Assembly Problems

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1 Introduction

A set of sellers own one unit each of an indivisible good. A buyer wants to purchase a subset of these units. Additionally, the units in the subset are required to constitute a path of a feasible length in a graph. The nodes of this graph represent units of the good, and edges between pair of nodes represent the complementarity of the pair in the production process used by the buyer. The sellers have non-negative valuations for the units they own. The buyer has a non-negative valuation for every subset of units on a feasible path. These valuations may be common knowledge or private information. An Assembly Problem is the exchange problem described by the graph, the minimal size of a feasible subset, and the valuations of the agents.

Efficient assembly is obtained easily in static models with complete information. It is the prospect of strategic delays or private information that makes the assembly problem interesting. Games of complete information multi-period bargaining are used to model the former, while static games of incomplete information are used to model the latter. This chapter provides a brief survey of the literature and discusses some recent results using these approaches.

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Practical examples of assembly problems include assembling of patent rights for manufacturing life-saving drugs, copyright of musical pieces for composing scores for movies or concerts, and land rights for industrial development among others (see [9] for a survey). In what follows, the problem of assembling land for industrial development is treated as a leading example. For detailed discussions of these applications, the interested reader may refer to Sarkar [28] and Gupta and Sarkar [8].

Assembly is a problem of exchange between a buyer and a set of sellers. Therefore, it is a general version of the bilateral trade problem investigated extensively in the literature on strategic bargaining as well as mechanism design. While the buyer cannot extract full surplus in the unique equilibrium of an alternate offer bilateral bargaining problem [27], private information prevents mechanisms for bilateral trade to be successful [16]. Till recently, the analysis of assembly problems was restricted to the case of fully complementary items. Consequently, the results available mimicked the negative results in the bilateral trade problem. This chapter provides a general model that accommodates various degrees of complementarity and substitutability—thus providing a set of results the nature of which ranges from negative to positive.

The next section provides a brief survey of the literature on assembly problems. A general model is described in the subsequent section which is exploited further to drive some of the results on the efficiency of equilibria. The final section discusses and compares the results under alternative approaches and indicates some directions for further research.

2 Literature

We explore two alternative assumptions about the information structure: (a) agents have complete information about the valuations of other agents and (b) agents have private information about their respective valuations. The natural way to model an assembly problem under the assumption (a) is strategic bargaining among the buyer and sellers, while the approach taken for characterizing satisfactory equilibrium outcomes under the assumption (b) is mechanism design.

In strategic bargaining games, agents on one side of the market propose prices, and those on the other side accept or reject. The range of price offers, sequencing of the offers, and length of the negotiation process are given by the bargaining protocol which is common knowledge (see [21]). Consider the one-period bilateral trade problem where the buyer makes the first offer which the seller may accept or reject. This game has a unique subgame perfect Nash equilibrium outcome if the seller accepts any offer that does not make him strictly worse-off: the buyer offers the seller his exact valuation, the seller accepts, and thus the buyer extracts the entire surplus. In contrast, in the infinite horizon alternate offer bargaining model due to Rubinstein [27], the buyer has to offer a strictly positive share of the surplus to the seller to avoid strategic delay. This share of surplus can be viewed as a cost of the holdout.

The holdout problem has been studied in the land assembly context [1–3, 6, 14, 15, 20, 25]. Secret offers [11, 19, 25] and the choice of bargaining order over sellers [12, 32] are two other topics of interest.

Roy Chowdhury and Sengupta [25] use a protocol which is a natural extension of the protocol by Rubinstein [27] to the assembly problem where all items are complementary. If the offers are public, the buyer who has an outside option extracts a higher share surplus relative to the buyer without an outside option. Holdout may be unavoidable when offers are less transparent even if the buyer has an outside option.

The general model introduced in the next section potentially accommodates more number of sellers than the number of items required by the buyer. The buyer may also require the purchased items to form a path on a given graph. This model allows for various degrees of complementarity. Secret offers or outside options are not explored here, and instead, the focus is on competition among sellers. It uses the public offer protocol due to Roy Chowdhury and Sengupta [25].

The holdout problem has been modeled using the Coalitional Bargaining approach. In the first of such models, Chatterjee et al. [4] studied sequential offers of n-person coalitional bargaining with transferable utility and time discounting. They showed that the efficient coalition may not form for a certain order of proposers. Ray and Vohra [24] study the same problem where externalities across coalitions are a possibility. Myerson [17] provides a complementary approach to coalitional bargaining, analyzing bargaining on networks, where edges between agents are used to model some specific relationship.

