, we have (Fig. 4)

$Niter = 11, \int_0^1 x(t)dt = 0.2500397$ and $\|x\|_{L^2(0,1)} = 0.3830179.$

Parallel algorithm

Case 4. When $b_1 = 0$ and $b_2 = 0$, we have (Table 12)

$Niter = 4, \int_0^1 x(t)dt = 0.0004276$ and $\|x\|_{L^2(0,1)} = 0.0001402.$

Case 5. When $b_1 = 0$ and $b_2 = 1$, we get (Fig. 5; Table 13)

$Niter = 512, \int_0^1 x(t)dt = 0.000403$ and $\|x\|_{L^2(0,1)} = 0.4611933.$

Case 6. When $b_1 = 0.25$ and $b_2 = 1$, we get $Niter = 88, \int_0^1 x(t)dt = 0.252749$ and $\|x\|_{L^2(0,1)} = 0.358893$ (Fig. 6).

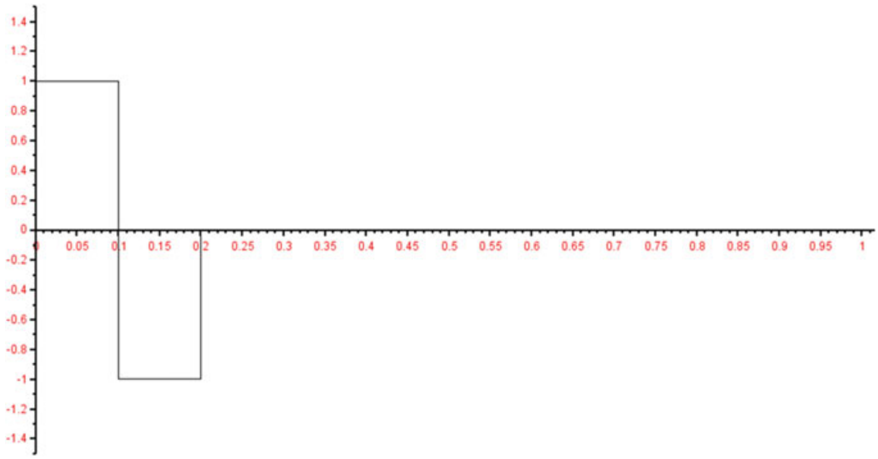

Fig. 3 Approximate solution function for Case 2

Table 11 Numerical results of Case 2

Niter	x(1)	x(2)	x(3)	x(4)	x(5)	x(6)	x(7)	x(8)	x(9)
2	1.0098324	2.9960256	2.9930493	2.990076	2.9871057	2.9841384	2.981174	2.9782126	2.9752543
4	0.9472163	−0.0845218	−0.0856985	−0.0868681	−0.0880307	−0.0891864	−0.0903352	−0.0914769	−0.0926118
8	0.9984449	−0.1512623	−0.1380361	−0.1251709	−0.112671	−0.1005409	−0.0887848	−0.0774068	−0.0664113
16	0.9991434	−0.220606	−0.1831627	−0.1497674	−0.1201408	−0.0940151	−0.0711342	−0.0512531	−0.0341377
32	0.999452	−0.3087523	−0.2294575	−0.1655353	−0.1146942	−0.0749078	−0.0443912	−0.0215781	−0.0051006
64	0.9996253	−0.4118675	−0.2658848	−0.16262	−0.0916611	−0.0447246	−0.0152871	0.0017283	0.0102183
128	0.9997435	−0.5137242	−0.2767604	−0.1357491	−0.0565577	−0.01573	0.0024609	0.0082385	0.0079835
256	0.9998231	−0.6026998	−0.2607462	−0.0973886	−0.0271324	−0.0018548	0.0040956	0.0033766	0.0015116
512	0.9998757	−0.6780957	−0.2312046	−0.0656672	−0.0131141	−0.000396	0.0010878	0.0007597	0.0007697
1024	0.9999121	−0.7427523	−0.1984342	−0.0436565	−0.0067157	0.0000298	0.0008224	0.0008921	0.0009158
2048	0.9999378	−0.7988458	−0.1643025	−0.0273685	−0.0029303	0.0003617	0.0006316	0.0006335	0.0006313
4096	0.9999562	−0.8478227	−0.1302435	−0.0157253	−0.0009744	0.0003862	0.0004457	0.0004421	0.0004419
8192	0.9999696	−0.8906296	−0.0971678	−0.0078404	−0.0001205	0.0002921	0.0002919	0.0002903	0.0002905
8192	0.9999696	−0.8906296	−0.0971678	−0.0078404	−0.0001205	0.0002921	0.0002919	0.0002903	0.0002905
16384	0.9999794	0.9278601	0.0658945	0.0029849	0.0001294	0.0001754	0.0001694	0.0001692	0.0001693
32768	0.9999867	−0.959926	−0.0371541	−0.0004706	0.000106	0.0000741	0.0000727	0.0000728	0.0001693
32768	0.9999867	−0.959926	−0.0371541	−0.0004706	0.000106	0.0000741	0.0000727	0.0000728	0.0000728
65536	0.9999922	−0.9875855	−0.0111779	0.0003745	0.0000054	−0.0000054	−0.0000049	−0.0000049	−0.0000049
131072	0.9989939	−0.9973224	−0.0016673	0.0004517	0.0000355	0.0000359	0.0000363	0.0000363	0.0000364

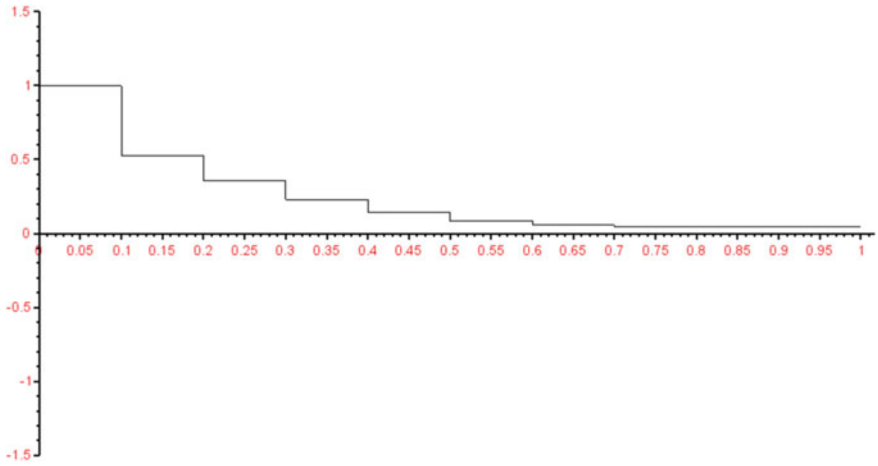

Fig. 4 Approximate solution function for Case 3

Table 12 Numerical results of Case 4

Niter	x(1)	x(2)	x(3)	x(4)	x(5)	x(6)	x(7)	x(8)	x(9)
2	0.0149867	0.0149762	0.0149688	0.0149643	0.0149628	0.0149643	0.0149688	0.0149762	0.0149867
4	0.0000442	0.0000334	0.0000287	0.0000289	0.0000327	0.0000388	0.0000461	0.0000533	0.0000591

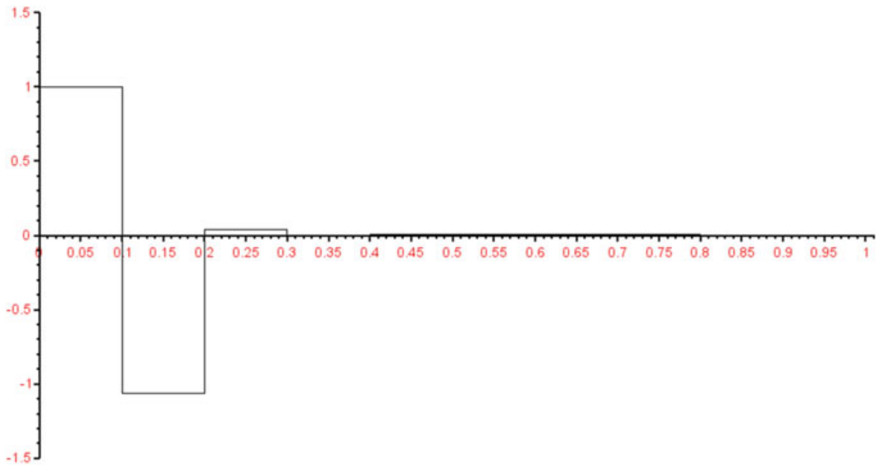

Fig. 5 Approximate solution function for Case 5

Table 13 Numerical results of Case 5

$Niter$	x(1)	x(2)	x(3)	x(4)	x(5)	x(6)	x(7)	x(8)	x(9)
2	0.0149867	0.0149762	0.0149688	0.0149643	0.0149628	0.0149643	0.0149688	0.0149867	0.0150001
4	0.9994321	−0.1132079	−0.1125981	−0.1119848	−0.1113686	−0.1107501	−0.11013	−0.1095089	−0.1088875
8	1.0001526	−0.118891	−0.1168356	−0.1147926	−0.1127624	−0.1107458	−0.1087433	−0.1067555	−0.1047829
16	0.9999083	−0.1305305	−0.1253511	−0.1202737	−0.1152977	−0.1104225	−0.1056472	−0.1009712	−0.0963939
32	0.9993987	−0.1539795	−0.1419818	−0.1305032	−0.1195291	−0.1090452	−0.0990374	−0.0894919	−0.0803952
64	0.9944281	−0.3229801	−0.2431531	−0.1768252	−0.1223319	−0.0781659	−0.0429656	−0.0155035	0.0053238
128	0.99648	−0.8206984	−0.1817345	−0.04063	0.0159882	0.0479732	0.0495674	0.0198897	−0.0232729
256	0.9987749	−0.9596925	−0.0543724	−0.0209714	−0.0110713	0.0055497	0.0197592	0.0225715	0.0113809
512	0.9992989	−1.0396428	0.0188623	−0.004233	0.0069186	0.0148218	0.0152512	0.0076823	−0.0037023

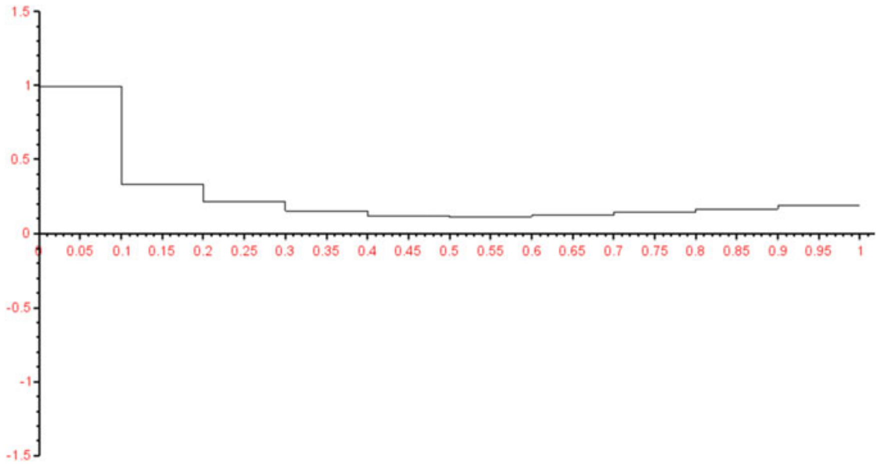

Fig. 6 Approximate solution function for Case 6

Note that, a slight deformation of the shape of the solution compared to Case 3 which can be explained by the same arguments as in the previous case, also for each algorithm, the behavior of the method is first tested with the data $b_1 = 0$ and $b_2 = 0$ to verify whether one will obtain the trivial solution $x = 0$. As related in the previous part with space $\mathbb{R}^3$, the cyclic algorithm is more precise as it needs more iterations to obtain the solution with the prescribed accuracy. The parallel algorithm, although much faster, may skip the solution due to the fact that it performs relatively large increments between the iterates. It was found that the cyclic algorithm is computationally more precise than the parallel algorithm which is much more faster and that their convergence can be controlled by adjusting the parameters θ_n and λ.

5 Conclusions

In this paper, we have proposed one parallel and one cyclic hybrid algorithm which combines the positive features of the (SEM). The choice of the intersection of the sets C^i_{n+1} instead of only one single set based on the furthest intermediate approximation from the current iterate in the parallel algorithm allowed to increase the precision of the computations and, therefore, to reduce the number of computations to get the required precision. The efficiency of the algorithms has been illustrated with some numerical experiments for discussing the (CSVIP).

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On a New Class of Interval-Valued Variational Control Problems

Savin Treanţă

Abstract A class of optimization problems that includes interval-valued multiple integral objective functionals is investigated in this paper. First, a new generalized convexity condition is defined for the functionals involved and it is proved that an interval-valued KT-pseudoinvex optimization problem is described so that every Kuhn–Tucker point represents an LU-optimal solution. Further, an optimization problem with modified interval-valued cost functional is introduced and an equivalence between the two considered control problems is established. Finally, a connection between an LU-optimal solution of the considered optimization problem and a saddle-point associated with the interval-valued Lagrange functional corresponding to the modified interval-valued optimization problem is studied.

1 Introduction

The concept of *convexity* has a crucial significance in the study of many mathematical models that describe phenomena from different branches of science. Due to the complexity of the environment, as many models of Physics, Economics, Mechanics or Neural Networks can no longer be described using only the classical definition of convexity. Consequently, numerous extensions and generalizations of convexity were necessary for the study of these mathematical models that describe practical phenomena (see, for instance, Hanson [4], Jeyakumar [6] and Antczak [2]). In order to investigate the connection between Kuhn–Tucker points and global minimizers, the notion of *invexity* formulated by Hanson [4] has been extended to *KT-invexity*. Further, this concept was adapted for wider classes of optimization problems (infinite, multiobjective and continuous-time programming problems, variational control problems) and, in this regard, we mention Arana-Jiménez et al. [3], Osuna-Gómez et al. [11] and de Oliveira et al. [10]. Moreover, a generalization of convexity was made in the context of Geometry (see Pini [12], Udrişte [19] and Rapcsák [13]), but also for the study of multidimensional variational problems governed by multiple

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and path-independent curvilinear integral functionals (see, for instance, Treanţă and Arana-Jiménez [16, 17], Treanţă [15], Mititelu and Treanţă [8] and Jayswal et al. [5]).

To manage the uncertainty data in many optimization problems, interval optimization is a scientific field increasingly used in applied mathematics. Therefore, by considering the application of interval analysis in different fields, in this paper, we extend the result established by Mond and Smart [9], according to Martin [7], Arana-Jiménez et al. [3] and Treanţă and Arana-Jiménez [16, 17]. More precisely, a new condition of generalized convexity is introduced such that all Kuhn–Tucker points to be LU-optimal solutions in the considered interval-valued optimization problem. Further, an optimization problem involving modified interval-valued cost functional is defined and an equivalence between the two considered control problems is established. Moreover, a connection between an LU-optimal solution of the considered interval-valued optimization problem and a saddle-point associated with the interval-valued Lagrange functional corresponding to the modified interval-valued optimization problem is studied. Finally, in Section 6, we conclude the paper.

2 Preliminaries

In this section, some preliminary results and working assumptions to be used in the sequel are introduced, as follows:

– $\Theta \subset R^m$ is a compact domain, with nonempty interior and the smooth boundary $\partial\Theta$, and $\theta = (\theta^\alpha)$, $\alpha = \overline{1, m}$, is a point in Θ;

– let $\mathcal{S}$ be the following state function space

$$\mathcal{S} = \left\{ s : \Theta \to R^n \mid \; \| s \| = \| s \|_\infty + \sum_{\alpha=1}^{m} \| s_\alpha \|_\infty, s = \text{ piecewise smooth functions} \right\},$$

where $s_\alpha := \dfrac{\partial s}{\partial \theta^\alpha}$;

– also, denote by C the space of piecewise continuous control functions $c : \Theta \to R^k$ with the uniform norm $\| \cdot \|_\infty$;

– $d\theta := d\theta^1 d\theta^2 \cdots d\theta^m$ represents the volume element on $R^m \supset \Theta$;

– for $\mathcal{K} := \Theta \times R^n \times R^k$, we define the following continuously differentiable functions

$$V = \left(V_\alpha^i\right) : \mathcal{K} \to R^{nm}, \quad i = \overline{1, n}, \; \alpha = \overline{1, m},$$

$$W = \left(W_\beta\right) : \mathcal{K} \to R^q, \quad \beta = \overline{1, q}$$

and we assume that the continuously differentiable functions

$$V_\alpha = \left(V_\alpha^i\right) : \mathcal{K} \to R^n, \quad \alpha = \overline{1, m}, \; i = \overline{1, n},$$

satisfy the following conditions of integrability

$$D_\zeta V_\alpha^i = D_\alpha V_\zeta^i, \quad \alpha \neq \zeta, \ \alpha, \zeta = \overline{1, m}, \ i = \overline{1, n},$$

with $D_\zeta := \dfrac{\partial}{\partial \theta^\zeta}$;

– to simplify writing, we use the notations $(\Lambda_{sc}) := \big(\theta, s(\theta), c(\theta), s^0(\theta), c^0(\theta)\big)$ and $\chi_{sc}(\theta) := (\theta, s(\theta), c(\theta))$;

– Einstein summation is assumed (for instance, $\mu^\beta \dfrac{\partial W_\beta}{\partial s^i} := \sum_{\beta=1}^{q} \mu^\beta \dfrac{\partial W_\beta}{\partial s^i}$);

– for $a = \big(a_1, \dots, a_p\big)$, $b = \big(b_1, \dots, b_p\big)$ in R^p, the following convention will be used throughout the paper:

$$a = b \Leftrightarrow a_i = b_i, \quad a \leq b \Leftrightarrow a_i \leq b_i,$$

$$a < b \Leftrightarrow a_i < b_i, \quad a \preceq b \Leftrightarrow a \leq b, \ a \neq b, \quad i = \overline{1, p}.$$

Next, we formulate the definitions for *invexity* and *pseudoinvexity* associated with multiple integral functionals. In this regard, consider

$$h : J^1(R^m, R^n) \times R^k \to R, \quad h = h\left(\theta, s(\theta), s_\alpha(\theta), c(\theta)\right),$$

a continuously differentiable function, where $J^1(R^m, R^n)$ is the jet bundle of first-order associated with R^m and R^n, that determines the following multiple integral scalar functional

$$H : \mathcal{S} \times C \to R, \quad H\left(s, c\right) = \int_\Theta h\left(\theta, s(\theta), s_\alpha(\theta), c(\theta)\right) d\theta.$$

Following Mititelu and Treanţă [8], Treanţă [15], Treanţă and Arana-Jiménez [16], [17], we have

Definition 1 If there exist a C^1-class function

$$\kappa : \Theta \times R^n \times R^k \times R^n \times R^k \to R^n, \quad \kappa = \kappa\left(\Lambda_{sc}\right) = \left(\kappa_i\left(\Lambda_{sc}\right)\right), \quad i = \overline{1, n},$$

satisfying $\kappa\left(\Lambda_{s^0c^0}\right) = 0, \ \forall \theta \in \Theta, \ \kappa|_{\partial\Theta} = 0$, and a C^0-class function

$$\pi : \Theta \times R^n \times R^k \times R^n \times R^k \to R^k, \quad \pi = \pi\left(\Lambda_{sc}\right) = \big(\pi_j\left(\Lambda_{sc}\right)\big), \quad j = \overline{1, k},$$

satisfying $\pi\left(\Lambda_{s^0c^0}\right) = 0, \ \forall \theta \in \Theta, \ \pi|_{\partial\Theta} = 0$, such that for every $(s, c) \in \mathcal{S} \times C$:

$$H\left(s, c\right) - H\left(s^0, c^0\right)$$

$$\geq \int_{\Theta} \left[h_s \left(\theta, s^0(\theta), s^0_\alpha(\theta), c^0(\theta) \right) \kappa + h_{s_\alpha} \left(\theta, s^0(\theta), s^0_\alpha(\theta), c^0(\theta) \right) D_\alpha \kappa \right] d\theta$$

$$+ \int_{\Theta} \left[h_c \left(\theta, s^0(\theta), s^0_\alpha(\theta), c^0(\theta) \right) \pi \right] d\theta,$$

then H is called *invex at* $\left(s^0, c^0\right) \in \mathcal{S} \times C$ *with respect to* κ *and* π.

Definition 2 If there exist a C^1-class function

$$\kappa : \Theta \times R^n \times R^k \times R^n \times R^k \to R^n, \quad \kappa = \kappa\left(\Lambda_{sc}\right) = \left(\kappa_i\left(\Lambda_{sc}\right)\right), \quad i = \overline{1, n},$$

satisfying $\kappa\left(\Lambda_{s^0 c^0}\right) = 0,\ \forall \theta \in \Theta,\ \kappa|_{\partial\Theta} = 0$, and a C^0-class function

$$\pi : \Theta \times R^n \times R^k \times R^n \times R^k \to R^k, \quad \pi = \pi\left(\Lambda_{sc}\right) = \left(\pi_j\left(\Lambda_{sc}\right)\right), \quad j = \overline{1, k},$$

satisfying $\pi\left(\Lambda_{s^0 c^0}\right) = 0,\ \forall \theta \in \Theta,\ \pi|_{\partial\Theta} = 0$, such that for every $(s, c) \in \mathcal{S} \times C$:

$$H(s, c) - H\left(s^0, c^0\right) < 0$$

$$\Rightarrow \int_{\Theta} \left[h_s \left(\theta, s^0(\theta), s^0_\alpha(\theta), c^0(\theta) \right) \kappa + h_{s_\alpha} \left(\theta, s^0(\theta), s^0_\alpha(\theta), c^0(\theta) \right) D_\alpha \kappa \right] d\theta$$

$$+ \int_{\Theta} \left[h_c \left(\theta, s^0(\theta), s^0_\alpha(\theta), c^0(\theta) \right) \pi \right] d\theta < 0,$$

then H is called *pseudoinvex at* $\left(s^0, c^0\right) \in \mathcal{S} \times C$ *with respect to* κ *and* π.

Let $\mathcal{I}$ be the set of closed and bounded real intervals. For a closed and bounded real interval $W = [w^L, w^U] \in \mathcal{I}$, let w^L and w^U be the lower and upper bounds of W, respectively. Further, we will use the following rules:

(1) $W = Z \Longrightarrow w^L = z^L$ and $w^U = z^U$;
(2) if $w^L = w^U = w$ then $W = [w, w] = w$;
(3) $W + Z = [w^L + z^L, w^U + z^U]$;
(5) $-W = -[w^L, w^U] = [-w^U, -w^L]$;
(5) $W - Z = [w^L - z^U, w^U - z^L]$;
(6) $k + W = [k + w^L, k + w^U],\ k \in R$;
(7) $kW = [kw^L, kw^U],\ k \in R,\ k \geq 0$;
(8) $kW = [kw^U, kw^L],\ k \in R,\ k < 0$.

Definition 3 ([14]) The relation $W \preceq_{LU} Z$ between $W, Z \in \mathcal{I}$ is valid $\Longleftrightarrow w^L \leq z^L$ and $w^U \leq z^U$.

Definition 4 ([14]) The relation $W \prec_{LU} Z$ between $W, Z \in \mathcal{I}$ is valid $\Longleftrightarrow W \preceq_{LU} Z$ and $W \neq Z$.

Definition 5 ([14]) A function $f : \mathcal{K} \to \mathcal{I}$, introduced as

$$f\chi_{sc}(\theta) = [f^L\chi_{sc}(\theta), f^U\chi_{sc}(\theta)], \quad \theta \in \Theta,$$

where $f^L\chi_{sc}(\theta)$ and $f^U\chi_{sc}(\theta)$ are real-valued functions, with

$$f^L\chi_{sc}(\theta) \leq f^U\chi_{sc}(\theta), \quad \theta \in \Theta,$$

is called *interval-valued function*.

In this paper, we study the following optimization problem, where the multiple integral objective scalar functional $F(s, c) = \int_{\Theta} f\chi_{sc}(\theta)d\theta$, $(s, c) \in \mathcal{S} \times C$, is considered as interval-valued:

$$(OP) \quad \min_{(s,c)} \left\{ \int_{\Theta} f\chi_{sc}(\theta)d\theta = \left[\int_{\Theta} f^L\chi_{sc}(\theta)d\theta, \int_{\Theta} f^U\chi_{sc}(\theta)d\theta \right] \right\}$$

subject to

$$\frac{\partial s^i}{\partial \theta^\alpha}(\theta) = V_\alpha^i\chi_{sc}(\theta), \quad i = \overline{1, n}, \ \alpha = \overline{1, m}, \ \theta \in \Theta, \tag{1}$$

$$W\chi_{sc}(\theta) \leq 0, \quad \theta \in \Theta, \tag{2}$$

$$s(\theta)|_{\partial\Theta} = \varphi(\theta) = \text{given}. \tag{3}$$

Denoyte by $\mathcal{X}$ the *set of all feasible solutions* for (OP)

$$\mathcal{X} = \{(s, c)\, |s \in \mathcal{S}, \ c \in C \ \text{ fulfilling } \ (1), \ (2) \text{ and } (3)\}.$$

Definition 6 ([14]) The point $(s^0, c^0) \in \mathcal{X}$ is said to be *LU-optimal solution* if there exists no other $(s, c) \in \mathcal{X}$ such that $F(s, c) \prec_{LU} F(s^0, c^0)$.

The necessary conditions of LU-optimality are provided by the following result:

Theorem 1 *([14]) Under constraint qualification assumptions, let $(s^0, c^0) \in \mathcal{X}$ be an LU-optimal solution of (OP). Then, for all $\theta \in \Theta$, except at discontinuities, there exists $\varrho : \Theta \to R^2$, $\varrho(\theta) = (\varrho^L(\theta), \varrho^U(\theta))$, $\mu : \Theta \to R^q$ and $\lambda : \Theta \to R^{nm}$, with $\mu(\theta) = (\mu^\beta(\theta)) \in R^q$, $\lambda(\theta) = (\lambda_i^\alpha(\theta)) \in R^{nm}$, such that:*

$$\varrho^L(\theta)\frac{\partial f^L}{\partial s^i}\chi_{s^0c^0}(\theta)+\varrho^U(\theta)\frac{\partial f^U}{\partial s^i}\chi_{s^0c^0}(\theta)+\lambda_i^\alpha(\theta)\frac{\partial V_\alpha^i}{\partial s^i}\chi_{s^0c^0}(\theta) \tag{4}$$

$$+\mu^\beta(\theta)\frac{\partial W_\beta}{\partial s^i}\chi_{s^0c^0}(\theta)+\frac{\partial \lambda_i^\alpha}{\partial \theta^\alpha}(\theta)=0, \quad i=\overline{1,n},$$

$$\varrho^L(\theta)\frac{\partial f^L}{\partial c^j}\chi_{s^0c^0}(\theta)+\varrho^U(\theta)\frac{\partial f^U}{\partial c^j}\chi_{s^0c^0}(\theta)+\lambda_i^\alpha(\theta)\frac{\partial V_\alpha^i}{\partial c^j}\chi_{s^0c^0}(\theta) \tag{5}$$

$$+\mu^\beta(\theta)\frac{\partial W_\beta}{\partial c^j}\chi_{s^0c^0}(\theta)=0, \quad j=\overline{1,k},$$

$$\mu^\beta(\theta)W_\beta\chi_{s^0c^0}(\theta)=0 \quad \text{(without summation)}, \quad (\varrho(\theta),\mu(\theta))\succeq 0. \tag{6}$$

Definition 7 ([14]) If $(\varrho^L,\varrho^U)>(0,0)$, an LU-optimal solution $(s^0,c^0)\in\mathcal{X}$ of (OP) is a *normal LU-optimal solution* in (OP).

Definition 8 The point $(s^0,c^0)\in\mathcal{X}$ is called *Kuhn-Tucker point* of (OP) if, for all $\theta\in\Theta$, except at discontinuities, there exist the multipliers $\mu:\Theta\to R^q$ and $\lambda:\Theta\to R^{nm}$, with $\mu(\theta)=(\mu^\beta(\theta))\in R^q$, $\lambda(\theta)=(\lambda_i^\alpha(\theta))\in R^{nm}$, such that:

$$\frac{\partial f^L}{\partial s^i}\chi_{s^0c^0}(\theta)+\frac{\partial f^U}{\partial s^i}\chi_{s^0c^0}(\theta)+\lambda_i^\alpha(\theta)\frac{\partial V_\alpha^i}{\partial s^i}\chi_{s^0c^0}(\theta) \tag{7}$$

$$+\mu^\beta(\theta)\frac{\partial W_\beta}{\partial s^i}\chi_{s^0c^0}(\theta)+\frac{\partial \lambda_i^\alpha}{\partial \theta^\alpha}(\theta)=0, \quad i=\overline{1,n},$$

$$\frac{\partial f^L}{\partial c^j}\chi_{s^0c^0}(\theta)+\frac{\partial f^U}{\partial c^j}\chi_{s^0c^0}(\theta)+\lambda_i^\alpha(\theta)\frac{\partial V_\alpha^i}{\partial c^j}\chi_{s^0c^0}(\theta) \tag{8}$$

$$+\mu^\beta(\theta)\frac{\partial W_\beta}{\partial c^j}\chi_{s^0c^0}(\theta)=0, \quad j=\overline{1,k},$$

$$\mu^\beta(\theta)W_\beta\chi_{s^0c^0}(\theta)=0 \quad (without\ summation), \quad \mu(\theta)\geq 0. \tag{9}$$

The next result formulates a connection between the normal LU-optimal solution of (OP) and Kuhn–Tucker point of (OP).

Theorem 2 *If $(s^0,c^0)\in\mathcal{X}$ represents a normal LU-optimal solution of (OP), then $(s^0,c^0)\in\mathcal{X}$ is a Kuhn–Tucker point of (OP).*

3 Interval-Valued KT-pseudoinvex Optimization Problems

In this section, the concept of *interval-valued KT-pseudoinvexity* corresponding to the optimization problem with interval-valued objective functional (OP) is introduced. Specifically, we generalize the *KT-pseudoinvexity* notion, introduced by Treanţă and Arana-Jiménez [16, 17], for interval-valued optimization problems. We prove that interval-valued KT-pseudoinvexity condition is a necessary and sufficient condition so that all Kuhn–Tucker points of (OP) to be LU-optimal solutions of (OP).

Definition 9 The interval-valued optimization problem (OP) is called *interval-valued KT-pseudoinvex at* $(s^0, c^0) \in \mathcal{X}$ if for all $\mu : \Theta \to R^q$, satisfying (9), and piecewise smooth functions $\lambda : \Theta \to R^{nm}$, there exist a C^1-class function

$$\kappa : \Theta \times (R^n \times R^k)^2 \times R^{nm} \times R^q \to R^n,$$

$$\kappa = \left(\kappa_i \left(\theta, s(\theta), c(\theta), s^0(\theta), c^0(\theta), \lambda(\theta), \mu(\theta)\right)\right), \quad i = \overline{1, n},$$

satisfying $\kappa \left(\theta, s^0(\theta), c^0(\theta), s^0(\theta), c^0(\theta), \lambda(\theta), \mu(\theta)\right) = 0, \ \forall \theta \in \Theta, \ \kappa|_{\partial\Theta} = 0$, and a C^0-class function

$$\pi : \Theta \times (R^n \times R^k)^2 \times R^{nm} \times R^q \to R^k,$$

$$\pi = \left(\pi_j \left(\theta, s(\theta), c(\theta), s^0(\theta), c^0(\theta), \lambda(\theta), \mu(\theta)\right)\right), \quad j = \overline{1, k},$$

satisfying $\pi \left(\theta, s^0(\theta), c^0(\theta), s^0(\theta), c^0(\theta), \lambda(\theta), \mu(\theta)\right) = 0, \ \forall \theta \in \Theta, \ \pi|_{\partial\Theta} = 0$, such that for all $(s, c) \in \mathcal{X}$:

$$F(s, c) \prec_{LU} F\left(s^0, c^0\right) \Rightarrow \mathcal{L}(\kappa, \pi) < 0,$$

where

$$\mathcal{L}(\kappa, \pi) := \int_\Theta \left\{\kappa \left(f_s^L \chi_{s^0c^0}(\theta) + f_s^U \chi_{s^0c^0}(\theta) + \lambda_i^\alpha(\theta) \left(V_\alpha^i\right)_s \chi_{s^0c^0}(\theta)\right)\right\} d\theta$$

$$+ \int_\Theta \left(\mu^\beta(\theta) \left(W_\beta\right)_s \chi_{s^0c^0}(\theta)\right) \kappa d\theta - \int_\Theta \{\lambda^\alpha(\theta) D_\alpha \kappa\} d\theta$$

$$+ \int_\Theta \left\{\pi \left(f_c^L \chi_{s^0c^0}(\theta) + f_c^U \chi_{s^0c^0}(\theta) + \lambda_i^\alpha(\theta) \left(V_\alpha^i\right)_c \chi_{s^0c^0}(\theta)\right)\right\} d\theta$$

$$+ \int_\Theta \left(\mu^\beta(\theta) \left(W_\beta\right)_c \chi_{s^0c^0}(\theta)\right) \pi d\theta.$$

Definition 10 If, for all $(s^0, c^0) \in \mathcal{X}$, the interval-valued optimization problem (OP) is interval-valued KT-pseudoinvex, then it is called *interval-valued KT-pseudoinvex.*

Theorem 3 *All Kuhn–Tucker points of (OP) are LU-optimal solutions of (OP) if and only if the interval-valued optimization problem (OP) is interval-valued KT-pseudoinvex.*

Proof "$\Longleftarrow$" Consider $(s^0, c^0) \in \mathcal{X}$ is a Kuhn-Tucker point of (OP). In consequence, there exist $\mu : \Theta \to R^q$ and piecewise smooth functions $\lambda : \Theta \to R^{nm}$ satisfying (7)–(9). By using the hypothesis, the interval-valued optimization problem (OP) is interval-valued KT-pseudoinvex. Thus, for μ (satisfying (9)) and λ, for all $(s, c) \in \mathcal{X}$, there exist the functions κ and π fulfilling the interval-valued KT-pseudoinvexity definition. Next, applying the hypothesis $\kappa|_{\partial\Theta} = 0$ and the flow-divergence formula, we get

$$\int_{\Theta} \kappa D_\alpha \lambda^\alpha(\theta) d\theta = -\int_{\Theta} \left[\lambda^\alpha(\theta) D_\alpha \kappa\right] d\theta.$$

Consequently, we obtain (see (7) and (8))

$$\mathcal{L}(\kappa, \pi) = \int_{\Theta} \left(f_s^L \chi_{s^0 c^0}(\theta) + f_s^U \chi_{s^0 c^0}(\theta) + \lambda_i^\alpha(\theta) \left(V_\alpha^i\right)_s \chi_{s^0 c^0}(\theta)\right) \kappa d\theta$$

$$+ \int_{\Theta} \left(\mu^\beta(\theta) \left(W_\beta\right)_s \chi_{s^0 c^0}(\theta) + D_\alpha \lambda^\alpha(\theta)\right) \kappa d\theta$$

$$+ \int_{\Theta} \left\{\pi \left(f_c^L \chi_{s^0 c^0}(\theta) + f_c^U \chi_{s^0 c^0}(\theta) + \lambda_i^\alpha(\theta) \left(V_\alpha^i\right)_c \chi_{s^0 c^0}(\theta)\right)\right\} d\theta$$

$$+ \int_{\Theta} \left(\mu^\beta(\theta) \left(W_\beta\right)_c \chi_{s^0 c^0}(\theta)\right) \pi d\theta = 0.$$

Since (OP) is interval-valued KT-pseudoinvex, it results $F(s, c) \succeq_{LU} F\left(s^0, c^0\right)$, for all $(s, c) \in \mathcal{X}$. Thus, $(s^0, c^0) \in \mathcal{X}$ is an LU-optimal solution of (OP). This completes the proof of implication "$\Longleftarrow$".

"$\Longrightarrow$" Let $(s, c),\ (s^0, c^0) \in \mathcal{X}$ be two feasible points in (OP), with $F(s, c) \prec_{LU} F\left(s^0, c^0\right)$, μ (satisfying (9)) and λ piecewise smooth functions. We must find two vector-valued functions κ and π, fulfilling the interval-valued KT-pseudoinvexity definition, such that $\mathcal{L}(\kappa, \pi) < 0$. Arguing by contradiction, suppose that the inequality $\mathcal{L}(\kappa, \pi) < 0$ is not satisfied for any vector-valued functions κ and π given as above. Putting $-\kappa,\ -\pi$ as arguments of $\mathcal{L}$, we obtain that $\mathcal{L}(\kappa, \pi) > 0$ is not satisfied. Consequently, for all vector-valued functions κ and π, we have $\mathcal{L}(\kappa, \pi) = 0$.

Let us consider $\pi = 0,\ \forall \theta \in \Theta$, and fix it as argument of $\mathcal{L}$. Therefore, $\mathcal{L}(\kappa, 0) = 0$, equivalent with

$$\int_{\Theta} \left[f_s^L \chi_{s^0c^0}(\theta) + f_s^U \chi_{s^0c^0}(\theta) + \lambda_i^\alpha(\theta) \left(V_\alpha^i\right)_s \chi_{s^0c^0}(\theta) + \mu^\beta(\theta) \left(W_\beta\right)_s \chi_{s^0c^0}(\theta) \right] \kappa d\theta$$

$$+ \int_{\Theta} \left[D_\alpha \lambda^\alpha(\theta) \right] \kappa d\theta = 0,$$

is satisfied, for all vector-valued functions κ. Using the Dubois-Raymond's Lemma (see Alekseev et al. [1]), we have

$$f_s^L \chi_{s^0c^0}(\theta) + f_s^U \chi_{s^0c^0}(\theta) + \lambda_i^\alpha(\theta) \left(V_\alpha^i\right)_s \chi_{s^0c^0}(\theta) \tag{10}$$

$$+ \mu^\beta(\theta) \left(W_\beta\right)_s \chi_{s^0c^0}(\theta) + D_\alpha \lambda^\alpha(\theta) = 0.$$

As above, if we fix $\kappa = 0, \ \forall \theta \in \Theta$, as argument of $\mathcal{L}$, we obtain

$$f_c^L \chi_{s^0c^0}(\theta) + f_c^U \chi_{s^0c^0}(\theta) + \lambda_i^\alpha(\theta) \left(V_\alpha^i\right)_c \chi_{s^0c^0}(\theta) + \mu^\beta(\theta) \left(W_\beta\right)_c \chi_{s^0c^0}(\theta) = 0. \tag{11}$$

Since $s^0, \ c^0, \ \lambda, \ \mu$ fulfils the conditions (9), (10) and (11), we conclude that $(s^0, c^0) \in \mathcal{X}$ is a Kuhn–Tucker point of (OP). By using the hypothesis, $(s^0, c^0) \in \mathcal{X}$ is an LU-optimal solution of (OP) and we get a contradiction. Consequently, the interval-valued optimization problem (OP) is interval-valued KT-pseudoinvex. This completes the proof. □

4 An Interval-Valued Optimization Problem Associated with (OP) with Modified Objective Functional

This section includes an interval-valued optimization problem associated with (OP) with modified objective functional. By considering a concrete application, it can be easily noticed that the interval-valued optimization problem associated with (OP) with modified objective functional is simpler to investigate than the original interval-valued optimization problem.

In the following, for $\kappa, \ \pi$ given as in Sect. 2 (see Definitions 1 and 2) and for an arbitrary $(s^0, c^0) \in \mathcal{X}$ of (OP), we introduce an interval-valued optimization problem corresponding to (OP) with modified objective functional, as follows:

$$(OP_{\kappa,\pi}(s^0, c^0)) \qquad \min_{(s,c)} \int_{\Theta} \left(f_s \chi_{s^0c^0}(\theta)\kappa + f_c \chi_{s^0c^0}(\theta)\pi \right) d\theta$$

subject to

$$\frac{\partial s^i}{\partial \theta^\alpha}(\theta) = V_\alpha^i \chi_{sc}(\theta), \quad i = \overline{1, n}, \ \alpha = \overline{1, m}, \ \theta \in \Theta, \tag{12}$$

$$W\chi_{sc}(\theta) \leq 0, \quad \theta \in \Theta, \tag{13}$$

$$s(\theta)|_{\partial\Theta} = \varphi(\theta) = \text{given}, \tag{14}$$

where

$$\int_{\Theta} (f_s \chi_{s^0c^0}(\theta)\kappa + f_c \chi_{s^0c^0}(\theta)\pi)\, d\theta$$

$$:= \left[\int_{\Theta} \left(f_s^L \chi_{s^0c^0}(\theta)\kappa + f_c^L \chi_{s^0c^0}(\theta)\pi \right) d\theta, \int_{\Theta} \left(f_s^U \chi_{s^0c^0}(\theta)\kappa + f_c^U \chi_{s^0c^0}(\theta)\pi \right) d\theta \right].$$

Remark 1 The set of feasible solutions in the aforementioned interval-valued optimization problem with modified objective functional is $\mathcal{X}$, as well.

Definition 11 The feasible solution $(\hat{s}, \hat{c}) \in \mathcal{X}$ is called *LU-optimal solution* of $(OP_{\kappa,\pi}(s^0, c^0))$ if the inequality

$$\int_{\Theta} (f_s \chi_{s^0c^0}(\theta)\kappa\ (\Lambda_{sc}) + f_c \chi_{s^0c^0}(\theta)\pi\ (\Lambda_{sc}))\, d\theta$$

$$\succeq_{LU} \int_{\Theta} (f_s \chi_{s^0c^0}(\theta)\kappa\ (\Lambda_{\hat{s}\hat{c}}) + f_c \chi_{s^0c^0}(\theta)\pi\ (\Lambda_{\hat{s}\hat{c}}))\, d\theta,$$

is fulfilled, for every $(s, c) \in \mathcal{X}$.

The *normal LU-optimal solution* notion in the considered interval-valued optimization problem with modified objective functional $(OP_{\kappa,\pi}(s^0, c^0))$ has the same significance as in Definition 7.

In the following, we set some results of equivalence for LU-optimal solutions of (OP) and $(OP_{\kappa,\pi}(s^0, c^0))$.

Theorem 4 *Consider* $\int_{\Theta} f^{\epsilon} \chi_{sc}(\theta) d\theta,\ \epsilon \in \{L, U\}$, *are pseudoinvex at* $(s^0, c^0) \in \mathcal{X}$ *with respect to* κ *and* π *and* $(s^0, c^0) \in \mathcal{X}$ *is an LU-optimal solution of* $(OP_{\kappa,\pi}(s^0, c^0))$. *Then* $(s^0, c^0) \in \mathcal{X}$ *is an LU-optimal solution of* (OP).

Proof Arguing by contradiction, consider $(s^0, c^0) \in \mathcal{X}$ isn't an LU-optimal solution for (OP). Consequently, there exists $(\overline{s}, \overline{c}) \in \mathcal{X}$ fulfilling

$$\int_{\Theta} f \chi_{\overline{s}\overline{c}}(\theta) d\theta \prec_{LU} \int_{\Theta} f \chi_{s^0c^0}(\theta) d\theta.$$

By hypothesis, $\int_{\Theta} f^{\epsilon} \chi_{sc}(\theta) d\theta, \epsilon \in \{L, U\}$, are pseudoinvex at $(s^0, c^0) \in \mathcal{X}$ with respect to κ and π. Therefore, the above inequality involves

$$\int_{\Theta} f_s \chi_{s^0c^0}(\theta)\kappa\ (\Lambda_{\overline{s}\overline{c}})\, d\theta + \int_{\Theta} f_c \chi_{s^0c^0}(\theta)\pi\ (\Lambda_{\overline{s}\overline{c}})\, d\theta \prec_{LU} [0, 0].$$

By using $\kappa\left(\Lambda_{s^0c^0}\right)=\pi\left(\Lambda_{s^0c^0}\right)=0$, we can write as follows:

$$\int_{\Theta} f_s \chi_{s^0c^0}(\theta)\kappa\left(\Lambda_{\overline{sc}}\right) d\theta + \int_{\Theta} f_c \chi_{s^0c^0}(\theta)\pi\left(\Lambda_{\overline{sc}}\right) d\theta$$

$$\prec_{LU} \int_{\Theta} f_s \chi_{s^0c^0}(\theta)\kappa\left(\Lambda_{s^0c^0}\right) d\theta + \int_{\Theta} f_c \chi_{s^0c^0}(\theta)\pi\left(\Lambda_{s^0c^0}\right) d\theta,$$

which contradicts the optimality of $(s^0, c^0) \in \mathcal{X}$ in $(OP_{\kappa,\pi}(s^0, c^0))$. In consequence, $(s^0, c^0) \in \mathcal{X}$ is LU-optimal solution of (OP). □

Theorem 5 *([18]) Let $(s^0, c^0) \in \mathcal{X}$ be a normal LU-optimal solution of (OP). Also, if $\int_{\Theta} \mu^{\beta}(\theta) W_{\beta} \chi_{sc}(\theta) d\theta$, $\int_{\Theta} \lambda_i^{\alpha}(\theta)\left(V_{\alpha}^i \chi_{sc}(\theta) - \frac{\partial s^i}{\partial \theta^{\alpha}}(\theta)\right) d\theta$ are invex at $(s^0, c^0) \in \mathcal{X}$ with respect to κ and π, then $(s^0, c^0) \in \mathcal{X}$ is an LU-optimal solution of $(OP_{\kappa,\pi}(s^0, c^0))$.*

5 Saddle-Point Optimality Criteria

In this section, we establish a relation between an LU-optimal solution of (OP) and a *saddle-point* corresponding to the *interval-valued Lagrange functional* of the considered interval-valued optimization problem associated with (OP) with modified objective functional $(OP_{\kappa,\pi}(s^0, c^0))$.

