Partial Invariance and Problems with Free Boundaries

V. V. Pukhnachev

Abstract The foundations of group analysis of differential equations were laid by S. Lie. This theory was essentially developed in works of L. V. Ovsiannikov, N. H. Ibragimov, their students, and followers. Notion of the partially invariant solution to the system of differential equations (Ovsiannikov 1964) substantially extended possibilities of exact solutions construction for multidimensional systems of differential equations admitting the Lie group. It is important to note that fundamental equations of continuum mechanics and physics fall in this class a priori as invariance principle of space, time, and moving medium there with respect to some group (Galilei, Lorenz, and others) are situated in the base of their derivation. It should be noticed that classical group analysis of differential equations has a local character. To apply this approach to initial boundary value problems, one need to provide the invariance properties of initial and boundary conditions. Author (1973) studied these properties for free boundary problems to the Navier–Stokes equations. Present chapter contains an example of partially invariant solution of these equations describing the motion of a rotating layer bounded by free surfaces.

1 Introduction

I had occasion to work with N. H. Ibragimov for 12 years before he moves from Novosibirsk to Ufa. We are of the same age, and we both are pupils of L. V. Ovsyannikov. Lev Vasil'evich supervised the theoretical department of our Institute, which included several research aspects, such as the group analysis of differential equations, mathematical problems of gas dynamics, and theory of problems with free boundaries. I managed to work in all these research fields, but the last one became the main aspect for me.

V. V. Pukhnachev (✉)
Lavrentyev Institute of Hydrodynamics, Siberian Branch, Russian Academy of Sciences, Novosibirsk State University, Novosibirsk 630090, Russia

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The Navier–Stokes equations are used as the basic mathematical model in fluid dynamics. By 1970, the main boundary value and initial boundary value problems for these equations with a fixed flow domain have been studied. However, there was not a single result for problems where the domain boundary or some part of this boundary is free. (Here, I mean problems with a free boundary in the exact formulation. Approximate models of motions with free boundaries have been developed since the paper published by Stokes in 1848. The problem of justification of approximate models had also to be solved.)

Recognizing that my efforts were insufficient to develop a general theory for three-dimensional problems, I decided to consider problems of smaller dimensions. A helpful fact was that the Navier–Stokes equations admit a wide Lie group G_∞. Let the equation of the free boundary Γ_t have the form $F(x, t) = 0$. If Γ_t is an invariant manifold of the group $H \subset G_\infty$, then the conditions on this surface are written in terms of invariants of the group H. My proof of this theorem was rather cumbersome, but Nail made it significantly simpler, and I am extremely grateful for that. This statement allows one to construct invariant solutions of the Navier–Stokes equations, which are preliminary matched with the conditions on the free boundary.

In early 1970s, N. Kh. Ibragimov and I worked on our doctor's dissertations, whereas V. M. Men'shchikov, one more pupil of L. V. Ovsyannikov, prepared his candidate's thesis. Lev Vasil'evich posed the following problem for him: Is it possible to continue the invariant solution of gas-dynamic equations through the shock wave? Men'shchikov answered this question positively under the condition that the equation of the strong discontinuity surface is an invariant manifold of the corresponding group in the space $x,\ t$.

Now I return to the year 1964, when L. V. Ovsyannikov made one of his main mathematical discoveries: he introduced the notion of a partially invariant solution of a system of differential equations. This notion is specific for systems and arises in a situation where the subgroup H of the group G admitted by the system has too few invariants for the invariant H-solution to exist. The procedure of constructing a partially invariant solution was described in detail in Ovsyannikov's book entitled *Group Analysis of Differential Equations* [1]. The procedure consists in splitting the original system into the resolving and automorphic subsystems. The first one relates only the invariants of the group H and contains a smaller number of independent variables than the original system. If the solution of this system is known, then the automorphic system is integrated in quadratures.

This discovery of L. V. Ovsyannikov significantly extended the possibility of constructing exact solutions of multidimensional systems of differential equations admitting the Lie group. It is important to note here than the fundamental equations of mechanics and physics of continuous media are *a priori* included into this class because their derivation is based on the principles of invariance of space and time and the moving medium with respect to a certain group (Galileo, Lorentz, and other groups).

The first examples of partially invariant solutions were obtained for gas-dynamic equations, and it seemed that their existence is a privilege of hyperbolic systems. In 1972, however, V. O. Bytev (our common pupil with L. V. Ovsyannikov) found an example of a partially invariant solution of unsteady two-dimensional Navier–Stokes equations, which do not have any particular kind at all. Later on, partially invariant solutions of systems of boundary layer equations, gravity-induced thermal convection equations, and other equations were obtained.

In 1973, I found that the theorem of invariance of conditions on the free boundary is also valid if the solution of the Navier–Stokes equations is only partially invariant. It is sufficient that the unknown boundary with the equation $F(x, t) = 0$ defines an invariant manifold in the space $x,\ t$. This fact made it possible to obtain new solutions of problems with a free boundary and with an interface of immiscible fluids.

2 Definition of the Partially Invariant Solution

The notion of a partially invariant solution of a system of differential equations was introduced by Ovsyannikov [2, 3]. The theory of partial invariance is described in Chap. VI of his monograph [1]. L. V. Ovsyannikov demonstrated that the possibility of constructing partial solutions of differential equations can be extended by eliminating the property of full invariance of the solution.

Let $G = G'(f)$ be a local r-parameter Lie group of transformations of the n-dimensional space X generated by the mapping $f\ : V \times O \to X$ of the product of the open set $V \in X$ and the neighborhood of zero of the parametric space of this group.