Mechanism Design theory lays down rules for "satisfactory" allocation in the presence of private information [26]. Myerson and Satterthwaite [16] have provided such a set of desirable properties: maximum welfare or gains at every allocation (ex-post efficiency), truthful reporting in expectation (interim incentive compatibility), participation in expectation (interim individual rationality), and balanced payments (budget balance).

Consider bilateral trade under private information. The double auction mechanism due to Chatterjee and Samuelson [5] is described as follows: trade takes place if the buyer's reported valuation exceeds that of the seller's, at a price equal to the average of these two reports. When all valuations are distributed uniformly over [0, 1], the double auction mechanism maximizes expected welfare subject to interim incentive compatibility and individual rationality [16]. But it is not efficient in the ex-post sense: the double auction mechanism forgoes some efficient trade opportunities.

Early literature on mechanism design for land assembly primarily look for *second-best* mechanisms in exchange models without any contiguity restrictions (e.g., see [7, 10, 23]). The question of existence of satisfactory mechanisms for general environments remained unresolved till recently.

Williams [31] finds that a satisfactory mechanism can be constructed if and only if there is a Groves mechanism for the problem that results in an expected budget surplus. In a closely related paper, Krishna and Perry [13] show that a *successful* mechanism can be constructed if and only if the VCG mechanism for the problem results in a positive expected budget surplus. The second half of the next section shows how the results due to Williams [31] and Krishna and Perry [13] can resolve the question of satisfactory mechanism design in the assembly problem. We primarily confine the discussion to the existence of first-best mechanisms in the independent private value settings following Sarkar [28–30].

3 The Models

For a given production process, the nature of complementarity among items held by the sellers can be modeled through a graph, say Γ. In any such graph, items, or equivalently, corresponding sellers, are represented by nodes. An edge connects a pair of nodes in this graph if the corresponding inputs are complementary in the buyer's production process. A path is a sequence of connected nodes. The buyer wants to purchase a path of the desired length, say k. This implies that the buyer can combine any k complementary inputs to produce output. We will denote a path by $\mathcal{P}$ and the corresponding sum of seller valuations by $\mathcal{S}$.

A seller is critical if he lies on every feasible path (see Fig. 1). This implies that the corresponding input is complementary with respect to every feasible production plan. If there is only one feasible path in Γ, all sellers in that path are critical. But if there are multiple feasible paths, a seller must belong to their intersection in order to qualify as critical. If there are multiple feasible paths, the number of critical sellers cannot exceed $k - 1$: not all sellers on a single path can be critical. Paths of length less than k that do not have an intersection with any feasible path can be excluded from the analysis, because the buyer's valuation over such paths is zero.

A classification of graphs with at least two feasible paths is useful in this context.

In cycles of order $k + 1$, referred to as $\Gamma^{\triangle}$ (see Fig. 2), every input on a feasible path can be substituted by another input on the graph.

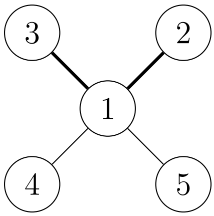


Fig. 1 A feasible path in the star graph when $k = 3$; seller 1 is critical

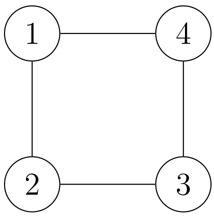


Fig. 2 A cycle of length 4

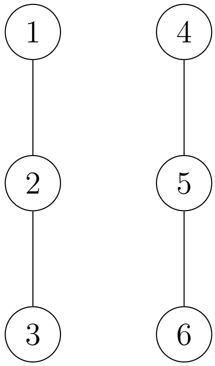
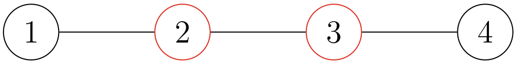


Fig. 3 Graph with disjoint feasible paths; k = 3

Fig. 4 A line graph with two critical sellers marked red; k = 3



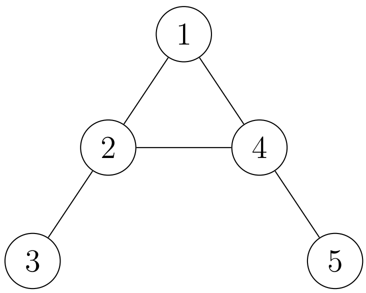
In graphs with two disjoint paths, referred to as Γ^D (see Fig. 3), no input is completely substitutable, but a feasible path can be substituted by another feasible path.

In graphs with critical sellers, referred to as Γ^* (see Figs. 4 and 1), inputs corresponding to critical sellers are not substitutable but those corresponding to non-critical sellers are substitutable in a limited sense.

Finally, consider graphs where (i) there is no cycle of length $k+1$, (ii) no two paths are disjoint, and (iii) the intersection of all feasible paths is empty, referred to as Γ^O (see Fig. 5), referred to as *oddball*. In such graphs, inputs in the intersection of two or more feasible paths cannot be substituted with respect to these feasible paths, but they are substitutable with respect to inputs on other feasible paths.