Definition 12 *The interval-valued Lagrange functional* corresponding to the considered interval-valued optimization problem with modified objective functional $(OP_{\kappa,\pi}(s^0, c^0))$ is defined as follows:

$$\mathcal{L}_{\kappa,\pi}(s, c; \lambda, \mu)$$

$$= \int_{\Theta} \Big(f_s \chi_{s^0c^0}(\theta)\kappa + f_c \chi_{s^0c^0}(\theta)\pi + \lambda_i^{\alpha}(\theta)\left(V_{\alpha}^i \chi_{sc}(\theta) - \frac{\partial s^i}{\partial \theta^{\alpha}}(\theta)\right)$$

$$+\mu^{\beta}(\theta) W_{\beta} \chi_{sc}(\theta)\Big) d\theta$$

$$:= \left[\mathcal{L}_{\kappa,\pi}^{L}(s, c; \lambda, \mu), \mathcal{L}_{\kappa,\pi}^{U}(s, c; \lambda, \mu)\right]$$

where, for $\epsilon \in \{L, U\}$, we have denoted

$$\mathcal{L}_{\kappa,\pi}^{\epsilon}(s, c; \lambda, \mu)$$

$$:= \int_{\Theta} \Big(f_s^{\epsilon} \chi_{s^0c^0}(\theta)\kappa + f_c^{\epsilon} \chi_{s^0c^0}(\theta)\pi + \lambda_i^{\alpha}(\theta)\left(V_{\alpha}^i \chi_{sc}(\theta) - \frac{\partial s^i}{\partial \theta^{\alpha}}(\theta)\right)$$

$$+\mu^\beta(\theta)W_\beta\chi_{sc}(\theta))d\theta.$$

Remark 2 We notice that

$$\mathcal{L}_{\kappa,\pi}(s, c; \lambda, \mu)$$

$$= \left[\int_\Theta \left(f_s^L \chi_{s^0c^0}(\theta)\kappa + f_c^L \chi_{s^0c^0}(\theta)\pi\right) d\theta, \int_\Theta \left(f_s^U \chi_{s^0c^0}(\theta)\kappa + f_c^U \chi_{s^0c^0}(\theta)\pi\right) d\theta\right]$$

$$+ \int_\Theta \left(\lambda_i^\alpha(\theta)\left(V_\alpha^i\chi_{sc}(\theta) - \frac{\partial s^i}{\partial\theta^\alpha}(\theta)\right) + \mu^\beta(\theta)W_\beta\chi_{sc}(\theta)\right) d\theta.$$

Definition 13 A point $(s^0, c^0; \overline{\lambda}, \overline{\mu}) \in \mathcal{X} \times \mathbb{R}^{nm} \times \mathbb{R}_+^q$ is called *saddle-point* of the interval-valued Lagrange functional $\mathcal{L}_{\kappa,\pi}(s, c; \lambda, \mu)$ corresponding to the modified interval-valued optimization problem $(OP_{\kappa,\pi}(s^0, c^0))$ if:

$$\mathcal{L}_{\kappa,\pi}(s^0, c^0; \lambda, \mu) \preceq_{LU} \mathcal{L}_{\kappa,\pi}(s^0, c^0; \overline{\lambda}, \overline{\mu}), \quad \forall(\lambda, \mu) \in \mathbb{R}^{nm} \times \mathbb{R}_+^q \tag{15}$$

$$\mathcal{L}_{\kappa,\pi}(s^0, c^0; \overline{\lambda}, \overline{\mu}) \preceq_{LU} \mathcal{L}_{\kappa,\pi}(s, c; \overline{\lambda}, \overline{\mu}), \quad \forall(s, c) \in \mathcal{X}. \tag{16}$$

Further, by using the above definitions, we set the following two characterization results:

Theorem 6 *Consider $(s^0, c^0; \overline{\lambda}, \overline{\mu}) \in \mathcal{X} \times \mathbb{R}^{nm} \times \mathbb{R}_+^q$ be a saddle-point of the interval-valued Lagrange functional $\mathcal{L}_{\kappa,\pi}(s, c; \lambda, \mu)$ corresponding to the interval-valued optimization problem $(OP_{\kappa,\pi}(s^0, c^0))$. Also, consider the functionals $\int_\Theta f^\epsilon \chi_{sc}(\theta)d\theta$, $\epsilon \in \{L, U\}$, are pseudoinvex at $(s^0, c^0) \in \mathcal{X}$ with respect to κ and π. Then $(s^0, c^0) \in \mathcal{X}$ is an LU-optimal solution of (OP).*

Proof Let $(s^0, c^0; \overline{\lambda}, \overline{\mu}) \in \mathcal{X} \times \mathbb{R}^{nm} \times \mathbb{R}_+^q$ be a saddle-point for the interval-valued Lagrange functional $\mathcal{L}_{\kappa,\pi}(s, c; \lambda, \mu)$ corresponding to the modified optimization problem $(OP_{\kappa,\pi}(s^0, c^0))$. Consequently, using (15), we get

$$\int_\Theta \left[f_s\chi_{s^0c^0}(\theta)\kappa\left(\theta, s^0(\theta), c^0(\theta), s^0(\theta), c^0(\theta)\right)\right] d\theta$$

$$+ \int_\Theta \left[f_c\chi_{s^0c^0}(\theta)\pi\left(\theta, s^0(\theta), c^0(\theta), s^0(\theta), c^0(\theta)\right)\right] d\theta$$

$$+ \int_\Theta \lambda_i^\alpha(\theta)\left[V_\alpha^i\chi_{s^0c^0}(\theta) - \frac{\partial s^i}{\partial\theta^\alpha}(\theta)\right] d\theta + \int_\Theta \mu^\beta(\theta)W_\beta\chi_{s^0c^0}(\theta)d\theta$$

$$\preceq_{LU} \int_\Theta \left[f_s\chi_{s^0c^0}(\theta)\kappa\left(\theta, s^0(\theta), c^0(\theta), s^0(\theta), c^0(\theta)\right)\right] d\theta$$

$$+\int_{\Theta}\left[f_c\chi_{s^0c^0}(\theta)\pi\left(\theta, s^0(\theta), c^0(\theta), s^0(\theta), c^0(\theta)\right)\right]d\theta$$

$$+\int_{\Theta}\overline{\lambda}_i^{\alpha}(\theta)\left[V_{\alpha}^i\chi_{s^0c^0}(\theta)-\frac{\partial s^i}{\partial\theta^{\alpha}}(\theta)\right]d\theta+\int_{\Theta}\overline{\mu}^{\beta}(\theta)W_{\beta}\chi_{s^0c^0}(\theta)d\theta,$$

for all $(\lambda, \mu)\in\mathbb{R}^{nm}\times\mathbb{R}^q_+$. Using the properties of κ and π, it results

$$\int_{\Theta}\lambda_i^{\alpha}(\theta)\left[V_{\alpha}^i\chi_{s^0c^0}(\theta)-\frac{\partial s^i}{\partial\theta^{\alpha}}(\theta)\right]d\theta+\int_{\Theta}\mu^{\beta}(\theta)W_{\beta}\chi_{s^0c^0}(\theta)d\theta$$

$$\leq\int_{\Theta}\overline{\lambda}_i^{\alpha}(\theta)\left[V_{\alpha}^i\chi_{s^0c^0}(\theta)-\frac{\partial s^i}{\partial\theta^{\alpha}}(\theta)\right]d\theta+\int_{\Theta}\overline{\mu}^{\beta}(\theta)W_{\beta}\chi_{s^0c^0}(\theta)d\theta,$$

for all $(\lambda, \mu)\in\mathbb{R}^{nm}\times\mathbb{R}^q_+$. Taking $\mu(\theta)=(\mu^{\beta}(\theta))=0,\ \beta=\overline{1,q}$, and using the feasibility of (s^0, c^0), we find

$$\int_{\Theta}\overline{\lambda}_i^{\alpha}(\theta)\left[V_{\alpha}^i\chi_{s^0c^0}(\theta)-\frac{\partial s^i}{\partial\theta^{\alpha}}(\theta)\right]d\theta+\int_{\Theta}\overline{\mu}^{\beta}(\theta)W_{\beta}\chi_{s^0c^0}(\theta)d\theta\geq 0. \tag{17}$$

Further, arguing by contradiction, suppose that $(s^0, c^0)\in\mathcal{X}$ isn't LU-optimal solution for (OP). Consequently, there exists $(\overline{s}, \overline{c})\in\mathcal{X}$ fulfilling

$$\int_{\Theta}f\chi_{\overline{s},\overline{c}}(\theta)d\theta\prec_{LU}\int_{\Theta}f\chi_{s^0c^0}(\theta)d\theta.$$

Since $\int_{\Theta}f^{\epsilon}\chi_{sc}(\theta)d\theta,\ \epsilon\in\{L, U\}$, are pseudoinvex at $(s^0, c^0)\in\mathcal{X}$ with respect to κ and π, it follows

$$\int_{\Theta}\left[f_s\chi_{s^0c^0}(\theta)\kappa\left(\theta, \overline{s}(\theta), \overline{c}(\theta), s^0(\theta), c^0(\theta)\right)\right]d\theta$$

$$+\int_{\Theta}\left[f_c\chi_{s^0c^0}(\theta)\pi\left(\theta, \overline{s}(\theta), \overline{c}(\theta), s^0(\theta), c^0(\theta)\right)\right]d\theta\prec_{LU}[0, 0].$$

By considering

$$\kappa\left(\theta, s^0(\theta), c^0(\theta), s^0(\theta), c^0(\theta)\right)=\pi\left(\theta, s^0(\theta), c^0(\theta), s^0(\theta), c^0(\theta)\right)=0,\ \forall t\in\Theta,$$

the above inequality becomes

$$\int_{\Theta}\left[f_s\chi_{s^0c^0}(\theta)\kappa\left(\theta, \overline{s}(\theta), \overline{c}(\theta), s^0(\theta), c^0(\theta)\right)\right]d\theta \tag{18}$$

$$+\int_\Theta \left[f_c\chi_{s^0c^0}(\theta)\pi\left(\theta,\overline{s}(\theta),\overline{c}(\theta),s^0(\theta),c^0(\theta)\right)\right]d\theta$$

$$\prec_{LU}\int_\Theta \left[f_s\chi_{s^0c^0}(\theta)\kappa\left(\theta,s^0(\theta),c^0(\theta),s^0(\theta),c^0(\theta)\right)\right]d\theta$$

$$+\int_\Theta \left[f_c\chi_{s^0c^0}(\theta)\pi\left(\theta,s^0(\theta),c^0(\theta),s^0(\theta),c^0(\theta)\right)\right]d\theta.$$

Since $(\overline{s},\overline{c})\in\mathcal{X}$, the following inequality hold

$$\int_\Theta \overline{\lambda}_i^\alpha(\theta)\left[V_\alpha^i\chi_{\overline{s},\overline{c}}(\theta)-\frac{\partial s^i}{\partial\theta^\alpha}(\theta)\right]d\theta+\int_\Theta \overline{\mu}^\beta(\theta)W_\beta\chi_{\overline{s},\overline{c}}(\theta)d\theta\le 0, \tag{19}$$

and, combining (17) and (19), it follows

$$\int_\Theta \overline{\lambda}_i^\alpha(\theta)\left[V_\alpha^i\chi_{\overline{s},\overline{c}}(\theta)-\frac{\partial s^i}{\partial\theta^\alpha}(\theta)\right]d\theta+\int_\Theta \overline{\mu}^\beta(\theta)W_\beta\chi_{\overline{s},\overline{c}}(\theta)d\theta \tag{20}$$

$$\le\int_\Theta \overline{\lambda}_i^\alpha(\theta)\left[V_\alpha^i\chi_{s^0c^0}(\theta)-\frac{\partial s^i}{\partial\theta^\alpha}(\theta)\right]d\theta+\int_\Theta \overline{\mu}^\beta(\theta)W_\beta\chi_{s^0c^0}(\theta)d\theta.$$

In the following, by using (18) and (20), we obtain

$$\int_\Theta \left[f_s\chi_{s^0c^0}(\theta)\kappa\left(\theta,\overline{s}(\theta),\overline{c}(\theta),s^0(\theta),c^0(\theta)\right)\right]d\theta$$

$$+\int_\Theta \left[f_c\chi_{s^0c^0}(\theta)\pi\left(\theta,\overline{s}(\theta),\overline{c}(\theta),s^0(\theta),c^0(\theta)\right)\right]d\theta$$

$$+\int_\Theta \overline{\lambda}_i^\alpha(\theta)\left[V_\alpha^i\chi_{\overline{s},\overline{c}}(\theta)-\frac{\partial s^i}{\partial\theta^\alpha}(\theta)\right]d\theta+\int_\Theta \overline{\mu}^\beta(\theta)W_\beta\chi_{\overline{s},\overline{c}}(\theta)d\theta$$

$$\prec_{LU}\int_\Theta \left[f_s\chi_{s^0c^0}(\theta)\kappa\left(\theta,s^0(\theta),c^0(\theta),s^0(\theta),c^0(\theta)\right)\right]d\theta$$

$$+\int_\Theta \left[f_c\chi_{s^0c^0}(\theta)\pi\left(\theta,s^0(\theta),c^0(\theta),s^0(\theta),c^0(\theta)\right)\right]d\theta$$

$$+\int_\Theta \overline{\lambda}_i^\alpha(\theta)\left[V_\alpha^i\chi_{s^0c^0}(\theta)-\frac{\partial s^i}{\partial\theta^\alpha}(\theta)\right]d\theta+\int_\Theta \overline{\mu}^\beta(\theta)W_\beta\chi_{s^0c^0}(\theta)d\theta,$$

which is a contradiction with (16) (see Definition 13). Consequently, $(s^0,c^0)\in\mathcal{X}$ is LU-optimal solution for (OP). □

Theorem 7 *Consider* $(s^0, c^0) \in \mathcal{X}$ *is a normal LU-optimal solution for* (OP) *and the multiple integral functional*

$$\int_{\Theta} \left\{ \overline{\mu}^{\beta}(\theta) W_{\beta} \chi_{sc}(\theta) + \overline{\lambda}_i^{\alpha}(\theta) \left[V_{\alpha}^{i} \chi_{sc}(\theta) - \frac{\partial s^i}{\partial \theta^{\alpha}}(\theta) \right] \right\} d\theta$$

is invex at $(s^0, c^0) \in \mathcal{X}$ *with respect to* κ *and* π*. Then* $(s^0, c^0; \overline{\lambda}, \overline{\mu}) \in \mathcal{X} \times \mathbb{R}^{nm} \times \mathbb{R}_{+}^{q}$ *is a saddle-point for the interval-valued Lagrange functional* $\mathcal{L}_{\kappa,\pi}(s, c; \lambda, \mu)$ *corresponding to the modified optimization problem* $(OP_{\kappa,\pi}(s^0, c^0))$.

Proof The proof is immediate. □

6 Conclusions

A class of optimization problems with interval-valued multiple integral objective functionals has been studied in this paper. More precisely, it has been shown that an interval-valued KT-pseudoinvex optimization problem is described so that every Kuhn–Tucker point is an LU-optimal solution. Further, an interval-valued optimization problem with modified objective functional has been introduced and an equivalence between the two considered control problems was established. Finally, a connection between an LU-optimal solution of (OP) and a saddle-point corresponding to the interval-valued Lagrange functional of the considered interval-valued optimization problem with modified objective functional $(OP_{\kappa,\pi}(s^0, c^0))$ was studied.

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Best Proximity Points for Multivalued Mappings Satisfying Z_σ-Proximal Contractions with Applications

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Abstract The purpose of this manuscript is to ensure best proximity point for multivalued maps satisfying Z_σ-proximal contraction under an influence of ϕ function on the context of metric spaces. Also, we discuss an example to display the validity of our work. At the end, we apply our main results to derive new best proximity point results on a metric space endowed with a partial ordering/graph.

1 Introduction and Preliminaries

Best proximity point results equip sufficient conditions that guarantee the existence of approximate type solutions, which are optimal as well. Indeed, in the study of fixed point theory the functional equation $\Omega \upsilon = \upsilon$ that is, $\rho(\Omega \upsilon, \upsilon) = 0$ has no solution for a non-self mapping $(\Omega : \mathcal{M} \to \mathcal{N})$, it is desirable to make an approximation solution υ the error of $\rho(\Omega \upsilon, \upsilon)$ is minimum. In light of that consideration $\rho(\Omega \upsilon, \upsilon) \geq \rho(\mathcal{M}, \mathcal{N})$, an absolute optimal approximate solution is an υ for which the error $\rho(\Omega \upsilon, \upsilon)$ assumes the least possible value $\rho(\mathcal{M}, \mathcal{N})$. As a hypothesis, best proximity point theorems supply sufficient conditions for the existence of an optimal approximate solution υ, known as a best proximity point of the mapping Ω that satisfying $\rho(\Omega \upsilon, \upsilon) = \rho(\mathcal{M}, \mathcal{N})$. Notice that, best proximity point theorem is a natural generalization of fixed point theorem. In the presence of self-mapping, a best proximity point becomes a fixed point.

Let (Ξ, ρ) be a metric space. Denote $N(\Xi)$, $CL(\Xi)$, $CB(\Xi)$ and $K(\Xi)$ by the class of all subsets of Ξ, the class of all closed subsets of Ξ, the class of all nonempty closed and bounded subsets of Ξ and the class of all nonempty compact subsets of

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Ξ, respectively. Define the generalized Hausdorff metric H induced by ρ on $CL(\Xi)$ as follows:

$$H(\mathcal{M}, \mathcal{N}) = \begin{cases} \max\{\sup_{\upsilon_1 \in \mathcal{M}} \rho(\upsilon_1, \mathcal{N}), \sup_{\upsilon_2 \in \mathcal{N}} \rho(\upsilon_2, \mathcal{M})\}, & \text{maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $\mathcal{M}, \mathcal{N} \in CL(\Xi)$. For $\mathcal{M}, \mathcal{N} \subseteq \Xi$, we use the following facts:

$$\begin{aligned} &\rho(\upsilon_1, \mathcal{N}) = \inf\{\rho(\upsilon_1, \upsilon_2) : \upsilon_2 \in \mathcal{N}\}; \\ &P_{\mathcal{M}}(\upsilon_1) = \{\upsilon_2 \in \mathcal{N} : \rho(\upsilon_1, \upsilon_2) = \rho(\upsilon_1, \mathcal{M})\}; \\ &\Lambda := \rho(\mathcal{M}, \mathcal{N}) = \inf\{\rho(\upsilon_1, \upsilon_2) : \upsilon_1 \in \mathcal{M},\ \upsilon_2 \in \mathcal{N}\}; \\ &\mathcal{M}_0 = \{\upsilon_1 \in \mathcal{M} : \rho(\upsilon_1, \upsilon_2) = \rho(\mathcal{M}, \mathcal{N}) \text{ for some } \upsilon_2 \in \mathcal{N}\}; \\ &\mathcal{N}_0 = \{\upsilon_2 \in \mathcal{N} : \rho(\upsilon_1, \upsilon_2) = \rho(\mathcal{M}, \mathcal{N}) \text{ for some } \upsilon_1 \in \mathcal{M}\}. \end{aligned}$$

There are some sufficient conditions claimed the nonempty of $\mathcal{M}_0$ and $\mathcal{N}_0$. A such simple condition is that, $\breve{A}$ is compact and $\mathcal{N}$ is approximatively compact w.r.t. $\mathcal{M}$ (every sequence $\{\upsilon_i\}$ in $\mathcal{N}$ such that $\rho(u, \upsilon_i) \to \rho(u, \mathcal{N})$ for some u in $\mathcal{M}$ should have a convergent subsequence).

A point $\upsilon^* \in \mathcal{M}$ is said to be a best proximity point of mapping $\Omega : \mathcal{M} \to CL(\mathcal{N})$, if $\rho(\upsilon^*, \Omega\upsilon^*) = \rho(\mathcal{M}, \mathcal{N})$.

Definition 1 ([23]) Let $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty subsets of a metric space (Ξ, ρ) with $\mathcal{M}_0 \neq \emptyset$. Then the pair $(\mathcal{M}, \mathcal{N})$ is said to have the weak P-property if and only if for any $\upsilon_1, \upsilon_2 \in \mathcal{M}_0$ and $u_1, u_2 \in \mathcal{N}_0$,

$$\begin{cases} \rho(\upsilon_1, u_1) = \rho(\mathcal{M}, \mathcal{N}) \\ \rho(\upsilon_2, u_2) = \rho(\mathcal{M}, \mathcal{N}) \end{cases} \implies \rho(\upsilon_1, \upsilon_2) \leq \rho(u_1, u_2).$$

Jleli and Samet [16] introduced the concept of σ-proximal admissible for a non-self-mapping as following:

Definition 2 ([16]) Let $\mathcal{M}$ and $\mathcal{N}$ be two nonempty subsets of a metric space (Ξ, ρ). Let $\Omega : \mathcal{M} \to \mathcal{N}$ and $\sigma : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ be given mappings. The mapping Ω is said to be σ-proximal admissible, if for all $\upsilon_1, \upsilon_2, u_1, u_2 \in \mathcal{M}$,

$$\begin{cases} \sigma(\upsilon_1, \upsilon_2) \geq 1 \\ \rho(u_1, \Omega\upsilon_1) = \rho(\mathcal{M}, \mathcal{N}) \\ \rho(u_2, \Omega\upsilon_2) = \rho(\mathcal{M}, \mathcal{N}) \end{cases} \implies \sigma(u_1, u_2) \geq 1.$$

Definition 3 ([9]) Let $\mathcal{M}, \mathcal{N}$ be two nonempty subsets of a metric space (Ξ, ρ) and $\sigma : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ a given function. A mapping $\Omega : \mathcal{M} \to CL(\mathcal{N})$ is called multivalued σ-proximal admissible, if for all $\upsilon_1, \upsilon_2, u_1, u_2 \in \mathcal{M},\ y_1 \in \Omega\upsilon_1$ and $y_2 \in \Omega\upsilon_2$,

$$\begin{cases} \sigma(\upsilon_1, \upsilon_2) \geq 1 \\ \rho(u_1, y_1) = \rho(\mathcal{M}, \mathcal{N}) \\ \rho(u_2, y_2) = \rho(\mathcal{M}, \mathcal{N}) \end{cases} \implies \sigma(u_1, u_2) \geq 1.$$

Lemma 1 ([9]) *Let $\mathcal{M}$ and $\mathcal{N}$ be two nonempty closed subsets of a metric space (Ξ, ρ) and $\Omega : \mathcal{M} \to CL(\mathcal{N})$ be a multivalued mapping. Then for $a, b \in \mathcal{M}$ and $u \in \Omega a$, there exists $v \in \Omega b$ such that $\rho(u, v) \leq \eta H(\Omega a, \Omega b)$ where $\eta \geq 1$.*

There after, many researchers worked on existence of best proximity point results for single-valued and multivalued mappings satisfying different classes of contractive conditions (see, [1, 3, 4, 7, 8, 11–13, 16–21, 23–25]).

Hussain et al. [10] considered the following class of mappings:

$$\mathcal{Z} = \{\vartheta : [0, +\infty) \to [1, +\infty) \text{ satisfies } (\vartheta_1) - (\vartheta_5)\}$$

where

- (ϑ_1) ϑ is nondecreasing;
- (ϑ_2) $\vartheta(s) = 1$ if and only if $s = 0$;
- (ϑ_3) for every $\{s_n\}$ in $(0, +\infty)$, $\lim_{n\to+\infty} \vartheta(s_n) = 1$ if and only if $\lim_{n\to+\infty} s_n = 0$;
- (ϑ_4) there exist $r \in (0, 1)$ and $\ell \in (0, +\infty]$ such that $\lim_{s\to 0^+} \frac{\vartheta(s)-1}{s^r} = \ell$;
- (ϑ_5) $\vartheta(s_1 + s_2) \leq \vartheta(s_1)\vartheta(s_2)$.

Example 1 Let $\vartheta_1, \vartheta_2 : [0, +\infty) \to [1, +\infty)$ be defined by $\vartheta_1(s) = e^{\sqrt{s}}$ and $\vartheta_2(s) = 5^{\sqrt{s}}$, respectively. Then $\vartheta_1, \vartheta_2 \in \mathcal{Z}$.

Denote $\mathbb{R}^+ := [0, +\infty)$ and define the following class of mappings, which was considered in [6].

$$\Phi = \{\phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R} \mid \phi \text{ satisfies } \phi(r_1, r_2) \leq \frac{1}{2} r_1 - r_2\}.$$

The following functions ϕ_1 and ϕ_2 are elements of Φ:

(i) $\phi_1 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $\phi_1(r_1, r_2) = v(r_1) - u(r_2)$, where $v, u : \mathbb{R}^+ \to \mathbb{R}^+$ are given as $v(r_1) = \frac{r_1}{2}$ and $u(r_2) = r_2$.
(ii) $\phi_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $\phi_2(r_1, r_2) = \frac{r_1}{2} - \frac{v(r_1, r_2)}{u(r_1, r_2)} r_2$, where $v, u : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are given as $v(r_1, r_2) = r_1 r_2$ and $u(r_1, r_2) = r_1 r_2 + r_2$ for all $r_1, r_2 > 0$.

Theorem 1 ([10]) *Let (Ξ, ρ) be a complete metric space and $\Omega : \Xi \to \Xi$ be continuous mapping. If there exist $\vartheta \in \mathcal{Z}$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \mathbb{R}^+$ with $0 \leq \gamma_1 + \gamma_2 + \gamma_3 + 2\gamma_4 < 1$ such that*

$$\vartheta(\rho(\Omega\upsilon_1, \Omega\upsilon_2)) \leq [\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\gamma_1} [\vartheta(\rho(\upsilon_1, \Omega\upsilon_1))]^{\gamma_2} [\vartheta(\rho(\upsilon_2, \Omega\upsilon_2))]^{\gamma_3} \cdot [\vartheta(\rho(\upsilon_1, \Omega\upsilon_2) + \rho(\upsilon_2, \Omega\upsilon_1))]^{\gamma_4},$$

for all $\upsilon_1, \upsilon_2 \in \Xi$. Then Ω has a unique fixed point.

Definition 4 Let $\Omega : \Xi \to CL(\Sigma)$ be a multivalued mapping, where (Ξ, σ), (Σ, ρ) are two metric spaces and H is the Hausdorff metric on $CL(\Sigma)$. The mapping Ω is said to be continuous at $u \in \Xi$, if $H(\Omega u, \Omega u_n) \to 0$ whenever $\sigma(u, u_n) \to 0$ as $n \to +\infty$.

2 Multivalued Suzuki-Type Z_σ-Contractions

We begin this section with the following definition.

Definition 5 Let $\mathcal{M}$ and $\mathcal{N}$ be two nonempty closed subset of a metric space (Ξ, ρ) and $\sigma : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$. A multivalued mapping $\Omega : \mathcal{M} \to CL(\mathcal{N})$ is said to be multivalued Suzuki-type Z_σ-contraction, if there exist $\phi \in \Phi$, $\vartheta \in \mathcal{Z}$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}^+$ with $0 \leq \gamma_1 + \gamma_2 + \gamma_3 < 1$ such that

$$\begin{aligned} &\phi[\rho(\upsilon, \Omega\nu) - \Lambda, \rho(\upsilon, \nu)] < 0 \implies \\ &\vartheta(H(\Omega\upsilon, \Omega\nu)) \leq [\vartheta(\rho(\upsilon, \nu))]^{\gamma_1}[\vartheta(\rho(\upsilon, \Omega\upsilon) - \Lambda)]^{\gamma_2}[\vartheta(\rho(\nu, \Omega\nu) - \Lambda)]^{\gamma_3}, \end{aligned} \tag{1}$$

for all $\upsilon, \nu \in \mathcal{M}$ with $\sigma(\upsilon, \nu) \geq 1$ and $H(\Omega\upsilon, \Omega\nu) > 0$.

Now, we can state the first result of this paper.

Theorem 2 *Let (Ξ, ρ) be a complete metric space and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty closed subsets of Ξ such that $\mathcal{M}_0$ is nonempty. Let $\Omega : \mathcal{M} \to K(\mathcal{N})$ be a continuous multivalued mapping such that*

(i) $\Omega\upsilon \subseteq \mathcal{N}_0$ *for all* $\upsilon \in \mathcal{M}_0$ *and* $(\mathcal{M}, \mathcal{N})$ *satisfies the weak P-property;*
(ii) Ω *is multivalued σ-proximal admissible;*
(iii) *there exist* $\upsilon_0, \upsilon_1 \in \mathcal{M}_0$ *and* $\nu_0 \in \Omega\upsilon_0$ *such that* $\rho(\upsilon_1, \nu_0) = \Lambda$ *and* $\sigma(\upsilon_0, \upsilon_1) \geq 1$*;*
(iv) Ω *is multivalued Suzuki-type Z_σ-contraction.*

Then Ω has a best proximity point in $\mathcal{M}$.

Proof Owing to (iii), there exist $\upsilon_0, \upsilon_1 \in \mathcal{M}_0$ and $\nu_0 \in \Omega\upsilon_0$ such that

$$\rho(\upsilon_1, \nu_0) = \Lambda \quad \text{and} \quad \sigma(\upsilon_0, \upsilon_1) \geq 1.$$

If $\nu_0 \in \Omega\upsilon_1$, then

$$\Lambda \leq \rho(\upsilon_1, \Omega\upsilon_1) \leq \rho(\upsilon_1, \nu_0) = \Lambda,$$

and so υ_1 is the required point. Due to this trivial way, let $\nu_0 \notin \Omega\upsilon_1$ and hence $H(\Omega\upsilon_0, \Omega\upsilon_1) > 0$. On the other side, since $\nu_0 \in \Omega\upsilon_0$, we get

$$\rho(\upsilon_0, \Omega\upsilon_0) \leq \rho(\upsilon_0, \nu_0) \leq \rho(\upsilon_0, \upsilon_1) + \rho(\upsilon_1, \nu_0) = \rho(\upsilon_0, \upsilon_1) + \Lambda,$$

and thus $\rho(\upsilon_0, \Omega\upsilon_0) - \Lambda \le \rho(\upsilon_0, \upsilon_1)$. Then,

$$\begin{aligned}\phi[\rho(\upsilon_0, \Omega\upsilon_0) - \Lambda, \rho(\upsilon_0, \upsilon_1)] &\le \frac{1}{2}(\rho(\upsilon_0, \Omega\upsilon_0) - \Lambda) - \rho(\upsilon_0, \upsilon_1)\\ &< (\rho(\upsilon_0, \Omega\upsilon_0) - \Lambda) - \rho(\upsilon_0, \upsilon_1)\\ &\le \rho(\upsilon_0, \upsilon_1) - \rho(\upsilon_0, \upsilon_1)\\ &= 0.\end{aligned}$$

Since $\Omega\upsilon_1$ is compact, there exists $\nu_1 \in \Omega\upsilon_1$ such that $\rho(\nu_0, \Omega\upsilon_1) = \rho(\nu_0, \nu_1)$. Also, as $0 < \rho(\nu_0, \Omega\upsilon_1) \le H(\Omega\upsilon_0, \Omega\upsilon_1)$, from (ϑ_1) and condition (iv), we have

$$\begin{aligned}\vartheta(\rho(\nu_0, \nu_1)) &= \vartheta(\rho(\nu_0, \Omega\upsilon_1)) \le \vartheta(H(\Omega\upsilon_0, \Omega\upsilon_1))\\ &\le [\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_1}[\vartheta(\rho(\upsilon_0, \Omega\upsilon_0) - \Lambda)]^{\gamma_2}[\vartheta(\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda)]^{\gamma_3}\\ &\le [\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_1}[\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_2}[\vartheta(\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda)]^{\gamma_3}.\end{aligned} \tag{2}$$

By $\nu_1 \in \Omega\upsilon_1 \subseteq \mathcal{N}_0$, there exists $\upsilon_2 \in \mathcal{M}_0$ such that $\rho(\upsilon_2, \nu_1) = \Lambda$. Since $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property, we deduce that $\rho(\upsilon_1, \upsilon_2) \le \rho(\nu_0, \nu_1)$. Also, since $\nu_1 \in \Omega\upsilon_1$, we get

$$\rho(\upsilon_1, \Omega\upsilon_1) \le \rho(\upsilon_1, \nu_1) \le \rho(\upsilon_1, \upsilon_2) + \rho(\upsilon_2, \nu_1) = \rho(\upsilon_1, \upsilon_2) + \Lambda,$$

and thus

$$\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda \le \rho(\upsilon_1, \upsilon_2). \tag{3}$$

If $\upsilon_1 = \upsilon_2$, then υ_1 is the required best proximity point of Ω. Assume that $\upsilon_1 \ne \upsilon_2$. By (2), we obtain

$$\begin{aligned}\vartheta(\rho(\upsilon_1, \upsilon_2)) &\le \vartheta(\rho(\nu_0, \nu_1))\\ &\le [\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_1+\gamma_2}[\vartheta(\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda)]^{\gamma_3}\\ &\le [\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_1+\gamma_2}[\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\gamma_3},\end{aligned}$$

which implies that

$$\vartheta(\rho(\upsilon_1, \upsilon_2)) \le \vartheta(\rho(\nu_0, \nu_1)) \le [\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\frac{\gamma_1+\gamma_2}{1-\gamma_3}}. \tag{4}$$

Now, $\upsilon_0, \upsilon_1, \upsilon_2 \in \mathcal{M}_0 \subseteq \mathcal{M}$ and $\nu_0 \in \Omega\upsilon_0$, $\nu_1 \in \Omega\upsilon_1$ such that $\sigma(\upsilon_0, \upsilon_1) \ge 1$, $\rho(\upsilon_1, \nu_0) = \Lambda$, $\rho(\upsilon_2, \nu_1) = \Lambda$. Then, it follows from condition (ii) that $\sigma(\upsilon_1, \upsilon_2) \ge 1$. Thus, we have

$$\rho(\upsilon_2, \nu_1) = \Lambda \quad \text{and} \quad \sigma(\upsilon_1, \upsilon_2) \ge 1.$$

By the processes analogous to the above, we get that $\nu_1 \notin \Omega\upsilon_2$ and hence $H(\Omega\upsilon_1, \Omega\upsilon_2) > 0$. By using (3), we have

$$\begin{aligned}\phi[\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda, \rho(\upsilon_1, \upsilon_2)] &\leq \frac{1}{2}(\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda) - \rho(\upsilon_1, \upsilon_2) \\ &< (\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda) - \rho(\upsilon_1, \upsilon_2) \\ &\leq \rho(\upsilon_1, \upsilon_2) - \rho(\upsilon_1, \upsilon_2) \\ &= 0.\end{aligned}$$

Since $\Omega\upsilon_2$ is compact, there exists $\nu_2 \in \Omega\upsilon_2$ such that $\rho(\nu_1, \Omega\upsilon_2) = \rho(\nu_1, \nu_2)$. Also, as $0 < \rho(\nu_1, \Omega\upsilon_2) \leq H(\Omega\upsilon_1, \Omega\upsilon_2)$, from (ϑ_1) and condition (iv), we have

$$\begin{aligned}\vartheta(\rho(\nu_1, \nu_2)) &= \vartheta(\rho(\nu_1, \Omega\upsilon_2)) \leq \vartheta(H(\Omega\upsilon_1, \Omega\upsilon_2)) \\ &\leq [\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\gamma_1}[\vartheta(\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda)]^{\gamma_2}[\vartheta(\rho(\upsilon_2, \Omega\upsilon_2) - \Lambda)]^{\gamma_3} \\ &\leq [\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\gamma_1}[\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\gamma_2}[\vartheta(\rho(\upsilon_2, \Omega\upsilon_2) - \Lambda)]^{\gamma_3}. \qquad (5)\end{aligned}$$

By $\nu_2 \in \Omega\upsilon_2 \subseteq \mathcal{N}_0$, there exists $\upsilon_3 \in \mathcal{M}_0$ such that $\rho(\upsilon_3, \nu_2) = \Lambda$. Since $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property, we deduce that $\rho(\upsilon_2, \upsilon_3) \leq \rho(\nu_1, \nu_2)$. Also, since $\nu_2 \in \Omega\upsilon_2$, we get

$$\rho(\upsilon_2, \Omega\upsilon_2) \leq \rho(\upsilon_2, \nu_2) \leq \rho(\upsilon_2, \upsilon_3) + \rho(\upsilon_3, \nu_2) = \rho(\upsilon_2, \upsilon_3) + \Lambda,$$

and thus

$$\rho(\upsilon_2, \Omega\upsilon_2) - \Lambda \leq \rho(\upsilon_2, \upsilon_3). \qquad (6)$$

If $\upsilon_2 = \upsilon_3$, then υ_2 is the required best proximity point of Ω. Assume that $\upsilon_2 \neq \upsilon_3$. By using (6) in (5), we obtain

$$\begin{aligned}\vartheta(\rho(\upsilon_2, \upsilon_3)) &\leq \vartheta(\rho(\nu_1, \nu_2)) \\ &\leq [\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\gamma_1+\gamma_2}[\vartheta(\rho(\upsilon_2, \Omega\upsilon_2) - \Lambda)]^{\gamma_3} \\ &\leq [\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\gamma_1+\gamma_2}[\vartheta(\rho(\upsilon_2, \upsilon_3))]^{\gamma_3},\end{aligned}$$

which implies that

$$\vartheta(\rho(\upsilon_2, \upsilon_3) \leq \vartheta(\rho(\nu_1, \nu_2)) \leq [\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\frac{\gamma_1+\gamma_2}{1-\gamma_3}}. \qquad (7)$$

Now, $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathcal{M}_0 \subseteq \mathcal{M}$ and $\nu_1 \in \Omega\upsilon_1$, $\nu_2 \in \Omega\upsilon_2$ such that $\sigma(\upsilon_1, \upsilon_2) \geq 1$, $\rho(\upsilon_2, \nu_1) = \Lambda$, $\rho(\upsilon_3, \nu_2) = \Lambda$. Then, it follows from condition (ii) that $\sigma(\upsilon_2, \upsilon_3) \geq 1$. Thus, we have

$$\rho(\upsilon_3, \nu_2) = \Lambda \quad \text{and} \quad \sigma(\upsilon_2, \upsilon_3) \geq 1.$$

Continuing this process, we construct two sequences $\{\upsilon_n\}$ and $\{\nu_n\}$, respectively in $\mathcal{M}_0 \subseteq \mathcal{M}$ and $\mathcal{N}_0 \subseteq \mathcal{N}$ such that for $n = 0, 1, 2, \ldots,$

(a) $\nu_n \in \Omega\upsilon_n$ and $\nu_n \notin \Omega\upsilon_{n+1}$;
(b) $\sigma(\upsilon_n, \upsilon_{n+1}) \geq 1$ and $\upsilon_n \neq \upsilon_{n+1}$;
(c) $\rho(\upsilon_{n+1}, \nu_n) = \Lambda$ and

$$1 < \vartheta(\rho(\upsilon_n, \upsilon_{n+1})) \leq \vartheta(\rho(\nu_{n-1}, \nu_n)) \leq [\vartheta(\rho(\upsilon_{n-1}, \upsilon_n))]^{\frac{\gamma_1+\gamma_2}{1-\gamma_3}}, \tag{8}$$

which implies

$$1 < \vartheta(\rho(\upsilon_n, \upsilon_{n+1})) \leq \vartheta(\rho(\nu_{n-1}, \nu_n)) \leq [\vartheta(\rho(\upsilon_0, \upsilon_1))]^{h^n}, \tag{9}$$

where $h = \frac{\gamma_1+\gamma_2}{1-\gamma_3} < 1$. Taking limit as $n \to +\infty$ in (9), we get $\lim_{n\to+\infty} \vartheta(\rho(\upsilon_n, \upsilon_{n+1})) = 1$ and so

$$\lim_{n\to+\infty} \rho(\upsilon_n, \upsilon_{n+1}) = 0. \tag{10}$$

Next, we prove that $\{\upsilon_n\}$ is a Cauchy sequence in $\mathcal{M}_0$. Setting $\delta_n{:=}\rho(\upsilon_n,\upsilon_{n+1})$, from (ϑ_4), there exist $r \in (0, 1)$ and $\ell \in (0, +\infty]$ such that

$$\lim_{n\to+\infty} \frac{\vartheta(\delta_n) - 1}{(\delta_n)^r} = \ell.$$

Take $\lambda \in (0, \ell)$. From the definition of limit, there exists $n_0 \in \mathbb{N}$ such that

$$[\delta_n]^r \leq \lambda^{-1}[\vartheta(\delta_n) - 1], \quad \text{for all } n > n_0.$$

Using (9) and the above inequality, we deduce

$$n[\delta_n]^r \leq \lambda^{-1} n([\vartheta(\delta_0)]^{h^n} - 1), \quad \text{for all } n > n_0.$$

This implies that

$$\lim_{n\to+\infty} n[\delta_n]^r = \lim_{n\to+\infty} n[\rho(\upsilon_n, \upsilon_{n+1})]^r = 0.$$

Hence, there exists $n_1 \in \mathbb{N}$ such that

$$\rho(\upsilon_n, \upsilon_{n+1}) \leq \frac{1}{n^{1/r}}, \quad \text{for all } n > n_1. \tag{11}$$

Let $p > n > n_1$. Then using the triangular inequality and (11), we get

$$\rho(\upsilon_n, \upsilon_p) \leq \sum_{j=n}^{p-1} \rho(\upsilon_j, \upsilon_{j+1}) \leq \sum_{j=n}^{p-1} \frac{1}{j^{1/r}} < \sum_{j=n}^{+\infty} \frac{1}{j^{1/r}}.$$

Due to the convergence of the series $\sum_{j=n}^{+\infty} \frac{1}{j^{1/r}}$, we deduce that $\{\upsilon_n\}$ is a Cauchy sequence in $\mathcal{M}$. By the similar processes and (9), we can easily prove that $\{\nu_n\}$ is a Cauchy sequence in $\mathcal{N}$. Since $\mathcal{M}$ and $\mathcal{N}$ are closed subsets of the complete metric space (Ξ, ρ), there exist $\upsilon^* \in \mathcal{M}$ and $\nu^* \in \mathcal{N}$ such that $\upsilon_n \to \upsilon^*$ and $\nu_n \to \nu^*$ as $n \to +\infty$. From (c), we know that

$$\rho(\upsilon_{n+1}, \nu_n) = \Lambda, \quad \text{for all } n = 0, 1, 2, \ldots.$$

Letting $n \to +\infty$, we get that

$$\rho(\upsilon^*, \nu^*) = \Lambda. \tag{12}$$

Now, we claim that $\nu^* \in \Omega\upsilon^*$. Since $\nu_n \in \Omega\upsilon_n$, we have

$$\rho(\nu_n, \Omega\upsilon^*) \leq H(\Omega\upsilon_n, \Omega\upsilon^*).$$

Taking limit as $n \to +\infty$ in the above inequality and using the continuity of Ω, we get

$$\rho(\nu^*, \Omega\upsilon^*) = \lim_{n\to+\infty} \rho(\nu_n, \Omega\upsilon^*) \leq \lim_{n\to+\infty} H(\Omega\upsilon_n, \Omega\upsilon^*) = 0.$$

Since $\Omega\upsilon^*$ is compact, then $\Omega\upsilon^*$ is closed. Hence, $\rho(\nu^*, \Omega\upsilon^*) = 0$ implies $\nu^* \in \Omega\upsilon^*$.

Now, using (12), we have

$$\rho(\upsilon^*, \Omega\upsilon^*) \leq \rho(\upsilon^*, \nu^*) = \Lambda = \rho(\mathcal{M}, \mathcal{N}) \leq \rho(\upsilon^*, \Omega\upsilon^*),$$

which implies that $\rho(\upsilon^*, \Omega\upsilon^*) = \rho(\mathcal{M}, \mathcal{N})$ and this completes the proof. □

In the next theorem, we replace $K(\mathcal{N})$ with $CB(\mathcal{N})$ by considering the following additional condition:

(C) ϑ is right continuous.