The orbit of the manifold $\Psi \in V$ is understood as a set $f(\Psi, O)$ of all possible points $x \in \Psi$. In other words, the orbit $f(\Psi, O)$ of the manifold Ψ is the sum of the orbits of all points of this manifold. There is an alternative: the orbit $f(\Psi, O)$ either contains a certain open set of the space X, or is a manifold in this space with a dimension smaller than $\dim X = n$. If

$$\dim f(\Psi, O) < n, \tag{1}$$

then Ψ is called the proper subspace of the space X. If the manifold orbit satisfies inequality (1), then the group G is intransitive. The following inclusion is valid for any invariant manifold Φ of the group G containing the manifold $\Psi : f(\Psi, O)$. Therefore, the orbit $f(\Psi, O)$ is the smallest invariant manifold of the group G containing Ψ. Clearly, if Ψ itself is an invariant manifold of the group G, then $f(\Psi, O) = \Psi$.

The rank of the manifold Ψ with respect to the group G is understood as the rank of its orbit $f(\Psi, O)$. This rank is considered as a function of the pair (Ψ, G) and is denoted by $\rho(\Psi, G)$. The defect of the manifold Ψ with respect to the group G is the difference between the dimension of its orbit $f(\Psi, O)$ and the dimension of the manifold Ψ. Being considered as a function of the pair (Ψ, G), this defect is denoted by $\delta(\Psi, G)$, so that

$$\delta(\Psi, G) = \dim f(\Psi, O) - \dim \Psi. \tag{2}$$

The number $\delta(\Psi, G)$ shows how far the manifold Ψ is from the invariant manifold. The equality $\delta(\Psi, G) = 0$ is a criterion of invariance of the manifold Ψ. The manifold Ψ for which $\delta(\Psi, G) > 0$ is called a partially invariant manifold of the group G with the invariance defect equal to $\delta(\Psi, G)$. It is inconvenient to use formula (2) for calculating the defect because it implies that either the orbit dimension dim $f(\Psi, O)$, or the rank of this orbit is known. Ovsyannikov [1] derived a formula for defect calculation, where the defect is expressed via the mapping $\psi(x) = 0$ defining the manifold Ψ in the space X.

Let us consider a system of differential equations SE. The solution $u \in SE$ is called a partially invariant solution if the manifold U defined by the relations $u = u(x)$ is a partially invariant manifold of the group H admitted by the system SE. In this case, the rank $\rho = \rho(U, H)$ and the defect $\delta = \delta(U, H)$ are called the rank and defect (of invariance) of this partially invariant solution U, respectively .

The algorithm of constructing the partially invariant solution was described in [1]. It consists of constructing two systems (resolving and automorphic) based on the system SE. The resolving system relates the invariants of the group H. It is simpler than the original system because it contains a smaller number of independent variables and sought functions. In turn, the automorphic system is a (consistent) overdetermined system and can be easily solved in most cases. It should be emphasized that the concept of a partially invariant solution is specific for system of differential equations. Examples of partially invariant solutions of gas-dynamic equations can be found in [1, 4, 5].

Let $X = \mathrm{R}^2\ (x, t)$ and let the variables t and x be treated as the time and distance, respectively. The solution $u = v(x - ct)$, where $c = const$, is called a traveling wave, and c is the traveling wave velocity. The existence of such solutions of the system of gas-dynamic equations is caused by the group of translations in terms of the variables and admitted by this system. Obviously, the traveling wave is an invariant solution of these equations. In general three-dimensional case, the system of gas-dynamic equations admits a group of translations along the coordinate axes and time. The notion of a double wave is an extension of the notion of the traveling wave to the case of two-dimensional motions. The system of equations of two-dimensional isentropic motion of a gas relates its velocity components $u(x, y, t)$, $v(x, y, t)$ and density $\rho(x, y, t)$. The solution of this system is called a double wave if $\rho = r(u, v)$. This solution is a partially invariant solution of this system of equations of rank 2 and defect 1 with respect to the group of translations over the axes x, y, t. Finding this solution is reduced to integrating the system of equations with two independent variables u, v.

Yanenko [6] started systematic investigations of the notion of multiple, in particular, double and triple traveling waves. Such waves are described by partially invariant solutions of gas-dynamic equations [1]. The theory of multiple waves is a significant part of the monograph [7].

3 Equations of Hydrodynamics and Their Group Properties

The main mathematical model in hydrodynamics is the Navier–Stokes equations. For an incompressible fluid moving in a potential field of external forces, these equations have the following form [8, 9]:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\rho^{-1} \nabla p + \nu \Delta\, \mathbf{v}, \quad \nabla \cdot\, \mathbf{v} = 0. \tag{3}$$

In system (3), $\mathbf{v}(x, t) = (v_1, v_2, v_3)$ is the fluid velocity vector in the initial inertial coordinate system, $x = (x_1, x_2, x_3)$, t is the time, and $p(x, t)$ is a modified pressure related to the true pressure p_g by the equality $p = p_g - \rho G(x, t)$, where G is the potential of acceleration of external forces. It should be noted that the most important fields from the viewpoint of applications, i.e., the gravity field and the field of centrifugal forces, are potential fields. The fluid density $\rho > 0$ is assumed to be constant, as well as the kinematic viscosity coefficient $\nu > 0$. The gradient over the variables x_1, x_2, x_3 is denoted by ∇, so that $\nabla\, \mathbf{v}$ is the tensor with the elements $(\nabla \mathbf{v})_{jk} = \partial v_k / \partial x_j$, and $\nabla \cdot \mathbf{v}$ is the divergence of the vector $\mathbf{v}$.