Facts 1–5 below imply that single component graphs with (a) critical sellers, (b) $k+1$-cycle, (c) disjoint paths, and (d) oddball are four mutually exclusive and exhaustive categories. A graph may have multiple components from different classes.

Fig. 5 An oddball graph, $n = 5, k = 3$



- **Fact 1:** All sellers on a single path of length k are critical, regardless of whether this path is a cycle.
- **Fact 2:** The number of critical sellers on a single path reduces with its length.
- **Fact 3:** No cycle of length more than k has a critical seller.
- **Fact 4:** Cycles of length $2k$ or more have at least two disjoint feasible paths and hence, no critical seller.
- **Fact 5:** The oddball class covers all cycles of length larger than $k + 1$ but smaller than $2k$. Further, since every pair of feasible paths on an oddball graph intersect at least once, it also covers graphs containing cycles of length less than or equal to k.

In a complete information environment, the valuation of the buyer and sellers is given by a vector $\mathbf{v} \equiv (v_0, v_1, \dots, v_n)$. The first component of this vector denotes the valuation of the buyer for a path of length k or more, and other components denote the valuation of the respective sellers for their items. In a private information environment, agents only know their own valuation and the support of the valuations of other agents; a commonly known prior μ describes their beliefs over possible valuation profiles.

We assume that there exists a path $\mathcal{P} \in \Gamma$, such that it results in a positive surplus: $v_0 - \sum_{i \in \mathcal{P}} v_i > 0$. Given such a graph Γ, the expression $\max_{\mathcal{P} \in \Gamma} \left(v_0 - \sum_{i \in \mathcal{P}} v_i \right)$ is referred to as "full surplus" or "efficient surplus".

In complete information strategic bargaining games, a discount factor is applied to compare payoffs that arise in different time periods. We assume that the agents use the same discounting factor $\delta \in [0, 1]$.

An assembly problem with complete information is a tuple: $\langle \Gamma, k, \mathbf{v}, \delta \rangle$. When Γ is a complete graph of order n, an assembly problem is referred to by the tuple $\langle n, k, \mathbf{v}, \delta \rangle$. An assembly problem with private information is a triple: $\langle \Gamma, k, \mu \rangle$.

The results on complete information bargaining and mechanism design for the assembly problem are discussed in the next two subsections. Only a brief sketch of the argument is provided below each result. The interested reader is referred to the original papers for detailed proofs.

3.1 Bargaining with Complete Information

The bargaining protocol due to Rubinstein [27] and its different extended versions have been used in many contexts. A slightly general version of this protocol due to Roy Chowdhury and Sengupta [25] can be described as follows. In each period, active agents on one side of the market make offers of surplus shares to the other side—this gives rise to two alternative cases, where buyers make offers in odd periods and sellers in even periods and vice versa. The offers made are either accepted or rejected. If accepted, the deal is implemented, i.e., the seller sells his item at the agreed offer and leaves the market with his payment immediately. The game proceeds with the reduced set of agents. The ones making offers in the previous period now take on the role of responders. Offers are made and are either accepted or rejected. And so the game proceeds till the buyer is able to pick up at least one feasible path.

There are usually multiple equilibria in multiagent bargaining problems like assembly, some of which may be non-stationary. The nature of the equilibria also depends on which side of the market proposes first. In the discussion below, our focus will be to characterize bounds on equilibrium surplus shares under this protocol. Consequently, we are able to avoid details like stationarity or sequencing of the offers.

It is a standard practice in bargaining theory to express payoffs in terms of surplus shares instead of net payoffs. For instance, in the bilateral trade game when the buyer has valuation v_0 and seller v_1, the surplus realized on trade is $v_0 - v_1$. If the surplus shares in an equilibrium are α and $1 - \alpha$, the net payoffs are $\alpha(v_0 - v_1)$ and $(1 - \alpha)(v_0 - v_1)$—indicating that trade takes place at the price of $v_1 + (1 - \alpha)(v_0 - v_1)$, which the buyer pays and the seller receives.

The buyer can utilize negative surplus offers to exclude some sellers from the bargaining process, i.e., choose the sellers to bargain with in each period. Notice that a seller will not possibly make a negative offer to the buyer in our setting, since it delays trade with the buyer or eliminates the prospect of a trade. Bilateral bargaining models like that by Rubinstein [27] do not use this feature, while multilateral models like Roy Chowdhury and Sengupta [25] do.

The bilateral game studied by Rubinstein [27] is a special assembly problem with $n = k = 1$. Here, the only seller present is critical. The Subgame Perfect Nash Equilibrium of this game, which is now a standard result, is presented below.

