Recent Results on Strategy-Proofness of Random Social Choice Functions

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1 Introduction

Randomization has long been recognized as a useful device to achieve fairness in allocation problems. For example, in a cricket match, the first use of the pitch for batting may be important for the determination of the result, and a flip of coin is the device used to decide this issue. In addition to resolving fairness, randomization is also useful for incentivizing people to reveal their private information truthfully in mechanism design problems. In this essay, we will briefly survey some of the main results in randomized mechanism design problem in the context of voting models.

A voting model is one where individuals/agents/voters have to choose one among a number of alternatives or candidates. Each individual has a ranking or preference over all alternatives and a *(deterministic) social choice function* picks an alternative for every tuple of individual ranking of alternatives. An important feature of the voting model is that monetary payments or transfers are not permitted—this assumption is entirely in keeping with our understanding of voting. Individual preferences are private information and are known only to the individuals themselves. A social choice function is *strategy-proof* if no individual can gain by misrepresenting her preference. A fundamental question in mechanism design theory is the following: what is the set of strategy-proof social choice functions? If a social choice function is not strategy-proof, there are strong grounds to conclude that the social goals represented by the social choice functions are unattainable in the presence of private information.

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The classical result on strategy-proofness is the Gibbard-Satterthwaite Theorem [25, 45]. According to the theorem, a social choice function is strategy-proof and satisfies a mild range condition only if it is *dictatorial*. Thus, there exists an agent whose most preferred alternative is always chosen. This result relies on the assumption of an *unrestricted domain*, i.e., it is assumed that every individual can have *any* preference over the alternatives. However, in several situations, it is natural to assume that individuals can never have certain preferences. In an extreme case, all individuals have a single ranking; the problem is then rendered trivial because individuals do not have any opportunity for misrepresentation. In general, considering plausible restrictions on possible preferences, called the *restricted domain* approach to the mechanism design problem, has produced important insights. For instance, the domain of *single-peaked* preferences admits a variety of well-behaved strategy-proof social choice functions (see Sect. 4.2 for further details).

There has been a great deal of research since the 1970s on the structure of strategy-proof (deterministic) social choice functions on both restricted and unrestricted domains. In contrast, there has been much less work on strategy-proof random social choice functions. There has, however, been some recent literature addressing these issues, and our goal in this paper is to survey some of these results. We focus mainly on two questions. Does randomization help in escaping the well-known negative results such as the Gibbard-Satterthwaite Theorem? Secondly, in restricted domain environments, does randomization further enrich the class of well-behaved deterministic social choice functions?

The paper is organized as follows. Section 2 introduces the problem and discusses various properties of random social choice functions. Sections 3 and 4 present results on unrestricted domains and various restricted domains, respectively. Section 5 discusses the deterministic extreme point property while Sect. 6 concludes.

2 Preliminaries

We consider a society consisting of a (finite) set of individuals $N = \{1, \ldots, n\}$ with at least two individuals. Except in Sect. 4.5, the set A is assumed to be finite. The set of alternatives or candidates is A with $|A| \geq 2$. Society faces the problem of choosing a probability distribution over alternatives based on the "preferences" of individuals in the society.

For notational convenience, we do not use braces for singleton sets whenever it is clear from the context; for instance, we denote the set $\{i\}$ by i.

2.1 Preferences

A complete, reflexive, asymmetric, and transitive binary relation over A (also called a linear order) is called a *preference*. A preference can be viewed as a strict ranking over the alternatives. We denote by $\mathbb{L}(A)$ the set of all preferences over A. For $P \in \mathbb{L}(A)$ and $a, b \in A$, aPb is interpreted as "a is strictly preferred to b according to P". For $P \in \mathbb{L}(A)$ and $1 \leq k \leq m$, by $r_k(P)$ we denote the k-th ranked alternative in P, i.e., $r_k(P) = a$ if and only if $|\{b \in A \mid bPa\}| = k$. We denote the top-ranked alternative of a preference P by $\tau(P)$ (instead of $r_1(P)$). For $P \in \mathscr{D}$ and $a \in A$, the *upper contour set* of a at P, denoted by $U(a, P)$, is defined as the set of alternatives that are as good as a in P, that is, $U(a, P) = \{b \in A \mid bPa\}$. We call a set U an upper contour set at a preference P if it is the upper contour set of some alternative at P.

A set of admissible preferences (henceforth referred to as a *domain*) is denoted by $\mathscr{D} \subseteq \mathbb{L}(A)$. For $a \in A$, we denote by $\mathscr{D}^a$ the preferences in $\mathscr{D}$ that have a as the top-ranked alternative. For a domain $\mathscr{D}$, the *top-set* of $\mathscr{D}$, denoted by $\tau(\mathscr{D})$, is the set of alternatives that appear as a top-ranked alternative in some preference in $\mathscr{D}$, that is, $\tau(\mathscr{D}) = \cup_{P \in \mathscr{D}} \tau(P)$.

A *preference profile* (or simply a profile), denoted by $P_N = (P_1, \ldots, P_n)$, is an element of $\mathscr{D}^n = \mathscr{D} \times \cdots \times \mathscr{D}$ that represents a collection of preferences one for each individual.

2.2 *Random Social Choice Functions*

In this section, we define social choice functions and discuss their properties. We denote the set of probability distributions over A by ΔA. A **random social choice function (RSCF)** is a function $\varphi : \mathscr{D}^n \to \Delta A$ that assigns a probability distribution or lottery over A at every profile. For $a \in A$ and $P_N \in \mathscr{D}^n$, the probability of a at the outcome $\varphi(P_N)$ is denoted by $\varphi_a(P_N)$, and for $B \subseteq A$, the total probability of the alternatives in B at $\varphi(P_N)$ is denoted by $\varphi_B(P_N) = \sum_{a \in B} \varphi_a(P_N)$. Some examples of RSCFs are provided below.

Example 1 (*RSCFs based on scoring rules*) A score vector s is an m-dimensional vector $(s_1, s_2, \ldots, s_m)$ such that $s_1 \geq s_2 \geq \cdots \geq s_m$ with $s_1 > s_m$.[1] For any individual i, any preference P_i, and any alternative a, the score assigned by i to a in P_i is $s(a, P_i) = s_k$ where k is the rank of a in P_i, i.e., $r_k(P_i) = a$. The score of a at profile P_N is $\boldsymbol{s}(a, P_N) = \sum_{i \in N} s_i(a, P_i)$. We now define two RSCFs based on score vectors (for other such RSCFs see [7])

The Proportional Scoring Rule φ^{PS}: for all $a \in A$ and profiles P_N,

$$\varphi_a^{PS}(P_N) = \frac{\boldsymbol{s}(a, P_N)}{\sum_{a \in A} \boldsymbol{s}(a, P_N)}.$$

Let $M(P_N)$ denote the set of alternatives that attain the maximum score at profile P_N, i.e., $M(P_N) = \arg\max_{a \in A} \boldsymbol{s}(a, P_N)$.

[1] Well-known score vectors are the *Plurality vector* $(1, 0, \ldots, 0)$ and the *Borda vector* $(m-1, m-2, \ldots, 0)$.

Table 1 The proportional scoring rule

1\2	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	$(\frac{6}{12}, \frac{4}{12}, \frac{2}{12})$	$(\frac{6}{12}, \frac{3}{12}, \frac{3}{12})$	$(\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$	$(\frac{4}{12}, \frac{5}{12}, \frac{3}{12})$	$(\frac{5}{12}, \frac{3}{12}, \frac{4}{12})$	$(\frac{4}{12}, \frac{4}{12}, \frac{4}{12})$
acb	$(\frac{6}{12}, \frac{3}{12}, \frac{3}{12})$	$(\frac{6}{12}, \frac{2}{12}, \frac{4}{12})$	$(\frac{5}{12}, \frac{4}{12}, \frac{3}{12})$	$(\frac{4}{12}, \frac{4}{12}, \frac{4}{12})$	$(\frac{5}{12}, \frac{2}{12}, \frac{5}{12})$	$(\frac{4}{12}, \frac{3}{12}, \frac{5}{12})$
bac	$(\frac{5}{12}, \frac{5}{12}, \frac{2}{12})$	$(\frac{5}{12}, \frac{4}{12}, \frac{3}{12})$	$(\frac{4}{12}, \frac{6}{12}, \frac{2}{12})$	$(\frac{3}{12}, \frac{6}{12}, \frac{3}{12})$	$(\frac{4}{12}, \frac{4}{12}, \frac{4}{12})$	$(\frac{3}{12}, \frac{5}{12}, \frac{4}{12})$
bca	$(\frac{4}{12}, \frac{5}{12}, \frac{3}{12})$	$(\frac{4}{12}, \frac{4}{12}, \frac{4}{12})$	$(\frac{3}{12}, \frac{6}{12}, \frac{3}{12})$	$(\frac{2}{12}, \frac{6}{12}, \frac{4}{12})$	$(\frac{3}{12}, \frac{4}{12}, \frac{5}{12})$	$(\frac{2}{12}, \frac{5}{12}, \frac{5}{12})$
cab	$(\frac{5}{12}, \frac{3}{12}, \frac{4}{12})$	$(\frac{5}{12}, \frac{2}{12}, \frac{5}{12})$	$(\frac{4}{12}, \frac{4}{12}, \frac{4}{12})$	$(\frac{3}{12}, \frac{4}{12}, \frac{5}{12})$	$(\frac{4}{12}, \frac{2}{12}, \frac{6}{12})$	$(\frac{3}{12}, \frac{3}{12}, \frac{6}{12})$
cba	$(\frac{4}{12}, \frac{4}{12}, \frac{4}{12})$	$(\frac{4}{12}, \frac{3}{12}, \frac{5}{12})$	$(\frac{3}{12}, \frac{5}{12}, \frac{4}{12})$	$(\frac{2}{12}, \frac{5}{12}, \frac{5}{12})$	$(\frac{3}{12}, \frac{3}{12}, \frac{6}{12})$	$(\frac{2}{12}, \frac{4}{12}, \frac{6}{12})$