The widest group G_∞ admitted by system (3) was calculated by Yu.A. Danilov [10]. However, his paper was published as a preprint, which has limited access, and the result obtained by Danilov was repeated twice [11, 12]. The Lie algebra L_∞ corresponding to the group G_∞ is generated by the infinitesimal operators

$$Z = 2t\partial_t + \sum_{i=1}^{3} \left(x_i \partial_{x_i} - v_i \partial_{v_i} \right) - 2p\partial_p, \quad X_0 = \partial_t, \tag{4}$$

$$X_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + v_k \partial_{v_l} - v_l \partial_{v_k}; \quad k, l = 1, 2, 3; \quad k < l,$$

$$\Phi = \varphi \partial_p, \quad \Psi_k = \psi_k \partial_{x_k} + \dot{\psi}_k \partial_{v_k} - \rho x_k \ddot{\psi}_k \partial_p; \quad k = 1, 2, 3.$$

Here, ψ_i and φ are arbitrary (of class C^∞) functions of time, and the dot means differentiation with respect to t. Thus, the group admitted by system (3) is infinite-dimensional.

Assuming that $\nu = 0$ in system (3), we obtain a system of the Euler equations

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\rho^{-1} \nabla p, \quad \nabla \cdot\, \mathbf{v} = 0, \tag{5}$$

which describes the motion of an ideal incompressible fluid. The group $\hat{G}_\infty$ admitted by this system is a direct product of the group G_∞ and the dilation group with the operator

$$\hat{Z} = t\partial_t + \sum_{i=1}^{3} x_i \partial_{x_i}. \tag{6}$$

The algebra corresponding to the group $\hat{G}_\infty$ is denoted by $\hat{L}_\infty$.

The presence of the dilation operator Z in the algebra L_∞ means scaling invariance of Eq. (3). This important property forms the basis of physical modeling of viscous fluid flows. The set of three operators X_{kl} generates a group of consistent rotations in the space of coordinates and in the space of velocities admitted by system (3). This property reflects the absence of preferential directions in the spaces mentioned above. It should be noted that the existence of axisymmetric solutions of the Navier–Stokes equations is directly related to the fact that the algebra L_∞ contains rotation operators, as well as the existence of steady solutions of these equations is related to the presence of the translation operator in terms of time X_0 in L_∞.

The operators Φ, Ψ_i are specific for equations of incompressible fluid dynamics. The first of them implements the possibility of adding an arbitrary function of time to the pressure without changing the equations of motion. This fact is consistent with the statement that the pressure in an incompressible medium is not a thermodynamic variable [8]. The operator Ψ_i $(i = 1, 2, 3)$ corresponds to the transformation of the transition to a new coordinate system (which is non-inertial, generally speaking), which moves along the axis x_i with a velocity $\dot{\psi}(t)$ with respect to the initial system. In this case, there appears an additional term ψ_i in the ith momentum Eq. (3), i.e., acceleration of the inertia force, which is compensated by adding the function $-\rho x_i \ddot{\psi}_i$ to the pressure.

Assuming if $\Psi_i = 1$ $(i = 1, 2, 3)$ and $\Psi_i = t$ in system (4), we obtain the operators

$$X_i = \partial_{x_i}, \quad Y_i = t\partial_{x_i} + \partial_{v_i} \quad (i = 1, 2, 3). \tag{7}$$

The set of the operators X_0, X_i, Y_i, X_{jk} forms a ten-parameter Lie algebra L_{10}. The corresponding group G_{10} is called the Galileo group. The presence of operators of translations along the coordinates x_i in the algebra L_{10} is a consequence of space homogeneity. The presence of Galileo translation operators Y_i reflects the fact that the fluid motion laws are independent of the choice of the inertial coordinate system. Supplementing the operators of L_{10} with the dilation operator Z, we obtain an eleven-parameter Lie algebra L_{11}. The corresponding Lie group G_{11} is called the extended Galileo group. Supplementing the algebra L_{11} with the operator $\hat{Z}$ (6), we obtain a twelve-parameter group G_{12}. The groups G_{10} and G_{11} (G_{10} and G_{12}) play an important role in studying invariant and partially invariant solutions of problems with a free boundary for the Navier–Stokes (Euler) equations.

4 Problems with Free Boundaries

The notion of the free boundary of the fluid is an idealized interface of two immiscible fluids if the density of one fluid is much smaller than the density of the other fluid. A typical example of such a situation is the water-air interface. At small velocities of air, it is possible to neglect its dynamic action on water, and the atmospheric pressure p_a can be imposed on the free surface Γ_t. The subscript $t$in Γ_t characterizes the dependence of the free surface shape on time.

Let the free boundary Γ_t be defined by the equation $F(x,t) = 0$. The conditions for the Navier–Stokes equations (3) on this surface have the form

$$F_t + \mathbf{v} \cdot \nabla F = 0 \text{ for } F = 0, \tag{8}$$

$$(p_a - p_g)\,\mathbf{n} + 2\rho\nu D \cdot \mathbf{n} = -2\sigma K\,\mathbf{n} \text{ for } F = 0. \tag{9}$$

Here, p_g is the pressure in the fluid, D is the strain rate tensor, $2D_{ij} = \partial v_i/\partial x_j + \partial v_j/\partial x_{ij}$ $(i, j = 1, 2, 3)$, K is the mean curvature of the surface Γ_t, $\mathbf{n}$ is the unit vector of the external normal to this surface, and $\sigma \geq 0$ is the surface tension coefficient. Condition (8) means that the surface Γ_t is a Lagrangian surface so that the velocity of its motion in the direction $\mathbf{n}$ coincides with the normal component of the fluid velocity. This condition is called the kinematic condition. Condition (9) is called the dynamic condition. It reflects the fact that the shear stress on the free surface of the fluid is equal to zero, and the difference between the normal stress and atmospheric pressure is equal to the capillary pressure.