Theorem 1 ([27]) *Consider the model where the buyer bargains with one seller for one input:* $\langle n = 1, k = 1, v_0 > v_1, \delta\rangle$. *There is a unique SPNE of the model described as follows:*

> *Agent i proposes a share* $\frac{\delta}{1+\delta}$ *of the surplus to j whenever she has to propose, and accept any share at least equal to* $\frac{\delta}{1+\delta}$ *whenever j has to propose.*

The game ends in the first period itself, with agent i proposing $\frac{\delta}{1+\delta}$ *to the seller and the seller accepting it.*

To see that the strategies proposed above constitute an equilibrium, apply the "one-shot deviation principle": no agent can gain by deviating from these strategies in any period for one period and conforming in the preceding and succeeding periods. If agent i proposes a higher share, it will be rejected and the play in the succeeding periods can only guarantee a lower payoff; if she proposes a lower share, it will be accepted immediately. Accepting lower shares is not profitable. Proving the uniqueness of this equilibrium is a rather involved exercise (see [22]).

The model studied by Roy Chowdhury and Sengupta [25] is a special assembly problem with $n = k \geq 2$ and all seller valuations are identical. Since the buyer wants all n plots, all sellers are critical here.

Theorem 2 ([25]) *Consider the model* $\langle n \geq 2, k = n, v_1 \leq \cdots \leq v_n, v_0 > \sum_{i=1}^{n} v_i, \delta\rangle$. *The buyer's equilibrium payoff cannot be more than* $\frac{1-\delta}{1+\delta}$ *of the full surplus for any* $\delta > 0$.

The proof of this result for $n = k = 2$ shows profitable deviations for one of the agents when the bound $\frac{1-\delta}{1+\delta}$ is crossed. An induction argument is used for the general case.

Both of these results correspond to the situation where all sellers are critical. The result below, in contrast, shows the possibility of full surplus extraction when there is no critical seller on the underlying graph, and seller valuations are identical.

Theorem 3 ([8]) *Consider* $\langle \Gamma, k, \mathbf{v}, \delta \rangle$ *where* Γ *has at least two different feasible paths and* $\mathbf{v}$ *is any arbitrary valuation profile.There exists* $\bar{\delta} \in [0, 1)$ *such that for all* $\delta > \bar{\delta}$ *the buyer extracts full surplus in at most two periods in an equilibrium if and only if*

- $\Gamma \neq \Gamma^*$*, i.e., there does not exist a critical seller in the underlying graph, and*
- $\mathcal{S}_1 = \mathcal{S}_2$*, i.e., there exist at least two paths with the minimum sum of valuations.*

This result characterizes equilibrium outcomes when the valuations of sellers are equal and the underlying graph does not contain a critical seller, i.e., either the graph has a $k + 1$-cycle, or it has at least a pair of disjoint paths, or it is an oddball graph. These three graphs have special properties—each node on a feasible path is substitutable by another node in a $k + 1$ cycle, every path is substitutable by another path in a graph with a pair of disjoint paths, and each node in a feasible path is substitutable by a set of nodes in an oddball graph. The first class of graphs exhibits full substitutability, while the other two exhibit limited substitutability. Consider the first class of graphs. If the buyer is the first to make offers, she can make offers of zero surplus shares to all sellers on a chosen feasible path in an equilibrium: any seller rejecting such offers must compete with corresponding substitute sellers in the next period. If the sellers are making first offers, competition ensures that sellers make no positive claims. Consequently, full surplus extraction takes place in the first period itself. In the other two classes, the buyer may be required to exclude all sellers in the first period, to achieve full surplus extraction in the second period.

The buyer cannot extract full surplus when the underlying graph contains at least one critical seller.

Theorem 4 ([8]) *Suppose* $\Gamma = \Gamma^*$. *The buyer cannot extract full surplus in an equilibrium.*

This result is obtained since at least one of the critical sellers can keep rejecting the offers of the buyer till all other sellers have accepted. He can then claim a positive surplus share in the ensuing subgame, by Theorem 1.

When seller valuations are not equal, the sum of seller valuations may differ over paths. The path corresponding to the least sum of seller valuations is efficient in the sense that it corresponds to the highest potential surplus. It follows that if possible, the buyer would prefer to purchase the efficient path.

Let $\mathcal{P}_i$ denote the path corresponding to the i-th smallest sum of valuations on a path in Γ. We will refer to a set of assembly problems as *rich* if there does not exist two disjoint paths $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{S}_1 = \mathcal{S}_2$. Suppose the richness condition is not satisfied. The buyer, if offering first, can offer negative surplus shares to all sellers who reject such offers. In the next period, sellers on $\mathcal{P}_1$ and $\mathcal{P}_2$ cannot claim any surplus: the buyer extracts full surplus in the second period. If the sellers are making offers first, sellers on these two paths cannot claim any surplus share.