Table 2 The uniform maximal scoring rule

1\2	*abc*	*acb*	*bac*	*bca*	*cab*	*cba*
abc	$(1, 0, 0)$	$(1, 0, 0)$	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(0, 1, 0)$	$(1, 0, 0)$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
acb	$(1, 0, 0)$	$(1, 0, 0)$	$(1, 0, 0)$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{2}, 0, \frac{1}{2})$	$(0, 0, 1)$
bac	$(\frac{1}{2}, \frac{1}{2}, 0)$	$(1, 0, 0)$	$(0, 1, 0)$	$(0, 1, 0)$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(0, 1, 0)$
bca	$(0, 1, 0)$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(0, 1, 0)$	$(0, 1, 0)$	$(0, 0, 1)$	$(0, \frac{1}{2}, \frac{1}{2})$
cab	$(1, 0, 0)$	$(\frac{1}{2}, 0, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$
cba	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(0, 0, 1)$	$(0, 1, 0)$	$(0, \frac{1}{2}, \frac{1}{2})$	$(0, 0, 1)$	$(0, 0, 1)$

The Uniform Maximal Scoring Rule φ^{UMS}: for all $a \in A$ and profiles P_N,

$$\varphi_a^{UMS}(P_N) = \begin{cases} \frac{1}{|M(P_N)|} & \text{if } a \in M(P_N), \\ 0 & \text{otherwise.} \end{cases}$$

Tables 1 and 2 illustrate the Proportional Scoring Rule and the Uniform Maximal Scoring Rule, respectively, in the case where $N = \{1, 2\}$, $A = \{a, b, c\}$, and s is the Borda score vector.

A RSCF is a **deterministic social choice function (DSCF)** if it selects a degenerate probability distribution at every profile. Formally, an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is a DSCF if $\varphi_a(P_N) \in \{0, 1\}$ for all $a \in A$ and all $P_N \in \mathscr{D}^n$. For convenience, we shall sometimes write a DSCF as a function $f : \mathscr{D}^n \to A$.

In our model, as in the standard model of mechanism design, an individual's preference is her *private information*, i.e., known only to herself. Since the mechanism designer goals depend on this information, it must be elicited from the individuals. The property of *strategy-proofness* ensures that individuals have the correct incentives to report their true preferences. Specifically, she cannot obtain a strictly better alternative according to her true preference irrespective of her beliefs about the reports of other individuals. In game-theoretic terms, truth-telling constitutes a dominant strategy for every individual in the direct revelation game.

Strategy-proofness for a DSCF can be defined straightforwardly along the lines of the discussion in the preceding paragraph: a DSCF $f : \mathscr{D}^n \to A$ is strategy-proof if either $f(P_i, P_{-i}) = f(P_i', P_{-i})$ or $f(P_i, P_i)P_i f(P_i', P_{-i})$ for all $P_i, P_i' \in \mathscr{D}$ for all $P_{-i} \in \mathscr{D}^{n-1}$ and all individuals i. Consider an individual i whose true preference is P_i and "believes" that all other individuals will announce $P_{-i} \in \mathscr{D}^{n-1}$. If she is truthful, the outcome is $f(P_i, P_{-i})$. On the other hand, suppose she considers manipulating or misrepresenting her preference as P_i', the new outcome is $f(P_i', P_{-i})$. If f is strategy-proof, the misrepresentation will either keep the outcome unchanged or lead to a worse outcome according to her true preference P_i. Importantly, i cannot gain by the misrepresentation no matter what she believes about the preferences of others.

There are some conceptual difficulties in extending the same idea to RSCFs. The strategy-proofness property involves the comparison of the outputs of a DSCF or RSCF at two profiles—one where the individual is truthful and the other, where she misrepresents her preference. In the case of a DSCF, these two outputs are alternatives and can be compared using the individual's (true) preference. However, in the case of a RSCF, the relevant outputs are lotteries and it is not obvious how preferences over alternatives can be extended to rankings over lotteries.

In some cases, there is a natural way to evaluate lotteries given an individual's preferences. Suppose $A = \{a, b, c\}$, and an individual has the preference $P = abc$.[2] Consider the lotteries $p = (0.5, 0.3, 0.2)$ and $q = (0.6, 0.35, 0.05)$.[3] Observe that q can be obtained from p by transferring probabilities from lower to higher ranked alternatives. Therefore, requiring the individual to prefer q to p would appear entirely reasonable. However, this argument cannot be applied while comparing p with $r = (0.4, 0.5, 0.1)$. Here, probabilities are simultaneously shifted from lower to higher ranked alternatives *and* from higher to lower ranked alternatives.

In this essay, we focus on the *stochastic dominance* approach introduced in Gibbard [26]. Following Von Neumann and Morgenstern [46], the standard approach to lottery comparisons is via expected utility comparisons: thus, lottery q is preferred to lottery p if the expected utility from q is greater than the expected utility from p. The difficulty in following this approach is that inputs to the RSCF are preferences (ordinal rankings) rather than utility functions. A natural way to deal with this issue is to consider *utility representations* of preferences. For example, a utility representation of the preference $P = abc$, consists of real numbers $u(a)$, $u(b)$, and $u(c)$ with $u(a) > u(b) > u(c)$. Observe that for any such representation, the expected utility from q is greater than the expected utility from p. However, the expected utility from p can be greater or less than that of r depending on the utility representation chosen.[4] According to the stochastic dominance criterion, the expected utility of the

[2] By $P = abc$, we mean the preference where a is the top-ranked, b is the second-ranked, and c is the third-ranked alternative.

[3] By (p_1, p_2, p_3), we denote the lottery where a, b, and c receive probabilities p_1, p_2, and p_3, respectively.

[4] To see this, choose $u(a) = 1$ and $u(c) = 0$. If $u(b)$ is close to one, r will have a higher expected utility than p. The opposite will be true if $u(b)$ is chosen sufficiently close to zero.

lottery obtained from truth-telling must not be lower than the expected utility of any lottery arising from misrepresentation of preferences for *any* representation of true preferences. This is stated formally below.[5]

Let P be a preference ordering. The function $u : A \to \Re$ is a *utility representation* of P if $u(a) > u(b)$ whenever aPb. The RSCF φ is **stochastic dominance strategy-proof** if $\sum_{a \in A} u(a)\varphi_a(P_i, P_{-i}) \geq \sum_{a \in A} u(a)\varphi_a(P_i', P_{-i})$ for all $P_i, P_i' \in \mathscr{D}$, for all $P_{-i} \in \mathscr{D}^{n-1}$, and for all utility representations u of P_i. This notion of strategy-proofness places a heavy burden on the truth-telling lottery. In the example discussed previously, φ will fail to be strategy-proof if p and r arise from truth-telling and misrepresentation, respectively, because *there exists* a utility representation of abc according to which r has a higher expected utility than p. Thus, we may be confident that a RSCF that is strategy-proof in this sense will induce individuals to be truthful. However, we may be excessively cautious in eliminating from consideration RSCFs that fail to satisfy this property. A weaker notion of strategy-proofness would only require the expected utility from the lottery from truth-telling not be smaller than that from misrepresentation for all utility representation of the true preference. In the previous example, a RSCF which produced p and q from truth-telling and misrepresentation, respectively, would fail strategy-proofness. However, it would not violate the condition if misrepresentation yielded r instead of q.[6]

We now present an alternative formulation of stochastic dominance strategy-proofness. The lottery p **stochastically dominates** lottery q at a preference P if $p(U) \geq q(U)$ for all upper contour sets U of P. Another equivalent way to define stochastic dominance is as follows. A RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **stochastic dominance strategy-proof** if $\varphi(P_i, P_{-i})$ stochastically dominates $\varphi(P_i', P_{-i})$ for all $P_i, P_i' \in \mathscr{D}$, for all $P_{-i} \in \mathscr{D}^{n-1}$ and all individuals i. It is straightforward to verify that the two notions of stochastic dominance strategy-proofness are equivalent and reduce to the notion of strategy-proofness for DSCFs. Henceforth, we shall refer to stochastic dominance strategy-proofness simply as strategy-proofness. If a RSCF is not strategy-proof, we shall say it is *manipulable*.

The proportional scoring rule is strategy-proof, while the uniform maximal scoring rule is not. For instance, individual 2 can manipulate φ^{UMS} at the profile (abc, cba) via the preference cab as $\varphi_U(abc, cba) > \varphi_U(abc, cab)$ for the upper contour set $U = c$ of the preference cba (see Table 2).