Then it is assumed that $\sigma = const$. This assumption is valid for isothermal motions in the absence of surfactants. The conditions on the interface of immiscible fluids in the case of their non-isothermal motion on the basis of thermodynamics of the Gibbs surface were derived in [13, 14] (Chap. II). Conditions (8) and (9) are obtained from the general conditions as a result of a limiting transition. In what follows, the motion is assumed to be isothermal.

Let us now consider the system of the Euler equations (5). For this system, the conditions on the free boundary have the form of Eq. (8) and

$$p_g - p_a = 2\sigma K, \quad x \in \Gamma_t. \tag{10}$$

Now we pass to considering the motion of two immiscible viscous incompressible fluids. The motion occurs in the domain $\Omega_t \subset R^3$, which is divided by a smooth surface Γ_t into two subdomains Ω_{1t} and Ω_{2t}. In each subdomain, let the functions $\mathbf{v}_1, p_1$ and $\mathbf{v}_2, p_2$ satisfying Eq. (3) with replacement of the coefficients ν, ρ by ν_1, ρ_1 and ν_2, ρ_2, respectively, be defined. At each point Γ_t at any time, we assume that there exist the limiting values of the functions $\mathbf{v}_i, p_i$ and their first derivatives with respect to all variables from the subdomains Ω_{1t} and Ω_{2t} It turns out that these set of functions cannot be arbitrary: they have to be related by appropriate expressions following from conservation laws and thermodynamic postulates.

The first relations have a kinematic character. They are based on the fact that the surface Γ_t is the Lagrangian (or material) surface. Thus, we avoid considering such processes as dissolving of one of the contacting fluids in the other, condensation, and evaporation, i.e., mass transfer through the interface is prohibited.

Let us use $\mathbf{n}$ to denote the unit vector of the normal to the surface Γ_t directed to the domain Ω_{2t} and V_n to denote the velocity of motion of the surface Γ_t in the direction of the normal $\mathbf{n}$. The fact that this surface is material is expressed by the following equalities [13, 14]:

$$\mathbf{v_1} \cdot \mathbf{n} = \mathbf{v}_2 \cdot \mathbf{n} = V_n, \quad x \in \Gamma_t. \tag{11}$$

Equalities (11) and continuity equation (the second equation of system (3)) ensure the validity of the integral law of mass conservation in an arbitrary material subdomain of the domain $\Omega_{1t} \cup \Omega_{2t}$.

The integral law of momentum conservation across the interface yields the following expression [13, 14]:

$$(-p_1 + p_2)\,\mathbf{n} + 2(\rho_1 \nu_1 D_1 - \rho_2 \nu_2 D_2) \cdot \mathbf{n} = -2\sigma K \mathbf{n}, \quad x \in \Gamma_t. \tag{12}$$

Here, D_i $(i = 1, 2)$ is the strain rate tensor corresponding to the velocity vector $\mathbf{v}_i$, and K is the mean curvature of the surface Γ_t (it is assumed that $K > 0$ if this surface is convex outward of the domain Ω_{2t}).

To conclude, we postulate the condition of continuity of the total velocity vector across the interface:

$$\mathbf{v}_1 = \mathbf{v}_2, \quad x \in \Gamma_t. \tag{13}$$

In fact, conditions (13) contain two additional scalar conditions because the continuity of the normal component is already implied in conditions (11).

5 Theorems of Invariance of Conditions on the Free Boundary

This paragraph deals with the properties of invariance of conditions (8)–(13) with respect to transformations that ensure conservation of the Navier–Stokes equations (3). For simplicity, we consider a situation without external forces. Then the function p_g involved into the dynamic condition (9) coincides with the function p involved into the momentum equation (the first equation of system (3)).

Let us consider the Euclidean space R^8 with the coordinates of its points being $x_1, x_2, x_3, t, v_1, v_2, v_3, p$. This space is subjected to the action of the Galileo group G_{10} with the basis operators X_0, X_j, Y_j, X_{ij} $(i, j = 1, 2, 3;\ i < j)$ defined by formulas (4). It is admitted by system (3). Let us consider a certain k–parameter subgroup H of the group G_{10}. Let $l \leq k$ be the maximum number of operators of this group that are not linearly related. It should be noted that $l < k$ only if $k \geq 3$ and H contains a group of rotations $\langle X_{12}, X_{13}, X_{23} \rangle$. We are interested only in intransitive groups G_{10}; in this case, $l < 8$.

Let I_α $(\alpha = 1, \ldots, 8 - l)$ be a complete set of functionally independent invariants of the group H. Let us use $m$to denote the rank of the matrix $(\partial I_\alpha / \partial v_\beta)$, where $\beta = 1, 2, 3, 4$, and it is assumed that $v_4 = p$. Clearly, $m \leq \min(4,\ 8 - l)$. In what follows, we consider only such groups H where $m < 8 - l$. Then there exist $n = 8 - l - m$ invariants of H that do not contain $\mathbf{v}$, p, which are the sought functions in system (3). Without loss of generality, we can assume that these invariants are $I_{m+1}, \ldots, I_{m+n}$. The condition $m < n$ is a necessary condition for the existence of an invariant H-solution of the above-mentioned system, and the number $n - m$ is the rank of this solution.