Theorem 5 ([8]) *Consider the rich class of assembly problems* $\langle \Gamma, k, \mathbf{v}, \delta \rangle$. *There does not exist any equilibrium where the buyer extracts full surplus.*

The proof of this result shows that at least one seller getting zero surplus share has a profitable deviation. Thus, full surplus extraction is not an equilibrium outcome.

By Theorem 4, the buyer cannot extract full surplus when the underlying graph contains critical sellers. The final results of this section provide bounds on the buyer's surplus share in such a problem.

Theorem 6 ([8]) *Consider an assembly problem* $\langle \Gamma, k, \mathbf{v}, \delta \rangle$ *with exactly one critical seller. In any equilibrium buyer's share of surplus cannot exceed* $\frac{1}{1+\delta}$.

Theorem 7 ([8]) *Consider an assembly problem* $\langle \Gamma, k, \mathbf{v}, \delta \rangle$ *with m critical sellers, where* $2 \leq m \leq k$. *In any equilibrium buyer's share of surplus cannot exceed* $\frac{1-\delta}{1+\delta}$.

The proof of Theorem 6 closely follows that of Theorem 4, while the proof of Theorem 7 follows that of Theorem 2. There exist assembly problems where these bounds are exactly achieved: for example, when $n = k = 1$, the corresponding bound is exactly achieved if the buyer is making the first offer (recall Theorem 1). It is also exactly achieved when Γ is a single line graph with three nodes, $k = 2$, and the buyer is making the first offer. When $n = k = 2$, the corresponding bound is exactly achieved if the buyer is making the first offer (recall Theorem 2). It is also exactly achieved when Γ is a single line graph with four nodes, $k = 3$, and the buyer is making the first offer.

3.2 Mechanism Design

Due to the well-known Revelation Principle (see [18]), it suffices to assume that the buyer and the sellers directly report their individual valuations to a central planner, who then decides allocations and payments according to a declared rule. A set of essential definitions is provided below.

A deterministic allocation $x \in \mathbb{R}^{n+1}$ is described as follows: for components $i = 1, \ldots, n$, x_i is -1 if seller i sells and 0 otherwise; $x_0 = 1$ if $\sum_{i=1}^{n} |x_i| \geq k$ and 0 otherwise. Let $\mathbb{X}$ be the set of all deterministic allocations.

Definition 1 (*Allocation Rule*) A deterministic allocation rule $P : [\underline{v_0}, \bar{v_0}] \times [\underline{v}, \bar{v}]^n \to \mathbb{X}$ maps each profile of reported values to a deterministic allocation.

For any agent j, $P_j(\mathbf{v})$ is the j-th component of $P(\mathbf{v})$.

Definition 2 (*Transfer Rule*) A transfer rule t is a map $t : [\underline{v_0}, \bar{v_0}] \times [\underline{v}, \bar{v}]^n \rightarrow \mathbb{R}^{n+1}$. If $t_j(\mathbf{v}) > 0$ (resp. $t_j(\mathbf{v}) < 0$), then agent j pays (resp. receives) the amount $t_j(\mathbf{v})$.

Definition 3 (*Payoffs*) Given a mechanism (P, t). The (ex-post) utility of agent j with valuation v_j reporting $\hat{v}_j$ in mechanism (P, t) is

$$U_j^{(P,t)}(\hat{v}_j, v_{-j}|v_j) = v_j P_j(\hat{v}_j, v_{-j}) - t_j(\hat{v}_j, v_{-j}).$$

For convenience, the superscript (P, t) in the notation will be henceforth dropped.

Bayesian incentive compatibility requires that truthful reporting is optimal for each agent and for each valuation in *expectation*. This expectation is computed with respect to the prior distribution of valuations of other agents.

Definition 4 (*Bayesian Incentive Compatibility*) A mechanism is Bayesian incentive compatible (BIC) if for all j,

$$E_{-j}U_j(v_j, v_{-j}|v_j) \geq E_{-j}U_j(\hat{v}_j, v_{-j}|v_j) \text{ for all } v_j \text{ and } \hat{v}_j,$$

where $E_{-j}(\cdot)$ denotes expectation taken over v_{-j}.

Definition 5 (*Interim Individual Rationality*) A mechanism is interim individually rational (IIR) if for all j,

$$E_{-j}U_j(v_j, v_{-j}|v_j) \geq 0 \quad \text{for all } v_j.$$

The rest of the chapter uses the notation $U_j(\mathbf{v})$ and $U_j(v_j)$ for the ex-post and interim utilities in an equilibrium, respectively. Also, E is used to denote expectation operator over profile $\mathbf{v}$.

