Complex Methods for Lie Symmetry Analysis

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Abstract When Lie developed symmetry analysis, he took the equations to be defined in the complex domain but did not explicitly use the entailed complex analyticity. Making it explicit necessitates the incorporation of the Cauchy–Riemann equations into the original system of equations, which modifies the symmetries of the system. This point was followed up by us, and some of our students, in a series of papers (and theses). It was found that complex methods, when they are applicable, provide more powerful tools for obtaining solutions and integrals of differential equations, even enabling us to find solutions of systems of differential equations that possess no symmetries. In this chapter we review the methods developed and then pose the crucial question that was begged in saying "when they are applicable." When *would* they be applicable and why, or how, does the complex method work? We indicate some lines to pursue to try to find the answers, or at least partial answers, to these questions.

Dedicated to the memory of one of the most innovative workers in the field of Symmetry Analysis, after Sophus Lie, Nail Hairullovich Ibragimov, who initiated many new methods for using Lie Analysis to deal with differential equations.
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1 Introduction

Before Lie, the usual method to solve differential equations (DEs) was by ad hoc approaches or by approximating it by a linear DE and solving the linear version. In general, the approximation will work well enough in some regime and become arbitrarily bad in other regimes. As such, it would be necessary to prove the existence of a solution and to determine the domain in which the approximation is good enough. Since these will be different for each DE, one is reduced to solving one DE at a time and not rely on any method for whole classes of DEs. Lie had wanted to do for DEs what Galois had done for polynomial equations, wherein he used the manifest symmetries of the roots of the equation to determine when the polynomial equation could be solved by means of radicals. Among the methods that had been used for solving some of the simpler DEs, there was the transformation of independent and dependent variables. Lie conceived of the idea of looking for invariance of the DE under such transformations [1–4] so that it could be determined when the DEs could be solved/reduced or transformed and then one can proceed to solve/reduce or transform the DEs. Lie used not only the groups of symmetries, but the algebra of the corresponding infinitesimal symmetry generators. The DEs are not necessarily single but could be systems (or we could say that they are *vector* DEs). Further, he did not restrict the domain of the DEs to be real, but allowed them to be complex.

Symmetry generators, inter alia, can be used to reduce the order of scalar ordinary differential equations (ODEs) or reduce the number of independent variables in partial differential equations (PDEs). Alternatively, the symmetries can be used to construct quantities that remain invariant under the transformation, thereby enabling a reduction of order or the number of independent variables [5, 6] by treating that combination as a new variable. If there are enough independent symmetries they can be used to fully solve the equation for scalar ODEs in the sense of providing an algebraic equation that constitutes an implicit solution, modulo quadratures or in the case of PDEs to construct invariant solutions. If the invariants contain derivatives in them, they are called *differential* invariants. If they depend only on dependent and/or independent variables, they are simply called “invariants.” In effect, what Lie had done was to take the space of independent variables on which the dependent variables were defined and extended, or enlarged, it to include also the dependent variables. In this new, extended space, we could perform the equivalent of coordinate transformations called *point transformations*. In this way of looking at it, it is natural to require invariance in the higher dimensional manifold under point transformations. What Lie wanted was that the DEs remain invariant under these transformations, thus visualizing them as “living on the manifold.”

To deal with DEs, we need to treat the derivatives *as if* they were independent variables and then constrain them in such a way that the DE is satisfied. The enlarged, or *prolonged* space of all the variables and their derivatives is also called the *jet space*. In this space, we restrict the transformations to be performed only in the original, non-prolonged, space. However, we could include any number of derivatives from the prolonged space that we choose. Thus, if we prolong to include the first derivative in our transformations but no more, we have "contact transformations." If we prolong further, we have "higher-order transformations," but there are no separate names for them. The name "contact" comes from the tangency requirement for the derivative to be met. Lie mainly restricted his analysis to point and contact transformations but subsequently others extended the Lie methods to higher order transformations. As such, the original transformations are called "Lie point transformations."