Let us now assume that the equation $F(x, t) = 0$ defines a non-singular invariant manifold of the group H. This means that F can be written in the form

$$F = Q[I_{m+1}(x, t), \ldots, I_{m+n}(x, t)]$$

with a certain function Q.

Theorem 1 *Let the free boundary $F(x, t) = 0$ be a non-singular invariant manifold of the subgroup $H \in G_{10}$. Then conditions (8) and (9) satisfied on this surface are also invariant with respect to H.*

Theorem 1 was put forward by Pukhnachev [15]. The proof of this theorem can be also found in [16]. At $\sigma \neq 0$, it turned out that the group G_{10} in the condition of Theorem 1 cannot be replaced by a wider subgroup of the infinite-dimensional group G_∞ admitted by system (3). This extension is possible if $\sigma = 0$. In this case, the statement of Theorem 1 remains valid if G_{10} is replaced with an extended Galileo group G_{11} by means of supplementing the generators of the group G_{10} with the dilation operator Z.

Let us now consider the conditions on the free boundary (8) and (10) for the Euler equations (7). If there are no external forces, it may be assumed that $p_g = p$. Here, we have an analog of Theorem 1; the proof is skipped.

As was noted earlier, the notion of the free surface is understood as an idealized interface of two immiscible fluids. Below, we formulate the properties of invariance of the conditions at the interface (11)–(13). Let us use G_{10}^{Φ} to denote the direct product of the Galileo group G_{10} and the group generated by the operator $\Phi = \varphi \partial_p$, where $\varphi(t) \in C^\infty$ is an arbitrary function.

Theorem 2 *Let the interface of immiscible fluids $F(x, t) = 0$ be a non-singular invariant manifold of the subgroup $H \in G_{10}^{\Phi}$. Then conditions (11)–(13) satisfied on this surface are also invariant with respect to H.*

If $\sigma = 0$ in condition (16), then the group G_{10}^{Φ} in the formulation of Theorem 2 can be replaced by the direct product of the group G_Φ and the extended Galileo group G_{11}. The maximum extension of the admitted group is observed at $\sigma = 0$ and identical densities of the contacting fluids.

Theorem 3 *Let us assume that $\rho_1 = \rho_2$ and $\sigma = 0$ in conditions (12). Moreover, let the interface $F(x, t) = 0$ be a non-singular invariant manifold of the subgroup $H \in G_\infty$. Then conditions (11)–(13) satisfied on this surface are also invariable with respect to H.*

Theorems 2 and 3 are new. They show that the invariant properties of the conditions at the interface of immiscible fluids are richer than similar properties of the conditions on the free boundary. This is caused by the fact that the concept of the interface is more natural from the physical viewpoint than the concept of the free surface. Theorems 2 and 3 are proved in accordance with the scheme used to prove Theorem 1; the proof is not provided here.

Theorem 1 was used to construct invariant solutions of the Navier–Stokes equations, which were preliminary matched with the conditions on the free boundary, which is an invariant manifold of the corresponding group [15, 16]. However, retaining the invariance of the free boundary, it is possible to alleviate the requirement to the solution of system (3): it can be partially invariant. Examples of such solutions of problems with a free boundary are provided in the next paragraphs.

6 Partially Invariant Solutions of the Navier–Stokes Equations

It is not an exaggeration to say that the initial trend of studying the Navier–Stokes equations was to obtain their exact solutions. Here, we should mention the solution of Hiemenz [17], which describes the flow near the stagnation point, and also the solution of Karman, which describes the motion in a half-space induced by plane rotation [18]. There is a popular opinion that these both steady solutions are self-similar solutions of system (3). In reality, these solutions have a group-theoretical nature, but it is more complicated. Petrova et al. [19] considered a problem of unsteady motion of a fluid near the stagnation point. It turned out that the solution of this problem is a partially invariant solution of system (3) of rank 2 and defect 2 with respect to the group generated by the operators $X_1 = \partial_{x_1}$ and $Y_1 = t\partial_{x_1} + \partial_{v_1}$. The corresponding resolving system inherits some of the group properties of system (3). The solution obtained by Hiemenz is an invariant solution of the resolving system with respect to the operator $X_0 = \partial_t$.

A similar situation is observed for the Karman solution. To write down this solution, we use system (3) in cylindrical coordinates $r = (x_1^2 + x_2^2)^{1/2}$, $\phi = \arctan(x_2/x_1)$, $z = x_3$:

$$\frac{dv_r}{dt} - \frac{v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\big(\Delta v_r - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} - \frac{v_r}{r^2}\big), \tag{14}$$

$$\frac{dv_\phi}{dt} + \frac{v_r v_\phi}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \nu\big(\Delta v_\phi + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r^2}\big),$$

$$\frac{dv_z}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \Delta v_z, \quad \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} = 0,$$

$$\frac{dv_z}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \Delta v_z, \quad \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0.$$

Here

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\phi}{r} \frac{\partial}{\partial \phi} + v_z \frac{\partial}{\partial z}, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}.$$