The next property of a RSCF ensures that it is minimally responsive to the preferences of individuals. This property requires an alternative to be chosen with probability one if this alternative is top-ranked by all individuals. Formally, a RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies **unanimity** if for all $a \in A$ and all $P_N \in \mathscr{D}^n$, $[\tau(P_i) =$

[5] It is important to emphasize that cardinal mechanisms are not considered here as the private information of agents is not cardinal (i.e., utility functions). There is a small literature on strategy-proof RSCFs on cardinal preferences (see [19, 20, 27] for details).

[6] For further discussion of alternate notions of strategy-proofness, see [3–6, 9].

a for all $i \in N] \Rightarrow [\varphi_a(P_N) = 1]$.[7] Note that the proportional scoring rule is not unanimous, whereas the uniform maximal scoring rule is unanimous.

There is a natural way to generate "new" RSCFs from any given collection of RSCFs. Let φ^j, $j = 1, \ldots, K$ be a collection of RSCFs and let λ^j, $j = 1, \ldots, K$ be non-negative real numbers such that $\sum_{j=1}^{K} \lambda_j = 1$. Define $\varphi = \sum_{j=1}^{K} \lambda^j \varphi^j$ where $\varphi_a(P_N) = \sum_{j=1}^{K} \lambda^j \varphi_a^j(P_N)$ for all $P_N \in \mathscr{D}^n$ and all $a \in A$. We shall refer to φ as a *convex combination* of the RSCFs φ^j. Since the convex combination of a collection of lotteries is a lottery, φ is a RSCF. We record some important properties of convex combinations of RSCFs below. They can be easily verified and are stated without proof.

Remark 2.1 Let φ be a convex combination of φ^j, $j = 1, \ldots, K$. If each φ^j is strategy-proof and satisfies unanimity, then φ is strategy-proof and satisfies unanimity.

The set of strategy-proof RSCFs satisfying unanimity is, therefore, a convex set. This set can, therefore, be characterized by its extreme points. Note that the RSCFs φ^j could be deterministic. Since DSCFs cannot be written as convex combinations of other RSCFs, it follows that strategy-proof DSCFs satisfying unanimity are extreme points of the set of strategy-proof RSCFs satisfying unanimity. A question of considerable theoretical and conceptual interest is whether they are the *only* extreme points. We shall discuss this issue in greater detail in Sect. 5.

3 Results on the Unrestricted Domain

In this section, we present characterization results for unanimous and strategy-proof RSCFs on the unrestricted domain. A domain $\mathscr{D}$ is **unrestricted** if it contains *all* preferences over A, i.e., $\mathscr{D} = \mathbb{L}(A)$. We distinguish two cases based on the number of alternatives in A.

3.1 The Case of Two Alternatives

An important class of social choice problems is concerned with the case of two alternatives. Among such problems are those where individuals have to vote Yes or No to a proposal, to Approve or Disapprove a resolution or if there are two candidates in an election.

We introduce a class of DSCFs on the unrestricted domain with two alternatives. A *committee* $\mathscr{W}$ is a set of subsets of N such that:

[7] It is worth mentioning that under strategy-proofness, unanimity can be weakened in the following way: a RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies unanimity if for all $a \in A$ and all $P_N \in \mathscr{D}^n$, $[P_i = P_j$ for all $i, j \in N] \Rightarrow [\varphi_a(P_N) = 1]$ where $a = \tau(P_i)$ for some (and hence, all) $i \in N$.

(i) $\emptyset \notin \mathscr{W}$ and $N \in \mathscr{W}$,
(ii) for all $S, T \subseteq N$, if $S \subseteq T$ and $S \in \mathscr{W}$, then $T \in \mathscr{W}$.

The elements of $\mathscr{W}$ are called *winning coalitions*, and other subsets of N are called *losing coalitions*.

Let us assume $A = \{a, b\}$. For $P_N \in \mathscr{D}^n$, by $N^a(P_N)$, we denote the set of individuals $i \in N$ who have a as their top-ranked alternative, that is, $\tau(P_i) = a$. For a committee $\mathscr{W}$, a DSCF $f_{\mathscr{W}}$ is called a **voting by committees** rule with respect to a and b if at any profile P_N, a is chosen as the outcome if and only all members of *some* winning coalition vote for a, that is, if for every $P_N \in \mathscr{D}^n$,

$$f_{\mathscr{W}}(P_N) = \begin{cases} a \text{ if } N^a(P_N) \in \mathscr{W} \\ b \text{ if } N^a(P_N) \notin \mathscr{W}. \end{cases}$$

Voting by Committees is a rich class of rules. It includes *majority voting* where a coalition is winning only if it contains at least half the members of the society, the *unanimity rule* where only the coalition of all individuals is winning, and *dictatorship* where a coalition is winning if and only if it contains a specific individual called the *dictator*.

A RSCF is called a *random voting by committees* rule with respect to a and b if it is a convex combination of voting by committees rules with respect to the same alternatives.

Theorem 1 ([33, 35]) *A RSCF on a domain over two alternatives is unanimous and strategy-proof if and only if it is a random voting by committees rule.*

3.2 The Case of More Than Two Alternatives

It is well-known in social choice theory that the set of strategy-proof DSCFs shrinks dramatically if the set of alternatives increases beyond two. According to the celebrated **Gibbard-Satterthwaite** Theorem, every strategy-proof DSCF satisfying unanimity must be dictatorial. Formally, a DSCF $f : \mathscr{D}^n \to A$ is **dictatorial** or is a **dictatorship** if there is an individual $i \in N$ called the dictator such that f selects the top-ranked alternative of i at every profile P_N, i.e., $f(P_N) = \tau(P_i)$ for all $P_N \in \mathscr{D}^n$. Thus, all the well-behaved rules such as majority rule are no longer strategy-proof once there are at least three alternatives. Gibbard [26] provides a complete answer to the following question: does the negative result for DSCFs extend to RSCFs as well?

A RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **unilateral** if (i) φ is strategy-proof and (ii) there exists an individual $i \in N$ such that $\varphi(P_N) = \varphi(P'_N)$ for all profiles P_N, P'_N such that $P_i = P'_i$. In other words, a RSCF is a unilateral if it is strategy-proof and depends only on the preference of a single individual. An example of a unilateral is a RSCF that gives probability 0.5, 0.4, and 0.1 to individual i's first-ranked, second-ranked, and third-ranked alternatives, respectively. A **duple** is a RSCF φ if (i) φ is strategy-proof and (ii) there exist alternatives $a, b \in A$ such that for all profiles P_N, $\varphi_c(P_N) > 0$ only if $c \in \{a, b\}$. A duple is a strategy-proof RSCF that gives positive probability to at most two fixed alternatives at any profile.

Remark 2.1 implies that convex combinations of unilaterals and duples are strategy-proof. Gibbard [26] shows that the converse is also true: every strategy-proof RSCF (on the unrestricted domain) is a convex combination of unilaterals and duples.

Theorem 2 ([26]) *Assume $m \geq 3$. A RSCF on the unrestricted domain is strategy-proof if and only if it is a convex combination of unilateral and duple rules.*

The class of strategy-proof RSCFs on the unrestricted domain is not as restricted as the class of strategy-proof DSCFs. Although a unilateral only considers the preference of a single individual while assigning probabilities, unilaterals for different individuals can be combined (using convex combinations) to generate a more acceptable RSCF which is also strategy-proof. Similarly, duples over different pairs of alternatives can be combined to produce RSCFs that have full support at every profile. Unilaterals and duples can be combined as well. Consequently, the class of strategy-proof RSCFs is "large" and includes many RSCFs that have attractive features from an ethical point of view (unlike dictatorship, for example). One such RSCF is the Proportional Scoring Rule in Example 1 which can be expressed as a convex combination of unilaterals and duples. Further examples and results can be found in Barbera [7].

The discussion in the previous paragraph is subject to an important caveat. A duple does not satisfy unanimity since it assigns zero probability to all except two alternatives. Nor can duples be convexified in a manner that the resulting RSCF satisfies unanimity. A unilateral satisfies unanimity only if the first-ranked alternative of an individual gets probability one. Recall that a RSCF is a random dictatorship if it is a convex combination of dictatorial DSCFs. Combining these observations, we obtain the following result.

Theorem 3 ([26]) *Assume $m \geq 3$. A RSCF on the unrestricted domain is strategy-proof and satisfies unanimity if only if it is a random dictatorship.*

We provide a proof of this result in the case where there are two individuals. An induction argument can be used to extend the argument to an arbitrary number of individuals.[8]

Proof It is left to the reader to verify that every random dictatorship is unanimous and strategy-proof. We prove the converse. Let $N = \{1, 2\}$. Assume that $\varphi : [\mathbb{L}(A)]^2 \to \Delta A$ satisfies unanimity and strategy-proofness.

Lemma 1 *Let $(P_1, P_2) \in [\mathbb{L}(A)]^2$ be such that $\tau(P_1) \neq \tau(P_2)$. Then $[\varphi_a(P_1, P_2) > 0] \implies [a \in \{\tau(P_1), \tau(P_2)\}]$.*

[8] Duggan [17] provides a geometric proof of the result.