Theorem 3 *Let (Ξ, ρ) be a complete metric space and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty closed subsets of Ξ such that $\mathcal{M}_0$ is nonempty. Let $\Omega : \mathcal{M} \to CB(\mathcal{N})$ be a continuous multivalued mapping such that the conditions (i)–(iv) in Theorem 2 and the assumption (C) are satisfied. Then Ω has a best proximity point in $\mathcal{M}$.*

Proof By (iii), there exist $\upsilon_0, \upsilon_1 \in \mathcal{M}_0$ and $\nu_0 \in \Omega\upsilon_0$ such that

$$\rho(\upsilon_1, \nu_0) = \Lambda \quad \text{and} \quad \sigma(\upsilon_0, \upsilon_1) \geq 1.$$

Next, suppose that $\nu_0 \notin \Omega\upsilon_1$ and hence $H(\Omega\upsilon_0, \Omega\upsilon_1) > 0$. Since $\nu_0 \in \Omega\upsilon_0$, we get

$$\rho(\upsilon_0, \Omega\upsilon_0) \leq \rho(\upsilon_0, \nu_0) \leq \rho(\upsilon_0, \upsilon_1) + \rho(\upsilon_1, \nu_0) = \rho(\upsilon_0, \upsilon_1) + \Lambda,$$

and thus $\rho(\upsilon_0, \Omega\upsilon_0) - \Lambda \leq \rho(\upsilon_0, \upsilon_1)$. Then,

$$\begin{aligned}\phi[\rho(\upsilon_0, \Omega\upsilon_0) - \Lambda, \rho(\upsilon_0, \upsilon_1)] &\leq \frac{1}{2}(\rho(\upsilon_0, \Omega\upsilon_0) - \Lambda) - \rho(\upsilon_0, \upsilon_1)\\ &< (\rho(\upsilon_0, \Omega\upsilon_0) - \Lambda) - \rho(\upsilon_0, \upsilon_1)\\ &\leq \rho(\upsilon_0, \upsilon_1) - \rho(\upsilon_0, \upsilon_1)\\ &= 0.\end{aligned}$$

From the condition (iv), we have

$$\begin{aligned}\vartheta(H(\Omega\upsilon_0, \Omega\upsilon_1)) &\leq [\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_1}[\vartheta(\rho(\upsilon_0, \Omega\upsilon_0) - \Lambda)]^{\gamma_2}[\vartheta(\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda)]^{\gamma_3}\\ &\leq [\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_1}[\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_2}[\vartheta(\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda)]^{\gamma_3}.\end{aligned}$$

By the property of right continuity of $\vartheta \in \mathcal{Z}$, there exists a real number $\eta_1 > 1$ such that

$$\vartheta(\eta_1 H(\Omega\upsilon_0, \Omega\upsilon_1)) \leq [\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_1}[\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_2}[\vartheta(\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda)]^{\gamma_3}. \tag{13}$$

From $\rho(\nu_0, \Omega\upsilon_1) < \eta_1 H(\Omega\upsilon_0, \Omega\upsilon_1)$, by Lemma 1, there exists $\nu_1 \in \Omega\upsilon_1$ such that $\rho(\nu_0, \nu_1) \leq \eta_1 H(\Omega\upsilon_0, \Omega\upsilon_1)$. Then, using (ϑ_1), (13) and last inequality, we infer that

$$\begin{aligned}\vartheta(\rho(\nu_0, \nu_1)) &\leq \vartheta(\eta_1 H(\Omega\upsilon_0, \Omega\upsilon_1))\\ &\leq [\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_1}[\vartheta(\rho(\upsilon_0, \upsilon_1))]^{\gamma_2}[\vartheta(\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda)]^{\gamma_3}.\end{aligned} \tag{14}$$

By $\nu_1 \in \Omega\upsilon_1 \subseteq \mathcal{N}_0$, there exists $\upsilon_2 \in \mathcal{M}_0$ such that $\rho(\upsilon_2, \nu_1) = \Lambda$. Since $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property, we deduce that $\rho(\upsilon_1, \upsilon_2) \leq \rho(\nu_0, \nu_1)$. Also, since $\nu_1 \in \Omega\upsilon_1$, we get

$$\rho(\upsilon_1, \Omega\upsilon_1) \leq \rho(\upsilon_1, \nu_1) \leq \rho(\upsilon_1, \upsilon_2) + \rho(\upsilon_2, \nu_1) = \rho(\upsilon_1, \upsilon_2) + \Lambda,$$

and thus

$$\rho(\upsilon_1, \Omega\upsilon_1) - \Lambda \leq \rho(\upsilon_1, \upsilon_2). \tag{15}$$

If $\upsilon_1 = \upsilon_2$, then υ_1 is the required best proximity point of Ω. Assume that $\upsilon_1 \neq \upsilon_2$. By (14) and (15), we obtain

$$\begin{aligned}\vartheta(\rho(\upsilon_1,\upsilon_2)) &\le \vartheta(\rho(\nu_0,\nu_1)) \le \vartheta(\eta_1 H(\Omega\upsilon_0,\Omega\upsilon_1))\\ &\le [\vartheta(\rho(\upsilon_0,\upsilon_1))]^{\gamma_1+\gamma_2}[\vartheta(\rho(\upsilon_1,\Omega\upsilon_1)-\Lambda)]^{\gamma_3}\\ &\le [\vartheta(\rho(\upsilon_0,\upsilon_1))]^{\gamma_1+\gamma_2}[\vartheta(\rho(\upsilon_1,\upsilon_2))]^{\gamma_3},\end{aligned}$$

which implies that

$$\vartheta(\rho(\upsilon_1,\upsilon_2) \le \vartheta(\rho(\nu_0,\nu_1)) \le \vartheta(\eta_1 H(\Omega\upsilon_0,\Omega\upsilon_1)) \le [\vartheta(\rho(\upsilon_0,\upsilon_1))]^{\frac{\gamma_1+\gamma_2}{1-\gamma_3}}. \tag{16}$$

Now, $\upsilon_0,\upsilon_1,\upsilon_2 \in \mathcal{M}_0 \subseteq \mathcal{M}$ and $\nu_0 \in \Omega\upsilon_0$, $\nu_1 \in \Omega\upsilon_1$ such that $\sigma(\upsilon_0,\upsilon_1) \ge 1$, $\rho(\upsilon_1,\nu_0) = \Lambda$, $\rho(\upsilon_2,\nu_1) = \Lambda$. Then, it follows from condition (ii) that $\sigma(\upsilon_1,\upsilon_2) \ge 1$. Thus, we have

$$\rho(\upsilon_2,\nu_1) = \Lambda \quad \text{and} \quad \sigma(\upsilon_1,\upsilon_2) \ge 1.$$

Suppose that $\nu_1 \notin \Omega\upsilon_2$ and hence $H(\Omega\upsilon_1,\Omega\upsilon_2) > 0$. By using (15), we have

$$\begin{aligned}\phi[\rho(\upsilon_1,\Omega\upsilon_1)-\Lambda,\rho(\upsilon_1,\upsilon_2)] &\le \frac{1}{2}(\rho(\upsilon_1,\Omega\upsilon_1)-\Lambda)-\rho(\upsilon_1,\upsilon_2)\\ &< (\rho(\upsilon_1,\Omega\upsilon_1)-\Lambda)-\rho(\upsilon_1,\upsilon_2)\\ &\le \rho(\upsilon_1,\upsilon_2)-\rho(\upsilon_1,\upsilon_2)\\ &= 0.\end{aligned}$$

From the condition (iv), we have

$$\begin{aligned}\vartheta(H(\Omega\upsilon_1,\Omega\upsilon_2)) &\le [\vartheta(\rho(\upsilon_1,\upsilon_2))]^{\gamma_1}[\vartheta(\rho(\upsilon_1,\Omega\upsilon_1)-\Lambda)]^{\gamma_2}[\vartheta(\rho(\upsilon_2,\Omega\upsilon_2)-\Lambda)]^{\gamma_3}\\ &\le [\vartheta(\rho(\upsilon_1,\upsilon_2))]^{\gamma_1}[\vartheta(\rho(\upsilon_1,\upsilon_2))]^{\gamma_2}[\vartheta(\rho(\upsilon_2,\Omega\upsilon_2)-\Lambda)]^{\gamma_3}.\end{aligned}$$

By the property of right continuity of $\vartheta \in \mathcal{Z}$, there exists a real number $\eta_2 > 1$ such that

$$\vartheta(\eta_2 H(\Omega\upsilon_1,\Omega\upsilon_2)) \le [\vartheta(\rho(\upsilon_1,\upsilon_2))]^{\gamma_1}[\vartheta(\rho(\upsilon_1,\upsilon_2))]^{\gamma_2}[\vartheta(\rho(\upsilon_2,\Omega\upsilon_2)-\Lambda)]^{\gamma_3}. \tag{17}$$

From $\rho(\nu_1,\Omega\upsilon_2) < \eta_2 H(\Omega\upsilon_1,\Omega\upsilon_2)$, by Lemma 1, there exists $\nu_2 \in \Omega\upsilon_2$ such that $\rho(\nu_1,\nu_2) \le \eta_2 H(\Omega\upsilon_1,\Omega\upsilon_2)$. Then, using (ϑ_1), (17) and last inequality, we get

$$\begin{aligned}\vartheta(\rho(\nu_1,\nu_2)) &\le \vartheta(\eta_2 H(\Omega\upsilon_1,\Omega\upsilon_2))\\ &\le [\vartheta(\rho(\upsilon_1,\upsilon_2))]^{\gamma_1}[\vartheta(\rho(\upsilon_1,\upsilon_2))]^{\gamma_2}[\vartheta(\rho(\upsilon_2,\Omega\upsilon_2)-\Lambda)]^{\gamma_3}.\end{aligned} \tag{18}$$

By $\nu_2 \in \Omega\upsilon_2 \subseteq \mathcal{N}_0$, there exists $\upsilon_3 \in \mathcal{M}_0$ such that $\rho(\upsilon_3,\nu_2) = \Lambda$. Since $(\mathcal{M},\mathcal{N})$ satisfies the weak P-property, we deduce that $\rho(\upsilon_2,\upsilon_3) \le \rho(\nu_1,\nu_2)$. Also, since $\nu_2 \in \Omega\upsilon_2$, we get

$$\rho(\upsilon_2,\Omega\upsilon_2) \le \rho(\upsilon_2,\nu_2) \le \rho(\upsilon_2,\upsilon_3)+\rho(\upsilon_3,\nu_2) = \rho(\upsilon_2,\upsilon_3)+\Lambda,$$

and thus

$$\rho(\upsilon_2, \Omega\upsilon_2) - \Lambda \le \rho(\upsilon_2, \upsilon_3). \tag{19}$$

If $\upsilon_2 = \upsilon_3$, then υ_2 is the required best proximity point of Ω. Assume that $\upsilon_2 \ne \upsilon_3$. By using (18) and (19), we deduce

$$\begin{aligned}\vartheta(\rho(\upsilon_2, \upsilon_3)) &\le \vartheta(\rho(\nu_1, \nu_2)) \le \vartheta(\eta_2 H(\Omega\upsilon_1, \Omega\upsilon_2))\\ &\le [\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\gamma_1+\gamma_2}[\vartheta(\rho(\upsilon_2, \Omega\upsilon_2) - \Lambda)]^{\gamma_3}\\ &\le [\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\gamma_1+\gamma_2}[\vartheta(\rho(\upsilon_2, \upsilon_3))]^{\gamma_3},\end{aligned}$$

which implies that

$$\vartheta(\rho(\upsilon_2, \upsilon_3) \le \vartheta(\rho(\nu_1, \nu_2)) \le \vartheta(\eta_2 H(\Omega\upsilon_1, \Omega\upsilon_2)) \le [\vartheta(\rho(\upsilon_1, \upsilon_2))]^{\frac{\gamma_1+\gamma_2}{1-\gamma_3}}. \tag{20}$$

Now, $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathcal{M}_0 \subseteq \mathcal{M}$ and $\nu_1 \in \Omega\upsilon_1$, $\nu_2 \in \Omega\upsilon_2$ such that $\sigma(\upsilon_1, \upsilon_2) \ge 1$, $\rho(\upsilon_2, \nu_1) = \Lambda$, $\rho(\upsilon_3, \nu_2) = \Lambda$. Then, it follows from condition (ii) that $\sigma(\upsilon_2, \upsilon_3) \ge 1$. Thus, we have

$$\rho(\upsilon_3, \nu_2) = \Lambda \quad \text{and} \quad \sigma(\upsilon_2, \upsilon_3) \ge 1.$$

Continuing this process, we get $\eta_n \subseteq (1, +\infty)$, $\{\upsilon_n\} \subseteq \mathcal{M}_0$ and $\{\nu_n\} \subseteq \mathcal{N}_0$ such that

(a) $\nu_n \in \Omega\upsilon_n$ and $\nu_n \notin \Omega\upsilon_{n+1}$;
(b) $\sigma(\upsilon_n, \upsilon_{n+1}) \ge 1$ and $\upsilon_n \ne \upsilon_{n+1}$;
(c) $\rho(\upsilon_{n+1}, \nu_n) = \Lambda$ and

$$\begin{aligned}1 < \vartheta(\rho(\upsilon_n, \upsilon_{n+1})) &\le \vartheta(\rho(\nu_{n-1}, \nu_n)) \le \vartheta(\eta_n H(\Omega\upsilon_{n-1}, \Omega\upsilon_n))\\ &\le [\vartheta(\rho(\upsilon_{n-1}, \upsilon_n))]^{\frac{\gamma_1+\gamma_2}{1-\gamma_3}},\end{aligned} \tag{21}$$

for all n. Doing the same as we have done in Theorem 2, we obtain $\{\upsilon_n\}$ in $\mathcal{M}$ and $\{\nu_n\}$ in $\mathcal{N}$ as Cauchy sequences. Since $\mathcal{M}$ and $\mathcal{N}$ are closed subsets of the complete metric space (Ξ, ρ), there exist $\upsilon^* \in \mathcal{M}$ and $\nu^* \in \mathcal{N}$ such that $\upsilon_n \to \upsilon^*$ and $\nu_n \to \nu^*$ as $n \to +\infty$. The rest of the proof is like in the proof of Theorem 2. □

The next result can given by replacing the continuity of the mapping Ω with the following property:

(H) If $\{\upsilon_n\}$ is a sequence in $\mathcal{M}$ such that $\sigma(\upsilon_n, \upsilon_{n+1}) \ge 1$ for all n and $\upsilon_n \to \upsilon \in \mathcal{M}$ as $n \to +\infty$, then $\sigma(\upsilon_n, \upsilon) \ge 1$ for all n.

Theorem 4 *Let (Ξ, ρ) be a complete metric space and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty closed subsets of Ξ such that $\mathcal{M}_0$ is nonempty. Let $\Omega : \mathcal{M} \to K(\mathcal{N})$ be a multivalued mapping such that the conditions $(i) - (iv)$ in Theorem 2 and the property (H) are satisfied. Then Ω has a best proximity point in $\mathcal{M}$.*

Proof From Theorem 2, we have $\{\upsilon_n\} \subseteq \mathcal{M}_0$ and $\{\nu_n\} \subseteq \mathcal{N}_0$ such that for $n = 0, 1, 2, \ldots,$

(a) $\nu_n \in \Omega\upsilon_n$ and $\nu_n \notin \Omega\upsilon_{n+1}$;
(b) $\sigma(\upsilon_n, \upsilon_{n+1}) \geq 1$ and $\upsilon_n \neq \upsilon_{n+1}$;
(c) $\rho(\upsilon_{n+1}, \nu_n) = \Lambda$.

Also, there exist $\upsilon^* \in \mathcal{M}$ and $\nu^* \in \mathcal{N}$ such that $\upsilon_n \to \upsilon^*$ and $\nu_n \to \nu^*$ as $n \to +\infty$, and $\rho(\upsilon^*, \nu^*) = \Lambda$. Now, we prove that υ^* is a best proximity point of Ω. If $\Omega\upsilon_n = \Omega\upsilon^*$, then

$$\Lambda \leq \rho(\upsilon_{n+1}, \Omega\upsilon_n) \leq \rho(\upsilon_{n+1}, \nu_n) = \Lambda,$$

which yields that

$$\Lambda \leq \rho(\upsilon_{n+1}, \Omega\upsilon^*) \leq \Lambda,$$

for all $n \geq 1$. Letting $n \to +\infty$, we have

$$\Lambda \leq \rho(\upsilon^*, \Omega\upsilon^*) \leq \Lambda.$$

Hence υ^* is a best proximity point of Ω. Suppose that $\Omega\upsilon_n \neq \Omega\upsilon^*$ for all n. Due to (a), $\nu_n \in \Omega\upsilon_n$ such that

$$\rho(\upsilon_n, \Omega\upsilon_n) \leq \rho(\upsilon_n, \nu_n) \leq \rho(\upsilon_n, \upsilon_{n+1}) + \rho(\upsilon_{n+1}, \nu_n) = \rho(\upsilon_n, \upsilon_{n+1}) + \Lambda,$$

and thus $\rho(\upsilon_n, \Omega\upsilon_n) - \Lambda \leq \rho(\upsilon_n, \upsilon_{n+1})$. Then,

$$\begin{aligned}\phi[\rho(\upsilon_n, \Omega\upsilon_n) - \Lambda, \rho(\upsilon_n, \upsilon_{n+1})] &\leq \frac{1}{2}(\rho(\upsilon_n, \Omega\upsilon_n) - \Lambda) - \rho(\upsilon_n, \upsilon_{n+1}) \\ &< (\rho(\upsilon_n, \Omega\upsilon_n) - \Lambda) - \rho(\upsilon_n, \upsilon_{n+1}) \\ &\leq \rho(\upsilon_n, \upsilon_{n+1}) - \rho(\upsilon_n, \upsilon_{n+1}) \\ &= 0.\end{aligned}$$

From (b) and (iv), we get

$$\begin{aligned}\vartheta(\rho(\upsilon_{n+1}, \Omega\upsilon_{n+1})) &\leq \vartheta(H(\Omega\upsilon_n, \Omega\upsilon_{n+1})) \\ &\leq [\vartheta(\rho(\upsilon_n, \upsilon_{n+1}))]^{\gamma_1} [\vartheta(\rho(\upsilon_n, \Omega\upsilon_n) - \Lambda)]^{\gamma_2} [\vartheta(\rho(\upsilon_{n+1}, \Omega\upsilon_{n+1}) - \Lambda)]^{\gamma_3} \\ &\leq [\vartheta(\rho(\upsilon_n, \upsilon_{n+1}))]^{\gamma_1+\gamma_2} [\vartheta(\rho(\upsilon_{n+1}, \Omega\upsilon_{n+1}))]^{\gamma_3},\end{aligned}$$

which implies that

$$\vartheta(\rho(\upsilon_{n+1}, \Omega\upsilon_{n+1})) \leq [\vartheta(\rho(\upsilon_n, \upsilon_{n+1}))]^{\frac{\gamma_1+\gamma_2}{1-\gamma_3}} < \vartheta(\rho(\upsilon_n, \upsilon_{n+1})).$$

Hence,

$$\rho(\upsilon_{n+1}, \Omega\upsilon_{n+1}) < \rho(\upsilon_n, \upsilon_{n+1}). \tag{22}$$

If $\phi[\rho(\upsilon_n, \Omega\upsilon_n) - \Lambda, \rho(\upsilon_n, \upsilon^*)] \geq 0$, then $\frac{1}{2}\rho(\upsilon_n, \Omega\upsilon_n) - \Lambda \geq \rho(\upsilon_n, \upsilon^*)$ and so

$$\frac{1}{2}\rho(\upsilon_n, \Omega\upsilon_n) > \rho(\upsilon_n, \upsilon^*).$$

From (22), we obtain that

$$\begin{aligned}
\rho(\upsilon_n, \upsilon_{n+1}) &\leq \rho(\upsilon_n, \upsilon^*) + \rho(\upsilon^*, \upsilon_{n+1}) \\
&< \frac{1}{2}\rho(\upsilon_n, \Omega\upsilon_n) + \frac{1}{2}\rho(\upsilon_{n+1}, \Omega\upsilon_{n+1}) \\
&< \frac{1}{2}\rho(\upsilon_n, \upsilon_{n+1}) + \frac{1}{2}\rho(\upsilon_n, \upsilon_{n+1}) \\
&= \rho(\upsilon_n, \upsilon_{n+1}),
\end{aligned}$$

which is a contradiction. Hence, $\phi[\rho(\upsilon_n, \Omega\upsilon_n) - \Lambda, \rho(\upsilon_n, \upsilon^*)] < 0$ for all n. Assume that $\upsilon^* \notin \Omega\upsilon^*$. By the property (H), we have $\sigma(\upsilon_n, \upsilon^*) \geq 1$. Then,

$$\rho(\nu^*, \Omega\upsilon^*) \leq \rho(\nu^*, \nu_n) + \rho(\nu_n, \Omega\upsilon^*) \leq \rho(\nu^*, \nu_n) + H(\Omega\upsilon_n, \Omega\upsilon^*),$$

implies that

$$\vartheta(\rho(\nu^*, \Omega\upsilon^*)) \leq \vartheta(\rho(\nu^*, \nu_n))\vartheta(H(\Omega\upsilon_n, \Omega\upsilon^*)). \tag{23}$$

Consequently,

$$\begin{aligned}
&\vartheta(\rho(\nu^*, \nu_n))\vartheta(H(\Omega\upsilon_n, \Omega\upsilon^*)) \\
&\quad \leq \vartheta(\rho(\nu^*,\nu_n))[\vartheta(\rho(\upsilon_n,\upsilon_{n+1}))]^{\gamma_1}[\vartheta(\rho(\upsilon_n,\Omega\upsilon_n)-\Lambda)]^{\gamma_2}[\vartheta(\rho(\upsilon_{n+1},\Omega\upsilon_{n+1})-\Lambda)]^{\gamma_3} \\
&\quad \leq \vartheta(\rho(\nu^*, \nu_n))[\vartheta(\rho(\upsilon_n, \upsilon_{n+1}))]^{\gamma_1}[\vartheta(\rho(\upsilon_n, \upsilon_{n+1})]^{\gamma_2}[\vartheta(\rho(\upsilon_{n+1}, \upsilon_{n+2}))]^{\gamma_3}.
\end{aligned}$$

From (23), it follows that

$$\vartheta(\rho(\nu^*, \Omega\upsilon^*)) \leq \vartheta(\rho(\nu^*, \nu_n))[\vartheta(\rho(\upsilon_n, \upsilon_{n+1}))]^{\gamma_1+\gamma_2}[\vartheta(\rho(\upsilon_{n+1}, \upsilon_{n+2}))]^{\gamma_3}.$$

Taking limit as $n \to +\infty$, we infer that $\vartheta(\rho(\nu^*, \Omega\upsilon^*)) = 1$. From (ϑ_2), we have $\rho(\nu^*, \Omega\upsilon^*) = 0$. Hence,

$$\Lambda \leq \rho(\upsilon^*, \Omega\upsilon^*) \leq \rho(\upsilon^*, \nu^*) + \rho(\nu^*, \Omega\upsilon^*).$$

Since $\rho(\upsilon^*, \nu^*) = \Lambda$ and $\rho(\nu^*, \Omega\upsilon^*) = 0$, we obtain that $\rho(\upsilon^*, \Omega\upsilon^*) = \Lambda = \rho(\mathcal{M}, \mathcal{N})$ and this finishes the proof. □

Theorem 5 *Let (Ξ, ρ) be a complete metric space and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty closed subsets of Ξ such that $\mathcal{M}_0$ is nonempty. Let $\Omega : \mathcal{M} \to CB(\mathcal{N})$ be a multivalued mapping such that the conditions $(i) - (iv)$ in Theorem 2, and the conditions (C) and (H) are satisfied. Then Ω has a best proximity point in $\mathcal{M}$.*

Proof The proof can easily be done like Theorem 4 and so we omit the proof here.

Taking $\gamma_2 = \gamma_3 = 0$ in Theorem 2, we obtain the following result.

Corollary 1 *Let (Ξ, ρ) be a complete metric space and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty closed subsets of Ξ such that $\mathcal{M}_0$ is nonempty. Let $\Omega : \mathcal{M} \to K(\mathcal{N})$ $(CB(\mathcal{N}))$ be a multivalued mapping such that*

$$\phi[\rho(\upsilon, \Omega\nu) - \Lambda, \rho(\upsilon, \nu)] < 0 \implies \vartheta(H(\Omega\upsilon, \Omega\nu)) \leq [\vartheta(\rho(\upsilon, \nu))]^{\gamma_1}, \tag{24}$$

for all $\upsilon, \nu \in \Xi$ with $\sigma(\upsilon, \nu) \geq 1$ and $H(\Omega\upsilon, \Omega\nu) > 0$, where $\sigma : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$, $\phi \in \Phi$, $\vartheta \in \mathcal{Z}$ and $\gamma_1 \in [0, 1)$. Assume that the following assertions hold:

(i) *$\Omega\upsilon \subseteq \mathcal{N}_0$ for all $\upsilon \in \mathcal{M}_0$ and $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property;*
(ii) *Ω is multivalued σ-proximal admissible;*
(iii) *there exist $\upsilon_0, \upsilon_1 \in \mathcal{M}_0$ and $\nu_0 \in \Omega\upsilon_0$ such that $\rho(\upsilon_1, \nu_0) = \Lambda$ and $\sigma(\upsilon_0, \upsilon_1) \geq 1$;*
(iv) *Ω is continuous or property (H) holds ((C) holds).*

Then Ω has a best proximity point in $\mathcal{M}$.

Taking $\gamma_1 = 0$ in Theorem 2, we get the following result.

Corollary 2 *Let (Ξ, ρ) be a complete metric space and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty closed subsets of Ξ such that $\mathcal{M}_0$ is nonempty. Let $\Omega : \mathcal{M} \to K(\mathcal{N})$ $(CB(\mathcal{N}))$ be a multivalued mapping such that*

$$\begin{aligned}&\phi[\rho(\upsilon, \Omega\nu) - \Lambda, \rho(\upsilon, \nu)] < 0 \implies \\ &\vartheta(H(\Omega\upsilon, \Omega\nu)) \leq [\vartheta(\rho(\upsilon, \Omega\upsilon) - \Lambda)]^{\gamma_2}[\vartheta(\rho(\nu, \Omega\nu) - \Lambda)]^{\gamma_3},\end{aligned} \tag{25}$$

for all $\upsilon, \nu \in \mathcal{M}$ with $\sigma(\upsilon, \nu) \geq 1$ and $H(\Omega\upsilon, \Omega\nu) > 0$, where $\sigma : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$, $\phi \in \Phi$, $\vartheta \in \mathcal{Z}$ and $\gamma_2, \gamma_3 \in \mathbb{R}^+$ with $0 \leq \gamma_2 + \gamma_3 < 1$. If the assertions $(i) - (iv)$ in Corollary 1, then Ω has a best proximity point in $\mathcal{M}$.

If we take $\vartheta(r) = e^{\sqrt{r}}k$ in Theorem 2, then we have the following result.

Corollary 3 *Let (Ξ, ρ) be a complete metric space and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty closed subsets of Ξ such that $\mathcal{M}_0$ is nonempty. Let $\Omega : \mathcal{M} \to K(\mathcal{N})$ $(CB(\mathcal{N}))$ be a multivalued mapping such that*

$$\begin{aligned}&\phi[\rho(\upsilon, \Omega\nu) - \Lambda, \rho(\upsilon, \nu)] < 0 \implies \\ &\sqrt{H(\Omega\upsilon, \Omega\nu)} \leq \gamma_1\sqrt{\rho(\upsilon, \nu)} + \gamma_2\sqrt{\rho(\upsilon, \Omega\upsilon) - \Lambda} + \gamma_3\sqrt{\rho(\nu, \Omega\nu) - \Lambda},\end{aligned} \tag{26}$$

for all $\upsilon, \nu \in \mathcal{M}$ *with* $\sigma(\upsilon, \nu) \geq 1$ *and* $H(\Omega\upsilon, \Omega\nu) > 0$, *where* $\sigma : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$, $\phi \in \Phi$, *and* $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}^+$ *with* $0 \leq \gamma_1 + \gamma_2 + \gamma_3 < 1$. *If the assertions* $(i) - (iv)$ *in Corollary 1, then* Ω *has a best proximity point in* $\mathcal{M}$.

Example 2 Let $\Xi = \mathbb{R}^+ \times \mathbb{R}^+$ be endowed with the usual metric ρ. Consider,

$$\mathcal{M} = \{(\frac{1}{2}, \upsilon) : \upsilon \in \mathbb{R}^+\} \text{ and } \mathcal{N} = \{(0, \upsilon) : \upsilon \in \mathbb{R}^+\}.$$

Define $\Omega : \mathcal{M} \to CB(\mathcal{N})$ by

$$\Omega(\frac{1}{2}, \upsilon) = \begin{cases} \{(0, \frac{\upsilon}{2}) : \upsilon \in [0, p]\}, & p \leq 1, \\ \{(0, \upsilon^2) : \upsilon \in [p, +\infty)\}, & p > 1, \end{cases}$$

and a function $\sigma : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ as follows:

$$\sigma(\upsilon, \nu) = \begin{cases} 1, & \upsilon, \nu \in \{(\frac{1}{2}, p) : p \in [0, 1]\}, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that, $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{N}_0 = \mathcal{N}$ and $\Omega\upsilon \subseteq \mathcal{N}_0$ for every $\upsilon \in \mathcal{M}_0$. Moreover, $(\mathcal{M}, \mathcal{N})$ has weak P-property. Let $\sigma(\upsilon_1, \upsilon_2) \geq 1$ implies that $\upsilon_1, \upsilon_2 \in \{(\frac{1}{2}, \nu) : \nu \in [0, 1]\}$. Then,

$$\Omega\upsilon_1, \Omega\upsilon_2 \subseteq \{(0, \frac{\nu}{2}) : \nu \in [0, 1]\}.$$

Consider $y_1 \in \Omega\upsilon_0$, $y_2 \in \Omega\upsilon_1$ and $\nu_1, \nu_2 \in \mathcal{M}$ such that $d(\nu_1, y_1) = \rho(\mathcal{M}, \mathcal{N})$ and $\rho(\nu_2, y_2) = \rho(\mathcal{M}, \mathcal{N})$. Then, $\nu_1, \nu_2 \in \{(\frac{1}{2}, \nu) : \nu \in [0, \frac{1}{2}]\}$. Hence, $\sigma(\nu_1, \nu_2) \geq 1$ implies that Ω is multivalued σ-proximal admissible. For $\upsilon_0 = (\frac{1}{2}, 1) \in \mathcal{M}_0$ and $\nu_0 = (0, \frac{1}{2}) \in \Omega\upsilon_0$, we have $\upsilon_1 = (\frac{1}{2}, \frac{1}{2}) \in \mathcal{M}_0$ such that $\rho(\upsilon_1, \nu_0) = \rho(\mathcal{M}, \mathcal{N})$ and $\sigma(\upsilon_0, \upsilon_1) = 1$. Next, let $\phi(r, s) = \frac{r}{2} - s$, if $r, s \in [0, 1]$ and $\phi(r, s) = 2s$, otherwise. Clearly, $\phi[\rho(\upsilon, \Omega\nu) - \Lambda, \rho(\upsilon, \nu)] < 0$ if and only if $\upsilon, \nu \in \{(\frac{1}{2}, p) : p \in [0, 1]\}$. Then,

$$\begin{aligned} \vartheta(H(\Omega\upsilon, \Omega\nu)) &= \vartheta(\frac{|\upsilon - \nu|}{2}) \\ &= e^{\sqrt{\frac{1}{2}|\upsilon - \nu|}} \\ &= e^{\sqrt{\frac{1}{2}\rho(\upsilon, \nu)}} \\ &\leq [\vartheta(\rho(\upsilon, \nu))]^{\gamma_1}, \end{aligned}$$

for all $\upsilon, \nu \in \{(\frac{1}{2}, p) : p \in [0, 1]\}$. Hence, Ω is an multivalued Suzuki-type Z_σ-contraction with the setting $\vartheta(r) = e^{\sqrt{r}}$, $\gamma_1 = \sqrt{\frac{2}{3}}$ and $\gamma_2 = \gamma_3 = 0$. Moreover, if $\{\upsilon_n = (\frac{1}{2}, p_n)\}$ is a sequence in $\mathcal{M}$ such that $\sigma(\upsilon_n, \upsilon_{n+1}) \geq 1$ for all n and $\upsilon_n = (\frac{1}{2}, p_n) \to \upsilon = (\frac{1}{2}, p) \in \mathcal{M}$ as $n \to +\infty$, then $p_n \to p$. Hence, $p_n \in [0, 1]$ and

so $p \in [0, 1]$. Thus, $\upsilon_n \in \{(\frac{1}{2}, p_n) : p_n \in [0, 1]\}$ and $\upsilon \in \{(\frac{1}{2}, p) : p \in [0, 1]\}$. This implies that $\sigma(\upsilon_n, \upsilon) \geq 1$ for all n.
Consequently, all conditions of Theorem 5 are satisfied. Therefore, Ω has a best proximity point in $\mathcal{M}$ which is $(\frac{1}{2}, 0)$.

3 Some Applications

In this section, we give new best proximity point results on a metric space endowed with a partial ordering/graph, by using the results provided in the previous section. Define

$$\sigma\colon \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+, \quad \sigma(u, v) = \begin{cases} 1, & \text{if } u \preceq v, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 6 Let $\mathcal{M}, \mathcal{N}$ be two nonempty subsets of a partially ordered metric space $(\Xi, \preceq, \rho)$. A mapping $\Omega : \mathcal{M} \to CL(\mathcal{N})$ is called multivalued $\preceq$-proximal increasing, if for all $\upsilon_1, \upsilon_2, \nu_1, \nu_2 \in \mathcal{M}$, $y_1 \in \Omega\upsilon_1$ and $y_2 \in \Omega\upsilon_2$,

$$\begin{cases} \upsilon_1 \preceq \upsilon_2 \\ \rho(\nu_1, y_1) = \rho(\mathcal{M}, \mathcal{N}) \\ \rho(\nu_2, y_2) = \rho(\mathcal{M}, \mathcal{N}) \end{cases} \Longrightarrow \nu_1 \preceq \nu_2.$$

$(\preceq_H)$: If $\{\upsilon_n\}$ is a sequence in $\mathcal{M}$ such that $\upsilon_n \preceq \upsilon_{n+1}$ for all n and $\upsilon_n \to \upsilon \in \mathcal{M}$ as $n \to +\infty$, then $\upsilon_n \preceq \upsilon$ for all n.

Then the following result is a direct consequence of Theorems 2, 3, 4 and 5.

Theorem 6 *Let $(\Xi, \preceq, \rho)$ be a complete partially ordered metric space and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty closed subsets of Ξ such that $\mathcal{M}_0$ is nonempty. Let $\Omega : \mathcal{M} \to K(\mathcal{N})$ $(CB(\mathcal{N}))$ be a multivalued mapping such that*

$$\begin{aligned} &\phi[\rho(\upsilon, \Omega\nu) - \Lambda, \rho(\upsilon, \nu)] < 0 \implies \\ &\vartheta(H(\Omega\upsilon, \Omega\nu)) \leq [\vartheta(\rho(\upsilon, \nu))]^{\gamma_1}[\vartheta(\rho(\upsilon, \Omega\upsilon) - \Lambda)]^{\gamma_2}[\vartheta(\rho(\nu, \Omega\nu) - \Lambda)]^{\gamma_3}, \end{aligned} \tag{27}$$

for all $\upsilon, \nu \in \mathcal{M}$ with $\upsilon \preceq \nu$ and $H(\Omega\upsilon, \Omega\nu) > 0$, where $\phi \in \Phi$, $\vartheta \in \mathcal{Z}$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}^+$ with $0 \leq \gamma_1 + \gamma_2 + \gamma_3 < 1$. Assume that the following assertions hold:

(i) $\Omega\upsilon \subseteq \mathcal{N}_0$ for all $\upsilon \in \mathcal{M}_0$ and $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property;
(ii) Ω is multivalued $\preceq$-proximal increasing;
(iii) there exist $\upsilon_0, \upsilon_1 \in \mathcal{M}_0$ and $\nu_0 \in \Omega\upsilon_0$ such that $\rho(\upsilon_1, \nu_0) = \Lambda$ and $\upsilon_0 \preceq \upsilon_1$;
(iv) Ω is continuous or property $(\preceq_H)$ holds ((C) holds).

Then Ω has a best proximity point in $\mathcal{M}$.

Let Ξ be a nonempty set and ∇ designates the diagonal of Cartesian product $\Xi \times \Xi$ and $G = (V(G), \mathcal{E}(G))$ be a directed graph with no paralel edges in such a way that the set $V(G)$ of its vertices coincides with Ξ and $\nabla \subset \mathcal{E}(G)$, where $\mathcal{E}(G)$ is the set of the edges of the graph, which contains all loops, like that $\nabla \subseteq \Xi \times \Xi$. Notice that, a graph G is connected, if there is a path between any two vertices and it is weakly connected, if G is connected, where G is an undirected form of the graph G in which direction of edges have not any role. In a graph G, by antipole the direction of edges we obtain the graph G^{-1}, whose set of edges and set of vertices are given by:

$$\mathcal{E}(G^{-1}) = \{(\upsilon_1, \upsilon_2) \in \Xi \times \Xi : (\upsilon_1, \upsilon_2) \in \mathcal{E}(G)\} \text{ and } V(G) = V(G^{-1}). \quad (28)$$

In the presence of this manner, we have

$$\mathcal{E}(G) = \mathcal{E}(G) \cup \mathcal{E}(G^{-1}). \quad (29)$$

Define,

$$\sigma : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+, \quad \sigma(u, v) = \begin{cases} 1, & \text{if } (u, v) \in \mathcal{E}(G), \\ 0, & \text{otherwise.} \end{cases}$$

Definition 7 Let (Ξ, ρ) be a metric space endowed with a graph G and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty subsets of Ξ. A mapping $\Omega : \mathcal{M} \to CL(\mathcal{N})$ is called multivalued G-proximal, if for all $\upsilon_1, \upsilon_2, \nu_1, \nu_2 \in \mathcal{M}$, $y_1 \in \Omega\upsilon_1$ and $y_2 \in \Omega\upsilon_2$,

$$\begin{cases} (\upsilon_1, \upsilon_2) \in \mathcal{E}(G) \\ \rho(\nu_1, y_1) = \rho(\mathcal{M}, \mathcal{N}) \\ \rho(\nu_2, y_2) = \rho(\mathcal{M}, \mathcal{N}) \end{cases} \implies (\nu_1, \nu_2) \in \mathcal{E}(G).$$

(G_H) : If $\{\upsilon_n\}$ is a sequence in $\mathcal{M}$ such that $(\upsilon_n, \upsilon_{n+1}) \in \mathcal{E}(G)$ for all n and $\upsilon_n \to \upsilon \in \mathcal{M}$ as $n \to +\infty$, then $(\upsilon_n, \upsilon) \in \mathcal{E}(G)$ for all n.

Then the following result is a direct consequence of Theorems 2, 3, 4 and 5.

Theorem 7 *Let (Ξ, ρ) be a complete metric space endowed with a graph G and $(\mathcal{M}, \mathcal{N})$ be a pair of nonempty closed subsets of Ξ such that $\mathcal{M}_0$ is nonempty. Let $\Omega : \mathcal{M} \to K(\mathcal{N})$ $(CB(\mathcal{N}))$ be a multivalued mapping such that*

$$\begin{aligned} &\phi[\rho(\upsilon, \Omega\nu) - \Lambda, \rho(\upsilon, \nu)] < 0 \implies \\ &\vartheta(H(\Omega\upsilon, \Omega\nu)) \leq [\vartheta(\rho(\upsilon, \nu))]^{\gamma_1}[\vartheta(\rho(\upsilon, \Omega\upsilon) - \Lambda)]^{\gamma_2}[\vartheta(\rho(\nu, \Omega\nu) - \Lambda)]^{\gamma_3}, \end{aligned} \quad (30)$$

for all $\upsilon, \nu \in \mathcal{M}$ with $(\upsilon, \nu) \in \mathcal{E}(G)$ and $H(\Omega\upsilon, \Omega\nu) > 0$, where $\phi \in \Phi$, $\vartheta \in \mathcal{Z}$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}^+$ with $0 \leq \gamma_1 + \gamma_2 + \gamma_3 < 1$. Assume that the following assertions hold:

(i) $\Omega\upsilon \subseteq \mathcal{N}_0$ for all $\upsilon \in \mathcal{M}_0$ and $(\mathcal{M}, \mathcal{N})$ satisfies the weak P-property;

(ii) *Ω is multivalued G-proximal;*
(iii) *there exist $\upsilon_0, \upsilon_1 \in \mathcal{M}_0$ and $\nu_0 \in \Omega\upsilon_0$ such that $\rho(\upsilon_1, \nu_0) = \Lambda$ and $(\upsilon_0, \upsilon_1) \in \mathcal{E}(G)$;*
(iv) *Ω is continuous or property (G_H) holds ((C) holds).*

Then Ω has a best proximity point in $\mathcal{M}$.

4 Conclusion

In this study, we introduce the new class of multivalued Suzuki-type Z_σ-contractions under an influence of ϕ function. Within this framework, we have introduced new related best proximity point results in metric spaces. At the end, we have applied our main results to derive new best proximity point results on a metric space endowed with a partial ordering/graph. A nontrivial example has been constructed to support our main works. Herein, the presented theorems and corollaries cannot be directly procured from the correlative metric spaces version.

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Coincidence Best Proximity Point Results via w_p-Distance with Applications

Naeem Saleem

Abstract Fixed-point theorems in metric spaces have extensively applied to a wide variety of mathematical problems, these kinds of results based upon some strong suppositions in metric spaces. In the current chapter, some weakly Kannan type generalized and weakly Kannan type proximal contractive mappings are introduced. Using these contractive conditions, we provide coincidence best proximity point results in complete metric space using w_p-distance (which is a generalization of w-distance). Some applications are also provided to the theory of fixed-points in metric spaces with ordered structure. We elaborated our results with examples which shows that obtained results are potential generalizations of already existing results in the literature.

1 Introduction

Theory of fixed-points is one of the incredible assets in modern mathematics, according to F. Browder, "*who gave another impulse to the advanced fixed-point theory by means of the improvement of nonlinear functional analysis as a fundamental part of science*". This theory is applied to numerous fields of current interest in the analysis, with topological contemplations assuming a pivotal job, incorporating the relationship with degree hypothesis. The fixed-point theory of certain significant mappings has its own privilege because of its outcomes having constructive proof and applications in industrial fields, for example, image processing, physics, software engineering, economics, and telecommunication. In mathematics, several problems can be transformed into a fixed-point problem $Tx = x$, where $T : X \to X$ is an operator and X is an abstract space. The solution of an operator equation $Tx = x$ is known as *fixed-point* of the operator T. In 1922, Banach [1] proved a contraction principle for self mappings named *Banach fixed-point theorem*, several authors extended and generalized the Banach contraction principles by modifying the contractive conditions and generalizing the underlying metric space. To generalize contractive condition means to generalize the associated conditions on the operator T such that we can

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obtain a fixed-point of operator T but the situation becomes more complicated when the domain and range of an operator T become different. If $T : A \to B$ where A and B are nonempty and disjoint subsets of a metric space (X, d), then an equation $Tx = x$ need not have a solution in X and hence the need to obtain an optimal solution of the operator equation arises. This is achieved by reducing the distance between pre-image x and image Tx as a solution of the following minimization problem:

$$\min_{x \in X} d(x, Tx). \tag{1}$$

The element $x \in X$ which satisfy the corresponding minimization problem (1) is known as an approximate solution or approximate fixed-point of the operator equation $Tx = x$ if T is a non-self operator. Any $x \in A$ satisfying

$$d(x, Tx) = d(A, B) \tag{2}$$

is the solution of minimization problem (1) is known as an *approximate fixed-point* or *best proximity point* of the operator T, and the distance between sets A and B is defined as

$$d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}.$$

Several authors further generalized the Banach contraction principle for non-self mappings, for further details, readers can see [2–29].

On the other hand, Kada et al. [30] introduced the concept of w-distance on a metric space and obtained well-known Caristi fixed-point theorem, Ekland variational principle, and Takahashi existence theorem. Afterward, Suzuki and Takahashi [31] obtained a fixed-point theorem for multivalued mapping with respect to w-distance. This result is an improvement of Nadler's fixed-point theorem. Several fixed-point theorems have been proved by many researchers in metric spaces via w-distance, for example, see [32–35]. Kutbi and Sintunavarat introduced the concept of generalized w_*-contractive mapping and proved a fixed-point theorem for such mappings using the concept of α_*-admissible mapping in [36], which is a multivalued mapping version of α-admissible mapping defined in [37].

In this chapter, we are going to define the concept of w_p-distance, and we will obtain coincidence best proximity point and best proximity point results in complete metric space using w_p-distance.

2 Preliminaries

This section will serve as an introduction to some basic and foundational concepts of metric spaces, best proximity points, coincidence best proximity points, w-distance and its generalizations. This detailed discussion will give a brief overview of the results related to our main theorems.

Definition 1 Let A and B are nonempty subsets of a metric space (X, d), define

$$A_0 = \{a \in A : \text{ there exists some } b \in B \text{ such that } d(a, b) = d(A, B)\},$$
and
$$B_0 = \{b \in B : \text{there exists some } a \in A \text{ such that } d(a, b) = d(A, B)\},$$
where
$$d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}, \text{ (distance between sets } A \text{ and } B)\}.$$

Definition 2 ([38]) Let A and B are two nonempty subsets of a metric space (X, d), and the pair (A, B) is said to satisfy the P-property, if

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \text{ implies that } d(x_1, x_2) = d(y_1, y_2),$$

for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

Definition 3 ([5]) Let A and B are nonempty subsets of a metric space (X, d) and $T : A \to B$, an element $x^* \in A$ is called a best proximity point of the mapping T, if

$$d(x^*, Tx^*) = d(A, B).$$

Definition 4 ([39]) A mapping $T : A \to B$ is called an α-proximal admissible mapping, if there exists a mapping $\alpha : A \times A \to [0, \infty)$ such that

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \text{ implies that } \alpha(u_1, u_2) \geq 1,$$

for all u_1, u_2, x_1 and x_2 in A.

Definition 5 Let (X, d) be a metric space. A function $p : X \times X \to [0, \infty)$ is called a w-distance on X if

- $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$
- p is lower semi-continuous in its second variable, i.e., if $x \in X$ and $y_n \in y$ in X then $p(x, y) \leq \liminf p(x, y_n)$;
- For each $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ implies $d(x, y) \leq \epsilon$.

Definition 6 ([21]) A self mapping $g : A \to A$ is said to satisfy α_R-property if there exist a mapping $\alpha : A \times A \to [0, \infty)$ such that $\alpha(gx, gy) = 1$ implies that $\alpha(x, y) \geq 1, forall x, y \in A$.