The motion is called rotationally symmetric if the sought functions in system (14) are independent of the variable ϕ. The Karman solution refers to this class. It is described by the formulas

$$v_r = r\Omega F(\varsigma), \quad v_\phi = r\Omega G(\varsigma), \quad v_z = (\nu\Omega)^{1/2} H(\varsigma), \quad p = \rho\nu\Omega Q(\varsigma), \qquad (15)$$

where $\varsigma = (\Omega/\nu)^{1/2} z$. Substitution of Eq. (15) into Eq. (14) yields a system of ordinary differential equations for the functions F, G, H, Q:

$$F^2 - G^2 + F'H = F'', \quad 2FG + G'H = G'', \quad HH' = Q' + H'', \quad 2F + H' = 0. \qquad (16)$$

Let us impose the following boundary conditions on the solution of system (16):

$$F = 0, \quad G = 1, \quad H = 0 \ \text{at} \ \varsigma = 0, \quad F \to 0, \quad G \to 0 \ \text{as} \ \varsigma \to \infty. \qquad (17)$$

Then solution (15) describes the fluid motion in the half-space $z > 0$ induced by rotation of the bounding solid plane around the axis of symmetry with an angular velocity Ω.

Following [20], we demonstrate how the Karman solution can be obtained on the basis of group considerations. Let us consider a five-parameter subgroup H of the group C_∞ with the basis operators X_1, X_2, Y_1, Y_2, X_{12}. This subgroup corresponds to the partially invariant solution of system (3) of rank 2 and defect 2. Its invariant part in cylindrical coordinates has the form $v_z = h(z,t)$, $p = q(z,t)$. By virtue of the continuity equation, there exists a relationship between the radial and axial velocity components: $\partial v_r/\partial r + v_r/r + \partial v_z/\partial z = 0$. By requiring that the function v_r is bounded as $r \to 0$, we have $v_r = rf(z,t)$, where $f = -h/2$.

Now we substitute the expressions for v_r, v_z, p into the first equation of system (14) and consider the fact that the sought functions are independent of ϕ. As a result, we obtain the presentation of the circumferential velocity $v_\phi = rg(z,t)$, where g is expressed via the function h and its derivatives. Thus, the general presentation of the partially invariant solution of system (3) with respect to the group H, which is regular on the axis of symmetry, is

$$v_r = rf(z,t), \quad v = rg(z,t), \quad v_z = h(z,t), \quad p = q(z,t). \qquad (18)$$

The functions f, g, h, q satisfy the system of equations

$$\frac{\partial f}{\partial t} + h\frac{\partial f}{\partial z} + f^2 - g^2 = \nu\frac{\partial^2 f}{\partial z^2}, \quad \frac{\partial g}{\partial t} + h\frac{\partial g}{\partial z} + 2fg = \nu\frac{\partial^2 g}{\partial z^2},$$

$$\frac{\partial h}{\partial t} + h\frac{\partial h}{\partial z} = -\frac{1}{\rho}\frac{\partial q}{\partial z} + \nu\frac{\partial^2 h}{\partial z^2}, \quad 2f + \frac{\partial h}{\partial z} = 0. \tag{19}$$

System (19) inherits some part of the group properties of the original system (3), in particular, the translation with respect to time. The corresponding steady solution of system (19) coincides with the Karman solution (15) with accuracy to notations. We can say that the Karman solution is an invariant solution of a certain partially invariant sub-model of the Navier–Stokes equations, whereas a solution of the form of Eq. (18) is an unsteady analog of the Karman solution. It turns out that this solution can describe the process of layer spreading on a rotating plane [21] and [16], Chap. VII.

We require that conditions (10) and (11) should be satisfied on the invariant manifold $z = s(t)$ of the group H for solution (18). For this purpose, the unknown functions in system (19) have to be subjected to the boundary conditions

$$\frac{\partial f}{\partial z} = \frac{\partial g}{\partial z} = 0, \quad q - 2\rho\nu\frac{\partial h}{\partial z} = 0 \ \text{ for } \ z = s(t), \quad \frac{ds}{dt} = h[s(t), t] \ \text{ for } \ t > 0. \tag{20}$$

In addition, we impose the boundary conditions

$$f = 0, \quad g = \omega(t), \quad h = 0 \ \text{ for } \ z = 0, \quad t > 0. \tag{21}$$

Here, $\omega(t)$ is a specified function, $\omega(0) = 0, \quad \omega'(0) = 0$. The formulation of the problem with an unknown boundary for system (19) is closed by setting the initial conditions

$$f = g = 0, \quad h = 0, \quad s = s_0 > 0 \ \text{ for } \ t = 0. \tag{22}$$

Thus, we obtain the following problem: we have to find a function $s(t)$ and a solution of system (19) in the domain $S_T = \{z, t: \quad 0 < z < s(t), \quad 0 < t < T\}$ so that conditions (20)–(22) are satisfied. Let us assume that the function $\omega(t)$ belongs to the Hölder class $C^{1+\alpha/2}[0, T]$, where $0 < \alpha < 1$. Then there exists a unique solution of problem (19)–(22) for any $T > 0$ [16].

The resultant solution is interpreted as follows. At the initial time, the quiescent fluid occupies an infinite layer $0 < z < s_0$ whose lower boundary is a solid surface, whereas the upper boundary is free. Then the plane starts smooth rotation around the z axis with an angular velocity $\omega(t)$ and sets the fluid into the corresponding motion. The characteristic feature of the problem implies that the free boundary remains flat for all $t > 0$. This property was used to develop a technology of depositing coatings onto a flat disk (see [22] and the references therein). As the problem solution exists for all $t > 0$, it is of interest to study its behavior as $t \to \infty$. This was made in [23], where the asymptotic behavior of the solution was found for the case with the function $\omega(t) = At^n$ for large values of t (A and n are constants). In the same paper, results of the numerical solution of problem (19)–(22) were reported for several typical values of $\omega(t)$ (see also Chap. VII of the monograph [16]).