Proof Suppose not, i.e., suppose that there exists $P_1, P_2 \in \mathbb{L}(A)$ and $a, b \in A$ such that $\tau(P_1) = a \neq b = \tau(P_2)$ and $\varphi_a(P_1, P_2) + \varphi_b(P_1, P_2) < 1$. Let $\alpha = \varphi_a(P_1, P_2)$ and $\beta = \varphi_b(P_1, P_2)$. Let $P_1' = ab\cdots$ and $P_2' = ba\cdots$. Then strategy-proofness implies $\varphi_a(P_1', P_2) = \alpha$. Furthermore, it must be that $\varphi_a(P_1', P_2) + \varphi_b(P_1', P_2) = 1$ as otherwise voter 1 will manipulate via P_2 and thereby obtaining probability one on b by unanimity. Hence, $\varphi_b(P_1', P_2) = 1 - \alpha$. Note that strategy-proofness also implies $\varphi_b(P_1', P_2') = \varphi_b(P_1', P_2) = 1 - \alpha$ and $\varphi_a(P_1', P_2') = \alpha$.

By a symmetric argument, $\varphi_b(P_1', P_2') = \varphi_b(P_1, P_2') = \beta$ and $\varphi_a(P_1', P_2') = 1 - \beta$. Comparing the probabilities on a and b given by φ at the profile (P_1', P_2'), it follows that $\alpha + \beta = 1$ contradicting the earlier conclusion. ■

Lemma 2 *Let* $(P_1, P_2), (\bar{P}_1, \bar{P}_2) \in [\mathbb{L}(A)]^2$ *be such that* $\tau(P_1) = a \neq b = \tau(P_2)$ *and* $\tau(\bar{P}_1) = c \neq d = \tau(\bar{P}_2)$. *Then* $[\varphi_a(P_1, P_2) = \varphi_c(\bar{P}_1, \bar{P}_2)]$ *and* $[\varphi_b(P_1, P_2) = \varphi_d(\bar{P}_1, \bar{P}_2)]$.

Proof Let $P_1 = a\cdots, P_2 = b\cdots$. Let $(\hat{P}_1, \hat{P}_2)$ be an arbitrary profile where $\tau(\hat{P}_1) = a$ and $\tau(\hat{P}_2) = b$. Then strategy-proofness implies that $\varphi_a(\hat{P}_1, P_2) = \varphi_a(P_1, P_2)$. Lemma 1 implies $\varphi_b(\hat{P}_1, P_2) = \varphi_b(P_1, P_2)$. Now changing voter 2's ordering from P_2 to $\hat{P}_2$ and applying the same arguments, it follows that $\varphi_a(\hat{P}_1, \hat{P}_2) = \varphi_a(P_1, P_2)$ and $\varphi_b(\hat{P}_1, \hat{P}_2) = \varphi_b(P_1, P_2)$.

Assume that $c \neq b$. The argument in the previous paragraph implies that it can be assumed without loss of generality that c is the second-ranked outcome at P_1 (if a and c are distinct), i.e., it can be assumed that $P_1 = ac\cdots$. Let $\bar{P}_1 = ca\cdots$. Then strategy-proofness implies $\varphi_a(\bar{P}_1, P_2) + \varphi_c(\bar{P}_1, P_2) = \varphi_a(P_1, P_2) + \varphi_c(P_1, P_2)$. By Lemma 1, $\varphi_c(P_1, P_2) = \varphi_a(\bar{P}_1, P_2) = 0$. Hence, $\varphi_a(P_1, P_2) = \varphi_c(\bar{P}_1, P_2)$ while $\varphi_b(P_1, P_2) = \varphi_b(\bar{P}_1, P_2)$. Assume $b \neq d$. Switching voter 2's preferences from P_2 to $\bar{P}_2$ and applying the same argument as above, it follows that $\varphi_c(\bar{P}_1, P_2) = \varphi_c(\bar{P}_1, \bar{P}_2)$ while $\varphi_b(\bar{P}_1, P_2) = \varphi_d(\bar{P}_1, \bar{P}_2)$.

The arguments above can deal with all cases except the case where $c = b$ and $d = a$. Since $m \geq 3$, there exists $x \in A$ distinct from a and b. Let $\tilde{P}_1$ be such that $\tau(\tilde{P}_1) = x$. From earlier arguments, $\varphi_a(P_1, P_2) = \varphi_x(\tilde{P}_1, \bar{P}_2)$ and $\varphi_b(P_1, P_2) = \varphi_a(\tilde{P}_1, \bar{P}_2)$.

Applying these arguments again, it can be inferred that $\varphi_x(\tilde{P}_1, \bar{P}_2) = \varphi_b(\bar{P}_1, \bar{P}_2)$ and $\varphi_a(\tilde{P}_1, \bar{P}_2) = \varphi_a(\bar{P}_1, \bar{P}_2)$ establishing the Lemma. ■

Lemmas 1 and 2 above establish that φ is a random dictatorship. ■

We now proceed to examine the structure of strategy-proof RSCFs on restricted domains.

4 Results on Restricted Domains

In many mechanism design problems, the mechanism designer has a-priori information about the preferences of individuals. For instance, a and c may represent candidates with "extreme" positions while b is a "moderate" candidate. The designer may know (without preference revelation) that b always lies between a and c in the preferences of all individuals. As a consequence, RSCFs need to be defined only over a subset of the set of all preferences. The designer also has to consider a narrower class of preferences while checking for possible deviations from truth-telling. Of course, various types of restricted domains can be considered. In this section, we review results on several well-known restricted domains.

4.1 Dictatorial Domains

A domain $\mathscr{D}$ is a **dictatorial domain** if *every* unanimous and strategy-proof DSCF $f : \mathscr{D}^n \to A$ is dictatorial. Similarly, a domain $\mathscr{D}$ is a **random dictatorial domain** if *every* unanimous and strategy-proof RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is a random dictatorship. A random dictatorial domain is clearly a dictatorial domain. The unrestricted domain is both a dictatorial domain as well as a random dictatorship domain by virtue of the Gibbard-Satterthwaite Theorem and Theorem 3, respectively. This observation motivates the following general observation: is every dictatorial domain a random dictatorial domain?

Aswal et al. [1] provide sufficient conditions for a domain to be dictatorial in terms of a graph induced by the domain.[9] Two alternatives a and b are defined to be *linked* if there exist two preferences in the domain, one where a is ranked first and b is ranked second and another preference where the reverse is true. Consider a graph where the nodes are alternatives. There is an edge between two nodes if they are linked. Aswal et al. [1] show that a condition on this graph is sufficient for the underlying domain to be dictatorial. They refer to this as the *linked domain* condition. It can be described as follows: there is an ordering of the nodes such that the first two in the ordering are linked; in addition, every subsequent node is linked to at least two others that are predecessors of the node in the ordering. An implication of this result is that there are several domains other than the unrestricted domain that are dictatorial. These domains can be much smaller than the unrestricted domain (which has $m!$ orderings). There are, in fact, linked domains whose size is a linear function of m.

Chatterji et al. [14] investigate the relationship between linked domains and random dictatorial domains. They provide examples of linked domains that are not random dictatorial. In other words, there are domains where every DSCF that is strategy-proof and satisfies unanimity is dictatorial but admit RSCFs that are strategy-proof, satisfy unanimity but are not random dictatorships. In these domains, the randomization allows the mechanism designer to escape the straightjacket of (random) dictatorship. Chatterji et al. [14] also provide additional conditions on linked domains that make them random dictatorial domains. One such condition is the *hub condition* according to which there is a node that is linked to every other node in the graph. Examples suggest that strong conditions are required to make linked domains, random dictatorial domains.

We now consider several domains that are not random dictatorial domains.

[9] See also Sato [44] and Pramanik [36].

4.2 Single-Peaked Domains

Single-peaked preferences are the bedrock of the theory of political economy (see [2] for example). There is an underlying structure on alternatives with respect to which preferences are described. We proceed to details.

We let $A = \{a_1, \ldots, a_m\}$. There is a prior ordering $\prec$ on the elements of A given by $a_1 \prec \cdots \prec a_m$. We write $x \preceq y$ to mean that either $x \prec y$ or $x = y$. For all $a, b \in A$, we define $[a, b] = \{c \mid \text{ either } a \preceq c \preceq b \text{ or } b \preceq c \preceq a\}$ as the set of alternatives that lie "between" a and b. For any $B \subseteq A$, $[a, b]_B = [a, b] \cap B$ denotes the set of alternatives in B that lie in the interval $[a, b]$. Whenever we refer to the maximum or minimum of a subset of alternatives, we are referring to the maximum and minimum with respect to the ordering $\prec$. Whenever we write $\tau(\mathscr{D}) = \{b_1, \ldots, b_k\}$, we assume without loss of generality that $b_1 \prec \cdots \prec b_k$.

A preference P is **single-peaked** if for all $a, b \in A$, $[\tau(P) \preceq a \prec b$ or $b \prec a \preceq \tau(P)]$ implies aPb. A domain is called *single-peaked* if each preference in the domain is single-peaked and is called *maximal single-peaked* if it contains all single-peaked preferences.