3 Weakly Kannan Type Best Proximity Points via w_p-Distance

In this section, we will introduce w_p-distance, weakly Kannan type proximal contractions with w_p-distance and then we will provide best proximity point result for these mappings.

Definition 7 Let (X, d) be a metric space. A mapping $p : X \times X \to [0, \infty)$ is called a w_p-distance on X, if p satisfies the following properties:

(1) $p(a, c) \leq p(a, b) + p(b, c)$ for any $a, b, c \in X$,
(2) p is lower semi-continuous in its second variable, i.e., if $a \in X$ and $b_n \to b$ in X, then $p(a, b) \leq \liminf_n p(a, b_n)$,
(3) for each $\epsilon > 0$, there exists $\delta > 0$ such that $p(c, a) \leq \delta$ and $p(c, b) \leq \delta$ implies that $p(a, b) \leq \epsilon$, for any $a, b, c \in X$.

In the following definition, we will generalize the notation of A_0, B_0, and α-proximal admissible mapping using w_p-distance.

Definition 8 Let p be a w_p-distance defined on set X, where (X, d) is a metric space induced by a metric d. Also, suppose that A and B are nonempty subsets of X, define

$$A_{0,p} = \{a \in A : \text{ there exists some } b \in B \text{ such that } p(a, b) = p(A, B)\},$$
and
$$B_{0,p} = \{b \in B : \text{ there exists some } a \in A \text{ such that } p(a, b) = p(A, B)\},$$
where
$$p(A, B) = \inf\{p(a, b) : a \in A \text{ and } b \in B\}, \ (w_p\text{-distance between sets } A \text{ and } B),$$
where
$$p^*(x, y) = p(x, y) - p(A, B).$$

From now and onwards, X will represents a metric space (X, d), A, B are nonempty and disjoint subsets of X, and p will represents a w_p-distance defined on X (until otherwise stated).

Definition 9 Let $\alpha : A \times A \to [0, \infty)$ and $T : A \to B$ are mappings. A mapping T is called an α_p-proximal admissible mapping, if

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ p(u_1, Tx_1) = p(A, B) \\ p(u_2, Tx_2) = p(A, B) \end{array}\right\} \text{ implies that } \alpha(u_1, u_2) \geq 1,$$

for all u_1, u_2, x_1 and x_2 in A.

Definition 10 A class ϑ is consisting upon all continuous mapping $\theta : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying $\theta\,(a, b) = 0$ if and only if $a = 0 = b$.

Definition 11 Let $g : A \to A, T : A \to B$ and $\alpha : A \times A \to [0, \infty)$ are mappings. A pair of mappings (g, T) is weakly Kannan type generalized proximal contraction, if

$$\alpha(gx, gy)p(gu, gv) \leq \frac{(1-k)}{2}[p^*(Tx, gx) + p^*(Ty, gy)] - \theta[p^*(Tx, gx), p^*(Ty, gy)], \quad (3)$$

where $p(gu, Tx) = p(A, B) = p(gv, Ty), \theta \in \vartheta$ and $\alpha(x, y) \geq 1$, for all $u, v, x, y \in A$.

Definition 12 Let $\alpha : X \times X \to [0, \infty)$ be a mapping. A mapping $T : A \to B$ is a weakly Kannan type proximal contraction, if

$$\alpha(x, y)p(u, v) \leq \frac{(1-k)}{2}[p^*(Tx, x) + p^*(Ty, y)] - \theta[p^*(Tx, x), p^*(Ty, y)], \quad (4)$$

where $p(u, Tx) = p(A, B) = p(v, Ty), \theta \in \vartheta$ and $\alpha(x, y) \geq 1$, for all $u, v, x, y \in A$.

Remark 1 Note that, if $gx = I_A$ (an identity mapping over set A) then every weakly Kannan type generalized proximal contraction becomes weakly Kannan type proximal contraction.

Definition 13 Let $g : A \to A$ and $T : A \to B$ are mappings. An element $x^* \in A$ is a w_p-coincidence best proximity point of a pair of mappings (g, T), if

$$p(gx^*, Tx^*) = p(A, B).$$

Definition 14 Let $T : A \to B$ be a mapping. An element $x^* \in A$ is called a w_p-best proximity point of the mapping T, if

$$p(x^*, Tx^*) = p(A, B).$$

Theorem 1 *Let A and B are nonempty subsets of a complete metric space (X, d). Consider a pair of mappings (g, T) be a weakly Kannan type generalized proximal contraction, where T be an α_p-proximal admissible mapping and g be an one to one mapping which satisfies α_R-property with $A_{0,p}$ is nonempty and closed subset of A. If $T(A_{0,p}) \subseteq B_{0,p}$ and $A_{0,p} \subset g(A_{0,p})$ then there exists a unique w_p-coincidence best proximity point x^* of pair (g, T) in A.*

Proof Since $A_{0,p}$ is nonempty and T is weakly Kannan type generalized proximal contractive mapping. Thus, we can choose an element $x_0 \in A_{0,p}$, since $Tx_0 \in T(A_{0,p}) \subseteq B_{0,p}$, hence there exists x_1 in $A_{0,p} \subset g(A_{0,p})$, such that $p(gx_1, Tx_0) = p(A, B)$. As $x_1 \in A_{0,p}$ and $Tx_1 \in T(A_{0,p}) \subseteq B_{0,p}$, there exists x_2 in $A_{0,p}$ such that $p(gx_2, Tx_1) = p(A, B)$. Since mapping T is an α_p-proximal admissible and $\alpha(x_0, x_1) \geq 1$, then we have

$$\left.\begin{array}{l}\alpha(x_0, x_1) \geq 1\\ p(gx_1, Tx_0) = p(A, B)\\ p(gx_2, Tx_1) = p(A, B)\end{array}\right\} \text{ implies that } \alpha(gx_1, gx_2) \geq 1.$$

Since mapping g satisfies α_R-property, hence $\alpha(gx_1, gx_2) \geq 1$ implies $\alpha(x_1, x_2) \geq 1$. Since, T is weakly Kannan type generalized proximal mapping, so we have

$$\begin{aligned}
p(gx_1, gx_2) &\leq \alpha(gx_0, gx_1)p(gx_1, gx_2)\\
&\leq \frac{(1-k)}{2}[p^*(Tx_0, gx_0) + p^*(Tx_1, gx_1)] - \theta(p^*(Tx_0, gx_0), p^*(Tx_1, gx_1))
\end{aligned}$$

$$\begin{aligned}
&= \frac{(1-k)}{2}[p(Tx_0, gx_0) + p(Tx_1, gx_1)] - p(A, B) - \theta([p(Tx_0, gx_0) - p(A, B),\\
&\quad p(Tx_1, gx_1) - p(A, B)])\\
&\leq \frac{(1-k)}{2}[p(Tx_0, gx_1) + p(gx_1, gx_0) + p(Tx_1, gx_2) + p(gx_2, gx_1)] - p(A, B) -\\
&\quad \theta(p(Tx_0, gx_1) + p(gx_1, gx_0) - p(A, B), p(Tx_1, gx_2) + p(gx_2, gx_1) - p(A, B))\\
&\leq \frac{(1-k)}{2}[p(gx_1, gx_0) + p(gx_2, gx_1)] - \theta[p(gx_1, gx_0), p(gx_2, gx_1)]\\
&\leq \frac{1}{2}[p(gx_1, gx_0) + p(gx_2, gx_1)],
\end{aligned}$$

which can be written as

$$p(gx_1, gx_2) \leq \frac{1}{2}[p(gx_1, gx_0) + p(gx_2, gx_1)].$$

After simplification, we have

$$p(gx_1, gx_2) \leq p(gx_1, gx_0). \tag{5}$$

Similarly, for $x_2 \in A_{0,p} \subset g(A_{0,p})$ and $Tx_2 \in T(A_{0,p}) \subseteq B_{0,p}$, there exists x_3 in $A_{0,p}$ such that $p(gx_3, Tx_2) = p(A, B)$. Since T is an α_p-proximal admissible mapping, hence we have

$$\left.\begin{array}{l}\alpha(x_1, x_2) \geq 1\\ p(gx_2, Tx_1) = p(A, B)\\ p(gx_3, Tx_2) = p(A, B)\end{array}\right\} \text{ implies } \alpha(gx_2, gx_3) \geq 1.$$

Since mapping g satisfies α_R-property, hence $\alpha(x_2, x_3) \geq 1$. Since, T is a weakly Kannan type generalized proximal mapping, then

$$\begin{aligned}
p(gx_2, gx_3) &\leq \alpha(gx_1, gx_2)p(gx_2, gx_3)\\
&\leq \frac{(1-k)}{2}[p^*(Tx_1, gx_1) + p^*(Tx_2, gx_2)] - \theta(p^*(Tx_1, gx_1), p^*(Tx_2, gx_2))\\
&= \frac{(1-k)}{2}[p(Tx_1, gx_1) + p(Tx_2, gx_2)] - p(A, B) - \theta(p(Tx_1, gx_1) - p(A, B),\\
&\quad p(Tx_2, gx_2) - p(A, B))
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{(1-k)}{2}[p(Tx_1, gx_2) + p(gx_2, gx_1) + p(Tx_2, gx_3) + p(gx_3, gx_2)] - p(A, B) - \\
&\theta[p(Tx_1, gx_2) + p(gx_2, gx_1) - p(A, B), p(Tx_2, gx_3) + p(gx_3, gx_2) - p(A, B)] \\
&\le \frac{(1-k)}{2}[p(gx_2, gx_1) + p(gx_3, gx_2)] - \theta[p(gx_2, gx_1), p(gx_3, gx_2)] \\
&\le \frac{1}{2}[p(gx_2, gx_1) + p(gx_3, gx_2)] - \theta(p(gx_2, gx_1), p(gx_3, gx_2)) \\
&\le \frac{1}{2}[p(gx_2, gx_1) + p(gx_3, gx_2)].
\end{aligned}
$$

Further, we can write

$$p(gx_2, gx_3) \le \frac{1}{2}[p(gx_2, gx_1) + p(gx_3, gx_2)].$$

After simplification, we have

$$p(gx_2, gx_3) \le p(gx_2, gx_1), \tag{6}$$

which shows that the sequence $\{p(gx_2, gx_3)\}$ is a decreasing sequence and bounded below. According to the pattern followed in (5) and (6) and the mapping T is an α_p-proximal admissible mapping, so generally we have

$$\left.\begin{array}{l} \alpha(x_n, x_{n-1}) \ge 1 \\ p(gx_{n+1}, Tx_n) = p(A, B) \\ p(gx_n, Tx_{n-1}) = p(A, B) \end{array}\right\} \text{ implies that } \alpha(gx_{n+1}, gx_n) \ge 1, \text{ for all } n \ge 1.$$

Since mapping g satisfies α_R-property, hence $\alpha(x_{n+1}, x_n) \ge 1$. Since, T is weakly Kannan type generalized proximal contractive mapping, and we have

$$
\begin{aligned}
p(gx_{n+1}, gx_n) &\le \alpha(gx_n, gx_{n-1}) p(gx_{n+1}, gx_n) \\
&\le \frac{(1-k)}{2}[p^*(Tx_n, gx_n) + p^*(Tx_{n-1}, gx_{n-1})] \\
&\quad -\theta(p^*(Tx_n, gx_n), p^*(Tx_{n-1}, gx_{n-1})) \\
&= \frac{(1-k)}{2}[p(Tx_n, gx_n) + p(Tx_{n-1}, gx_{n-1})] \\
&\quad -p(A, B) - \theta(p(Tx_n, gx_n) - p(A, B), \\
&\quad p(Tx_{n-1}, gx_{n-1}) - p(A, B)).
\end{aligned}
$$

After simplification, we have

$$
\begin{aligned}
p(gx_{n+1}, gx_n) \le\ & \frac{(1-k)}{2}[p(Tx_n, gx_{n+1}) + p(gx_{n+1}, gx_n) + p(Tx_{n-1}, gx_n) + p(gx_n, gx_{n-1})] \\
& -p(A, B) - \theta[p(Tx_n, gx_{n+1}) + p(gx_{n+1}, gx_n) - p(A, B), p(Tx_{n-1}, gx_n) + \\
& p(gx_n, gx_{n-1}) - p(A, B)] \\
\le\ & \frac{(1-k)}{2}[p(gx_{n+1}, gx_n) + p(gx_n, gx_{n-1})] - p(A, B) -
\end{aligned}
$$

$$\begin{aligned} &\theta[p(gx_{n+1}, gx_n), p(gx_n, gx_{n-1})] \\ &\le \frac{1}{2}[p(gx_{n+1}, gx_n) + p(gx_n, gx_{n-1})] - \theta[p(gx_{n+1}, gx_n), p(gx_n, gx_{n-1})] \\ &\le \frac{1}{2}[p(gx_{n+1}, gx_n) + p(gx_n, gx_{n-1})]. \end{aligned} \tag{7}$$

Above inequality can be written as

$$p(gx_{n+1}, gx_n) \le \frac{1}{2}[p(gx_{n+1}, gx_n) + p(gx_n, gx_{n-1})].$$

After simplification, we have

$$p(gx_{n+1}, gx_n) \le p(gx_n, gx_{n-1}), \text{ for all } n \ge 1.$$

Therefore, the sequence $\{p(gx_{n+1}, gx_n)\}$ is monotone decreasing and bounded below, so there exists $r \ge 0$, such that

$$\lim_{n\to\infty} p\left(gx_{n+1,} gx_n\right) = r.$$

We have to show that $r = 0$. On contrary, suppose that $r > 0$. By taking limit as $n \to \infty$, on inequality (7), we have

$$r \le \frac{1}{2}[r + r] - \theta(r, r),$$

which implies that

$$0 \le \theta(r, r) \le 0,$$

which implies $\theta(r, r) = 0$, where $\theta \in \vartheta$, hence, $r = 0$, a contradiction to our supposition that $r > 0$. Hence $r = 0$, then we have

$$\lim_{n\to\infty} p\left(gx_{n+1,} gx_n\right) = 0.$$

Now, we will prove that $\lim_{n\to\infty} p\left(gx_{n,} gx_m\right) \to 0$ as $n \to \infty$. We claim that $p\left(gx_n, gx_m\right) \ne 0$. Now, suppose that there exists $\epsilon > 0$ and a subsequence $\{gx_{m(k)}\}$ and $\{gx_{n(k)}\}$. On contrary, suppose that $\{gx_n\}$ is not a Cauchy sequence in $A_{0,p}$ that is,

$$p\left(gx_{n(k)}, gx_{m(k)}\right) \ge \epsilon,$$

and

$$p\left(gx_{n(k)}, gx_{m(k)-1}\right) < \epsilon, \tag{8}$$

where $m_k > n_k > N \in \mathbb{N}$. Using inequalities (3) and (8) we have, $\epsilon \le p\left(gx_{n(k)}, gx_{m(k)}\right)$. Since the mapping T is α_p-proximal admissible mapping, so we have

$$\left.\begin{array}{l}\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1 \\ p(gx_{n(k)}, Tx_{n(k)-1}) = p(A, B) \\ p(gx_{m(k)}, Tx_{m(k)-1}) = p(A, B)\end{array}\right\} \text{ implies that } \alpha(gx_{n(k)}, gx_{m(k)}) \geq 1, \text{ for all } k.$$

Since mapping g satisfies α_R-property, hence $\alpha(x_{n(k)}, x_{m(k)}) \geq 1$. Since, T is weakly Kannan type generalized proximal contractive mapping and we have

$$\begin{aligned}
p(gx_{n(k)}, gx_{m(k)}) &\leq \alpha(gx_{n(k)-1}, gx_{m(k)-1})p(gx_{n(k)}, gx_{m(k)}) \\
&\leq \frac{(1-k)}{2}[p^*(Tx_{n(k)-1}, gx_{n(k)-1}) + p^*(Tx_{m(k)-1}, gx_{m(k)-1})] \\
&\quad -\theta(p^*(Tx_{n(k)-1}, gx_{n(k)-1}), p^*(Tx_{m(k)-1}, gx_{m(k)-1})) \\
&= \frac{(1-k)}{2}[p(Tx_{n(k)-1}, gx_{n(k)-1}) + p(Tx_{m(k)-1}, gx_{m(k)-1})] \\
&\quad -p(A, B) - \theta(p(Tx_{n(k)-1}, gx_{n(k)-1}) - p(A, B), \\
&\quad p(Tx_{m(k)-1}, gx_{m(k)-1}) - p(A, B)) \\
&\leq \frac{(1-k)}{2}[p(Tx_{n(k)-1}, gx_{n(k)}) + p(gx_{n(k)}, gx_{n(k)-1}) + \\
&\quad p(Tx_{m(k)-1}, gx_{m(k)}) + p(gx_{m(k)}, gx_{m(k)-1})] - p(A, B) \\
&\quad -\theta(p(Tx_{n(k)-1}, gx_{n(k)}) + p(gx_{n(k)}, gx_{n(k)-1}) - p(A, B), \\
&\quad p(Tx_{m(k)-1}, gx_{m(k)}) + p(gx_{m(k)}, gx_{m(k)-1}) - p(A, B)) \\
&\leq \frac{(1-k)}{2}[p(gx_{n(k)}, gx_{n(k)-1}) + p(gx_{m(k)}, gx_{m(k)-1})] - \\
&\quad \theta[p(gx_{n(k)}, gx_{n(k)-1}), p(gx_{m(k)}, gx_{m(k)-1})] \\
&\leq \frac{1}{2}[p(gx_{n(k)}, gx_{n(k)-1}) + p(gx_{m(k)}, gx_{m(k)-1})] - \\
&\quad \theta(p(gx_{n(k)}, gx_{n(k)-1}), p(gx_{m(k)}, gx_{m(k)-1})) \\
&\leq \frac{1}{2}[p(gx_{n(k)}, gx_{n(k)-1}) + p(gx_{m(k)}, gx_{m(k)-1})],
\end{aligned}$$

which can be written as

$$\epsilon \leq p(gx_{n(k)}, gx_{m(k)}) \leq \frac{1}{2}[p(gx_{n(k)}, gx_{n(k)-1}) + p(gx_{m(k)}, gx_{m(k)-1})] \to 0, \text{ as } k \to \infty,$$

which is a contradiction. Hence $\{gx_n\}$ is a Cauchy sequence in $A_{0,p}$, where $A_{0,p}$ is closed subset of complete metric space (X, d). Then we have $gx_n \to gu$ in $A_{0,p} \subset X$. Since g is a continuous and one to one mapping hence $x_n \to u$. Now, we have to show that u be a coincidence best proximity point of pair of mappings (g, T) and we have

$$p(Tu, gu) \leq \liminf p(Tu, gx_n).$$

Since

$$\begin{aligned} p(gx_n, Tu) &\leq p(gx_n, gx_{n+1}) + p(gx_{n+1}, Tx_n) + p(Tx_n, Tu) \\ &\leq p(gx_n, gx_{n+1}) + p(A, B) + p(Tx_n, Tu). \end{aligned}$$

After simplification, we have

$$p(Tx_n, Tu) \geq p(gx_n, Tu) - p(gx_n, gx_{n+1}) - p(A, B),$$

Since $x_n \to u$ and $p\,(x, .)$ is lower semi continuous. So

$$\liminf p(Tx_n, Tu) \geq \liminf p(gx_n, Tu) - 0 - p(A, B).$$

Since mapping T is continuous, hence we have

$$0 \geq p(gu, Tu) - p(A, B),$$

which further implies that

$$p(gu, Tu) = p(A, B).$$

Hence u is the coincidence best proximity point of pair of mapping (g, T).

Uniqueness: Let u and v are two distinct coincidence best proximity point of pair of mapping (g, T) such that $u \neq v$. Thus, we have

$$p(gu, gv) = r > 0.$$

Since, the mapping T is α_p-proximal admissible mapping, then

$$\left.\begin{array}{l} \alpha(u, v) \geq 1 \\ p(gu, Tu) = p(A, B) \\ p(gv, Tv) = p(A, B) \end{array}\right\} \text{ implies that } \alpha(gu, gv) \geq 1.$$

Since the pair of mapping (g, T) is weakly Kannan type generalized proximal admissible mapping thus by using (3) we get,

$$0 < r \leq 0$$

is a contradiction. So, $r = 0$ which gives $gu = gv$. Since mapping g is one to one, so we have a unique coincidence best proximity points of the pair of mapping (g, T).

In next result, we obtained a best proximity point result for a mapping satisfying weakly Kannan type proximal contraction.

Theorem 2 *Let (X, d) be a complete metric space, A and B are nonempty subsets of X. Suppose that $T : A \to B$ be an α_p-proximal admissible and weakly Kannan type proximal mapping with $A_{0,p}$ is nonempty. Assume that if there exist x_0 and x_1 in $A_{0,p}$ such that*

$$p(x_1, Tx_0) = p(A, B),\ \alpha(x_0, x_1) \geq 1$$

and $T(A_{0,p}) \subseteq B_{0,p}$ then there exists a unique w_p-best proximity point x^* in A.

Proof Let $x_0 \in A_{0,p}$ and $Tx_0 \in T(A_{0,p}) \subseteq B_{0,p}$, there exists x_1 in $A_{0,p}$ such that $p(x_1, Tx_0) = d(A, B)$, since $x_1 \in A_{0,p}$ and $Tx_1 \in T(A_{0,p}) \subseteq B_{0,p}$, there exists x_2 in $A_{0,p}$ such that $p(x_2, Tx_1) = d(A, B)$. Since $\alpha : A \times A \to [0, \infty)$, further we assumed that $\alpha(x_0, x_1) \geq 1$ and T is an α_p-proximal admissible mapping, we have

$$\left.\begin{array}{l}\alpha(x_0, x_1) \geq 1\\ p(x_1, Tx_0) = p(A, B)\\ p(x_2, Tx_1) = p(A, B)\end{array}\right\}\text{ implies that } \alpha(x_1, x_2) \geq 1,$$

and mapping T is weakly Kannan type proximal mapping, we have

$$\begin{aligned}
p(x_1, x_2) &\leq \alpha(x_0, x_1)p(x_1, x_2)\\
&\leq \frac{(1-k)}{2}[p^*(Tx_0, x_0) + p^*(Tx_1, x_1)] - \theta([p^*(Tx_0, x_0), p^*(x_1, Tx_1)])\\
&= \frac{(1-k)}{2}[p(Tx_0, x_0) + p(Tx_1, x_1)] - p(A, B) -\\
&\quad \theta([p(Tx_0, x_0) - p(A, B), p(Tx_1, x_1) - p(A, B)])\\
&\leq \frac{(1-k)}{2}[p(Tx_0, x_1) + p(x_1, x_0) + p(Tx_1, x_2) + p(x_2, x_1)] - p(A, B) -\\
&\quad \theta([p(Tx_0, x_1) + p(x_1, x_0) - p(A, B), p(Tx_1, x_2) + p(x_2, x_1) - p(A, B))\\
&\leq \frac{(1-k)}{2}[p(A, B) + p(x_1, x_0) + p(A, B) + p(x_2, x_1)] - p(A, B) -\\
&\quad \theta([p(x_1, x_0), p(x_2, x_1)])\\
&\leq \frac{(1-k)}{2}[p(x_1, x_0) + p(x_2, x_1)] - \theta([p(x_1, x_0), p(x_2, x_1)])\\
&\leq \frac{1}{2}[p(x_1, x_0) + p(x_2, x_1)]
\end{aligned}$$

the above inequality can be written as

$$p(x_1, x_2) \leq p(x_1, x_0), \tag{9}$$

hence $\{p(x_1, x_2)\}$ is a decreasing sequence and bounded below. Now, on the same lines, using the α_p-proximal admissibility and weakly Kannan type proximal mapping T, we have

$$p(x_2, x_3) \leq p(x_2, x_1). \tag{10}$$

Therefore, the sequence $\{p(x_2, x_3)\}$ is a decreasing and bounded below. Following on the same lines, we have the general form

$$\left.\begin{array}{l}\alpha(x_n, x_{n-1}) \geq 1\\ p(x_{n+1}, Tx_n) = p(A, B)\\ p(x_n, Tx_{n-1}) = p(A, B)\end{array}\right\}\text{ implies that } \alpha(x_{n+1}, x_n) \geq 1$$

and

$$\begin{aligned}
p(x_{n+1},x_n) &\le \alpha(x_n,x_{n-1})p(x_{n+1},x_n)\\
&\le \frac{(1-k)}{2}[p^*(Tx_n,x_n)+p^*(Tx_{n-1},x_{n-1})]-\theta([p^*(Tx_n,x_n),p^*(Tx_{n-1},x_{n-1},)])\\
&= \frac{(1-k)}{2}[p(Tx_n,x_n)+p(Tx_{n-1},x_{n-1})]-p(A,B)-\\
&\quad \theta(p(Tx_n,x_n)-p(A,B),p(Tx_{n-1},x_{n-1})-p(A,B))\\
&\le \frac{(1-k)}{2}[p(Tx_n,x_{n+1})+p(x_{n+1},x_n)+p(Tx_{n-1},x_n)+p(x_n,x_{n-1})] \qquad (11)\\
&\quad -p(A,B)-\theta(p(Tx_n,x_{n+1})+p(x_{n+1},x_n)-p(A,B),\\
&\quad p(Tx_{n-1},x_n)+p(x_n,x_{n-1})-p(A,B))\\
&\le \frac{(1-k)}{2}[p(A,B)+p(x_{n+1},x_n)+p(A,B)+p(x_n,x_{n-1})]-p(A,B)-\\
&\quad \theta(p(x_{n+1},x_n),p(x_n,x_{n-1}))\\
&\le \frac{1}{2}[p(x_{n+1},x_n)+p(x_n,x_{n-1})]-\theta(p(x_{n+1},x_n),p(x_n,x_{n-1}))\\
&\le \frac{1}{2}[p(x_{n+1},x_n)+p(x_n,x_{n-1})],
\end{aligned}$$

which can be written as

$$p(x_{n+1},x_n)\le p(x_n,x_{n-1}),\ \text{for all } n\ge 1. \qquad (12)$$

Therefore the sequence $\{p(x_{n+1},x_n)\}$ is monotone decreasing and bounded below so there exists $r\ge 0$ such that

$$\lim_{n\to\infty} p\left(x_{n+1,}x_n\right)=r.$$

On contrary suppose that $r>0$. By taking limit as $n\to\infty$ and using (12) we have

$$r\le \frac{1}{2}[r+r]-\theta(r,r),$$

implies

$$0\le\theta(r,r)\le 0,$$

hence $\theta(r,r)=0,$ using the definition of θ, which is contradiction. Hence, $r=0$ and we have

$$\lim_{n\to\infty} p\left(x_{n+1,}x_n\right)=0.$$

Now, we claim that the sequence $\{x_n\}$ is a Cauchy sequence. On contrary suppose that the sequence $\{x_n\}$ is not Cauchy sequence. Suppose that $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ are subsequence of $\{x_n\}$, and suppose that there exists a $\epsilon>0$ such that

$$p\left(x_{n(k)},x_{m(k)}\right)\ge\epsilon,$$

and

$$p\left(x_{n(k)}, x_{m(k)-1}\right) < \epsilon. \tag{13}$$

Consider $m_k > n_k > N \in \mathbb{N}$, and using (3) and (13) we obtain,

$$\left.\begin{array}{l}\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1\\ p(x_{n(k)}, Tx_{n(k)-1}) = p(A, B)\\ p(x_{m(k)}, Tx_{m(k)-1}) = p(A, B).\end{array}\right\} \text{implies that } \alpha(x_{n(k)}, x_{m(k)}) \geq 1.$$

As mapping T is weakly Kannan type proximal mapping, then we have

$$\begin{aligned}
p(x_{n(k)}, x_{m(k)}) &\leq \alpha(x_{n(k)-1}, x_{m(k)-1})p(x_{n(k)}, x_{m(k)})\\
&\leq \frac{(1-k)}{2}[p^*(Tx_{n(k)-1}, x_{n(k)-1}) + p^*(x_{m(k)-1}, Tx_{m(k)-1})] -\\
&\quad \theta([p^*(Tx_{n(k)-1}, x_{n(k)-1}), p^*(x_{m(k)-1}, Tx_{m(k)-1})])\\
&= \frac{(1-k)}{2}[p(Tx_{n(k)-1}, x_{n(k)-1}) + p(x_{m(k)-1}, Tx_{m(k)-1})] - p(A, B) -\\
&\quad \theta([p(Tx_{n(k)-1}, x_{n(k)-1}) - p(A, B), p(x_{m(k)-1}, Tx_{m(k)-1}) - p(A, B)])\\
&\leq \frac{(1-k)}{2}[p(Tx_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{n(k)-1}) + p(x_{m(k)}, Tx_{m(k)-1}) +\\
&\quad p(x_{m(k)}, x_{m(k)-1})] - p(A, B) - \theta(p(Tx_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{n(k)-1})\\
&\quad -p(A, B), p(x_{m(k)}, Tx_{m(k)-1}) + p(x_{m(k)}, x_{m(k)-1}) - p(A, B))\\
&\leq \frac{(1-k)}{2}[p(x_{n(k)}, x_{n(k)-1}) + p(x_{m(k)}, x_{m(k)-1})]\\
&\quad -\theta(p(x_{n(k)}, x_{n(k)-1}), p(x_{m(k)}, x_{m(k)-1}))\\
&\leq \frac{1}{2}[p(x_{n(k)}, x_{n(k)-1}) + p(x_{m(k)}, x_{m(k)-1})]\\
&\leq \frac{1}{2}[p(x_{n(k)}, x_{n(k)-1}) + p(x_{m(k)}, x_{m(k)-1}),]
\end{aligned}$$

which can be written as

$$\epsilon \leq p(x_{n(k)}, x_{m(k)}) \leq \frac{1}{2}[p(x_{n(k)}, x_{n(k)-1}) + p(x_{m(k)}, x_{m(k)-1})] \to 0, \text{ as } k \to \infty,$$

which is contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since $\{x_n\}$ is a Cauchy sequence in $A_{0,p}$, where $A_{0,p}$ is closed subset of complete metric space X, so there exists some $u \in A_{0,p}$ such that $x_n \to u$ in $A_{0,p}$. Now, we have to show that u is the best proximity point of T. Since $x_n \to u$ and $p(x, .)$ is lower semi-continuous and we have

$$p(Tu, u) \leq \liminf p(Tu, x_n),$$

We can write as

$$\begin{aligned}
p(x_n, Tu) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, Tx_n) + p(Tx_n, Tu)\\
&\leq p(x_n, x_{n+1}) + p(A, B) + p(Tx_n, Tu).
\end{aligned}$$

After simplification, we have

$$p(Tx_n, Tu) \geq p(x_n, Tu) - p(x_n, x_{n+1}) - p(A, B).$$

Since $x_n \to u$ and $p\ (x, .)$ is lower semi continuous. So

$$\liminf p(Tx_n, Tu) \geq \liminf p(x_n, Tu) - 0 - p(A, B),$$

hence we have

$$0 \geq p(u, Tu) - p(A, B),$$

which further implies that

$$p(u, Tu) = p(A, B).$$

Hence u is the best proximity point of mapping T.

Uniqueness: Let u and v are two distinct best proximity point of the mapping T such that $u \neq v$. Thus, we have

$$r = p(u, v) > 0.$$

Since mapping T is an α_p-admissible

$$\left.\begin{array}{l} \alpha(u, v) \geq 1 \\ p(u, Tu) = p(A, B) \\ p(v, Tv) = p(A, B) \end{array}\right\} \text{ implies that } \alpha(u, v) \geq 1.$$

Since mapping T is a weakly Kannan type and proximal admissible mapping thus by using (3) we get,

$$\begin{aligned} r &\leq \frac{1}{2}[r + r] - \theta(r, r) \\ r &\leq r - \theta(r, r), \end{aligned}$$

we have

$$\theta(r, r) \leq 0,$$

which is contradiction. So, $r = 0$, so we have a unique best proximity points of the mapping T.

4 Results in Partially Ordered Metric Space

In this section, we will discuss the best proximity point results for weakly Kannan type generalized ordered proximal contraction and weakly Kannan type ordered proximal contraction in partially ordered metric space. From now and onward, we will consider the following notion:

$$\Delta = \{(x, y) \in A_0 \times A_0 : x \preceq y \text{ or } y \preceq x\}.$$

Definition 15 ([40]) Let X be a nonempty set. Then $(X, d, \preceq)$ is called a partially ordered metric space if the following conditions are satisfied:

1. d is a metric on X.
2. $\preceq$ is a partial order on X.

Definition 16 *(*[40]*)* A mapping $T : A \to B$ is said to be order preserving if and only if

$$x_1 \preceq x_2 \text{ implies } Tx_1 \preceq Tx_2 \text{ for all } x_1, x_2 \in A.$$

Definition 17 ([40]) A mapping $T : A \to B$ is said to be partially order preserving if and only if

$$\left.\begin{array}{c} x_1 \preceq x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \text{ implies that } u_1 \preceq u_2,$$

for all $u_1, u_2, x_1, x_2 \in A$.

Definition 18 Let $g : A \to A, T : A \to B$ and $\alpha : A \times A \to [0, \infty)$ are mappings, where A and B are nonempty subsets of a partially ordered metric space $(X, d, \preceq)$. A pair of mappings (g, T) is said to be weakly Kannan type generalized ordered proximal contraction, if

$$\left.\begin{array}{c} p(gx, Tu) = p(A, B) \\ p(gy, Tv) = p(A, B) \end{array}\right\} \text{ implies that}$$
$$\alpha(gx, gy)p(gu, gv) \leq \frac{(1-k)}{2}[p^*(Tx, gx) + p^*(Ty, gy)] - \theta(p^*(Tx, gx), p^*(Ty, gy)),$$

where $\theta \in \vartheta$, $k \in (0, 1)$ and for all $(u, v), (x, y) \in \Delta$.

Definition 19 Let $\alpha : A \times A \to [0, \infty)$be a mapping. A mapping $T : A \to B$ is said to be weakly Kannan type ordered proximal contraction, if

$$\left.\begin{array}{c} p(x, Tu) = p(A, B) \\ p(y, Tv) = p(A, B) \end{array}\right\} \text{ implies that}$$
$$\alpha(x, y)p(u, v) \leq \frac{(1-k)}{2}[p^*(Tx, x) + p^*(Ty, y)] - \theta(p^*(Tx, x), p^*(Ty, y)),$$

where $\theta \in \vartheta$, $k \in (0, 1)$ and for all $(u, v), (x, y) \in \Delta$.

Remark 2 If $gx = I_A$ (an identity mapping over set A) then every weakly Kannan type generalized ordered proximal contraction will reduce to weakly Kannan type ordered proximal contraction.

Similarly, we can deduce the ordered version of previous results as following.

Theorem 3 *Let $T : A \to B$ be a proximal ordered preserving mapping and $g : A \to A$ be an one to one mapping on A satisfies the α_R-property, where A and B are nonempty subsets of a complete partially ordered metric space $(X, d, \preceq)$. Also, $T(A_{0,p}) \subseteq B_{0,p}$ and $A_{0,p} \subset g(A_{0,p})$, where p is a w_p-distance with $A_{0,p}$ is nonempty subset of A. The pair of mappings (g, T) is a weakly Kannan type generalized ordered proximal contraction. If there exist some $x_0, x_1 \in A_{0,p}$ such that*

$$p(gx_1, Tx_0) = p(A, B) \text{ and } (x_0, x_1) \in \Delta,$$

then there exists a unique w_p-coincidence best proximity point x^* in A.

Proof Define $\alpha_p : A \times A \to [0, \infty)$, as

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Since T is an α-proximal admissible mapping, defined as

$$\left.\begin{array}{c} \alpha(x_1, x_2) \geq 1 \\ p(gu_1, Tx_1) = p(A, B) \\ p(gu_2, Tx_2) = p(A, B) \end{array}\right\} \text{ implies that } \alpha(gu_1, gu_2) \geq 1.$$

Since g satisfy α_R-property, so $\alpha(u_1, u_2) \geq 1$ equivalently, we have

$$\begin{cases} (x_1, x_2) \in \Delta \\ p(gu_1, Tx_1) = p(A, B), \\ p(gu_2, Tx_2) = p(A, B). \end{cases}$$

Since T is proximally ordered preserving and we have $(u_1, u_2) \in \Delta$, so we have

$$p(gx_1, Tx_0) = p(A, B) \text{ and } \alpha(x_0, x_1) \geq 1.$$

Note that, if $(x, y) \in \Delta$ then $\alpha(x, y) = 1$ otherwise, $\alpha(x, y) = 0$. Since the pair of mapping (g, T) satisfy weakly Kannan type generalized ordered proximal mapping, and we have

$$\left.\begin{array}{c} \alpha(x, y) \geq 1 \\ p(gx, Tu) = p(A, B) \\ p(gy, Tv) = p(A, B) \end{array}\right\} \text{ implies that}$$

$$p(gu, gv) \leq \frac{(1-k)}{2}[p^*(Tx, gx) + p^*(Ty, gy)] - \theta(p^*(Tx, gx), p^*(Ty, gy)).$$

Consider $\{x_n\}$ be a sequence such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ with $x_n \to x$ as $n \to \infty$, hence $(x_n, x_{n+1}) \in \Delta$, for all $n \in \mathbb{N} \cup \{0\}$, with $x_n \to x$ as $n \to \infty$. Hence all conditions of Theorem (1) holds and unique w_p-coincidence best proximity point of mappings (g, T) exist.

Similarly, we can prove the following theorem.

Theorem 4 *Let $T : A \to B$ be a proximal ordered preserving and weakly Kannan type ordered proximal mapping, where A and B are nonempty subsets of a complete partially ordered metric space $(X, d, \preceq)$. Also, $T(A_{0,p}) \subseteq B_{0,p}$ and $A_{0,p}$ is nonempty subset of A. If there exist some $x_0, x_1 \in A_{0,p}$ such that*

$$p(x_1, Tx_0) = p(A, B) \text{ and } (x_0, x_1) \in \Delta,$$

then there exists a unique w_p-best proximity point x^* of T in A.

5 Application to Fixed-Point Theory

In this section, we will prove some results related to the fixed-point theory for weakly Kannan type generalized contraction. Here, if we consider $A = B = X$, then we have the following definitions.

Definition 20 A self-mapping T on X is said to commute with respect to self-mapping g on X, if

$$gu = Tx \text{ and } gv = Ty \text{ implies that } gx = Ty \text{ and } gy = Tx.$$

Example 1 Let$X = \{0, 1, 2, 3, 4\}$ and mappings $g, T : X \to X$ are defined as,

$$g(x) = \begin{cases} 1 + x & x \in \{0, 1\} \\ x - 1 & x \in \{2, 3, 4\} \end{cases} \text{ and } T(x) = \begin{cases} 1 & x = 3 \\ x & x \in \{2, 4\} \\ 1 + x & x \in \{0, 1\}. \end{cases}$$

If we take $u = 0, x = 3, v = 1$ and $y = 2$ then the mapping T commute with respect to g.

Definition 21 Let $\alpha : X \times X \to [0, \infty)$ and $g, T : X \to X$ are mapping. A pair of mappings (g, T) is said to be weakly Kannan type generalized contraction, if

$$\alpha(Tx, Ty)p(Tx, Ty) \leq \frac{(1-k)}{2}[p^*(Tx, Ty) + p^*(Ty, Tx)] - \theta(p^*(Tx, Ty), p^*(Ty, Tx)),$$

where $\theta \in \vartheta, k \in (0, 1)$ and for all $(x, y) \in X$.

Definition 22 Let $\alpha : X \times X \to [0, \infty)$ be a mapping. A mapping $T : X \to X$ is said to be weakly Kannan type contraction, if

$$\alpha(x, y)p(x, y) \leq \frac{(1-k)}{2}[p^*(Tx, Ty) + p^*(Ty, Tx)] - \theta(p^*(Tx, Ty), p^*(Ty, Tx)),$$

where $\theta \in \vartheta$, $k \in (0, 1)$ and for all $(x, y) \in X$.

Now, from Theorem (1) and (2), we can deduce the following results.

Theorem 5 *Let g and $T$are continuous self-mapping on a complete metric space (X, d) and pair of mappings (g, T) be a weakly Kannan type generalized contraction, further mapping T commute with respect to mapping g. If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ then pair (g, T) has a unique coincidence point.*

Proof If we take $A = B = X$ in Theorem (1), since self-mapping T commute with respect to mapping g then every weakly Kannan type generalized proximal contraction becomes weakly Kannan type generalized contraction, for self-mapping every α_p-proximal admissible mapping is a α_p-admissible mapping, all conditions of Theorem (1) are satisfied, so according to Theorem (1), we can find $x \in X$ as a coincidence best proximity point of pair of mapping(g, T), as

$$p(gx, Tx) = p(A, B)$$

butin the case of self-mapping, $p(A, B) = 0 = p(gx, Tx)$, from above equation, (in the case of self-mapping) every weakly Kannan type generalized contraction mappings (g, T) has a unique coincidence point of $gx = Tx$.

Theorem 6 *Let (X, d) be a complete metric space and $T : X \to X$ be a continuous weakly Kannan type contraction, further if there exists $x_0 \in C$ with $\alpha(x_0, Tx_0) \geq 1$ then T has a unique fixed-point.*

Proof If we take $A = B = X$ in Theorem (2) then every weakly Kannan type proximal contraction becomes weakly Kannan type contraction and every α-proximal admissible mapping becomes α-admissible mapping, all conditions of Theorem (2) are satisfied, so according to Theorem (2), we can find x a best proximity point of mapping T such that

$$p(x, Tx) = p(A, B).$$

In case of self-mapping, as $A = B = X$ then $p(A, B) = 0 = p(x, Tx)$, from the above equation, we can say that every weakly Kannan type contractive mapping T has a unique fixed-point.

Definition 23 A pair of self mappings (g, T) on X is said to be a weakly Kannan type generalized ordered contraction, if

$$\alpha(gx, gy)p(Tx, Ty) \leq \frac{(1-k)}{2}[p^*(Tx, Ty) + p^*(Ty, Tx)] - \theta(p^*(Tx, Ty), p^*(Ty, Tx)),$$

where $\theta \in \vartheta$, $k \in (0, 1)$, for all $(x, y) \in \Delta$.

Definition 24 Let $\alpha : X \times X \to [0, \infty)$ be a mapping. A mapping $T : X \to X$ is said to be weakly Kannan type ordered contraction, if

$$\alpha(x, y)p(Tx, Ty) \leq \frac{(1-k)}{2}[p^*(Tx, Ty) + p^*(Ty, Tx)] - \theta(p^*(Tx, Ty), p^*(Ty, Tx)),$$

where $\theta \in \vartheta$, $k \in (0, 1)$, for all $(x, y) \in \Delta$.

Theorem 7 *Let $\alpha : X \times X \to [0, \infty)$, $g : X \to X$ and $T : X \to X$ are mapping and $(X, d, \preceq)$ is a partially ordered complete metric space and (g, T) be a pair of continuous weakly Kannan type generalized ordered contraction and mapping T commute with respect to mapping g. If $x_0 \in X$ and $(x_0, Tx_0) \in \Delta$, then pair (g, T) has a unique coincidence point.*

Proof Following the same lines of proof of Theorem (3), and taking in account for self-mapping such that $(x_0, Tx_0) \in \Delta$, we have $\alpha(x_0, Tx_0) = 1$, then every weakly Kannan type generalized ordered proximal contraction becomes weakly Kannan type generalized ordered contraction. Since mapping T commute with respect to mapping g and remaining conditions of Theorem (3) also holds. Then pair (g, T) has a unique coincidence point.

Theorem 8 *Let $(X, d_b, \preceq)$ is a complete partially ordered metric space and $T : X \to X$ is a weakly Kannan type ordered contraction satisfying the condition of Theorem (7), then T has a unique fixed-point.*

6 Conclusion

In this chapter, we introduced the concept of weakly Kannan type generalized proximal contraction and weakly Kannan type proximal contraction mapping and we obtained some coincidence and best proximity point results in complete metric space using w_p-distance, which extends and generalized the already exiting results in literature [41, 42]. Some applications in fixed-point theory and ordered metric spaces are also discussed.

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Application of Fixed Point Iterative Methods to Construct Fractals and Anti-fractals

Sudesh Kumari, Ashish Nandal, and Renu Chugh

Abstract In the present chapter, we demonstrate an application of fixed point iterative methods to construct fractals (Mandelbrot and Julia sets) and anti-fractals (tricorns, multicorns and Anti-Julia sets) for the complex polynomials and antipolynomials of the type $F_d(z) = z^n + d$ and $A_d(z) = \bar{z}^m + d$, respectively, where $d \in \mathbb{C}$ and $n, m \geq 2$. We derive some escape criteria to generate fractals and anti-fractals by adopting the Suantai type iterative method. Moreover, we graphically visualize and examine the dynamics of these fractals and anti-fractals for certain complex polynomials and antipolynomials, respectively. Several beautiful aesthetic patterns have been obtained which explore the geometry of fractals and anti-fractals and therefore enrich the theory of fixed points.

1 Introduction

Fixed point theory has been applied to investigate various nonlinear phenomena such as complex graphics, biology, physics and geometry [1–4]. There are several iterative methods in the literature for which the fixed points of operators have been approximated over the years by various authors. Some of well-known fixed point iterative methods are Mann [5], Ishikawa [6], Khan [7], Noor [8], Suantai [9], SP [10], Agarwal [11] and CR [12].These fixed point iterative methods have been used to generate various complex graphics like fractals and anti-fractals, e.g., the Mann iteration [13–16], Ishikawa-iteration [17, 18], S-iteration [19, 20], Noor-iteration [21, 22], CR-iteration [23, 24] and SP-iteration [25–27]. Thus, fixed point theory plays a prominent role to construct beautiful graphics of fractals and anti-fractals

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where fixed point iterative methods are used. Fixed point iterative methods assist in constructing beautiful graphics of fractals and anti-fractals that have been used in image encryption [28] or compression [29], cryptography [30] and art and design [31]. The applications of fractal theory in the fields of electrical and electronics engineering revolutionized the industry of security control system, capacitors, radar system, radio and antennae for wireless system [32, 33]. Moreover, architects and engineers apply fractal theory to sketch and design the maps of different projects [34]. Consequently, fractals and anti-fractals enriched the theory of fixed points in the form of aesthetic patterns of complex graphics [1].