Now we construct an example of a partially invariant solution of system (3), which describes plane motion with an internal interface. In what follows, x and y are the Cartesian coordinates on the plane, while u and v are the corresponding components of the velocity vector. It is assumed that both fluids have an identical density, $\rho_1 = \rho_2 = \rho$, but different viscosities, ν_1 and ν_2. The boundaries of the flow domain $\Pi_T = \{x, y, t : x \in \mathrm{R},\ 0 < y < l(t),\ 0 < t < T\}$ are solid impermeable walls. One of them, $y = 0$, is stationary, while the other one, $y = l(t)$, moves along the y axis. The line $y = s(t)$ is the interface between the fluids. The band $0 < y < s(t)$ is occupied by the fluid indicated by the subscript 1, and the band $s(t) < y < l(t)$ is occupied by the fluid indicated by the subscript 2. At the initial time, both fluids are at rest.

Let us consider a subgroup of the group G_∞ generated by the operators $X = \partial_x$ and $Y = t\partial_x + \partial_u$. It corresponds to a partially invariant solution of system (3) of rank 2 and defect 2 of the form

$$u_i = -x\frac{\partial v_i}{\partial y}, \quad v_i = v_i(y,t), \quad p = \frac{\rho}{2}a(t)x^2 + m(y,t), \quad i = 1, 2, \tag{23}$$

where a is a given function of t. (We confine ourselves to solutions of system (3), where the functions v_i and p are even functions of the variable x, while u_i are odd functions of this variable). Substituting expressions (23) into system (3), we obtain equations satisfied by the functions v_1, v_2, and m:

$$\frac{\partial^2 v_1}{\partial y \partial t} + v_1\frac{\partial^2 v_1}{\partial y^2} - \Big(\frac{\partial v_1}{\partial y}\Big)^2 = \nu_1\frac{\partial^3 v_1}{\partial y^3} + a(t) \ \text{ for } \ 0 < y < s(t), \ \ 0 < t < T, \tag{24}$$

$$\frac{\partial^2 v_2}{\partial y \partial t} + v_2\frac{\partial^2 v_2}{\partial y^2} - \Big(\frac{\partial v_2}{\partial y}\Big)^2 = \nu_2\frac{\partial^3 v_2}{\partial y^3} + a(t) \ \text{ for } \ s(t) < y < l(t), \ \ 0 < t < T,$$

$$\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial y} = -\frac{1}{\rho}\frac{\partial m}{\partial y} + \nu_1\frac{\partial^2 v_1}{\partial y^2} \ \text{ for } \ 0 < y < s(t), \ \ 0 < t < T, \tag{25}$$

$$\frac{\partial v_2}{\partial t} + v_2\frac{\partial v_2}{\partial y} = -\frac{1}{\rho}\frac{\partial m}{\partial y} + \nu_2\frac{\partial^2 v_2}{\partial y^2} \ \text{ for } \ s(t) < y < l(t), \ \ 0 < t < T.$$

The boundary conditions on the solid regions of the boundary of the domain Π_T (no-slip conditions) have the form

$$v_1 = 0 \ \text{ for } \ y = 0, \ \ 0 \le t \le T, \quad v_2 = \frac{dl}{dt} \ \text{ if } \ y = l(t), \ \ 0 \le t \le T. \tag{26}$$

Based on relations (11)–(13), the boundary conditions on the interface between the fluids are written as

$$v_1 = v_2, \quad v_1 \frac{\partial v_1}{\partial y} = v_2 \frac{\partial v_2}{\partial y}, \quad v_1 \frac{\partial^2 v_1}{\partial y^2} = v_2 \frac{\partial^2 v_2}{\partial y^2} \quad \text{for} \quad y = s(t), \quad 0 \le t \le T, \tag{27}$$

$$\frac{ds}{dt} = v_1[s(t), t] = v_2[s(t), t] \ \text{ for } \ 0 < t < T. \tag{28}$$

They are supplemented with the initial conditions

$$v_1(y, 0) = 0 \ \text{ for } \ 0 \le y \le s_0, \quad v_2(y, 0) = 0 \ \text{ if } \ s_0 \le y \le l_0, s(0) = s_0, \tag{29}$$

where $l_0 = l(0), \quad s_0 \in (0, l_0)$ is a specified constant.

The problem with an unknown boundary is formulated. We have to determine a function $s(t)$ and a solution v_1, v_2, m of system (24), (25) that satisfies conditions (8)–(29). It should be noted that relations (24), (26)–(29) form a closed system for finding the functions v_1, v_2, and s. If these functions are found, then the remaining sought function m is reconstructed by a quadrature from Eq. (25). The additive function of time arising thereby makes it possible to ensure the continuity of the function m across the interface.

Problem (24)–(29) is rather non-standard. At the moment, the uniqueness of its classical solution can be guaranteed. To prove the existence theorem, it is reasonable to pass in this problem to the Lagrangian coordinates in which the flow domain is fixed. It can be expected that problem (24)–(29) does have a solution, at least, for a sufficiently small value $T > 0$.