A preference is single-peaked if there exists a unique alternative that is first-ranked (sometimes referred to as the *peak*). Moving farther away from the peak in any direction leads to a decline in preferences. Consider the problem of finding a location on a street to build a public facility such as a hospital or school. Every individual has a unique location on the street which is her peak. While comparing two possible locations for the public good on the same side of her peak, she strictly prefers the location closer to her peak. The street can also be interpreted as the political spectrum. If a and b are two political candidates with $a \prec b$, then a is more "left-wing" than b. If a voter's preferences are single-peaked and her peak (or ideal candidate) c is more left-wing than a, i.e., $c \prec a$, then she will prefer candidate a to b. If on the other hand, the voter's peak is b, she will prefer a to c. Figure 1 is a diagrammatic representation of a single-peaked preference.

An important class of DSCFs on single-peaked domains is *min-max rules*. These rules were introduced in Moulin [30] and constitute the set of all unanimous and strategy-proof DSCFs on the maximal single-peaked domain. Min-max rules are based on a class of parameters, one for each subset of individuals, which we denote

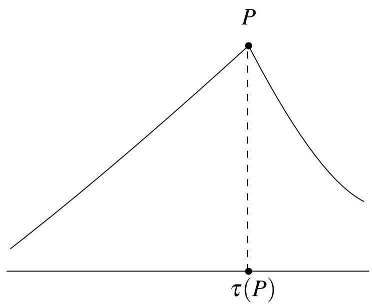

Fig. 1 A graphic illustration of a single-peaked preference

Table 3 Parameters of the min-max rule f

β	β_1	β_2	β_3	$\beta_{\{1,2\}}$	$\beta_{\{1,3\}}$	$\beta_{\{2,3\}}$
	a_8	a_9	a_7	a_4	a_5	a_2

by $(\beta_S)_{S\subseteq N}$. These parameters are required to satisfy some boundary conditions and some monotonicity properties. As the name suggests, the outcome at any profile is calculated by taking suitable minima and maxima of the top-ranked alternatives at the profile and the parameters.

Definition 4.1 A DSCF f on $\mathscr{D}^n$ is called a **min-max** rule if for all $S \subseteq N$, there exists $\beta_S \in A$ satisfying

$$\beta_\emptyset = a_m,\ \ \beta_N = a_1,\ \text{ and } \beta_T \preceq \beta_S \text{ for all } S \subseteq T$$

such that

$$f(P_N) = \min_{S\subseteq N}\left[\max_{i\in S}\{\tau(P_i), \beta_S\}\right].$$

A property that occurs frequently in social choice theory is *tops-onlyness*. A RSCF is tops-only if its outcome at a profile depends only on the top-ranked alternatives in that profile. Two profiles $P_N, P'_N \in \mathscr{D}^n$ are *tops-equivalent* if each individual has the same top-ranked alternative in the two profiles, i.e., $\tau(P_i) = \tau(P'_i)$ for all $i \in N$. A RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **tops-only** if $\varphi(P_N) = \varphi(P'_N)$ for all tops-equivalent $P_N, P'_N \in \mathscr{D}^n$. Note that min-max rules are tops-only by definition. In what follows, we provide an example of a min-max rule.

Example 2 Let $A = \{a_1, \ldots, a_{10}\}$ and $N = \{1, 2, 3\}$. Consider the min-max rule, say f, with parameters as given in Table 3.

The outcome of the min-max rule at the profile (a_5, a_3, a_8), where a_5, a_3, and a_8 are the top-ranked alternatives of individuals 1, 2, and 3, respectively, is determined as follows.

$$\begin{aligned} f(P_N) &= \min_{S\subseteq\{1,2,3\}}\left[\max_{i\in S}\{P_i(1), \beta_S\}\right] \\ &= \min\big[\max\{\beta_\emptyset\}, \max\{\tau(P_1), \beta_1\}, \max\{\tau(P_2), \beta_2\}, \max\{\tau(P_3), \beta_3\}, \\ &\qquad \max\{\tau(P_1), \tau(P_2), \beta_{\{1,2\}}\}, \max\{\tau(P_1), \tau(P_3), \beta_{\{1,3\}}\}, \max\{\tau(P_2), \tau(P_3), \beta_{\{2,3\}}\}, \\ &\qquad \max\{\tau(P_1), \tau(P_2), \tau(P_3), \beta_{\{1,2,3\}}\}\big] \\ &= \min\left[a_{10}, a_8, a_9, a_8, a_5, a_8, a_8, a_8\right] \\ &= a_5. \square \end{aligned}$$

It is shown in Moulin [30] and Weymark [48] that a DSCF on the *maximal* single-peaked domain is unanimous and strategy-proof if and only if it is a min-max rule. In this section, we present results for RSCFs for a large class of single-peaked domains, which we call *top-connected* single-peaked domains.

For a domain $\mathscr{D}$, the *top-interval* $I(\mathscr{D})$ is the set of alternatives $[\min(\tau(\mathscr{D})), \max(\tau(\mathscr{D}))]$.

Definition 4.2 A single-peaked domain $\mathscr{D}$ is **top-connected** if for every two consecutive alternatives a_r and a_s in $\tau(\mathscr{D})$ with $\min(\tau(\mathscr{D})) \preceq a_r \prec a_s \preceq \max(\tau(\mathscr{D}))$, there exist $P \in \mathscr{D}^{a_r}$ and $P' \in \mathscr{D}^{a_s}$ such that $a_s P a_{r-1}$ if $a_{r-1} \in I(\mathscr{D})$ and $a_r P' a_{s+1}$ if $a_{s+1} \in I(\mathscr{D})$.

Observe that some alternative may not appear as a top-ranked alternative in any preference in a top-connected single-peaked domain, in other words, the top-set of such a domain does not necessarily contain all alternatives.

Remark 4.1 Note that top-connectedness does not impose any restriction (except from single-peakedness) on any preference with the top-ranked alternative as $\min(\tau(\mathscr{D}))$ or $\max(\tau(\mathscr{D}))$. To see this, take, for instance, $\min(\tau(\mathscr{D})) = a_r \prec a_s \preceq \max(\tau(\mathscr{D}))$. Definition 4.2 says that there must exist a single-peaked preference $P \in \mathscr{D}^{a_r}$ such that $a_s P a_{r-1}$ if $a_{r-1} \in I(\mathscr{D})$. However, since $a_r = \min(\tau(\mathscr{D}))$, it must be that $a_{r-1} \notin I(\mathscr{D})$. Therefore, this condition does not apply to P. Similar logic applies to any preference with the top-ranked alternative as $\max(\tau(\mathscr{D}))$.

For a sequence of alternatives $b_1, \ldots, b_k$, denote by $\langle b_1, \ldots, b_k \rangle \ldots$ a preference where $P(l) = b_l$ for all $l = 1, \ldots, k$. Then, the top-connectedness property of a domain $\mathscr{D}$ assures that for every two consecutive alternatives a_r and a_s in $\tau(\mathscr{D})$ with $\min(\tau(\mathscr{D})) \preceq a_r \prec a_s \preceq \max(\tau(\mathscr{D}))$, there are two single-peaked preferences P and P' such that $P = \langle a_r, a_{r+1}, \ldots, a_{s-1}, a_s \rangle \ldots$ if $a_{r-1} \in I(\mathscr{D})$ and $P' = \langle a_s, a_{s-1}, \ldots, a_{r+1}, a_r \rangle \ldots$ if $a_{s+1} \in I(\mathscr{D})$. For example, if $A = \{a_1, \ldots, a_{15}\}$ and $\tau(\mathscr{D}) = \{a_3, a_4, a_5, a_8, a_{10}\}$, then top-connectedness ensures, for instance, that preferences such as $\langle a_5, a_6, a_7, a_8 \rangle \ldots$ and $\langle a_8, a_7, a_6, a_5 \rangle \ldots$ are present in the domain. Note that as we mention in Remark 4.1, top-connectedness does not impose any restriction (except from single-peakedness) on the preferences with top-ranked alternatives a_3 or a_{10}. Thus, the top-connectedness property of a domain $\mathscr{D}$ guarantees that for every two consecutive alternatives a_r and a_s in $\tau(\mathscr{D})$ with $\min(\tau(\mathscr{D})) \preceq a_r \prec a_s \preceq \max(\tau(\mathscr{D}))$, there are two single-peaked preferences P and P' such that $P|_{I(\mathscr{D})} = \langle a_r, a_{r+1}, \ldots, a_{s-1}, a_s \rangle \ldots$ and $P'|_{I(\mathscr{D})} = \langle a_s, a_{s-1}, \ldots, a_{r+1}, a_r \rangle \ldots$.

We provide an example of a top-connected single-peaked domain in Example 3.

Example 3 Let $A = \{a_1, \ldots, a_{10}\}$ be the set of alternatives. Consider the top-connected single-peaked domain $\mathscr{D} = \{P_1, \ldots, P_9\}$ given in Table 4. Here, $\tau(\mathscr{D}) = \{a_3, a_4, a_7, a_9\}$.