Fractals are defined as objects having irregular structures that cannot be completely described in Euclidean geometric language. Benoit B. Mandelbrot [35] described fractal as a fragmented geometric shape that contains congruent pieces, each of which is a reduced size copy of the original one. In 1918, French mathematician Gaston Julia [36] firstly introduced the iterative procedure for complex polynomial $z^2 + c;\ c \in \mathbb{C}$ and derived the Julia set. Afterward, the work of Julia was extended by French Mathematician Mandelbrot [35] and obtained beautiful graphics named as Mandelbrot sets with the help of computers. Julia sets and Mandelbrot sets are examples of classical fractals. Mandelbrot and Julia sets were extended from the complex numbers to quaternions [37], bicomplex numbers [38], tricomplex numbers [39], etc. Rani and Kumar [13, 14] defined superior iterate to obtain superior Mandelbrot sets and Julia sets for complex valued polynomials. In 2009, Rochon [40] generated Mandelbrot sets in bicomplex plane. Thereafter, Wang et al. [41–44] extended the work of Rochon [40] to generate the graphics of fractals. In 2010, Chauhan et al. [17, 45] used the Ishikawa-iteration to study dynamics of superior Julia and superior Mandelbrot sets. Ashish et al. [21] applied the Noor iterative method to generate Julia and Mandelbrot sets. Kang et al. [46] investigated the modified Ishikawa process and S-iteration to study the relative superior Mandelbrot sets. Kumari et al. [25–27] used four-step iterative methods to construct fractals. Recently, Abbas et al. [47] used the Picard-Ishikawa type iterative method to generate fractals.

Moreover, anti-fractals like tricorns, multicorns and Anti-Julia sets are the dynamics of antiholomorphic complex polynomials of the form $A_d(z) = \bar{z}^m + d;\ m \geq 2$ and $d \in \mathbb{C}$ [48]. The term "tricorn" was coined by Milnor. Milnor [49] and Branner [50] found multicorns in a real slice of cubic connectedness locus. Lau and Schleicher [51] analyzed the symmetries of tricorns and multicorns. Main features of tricorns and multicorns were explained by Nakane and Schleicher [48] together with beautiful figures. The Mann iteration was considered by Rani [15, 16] to investigate multicorns and Anti-Julia sets for complex polynomials $\bar{z}^m + d;\ m \geq 2$. Some fixed point results for anti-fractals had been proved by Mishra et al. [18] by using the Ishikawa iterate with s-convexity. Further, the dynamics of the anti-fractals had been analyzed by Chugh et al. [22], Chauhan et al. [52], Kang et al. [19], Partap et al. [53], Kwun et al. [23], Li et al. [24] and Chen et al. [20] by applying various fixed point iterative methods.

In the present chapter, the graphical behavior of fractals and anti-fractals have been discussed and visualized for the complex polynomials of the form $F_d(z) = z^n + d$ and $A_d(z) = \bar{z}^m + d$, respectively, where $d \in \mathbb{C}$ and $n, m \geq 2$ via the Suantai type fixed point iterative method. The Suantai type iterative method was introduced by Suantai [9] in 2005. The significance of the Suantai type iterative procedure lies in the fact that it includes several important iterative procedures like Picard, Mann, Ishikawa, Khan, Noor, etc. It has been observed that fractals (Mandelbrot and Julia sets) and anti-fractals (tricorns, multicorns and Anti-Julia sets) generated by us have comparatively distinctive shapes to already generated fractals and anti-fractals in the literature.

2 Preliminaries

This section is dedicated to some basic definitions which are prerequisites for further work.

Definition 1 (*Orbit*) [54]. The orbit of a point $x_0 \in \mathbb{C}$ under a mapping $g\colon \mathbb{C} \to \mathbb{C}$ is defined as a sequence of points

$$x_0,\ x_1 = g(x_0),\ x_2 = g^2(x_0), \ldots,\ x_n = g^n(x_0), \ldots .$$

Definition 2 (*Julia Set*) [35]. The Julia set of a function $g\colon \mathbb{C} \to \mathbb{C}$ is the boundary of the set of points $z \in \mathbb{C}$ that tends to infinity under repeated iteration by $g(z)$, i.e., for a function g, the Julia set is given by

$$J(g) = \partial\{z \in \mathbb{C} : g^n(z) \to \infty \ as\ n \to \infty\},$$

where $g^n(z)$ denotes the nth iteration of function g.

Definition 3 (*Mandelbrot Set*) [36]. The Mandelbrot set M is defined as the collection of all numbers $z \in \mathbb{C}$ for which the Julia set remains connected, i.e.,

$$M = \{z \in \mathbb{C} : J(g)\ is\ connected\}.$$

Definition 4 (*Multicorn*) [54]. The multicorn A_d for the function $A_d(z) = \bar{z}^m + d;\ m \geq 2$ is described as the collection of all $d \in \mathbb{C}$ such that the orbit of the point 0 is bounded, i.e.,

$$A_d = \{d \in \mathbb{C} : A_d^n(0)\ does\ not\ tend\ to\ \infty\},$$

where $A_d^n(0)$ represents the nth iteration of the function $A_d(z)$. Equivalently, multicorns can be defined as the connectedness of loci for higher degree antiholomorphic polynomials $A_d(z) = \bar{z}^m + d$. It is notable that tricorns are the reduced form of multicorns when m = 2.

Definition 5 (*Suantai Type Orbit*) [9] Let $Z \subseteq \mathbb{C}$ and $F_d\colon Z \to Z$ be a self map. A sequence $\{z_j\}$ of iterations is known as the Suantai type orbit for an initial point

$z_0 \in Z$ if it is defined by

$$\begin{aligned} z_{j+1} &= (1-\lambda_j-\mu_j)\, z_j + \lambda_j F_d(y_j) + \mu_j\; F_d(z_j), \\ y_j &= (1-\alpha_j-\beta_j)\, z_j + \alpha_j F_d(x_j) + \beta_j F_d(z_j), \\ x_j &= (1-\nu_j)\, z_j + \nu_j F_d(z_j),\;\; j = 0, 1, 2, \ldots, \end{aligned} \tag{1}$$

where $\lambda_j, \mu_j, \nu_j, \alpha_j, \beta_j \in [0, 1]$, $\lambda_j + \mu_j \in [0, 1]$, $\alpha_j + \beta_j \in [0, 1]$ for all $j \in \mathbb{N}$ and $\sum_{j=0}^{\infty}(\lambda_j + \mu_j) = \infty$.

Remark 1 The Suantai type orbit reduces to the

- Noor orbit when $\mu_j = \beta_j = 0,\; \lambda_j = \alpha_j = \nu_j \neq 0$.
- Khan orbit when $\lambda_j = 1,\; \mu_j = \alpha_j = \nu_j = 0,\; \beta_j \neq 0$.
- Ishikawa orbit when $\mu_j = \alpha_j = \nu_j = 0,\; \beta_j \neq 0,\; \lambda_j \neq 0$.
- Mann orbit when $\lambda_j = \alpha_j = \beta_j = \nu_j = 0,\; \mu_j \neq 0$.
- Picard orbit when $\mu_j = 1,\; \lambda_j = \alpha_j = \beta_j = \nu_j = 0$.

Throughout the present chapter, we assume that $z_0 = z,\;\; y_0 = y,\;\; x_0 = x, wherex, y, z \in \mathbb{C}$ and $\lambda_j = \lambda,\;\; \mu_j = \mu,\;\; \nu_j = \nu,\;\; \alpha_j = \alpha,\;\; \beta_j = \beta$ where $\lambda,\; \mu,\; \nu,\; \alpha,\; \beta \in (0, 1]$.

3 Main Results

The Suantai type orbit for any initial point $z_0 \in \mathbb{C}$ can be expressed as follows :

$$\begin{aligned} z_{j+1} &= (1-\lambda-\mu)\, z_j + \lambda F_d(y_j) + \mu\; F_d(z_j), \\ y_j &= (1-\alpha-\beta)\, z_j + \alpha F_d(x_j) + \beta F_d(z_j), \\ x_j &= (1-\nu)\, z_j + \nu F_d(z_j),\;\; j = 0, 1, 2, \ldots, \end{aligned} \tag{2}$$

where $F_d(z_j)$ can be a quadratic, cubic or higher degree complex polynomial and $0 < \lambda,\; \mu,\; \nu,\; \alpha,\; \beta \leq 1$.

Various techniques including escape criterion, iterated function systems, random fractals, etc. have been adopted to generate and analyze fractals. The escape criterion has a renowned place in the generation of fractals. The escape criterion is a stopping criterion that depends on the number of iterations required to find out whether the orbit of an initial point tends to infinity or not. This criterion is proved as an appropriate mechanism to demonstrate some attributes of a dynamic system using iteration procedures. Now, we prove the escape criterion for quadratic, cubic and higher degree complex polynomials to construct Julia and Mandelbrot sets under the Suantai type orbit.

3.1 Escape Criterion for Quadratic Complex Polynomials

Let $F_d(z) = z^2 + d$ be a quadratic complex polynomial, where d is a complex number, then escape criterion for quadratic complex polynomial is given by the following.

Theorem 1 *Assume that* $|z| \geq |d| > \max\left\{\frac{2}{|\lambda|-|\mu|}, \frac{2}{|\alpha|-|\beta|}, \frac{2}{|\nu|}\right\}$, *where* $\lambda, \mu, \nu, \alpha, \beta \in (0, 1]$ *and* d *is a complex number. Consider the sequence* $\{z_j\}_{j\in\mathbb{N}}$ *as given in (2), then the orbit of* $|z|$ *tends to infinity, i.e.,* $|z_j| \to \infty$ *as* $j \to \infty$.

Proof From (2), consider

$$|x| = |(1-\nu)z + \nu F_d(z)|.$$

As $F_d(z) = z^2 + d$, we have

$$\begin{aligned} |x| &= |(1-\nu)z + \nu(z^2 + d)| \\ &\geq |(1-\nu)z + \nu z^2| - |\nu d|. \end{aligned}$$

As $|z| \geq |d|$, we obtain

$$\begin{aligned} |x| &\geq |(1-\nu)\, z + \nu z^2| - |\nu||z| \\ &\geq |\nu z^2| - |(1-\nu)z| - |\nu||z| \\ &= |\nu||z|^2 - |z| + |\nu||z| - |\nu||z| \\ &= |z|(|\nu||z| - 1). \end{aligned}$$

Therefore,

$$|x| \geq |z|(|\nu||z| - 1). \tag{3}$$

The assumption $|z| > \frac{2}{|\nu|}$ gives

$$|\nu|\,|z| - 1 > 1. \tag{4}$$

Now, (3) becomes

$$|x| > |z|. \tag{5}$$

Also, from (2),

$$\begin{aligned} |y| &= |(1-\alpha-\beta)z + \alpha F_d(x) + \beta F_d(z)| \\ &= |(1-\alpha-\beta)z + \alpha(x^2 + d) + \beta(z^2 + d)| \\ &\geq |\alpha(x^2 + d)| - |\beta(z^2 + d)| - |(1-\alpha-\beta)z| \\ &\geq |\alpha||x|^2 - |\alpha||d| - |\beta||z|^2 - |\beta||d| - |(1-\alpha-\beta)z|. \end{aligned}$$

Using the assumption that $|z| \geq |d|$, we get

$$|y| \geq |\alpha||x|^2 - |\alpha||z| - |\beta||z|^2 - |\beta||z| - |(1-\alpha-\beta)z|.$$

From (5), we have

$$\begin{aligned}|y| &> |\alpha||z|^2 - |\alpha||z| - |\beta||z|^2 - |\beta||z| - |(1-\alpha-\beta)z| \\ &= |z|^2(|\alpha| - |\beta|) - |z|.\end{aligned}$$

Thus,

$$|y| > |z|\big((|\alpha| - |\beta|)|z| - 1\big). \tag{6}$$

Also, since $|z| > \frac{2}{|\alpha|-|\beta|}$, this gives

$$\big((|\alpha| - |\beta|)|z| - 1\big) > 1. \tag{7}$$

Then, (6) reduces to

$$|y| > |z|. \tag{8}$$

Now, for $z_{j+1} = (1-\lambda-\mu)z_j + \lambda F_d(y_j) + \mu F_d(z_j)$, we have

$$\begin{aligned}|z_1| &= |(1-\lambda-\mu)z + \lambda F_d(y) + \mu F_d(z)| \\ &= |(1-\lambda-\mu)z + \lambda(y^2+d) + \mu(z^2+d)| \\ &\geq |\lambda(y^2+d)| - |\mu(z^2+d)| - |(1-\lambda-\mu)z| \\ &\geq |\lambda||y|^2 - |\lambda||d| - |\mu||z|^2 - |\mu||d| - |(1-\lambda-\mu)z|.\end{aligned}$$

Using the assumption $|z| \geq |d|$, we obtain

$$|z_1| \geq |\lambda||y|^2 - |\lambda||z| - |\mu||z|^2 - |\mu||z| - |(1-\lambda-\mu)z|.$$

Now, (8) gives

$$\begin{aligned}|z_1| &> |\lambda||z|^2 - |\lambda||z| - |\mu||z|^2 - |\mu||z| - |(1-\lambda-\mu)z| \\ &= |z|^2(|\lambda| - |\mu|) - |z| \\ &= |z|\{(|\lambda| - |\mu|)|z| - 1\}.\end{aligned}$$

Since $|z| > \frac{2}{|\lambda|-|\mu|}$, we have $(|\lambda| - |\mu|)|z| - 1 > 1$. Thus, there exists a real number $\sigma > 0$ such that $(|\lambda| - |\mu|)|z| - 1 > \sigma + 1 > 1$.
Consequently, this gives

$$|z_1| > (\sigma + 1)|z|.$$

Particularly,

$$|z_1| > |z|.$$

Continuing this process j times, we have

$$|z_j| > (1+\sigma)^j |z|.$$

Hence, $|z_j| \to \infty$ as $j \to \infty$.

The escape criterion proved in Theorem 1 gives us a little more information. In the proof, we have used only the fact that $|z| \geq |d|$ and $|d| > \frac{2}{|\lambda|-|\mu|}$, $|d| > \frac{2}{|\alpha|-|\beta|}$ and $|d| > \frac{2}{|\nu|}$. Thus, the following corollary is obtained as a refinement of Theorem 1.

Corollary 1 *Assume that* $|z| > \max\left\{|d|, \frac{2}{|\lambda|-|\mu|}, \frac{2}{|\alpha|-|\beta|}, \frac{2}{|\nu|}\right\}$, *then* $|z_j| \to \infty$ *as* $j \to \infty$.

3.2 Escape Criterion for Cubic Complex Polynomials

Now, we derive the following escape criterion for a cubic complex polynomial $F_d(z) = z^3 + d$ where $d \in \mathbb{C}$.

Theorem 2 *Suppose* $|z| \geq |d| > max\left\{\left(\frac{2}{|\nu|}\right)^{1/2}, \left(\frac{2}{|\alpha|-|\beta|}\right)^{1/2}, \left(\frac{2}{|\lambda|-|\mu|}\right)^{1/2}\right\}$ *where* $\lambda, \mu, \nu, \alpha, \beta \in (0, 1]$ *and d is a complex number. Consider the sequence* $\{z_j\}_{j\in\mathbb{N}}$ *as defined in (2), then* $|z_j| \to \infty$ *as* $j \to \infty$.

Proof From (2), consider

$$|x| = |(1-\nu)z + \nu F_d(z)|.$$

As $F_d(z) = z^3 + d$, we have

$$\begin{aligned} |x| &= |(1-\nu)z + \nu(z^3 + d)| \\ &\geq |(1-\nu)z + \nu z^3| - |\nu\, d|. \end{aligned}$$

As $|z| \geq |d|$, we obtain

$$\begin{aligned} |x| &\geq |(1-\nu)\, z + \nu z^3| - |\nu||z| \\ &\geq |\nu z^3| - |(1-\nu)z| - |\nu||z| \\ &= |\nu||z|^3 - |z| + |\nu||z| - |\nu||z| \\ &= |z|(|\nu||z|^2 - 1). \end{aligned}$$

Therefore,

$$|x| \geq |z|(|\nu||z|^2 - 1). \tag{9}$$

The assumption $|z| > (\frac{2}{|\nu|})^{1/2}$ gives

$$|\nu||z|^2 - 1 > 1. \tag{10}$$

Now, (9) becomes

$$|x| > |z|. \tag{11}$$

Also, from (2),

$$\begin{aligned} |y| &= |(1-\alpha-\beta)z + \alpha F_d(x) + \beta F_d(z)| \\ &= |(1-\alpha-\beta)z + \alpha(x^3+d) + \beta(z^3+d)| \\ &\geq |\alpha(x^3+d)| - |\beta(z^3+d)| - |(1-\alpha-\beta)z| \\ &\geq |\alpha||x|^3 - |\alpha||d| - |\beta||z|^3 - |\beta||d| - |(1-\alpha-\beta)z|. \end{aligned}$$

Using the assumption that $|z| \geq |d|$, we obtain

$$|y| \geq |\alpha||x|^3 - |\alpha||z| - |\beta||z|^3 - |\beta||z| - |(1-\alpha-\beta)z|.$$

From (11), we have

$$\begin{aligned} |y| &> |\alpha||z|^3 - |\alpha||z| - |\beta||z|^3 - |\beta||z| - |(1-\alpha-\beta)z| \\ &= |z|^3(|\alpha| - |\beta|) - |z|. \end{aligned}$$

Thus,

$$|y| > |z|\big((|\alpha| - |\beta|)|z|^2 - 1\big). \tag{12}$$

Also, since $|z| > (\frac{2}{|\alpha|-|\beta|})^{1/2}$, this gives

$$(|\alpha| - |\beta|)|z|^2 - 1 > 1. \tag{13}$$

Then, (12) becomes

$$|y| > |z|. \tag{14}$$

Finally, we have

$$
\begin{aligned}
|z_1| &= |(1-\lambda-\mu)z + \lambda F_d(y) + \mu F_d(z)| \\
&= |(1-\lambda-\mu)z + \lambda(y^3+d) + \mu(z^3+d)| \\
&\geq |\lambda(y^3+d)| - |\mu(z^3+d)| - |(1-\lambda-\mu)z| \\
&\geq |\lambda||y|^3 - |\lambda||d| - |\mu||z|^3 - |\mu||d| - |(1-\lambda-\mu)z|.
\end{aligned}
$$

The assumption $|z| \geq |d|$ gives

$$|z_1| \geq |\lambda||y|^3 - |\lambda||z| - |\mu||z|^3 - |\mu||z| - |(1-\lambda-\mu)z|.$$

Now, (14) gives

$$
\begin{aligned}
|z_1| &> |\lambda||z|^3 - |\lambda||z| - |\mu||z|^3 - |\mu||z| - |(1-\lambda-\mu)z| \\
&= |z|^3(|\lambda| - |\mu|) - |z| \\
&= |z|\{(|\lambda| - |\mu|)|z|^2 - 1\}.
\end{aligned}
$$

Since $|z| > (\frac{2}{|\lambda|-|\mu|})^{1/2}$, we have $(|\lambda| - |\mu|)|z|^2 - 1 > 1$. Thus, there exists a real number $\sigma > 0$ such that $(|\lambda| - |\mu|)|z|^2 - 1 > \sigma + 1 > 1$.
Consequently, this gives

$$|z_1| > (\sigma + 1)|z|.$$

In particular,

$$|z_1| > |z|.$$

Repeating this process j times, we have

$$|z_j| > (1+\sigma)^j |z|.$$

Hence, $|z_j| \to \infty$ as $j \to \infty$.

3.3 Escape Criterion for General Complex Polynomials

Now, we prove the following theorem for general complex polynomials $F_d(z) = z^n + d;\ n = 2, 3, \ldots,$ where $d \in \mathbb{C}$ as a general escape criterion:

Theorem 3 *Assume*

$$|z| \geq |d| > \max\Big\{\Big(\frac{2}{|\nu|}\Big)^{1/n-1}, \Big(\frac{2}{|\alpha| - |\beta|}\Big)^{1/n-1}, \Big(\frac{2}{|\lambda| - |\mu|}\Big)^{1/n-1}\Big\}$$

where $\lambda, \mu, \nu, \alpha, \beta \in (0, 1]$. *Define a sequence* $\{z_j\}_{j\in\mathbb{N}}$ *as given in (2) where* $z_0 = z,\ y_0 = y$ *and* $x_0 = x$. *Then* $|z_j| \to \infty$ *as* $j \to \infty$.

Proof We prove the theorem by using the method of induction. For $n = 2,\ F_d(z) = z^2 + d$, the escape criterion takes the form

$$|z| \geq |d| > \max\{\frac{2}{|\lambda| - |\mu|}, \frac{2}{|\alpha| - |\beta|}, \frac{2}{|\nu|}\}.$$

Thus, from Theorem 1, $|z_j| \to \infty$ as $j \to \infty$. Similarly, for n=3, we get $F_d(z) = z^3 + d$. Then, the escape criterion is

$$|z| > \max\{|c|, (\frac{2(1+|a|)}{s\alpha})^{1/2}, (\frac{(2(1+|a|)}{s\beta})^{1/2}, (\frac{2(1+|a|)}{s\gamma})^{1/2}\}.$$

Theorem 2 implies that $|z_j| \to \infty$ as $j \to \infty$. Hence, the result holds for $n = 2, 3$. Now, let the results hold for any n. We shall prove the result for $n + 1$. Let us assume that
$F_d(z) = z^{n+1} + d$ and $|z| \geq |d| > \max\left\{\left(\frac{2}{|\nu|}\right)^{1/n},\ \left(\frac{2}{|\alpha|-|\beta|}\right)^{1/n},\ \left(\frac{2}{|\lambda|-|\mu|}\right)^{1/n}\right\}$.
Then, from (2), consider

$$\begin{aligned} |x| &= |(1-\nu)z + \nu F_d(z)| \\ &= |(1-\nu)z + \nu(z^{n+1} + d)| \\ &\geq |(1-\nu)z + \nu z^{n+1}| - |\nu\, d|. \end{aligned}$$

The assumption $|z| \geq |d|$ yields

$$\begin{aligned} |x| &\geq |(1-\nu)\, z + \nu z^{n+1}| - |\nu||z| \\ &\geq |\nu z^{n+1}| - |(1-\nu)z| - |\nu||z| \\ &= |\nu||z|^{n+1} - |z| + |\nu||z| - |\nu||z| \\ &= |z|(|\nu||z|^n - 1). \end{aligned}$$

Thus,

$$|x| \geq |z|(|\nu||z|^n - 1). \tag{15}$$

The assumption $|z| > (\frac{2}{|\nu|})^{1/n}$ implies

$$|\nu||z|^n - 1 > 1. \tag{16}$$

Now, (15) becomes

$$|x| > |z|. \tag{17}$$

Also, from (2),

$$\begin{aligned}|y| &= |(1-\alpha-\beta)z + \alpha F_d(x) + \beta F_d(z)|\\ &= |(1-\alpha-\beta)z + \alpha(x^{n+1}+d) + \beta(z^{n+1}+d)|\\ &\geq |\alpha||x|^{n+1} - |\alpha||d| - |\beta||z|^{n+1} - |\beta||d| - |(1-\alpha-\beta)z|.\end{aligned}$$

Using the assumption $|z| \geq |d|$ and (17), we have

$$\begin{aligned}|y| &> |\alpha||z|^{n+1} - |\alpha||z| - |\beta||z|^{n+1} - |\beta||z| - |(1-\alpha-\beta)z|\\ &= |z|^{n+1}(|\alpha|-|\beta|) - |z|.\end{aligned}$$

Thus,

$$|y| > |z|\big((|\alpha|-|\beta|)|z|^n - 1\big). \tag{18}$$

Also, the assumption $|z| > (\frac{2}{|\alpha|-|\beta|})^{1/n}$ gives

$$(|\alpha|-|\beta|)|z|^n - 1 > 1. \tag{19}$$

Then, (18) becomes

$$|y| > |z|. \tag{20}$$

Finally, consider

$$\begin{aligned}|z_1| &= |(1-\lambda-\mu)z + \lambda F_d(y) + \mu F_d(z)|\\ &= |(1-\lambda-\mu)z + \lambda(y^{n+1}+d) + \mu(z^{n+1}+d)|\\ &\geq |\lambda||y|^{n+1} - |\lambda||d| - |\mu||z|^{n+1} - |\mu||d| - |(1-\lambda-\mu)z|.\end{aligned}$$

The assumption $|z| \geq |d|$ implies

$$|z_1| \geq |\lambda||y|^{n+1} - |\lambda||z| - |\mu||z|^{n+1} - |\mu||z| - |(1-\lambda-\mu)z|. \tag{21}$$

Using (20) in (21), we obtain

$$\begin{aligned}|z_1| &> |\lambda||z|^{n+1} - |\lambda||z| - |\mu||z|^{n+1} - |\mu||z| - |(1-\lambda-\mu)z|\\ &= |z|^{n+1}(|\lambda|-|\mu|) - |z|\\ &= |z|\{(|\lambda|-|\mu|)|z|^n - 1\}.\end{aligned}$$

Moreover, our assumption $|z| > (\frac{2}{|\lambda|-|\mu|})^{1/n}$ gives that $(|\lambda|-|\mu|)|z|^n - 1 > 1$. Thus, there exists a real number $\sigma > 0$ such that $(|\lambda|-|\mu|)|z|^n - 1 > \sigma + 1 > 1$. Finally, this gives

$$|z_1| > (\sigma+1)|z|.$$

In particular,

$$|z_1| > |z|.$$

Repeating this process in the same manner, we have

$$|z_j| > (1+\sigma)^j |z|.$$

Therefore, $|z_j| \to \infty$ as $j \to \infty$. Thus, result is true for $n+1$. Hence, result holds for any n.

Corollary 2 *Suppose that if for some $p \geq 0$ and $k \geq 2$,*

$$|z_p| > \max\{|d|, \big(\frac{2}{|\nu|}\big)^{1/k-1}, \big(\frac{2}{|\alpha|-|\beta|}\big)^{1/k-1}, \big(\frac{2}{|\lambda|-|\mu|}\big)^{1/k-1}\},$$

then $|z_p| > (1+\sigma)^p |z|$ and $|z_p| \to \infty$ as $p \to \infty$.

Using this corollary, we obtain an algorithm to generate connected Julia sets of general complex polynomials $F_d(z)$; $d \in \mathbb{C}$. If for some j, $\{z_j\}$ lies outside the circle of radius $\max\{|d|, \big(\frac{2}{|\nu|}\big)^{1/p-1}, \big(\frac{2}{|\alpha|-|\beta|}\big)^{1/p-1}, \big(\frac{2}{|\lambda|-|\mu|}\big)^{1/p-1}\}$, then the orbit of z escapes to infinity, which means the point z does not lie in the connected Julia set. If $\{z_j\}$ does not exceed this bound, then by definition, z lies in the connected Julia set and collection of such points is known as the Mandelbrot set.

4 Algorithm for Generating Fractals

Now, we provide an algorithm to construct all fractals by using general escape criterion.

1. **Setup**:

   Choose a complex number $d = l + m\iota$.
   Initialize values of variables $\lambda,\ \mu,\ \nu,\ \alpha,\ \beta$.
   Take $z_0 = x + y\iota$ as an initial point.
2. **Iterate**:

$$\begin{aligned} z_{j+1} &= (1-\lambda-\mu)\, z_j + \lambda F_d(y_j) + \mu\, F_d(z_j), \\ y_j &= (1-\alpha-\beta)\, z_j + \alpha F_d(x_j) + \beta F_d(z_j), \\ x_j &= (1-\nu)\, z_j + \nu F_d(z_j),\ \ j = 0, 1, 2, \ldots, \end{aligned}$$

   where $F_d(z) = z^n + d,\ n = 2, 3, \ldots.$

3. **Stop**:

$|z_j| >$ escape radius

$$= \max\left\{|d|, \left(\frac{2}{|\nu|}\right)^{1/n-1}, \left(\frac{2}{|\alpha|-|\beta|}\right)^{1/n-1}, \left(\frac{2}{|\lambda|-|\mu|}\right)^{1/n-1}\right\}.$$

4. **Count**: The number of iterations to escape.
5. **Color**: Depends on the number of iterations required to escape.

5 Construction of Mandelbrot Sets in the Suantai Type Orbit

This section deals with the construction of Mandelbrot and Julia sets for quadratic and higher degree complex polynomials. The color and shape variations exhibit in the figures when we change the parameters. We have used a maximum of 15 number of iterations to visualize the fractals. With the help of the above algorithm, we construct the following Mandelbrot and Julia sets in the Suantai type orbit by using software Mathematica 11.0.

5.1 Mandelbrot Sets for Quadratic Polynomials

We consider quadratic complex polynomial $F_d(z) = z^2 + d;\ d \in \mathbb{C}$ and vary the values of parameters $\lambda, \mu, \nu, \alpha, \beta \in (0, 1]$ to observe the effect of parameters on shapes of quadratic Mandelbrot sets.

In Figs. 1, 2, 3, 4, 5 and 6, we generate Quadratic Mandelbrot sets by taking different values of parameters $\lambda,\ \mu,\ \nu,\ \alpha,\ \beta$ and observe that the shape becomes fattier when we increase the values of parameters. The Mandelbrot set 1 represents a classical Cardioid together with a large bulb on its left side. Also, infinite many small bulbs are attached around its perimeter where each bulb contains its own antennas. Figures 3 and 4 demonstrate the different Mandelbrot sets containing different bulbs. Figure 5 represents a fattier Mandelbrot set while a lengthy Mandelbrot set is represented by Fig. 6. We construct different shapes of quadratic Mandelbrot sets which all are symmetrical about x-axis.

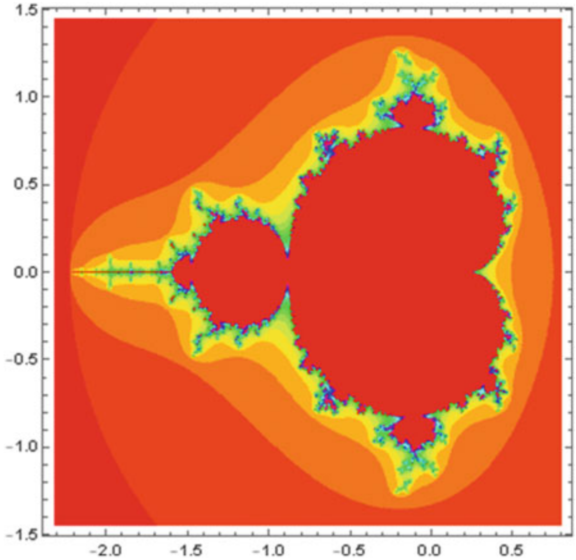


Fig. 1 Quadratic Mandelbrot set for $\lambda = 0.2,\ \mu = 0.7, \nu = 0.9,\ \alpha = 0.1,\ \beta = 0.8$

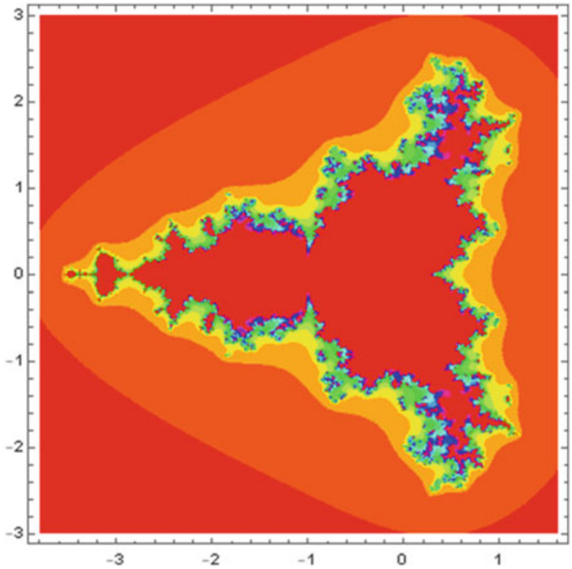


Fig. 2 Quadratic Mandelbrot set for $\lambda = 0.9,\ \mu = 0.04,\ \nu = 0.9,\ \alpha = 0.09,\ \beta = 0.01$

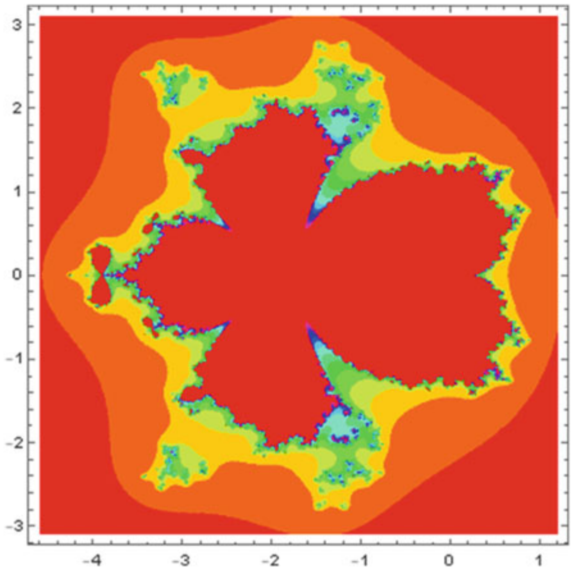


Fig. 3 Quadratic Mandelbrot set for $\lambda = 0.58,\ \mu = 0.18,\ \nu = 0.3,\ \alpha = 0.58,\ \beta = 0.37$

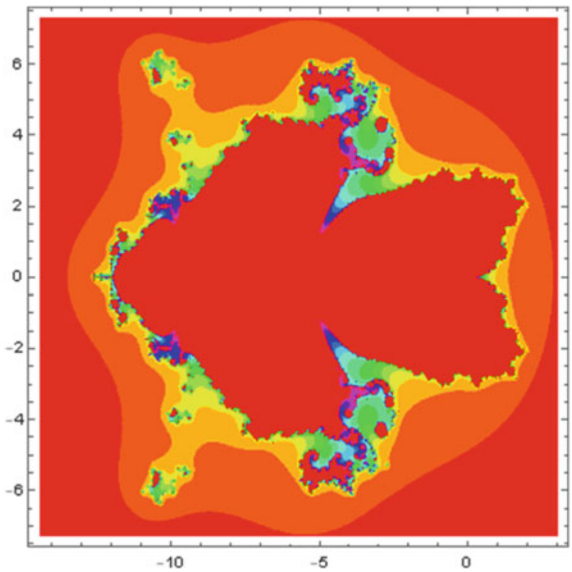


Fig. 4 Quadratic Mandelbrot set for $\lambda = 0.3,\ \mu = 0.1,\ \nu = 0.09,\ \alpha = 0.3,\ \beta = 0.12$

Fig. 5 Quadratic Mandelbrot set for $\lambda = 0.08,\ \mu = 0.01,\ \nu = 0.9,\ \alpha = 0.38,\ \beta = 0.19$

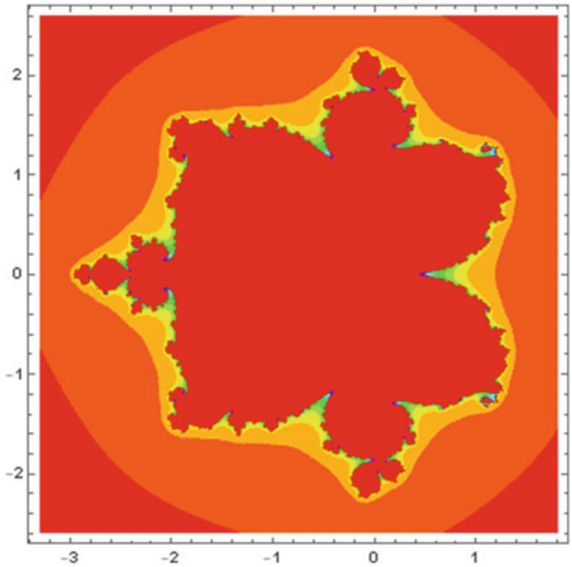
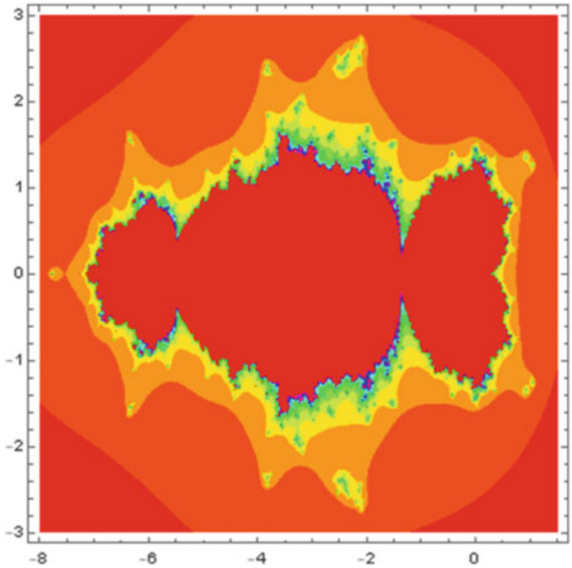


Fig. 6 Quadratic Mandelbrot set for $\lambda = 0.78,\ \mu = 0.04,\ \nu = 0.1,\ \alpha = 0.76,\ \beta = 0.03$



5.2 Mandelbrot Sets for Higher Degree Polynomials

To visualize the Mandelbrot sets for higher degree complex polynomials, we consider the general polynomial $F_d(z) = z^n + d$; $n \geq 3$, $d \in \mathbb{C}$ and taking different values of parameters $\lambda, \mu, \nu, \alpha, \beta \in (0, 1]$. We observe that these fractals have reflection symmetry with respect to both the axis, i.e., x-axis and y-axis.

- For the cubic polynomial $F_d(z) = z^3 + d$, the cubic Mandelbrot sets have been visualized in Figs. 7 and 8 by taking different values of parameters which look like decorated coupled urns.
- In Fig. 9, the Mandelbrot set for fifth-degree polynomial $F_d(z) = z^5 + d$ is shown while Fig. 10 represents a Mandelbrot set for tenth-degree polynomial $F_c(z) = z^{10} + d$.
- Mandelbrot sets for higher values of n, i.e., $n = 15$ and $n = 20$ are visualized by Figs. 11 and 12, respectively, which are similar to a circular saw.

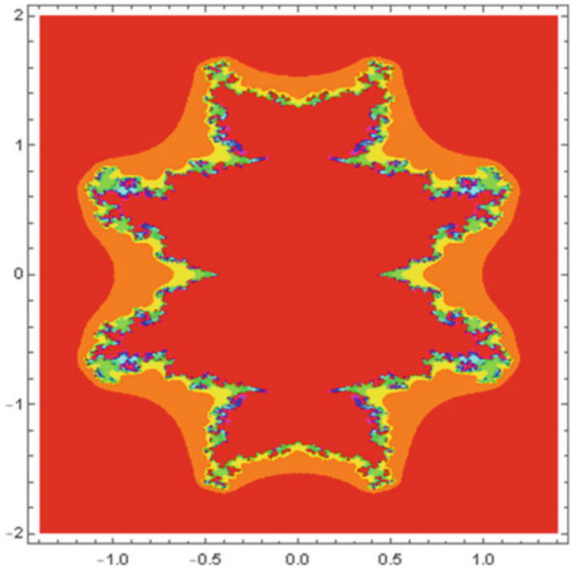


Fig. 7 Cubic Mandelbrot set for $\lambda = 0.9,\ \mu = 0.04,\ \nu = 0.9,\ \alpha = 0.09,\ \beta = 0.01$

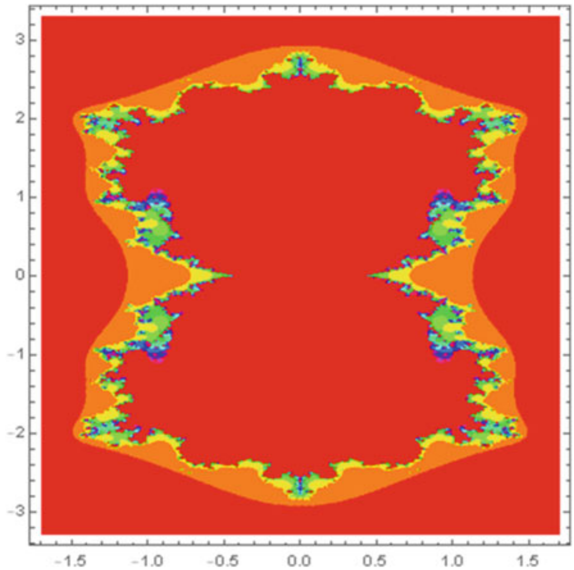


Fig. 8 Cubic Mandelbrot set for $\lambda = 0.53$, $\mu = 0.42$, $\nu = 0.08$, $\alpha = 0.33$, $\beta = 0.01$

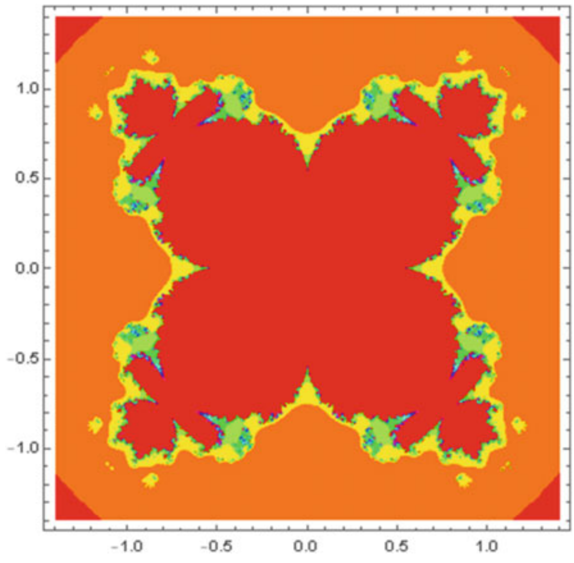


Fig. 9 Fifth-order Mandelbrot set for $\lambda = 0.79,\ \mu = 0.12,\ \nu = 0.3,\ \alpha = 0.56,\ \beta = 0.37$

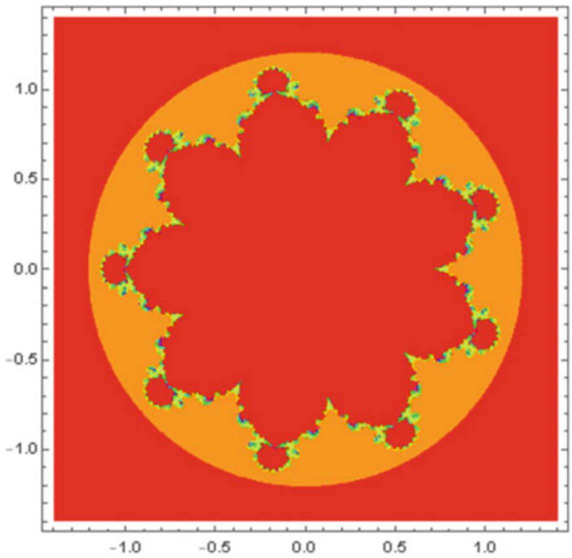


Fig. 10 Tenth-order Mandelbrot set for $\lambda = 0.69,\ \mu = 0.17,\ \nu = 0.58,\ \alpha = 0.68,\ \beta = 0.31$

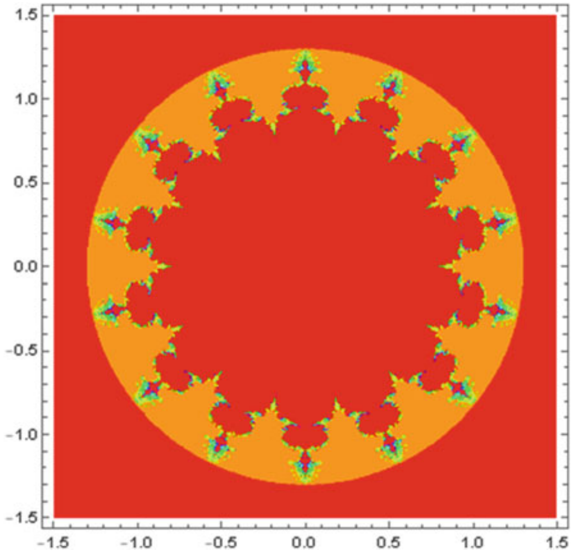


Fig. 11 Fifteenth-order Mandelbrot set for $\lambda = 0.6,\ \mu = 0.04,\ \nu = 0.05,\ \alpha = 0.6,\ \beta = 0.01$

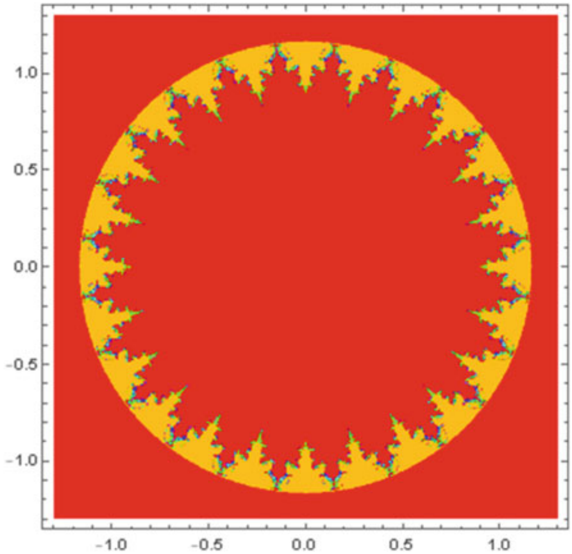


Fig. 12 Twenty-fifth-order Mandelbrot set for $\lambda = 0.6,\ \mu = 0.04,\ \nu = 0.05,\ \alpha = 0.09,\ \beta = 0.01$

6 Construction of Julia Sets in the Suantai Type Orbit

In this section, we generate Julia sets for quadratic and higher degree complex polynomials.