Among the various methods of using symmetries to solve DEs is the transformation of the DE to linear form, which is of special interest. The classic example for this is the Bernoulli equation, a first-order ODE in which the dependent variable appears to the nth power. However, there was no general procedure available for nonlinear DEs, especially higher order ones. If one can tell when a DE can, in principle, can be transformed to linear form, even without finding the required transformations and converting to linear form, we can say a lot about the solutions of the DE. For example, if it is an ODE, we know how many independent solutions there are, without having obtained the solutions. By requiring that the given DE transform to a chosen canonical form of a linear DE, we can arrive at conditions that the given DE must satisfy. Lie did this for second-order scalar ODEs [5] and demonstrated that the ODE must have eight infinitesimal symmetry generators that would constitute a Lie algebra as well as conditions on the DE. Then he looked at the maximal algebra admitted. He did not go further but others carried the work forward for higher order ODEs and for vector ODEs, using contact transformations and even Lie's original method. Equations that can be transformed to linear form are called *linearizable* and the process of transforming a DE to linear form by transformations of the dependent and independent variables is called *linearization* via point transformation. Note that this is *not* an approximation of the DE by a linear one, but a transformation that gives the exact solution of the DE. If it is a linearizable PDE, it has infinitely many linearly independent solutions. Consequently, there is no way that we can make the type of general statement that we could for ODEs. We then need boundary conditions to be able to arrive at a meaningful, unique, solution, or other invariant criteria. This entails that the conditions satisfy the symmetries of the PDE. Thus, for PDEs, invariants are especially useful although these needed further generalizations.

In this chapter we review work on a line that Lie did *not* take, namely making explicit use of complex analyticity. Recall that if a complex function of a complex variable is once differentiable in an open domain it is analytic in that domain, which entails infinite differentiability. This would not hold for real functions of real variables. While this fact simplifies statements of Lie's requirements for the DEs to be amenable to his symmetry methods, it is not obvious that it can make a fundamental difference to the procedures used to solve the equations. We demonstrate that it does so, provided certain additional conditions are met. Though there are explicit checks for when these complex methods *can be used*, there is no complete understanding of when they would be *useful*.

The plan of the chapter is as follows. In the next section, we give the preliminary background for Lie symmetry analysis and some basic geometry used in it, including contact and higher order symmetries. In section three, we review the fundamentals of the complex method. In the subsequent sections, we review its application for linearization and for Noether symmetries and their integrals. It is shown that the complex methods extend the applicability of symmetry analysis beyond the usual methods. In section seven, we present insights regarding the working of the complex methods obtained by iterative splitting of a scalar ODE. In the concluding section, we summarize the work reviewed and present the fundamental questions that need to be addressed so as to understand *why* complex methods work.

2 Preliminaries

For completeness, we give basic definitions despite the likelihood that the reader already knows them, in the hope that he/she will bear with us. At least they will be useful to establish notation. If there are l independent variables represented as a vector $\mathbf{x}$ and m dependent variables represented by $\mathbf{y}$, a *Lie point symmetry generator* is the operator

$$\mathbf{X} = \mathbf{A}(\mathbf{x}, \mathbf{y}).\nabla_{\mathbf{x}} + \mathbf{B}(\mathbf{x}, \mathbf{y}).\nabla_{\mathbf{y}} \ , \tag{1}$$

or using indices a for the independent variables and i for the dependent variables

$$\mathbf{X} = A^a(x^b, y^i)\frac{\partial}{\partial x^a} + B^i(x^a, y^j)\frac{\partial}{\partial y^i} \ , \tag{2}$$

where we have used the Einstein summation convention that repeated indices are summed over. Further, if the DE is of order n, we need to prolong the space and the generators to incorporate all the derivatives of the dependent variables with respect to the independent variables. For ODEs,

$$\mathbf{X}^{[n]} = A(x, y^i)\frac{\partial}{\partial x} + B^i(x, y^j)\frac{\partial}{\partial y^i} + B^{i\ [1]}(x, y^j, y^{j\ \prime})\frac{\partial}{\partial y^{i\ \prime}} + \cdots \ , \tag{3}$$

where

$$B^{i\ [p]} = D_x B^{i\ [p-1]} - y^{i\prime} D_x A \ , \tag{4}$$

$B^{i\ [0]}$ simply being B^i and D_x is the *total derivative in the prolonged space*,

$$D_x = \frac{\partial}{\partial x} + y^{i\ \prime}\frac{\partial}{\partial y^i} + \cdots + y^{i\ (p)}\frac{\partial}{\partial y^{i\ (p-1)}} \ . \tag{5}$$

For PDEs, the A would have to be replaced by $\mathbf{A}$ and the partial derivative with respect to x by $\nabla_{\mathbf{x}}$. While the former can be easily converted to index notation as A^i, the latter becomes somewhat involved in converting. The real problem in writing is for the $y^{i\ [p]}$, which would be a partial derivative with respect to x^a *to all orders* up to p. The set of all prolonged symmetry generators forms a Lie algebra and the symmetry group determines what reduction of the DE there can be. A system of

m ODEs of order n, $E^i(x, y^j; y^{j\,\prime}, \ldots, y^{j\,[n]}) = 0$, is said to be *symmetric* under the transformation generated by $\mathbf{X}$ if $\mathbf{X}^{[n]}E^i = 0$, when restricted to the solutions of $E^i = 0$. This is denoted by putting "$|_{\mathbf{E}=0}$" after the above equation. The generalization to PDEs is as before, with the corresponding complications.