Definition 6 (*Efficiency*) An allocation rule P is ex-post efficient if for all $\mathbf{v}$,

$$\sum_j v_j P_j(\mathbf{v}) \geq \sum_j v_j P'_j(\mathbf{v}) \text{ for any allocation rule } P'.$$

To define ex-post efficient allocations in this setting, denote the feasible paths in Γ by $\mathcal{P}_1, \ldots, \mathcal{P}_q$ where $q \geq 1$. Consider a valuation profile $\mathbf{v}$. The sum of valuations in path $\mathcal{P}_i$ will be denoted by $S_i(\mathbf{v})$, $i = 1, \ldots, q$. These sums are ordered as follows: $S_{[1]}(\mathbf{v}) \leq \cdots \leq S_{[q]}(\mathbf{v})$. The paths corresponding to these sums are denoted by $\mathcal{P}_{[1]}(\mathbf{v}), \ldots, P_{[q]}(\mathbf{v})$, respectively. Efficiency requires trade to take place with sellers in $\mathcal{P}_{[1]}(\mathbf{v})$ if $v_0 > S_{[1]}(\mathbf{v})$; if $v_0 \leq S_{[1]}(\mathbf{v})$, then trade does not occur. For example, in the graph in Fig. 6, there are two feasible paths $\{1-2-3\}$ and $\{2-3-4\}$ when $k = 3$. If the valuations of the sellers are as indicated in the diagram, efficiency requires trade with sellers 1, 2, and 3 if $v_0 > 19$. Since the subsequent analysis will not require any special treatment of tie-breaking, any rule satisfying the condition above is called an efficient rule, denoted by P^*.

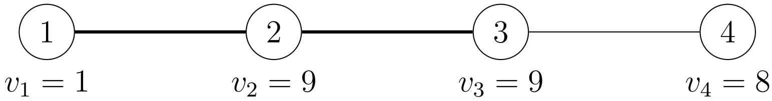


Fig. 6 $\mathcal{P}_{[1]}(\mathbf{v})$

Definition 7 (*Budget Balance*) A mechanism (P, t) satisfies ex-post budget balance if, for all $\mathbf{v}$,

$$\sum_{j=0}^{n} t_j(\mathbf{v}) = 0. \tag{1}$$

A mechanism achieves the *first-best* if it satisfies efficiency, IIR, and BB. A mechanism is *successful* if (a) it is BIC with respect to some prior μ and (b) it achieves the first-best.

Part A of the following result provides a sufficient condition and a weaker necessary condition for the existence of a successful mechanism when the number of feasible paths in the underlying graph is more than one. Part B states that no successful mechanism exists when there is only one feasible path.

Theorem 8 ([29])

A. *Let $\langle \Gamma, k, \mu \rangle$ be an assembly problem with $q > 1$.*

I. Suppose μ satisfies the following condition:

$$\underline{v}_0 \geq E\left(\sum_{i \in \mathcal{P}_{[1]}(\mathbf{v})} S_{[1]}(\bar{v}, v_{-i}) \right) - (k-1) E\left(S_{[1]}(\mathbf{v}) \right). \tag{2}$$

Then there exists a successful mechanism with respect to μ.

II. Suppose there exists a successful mechanism with respect to μ. Then the following holds:

$$\underline{v}_0 > E\left(\sum_{i \in \mathcal{P}_{[1]}(\mathbf{v})} S_{[1]}(\bar{v}, v_{-i}) - (k-1) S_{[1]}(\mathbf{v}) \middle| \mathbf{v} \in \tilde{V} \right), \tag{3}$$

where

$$\tilde{V} = \{\mathbf{v} \in [\underline{v}_0, \bar{v}_0] \times [\underline{v}, \bar{v}]^n : \underline{v}_0 > S_{[1]}(v) \text{ and } v_0 > S_{[1]}(\bar{v}, v_{-i}) \text{ for all } i \in \mathcal{P}_{[1]}(\mathbf{v})\}.$$

B. *Let $\langle \Gamma, k, \mu \rangle$ be an assembly problem with $q = 1$. The Myerson-Satterthwaite negative result applies, i.e., there does not exist any successful mechanism.*

This result is proved using the WKP condition due to Williams [31] and Krishna and Perry [13]: there exists a successful mechanism if and only if the well-known

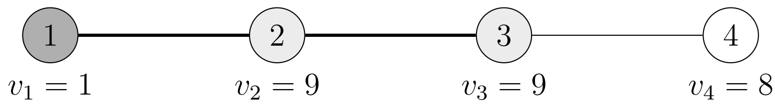


Fig. 7 Pivotal seller in darker shade and non-pivotal sellers in lighter shade at $\mathbf{v}$

VCG mechanism runs an expected budget surplus. The discussion below provides further interpretation.