It is worth noting that the number of preferences in a top-connected single-peaked domain can range from $2|\tau(\mathscr{D})| - 1$ to 2^{m-1}. Thus, the class of such domains is quite large. It should be further noted that any single-peaked domain $\mathscr{D}$ with $|\tau(\mathscr{D})| = 2$ is a top-connected single-peaked domain. This is because top-connectedness does not impose any condition on the preferences with top-ranked alternatives $\min(\tau(\mathscr{D}))$ or $\max(\tau(\mathscr{D}))$.

Table 4 Preference domain for Example 3

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
a_3	a_3	a_4	a_4	a_4	a_7	a_7	a_9	a_9
a_4	a_2	a_3	a_5	a_5	a_6	a_8	a_{10}	a_8
a_5	a_4	a_2	a_6	a_6	a_5	a_9	a_8	a_7
a_2	a_1	a_5	a_3	a_7	a_4	a_6	a_7	a_6
a_1	a_5	a_6	a_7	a_3	a_3	a_5	a_6	a_{10}
a_6	a_6	a_1	a_8	a_2	a_2	a_4	a_5	a_5
a_7	a_7	a_7	a_9	a_8	a_8	a_3	a_4	a_4
a_8	a_8	a_8	a_{10}	a_9	a_9	a_{10}	a_3	a_3
a_9	a_9	a_9	a_2	a_1	a_1	a_2	a_2	a_2
a_{10}	a_{10}	a_{10}	a_1	a_{10}	a_{10}	a_1	a_1	a_1

Our next theorem provides a characterization of unanimous and strategy-proof RSCFs on top-connected single-peaked domains. A *random min-max* rule is a convex combination of min-max rules.

Theorem 4 ([40]) *A RSCF on a top-connected single-peaked domain is unanimous and strategy-proof if and only if it is a random min-max rule.*

Ehlers et al. [21] consider the case where the set of alternatives is continuous, say the interval [0, 1]. They provide a different characterization of unanimous and strategy-proof RSCFs on the maximal single-peaked domain by means of a class of RSCFs called *probabilistic fixed ballot rule* (PFBR). Below, we define these rules for the case of finitely many alternatives.

A PFBR φ is based on a collection of parameters $(\beta_S)_{S \subseteq N}$, called **probabilistic ballots**. Each probabilistic ballot β_S, which is associated to the coalition $S \subseteq N$, is a probability distribution on A satisfying the following two properties.

- **Ballot unanimity**: β_N assigns probability 1 to a_1, and $\beta_\emptyset$ assigns probability 1 to a_m.
- **Monotonicity**: probabilities according to β_S move toward left as S gets bigger, i.e., $\beta_S([a_1, a_k]) \leq \beta_T([a_1, a_k])$ for all $S \subset T$ and all $a_k \in A$.[10]

For example, suppose there are two individuals $\{1, 2\}$ and four alternatives $\{a_1, a_2, a_3, a_4\}$. A choice of probabilistic ballots could be $\beta_\emptyset = (0, 0, 0, 1)$, $\beta_{\{1\}} = (0.5, 0.2, 0.1, 0.2)$, $\beta_{\{2\}} = (0.4, 0.3, 0.2, 0.1)$, and $\beta_N = (1, 0, 0, 0)$.[11]

A PFBR φ w.r.t. a collection of probabilistic ballots $(\beta_S)_{S \subseteq N}$ works as follows. For each $1 \leq k \leq m$, let $S(k, P_N) = \{i \in N : \tau(R_i) \preceq a_k\}$ be the set of individuals whose peaks are not to the right of a_k. Consider an arbitrary profile P_N and an arbitrary alternative a_k. We induce the probabilities $\beta_{S(k,P_N)}([a_1, a_k])$ and $\beta_{S(k-1,P_N)}([a_1, a_{k-1}])$. If

[10] For a subset B of A, we denote the probability of B according to β_S by $\beta_S(B)$.

[11] Here (x, y, w, z) is the probability distribution where a_1, a_2, a_3, and a_4 receive probabilities x, y, w, and z, respectively.

$a_k = a_1$, then set $\beta_{S(0,P_N)}([a_1, a_0]) = 0$. The probability of the alternative a_k selected at the profile P_N is defined as the difference between these two probabilities, i.e., $\varphi_{a_k}(P_N) = \beta_{S(k,P_N)}([a_1, a_k]) - \beta_{S(k-1,P_N)}([a_1, a_{k-1}])$.[12] Consider, for example, the PFBR φ w.r.t. the parameters described in the previous paragraph. Let $P_N = (P_1, P_2)$ be a profile where $\tau(P_1) = a_2$ and $\tau(P_2) = a_4$. Then,

$$\begin{aligned}
\varphi_{a_1}(P_N) &= \beta_{S(1,P_N)}([a_1, a_1]) - 0 = 0,\\
\varphi_{a_2}(P_N) &= \beta_{S(2,P_N)}([a_1, a_2]) - \beta_{S(1,P_N)}([a_1, a_1])\\
&= \beta_{\{1\}}([a_1, a_2]) - \beta_{\emptyset}([a_1, a_1]) = 0.7 - 0 = 0.7,\\
\varphi_{a_3}(P_N) &= \beta_{S(3,P_N)}([a_1, a_3]) - \beta_{S(2,P_N)}([a_1, a_2])\\
&= \beta_{\{1\}}([a_1, a_3]) - \beta_{\{1\}}([a_1, a_2]) = 0.8 - 0.7 = 0.1, \text{ and}\\
\varphi_{a_4}(P_N) &= \beta_{S(4,P_N)}([a_1, a_4]) - \beta_{S(3,P_N)}([a_1, a_3])\\
&= \beta_N([a_1, a_4])\beta_{\{1\}}([a_1, a_3]) = 1 - 0.8 = 0.2.
\end{aligned}$$

Clearly, the PFBR satisfies the tops-only property.

It is important to note that the probabilistic ballot β_S for a coalition $S \subseteq N$ represents the outcome of φ at the "boundary profile" where individuals in S have the preference $\underline{P}_i = (a_1 \cdots a_{k-1}\, a_k \cdots a_m)$, while the others have the preference $\overline{P}_i = (a_m \cdots a_k\, a_{k-1} \cdots a_1)$. We call such a profile a *S-boundary profile*.[13] If a PFBR φ is unanimous, then it follows that $\beta_\emptyset$ assigns probability 1 to a_m and β_N assigns probability 1 to a_1, which in turn implies ballot unanimity. We now argue that $(\beta_S)_{S\subseteq N}$ is monotonic if φ is strategy-proof. Consider a proper subset $S \subset N$ and $i \in N \setminus S$. Let P_N and P'_N be the S-boundary and $S \cup \{i\}$-boundary profiles, respectively. In other words, only individual i changes her preference $\underline{P}_i$ in the $S \cup \{i\}$-boundary profile to $\overline{P}_i$. Strategy-proofness of φ implies that the probability of each upper contour set of $\underline{P}_i$ is weakly increased from $\varphi(P_N)$ to $\varphi(P'_N)$. Since the interval $[a_1, a_k]$ coincides with the upper contour set of a_k at $\underline{P}_i$, it follows that $\beta_S([a_1, a_k]) \leq \beta_{S\cup\{i\}}([a_1, a_k])$. Monotonicity of $(\beta_S)_{S\subseteq N}$ follows from the repeated application of this argument.

The outcome of a PFBR at any profile is uniquely determined by its outcomes at boundary profiles. It is shown in Ehlers et al. [21] that every PFBR is unanimous and strategy-proof on the single-peaked domain. Thus, unanimity and strategy-proofness of a PFBR at every profile can be ensured by imposing those conditions only on boundary profiles.

The deterministic versions of PFBRs can be obtained by additionally requiring the probabilistic ballots be degenerate, i.e., $\beta_S(a_k) \in \{0, 1\}$ for all $S \subseteq N$ and $a_k \in A$. These DSCFs were introduced by Moulin [30]; we refer to these as *Fixed Ballot*

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[12] Since $S(k-1, P) \subseteq S(k, P_N)$ and $[a_1, a_{k-1}] \subset [a_1, a_k]$, monotonicity ensures $\varphi_{a_k}(P_N) = \beta_{S(k,P_N)}([a_1, a_k]) - \beta_{S(k-1,P_N)}([a_1, a_{k-1}]) \geq 0$. Moreover, note that $\sum_{k=1}^{m} \varphi_{a_k}(P_N) = \sum_{k=1}^{m} \beta_{S(k,P_N)}([a_1, a_k]) - \beta_{S(k-1,P_N)}([a_1, a_{k-1}]) = \beta_{S(m,P_N)}([a_1, a_m]) = 1$. Therefore, $\varphi(P_N) \in \Delta(A)$ and φ is a well-defined RSCF.

[13] Note that for every $S \subseteq N$, there is a unique S-boundary profile.

Rules (or FBRs).[14] Moulin [30] showed that a DSCF is unanimous, tops-only, and strategy-proof on the single-peaked domain if and only if it is an FBR. It can be easily verified that an arbitrary mixture of FBRs is unanimous and strategy-proof on the single-peaked domain and is a PFBR. Theorem 3 of [34] and Theorem 5 of [37] prove that the converse is also true.

Below, we present the formal definition of PFBRs.