7 Example of Partially Invariant Solution of the Euler Equations

Below we study a rotationally symmetric solution of the Euler equations (5) in a cylindrical layer $\Omega_T = \{r, z, t : \ b < r < s(t), \ z \in \mathrm{R}, \ 0 < t < T\}$. The equations of motion are obtained from Eq. (14) by assuming that $\nu = 0$ and taking into account that the sought functions are independent of the variables ϕ:

$$\begin{gathered}\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \quad \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + v_z \frac{\partial v_\phi}{\partial z} + \frac{v_r v_\phi}{r} = 0, \\ \frac{\partial v_z}{\partial r} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}, \quad \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0.\end{gathered} \tag{30}$$

System (30) admits a group with operators ∂_z and $t\partial_z + \partial_{v_z}$, which corresponds to its partially invariant solution of rank 2 and defect 1 of the form

$$v_r = u(r,t),\quad v_\phi = v(r,t),\quad v_z = -z\Big(\frac{\partial u}{\partial r} + \frac{u}{r}\Big),\quad p = p(r,t). \tag{31}$$

Substituting expressions (31) into Eq. (30), we obtain the resolving system of equations for the sought invariant functions, which can be conveniently written in the form

$$\frac{\partial u}{\partial r} + \frac{u}{r} = L,\quad \frac{\partial L}{\partial t} + u\frac{\partial L}{\partial r} - L^2 = 0,\quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r} = 0, \tag{32}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0.$$

Let us assume that the layer boundary $r = b$ is a permeable surface. It is subjected to the boundary condition

$$u = c(t),\quad r = b,\quad z \in \mathrm{R},\quad 0 < t < T, \tag{33}$$

where $c(t)$ is a prescribed function. The boundary $r = s(t)$ is assumed to be free. It is subjected to the conditions

$$p = p_a + \frac{\sigma}{s(t)},\quad r = s(t),\quad \frac{ds}{dt} = v[s(t),t],\quad 0 < t < T, \tag{34}$$

which follow from conditions (8) and (10). Moreover, the following initial conditions are imposed:

$$u(r,0) = u_0(r),\quad v(r,0) = v_0(r),\quad b \le r \le s_0,\quad s(0) = s_0. \tag{35}$$

Here, $u_0(r)$ and $v_0(r)$ are specified functions, and $s_0 > b$ is a prescribed constant.

System (32) has a recurrent structure: its first two equations are separated from the others. An effective analysis of problem (32)–(35) is reached by means of the transition to the Lagrangian coordinate ξ instead of r. The relationship between r and ξ is determined by solving the Cauchy problem

$$\frac{dr}{dt} = u(r,t),\quad t > 0;\quad r = \xi,\quad t = 0. \tag{36}$$

The following notations are introduced:

$$u[r(\xi,t),t] = U(\xi,t),\quad L[r(\xi,t),t] = \Lambda(\xi,t),$$

$$v[r(\xi,t),t] = V(\xi,t),\quad p[r(\xi,t),t] = P(\xi,t).$$

Here, $r(\xi,t)$ is the solution of the Cauchy problem (36). We denote $u_0'(\xi) + u_0(\xi)/\xi = a(\xi)$. In the new variables, system (32) takes the form

$$\frac{\partial \Lambda}{\partial t} = \Lambda^2, \quad r\frac{\partial^2 r}{\partial r \partial t} + \frac{\partial r}{\partial \xi}\frac{\partial r}{\partial t} = r\frac{\partial r}{\partial \xi}\Lambda, \tag{37}$$

$$\frac{\partial V}{\partial t} + \frac{UV}{r} = 0, \quad \frac{\partial U}{\partial t} - \frac{V^2}{r} + \frac{1}{\rho}\Big(\frac{\partial r}{\partial \xi}\Big)^{-1}\frac{\partial P}{\partial \xi} = 0. \tag{38}$$

The solution of the first equation of (37) with the initial condition $\Lambda(\xi, 0) = a(\xi)$ has the form

$$\Lambda = -\frac{a(\xi)}{a(\xi)t + 1}. \tag{39}$$

The second equation of (37) can be written as

$$\frac{\partial}{\partial t}\left(\frac{\partial r^2}{\partial \xi}\right) = \Lambda\left(\frac{\partial r^2}{\partial \xi}\right),$$

after which it is easily integrated with allowance for equalities (39) and $r = \xi$ at $t = 0$:

$$r(\xi, t) = \Big[\int\limits_{\varsigma(t)}^{\xi} \frac{2\eta d\eta}{a(\eta)t + 1} + b^2\Big]^{1/2}. \tag{40}$$

Here, the function $\varsigma(t)$ determines the image of the fixed boundary $r = b$ of the flow domain in passing to the Lagrangian coordinates. From Eq. (40) and the equalities $r[\varsigma(t), t] = b$, $r_t[\varsigma(t), t] = c(t)$, we obtain the Cauchy problem for finding the function $\varsigma(t)$:

$$2\varsigma\frac{d\varsigma}{dt} = -bc(t)[a(\varsigma)t + 1], \quad t > 0; \quad \varsigma = b, \quad t = 0.$$

In accordance with the second condition of (34), the image of the free boundary $r = s(t)$ on the plane of the Lagrangian coordinates is a segment of the straight line $\xi = s_0$.

Using formulas (40) and $U = r_t(\xi, t)$, we find the functions V and P with the help of quadratures from Eq. (38). In this case, the function P is determined with the accuracy up to the additive function of time. Thus, the first condition of (34) can be satisfied on the free boundary $r = s(t)$. As a result, we obtain a parametric representation of the solution of the problem with a cylindrical free surface for the Euler equations.

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