Suppose $q > 1$ and efficiency requires trade to take place at a profile $\mathbf{v}$. A seller is called successful at this profile if he trades under the efficient rule. A successful seller $i \in \mathcal{P}_{[1]}(\mathbf{v})$ is trade-pivotal at $\mathbf{v}$ if trade is not efficient at $(\bar{v}, v_{-i})$, i.e., when seller i reports his highest possible valuation. Trade-pivotality is illustrated in the examples below. Let $n = 4$, $k = 3$, and $q = 2$. The supports of the prior distributions are as follows: $\underline{v}_0 = 25$, $\bar{v}_0 = 35$, $\underline{v} = 0$, $\bar{v} = 10$. Let $v_0 = 26$.

Recall from the example corresponding to Fig. 6 that sellers 1, 2, and 3 trade at $\mathbf{v}$. If seller 1's valuation is 10 instead of 1, the sum of the valuations on paths $\{1 - 2 - 3\}$ and $\{2 - 3 - 4\}$ are 28 and 26, respectively. Hence, trade does not take place at $(10, v_{-1})$, i.e., seller 1 is trade-pivotal at $\mathbf{v}$. But sellers 2 and 3 are not trade-pivotal at $\mathbf{v}$: if seller 2 has a valuation of 10, the sum of valuations on $\{1 - 2 - 3\}$ is 20 and trade can take place at $(10, v_{-2})$; same follows for seller 3. Pivotal and non-pivotal sellers and the efficient feasible path are shown in Fig. 7.

Now consider the profile $v'_0 = 28$, , $v'_1 = 1$, $v'_2 = 2$, $v'_3 = 3$, and $v'_4 = 2$. Trade takes place at $\mathbf{v}'$ with sellers 1, 2, and 3. Note that trade also takes place when the buyer's valuation is the lowest possible, i.e., $\underline{v}_0 = 25$. Furthermore, no successful seller at $\mathbf{v}'$ is trade-pivotal: if seller 1 reports a valuation of 10, efficiency requires trade with sellers 2, 3, and 4; if sellers 2 or 3 report a valuation of 10, efficiency requires trade with sellers 1, 2, and 3. This is illustrated in Fig. 8.

In the statement of Theorem 8, the set $\tilde{V}$ is the set of profiles v such that (i) it is efficient to trade at $(\underline{v}_0, v_{-0})$ and, therefore, also at $\mathbf{v}$, and (ii) all successful sellers are non-pivotal at $\mathbf{v}$. Hence, $\mathbf{v}' \in \tilde{V}$ but $\mathbf{v} \notin \tilde{V}$.

Pick $\mathbf{v} \in \tilde{V}$ and a successful seller i. Suppose i's valuation changes to $\bar{v}$. Since i is not trade-pivotal, trade still takes place and the sum of valuations of the successful sellers in the profile $(\bar{v}, v_{-i})$ is $S_{[1]}(\bar{v}, v_{-i})$. The sum of valuations of all other successful sellers at v is $S_{[1]}(\mathbf{v}) - v_i$. The difference of these two terms, summed over all successful sellers, is $\sum_{i \in \mathcal{P}_{[1]}(\mathbf{v})} S_{[1]}(\bar{v}, v_{-i}) - (k - 1)S_{[1]}(\mathbf{v})$. Part A-I of Theorem 8 states that there exists a successful mechanism if the expectation of this term

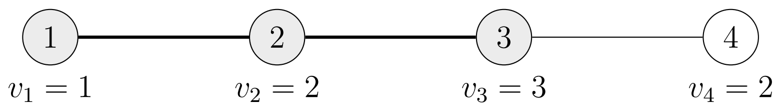


Fig. 8 Shaded nodes representing non-pivotal sellers at $\mathbf{v}'$

is at most $\underline{v}_0$. Part A-II states that if there exists a successful mechanism, then the expectation of this term, conditional on the profile belonging to $\tilde{V}$, is less than $\underline{v}_0$.

For illustration, consider the simple example: sellers are located on a complete graph and seller valuations are distributed uniformly in [0, 1]. If $\underline{v}_0 \geq \frac{k(k+1)}{n+1}$, then the existence of BIC mechanisms that achieve the first-best is guaranteed. For instance, if $n = 2$ and $k = 1$, $\underline{v}_0 \geq \frac{2}{3}$ is the required condition. Since $\frac{k(k+1)}{n+1} \to 0$ as $n \to \infty$, it becomes easier to satisfy the sufficient condition as the number of sellers increase.