6.1 Julia Sets for Quadratic Complex Polynomials

Let us consider the quadratic complex polynomial $F_d(z) = z^2 + d\ ;\ d \in \mathbb{C}$ and different values of parameters $\lambda,\ \mu,\ \nu,\ \alpha,\ \beta \in (0, 1]$ to discuss and visualize the quadratic Julia sets.

- In Figs. 13 and 14, beautiful graphics of Quadratic Julia sets are constructed. Figure 13 is a spiral type Julia set. Both the Julia sets have very nice aesthetic patterns to be used for designing purpose.
- Dragon types quadratic Julia sets have been represented by Figs. 15 and 16.

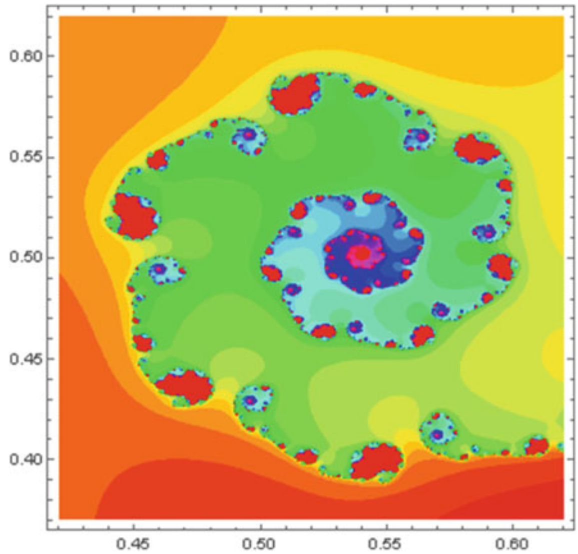


Fig. 13 Quadratic Julia set for $\lambda = 0.1,\ \mu = 0.4, \nu = 0.7,\ \alpha = 0.19,\ \beta = 0.8,\ d = 0.5 - 0.04\iota$

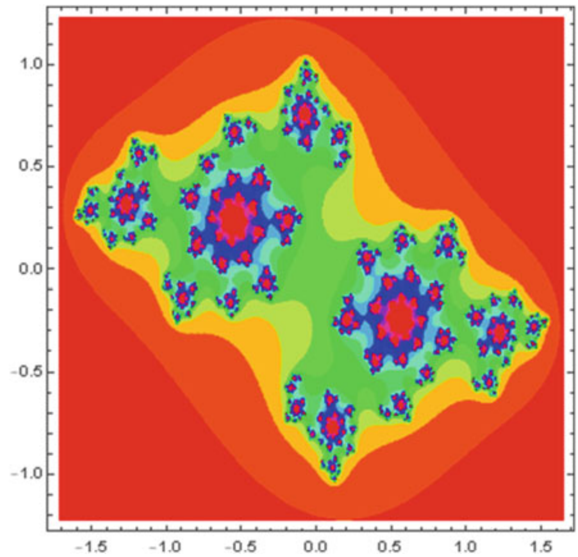


Fig. 14 Quadratic Julia set for $\lambda = 0.95,\ \mu = 0.04, \nu = 0.8,\ \alpha = 0.09,\ \beta = 0.9,\ d = -0.8 + 0.05\iota$

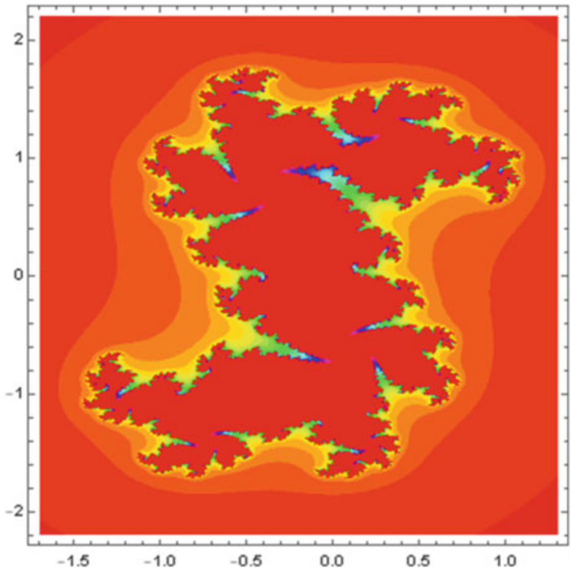


Fig. 15 Quadratic Julia set for $\lambda = 0.58,\ \mu = 0.38, \nu = 0.89,\ \alpha = 0.1,\ \beta = 0.18,\ d = 0.5 - 0.5\iota$

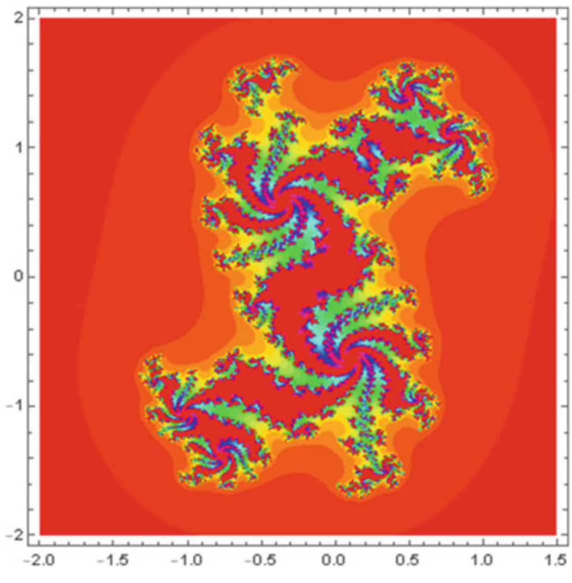


Fig. 16 Quadratic Julia set for $\lambda = 0.71,\ \mu = 0.25, \nu = 0.8,\ \alpha = 0.52,\ \beta = 0.37,\ d = 0.5 - 0.5\iota$

6.2 Julia Sets for Higher Degree Polynomials

Here, we visualize and study the dynamics of Julia sets for higher degree complex polynomial $F_d(z) = z^n + d\,;\, n \geq 3,\ d \in \mathbb{C}$ by taking different values of parameters $\lambda,\ \mu,\ \nu,\ \alpha,\ \beta \in (0, 1]$.

- Cubic Mandelbrot sets for the polynomial $F_d(z) = z^3 + c$ are presented by Figs. 17, 18 and 19 by choosing different values of parameters.
- The Mandelbrot sets for fourth- and fifth-degree polynomials are shown in Figs. 20 and 21, respectively. Figure 21 somewhat resembles with the shape of coronavirus. Further, the Mandelbrot set for tenth-degree polynomial $F_d(z) = z^{10} + d$ is represented in Fig. 22.

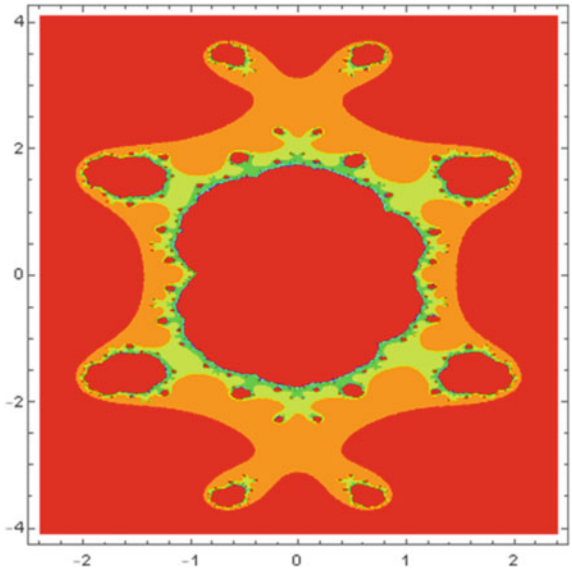


Fig. 17 Cubic Julia set for $\lambda = 0.9,\ \mu = 0.001, \nu = 0.115,\ \alpha = 0.1,\ \beta = 0.03,\ d = -0.115 - 0.05\iota$

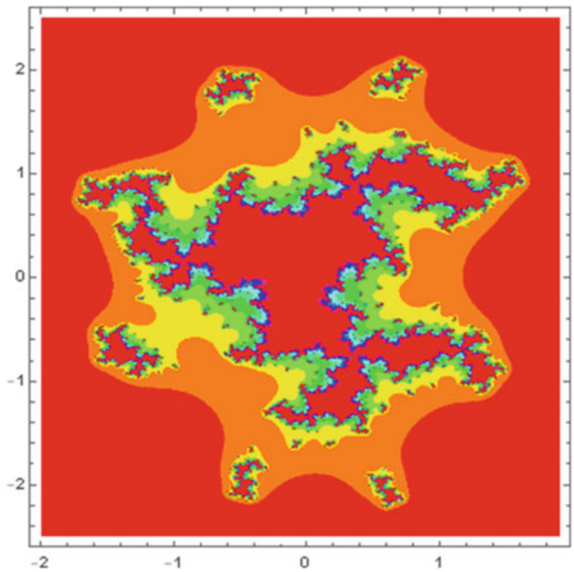


Fig. 18 Cubic Julia set for $\lambda = 0.9,\ \mu = 0.01, \nu = 0.4,\ \alpha = 0.09,\ \beta = 0.01,\ d = 0.9 - 0.9\iota$

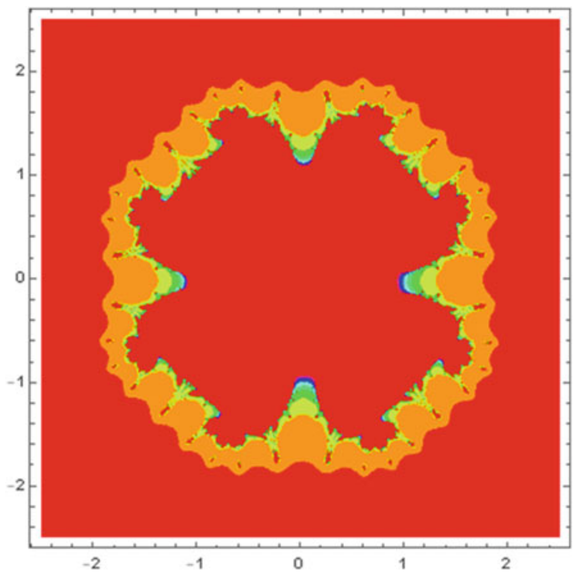


Fig. 19 Cubic Julia set for $\lambda = 0.048,\ \mu = 0.034, \nu = 0.79,\ \alpha = 0.031,\ \beta = 0.019,\ d = -0.18 - 0.18\iota$

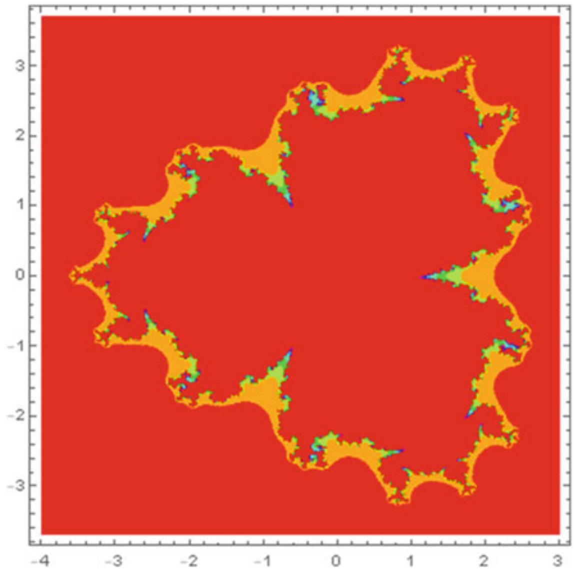


Fig. 20 Fourth-order Julia set for $\lambda = 0.05, \mu = 0.001, \nu = 0.05, \alpha = 0.01, \beta = 0.001, d = 0.2 - 0.2\iota$

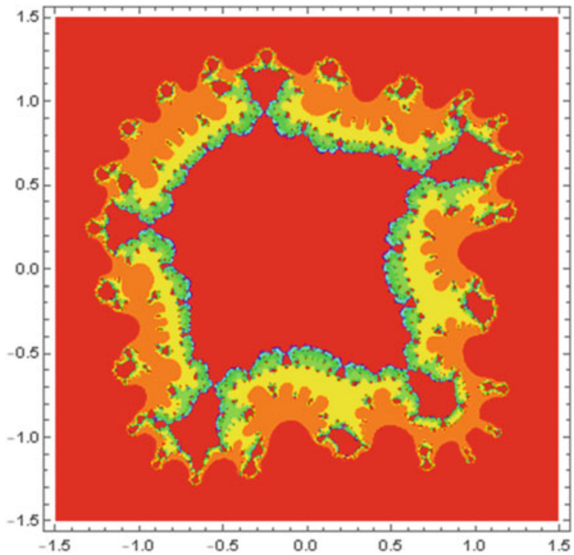


Fig. 21 Fifth-order Julia set for $\lambda = 0.9,\ \mu = 0.01, \nu = 0.4,\ \alpha = 0.09,\ \beta = 0.01,\ d = 0.9 - 0.9\iota$

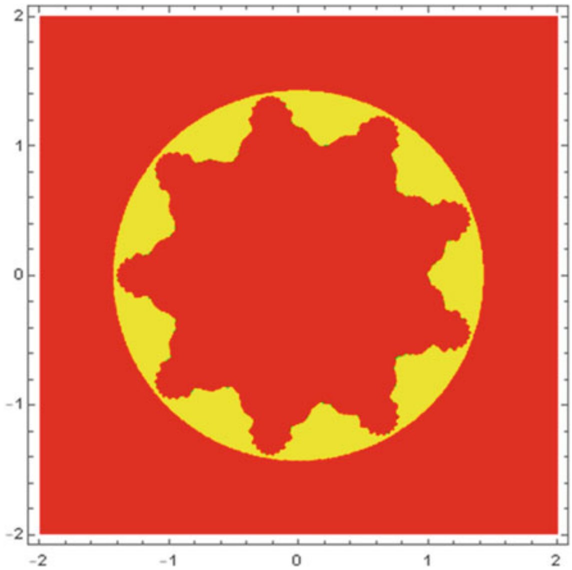


Fig. 22 Tenth-order Julia set for $\lambda = 0.5,\ \mu = 0.2,\ \nu = 0.08,\ \alpha = 0.4,\ \beta = 0.001,\ d = 0.1 - 0.1\iota$

7 Generation of Anti-fractals in the Suantai Type Orbit

In this section, we provide a general escape criterion to analyze and construct anti-fractals for antiholomorphic polynomials of the form $A_d(z) = \bar{z}^m + d;\ m \geq 2$ under the Suantai type orbit.

Theorem 4 *Assume*

$$|z| \geq |d| > \max\Big\{\Big(\frac{2}{|\nu|}\Big)^{1/m-1}, \Big(\frac{2}{|\alpha| - |\beta|}\Big)^{1/m-1}, \Big(\frac{2}{|\lambda| - |\mu|}\Big)^{1/m-1}\Big\},$$

where $\lambda, \mu, \nu, \alpha, \beta \in (0, 1]$. *Define a sequence* $\{z_j\}_{j\in\mathbb{N}}$ *given by*

$$\begin{aligned} z_{j+1} &= (1 - \lambda - \mu)\ z_j + \lambda F_d(\bar{y}_j) + \mu\ F_d(\bar{z}_j), \\ y_j &= (1 - \alpha - \beta)\ z_j + \alpha F_d(\bar{x}_j) + \beta F_d(\bar{z}_j), \\ x_j &= (1 - \nu)\ z_j + \nu F_d(\bar{z}_j),\ \ j = 0, 1, 2, \ldots, \end{aligned} \tag{22}$$

where $F_d(\bar{z}) = \bar{z}^m + d,\ z_0 = z,\ y_0 = y$ *and* $x_0 = x$. *Then* $|z_j| \to \infty$ *as* $j \to \infty$.

Proof From (22), consider

$$\begin{aligned} |x| &= |(1 - \nu)z + \nu F_d(\bar{z})| \\ &= |(1 - \nu)z + \nu(\bar{z}^m + d)| \\ &\geq |(1 - \nu)z + \nu\bar{z}^m| - |\nu\ d|. \end{aligned}$$

The assumption $|z| \geq |d|$ gives

$$\begin{aligned} |x| &\geq |(1-\nu)\, z + \nu \bar{z}^m| - |\nu||z| \\ &\geq |\nu \bar{z}^m| - |(1-\nu) z| - |\nu||z| \\ &= |\nu||\bar{z}|^m - |z| + |\nu||z| - |\nu||z| \\ &= |z|(|\nu||z|^{m-1} - 1), \qquad \because |\bar{z}| = |z|. \end{aligned}$$

Therefore,

$$|x| \geq |z|(|\nu||z|^{m-1} - 1). \tag{23}$$

The assumption $|z| > (\frac{2}{|\nu|})^{1/m-1}$ implies

$$|\nu||z|^{m-1} - 1 > 1. \tag{24}$$

Using (24), (23) yields

$$|x| > |z|. \tag{25}$$

Also,

$$\begin{aligned} |y| &= |(1-\alpha-\beta)z + \alpha F_d(\bar{x}) + \beta F_d(\bar{z})| \\ &= |(1-\alpha-\beta)z + \alpha(\bar{x}^m + d) + \beta(\bar{z}^m + d)| \\ &\geq |(1-\alpha-\beta)z + \alpha(\bar{x}^m + d)| - |\beta(\bar{z}^m + d)| \\ &\geq |\alpha(\bar{x}^m + d)| - |\beta(\bar{z}^m + d)| - |(1-\alpha-\beta)z| \\ &\geq |\alpha||\bar{x}^m| - |\alpha||d| - |\beta||\bar{z}^m| - |\beta||d| - |(1-\alpha-\beta)z| \\ &\geq |\alpha||\bar{x}|^m - |\alpha||d| - |\beta||\bar{z}|^m - |\beta||d| - |(1-\alpha-\beta)z|. \end{aligned}$$

Since $|\bar{x}|^m = |x|^m$ and $|\bar{z}|^m = |z|^m$, we get

$$|y| \geq |\alpha||x|^m - |\alpha||d| - |\beta||z|^m - |\beta||d| - |(1-\alpha-\beta)z|. \tag{26}$$

Using (25) and assumption $|z| \geq |d|$ in (26), we obtain

$$\begin{aligned} |y| &\geq |\alpha||z|^m - |\alpha||z| - |\beta||z|^m - |\beta||z| - |z| + |\alpha||z| + |\beta||z| \\ &= |\alpha||z|^m - |\beta||z|^m - |z| \\ &= |z|^m(|\alpha| - |\beta|) - |z| \\ &= |z|(|z|^{m-1}(|\alpha| - |\beta|) - 1). \end{aligned}$$

Thus,

$$|y| \geq |z|(|z|^{m-1}(|\alpha| - |\beta|) - 1). \tag{27}$$

Also, the assumption $|z| > (\frac{2}{|\alpha|-|\beta|})^{1/m-1}$ gives

$$(|\alpha| - |\beta|)|z|^{m-1} - 1 > 1. \tag{28}$$

Using (28) in (27), we obtain

$$|y| > |z|. \tag{29}$$

Finally, consider

$$\begin{aligned}
|z_1| &= |(1-\lambda-\mu)z + \lambda F_d(\bar{y}) + \mu F_d(\bar{z})| \\
&= |(1-\lambda-\mu)z + \lambda(\bar{y}^m + d) + \mu(\bar{z}^m + d)| \\
&\geq |(1-\lambda-\mu)z + \lambda(\bar{y}^m + d)| - |\mu(\bar{z}^m + d)| \\
&\geq |\lambda(\bar{y}^m + d)| - |\mu(\bar{z}^m + d)| - |(1-\lambda-\mu)z| \\
&\geq |\lambda\bar{y}^m| - |\lambda d| - |\mu\bar{z}^m| - |\mu d| - |(1-\lambda-\mu)z| \\
&= |\lambda||\bar{y}|^m - |\lambda||d| - |\mu||\bar{z}|^m - |\mu||d| - |(1-\lambda-\mu)z| \\
&= |\lambda||y|^m - |\lambda||d| - |\mu||z|^m - |\mu||d| - |(1-\lambda-\mu)z|.
\end{aligned}$$

Thus,

$$|z_1| \geq |\lambda||y|^m - |\lambda||d| - |\mu||z|^m - |\mu||d| - |(1-\lambda-\mu)z|. \tag{30}$$

Using the assumption $|z| \geq |d|$ and (29), (30) becomes

$$\begin{aligned}
|z_1| &\geq |\lambda||z|^m - |\lambda||z| - |\mu||z|^m - |\mu||z| - |(1-\lambda-\mu)z| \\
&= |\lambda||z|^m - |\mu||z|^m - |z| \\
&= |z|(|z|^{m-1}(|\lambda| - |\mu|) - 1).
\end{aligned}$$

Also, our assumption $|z| > (\frac{2}{|\lambda|-|\mu|})^{1/m-1}$ implies that $(|\lambda| - |\mu|)|z|^{m-1} - 1 > 1$. Thus, there exists a real number $\sigma > 0$ such that $(|\lambda| - |\mu|)|z|^{m-1} - 1 > \sigma + 1 > 1$. Hence, this gives

$$|z_1| > (\sigma + 1)|z|.$$

Continuing this process in the same manner, we have

$$|z_2| > (1+\sigma)^2|z|,$$

$$|z_3| > (1+\sigma)^3|z|,$$

$$\vdots$$

$$|z_j| > (1+\sigma)^j|z|.$$

Therefore, $|z_j| \to \infty$ as $j \to \infty$.

The following subsections contain tricorns and multicorns for antiholomorphic polynomial $A_d(z) = \bar{z}^m + d;\ m \geq 2$ via the Suantai type orbit. We use the escape criterion (Theorem 4) to construct the images of anti-fractals using the software Mathematica 11.0. A maximum of 10 number of iterations have been used to construct the following anti-fractals.

7.1 Construction of Tricorns and Multicorns

Here, we visualize the graphics of anti-fractals like tricorns and multicorns for antiholomorphic polynomial $A_d(z) = \bar{z}^m + d;\ m \geq 2$ by considering different values of parameters $\lambda,\ \mu,\ \nu,\ \alpha,\ \beta \in (0, 1]$.

- In Figs. 23, 24 and 25, tricorns are visualized which demonstrate the three cornered nature of tricorns. We demonstrate the multicorns for $m = 3$ in Figs. 26 and 27 which have four different corners.
- It is surprising to see that for $m = 4$, a star-shaped multicorn is constructed as shown by Fig. 28.
- Multicorns for $m = 15$ and $m = 25$ are constructed in Figs. 29 and 30, respectively. Figure 30 takes the form of a circular saw.
- We observe that multicorn is symmetric around both the axis, i.e., x-axis and y-axis when m is odd and it is symmetric only along x-axis when m is even.
- It is observed that each tricorn and multicorn for the antipolynomial $\bar{z}^m + d$ has $m + 1$ branches.

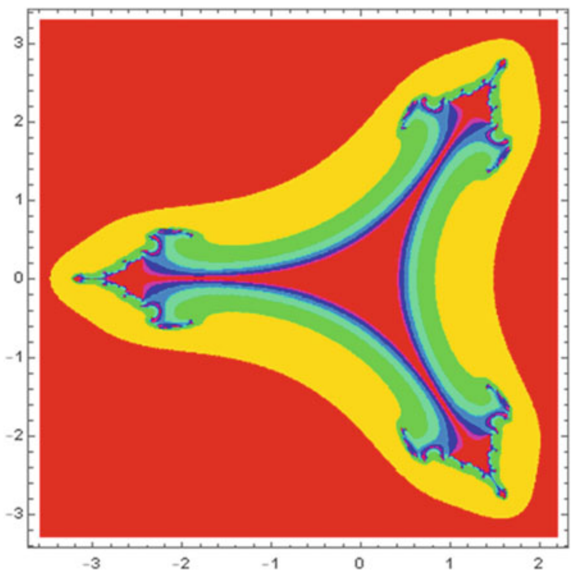


Fig. 23 Tricorn set for $\lambda = 0.2,\ \mu = 0.04,\ \nu = 0.6,\ \alpha = 0.68,\ \beta = 0.2$

Fig. 24 Tricorn set for $\lambda = 0.01,\ \mu = 0.07,\ \nu = 0.9,\ \alpha = 0.07,\ \beta = 0.02$

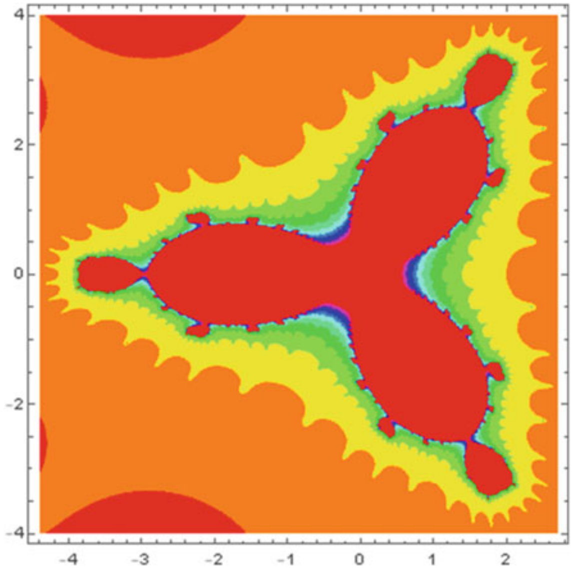
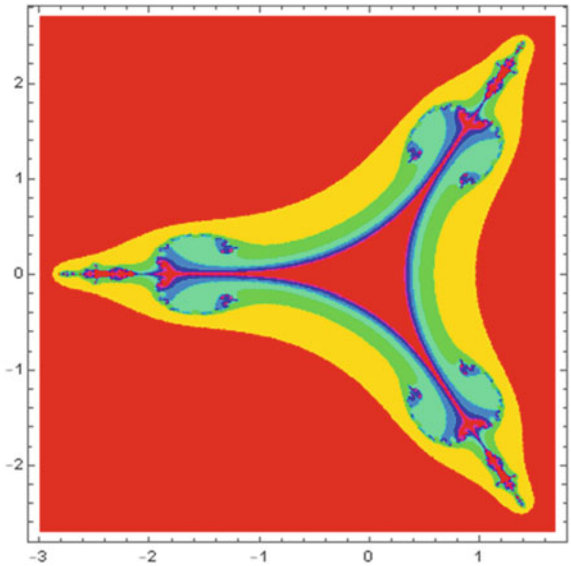


Fig. 25 Tricorn set for $\lambda = 0.57,\ \mu = 0.09,\ \nu = 0.77,\ \alpha = 0.48,\ \beta = 0.027$



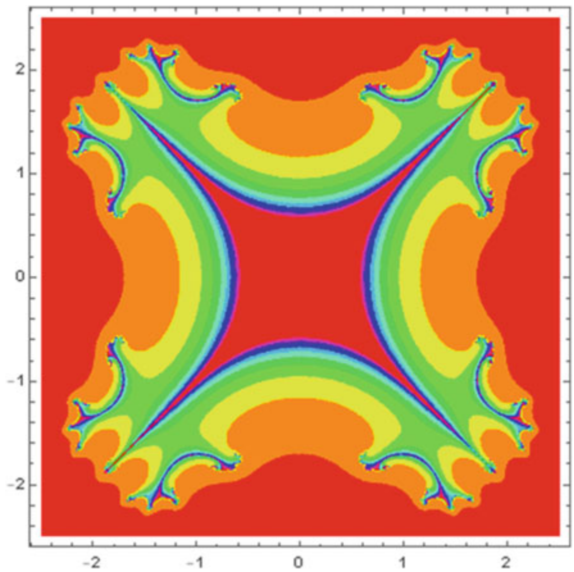


Fig. 26 Third-order Multicorn for $\lambda = 0.09,\ \mu = 0.09,\ \nu = 0.3,\ \alpha = 0.07,\ \beta = 0.02$

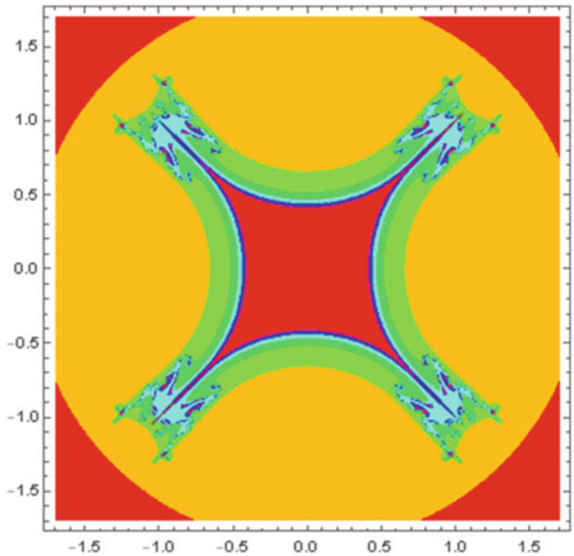


Fig. 27 Third-order Multicorn for $\lambda = 0.72, \mu = 0.007, \nu = 0.58, \alpha = 0.95, \beta = 0.001$

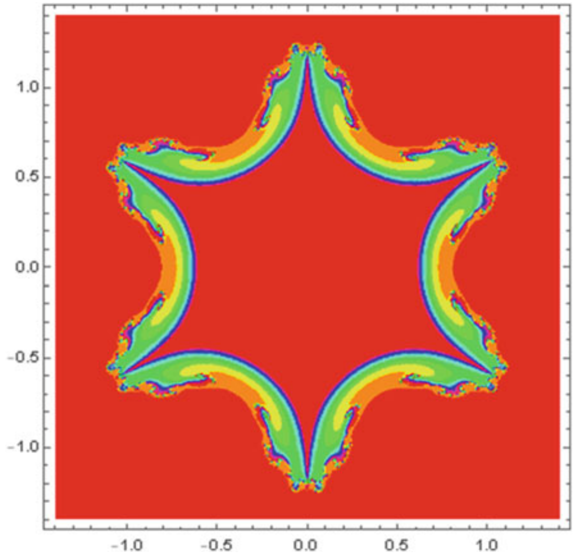


Fig. 28 Fifth -order Multicorn for $\lambda = 0.07,\ \mu = 0.01,\ \nu = 0.8,\ \alpha = 0.7,\ \beta = 0.02$

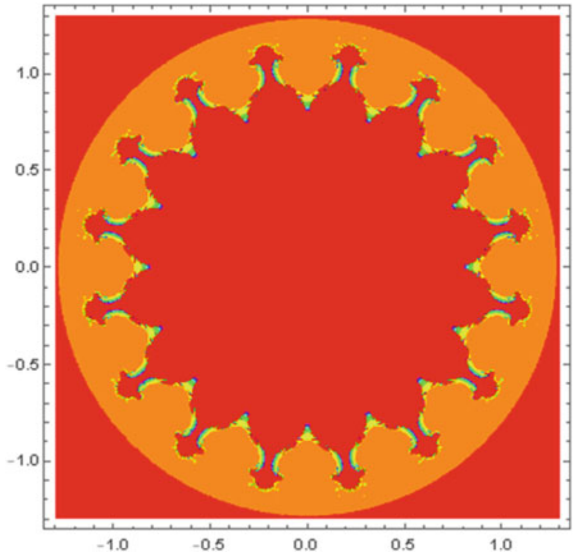


Fig. 29 Fifteenth-order Multicorn for $\lambda = 0.07,\ \mu = 0.01,\ \nu = 0.08,\ \alpha = 0.9,\ \beta = 0.09$

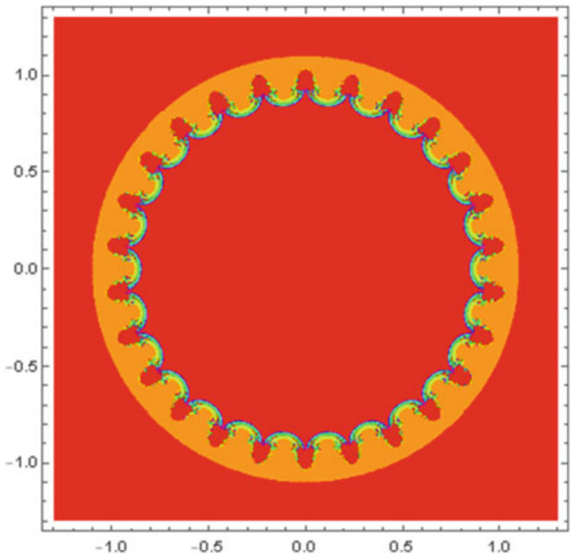


Fig. 30 Twenty-fifth-order Multicorn for $\lambda = 0.3,\ \mu = 0.1,\ \nu = 0.9,\ \alpha = 0.71,\ \beta = 0.25$

8 Construction of Anti-Julia Sets

Now, we construct Anti-Julia sets for the antipolynomial $A_d(z) = \bar{z}^m + d;\ m \geq 2$ by taking different values of parameters $\lambda, \mu, \nu, \alpha, \beta \in (0, 1]$ and $d \in \mathbb{C}$ by using software Mathematica 11.0.

- Quadratic Anti-Julia sets have been constructed in Figs. 31, 32 and 33 by taking different values of parameters.
- We construct cubic Anti-Julia sets in Figs. 34, 35 and 36 which maintain symmetry with respect to origin and have four folds. In Fig. 34, Anti-Julia set depicts four corners as a trident, a divine symbol named Trishula.
- Anti-Julia sets for $m = 5$ and $m = 10$ are visualized in Figs. 37 and 38, respectively. These graphics maintain symmetry with respect to origin and have 6 and 11 branches, respectively.

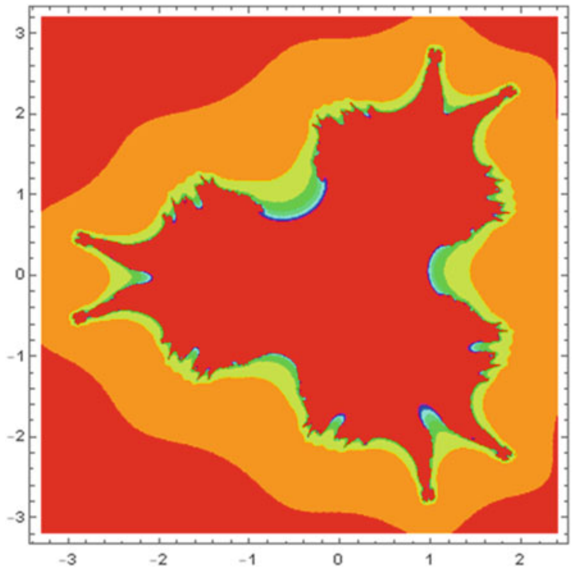


Fig. 31 Quadratic Anti-Julia set for $\lambda = 0.6,\ \mu = 0.1, \nu = 0.9,\ \alpha = 0.09,\ \beta = 0.01,\ d = 0.04 + 0.2\iota$

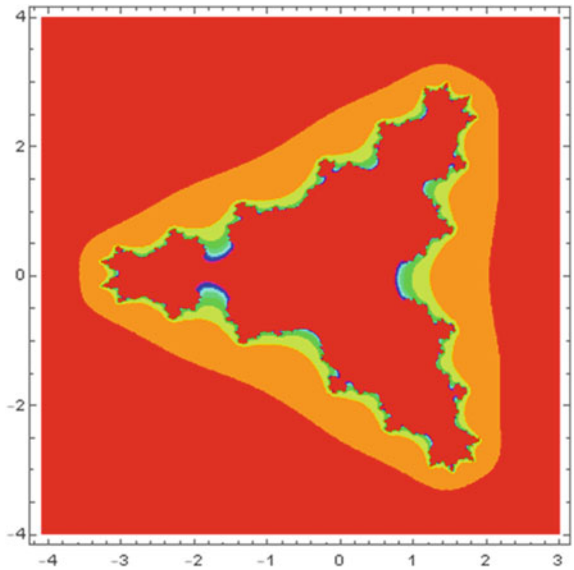


Fig. 32 Quadratic Anti-Julia set for $\lambda = 0.72,\ \mu = 0.05, \nu = 0.58,\ \alpha = 0.18,\ \beta = 0.008,\ d = 0.2 - 0.2\iota$

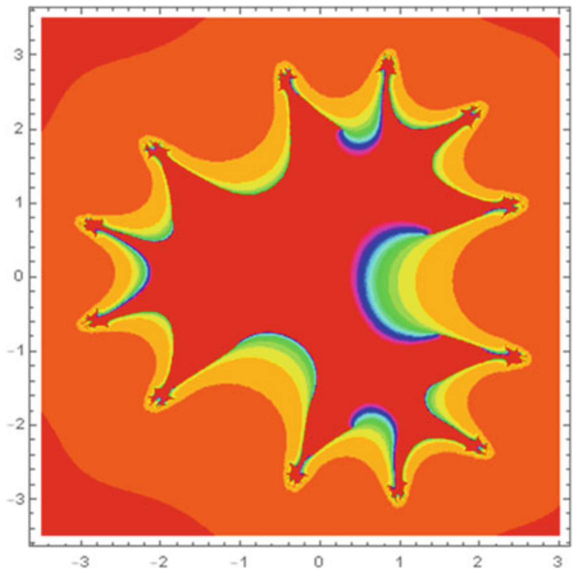


Fig. 33 Quadratic Anti-Julia set for $\lambda = 0.156,\ \mu = 0.126, \nu = 0.83,\ \alpha = 0.14,\ \beta = 0.131,\ d = 0.34 - 0.34\iota$

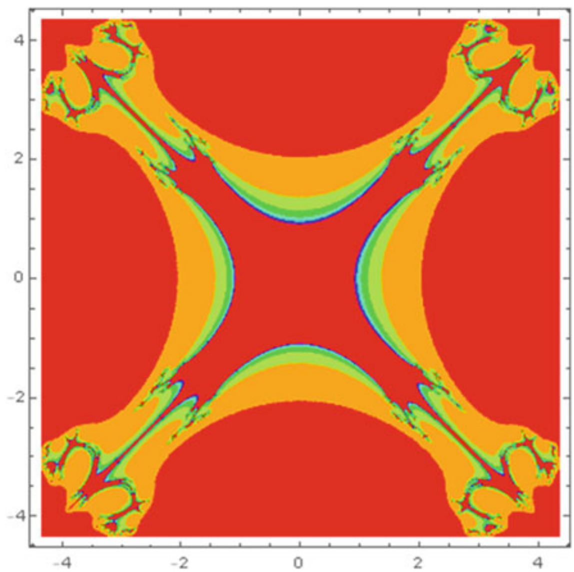


Fig. 34 Cubic Anti-Julia set for $\lambda = 0.3,\ \mu = 0.04, \nu = 0.05,\ \alpha = 0.05,\ \beta = 0.02,\ d = 0.02 + 0.2\iota$

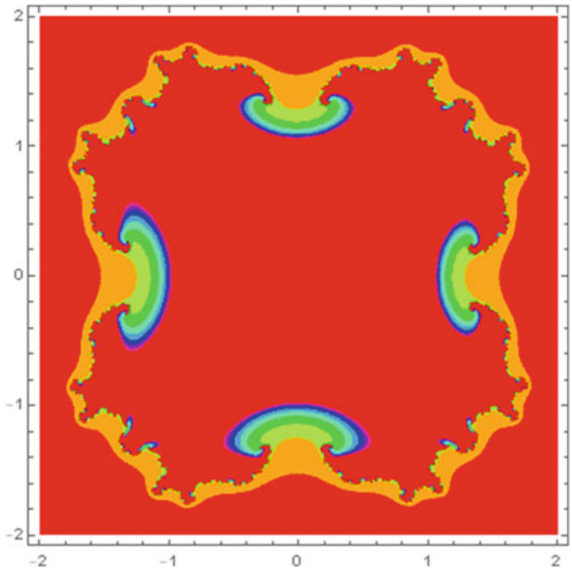


Fig. 35 Cubic Anti-Julia set for $\lambda = 0.05,\ \mu = 0.01, \nu = 0.6,\ \alpha = 0.1,\ \beta = 0.04,\ d = -0.1 - 0.1\iota$

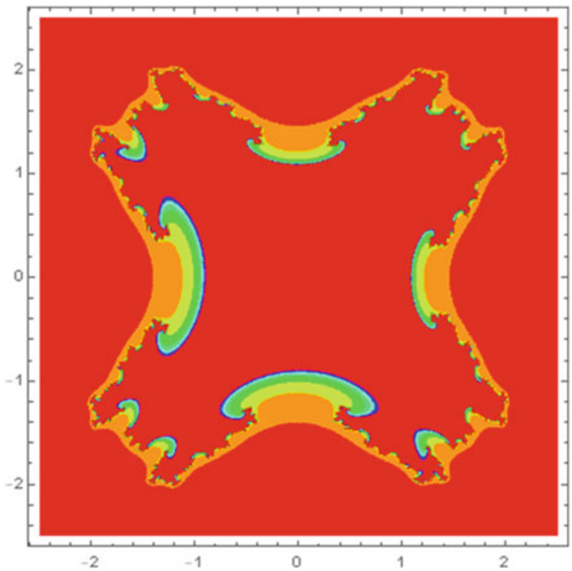


Fig. 36 Cubic Anti-Julia set for $\lambda = 0.25,\ \mu = 0.01, \nu = 0.28,\ \alpha = 0.15,\ \beta = 0.01,\ d = -0.19 - 0.19\iota$

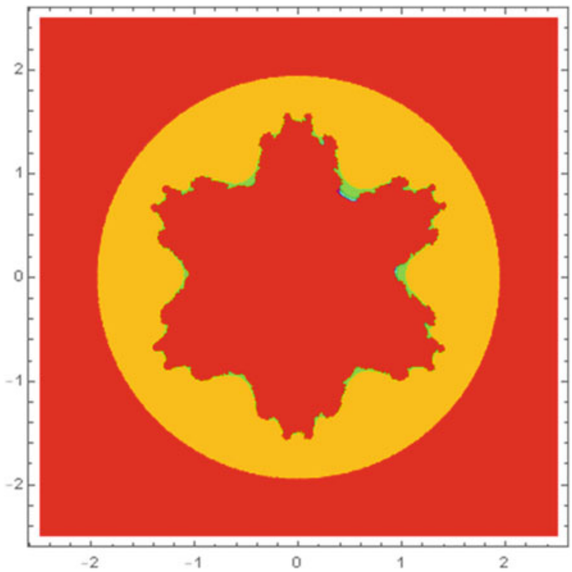


Fig. 37 Fifth-order Anti-julia set for $\lambda = 0.45,\ \mu = 0.01, \nu = 0.28,\ \alpha = 0.15,\ \beta = 0.01,\ d = 0.25 - 0.25\iota$

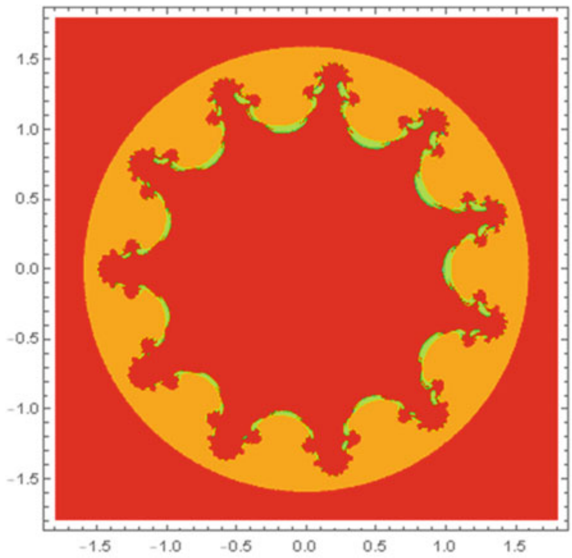


Fig. 38 Tenth-order Anti-julia set for $\lambda = 0.3,\ \mu = 0.04, \nu = 0.05,\ \alpha = 0.05,\ \beta = 0.02,\ d = 0.2 + 0.2\iota$

9 Conclusion

In this chapter, we used the Suantai type orbit as an application of fixed point iterative methods to examine the behaviors of complex polynomials and antipolynomials. We proved some escape criteria to generate fractals and anti-fractals. Some tempting graphics of fractals and anti-fractals have been constructed by choosing different values of parameters $\lambda,\ \mu,\ \nu,\ \alpha,\ \beta \in (0, 1]$. We noticed that eminent changes in the shapes of fractals and anti-fractals can be observed with the variation of parameters $\lambda,\ \mu,\ \nu,\ \alpha,\ \beta$. We observed that for higher degree polynomials, the fractals and anti-fractals have the rotational symmetry along the origin and their shapes look similar to a circular saw. Our results might be very useful in generating automatic aesthetic patterns for graphic designers.

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Nonexpansive Mappings, Their Extensions, and Generalizations in Banach Spaces

Rajendra Pant, Rahul Shukla, and Prashant Patel

Abstract This is a survey chapter. We present a brief development of fixed point theory for nonexpansive type mappings in Banach spaces.