Definition 4.3 A RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is called a **Probabilistic Fixed Ballot Rule** (or **PFBR**) if there exists a collection of probabilistic ballots $(\beta_S)_{S \subseteq N}$ satisfying ballot unanimity and monotonicity such that for all $P_N \in \mathscr{D}^n$ and $a_k \in A$, we have

$$\varphi_{a_k}(P_N) = \beta_{S(k,P_N)}([a_1, a_k]) - \beta_{S(k-1,P_N)}([a_1, a_{k+1}]),$$

where $\beta_{S(0,P_N)}([a_1, a_0]) = 0$.

Theorem 5 ([21]) *A RSCF on the maximal single-peaked domain is unanimous and strategy-proof if and only if it is a PFBR.*

It follows from Theorem 5 and Theorem 4 that every PFBR is a random min-max rule and vice versa.[15]

4.3 Single-Dipped Domains

Single-dipped preferences are the reverse of single-peaked preferences. In the latter, preferences decline as one moves farther away from its peak. On the other hand, preferences increase in single-dipped preferences as one moves farther away from its “dip”. These preferences are appropriate for the location of “public bads” such as nuclear plants and garbage dumps. All individuals want such facilities to be located as far away as possible from their location.

A preference P is **single-dipped** if it has a unique minimal element $d(P)$, the *dip* of P, such that for all $a, b \in A$, $[d(P) \preceq a \prec b$ or $b \prec a \preceq d(P)] \Rightarrow bPa$. A domain is *single-dipped* if each preference in it is single-dipped. A single-dipped preference is illustrated in Fig. 2.

Peters et al. [33] introduce the notion of binary restricted domains and show that every unanimous and strategy-proof RSCF on a binary restricted domain is a convex combination of unanimous and strategy-proof DSCFs on the same domain. It is easy to verify that every top-connected single-dipped domain is a binary restricted domain.

[14] Moulin [30] called these Augmented Median Voter Rules, while [8] called these Generalized Median Voter Schemes. For an FBR φ, the subtraction form in Definition 4.3 can be simplified to a min-max form (see Definition 10.3 in [31]).

[15] In a related paper, Dutta et al. [18] analyzed the structure of strategy-proof and unanimous RSCFs on domains containing strictly convex continuous single-peaked preferences on a convex subset of Euclidean space. They show that every strategy-proof and unanimous RSCF on any such domain must be a random dictatorship.

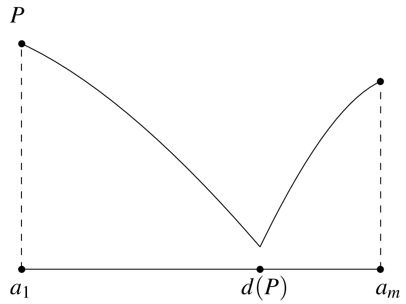

Fig. 2 An illustration of a single-dipped preference

It follows that every strategy-proof RSCF satisfying unanimity on a top-connected single-dipped domain is a random voting by committees rules (recall the definition of such rules in Sect. 3.1) with respect to the boundary alternatives a_1 and a_m.

Theorem 6 ([33]) *A RSCF on a top-connected single-dipped domain is strategy-proof and satisfies unanimity if and only if it is a random voting by committees rule with respect to* a_1 *and* a_m.

It follows from Theorem 6 that any strategy-proof RSCF that satisfies unanimity on a top-connected single-dipped domain can assign positive probability to only the "boundary" alternatives a_1 and a_m.

4.4 Single-Crossing Domains

The *single-crossing* property is a familiar one in economic theory.[16] It appears frequently in models of income taxation and redistribution [29, 38], local public goods and stratification [22, 23, 47], and coalition formation [16, 28]. A more detailed discussion of applications and other issues can be found in Saporiti [42].

A domain $\mathscr{D}$ is a **single-crossing** domain if there exists an ordering $\prec$ over A and an ordering $\lhd$ over $\mathscr{D}$ such that for all $a, b \in A$ and all $P, P' \in \mathscr{D}$, $[a \prec b, P \lhd P',$ and $bPa] \implies bP'a$. Preferences in a single-crossing domain can be ordered in such a way that every pair of alternatives switch their relative ranking at most once along the ordering. A single-crossing domain $\bar{\mathscr{D}}$ is *maximal* if there does not exist another single-crossing domain that is a strict superset of $\bar{\mathscr{D}}$. Note that a maximal single-crossing domain with m alternatives contains $m(m-1)/2+1$ preferences.[17] A domain $\mathscr{D}$ is *successive single-crossing* if there is a maximal single-crossing domain $\bar{\mathscr{D}}$ with respect to some ordering $\lhd$ and two preferences $P', P'' \in \bar{\mathscr{D}}$ with $P' \unlhd P''$ such that $\mathscr{D} = \{P \in \bar{\mathscr{D}} \mid P' \unlhd P \unlhd P''\}$.[18]

[16] See, for example, Romer [39], p. 181, and Austen-Smith and Banks [2], pp. 114–115.

[17] For details, see Saporiti [42].

[18] By $P \unlhd P'$, we mean either $P = P'$ or $P \lhd P'$.

Examples of a maximal single-crossing domain and a successive single-crossing domain with five alternatives are shown below.

Example 4 Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4, a_5\}$ with the prior order $a_1 \prec \cdots \prec a_5$. The domain $\bar{\mathscr{D}} = \{a_1a_2a_3a_4a_5,\ a_2a_1a_3a_4a_5,\ a_2a_3a_1a_4a_5,\ a_2a_3a_4a_1a_5,\ a_2a_4a_3a_1a_5,\ a_4a_2a_3a_1a_5,\ a_4a_2a_3a_5a_1,\ a_4a_3a_2a_5a_1, a_4a_3a_5a_2a_1,\ a_4a_5a_3a_2a_1,\ a_5a_4a_3a_2a_1\}$ is a maximal single-crossing domain with respect to the ordering $\lhd$ given by $a_1a_2a_3a_4a_5 \lhd a_2a_1a_3a_4a_5 \lhd a_2a_3a_1a_4a_5 \lhd a_2a_3a_4a_1a_5 \lhd a_2a_4a_3a_1a_5 \lhd a_4a_2a_3a_1a_5 \lhd a_4a_2a_3a_5a_1 \lhd a_4a_3a_2a_5a_1 \lhd a_4a_3a_5a_2a_1 \lhd a_4a_5a_3a_2a_1 \lhd a_5a_4a_3a_2a_1$ since every pair of alternatives change their relative ordering at most once along this ordering. Note that the cardinality of A is 5 and that of $\bar{\mathscr{D}}$ is $5(5-1)/2+1 = 11$. The domain $\mathscr{D} = \{a_1a_2a_3a_4a_5,\ a_2a_1a_3a_4a_5,\ a_2a_3a_1a_4a_5,\ a_2a_3a_4a_1a_5,\ a_2a_4a_3a_1a_5,\ a_4a_2a_3a_1a_5\}$ is a successive single-crossing domain since it contains all the preferences between $a_1a_2a_3a_4a_5$ and $a_4a_2a_3a_1a_5$ in the maximal single-crossing domain $\bar{\mathscr{D}}$. □

In what follows, we introduce a restricted version of min-max rules called tops-restricted min-max rule. For such a min-max rule, all the parameters are required to come from the top-set of the domain. Formally, a DSCF $f : \mathscr{D}^n \to A$ is a **tops-restricted min-max (TM)** rule if for all $S \subseteq N$, there exists $\beta_S \in \tau(\mathscr{D})$ satisfying the conditions that $\beta_\emptyset = \max(\tau(\mathscr{D}))$, $\beta_N = \min(\tau(\mathscr{D}))$, and $\beta_T \preceq \beta_S$ for all $S \subseteq T$ such that

$$f(P_N) = \min_{S \subseteq N}\left[\max_{i \in S}\{\tau(P_i), \beta_S\}\right].$$

Note that if $\tau(\mathscr{D}) = A$, then a TM rule becomes a min-max rule. For an example of a TM rule, consider the DSCF f in Example 2 and a domain $\mathscr{D}$ with $\tau(\mathscr{D}) = \{a_2, a_3, a_4, a_5, a_7, a_8, a_9\}$. Since all parameters of f take values in $\tau(\mathscr{D})$, f becomes a TM rule on $\mathscr{D}$.

It is worth noting that the outcome of a min-max rule at a profile is either some top-ranked alternative at that profile or some parameter value (that is, β_S for some $S \subseteq N$). Since for a TM rule f, all these alternatives must be in the top-set of the corresponding domain, its outcome also lies in the same set, that is, $f(P_N) \in \tau(\mathscr{D})$ for all $P_N \in \mathscr{D}^n$.

A crucial property of a single-crossing domain is that the outcome of a unanimous and strategy-proof DSCF always lies in the top-set of the domain. This implies that one can restrict a single-crossing domain to its top-set for the purpose of analyzing unanimous and strategy-proof DSCFs on it. It can be verified that a single-crossing domain restricted to its top-set is a top-connected single-peaked domain. Therefore, by Theorem 4, it follows that a DSCF on a single-crossing domain is unanimous and strategy-proof if and only if it is a TM rule. These results are formally proved in Saporiti [43].[19] Subsequently, [41] have shown that these properties hold for RSCFs on single-crossing domains as well and provide a characterization of unanimous and strategy-proof RSCFs on these domains.