To examine the role of critical sellers with respect to existence of a successful mechanism, let $c(\Gamma)$ denote the set of critical sellers in $\langle \Gamma, k, \mu \rangle$. If $q > 1$, then $|c(\Gamma)| \leq k - 1$. Conditions (2) and (3) can be reformulated to account for critical nodes.

Theorem 9 ([29]) *Let $\langle \Gamma, k, \mu \rangle$ be an assembly problem with $q > 1$.*

I. Suppose μ satisfies the following condition:

$$\underline{v}_0 \geq |c(\Gamma)|\bar{v} + E\left(\sum_{i \in \mathcal{P}_{[1]}(\mathbf{v}) \setminus c(\Gamma)} \big(S_{[1]}(\bar{v}, v_{-i}) + v_i\big) - (k - |c(\Gamma)|) S_{[1]}(\mathbf{v})\right). \tag{4}$$

Then there exists a successful mechanism with respect to μ.

II. Suppose there exists a successful mechanism with respect to μ. Then the following holds:

$$\underline{v}_0 > |c(\Gamma)|\bar{v} + E\left(\sum_{i \in \mathcal{P}_{[1]}(\mathbf{v}) \setminus c(\Gamma)} \big(S_{[1]}(\bar{v}, v_{-i}) + v_i\big) - (k - |c(\Gamma)|) S_{[1]}(\mathbf{v}) \middle| \mathbf{v} \in \tilde{V}\right), \tag{5}$$

where

$$\tilde{V} = \{\mathbf{v} \in [\underline{v}_0, \bar{v}_0] \times [\underline{v}, \bar{v}]^n : \underline{v}_0 > S_{[1]}(\mathbf{v}) \text{ and } v_0 > S_{[1]}(\bar{v}, v_{-i}) \text{ for all } i \in \mathcal{P}_{[1]}(\mathbf{v})\}.$$

Corollary 1 *Suppose there exists a successful mechanism with respect to μ. Then*

$$\underline{v}_0 > |c(\Gamma)|\bar{v}. \tag{6}$$

Theorem 9 and Corollary 1 state that the count of critical nodes puts a lower bound on the support of the buyer's valuation essential for the existence of a successful mechanism.

4 Discussion

The results presented in Sect. 3.1 show the bearing of the degree of complementarity among inputs and asymmetry of valuations on full surplus extraction in the assembly problem. While the presence of critical sellers or sufficient asymmetry in seller valuations prevents full surplus extraction, even limited substitutability enables the buyer to extract full surplus in two periods, provided she is sufficiently patient. The number of critical sellers present in the problem also imposes an upper bound on the volume of surplus that can be extracted by the buyer.

In the light of these results, the formation of seller coalitions becomes one of the possible explanations of a holdout in the assembly problem. Consider the following example for an illustration. In the problem where one item is required and two sellers are present, by making alternate offers to one of the sellers according to the equilibrium strategy specified in Theorem 1 and by excluding the other seller using negative offers, the buyer can assure herself $\frac{\delta}{1+\delta}$ share of the full surplus. If sellers are allowed to use trigger strategies, there exists an equilibrium where both sellers collude to claim $\frac{1}{1+\delta}$ of the full surplus and the buyer picks one of them with equal probability provided $\delta > \frac{1}{\sqrt{2}}$. This equilibrium is sustained by the following trigger strategy: if any seller deviates by charging less than $\frac{1}{1+\delta}$, the other seller charges zero surplus share in the subsequent period. The buyer then rejects the deviating seller's offer and chooses to purchase from the other seller. The collusive payoff $\frac{1}{2(1+\delta)}$ is greater than the non-collusive payoff $1-\delta$ if $\delta > \frac{1}{\sqrt{2}}$. In this equilibrium, both sellers get a positive expected payoff. If $\delta < \frac{1}{\sqrt{2}}$, sellers compete and earn zero surplus shares in the equilibrium. A complete characterization of possible coalitions and corresponding surplus shares in assembly problems is an open agenda for future work.

The results in Sect. 3.2 are not strategically informative, since the discussion here involves direct mechanisms. But the role of critical sellers turns out to be prominent here as well.

The nature of the first-best mechanism is not described in these results. As shown by Krishna and Perry [13], it is essentially a projection of the well-known VCG payments in the space of balanced transfers. Implementing the first-best requires knowledge of the prior, and hence turn out to be costly in the informational requirement in many real-life applications. This remark is also applicable to the optimal mechanism (see [28]).

The VCG mechanism itself exhibits several good properties like ex-post efficiency, dominant strategy incentive compatibility, and ex-post individual rationality. Further, it is also ex-post budget balanced in the limit as the number of sellers becomes large (see [30]), making it attractive for many applications with independent private values.

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