1 Introduction

Nonexpansive mappings are natural generalization of contraction mappings. These mappings are important due to their connection with the monotonicity methods and also appear in applications for initial value, variational inequality, optimization, equilibrium, and many other problems in nonlinear analysis [56]. It is well-known that a nonexpansive self-mapping of a complete metric space need not have a fixed point.

Example 1 Let $(\ell^1, \|.\|_1)$ be the Banach space of all real absolutely summable sequences and

$$K = \left\{ x = (x_1, x_2, \dots) : x_n \geq 0 \text{ for all } n \text{ and } \sum_{n=1}^{\infty} x_n = 1 \right\},$$

a closed bounded subset of ℓ^1. Let $T : K \to K$ be defined by $T(x) = (0, x_1, x_2, \dots)$. Then T is a fixed point free nonexpansive mapping on K.

Also, the sequence of iterates (the Picard sequence) of a nonexpansive mapping may not converge to a fixed point of the mapping, unlike the contraction mappings. Therefore the study of existence and convergence of fixed points of nonexpansive

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mappings is an important subject. In this chapter, we present some important existence and convergence results for nonexpansive mappings, their extensions, and generalizations.

2 Preliminaries

Throughout this chapter, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{N}$ the set of all positive integers.

Definition 1 [13, 22]. The *modulus of convexity* of Banach space X is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : x, y \in X,\ \|x\|,\ \|y\| \leq 1,\ \|x - y\| \geq \varepsilon\right\}.$$

The *characteristic of convexity* of X is defined as

$$\varepsilon_0(X) = \sup\{\varepsilon : \delta(\varepsilon) = 0\}.$$

Definition 2 [25]. A Banach space X is said to be *uniformly convex in every direction* (*UCED*, for short) if for $\varepsilon \in (0, 2]$ and $z \in X$ with $\|z\| = 1$, there exists $\delta(\varepsilon, z) > 0$ such that

$$\left\|\frac{x + y}{2}\right\| \leq 1 - \delta(\varepsilon, z) \text{ for all } x, y \in X \text{ with } \|x\| \leq 1 \text{ and } \|y\| \leq 1 \text{ and }$$
$$x - y \in \{tz : t \in [-2, -\varepsilon] \cup [\varepsilon, 2]\}.$$

The Banach space X is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$ there exists $\delta > 0$ such that

$$\left\|\frac{x + y}{2}\right\| \leq 1 - \delta \text{ for all } x, y \in X \text{ with } \|x\| = \|y\| \leq 1 \text{ and } \|x - y\| \geq \varepsilon.$$

It is obvious that uniform convexity implies UCED.

Definition 3 [49]. A Banach space X is said to be nearly uniformly convex if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|x_n\| \leq 1$, and $\|x_n - x_m\| \geq \varepsilon$ for $m \neq n$, then there is an $N \geq 1$ and scalars $\lambda_1, \cdots, \lambda_N \geq 0$ with $\sum_{n=1}^{N} \lambda_n = 1$ such that $\|\lambda_n x_n\| \leq 1 - \delta$.

Definition 4 [39]. Let K be a nonempty subset of a Banach space X. For $x \in X$ define

$$r_x(K) = \sup\{\|x - y\| : y \in K\} \text{ and } r(K) = \inf\{r_x(K) : x \in K\}.$$

The number $r(K)$ is called the Chebyshev radius of K.

Definition 5 [18, 39]. The character of a Banach space X is defined as

$$\kappa(X) = \sup\{c > 0 : r(B(0,1) \cap B(x,d)) < 1 \text{ and } \|x\| = 1\},$$

where $B(x,d) \subset X$ is the open ball centered at x with radius d.

Definition 6 [25]. Let K be a nonempty subset of a Banach space X. A point $x \in K \subseteq X$ is said to be diametral if $r_x(K) = \operatorname{diam}(K)$ (diameter). The point x is said to be nondiametral if it is not diametral. A convex subset K of X is said to have normal structure if each bounded, convex subset D of K with diam $D > 0$ contains a nondiametral point.

Definition 7 [25]. Let A be a nonempty subset of a Banach space X. The *convex hull* of A, is the smallest convex set containing A, that is,

$$\operatorname{conv}(A) = \cap\{K \subset X : K \supset A,\ K \text{ is a convex set}\},$$

and the *closed convex hull* of A, is defined as

$$\overline{\operatorname{conv}}(A) = \cap\{K \subset X : K \supset A,\ K \text{ is closed convex in } X\}.$$

Definition 8 [14, 15, 25]. Let K be a nonempty subset of a Banach space X, and $\{x_n\}$ a bounded sequence in X. The *asymptotic radius* of $\{x_n\}$ at a point x is defined by

$$r(x, \{x_n\}) := \limsup_{n\to\infty} \|x_n - x\|.$$

The asymptotic radius of $\{x_n\}$ with respect to K is defined by

$$r(K, \{x_n\}) := \inf\{r(x, \{x_n\}) : x \in K\}.$$

The asymptotic center of $\{x_n\}$ with respect to K is defined by

$$A(K, \{x_n\}) := \{x \in K : r(x, \{x_n\}) = r(K, \{x_n\}).$$

Remark 1 We note that

(a) If K is a nonempty closed convex subset of a uniformly convex Banach space X, then the asymptotic center of every bounded sequence $\{x_n\}$ in X relative to K is singleton.
(b) Further, if K is a nonempty weakly compact convex subset of a UCED Banach space X, then the asymptotic center of every bounded sequence $\{x_n\}$ in X relative to K is singleton [25, 33].

Definition 9 [3]. Let X be a Banach space and X^* its dual. Then the multivalued mapping $J : X \to 2^{X^*}$ defined by

$$J(x) := \{h \in X^* : \langle x, h\rangle = \|x\|^2 = \|h\|^2\},$$

is called the *normalized duality mapping*.

Definition 10 [3]. Let X be a Banach space and $S_X = \{x \in X : \|x\| = 1\}$ the unit sphere of X. Then

(1) the norm of X is said to be *Gâteaux differentiable* at point $x \in S_X$ if for $y \in S_X$

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists. The norm of X is said to Gâteaux differentiable if it is Gâteaux differentiable at each point of S_X.

(2) the Banach space X is said to be *smooth* if the limit (1) exists for all $x, y \in S_X$.

(3) the norm of the Banach space X is said to be *Frećhet differentiable* if for each $x, y \in S_X$, the limit (1) exists uniformly.

It is known that if X is smooth then the duality mapping J is single valued [25]. In this case, for all $x, f \in X$

$$\langle f, J(x)\rangle + \frac{1}{2}\|x\|^2 \le \frac{1}{2}\|x + f\|^2 \le \langle f, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|f\|),$$

where $J(x)$ is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x \in X$ and b is an increasing function defined on $[0, \infty)$ such that $\lim_{t\to 0} \frac{b(t)}{t} = 0$.

Definition 11 [45]. A Banach space X is said to satisfy *Opial property* if, for every weakly convergent sequence $\{x_n\}$ with weak limit $x \in X$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \in X$ with $x \neq y$.

All finite dimensional Banach spaces, all Hilbert spaces and ℓ^p $(1 < p < \infty)$ satisfy the Opial property. But L^p $(1 < p < \infty,\ p \neq 2)$ do not satisfy the Opial property [13].

Definition 12 [31]. A Banach space X is said to be uniformly nonsquare if there exists $\delta \in (0, 1)$ such that for any $x, y \in S_X$ either $\frac{\|x+y\|}{2} \le 1 - \delta$ or $\frac{\|x-y\|}{2} \le 1 - \delta$. The constant $J(X)$ defined by

$$J(X) = \sup\{\min(\|x + y\|, \|x - y\|) : x, y \in S_X\},$$

is called the nonsquare or James constant of X. Here S_X is as in Definition 10.

Definition 13 [54]. Let K be a nonempty subset of a Banach space X. A function $f : K \to \mathbb{R}$ is said to be convex if for all $x, y \in K$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

The function f is said to be quasi-convex if for all $x, y \in K$ and $\lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda)y) < \max\{f(x), f(y)\}.$$

Definition 14 [5].

(a) Let K be a nonempty subset of a Banach space X. A function $g : K \to \mathbb{R}$ is said to be coercive if $g(z_n) \to \infty$ whenever $\{z_n\}$ is a sequence in K such that $\|z_n\| \to \infty$.
(b) Let $(\ell^\infty, \|.\|_\infty)$ be the Banach space of bounded real sequences. Then there exists a bounded linear functional μ on ℓ^∞ such that for all $n \in \mathbb{N}$,
   (i) if $\{t_n\} \in \ell^\infty$ and $t_n \geq 0$ then $\mu(\{t_n\}) \geq 0$,
   (ii) if $t_n = 1$ then $\mu(\{t_n\}) = 1$,
   (iii) $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\} \in \ell^\infty$.

The functional μ is said to be a Banach limit [55].

Proposition 1 *[55]. Let K be a nonempty closed convex subset of a reflexive Banach space X and $g : K \to \mathbb{R}$ a convex, continuous, and coercive function. Then there exists $x \in K$ such that $g(x) = \inf g(K)$.*

Definition 15 [13, 25]. Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be nonexpansive if for all $x, y \in K$,

$$\|T(x) - T(y)\| \leq \|x - y\|.$$

A point $p \in K$ is said to be a fixed point of T if $T(p) = p$. We denote the set of all fixed points of T by $F(T)$.
The mapping T is said to be quasi-nonexpansive provided that it has a fixed point in K and for each fixed point $p \in K$ and every $y \in K$,

$$\|T(p) - T(y)\| = \|p - T(y)\| \leq \|p - y\|.$$

Definition 16 [25]. Let K be a nonempty subset of a Banach space X and $T : K \to K$ a mapping. A sequence $\{x_n\}$ in K is said to be approximate fixed point sequence (in short, a.f.p.s.) for T if $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$.

Lemma 1 (Zorn's Lemma). *Let $A \neq \emptyset$ be a partially ordered set. If every chain $C \subset A$ has an upper bound, then A has a maximal element.*

3 Fixed Point Theorems for Nonexpansive Mappings

In 1965, the study of existence of fixed points of nonexpansive mappings was initiated by Browder [9], Göhde [27] and Kirk [36], independently.

Theorem 1 (Browder [9] and Göhde [27]). *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Then every nonexpansive mapping $T : K \to K$ has a fixed point.*

We present a simpler proof of the above theorem which has been taken from [21, 50].

Proof Fix $y_0 \in K$ and for each $n \in \mathbb{N}$, define the mapping $T_n : K \to K$ by

$$T_n(x) = \left(1 - \frac{1}{n}\right) T(x) + \left(\frac{1}{n}\right) y_0 \text{ for all } x \in K.$$

Then for all $x, y \in K$, we have

$$\|T_n(x) - T_n(y)\| = \left(1 - \frac{1}{n}\right) \|T(x) - T(y)\| \leq \left(1 - \frac{1}{n}\right) \|x - y\|.$$

Thus T_n is a contraction mapping for each $n \in \mathbb{N}$. Now, by Banach contraction principle, T_n has a fixed point $x_n \in K$. Since K is bounded, we get

$$\lim_{n\to\infty} \|x_n - T(x_n)\| = \lim_{n\to\infty} \frac{1}{n} \|y_0 - T(x_n)\| = 0.$$

Let $r = r(K, \{x_n\})$ and $\{z\} = A(K, \{x_n\})$. By the triangle inequality and nonexpansiveness of the mapping T, we get

$$\begin{aligned} \|T(z) - x_n\| &\leq \|T(z) - T(x_n)\| + \|T(x_n) - x_n\| \\ &\leq \|z - x_n\| + \|T(x_n) - x_n\|. \end{aligned}$$

Taking lim sup on both sides of the above inequality, we have

$$r(T(z), \{x_n\}) \leq r(z, \{x_n\}) = r.$$

This implies that $T(z) \in A(K, \{x_n\})$, and $T(z) = z$.

Theorem 2 (Kirk [36]). *Let K be a nonempty weakly compact convex subset of a Banach space X with normal structure. Then every nonexpansive mapping $T : K \to K$ has a fixed point.*

The proof of Theorem 2, we present here, is taken from [50].

Proof Let $\mathfrak{A}$ be a family of all nonempty weakly closed convex T-invariant subsets of K. It is clear that the relation on $\mathfrak{A}$ defined by

$$C_1 \preceq C_2 \Leftrightarrow C_1 \supset C_2$$

generates a partial order. We consider any chain $\mathfrak{C} \subset \mathfrak{A}$ and the set

$$C^o = \bigcap_{C \in \mathfrak{C}} C.$$

Since C is weakly compact and the family $\mathfrak{C}$ has the finite intersection property, $C^o \neq \emptyset$. Further, C^o is weakly closed bounded convex T-invariant subset of K. So, C^o is an upper bound for the chain $\mathfrak{C}$. By Zorn's lemma, there exists a maximal element $D^o \in \mathfrak{A}$. Take

$$D^{oo} = \overline{\text{conv}} T(D^o) \subset D^o.$$

Then,

$$T(D^{oo}) \subset T(D^o) \subset \overline{\text{conv}} T(D^o) = D^{oo}.$$

Therefore $D^{oo} \in \mathfrak{A}$. By maximality of D^o, we get $D^o = D^{oo}$. Thus

$$\overline{\text{conv}} T(D^o) = D^o.$$

Since D^o is weakly compact, we have

$$M(D^o) = \{z \in D^o : r_z(D^o) = r(D^o)\} \neq \emptyset.$$

Therefore $\exists z \in D^o$ such that $r_z(D^o) = r(D^o)$. For each $y \in D^o$, we have

$$\|T(z) - T(y)\| \leq \|z - y\| \leq r_z(D^o) = r(D^o).$$

Thus

$$T(D^o) \subset B(T(z), r(D^o))$$

and

$$D^o = \overline{\text{conv}} T(D^o) \subset B(T(z), r(D^o)).$$

Therefore,

$$r_{Tz}(D^o) = r_z(D^o) = r(D^o)$$

and $M(D^o)$ is T-invariant. The set D^o is minimal invariant, hence

$$M(D^o) = D^o.$$

Since D^o has normal structure, it follows that $\text{diam}(D^o) = 0$. Thus,

$$D^o = \{z\},$$

and z is a fixed point of T.

In 2006, García-Falset et al. obtained the following important generalization of Browder-Göhde and Kirk theorems for the existence of fixed points of a nonexpansive mapping.

Theorem 3 *[19]. Let K be a nonempty bounded closed convex subset a uniformly nonsquare Banach space X. Then every nonexpansive mapping $T : K \to K$ has a fixed point.*

4 Some Extensions and Generalizations of Nonexpansive Mappings

In [32], Kannan extended nonexpansive mappings as follows and obtained some fixed point results.

Definition 17 Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be *Kannan-nonexpansive* if for all $x, y \in X$,

$$\|T(x) - T(y)\| \leq \frac{1}{2}\{\|x - T(x)\| + \|y - T(y)\|\}.$$

Theorem 4 *[32]. Let K be a nonempty bounded closed and convex subset of a reflexive Banach space X and $T : K \to K$ a Kannan-nonexpansive mapping. If $\sup_{y \in F} \|y - T(y)\| < diam(F)$ for every nonempty bounded closed convex subset F of K, containing more than one element and mapped into itself by T. Then T has a unique fixed point in K.*

In 1972, Goebel and Kirk [24] introduced the notion of asymptotically nonexpansive mappings.

Definition 18 Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if for each $x, y \in K$,

$$\|T^n(x) - T^n(y)\| \leq k_n \|x - y\|,$$

where $\{k_n\}$ is a sequence of real numbers such that $k_n \to 1$ as $n \to \infty$.

The following example shows that an asymptotically nonexpansive mapping need not be nonexpansive.

Example 2 [24]. Let B be a unit ball in the Hilbert space ℓ^2 and $T : B \to B$ defined by

$$T(x) = (0, x_1^2, A_2 x_2, A_3 x_3, \ldots),$$

where A_i is a sequence of numbers such that $0 < A_i < 1$ and $\prod_{i=2}^{\infty} A_i = \frac{1}{2}$. Then T is an asymptotically nonexpansive mapping but not nonexpansive.

Theorem 5 *[24]. Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T : K \to K$ an asymptotically nonexpansive mapping. Then T has a fixed point.*

Proof For each $x \in K$ and $r > 0$ let $S(x, r)$ denote the spherical ball centered at x with radius r. Let $y \in K$ be fixed, and let the set R_y, consist of those numbers ρ for which there exists an integer k such that

$$K \cap \left(\bigcap_{i=k}^{\infty} S(T^i(y), \rho) \right) \neq \emptyset.$$

If d is the diameter of K then $d \in R_y$, and $R_y \neq \emptyset$. Let $\rho_0 = \text{g.l.b. } R_y$, and for each $\varepsilon > 0$ define

$$C_\varepsilon = \bigcup_{k=1}^{\infty} (\bigcap_{i=k}^{\infty} S(T^i(y), \rho_0 + \varepsilon)).$$

So, for every $\varepsilon > 0$ the sets $C_\varepsilon \cap K$ are nonempty and convex. By reflexivity of X, we get

$$C = \bigcap_{\varepsilon > 0} (\overline{C_\varepsilon} \cap K) \neq \emptyset.$$

Note that for $x \in C$ and $\eta > 0$ there exists an integer N such that if $i \geq N$, then

$$\|x - T^i(y)\| \leq \rho_0 + \eta.$$

Now let $x \in C$ and suppose the sequence $\{T^n(x)\}$ does not converge to x (i.e., suppose $T(x) \neq x$). Then there exists $\varepsilon > 0$ and a subsequence $\{T^{n_i}(x)\}$ of $\{T^n(x)\}$ such that $\|T^{n_i}(x) - x\| \geq \varepsilon, \quad i \in \mathbb{N}$. Now, for $m > n$,

$$\|T^n(x) - T^m(x)\| \leq k_n \|x - T^{m-n}(x)\|.$$

Assume $\rho_0 > 0$ and choose $\alpha > 0$ such that

$$\left(1 - \delta\left(\frac{\varepsilon}{\rho_0 + \alpha}\right)\right)(\rho_0 + \alpha) < \rho_0.$$

Choose n so that

$$\|x - T^n(x)\| \geq \varepsilon \text{ and } k_n \left(\rho_0 + \frac{\alpha}{2}\right) \leq \rho_0 + \alpha.$$

If $N \geq n$ is sufficiently large, then $m > N$ implies

$$\|x - T^{m-n}(y)\| \leq \rho_0 + \frac{\alpha}{2}.$$

Now, we have

$$\|T^n(x) - T^m(y)\| \le k_n\|x - T^{m-n}(y)\| \le \rho_0 + \alpha \text{ and } \|x - T^m(y)\| \le \rho_0 + \alpha.$$

By uniform convexity of X, if $m > N$,

$$\left\| \frac{x + T^n(x)}{2} - T^m(y) \right\| \le \left(1 - \delta\left(\frac{\varepsilon}{\rho_0 + \alpha}\right)\right)(\rho_0 + \alpha) < \rho_0,$$

which contradicts the definition of ρ_0. Therefore we conclude $\rho_0 = 0$ or $T(x) = x$. If $\rho_0 = 0$, then $\{T^n(y)\}$ is a Cauchy sequence and $T^n(y) \to x = T(x)$ as $n \to \infty$. Therefore the set C consists of a single point which is a fixed point of T.

In 1974, Kirk [37] extended the concept of asymptotically nonexpansive mappings to asymptotically nonexpansive type mappings as follows and obtained a fixed point theorem for these mapping.

Definition 19 A mapping $T : K \to K$ is said to be asymptotically nonexpansive type if for each $x \in K$

$$\limsup_{i\to\infty} \left\{ \sup_{y\in K} \left[\|T^i(x) - T^i(y)\| - \|x - y\|\right] \right\} \le 0. \tag{2}$$

Theorem 6 *[37]. Let K be a nonempty bounded closed convex subset of a Banach space X for which $\varepsilon_0(X) < 1$. Suppose $T : K \to K$ is an asymptotically nonexpansive type mapping such that T^N is continuous for some positive integer N. Then T has a fixed point in K.*

Proof Let $x \in K$ be fixed. There exists a number $\rho_0 = \rho_0(x) \ge 0$ which is minimal with respect to the property: for each $\varepsilon > 0$ there exists an integer K such that

$$K \cap \left(\bigcap_{i=k}^{\infty} S(T^i(x); \rho_0 + \varepsilon) \right) \ne \emptyset.$$

Let

$$C_\varepsilon = \bigcup_{k=1}^{\infty} \left(\bigcap_{i=k}^{\infty} S(T^i(x); \rho_0 + \varepsilon) \right).$$

Then the set C_ε is nonempty bounded and convex, hence by reflexivity of X the closure $\overline{C_\varepsilon}$ of C_ε is weakly compact and

$$C = \bigcap_{\varepsilon>0} (\overline{C_\varepsilon} \cap K) \ne \emptyset.$$

Now let $z \in C$, and

$$d(z) = \limsup_{i\to\infty} \|z - T^i(z)\|.$$

If $\rho_0(x) = 0$, then clearly $T^n(x) \to z$ as $n \to \infty$. Let $\eta > 0$ and using (2), choose M so that $i \geq M$ implies

$$\sup_{y \in K} \left[\|T^i(z) - T^i(y)\| - \|z - y\| \right] \leq \frac{1}{3}\eta.$$

Given $i \geq M$, since $T^n(x) \to z$ there exists $m > i$ such that

$$\|T^m(x) - z\| \leq \frac{1}{3}\eta \text{ and } \|T^{m-i}(x) - z\| \leq \frac{1}{3}\eta.$$

Thus if $i \geq M$,

$$\begin{aligned} \|z - T^i(z)\| &\leq \|z - T^m(x)\| + \|T^m(x) - T^i(z)\| \\ &\leq \|z - T^m(x)\| + \|T^i(z) - T^i(T^{m-i}(x))\| - \|z - T^{m-i}(x)\| \\ &+ \|z - T^{m-i}(x)\| \\ &\leq \frac{1}{3}\eta + \sup_{y \in K} \left[\|T^i(z) - T^i(y)\| - \|z - y\| \right] + \frac{1}{3}\eta \\ &\leq \eta. \end{aligned}$$

This proves $T^n(z) \to z$ as $n \to \infty$, that is, $d(z) = 0$. But $d(z) = 0$ implies $T^{Ni}(z) \to z$ as $i \to \infty$ and by continuity of T^N, we have $T^N(z) = z$. Thus

$$T(z) = T(T^{Ni}(z)) = T^{Ni+1}(z) \to z \text{ as } i \to \infty, \tag{3}$$

and $T(z) = z$. Therefore we may assume $\rho_0(x) > 0$ and $d(z) > 0$.

Now let $\varepsilon > 0$, $\varepsilon \leq d(z)$. By the definition of ρ_0 there exists an integer N^* such that if $i \geq N^*$ then

$$\|z - T^i(x)\| \leq \rho_0 + \varepsilon,$$

and by (2) there exists N^{**} such that if $i \geq N^{**}$ then

$$\sup_{y \in K} [\|T^i(z) - T^i(y)\| - \|z - y\|] \leq \varepsilon.$$

Choose j so that $j \geq N^{**}$ and so that

$$\|z - T^j(z)\| \geq d(z) - \varepsilon.$$

Thus if $i - j \geq N^*$,

$$\begin{aligned}\|T^j(z) - T^i(x)\| &= \left\{\|T^j(z) - T^j(T^{i-j}(x))\| - \|z - T^{i-j}(x)\|\right\} + \|z - T^{i-j}(x)\| \\ &\leq \varepsilon + (\rho_0 + \varepsilon) \\ &= \rho_0 + 2\varepsilon.\end{aligned}$$

Let $w = \frac{1}{2}(z + T^j(z))$. By the modulus of convexity of X, we have

$$\|w - T^i(x)\| \leq \left(1 - \delta\left(\frac{d(z) - \varepsilon}{\rho_0 + 2\varepsilon}\right)\right)(\rho_0 + 2\varepsilon), \quad i \geq N^* + j.$$

By the minimality of ρ_0, we have

$$\rho_0 \leq \left(1 - \delta\left(\frac{d(z) - \varepsilon}{\rho_0 + 2\varepsilon}\right)\right)(\rho_0 + 2\varepsilon).$$

Letting $\varepsilon \to 0$, we get

$$\rho_0 \leq \left(1 - \delta\left(\frac{d(z)}{\rho_0}\right)\right)\rho_0.$$

This implies $1 - \delta\left(\frac{d(z)}{\rho_0}\right) \geq 1$ and hence $\delta\left(\frac{d(z)}{\rho_0}\right) = 0$. It follows from the definition of ε_0 that $\frac{d(z)}{\rho_0} \leq \varepsilon_0$. So,

$$d(z) \leq \varepsilon_0 \rho_0(x).$$

Let $d(x) = \limsup_{i\to\infty} \|x - T^i(x)\|$. Then we have $\rho_0(x) \leq d(x)$, and

$$d(z) \leq \varepsilon_0 d(x). \tag{4}$$

Also notice that $\|z - x\| \leq d(x) + \rho_0(x) \leq 2d(x)$.

Now, fix $x_0 \in K$ and define the sequence $\{x_n\}$ by $x_{n+1} = z(x_n), n \in \mathbb{N} \cup \{0\}$, where $z(x_n)$ is obtained from x_n in the same manner as $z(x)$ from x. If for any n we have $\rho(x_n) = 0$ then, as seen above, $T(x_{n+1}) = x_{n+1}$. Otherwise, we have by (4)

$$\|x_{n+1} - x_n\| \leq 2d(x_n) \leq 2(\varepsilon_0)^n d(x_0).$$

Since $\varepsilon_0 < 1$, the sequence $\{x_n\}$ is Cauchy. Thus there exists $y \in K$ such that $x_n \to y$ as $n \to \infty$. Also

$$\begin{aligned}\|y - T^i(y)\| &\leq \|y - x_n\| + \|x_n - T^i(x_n)\| + \|T^i(x_n) - T^i(y)\| \\ &\leq \|y - x_n\| + \|x_n - T^i(x_n)\| + \left[\|T^i(x_n) - T^i(y)\| - \|x_n - y\|\right] \\ &\quad + \|x_n - y\|.\end{aligned}$$

Thus

$$\begin{aligned} d(y) &= \limsup_{i\to\infty} \|y - T^i(y)\| \\ &\leq \limsup_{i\to\infty} 2\|x_n - y\| + \limsup_{i\to\infty} \|x_n - T^i(x_n)\| \\ &+ \limsup_{i\to\infty} \left[\|T^i(x_n) - T^i(y)\| - \|x_n - y\|\right] \\ &\leq d(x_n) + 2\|x_n - y\|. \end{aligned}$$

Since $x_n \to y$ and $d(x_n) \to 0$ as $n \to \infty$, $d(y) = 0$. But as seen before (3), this implies $T(y) = y$.

Using the concept of asymptotic center Edelstein [15] obtained the following fixed point theorem.

Theorem 7 *Let K be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T : K \to K$ be a mapping such that $\{T^n(x)\}$ is bounded for some $x \in K$. Let c be the asymptotic center of $\{T^n(x)\}$ with respect to K and there exists N such that*

$$\|T(c) - T^n(x)\| \leq \|c - T^{n-1}(x)\|$$

for $n \geq N$, then c is a fixed point of T.

Kar and Veeramani [33] generalized certain results from Edelstein [15], Goebel and Kirk [24], and Kirk and Xu [38] as follows.

Theorem 8 *Let K be a nonempty closed and convex subset of a uniformly convex Banach space X. Let $T : K \to K$ be a mapping such that $\{T^n(x_0)\}$ is bounded for some $x_0 \in K$. Suppose that $K_0 \subset K$ is a nonempty closed convex set which is invariant under T. Let c be the asymptotic center of $\{T^n(x_0)\}$ with respect to K_0 and there exists N such that*

$$\|T^p(c) - T^{n+p}(x_0)\| \leq k_{np}\|c - T^n(x_0)\|$$

for all $n, p \geq N$, where $\lim_{p\to\infty} \lim_{n\to\infty} k_{np} = 1$.
Then $T^p(c)$ converges strongly to c. Further, if T is continuous at c then c is a fixed point of T.

In 1973, Goebel et al. [26], considered the following general class of nonexpansive mappings.

Definition 20 [26]. Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be a generalized nonexpansive mapping if for every $x, y \in K$

$$\begin{aligned} \|T(x) - T(y)\| \leq a_1\|x - y\| + a_2\|T(x) - x\| + a_3\|T(y) - y\| + a_4\|T(y) - x\| \\ + a_5\|T(x) - y\|, \end{aligned} \tag{5}$$

where $a_i \geq 0$ and $\sum_{i=1}^{5} a_i = 1$.

Theorem 9 *[8]. Let K be a nonempty weakly compact convex subset of a Banach space X with normal structure and $T : K \to K$ a generalized nonexpansive mapping. Then T has a fixed point in K.*

In 1983, Jaggi [30] introduced a generalization of nonexpansive mapping. This class of mappings is known as *Jaggi-nonexpansive* mappings [16].

Definition 21 Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be *Jaggi-nonexpansive* if for every T-invariant closed convex subset E of K with at least two points and for every $x \in E$,

$$\sup\{\|T(x) - T(y)\| : y \in E\} \leq \sup\{\|x - y\| : y \in E\}.$$

The following example shows that a Jaggi-nonexpansive mapping need not be quasi-nonexpansive.

Example 3 [16]. Let $K = [0, 1]$ be a subset of $\mathbb{R}$ endowed with the usual norm. Define $T : K \to K$ by

$$T(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \subset K, \\ 1 - x, & \text{if } x \in K,\ x \notin \mathbb{Q}, \end{cases}$$

where $\mathbb{Q}$ is the set of all rational numbers. Then it is easy to verify that the mapping T is Jaggi-nonexpansive mapping but not quasi-nonexpansive.

Theorem 10 *[30]. Let K be a nonempty bounded closed convex subset of a reflexive Banach space X with normal structure and $T : K \to K$ a Jaggi-nonexpansive mapping. Then T has a fixed point in K.*

Proof Let $\mathcal{A}$ be a family of closed convex subsets H of K with $T(H) \subset H$. The family $\mathcal{A}$ is nonempty as $K \in \mathcal{A}$. Using Zorn's lemma and reflexivity of X, $\mathcal{A}$ has a minimal element E. If E is singleton, then $T(E) \subset E$ implies that T has a fixed point. Thus let E has at least two elements. As K has normal structure, we can find an element $c \in E$ such that

$$\sup_{t \in E} \|c - t\| = d \text{ (say)} < \operatorname{diam}(E).$$

Also, the closed convex hull $\overline{\operatorname{conv}}(T(E))$ of $T(E)$ is contained in E and belongs to $\mathcal{A}$. Therefore $\overline{\operatorname{conv}}(T(E)) = E$. Let

$$F = \{z \in E : \sup_{t \in E} \|z - t\| \leq d\}.$$

As $c \in F$, so F is nonempty. We claim that $T(F) \subset F$. For any $z \in F$, we see that

$$\sup_{t \in E} \|T(z) - T(t)\| \leq \sup_{t \in E} \|z - t\| \leq d.$$

This implies that $T(E) \subset \overline{B}(T(z), d)$, a closed ball centered at $T(z)$ with radius d. Therefore, $E = \overline{\text{conv}}(T(E))$ is also contained in $\overline{B}(T(z), d)$. Consequently, $\sup_{t \in E} \|T(z) - t\| \le d$. Thus $T(z) \in F$. It is easy to verify that F is closed and convex. Lastly

$$\begin{aligned}\text{diam}(F) &= \sup_{x,y \in F} \|x - y\| \\ &\le \sup_{x \in F, y \in E} \|x - y\|.\end{aligned}$$

Also, for each $x \in F$,

$$\sup_{y \in E} \|x - y\| \le d < \text{diam}(E).$$

Therefore

$$\text{diam}(F) \le \sup_{x \in F, y \in E} \|x - y\| < \text{diam}(E).$$

This implies that F is a proper subset of E, which is a contradiction.

Remark 2 In [34], Kassay showed that the converse of the above theorem is also true. More precisely, a reflexive Banach space having normal structure can be characterized by the fixed point property for Jaggi-nonexpansive mappings. For more details on Jaggi-nonexpansive mappings, we refer to [16].

In 2007, Goebel and Pineda [23] introduced the concept of mean type α-nonexpansive mappings. We say $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index if $n \ge 1$, $\alpha_1 > 0$, $\alpha_n > 0$, $\alpha_i \ge 0$ and $\sum_{i=1}^{n} \alpha_i = 1$.

Definition 22 Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be *mean type α-nonexpansive* if for all $x, y \in K$

$$\sum_{i=1}^{n} \alpha_i \|T^i(x) - T^i(y)\| \le \|x - y\|.$$

Example 4 [23]. Let B be a closed unit ball in $(\ell^1, \|.\|_1)$ and $\tau : [-1, 1] \to [-1, 1]$ a function defined by

$$\tau(t) = \begin{cases} 2t + 1, & \text{if } t \in [-1, -\frac{1}{2}), \\ 0, & \text{if } t \in [-\frac{1}{2}, \frac{1}{2}), \\ 2t - 1, & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Define a mapping $T : B \to B$ by

$$T(x_1, x_2, \ldots) = \left(\tau(x_2), \frac{2}{3}x_3, x_4, x_5, \ldots\right).$$

Then T is a mean type α-nonexpansive with $\alpha = (\frac{1}{2}, \frac{1}{2})$ but not nonexpansive.

Theorem 11 *[17, 23]. Let K be a nonempty bounded closed convex subset of a Banach space X and $T : K \to K$ a mean type (α_1, α_2)-nonexpansive mapping. Then T has an a.f.p.s., provided that $\alpha_1 \geq \frac{1}{2}$.*

Proof Fix $\varepsilon > 0$. Since T_α is a nonexpansive self-mapping on K, $\inf_K \|T_\alpha(x) - x\| = 0$. Thus there exists $x_\varepsilon \in K$ for which $\|T_\alpha(x_\varepsilon) - x_\varepsilon\| \leq \alpha_2\varepsilon$. For T is (α_1, α_2)-nonexpansive, we have

$$\begin{aligned}
\alpha_1\|T^2(x_\varepsilon) - T(x_\varepsilon)\| + \alpha_2\|T^3(x_\varepsilon) - T^2(x_\varepsilon)\| &\leq \|T(x_\varepsilon) - x_\varepsilon\| \\
&= \|T(x_\varepsilon) - T_\alpha(x_\varepsilon) + T_\alpha(x_\varepsilon) - x_\varepsilon\| \\
&\leq \|T(x_\varepsilon) - T_\alpha(x_\varepsilon)\| + \|T_\alpha(x_\varepsilon) - x_\varepsilon\| \\
&\leq \|(1-\alpha_1)T(x_\varepsilon) - \alpha_2 T^2(x_\varepsilon)\| + \alpha_2\varepsilon \\
&= \alpha_2\|T(x_\varepsilon) - T^2(x_\varepsilon)\| + \alpha_2\varepsilon.
\end{aligned}$$

Thus,

$$\begin{aligned}
&(\alpha_1 - \alpha_2)\|T(x_\varepsilon) - T^2(x_\varepsilon)\| + \alpha_2\|T^3(x_\varepsilon) - T^2(x_\varepsilon)\| \leq \alpha_2\varepsilon \\
&\iff (2\alpha_1 - 1)\|T(x_\varepsilon) - T^2(x_\varepsilon)\| + \alpha_2\|T^3(x_\varepsilon) - T^2(x_\varepsilon)\| \leq \alpha_2\varepsilon.
\end{aligned}$$

Since $\alpha_1 \geq \frac{1}{2}$, we know $2\alpha_1 - 1 \geq 0$, so $\|T(z_\varepsilon) - z_\varepsilon\| \leq \varepsilon$, where $z_\varepsilon = T^2(x_\varepsilon) \in K$.

Theorem 12 *[18]. Let K be a nonempty bounded closed convex subset of a Banach space for which $\varepsilon_0(X) < 1$ and $T : K \to K$ an (α, p)-nonexpansive mapping with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $p \geq 1$ such that*

$$\left(\sum_{j=1}^{n}\left(\sum_{i=j}^{n}\alpha_i\right)\right)^{1/p} < \kappa(X).$$

Then T has a fixed point.

In 2008, Kirk and Xu [38] introduced the notion of *pointwise eventually nonexpansive mappings*.

Definition 23 Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be pointwise eventually nonexpansive if for each $x \in K$ there exists $N(x) \in \mathbb{N}$ such that if $n \geq N(x)$,

$$\|T^n(x) - T^n(y)\| \leq \|x - y\|$$

for each $y \in K$.

Butsan et al. [12] obtained the following result for pointwise eventually nonexpansive mappings.

Theorem 13 *[12]. Let K be a nonempty compact convex subset of a nearly uniformly convex Banach space X. Then every pointwise eventually nonexpansive mapping $T : K \to K$ has a fixed point.*

In 2010, Nicolae [43] introduced the concept of nonexpansive mapping with respect to orbits (wrt orbits, in short).

Definition 24 Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be nonexpansive wrt orbits if for all $x, y \in K$,

$$\|T(x) - T(y)\| \leq \sup_{y \in O_T(y)} \|x - y\|,$$

where $O_T(y) := \{T^n(y) : n \in \mathbb{N} \cup \{0\}\}$.

Proposition 2 *[16]. Let K be a nonempty subset of a Banach space X and $T : K \to K$ a nonexpansive wrt orbits. Then T is a Jaggi-nonexpansive as well as quasi-nonexpansive mapping.*

Remark 3 The converse of the above proposition does not hold. In fact, the mapping considered in Example 3 is Jaggi-nonexpansive mapping but it fails to be quasi-nonexpansive mapping and therefore it can not be nonexpansive wrt orbits.

Amini-Harandi et al. [4] obtained a fixed point theorem for nonexpansive mappings wrt orbits using weak normal structure in a Banach space.

Theorem 14 *[4]. Let K be a nonempty weakly compact convex subset of a Banach space X. Then, X has weak normal structure if and only if every nonexpansive mapping wrt orbits $T : K \to K$ has a fixed point in K.*

Remark 4 [16]. In view of Proposition 2 it is evident that Theorem 14 is indeed, a corollary of Theorem 10.

In 2011, Aoyama and Kohsaka [5] introduced the notion of an α-nonexpansive mapping.

Definition 25 Let K be a nonempty subset of a Banach space X and $\alpha < 1$ a real number. A mapping $T : K \to K$ is said to be α-nonexpansive if for all $x, y \in K$,

$$\|T(x) - T(y)\|^2 \leq \alpha\|T(x) - y\|^2 + \alpha\|T(y) - x\|^2 + (1 - 2\alpha)\|x - y\|^2.$$

Although, α-nonexpansive mappings are defined for any real number $\alpha < 1$, Ariza-Ruiz et al. [6] pointed out that this concept is trivial for $\alpha < 0$.

Theorem 15 *[5]. Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $T : K \to K$ an α-nonexpansive mapping. Then $F(T)$ is nonempty if and only if there exists $x \in K$ such that $\{T^n(x)\}$ is bounded.*

Proof Fix $y \in K$. Since T is α-nonexpansive, for all $n \in \mathbb{N}$

$$\|T^{n+1}(x) - T(y)\|^2 \le \alpha\|T^{n+1}(x) - y\|^2 + \alpha\|T(y) - T^n(x)\|^2 + (1-2\alpha)\|T^n(x) - y\|^2.$$

Let μ be a Banach limit. Then

$$\begin{aligned}\mu_n\|T^n(x) - T(y)\|^2 \le \alpha\mu_n\|T^n(x) - y\|^2 + \alpha\mu_n\|T(y) - T^n(x)\|^2 \\ +(1-2\alpha)\mu_n\|T^n(x) - y\|^2,\end{aligned}$$

and hence

$$(1-\alpha)\mu_n\|T^n(x) - T(y)\|^2 \le (1-\alpha)\mu_n\|T^n(x) - y\|^2.$$

Since $1 - \alpha > 0$, we get

$$\mu_n\|T^n(x) - T(y)\|^2 \le \mu_n\|T^n(x) - y\|^2. \tag{6}$$

Let $g : K \to \mathbb{R}$ be the function defined by $g(y) = \mu_n\|T^n(x) - y\|^2$ for all $y \in K$. We show that g is a convex, continuous, and coercive function. In fact, the convexity of g is obvious. Let $\{y_m\}$ be a sequence in K such that $y_m \to y$. Then for all $n \in \mathbb{N}$,

$$\begin{aligned}&\left|\|T^n(x) - y_m\|^2 - \|T^n(x) - y\|^2\right| \\ &= \left|\|T^n(x) - y_m\| - \|T^n(x) - y\|\right| \left(\|T^n(x) - y_m\| + \|T^n(x) - y\|\right) \\ &\le \|y_m - y\| \sup\{\|T^n(x) - y_m\| + \|T^n(x) - y\| : m, n \in \mathbb{N}\}.\end{aligned}$$

This shows that $h : K \to \ell^\infty$ defined by

$$h(z) = \{\|T(x) - z\|^2, \|T^2(x) - z\|^2, \dots\}$$

for all $z \in K$ is continuous. Therefore $g = \mu \circ h$ is also continuous. Next we show that g is coercive. If $\{z_m\}$ is a sequence in K such that $\|z_m\| \to \infty$, then we have

$$\|T^n(x) - z_m\|^2 \ge (\|z_m\| - \|T^n(x)\|)^2 \ge \|z_m\| \left(\|z_m\| - 2\sup_{n\in\mathbb{N}}\|T^n(x)\|\right),$$

and hence $g(z_m) \to \infty$. From Proposition 1, there exists $u \in K$ such that $g(u) = \inf g(K)$. Since X is uniformly convex, such a point u is unique and $g(T(u)) \le g(u)$. Therefore u must be a fixed point of T.

In 2016, Llorens-Fuster [40] introduced the concept of orbitally nonexpansive mappings.

Definition 26 [40]. Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be orbitally nonexpansive if for every nonempty closed convex T-invariant subset D of K, there exists some $x_0 \in D$ such that

$$\limsup_{n\to\infty} \|T^n(x_0) - T(x)\| \le \limsup_{n\to\infty} \|T^n(x_0) - x\|$$

for every $x \in D$.

Example 5 [40]. Consider the Hilbert space $(\ell^2, \|.\|_2)$ and set

$$K := \{x = (x_1, x_2, \dots) \in \ell^2 : \ \|x\|_2 \le 1, \ x_n \ge 0, \ n \in \mathbb{N}\}.$$

Let $T : K \to K$ be a mapping defined by

$$T(x) = (x_1^2, x_2^2, \dots).$$

Then T is orbitally nonexpansive but not nonexpansive.

Theorem 16 *[8]. Let K be a nonempty convex subset of a Banach space X. Then X has normal structure if and only if for each non-constant bounded sequence $\{x_n\}$ in K, the function $g(x) = \limsup_{n\to\infty} \|x_n - x\|$ is not constant in conv$\{x_n\}$.*

Theorem 17 *[40]. Let K be a nonempty weakly compact convex subset of a Banach space X with normal structure and $T : K \to K$ be an orbitally nonexpansive mapping. Then T has a fixed point.*

Proof Since K is a weakly compact set, by Zorn's lemma there exists a nonempty closed convex, T-invariant subset C of K with no proper subsets enjoying these characteristics. From the definition of orbitally nonexpansive mapping, there exists $x_0 \in C$ such that for every $x \in C$,

$$\limsup_{n\to\infty} \|T^n(x_0) - T(x)\| \le \limsup_{n\to\infty} \|T^n(x_0) - x\|.$$

We consider two cases.

Case 1: There exists $z \in C$ such that $T^n(x_0) = z$ for n large enough. We claim that, in this case, z is a fixed point of T. Indeed,

$$\|z - T(z)\| = \limsup_{n\to\infty} \|T^n(x_0) - T(z)\| \le \limsup_{n\to\infty} \|T^n(x_0) - z\| = 0.$$

Case 2: The sequence $\{T^n(x_0)\}$ is bounded and not (eventually) constant. Since the Banach space X has normal structure, from Theorem 16, the real function $g : C \to [0, 1)$ defined by

$$g(x) := \limsup_{n\to\infty} \|x - T^n(x_0)\|$$

is not constant on conv$\{T^n(x_0) : n \in \mathbb{N}\}$. Then g takes at least two different real values, that is, there exist $v_1, v_2 \in$ conv$\{T^n(x_0) : n \in \mathbb{N}\} \subset C$ such that

$$r_1 := g(v_1) < g(v_2) := r_2.$$

Let $r := \frac{1}{2}(r_1 + r_2)$ and consider the set

$$M := \{x \in C : g(x) \le r\}.$$

It is easy to verify that M is nonempty, closed, and convex and $M \neq C$ because $v_2 \notin M$. Since T is an orbitally nonexpansive mapping, we have

$$g(T(x)) := \limsup_{n\to\infty} \|T(x) - T^n(x_0)\| \le \limsup_{n\to\infty} \|x - T^n(x_0)\| = g(x) \le r.$$

Thus, M is a nonempty closed convex and T-invariant subset of C with $M \neq C$, which is a contradiction to the minimality of C. So, Case 2 is not possible. This completes the proof.

In 2016, Bin Dehaish and Khamsi [7] considered the following class of nonexpansive mappings in partially ordered Banach spaces (see also [53]).

Definition 27 [7]. Let K be a nonempty subset of a partially ordered Banach space $(X, \|.\|, \preceq)$. Let $T : K \to K$ be a mapping. T is said to be monotone nonexpansive if

$$\|T(x) - T(y)\| \le \|x - y\| \text{ and } T(x) \preceq T(y)$$

for $x, y \in C$ such that $x \preceq y$.