[19] Saporiti [43] uses the term *augmented representative voter schemes* for TM rules.

A RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is a **tops-restricted random min-max (TRM)** rule if φ can be written as a convex combination of some TM rules on $\mathscr{D}^n$. As we have explained earlier, if $\tau(\mathscr{D}) = A$, then a TRM rule $\varphi : \mathscr{D}^n \to \Delta A$ becomes a random min-max rule.

Theorem 7 ([41]) *A RSCF on a successive single-crossing domain is unanimous and strategy-proof if and only if it is a tops-restricted random min-max rule.*

4.5 Euclidean Domains

Euclidean preferences are a special case of single-peaked preferences. Alternatives are located on the real line (or the unit interval without loss of generality) and $\prec$ is the natural order $<$ on the real numbers. We let the set of alternatives A be the interval $[0, 1]$. A preference P is **Euclidean** if there is $x \in [0, 1]$, such that $\tau(P) = x$ and for all alternatives $a, b \in A$, $|x - a| < |x - b|$ implies aPb. A domain is *Euclidean* if it contains *all* Euclidean preferences.

A preference is Euclidean if an alternative a is preferred to another alternative b only if the distance from a to the peak is smaller than the distance of b to the peak. If both a and b lie on the same side of the peak, then single-peakedness would imply that the alternative closer to the peak would be preferred. However, Euclidean preferences also compare alternatives on different sides of the peak unlike single-peakedness. Euclidean preferences are determined completely by the peak of an individual's peak. Consequently, the domain of Euclidean preferences is a strict subset of the set of the maximal single-peaked domain.

Since the Euclidean domain is a strict subset of the maximal single-peaked domain, the possibility that there are unanimous and strategy-proof DSCFs on the domain apart from the min-max rules cannot be excluded. However, [11] show that this case does not arise: a DSCF on the Euclidean domain is unanimous and strategy-proof if and only if it is a min-max rule. Furthermore, [40] show that the same holds even for RSCFs on the Euclidean domain.

Theorem 8 ([40]) *A RSCF on the Euclidean domain is unanimous and strategy-proof if and only if it is a random min-max rule.*

4.6 Dichotomous Domains

Dichotomous preferences are a generalization of binary preferences. There are an arbitrary number of alternatives but each alternative can belong to exactly one of two indifference classes—a "good" set and a "bad" set. An important point of departure from our earlier discussion is that an individual can be indifferent between alternatives. A *dichotomous* domain is the set of all dichotomous preferences. Dichotomous domains have been studied extensively in Bogomolnaia et al. [10].

A dichotomous preference for individual i can be represented by a subset X_i of A with the interpretation that X_i is the good set of i. A profile is n-tuple $(X_1, \ldots, X_n)$. Let $\mathscr{X}^n$ denote the set of all profiles.

A characterization of strategy-proof DSCFs satisfying unanimity remains an open and difficult problem. However, [24] provide a necessary condition, called the pair-triple property, for a strategy-proof RSCF to be representable as a convex combination of strategy-proof DSCFs.

A RSCF $\varphi : \mathscr{X}^n \to \Delta A$ satisfies the **Pair-Triple (PT) Property** if for all $i, j \in N$, all $a, b, c \in A$, and all $X_{-\{i,j\}} \in \mathscr{X}^{n-2}$, we have

$$\varphi_a(\{b\}, \{c\}, X_{-\{i,j\}}) + \varphi_b(\{c\}, \{a\}, X_{-\{i,j\}}) + \varphi_c(\{a\}, \{b\}, X_{-\{i,j\}}) \leq 1.$$

In the notation above, the first component of a profile denotes the preference of individual i and the second one denotes that of individual j.

Theorem 9 ([24]) *A strategy-proof RSCF on the dichotomous domain satisfying unanimity can be represented as a convex combination of strategy-proof DSCF satisfying unanimity only if it satisfies the PT property. In the case of three alternatives, the converse also holds.*

A more complete result on the structure of strategy-proof RSCFs satisfying unanimity on dichotomous domains is not yet available.

4.7 Additional Literature

In this subsection, we briefly review some related results in the literature.

Chatterji and Sen [13] provide conditions on a domain which ensure that every unanimous and strategy-proof DSCF on it has the tops-only property. Subsequently, [15] consider the same problem for RSCFs. They identify two conditions, the *interior property* and the *exterior property*, and show that on every domain satisfying these two properties, a strategy-proof RSCF satisfying unanimity also satisfies the tops-only property. This result is particularly useful in characterizing strategy-proof RSCFS on various domains.

Chatterji et al. [12] investigate *hybrid domains*. Given an ordering $\prec$ over the alternatives, a preference is *hybrid* if there exist *threshold* alternatives $a_{\underline{k}}$ and $a_{\overline{k}}$ with $a_{\underline{k}} \prec a_{\overline{k}}$ such that preferences over the alternatives in the interval between $a_{\underline{k}}$ and $a_{\overline{k}}$ are "unrestricted" relative to each other, while preferences over other alternatives retain features of single-peakedness. Thus, the set A can be decomposed into three parts: left interval $L = \{a_1, \ldots, a_{\underline{k}}\}$, right interval $R = \{a_{\overline{k}}, \ldots, a_m\}$, and middle interval $M = \{a_{\underline{k}}, \ldots, a_{\overline{k}}\}$. Formally, a preference is $(\underline{k}, \overline{k})$-*hybrid* if the following holds: (i) for a voter whose best alternative lies in L (respectively in R), preferences over alternatives in the set $L \cup R$ are conventionally single-peaked, while preferences over alternatives in M are arbitrary subject to the restriction that the best alternative in

M is the left threshold $a_{\underline{k}}$ (respectively, right threshold $a_{\overline{k}}$), and (ii) for a voter whose peak lies in M, preferences restricted to $L \cup R$ are single-peaked but arbitrary over M. Observe that if $\underline{k} = 1$ and $\overline{k} = m$, then preferences are unrestricted, while the case where $\overline{k} - \underline{k} = 1$ coincides with the case of single-peaked preferences. They characterize all strategy-proof RSCFs satisfying unanimity on these domains.

Peters et al. [32] consider domains on graphs. In such domains, there is a graph with the alternatives as nodes with preferences declining as one moves away from the top-ranked alternative along any spanning tree of the graph. Note that if the underlying graph is a line graph, then the resulting domain becomes single-peaked. They characterize all strategy-proof RSCFs satisfying unanimity.

5 The Deterministic Extreme Point Property

In this subsection, we discuss the following issue: in what sense does randomization enlarge the possibilities for a mechanism designer? As we have noted earlier, an implication of Remark 2.1 is that a convex combination of strategy-proof DSCFs satisfying unanimity is a strategy-proof RSCF that satisfies unanimity. A domain $\mathscr{D}$ satisfies the **deterministic extreme point property (DEP)** if the converse is true: i.e., if every unanimous and strategy-proof RSCF can be written as a convex combination of unanimous and strategy-proof DSCFs. If a domain satisfies DEP, the only additional possibility afforded by randomization is that before the elicitation of preferences from individuals, the designer can pick a strategy-proof DSCF satisfying unanimity according to a fixed probability distribution. Thereafter, the designer simply follows the DSCF chosen. Such a procedure does not exhaust all possibilities if the domain does not satisfy DEP. In particular, there will exist strategy-proof RSCFs satisfying unanimity, where the designer will have to randomize over alternatives *after* the elicitation of preferences. For this reason, we regard DEP as a benchmark property for domains. Randomization expands the possibilities available to the designer only if the domain under consideration violates the DEP property.

The DEP property of a domain can be utilized in finding optimal mechanisms on it. Consider an optimization problem with incentive constraints and unanimity constraints. Since these are linear constraints, the constraint space is a polytope and the results identify its extreme points. If the objective function is linear, the DEP property implies that an optimal solution is a deterministic mechanism. This fact may help in finding optimal random mechanisms using the knowledge of the same for deterministic mechanisms as optimizing over an infinite set of random mechanisms may be harder than optimizing over a finite set of deterministic mechanisms. Further, the incentive constraints may simplify with deterministic mechanisms.

It follows from Theorems 1, 2, 3, 4, 6, 7, and 8 that several well-known domains of strict preferences, namely the unrestricted, single-peaked, single-dipped, single-crossing, and Euclidean domains, satisfy the DEP property. However, as we have seen, there are dictatorial domains [14] and hybrid domains [12] that are not random dictatorial. Peters et al. [32] show that DEP is satisfied for a domain on graph only when the underlying graph is a line, i.e., only when the domain is single-peaked. The dichotomous domain (a domain where indifference is permitted) also does not satisfy DEP. This conclusion follows from Theorem 9 since the TP property is not vacuous.

In spite of its theoretical significance, there is as yet, no general analysis of domains satisfying DEP. A more challenging open question is to characterize the extreme points of strategy-proof RSCFs satisfying unanimity in domains that do not satisfy DEP.

6 Conclusion

We have attempted to provide a brief survey of recent results pertaining to the structure of strategy-proof RSCFs on various preference domains. Although considerable progress has been made, some key issues, such as the precise relationship between strategy-proof DSCFs and strategy-proof RSCFs on a given domain, require further investigation.

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