Theorem 18 *[7]. Let $(X, \|.\|, \preceq)$ be a partially ordered UCED Banach space such that order intervals are closed and convex. Let K be a nonempty weakly compact convex subset of X not reduced to one point. Let $T : K \to K$ be a monotone nonexpansive mapping. Assume there exists $x_1 \in C$ such that x_1 and $T(x_1)$ are comparable. Then T has a fixed point.*

5 Suzuki-Type Generalized Nonexpansive Mappings

In 2008, Suzuki [54] introduced a new class of nonexpansive mappings and obtained some fixed point results.

Definition 28 Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to satisfy condition (C) if for all $x, y \in K$

$$\frac{1}{2}\|x - T(x)\| \le \|x - y\| \text{ implies } \|T(x) - T(y)\| \le \|x - y\|.$$

A mapping satisfying condition (C) is also known as Suzuki-type generalized nonexpansive mapping. This class of mappings need not be continuous.

Example 6 [54]. Let $K = [0, 3]$ be a subset of $\mathbb{R}$ endowed with the usual norm. Define $T : K \to K$ by

$$T(x) = \begin{cases} 0, & \text{if } x \neq 3, \\ 1, & \text{if } x = 3. \end{cases}$$

Then T satisfies condition (C). However, T is not continuous and therefore T is not a nonexpansive mapping.

Theorem 19 *[54]. Let K be a nonempty convex subset of a Banach space X and $T : K \to K$ a mapping satisfying the condition (C). Assume that either K is compact or K is weakly compact and X has the Opial property. Then T has a fixed point.*

García-Falset *et al.* [20] introduced a generalization of Suzuki-type generalized nonexpansive mappings as follows:

Definition 29 Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to satisfy condition (E_μ) if there exists $\mu \geq 1$ such that for all $x, y \in K$,

$$\|x - T(y)\| \leq \mu\|x - T(x)\| + \|x - y\|.$$

We say that T satisfies the condition (E) on K whenever T satisfies (E_μ) for some $\mu \geq 1$.

Theorem 20 *[20]. Let K be a nonempty subset of a Banach space X having the Opial property and $T : K \to X$ a mapping satisfying the condition (E). Suppose there exists an a.f.p.s. $\{x_n\}$ for T such that $x_n \to z \in K$. Then, $T(z) = z$.*

Llorens-Fuster and Gálvez [41] introduced the following notion of condition (L) and obtained a fixed to point theorem for a mapping satisfying condition (L).

Definition 30 Let K to be a nonempty subset of a Banach space X. A mapping $T : K \to K$ satisfies condition (L) on K provided that

(a) If a set $D \subset K$ is nonempty closed convex and T-invariant, (i.e., $T(D) \subset D$), then there exists an a.f.p.s. for T in D.
(b) For any a.f.p.s. $\{x_n\}$ of T in $K$and each $x \in K$,

$$\limsup_{n\to\infty} \|x_n - T(x)\| \leq \limsup_{n\to\infty} \|x_n - x\|.$$

Example 7 [41]. Let $K = \left[0, \frac{2}{3}\right]$ be a subset of $\mathbb{R}$ endowed with the usual norm. Define $T : K \to K$ by

$$T(x) = x^2.$$

One can verify that the mapping T satisfies condition (L), but it is neither a generalized nonexpansive mappings nor satisfies the condition (C).

Theorem 21 *[41]. Let K be a nonempty weakly compact convex subset of a Banach space X having normal structure and $T : K \to K$ a mapping satisfying condition (L). Then, T has a fixed point.*

Proof Let C be a minimal subset of K. Since T satisfies condition (L), there exists an a.f.p.s. $\{x_n\}$ for T in C. This sequence is either constant, and hence it consists of a fixed point of T, or it is non-constant. In this case, since X has normal structure, from Theorem 16, the real function $g : C \to [0, \infty)$ given by

$$g(x) = \limsup_{n\to\infty} \|x - x_n\|,$$

is not constant in $\text{conv}\{x_n : n \in \mathbb{N}\} \subset C$. Then, g takes at least two values. If r is an intermediate value, then the set

$$M := \{x \in C : g(x) \leq r\},$$

is nonempty, convex, and closed and $M \neq C$. From condition (L), the set M is also T-invariant which contradicts the minimality of C.

Recently, Pant and Shukla [47] introduced a wider class of nonexpansive mappings, which properly contains nonexpansive, Suzuki-type generalized nonexpansive mappings and partially extends firmly-nonexpansive and α-nonexpansive mappings.

Definition 31 [47]. Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is called a generalized α-nonexpansive if there exists an $\alpha \in [0, 1)$ such that for all $x, y \in K$,

$$\begin{aligned}&\frac{1}{2}\|x - T(x)\| \leq \|x - y\| \text{ implies}\\ &\|T(x) - T(y)\| \leq \alpha\|T(x) - y\| + \alpha\|x - T(y)\| + (1 - 2\alpha)\|x - y\|.\end{aligned}$$

Lemma 2 *[47]. Let K be a nonempty subset of a Banach space X and $T : K \to K$ a generalized α-nonexpansive mapping. Then for all $x, y \in K$,*

$$\|x - T(y)\| \leq \frac{(3 + \alpha)}{(1 - \alpha)}\|x - T(x)\| + \|x - y\|.$$

Theorem 22 *Let K be a nonempty weakly compact convex subset of a UCED Banach space X and $T : K \to K$ a generalized α-nonexpansive mapping. If T has an a.f.p.s. then T has a fixed point in K.*

Proof Let $\{x_n\}$ be an a.f.p.s. for T in K. Define a continuous convex function $f : K \to [0, \infty)$ by

$$f(x) = \limsup_{n\to\infty} \|x_n - x\|.$$

Since K is weakly compact and f is continuous, there exists a point $p \in K$ such that

$$f(p) = \min\{f(x) : x \in K\}.$$

By Lemma 2, we have

$$\|x_n - T(p)\| \le \frac{(3+\alpha)}{(1-\alpha)}\|x_n - T(x_n)\| + \|x_n - p\|.$$

So,

$$\limsup_{n\to\infty} \|x_n - T(p)\| \le \limsup_{n\to\infty} \left\{\frac{(3+\alpha)}{(1-\alpha)}\|x_n - T(x_n)\| + \|x_n - p\|\right\}.$$

Thus $f(T(p)) \le f(p)$. Since $f(p)$ is minimum, $f(T(p)) = f(p)$. For f is quasi-convex, we get

$$f(p) \le f\left(\frac{p + T(p)}{2}\right) < \max\{f(p), f(T(P))\} = f(p),$$

a contradiction, unless $T(p) = p$. This completes the proof.

In 2019, Pandey *et al.* [46] (see also [48]) further generalized the class of generalized α-nonexpansive mapping as follows:

Definition 32 Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be a *generalized α-Reich-Suzuki nonexpansive mapping* if there exists an $\alpha \in [0, 1)$ for each $x, y \in K$,

$$\frac{1}{2}\|x - T(x)\| \le \|x - y\| \text{ implies } \|T(x) - T(y)\| \le \max\{P(x, y), Q(x, y)\},$$

where

$$P(x, y) = \alpha\|T(x) - x\| + \alpha\|T(y) - y\| + (1 - 2\alpha)\|x - y\|;$$

and

$$Q(x, y) = \alpha\|T(x) - y\| + \alpha\|T(y) - x\| + (1 - 2\alpha)\|x - y\|.$$

Lemma 3 *[46]. Let K be a nonempty subset of a Banach space X and $T : K \to K$ a generalized α-Reich-Suzuki nonexpansive mapping. Then for each $x, y \in K$,*

$$\|x - T(y)\| \le \left(\frac{3+\alpha}{1-\alpha}\right)\|x - T(x)\| + \|x - y\|.$$

Proposition 3 *[20]. Let K be a nonempty closed convex subset of a Banach space X and $T : K \to X$ be a mapping satisfying the condition (E) with $F(T) \neq \emptyset$. Then T is quasi-nonexpansive.*

Remark 5 In view of the above lemma, a generalized α-Reich-Suzuki nonexpansive mapping satisfies the condition (E) with $\mu = \dfrac{(3+\alpha)}{(1-\alpha)}$. Therefore the class of mappings satisfying the condition (E) is larger.

Theorem 23 *[46]. Let K be a nonempty bounded closed subset of a Banach space X, and $T : K \to K$ a mapping satisfying the condition (E). Suppose that there is an a.f.p.s. for T such that asymptotic center is nonempty and compact. Then T has a fixed point.*

6 Convergence of Fixed Points of Nonexpansive Type Mappings

A well-known way to find a fixed point of a nonexpansive mapping T is to use a contraction to approximate it (Browder [10, 11]). More precisely, fix $z \in K$ and define a mapping $T_t : K \to K$ by $T_t(x) = tz + (1-t)T(x)$ for all $x \in K$ and given $t \in (0, 1)$. It is easy to see that T_t is a contraction on K and the classical Banach contraction principle assures that T_t has a unique fixed point $x_t \in K$, that is,

$$x_t = tz + (1-t)T(x_t).$$

To approximate fixed point of a nonlinear mapping, the simplest iteration process is the well-known Picard iteration process:

$$\begin{cases} x_1 \in K \\ x_{n+1} = T(x_n), \quad n \in \mathbb{N}. \end{cases}$$

However, the Picard iteration of a nonexpansive mapping T may not converge to a fixed point of T.

Example 8 Let $X = K = [0, 1]$ and $T : K \to K$ defined by $T(x) = 1 - x$ for all $x \in K$. Then T is nonexpansive and has a unique fixed point $\frac{1}{2}$. But for any $x_1 = a \neq \frac{1}{2}$ the Picard iteration yields an oscillatory sequence $a, 1-a, a, 1-a, \ldots$.

To overcome from these problems and to get better rate of convergence, a number of iteration processes have been introduced by many authors. Some prominent iteration processes are given below. The following iteration process is known as Mann iteration process [42]:

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T(x_n), \quad n \in \mathbb{N}, \end{cases}$$

where $\{\beta_n\}$ is a sequence in $[0, 1]$.

Theorem 24 *[28]. An infinite matrix (c_{mn}) is regular if and only if the following are true:*

(1) $\lim_{m\to\infty} c_{mn} = 0$ *for each* $n \in \mathbb{N}$,

(2) $\lim_{m\to\infty} \sum_{n=1}^{\infty} c_{mn} = 1$,

(3) $\sup_m \left\{ \sum_{n=1}^{\infty} |c_{mn}| \right\} \leq L < +\infty$ *for some* $L > 0$.

Theorem 25 *[42]. Let K be a compact convex subset of a Banach space X and $T : K \to K$ a continuous mapping. Let $\{x_n\}$ be the sequence of T-iterates generated by $x_1 \in K$. Define*

$$v_n = \sum_{k=1}^{n} a_{nk} x_k \text{ and } x_{n+1} = T(v_n),$$

where $A = (a_{nk})$ is the triangular matrix satisfying

(a) $a_{ij} \geq 0$ *for* $i, j \in \mathbb{N}$,
(b) $a_{ij} = 0$ *for all* $j > i$,
(c) $\sum_{j=1}^{i} a_{ij} = 1$ *for all* $i \in \mathbb{N}$.

If either of the sequences $\{x_n\}$ and $\{v_n\}$ converges, then the other also converges to the same point and their common limit is a fixed point of T.

Theorem 26 *[42]. Suppose neither $\{x_n\}$ nor $\{v_n\}$ defined in Theorem 25 is convergent. Let X be the set of all limit points of $\{x_n\}$ and V the set of all limit points of $\{v_n\}$. If A satisfies additionally $\lim_{n\to\infty} a_{nn} = 0$ and $\lim_{n\to\infty} \sum_{k=1}^{n} |a_{n+1k} - a_{nk}| = 0$, then X and V are closed connected sets.*

In 1979, Reich [51], obtained the following convergence results for Mann iteration process on very general settings.

Proposition 4 *[51]. Let K be a closed convex subset of a uniformly convex Banach space with Frećhet differentiable norm, and let $\{T_n : 1 \leq n < \infty\}$ be a family of nonexpansive self mappings of K with a nonempty common fixed point set F. If $x_1 \in K$ and $x_{n+1} = T_n(x_n)$ for $n \in \mathbb{N}$, then $\lim_{n\to\infty} \langle x_n, J(f_1 - f_2)\rangle$ exists for all $f_1 \neq f_2$ in F.*

Proof Let $a_n = \|tx_n + (1-t)f_1 - f_2\|$ for $t \in [0, 1]$, δ the modulus of convexity of the space,

$$M = \|x_1 - f_1\|, \gamma(r) = \left(\frac{M}{2}\right)\delta\left(\frac{4r}{M}\right), S_{n,m} = T_{n+m-1}T_{n+m-2}\cdots T_n,$$

and

$$b_{n,m} = \|S_{n,m}(tx_n - (1-t)f_1) - (tx_{n+m} + (1-t)f_1)\|.$$

Note that $a_{n,m} \leq b_{n,m} + a_n$. After some manipulation, we see that

$$\gamma(\|T(cx + (1-c)y) - cT(x) - (1-c)T(y)\|) \leq \|x - y\| - \|T(x) - T(y)\|$$

for all $c \in [0, 1]$, $\|x - y\| \leq M$ and nonexpansive $T : K \to K$. Hence

$$\gamma(b_{n,m}) \leq \|x_n - f_1\| - \|x_{n+m} - f_1\| \leq \varepsilon_n \to 0 \text{ as } n \to \infty.$$

Consequently,

$$\limsup_{n\to\infty} a_n \leq \liminf_{n\to\infty} a_n \text{ and } \lim_{n\to\infty} a_n = a$$

exists. Let $d_n = \langle x_n - f_1, J(f_1 - f_2)\rangle$. Given $\varepsilon > 0$ there exists $t \in (0, 1)$ such that $0 \leq \frac{a_n}{t} - d_n < \varepsilon$ for all $n \in \mathbb{N}$. Therefore $\limsup_{n\to\infty} d_n \leq \frac{a}{t}$, $\liminf_{n\to\infty} d_n \geq \frac{a}{t} - \varepsilon$, and the result follows.

Theorem 27 *[51]. Let K be a closed convex subset of a uniformly convex Banach space X with a Frećhet differentiable norm, $T : K \to K$ a nonexpansive mapping with a fixed point, and $\{\beta_n\}$ a real sequence such that $0 \leq \beta_n \leq 1$ and $\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$. Let $x_1 \in K$ and $x_{n+1} = \beta_n T(x_n) + (1 - \beta_n)x_n$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof Since $\sum_{n=1}^{\infty} \beta_n(1 - \beta_n) = \infty$, the sequence $\{x_n - T(x_n)\}$ converges strongly to zero. Therefore every weak subsequential limit of $\{x_n\}$ is a fixed point of T. Let f_1 and f_2 be two such limits. By Proposition 4 with $T_n = \beta_n T_n + (1 - \beta_n)I$,

$$\langle f_2, J(f_1 - f_2)\rangle = \langle f_1, J(f_1 - f_2)\rangle,$$

so that $f_1 = f_2$.

In 1974, Ishikawa [29] generalized Mann iteration process from one step to two step as follows:

$$\begin{cases} x_1 \in K \\ y_n = (1 - \beta_n)x_n + \beta_n T(x_n) \\ x_{n+1} = (1 - \gamma_n)x_n + \gamma_n T(y_n), \quad n \in \mathbb{N}, \end{cases}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in [0, 1].

Noor [44] introduced the following three step iteration process and studied the approximate solution of the variational inclusions in Hilbert spaces:

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T(y_n) \\ y_n = (1-\gamma_n)x_n + \gamma_n T(z_n) \\ z_n = (1-\delta_n)x_n + \delta_n T(x_n), \quad n \in \mathbb{N}, \end{cases}$$

where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $[0, 1]$.

In 2007, Agarwal *et al.* [2] introduced the following iteration process known as S-iteration process.

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\beta_n)T(x_n) + \beta_n T(y_n) \\ y_n = (1-\gamma_n)x_n + \gamma_n T(x_n), \quad n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the condition

$$\sum_{n=1}^{\infty} \beta_n \gamma_n (1-\gamma_n) = \infty.$$

Proposition 5 *[2]. Let K be a nonempty closed convex subset of a Banach space X and $T : K \to K$ a contraction mapping with Lipschitz constant k and a unique fixed point p. For $x_1 \in K$, let $\{x_n\}$ and $\{y_n\}$ be sequences defined in (7). Then for all $n \in \mathbb{N}$,*

$$\|x_{n+1} - p\| \le k[1 - (1-k)\beta_n \gamma_n]\|x_n - p\|.$$

Theorem 28 *[2]. Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $T : K \to K$ a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined by (7) with the restriction:*

$$\lim_{n\to\infty} \beta_n \gamma_n (1-\beta_n) \text{ exists and } \lim_{n\to\infty} \beta_n \gamma_n (1-\beta_n) \neq 0. \tag{8}$$

Then, for any arbitrary $x_1 \in K$, the sequence $\{\|x_n - T(x_n)\|\}$ converges to some constant $r_K(T) = \inf\{\|x - T(x)\| : x \in K\}$, which is independent of the choice of the initial value $x_1 \in K$.

Theorem 29 *[2]. Let X be a uniformly convex Banach space with a Frećhet differentiable norm or that satisfies Opial property. Let K be a nonempty closed convex subset of X and $T : K \to K$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (7) with the restriction (8). Then $\{x_n\}$ converges weakly to a fixed point of T.*

In 2014, Abbas and Nazir [1] introduced the following process and showed that it converges faster than Picard, Mann, Ishikawa, Noor and S-iteration process.

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1-\beta_n)T(y_n) + \beta_n T(z_n) \\ y_n = (1-\gamma_n)T(x_n) + \gamma_n T(z_n) \\ z_n = (1-\delta_n)x_n + \delta_n T(x_n), \quad n \in \mathbb{N}, \end{cases} \tag{9}$$

where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$.

Definition 33 [52]. Let K be a subset of a normed space X. A mapping $T : K \to K$ is said to satisfy Condition (I) if there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ satisfying $g(0) = 0$ and $g(r) > 0$ for all $r \in (0, \infty)$ such that $\|x - T(x)\| \geq g(\inf_{y \in F(T)} \|x - y\|)$ for all $x \in K$.

Now, we present a result for quasi-nonexpansive mappings and a sequence defined by (9).

Lemma 4 *Let K be a nonempty closed convex subset of a Banach space X and $T : K \to X$ be a quasi-nonexpansive mapping. Suppose $\{x_n\}$ is a sequence defined by (9). Then the following holds:*

(1) $\max\{\|x_{n+1} - p\|, \|y_n - p\|, \|z_n - p\|\} \leq \|x_n - p\|$ *for all* $n \in \mathbb{N}$;
(2) $\lim_{n \to \infty} \|x_n - p\|$ *exists.*

Proof By (9) and Proposition 3, we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\delta_n)x_n + \delta_n T(x_n) - p\| \\ &\leq (1-\delta_n)\|x_n - p\| + \delta_n\|T(x_n) - p\| \\ &\leq (1-\delta_n)\|x_n - p\| + \delta_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{10}$$

By (9), (10) and Proposition 3, we have

$$\begin{aligned} \|y_n - p\| &= \|(1-\gamma_n)T(x_n) + \gamma_n T(z_n) - p\| \\ &\leq (1-\gamma_n)\|T(x_n) - p\| + \gamma_n\|T(z_n) - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|z_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{11}$$

Again, using (9), (10),(11) and Proposition 3, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\beta_n)T(y_n) + \beta_n T(z_n) - p\| \\ &\leq (1-\beta_n)\|T(y_n) - p\| + \beta_n\|T(z_n) - p\| \\ &\leq (1-\beta_n)\|y_n - p\| + \beta_n\|z_n - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{12}$$

On combining (10), (11) and (12) proves (1). Also by (12) the sequence $\{\|x_n - p\|\}$ is a monotonic nonincreasing and bounded. Hence, $\lim_{n\to\infty} \|x_n - p\|$ exists.

Lemma 5 *[56, p.484]. Let X be a uniformly convex Banach space and $0 < a \le l_n \le b < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$ and $\lim_{n\to\infty} \|l_n x_n + (1 - l_n)y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

In [46], authors obtained some convergence results for mappings satisfying condition (E).

Theorem 30 *[46]. Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $T : K \to X$ be a mapping satisfying the condition (E). Suppose $\{x_n\}$ is a sequence defined by (9). Then $F(T) \ne \emptyset$ if and only if $\{x_n\}$ is a bounded a.f.p.s for T.*

Proof Let $\{x_n\}$ be a bounded sequence and $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$. Let $z \in A(K, \{x_n\})$. By the definition of asymptotic radius,

$$r(T(z), \{x_n\}) = \limsup_{n\to\infty} \|x_n - T(z)\|.$$

Since T satisfies the condition (E),

$$\begin{aligned} r(T(z), \{x_n\}) &= \limsup_{n\to\infty} \|x_n - T(z)\| \\ &\le \mu \limsup_{n\to\infty} \|T(x_n) - x_n\| + \limsup_{n\to\infty} \|x_n - z\| \\ &= r(z, \{x_n\}). \end{aligned}$$

By the uniqueness of asymptotic center of $\{x_n\}$, we have $T(z) = z$.

Conversely, let $F(T) \ne \emptyset$ and $z \in F(T)$. Then from Lemma 4, $\lim_{n\to\infty} \|x_n - z\|$ exists. Suppose

$$\lim_{n\to\infty} \|x_n - z\| = r. \tag{13}$$

By (13) and Proposition 3, we have

$$\limsup_{n\to\infty} \|T(x_n) - z\| \le r. \tag{14}$$

From (13) and (10),

$$\limsup_{n\to\infty} \|z_n - z\| \le \lim_{n\to\infty} \|x_n - z\| = r. \tag{15}$$

Again, by (11) and (13),

$$\limsup_{n\to\infty} \|y_n - z\| \le r. \tag{16}$$

Using (15) and Proposition 3,

$$\limsup_{n\to\infty} \|T(z_n) - z\| \le r. \tag{17}$$

Similarly,

$$\limsup_{n\to\infty} \|T(y_n) - z\| \le r. \tag{18}$$

By (9), (10), (11) and Proposition 3, we have

$$\begin{aligned}
\|x_{n+1} - z\| &= \|(1-\beta_n)T(y_n) + \beta_n T(z_n) - z\| \\
&\le (1-\beta_n)\|T(y_n) - z\| + \beta_n \|T(z_n) - z\| \\
&\le (1-\beta_n)\|y_n - z\| + \beta_n \|z_n - z\| \\
&\le (1-\beta_n)\|x_n - z\| + \beta_n \|x_n - z\| \\
&= \|x_n - z\|,
\end{aligned}$$

or

$$\|x_{n+1} - z\| \le \|(1-\beta_n)T(y_n) + \beta_n T(z_n) - z\| \le \|x_n - z\|, \tag{19}$$

it implies that

$$r \le \lim_{n\to\infty} \|(1-\beta_n)T(y_n) + \beta_n T(z_n) - z\| \le r.$$

Then,

$$\lim_{n\to\infty} \|(1-\beta_n)T(y_n) + \beta_n T(z_n) - z\| = r. \tag{20}$$

From (17), (18), (20) and Lemma 5, we get

$$\lim_{n\to\infty} \|T(y_n) - T(z_n)\| = 0. \tag{21}$$

Now by (9), we have

$$\begin{aligned}
\|x_{n+1} - T(z_n)\| &= \|(1-\beta_n)T(y_n) + \beta_n T(z_n) - T(z_n)\| \\
&\le (1-\beta_n)\|T(y_n) - T(z_n)\|.
\end{aligned}$$

Making $n \to \infty$ we get

$$\lim_{n\to\infty} \|x_{n+1} - T(z_n)\| = 0. \tag{22}$$

By the triangle inequality and (17), we have

$$\begin{aligned}\|x_{n+1} - z\| &\leq \|x_{n+1} - T(z_n)\| + \|T(z_n) - z\| \\ &\leq \|x_{n+1} - T(z_n)\| + \|z_n - z\|.\end{aligned}$$

By (13) and (22), we get

$$r \leq \liminf_{n\to\infty} \|z_n - z\|. \tag{23}$$

So, by (15) and (23) we have,

$$\lim_{n\to\infty} \|z_n - z\| = r. \tag{24}$$

Now, by (9), Proposition 3 and (13), we have

$$\begin{aligned}\|z_n - z\| &= \|(1-\delta_n)x_n + \delta_n T(x_n) - z\| \\ &\leq (1-\delta_n)\|x_n - z\| + \delta_n\|T(x_n) - z\| \\ &\leq (1-\delta_n)\|x_n - z\| + \delta_n\|x_n - z\| \\ &= \|x_n - z\|.\end{aligned} \tag{25}$$

So, making $n \to \infty$ and using equation (24), (13), we get

$$r \leq \lim_{n\to\infty} \|(1-\delta_n)(x_n - z) + \delta_n(T(x_n) - z)\| \leq r.$$

Therefore

$$\lim_{n\to\infty} \|(1-\delta_n)(x_n - z) + \delta_n(T(x_n) - z)\| = r. \tag{26}$$

By (13), (14), (26) and Lemma 5, we conclude that $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$.

Our next result is prefaced by the following Lemma.

Lemma 6 *[46]. Suppose that all the conditions of Theorem 30 are satisfied. Then* $\lim_{n\to\infty} \langle x_n, J(z_1 - z_2)\rangle$ *exists for any* $z_1, z_2 \in F(T)$*; in particular* $\langle x - y, J(z_1 - z_2)\rangle = 0$ *for all weak limits* x, y *of* $\{x_n\}$.

Proof It may be completed following the proof of Lemma 2.3 [35].

Theorem 31 *[46]. Let* X*,* K*,* T*, and* $\{x_n\}$ *be same as in Theorem 30. Assume that either* X *satisfies (a) the Opial property or has (b) a a Fréchet differential norm and* $I - T$ *is demiclosed at zero. If* $F(T) \neq \emptyset$*, then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Proof By Theorem 30, the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$. Uniform convexity of X implies reflexiveness of X so, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some $z \in K$. Suppose (a) holds. Then using the Opial property, it can be easily shown that the sequence $\{x_n\}$ converges weakly to z. Now suppose (b) holds. From Lemma 6, we have $\langle x - y, J(z - p)\rangle = 0$

for all $x, y \in \omega_w(x_n)$. By demiclosedness of $I - T$ at zero, we have $z, p \in F(T)$. Thus
$$\|z - p\|^2 = \langle z - p, J(z - p)\rangle = 0.$$

Therefore $z = p$.

The proof of the following theorem is elementary and therefore omitted.

Theorem 32 *[46]. Let T be a mapping on a closed convex subset K of a Banach space X satisfying the condition (E) with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence with $x_1 \in K$ defined by (9). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T))$ denotes the distance from x to $F(T)$.*

Finally, we present a strong convergence theorem.

Theorem 33 *[46]. Let X, K, T and $\{x_n\}$ be same as in Theorem 30. Let T satisfies condition (I) with $F(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof From Theorem 30, it follows that

$$\liminf_{n\to\infty} \|T(x_n) - x_n\| = 0. \tag{27}$$

Since T satisfies condition (I), we have

$$\|x_n - T(x_n)\| \geq g(d(x_n, F(T))).$$

From (27), we get
$$\liminf_{n\to\infty} g(d(x_n, F(T))) = 0.$$

Since $g : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $g(0) = 0$ and $g(r) > 0$ for all $r \in (0, \infty)$, therefore we have

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Therefore all the assumptions of Theorem 32 are satisfied and $\{x_n\}$ converges strongly to a fixed point of T.

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A Mathematical Model Using Fixed Point Theorem for Two-Choice Behavior of Rhesus Monkeys in a Noncontingent Environment

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Abstract Two-choice behavior of monkeys were studied in 1959 by Wilson and Rollin in a noncontingent environment to ascertain whether this intermediate species would exhibit behavior similar to that of humans or to that of paradise fish. In this chapter, our aim is to examine the conduct of Rhesus monkeys in such a noncontingent environment and to establish an appropriate mathematical framework for the same. We establish the existence and uniqueness of solution for this noncontingent environment model with the help of Banach's contraction principle.

1 Introduction

The application of mathematics, particularly that of probability theory, in the study of learning processes dates back to mid-twentieth century [3, 5–7]. In 1957, Mosteller [12] in his breakthrough paper demonstrates the use of mathematics through some examples and experiments. Through three experiments he illustrated the application of mathematics to present: (a) a synopsis of the method of learning in an experiment; (b) a subjective difference between two theoretical ideas; and (c) an investigation of the conformity between items one in theory and the other in practice. Mosteller also discussed a new mathematical problem that had arisen from such applications.

In several psychological experiments such as learning and recalling a list of words, it has been observed that the performance in recalling the words gets better with increased measure of practice. As such, there were attempts to describe these learning processes explicitly by finding suitable curves. In many cases, the curves like hyperbola, exponential curve, etc. were fitting the desired experiment. The significant features of these curves are that they increase monotonically with increased practice and reach an asymptote or ceiling when the best possible performance is achieved. But in reality, we observe that the performance in a certain trial is much weaker than the earlier trial. Therefore, the monotonically increasing character of the curve is not always in tune with the experiment.

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This unpredictable and inconsistent behavior of learning process encouraged the use of statistical or random processes instead of deterministic curves to achieve a better description of the experiment. The first statistical or stochastic model for learning may be credited to Thurstone [14]. Gulliksen and Wolfe [9] found modified and developed version of these learning curves.

Since 1949, several stochastic models have been developed those represent the reactions made by the subject in unvaried experiments. Upon receiving a stimulus, the subject makes a reaction and an outcome of this response takes place which is possibly a reward or a shock. It is supposed that in the beginning of each trial every probable reaction has its own probability of turning out and also that the event which happens during the trial alters the probabilities of these reactions from one trial to the next. While formulating the experiment mathematically, an operator which adjusts the probabilities in a prearranged way is the mathematical analogue of that event. Hence, in such a framework, the learning process is made of different probabilities of the reactions and the laws that alter them.

Mosteller [12] described the models discussed above with the help of some particular experiments. He discussed an escape-avoidance test performed on dogs by Solomon and Wynne [13] and another experiment that was performed on paradise fish by Bush and Wilson [4]. Recently, Turab and Sintunavarat modeled the two-choice behavior of the paradise fish [15, 16] and the traumatic avoidance learning model for dogs with the help of Banach's contraction principle [2]. In the current chapter, our aim is to describe the two-choice behavior of Rhesus monkeys in a noncontingent environment which were studied by Wilson and Rollin [17] and Wilson [18].

2 An Experiment with Rhesus Monkeys

Wilson and Rollin [17] investigated the two-choice behavior of Rhesus monkeys in a noncontingent environment. Ten untrained Rhesus monkeys were chosen as *S*s for the first part of the experiment out of which one died and the experiment continued with the remaining nine. The apparatus for the experiment was a Wisconsin General Test Apparatus (WGTA). When the door of the *S* was opened, he came across a black horizontal surface upon which there were two black plastic boxes separated by a transparent plexiglass barrier. The boxes were, in fact, covers for shallow food wells (Fig. 1).

The boxes were weighted and connected in such a way that if the front of one box was slightly opened, the other would fly open as well. When *S* selected a box by raising it, he could observe both food wells but could reach only the one he had selected.

A 7-day initial training was performed to enable the *S*s to get used to the apparatus, learn to open the food well, and get rewarded. After this training, various training series began and the experiment was completed in five parts. Each *S* was given 32

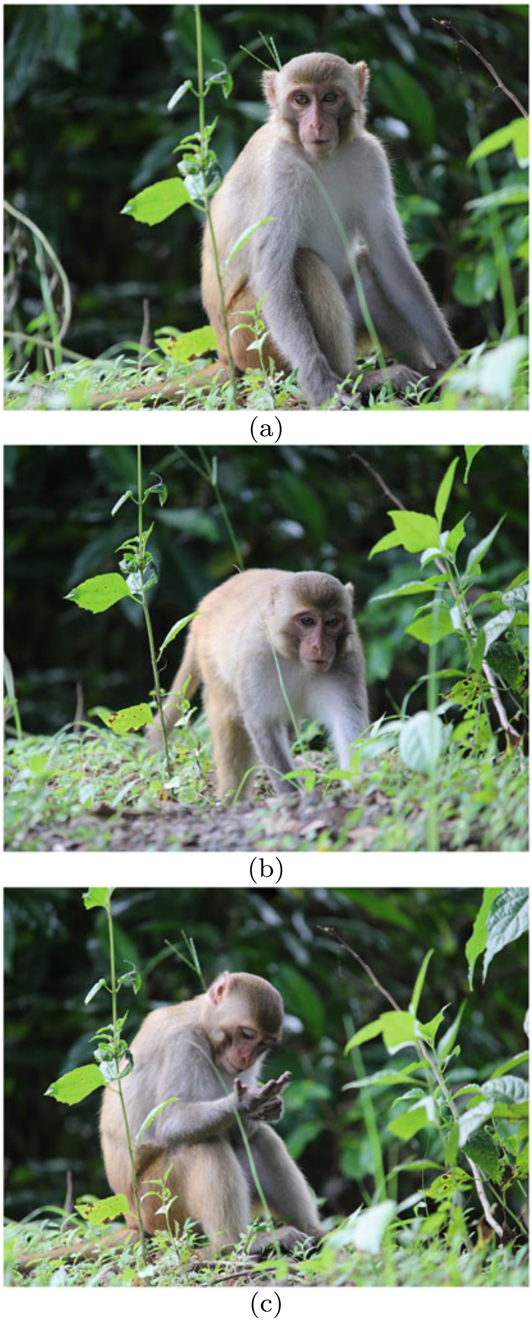

Fig. 1 Images of Rhesus monkeys captured by the author using Canon camera

massed trials on a daily basis. The *S*s were approximately 22 h hungry and the reward was a half-peanut.

In that experiment, on each trial *S* (which is the subject) could see (i) the outcome that resulted from his reaction; (ii) what outcome would have resulted if the other reaction was made. As such, a noncontingent situation was created successfully in that experiment. In spite of the noncontingent environment, in Part 1 of the experiment, all *S*s adopted at asymptote the “most rational” method of always reacting to the more often correct box. This result is different from the one obtained in noncontingent environment with paradise fish where *S*s approach asymptotes of either 0% or 100%.

The information model predicts that each *S* should choose the more favorable side 75% of the time. This is, of course, not in tune with the experiment where the *S*s choose the favorable side so regularly. But this fact does not go too much against the model possibly because our sample was not sufficiently large.

When rectification of an uncredited response was not allowed, it was observed that the *S*s quickly reached an asymptote of approximately 100% choice of the 0.75 side, regardless of their initial probabilities of choice of the more favorable side.

On the other hand, when immediate correction was permitted, most of the *S*s slowly increased their proportion of choices of the 0.75 side. The data do not conform the deduction that all *S*s were getting closer to an asymptote of 75% choice of the 0.75 side. Rather, they are consistent with the suggestion that with sufficient testing all would have chosen the 0.75 side approximately 100% of the time.

3 Modeling of the Problem Using Banach’s Fixed Point Theorem

It is assumed that in this experiment with Rhesus monkeys in a noncontingent situation, the habit formation model is reasonably fitting. A natural question arises that what choice by the monkey will bring about a steady state on the more favorable side? Such questions are of fundamental importance in the mathematical modeling of study of learning. To depict the experiment mathematically, we adopt a conventional approach.

We assume that M and N are two responses and both have satisfactory outcome but not always equal. When A occurs on trial n, the operator P is applied to x_n to produce x_{n+1}. Further, when B occurs, the operator Q is applied to obtain x_{n+1}. The operators P and Q are represented by the following:

$$Px = \alpha x + \gamma;$$

$$Qx = \beta x + \delta;$$

where $0 < \alpha \le \beta < 1$; $\gamma, \delta \in [0, 1]$, and $\gamma, \delta \le \beta$.

We further implement the assumption that the probabilities of responses M and N are x and $1 - x$, respectively. If x is the probability of M on some trial and M occurs, the new probability of M is Px. On the contrary, if N occurs, the new probability of M is Qx.

The inspection reveals that an entity observing this rule will generate one of the responses and give response only with the other (where the probability is 1). We denote this probability as (x, α, β).

If one trial is conducted, the new probability of the entity is Px (if M occurs) with probabilities x and $1 - x$. Thus, if M is the first trial, its new probability of acculturation by M is $\psi(\alpha x + \gamma)$. Again, if N is the first trial, the immersion of new probability by M is $\psi(\beta x + \delta)$. Considering the pertinent probabilities, we have following functional equation depicting the probabilities:

$$\psi(x, \alpha, \beta) = x\psi(\alpha x + \gamma) + (1 - x)\psi(\beta x + \delta). \tag{1}$$

4 Main Results

By $\mathcal{C}[0, 1]$ we denote the family of all continuous real-valued functions $\psi : [0, 1] \to \mathbb{R}$ such that $\psi(0) = 0$ and the norm on $\mathcal{C}[0, 1]$ is defined by

$$||\psi||_{\mathcal{C}} = \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|} < \infty,$$

for all $\psi \in \mathcal{C}[0, 1]$.

It can be proved that $\left(\mathcal{C}[0, 1], || \cdot ||_{\mathcal{C}}\right)$ is a Banach space.

Functional Eq. (1) may be written as

$$\psi(x) = x\psi(\alpha x + \gamma) + (1 - x)\psi(\beta x + \delta), \tag{2}$$

where $\psi \in \mathcal{C}[0, 1]$, $0 < \alpha \leq \beta < 1$, and $\gamma, \delta \in [0, 1]$ such that $\gamma, \delta \leq \beta$.

We shall apply fixed point theory to prove existence and uniqueness of a solution of functional Eq. (2).

The following result is of paramount importance in this context.

Theorem 1 *Let Δ be a closed and Γ-invariant subset of $\mathcal{C}[0, 1]$, i.e., $\Gamma(\Delta) \subseteq \Delta$, where $\Gamma : \Delta \to \Delta$ is defined as*

$$(\Gamma\psi)(x) = x\psi(\alpha x + \gamma) + (1 - x)\psi(\beta x + \delta),$$

for each $\psi \in \Delta$ and for all $x \in [0, 1]$. If $0 < \alpha \leq \beta < \frac{1}{6}$ and $\gamma, \delta \in [0, 1]$ such that $\gamma, \delta \leq \beta$, then Γ is a Banach contraction mapping.

Proof Let $\psi, \zeta \in \Delta$ and $x, y \in [0, 1]$ such that $x \neq y$. Then we have

$$\begin{aligned}
&\frac{\big|(\Gamma\psi - \Gamma\zeta)(x)\big| - \big|(\Gamma\psi - \Gamma\zeta)(y)\big|}{|x-y|} \\
=&\frac{1}{|x-y|}\big[\Gamma(\psi-\zeta)(x) - \Gamma(\psi-\zeta)(y)\big] \\
=&\Big|\frac{1}{|x-y|}\big[x(\psi-\zeta)(\alpha x+\gamma) + (1-x)(\psi-\zeta)(\beta x+\delta) \\
&- y(\psi-\zeta)(\alpha y+\gamma) + (1-y)(\psi-\zeta)(\beta y+\delta)\big]\Big| \\
=&\Big|\frac{1}{|x-y|}\big[\{x(\psi-\zeta)(\alpha x+\gamma) - x(\psi-\zeta)(\alpha y+\gamma)\} \\
&+ \{(1-x)(\psi-\zeta)(\beta x+\delta) - (1-x)(\psi-\zeta)(\beta y+\delta)\} \\
&+ \{x(\psi-\zeta)(\alpha y+\gamma) - y(\psi-\zeta)(\alpha y+\gamma)\} \\
&+ \{(1-x)(\psi-\zeta)(\beta y+\delta) + (1-y)(\psi-\zeta)(\beta y+\delta)\big]\Big| \\
\leq&\alpha x||\psi-\zeta|| + \beta(1-x)||\psi-\zeta|| + |(\psi-\zeta)(\alpha y+\gamma) - (\psi-\zeta)(0)| \\
&+ |(\psi-\zeta)(\beta y+\delta) - (\psi-\zeta)(0)| \\
\leq&\alpha x||\psi-\zeta|| + \beta(1-x)||\psi-\zeta|| + |\alpha y+\gamma|||\psi-\zeta|| + |\beta y+\delta|||\psi-\zeta|| \\
\leq&6\beta||\psi-\zeta||.
\end{aligned}$$

Since $0 < 6\beta < 1$, it follows that Γ is a Banach contraction mapping.

The next result gives us the existence and uniqueness of a solution of functional Eq. (1).

Theorem 2 *If* $6\beta < 1$*, then functional Eq. (1) has a unique solution and there exists a closed and* Γ*-invariant subset* Δ *of* $\mathcal{C}[0, 1]$*, where the operator* $\Gamma : \Delta \to \Delta$ *is defined as*

$$(\Gamma\psi)(x) = x\psi(\alpha x + \gamma) + (1 - x)\psi(\beta x + \delta),$$

for each $\psi \in \Delta$ *and for all* $x \in [0, 1]$*.*

Moreover, the iteration of functions $\{\psi_n\}$ *in* Δ *defined by*

$$(\psi_n)(x) = x\psi_{n-1}(\alpha x + \gamma) + (1 - x)\psi_{n-1}(\beta x + \delta),$$

for all $n \in \mathbb{N}$ *with* $\psi_0 \in \Delta$*, converges to the unique solution of functional Eq. (1) with respect to the metric induced by the norm* $||\cdot||_{\mathcal{C}}$*.*

Proof Δ being a closed subset of a Banach space $\mathcal{C}[0, 1]$, we can conclude that Δ is complete. The rest of the proof follows from the Banach contraction principle together with Theorem 1.

5 Some Consequences of the Main Result

In a two-choice behavior experiment of Rhesus monkeys, the directives given to the monkeys may result in producing the faith in it that if one response remains non-credited, then the other one must be rewarded. Such assumption by the subject of experiment may result from the administered commands or may arise without the experimenter suggestion. The scheme of the experiment is arranged in such a way that the assumption is correct, but more often than not the subject is not certain about it.

In such a situation, it is quite reasonable to believe that the rewarded response M has almost the unchanged impact on the behavior of the non-credited response N and conversely. Such a condition may be termed as equal alpha-beta condition ($\alpha = \beta$).

In that case, we can rewrite Eq. (1) as

$$\psi(x) = x\psi(\alpha x + \gamma) + (1 - x)\psi(\alpha x + \delta), \tag{3}$$

for all $x \in [0, 1]$, where $\psi \in \mathcal{C}[0, 1]$ and $0 < \alpha < 1$. Thus, we have the following consequences of our main result from the previous section.

Corollary 1 *Let Δ be a closed and Γ-invariant subset of $\mathcal{C}[0, 1]$, where $\Gamma : \Delta \to \Delta$ is defined as*

$$(\Gamma\psi)(x) = x\psi(\alpha x + \gamma) + (1 - x)\psi(\alpha x + \delta),$$

for each $f \in \Delta$ and for all $x \in [0, 1]$. If $0 < \alpha < \frac{1}{6}$ and $\gamma, \delta \in [0, 1]$ be such that $\gamma, \delta \leq \alpha$, then Γ is a Banach contraction mapping.

Corollary 2 *If $6\alpha < 1$, then functional Eq. (3) has a unique solution and there exists a closed and Γ-invariant subset Δ of $\mathcal{C}[0, 1]$, where the operator $\Gamma : \Delta \to \Delta$ is defined as*

$$(\Gamma\psi)(x) = x\psi(\alpha x + \gamma) + (1 - x)\psi(\alpha x + \delta),$$

for each $\psi \in \Delta$ and for all $x \in [0, 1]$.

Moreover, the iteration of functions $\{\psi_n\}$ in Δ defined by

$$(\psi_n)(x) = x\psi_{n-1}(\alpha x + \gamma) + (1 - x)\psi_{n-1}(\alpha x + \delta),$$

for all $n \in \mathbb{N}$ with $\psi_0 \in \Delta$, converges to the unique solution of functional Eq. (3) with respect to the metric induced by the norm $|| \cdot ||_{\mathcal{C}}$.

6 Discussion and Interpretation of the Problem in Hand

In each trial of an infinite sequence, we assume that the monkey may choose a box in one of the two ways. For the ease of interpretation, we specify the choices as M and N. Also suppose that on a given trial the probability of selecting M is x and that of N is $1 - x$ where $x \in [0, 1]$. If M is selected, the probability of choosing M in the next trial is $\alpha x + \gamma$, whereas if N is chosen, the probability of M in the next trial is $\beta x + \delta$ with $0 < \alpha \leq \beta < 1$ and $\gamma, \delta \leq \beta$.

Suppose that $\psi(x, \alpha, \beta)$ is the probability that an infinite sequence of trials terminates with selections of M. Thus, if x_n is the probability of selecting M on trial n, then the probability of M on the next trial is

$$x_{n+1} = \begin{cases} \alpha x + \gamma, & \text{if } M \text{ is the choice on trial } n \\ \beta x + \delta, & \text{if } N \text{ is the choice on trial } n. \end{cases}$$

7 Conclusion and Future Work

The investigation of stability of functional equations in connection with Ulam–Hyers and Ulam–Hyers–Rassias stability [1, 8, 10, 11] is of fundamental importance. The study of stability of the following functional equation calls for an interesting future study:

$$\psi(x) = x\psi(\alpha x + \gamma) + (1 - x)\psi(\beta x + \delta), \tag{4}$$

for all $x \in [0, 1]$, where $\psi \in \mathcal{C}[0, 1]$, $0 < \alpha \leq \beta < 1$, and $\gamma, \delta \in [0, 1]$ with $\gamma, \delta \leq \beta$.

Also, we have assumed the closedness of the set Δ in our study. It would be worth investigating if this condition can be relaxed.

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