A major activity arose of classifying the Lie point symmetry algebras of all second-order scalar ODEs. This was called the "classification problem for scalar second-order ODEs." The classification problem for higher order scalar ODEs rapidly becomes extremely difficult on account of the proliferation of possible cases and sub-cases of the allowed Lie algebras. Similarly, the classification problem for higher dimensional systems becomes even more complicated as the number of sub-cases proliferates even more. Increasing the order and dimension simultaneously makes the problem well high intractable. We do not go into this further here as the complex methods were not used for this purpose.

Since much of the complex work is motivated by considerations of linearization, it is necessary to very briefly review the key features of Lie's linearization procedure. By requiring that a scalar second-order semilinear ODE

$$y'' = f(x, y; y') \,, \tag{6}$$

be transformed under $p = p(x, y), q = q(x, y)$ to

$$q'' + A(p)q' + B(p)q + C(p) = 0 \,, \tag{7}$$

he showed that (6) would have to be of the form

$$y'' + a(x, y)y'^{\,3} + b(x, y)y'^{\,2} + c(x, y)y' + h(x, y) = 0 \,, \tag{8}$$

and would have to satisfy a system of four first-order conditions that the coefficients a, b, c, h and two auxiliary functions would have to satisfy. This is not as bad as it may sound, since one is not *solving* the coupled system of equations but merely verifying them. Nevertheless, the auxiliary functions complicate matters as they are arbitrary and would have to be guessed. Tressé [7] invariants eliminates the auxiliary functions via compatibility by taking derivatives and one obtains two second-order conditions to be satisfied by the coefficients, viz.

$$3(ac)_x + ha_y - 2bb_x - cb_y - 3a_{xx} - 2b_{xy} - c_{yy} = 0 \,, \tag{9}$$

$$3(hb)_y + ab_x - 2aa_y - ha_x - 3h_{yy} - 2c_{xy} - b_{xx} = 0 \,. \tag{10}$$

Tressé's formulation makes the application of the Lie conditions much easier.

Chern [8] did not use the Lie point transformations to linearize third-order scalar ODEs but incorporated the first derivative of the dependent variable in the coefficients of the operator, $A = A(x, y; y')$ and $B = B(x, y; y')$ in the scalar case for (2), to solve the problem. One must now ensure that under the transformation, the "derivative" used here corresponds to the derivative of the dependent variable by differentiation. Writing the transformation as $(x, y) \rightarrow (\overline{x}, \overline{y})$, the *contact* or *tangency* condition is

$$d\overline{y} - \overline{y}' d\overline{x} = \lambda(x, y; y')(dy - y' dx) , \tag{11}$$

where λ is an undetermined multiplier [5]. The contact transformations of Chern can be extended to systems of equations for several independent variables by re-inserting the indices, so that the condition becomes

$$d\overline{y}^i - \overline{y}^i_{,a} d\overline{x}^a = \lambda(x, y; y^i_{,a})(dy^i - y^i_{,a} dx^a) . \tag{12}$$

Lie had managed to prove that the second-order scalar ODE is linearizable if, and only if, it has eight Lie point symmetry generators. The Lie point symmetry algebra for order n scalar ODEs was obtained much later [9] and it was shown that Lie's theorem does not hold there, as there are three linearizable classes with $(n+1)$, $(n+2)$ or $(n+4)$ generators. For the third-order case, the canonical forms associated with those symmetries were made explicit [10]. The classes of linearizable second-order systems was also achieved at around the same time, first for two-dimensional systems and then for arbitrary m [11]. The linear classes for the two-dimensional system have 5, 6, 7, 8, or 15 generators and for arbitrary m, $2m+1, \ldots, (2m)^2 - 1$ symmetry generators [12, 13].

A question arises here, *why* is the $n = 2$ case special? The answer may lie in the geometric methods that had been developed and were used to linearize ODEs to which we now turn. To explain it, we need to establish the notation and concepts used there. For our purposes, we will be using a manifold with a metric tensor, g_{ab}, and inverse metric tensor, g^{ab}, defined on it, and assume that it is torsion-free so that the connection symbol is the Christoffel symbol in a coordinated basis (see Chaps. 2 and 3 of [14]),

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{bd,c} + g_{cd,b} - g_{bc,d}) , \tag{13}$$

where "$_{,c}$" stands for the partial derivative relative to x^c. This object comes from the differentiation of the basis vectors relating the tensor quantity in the manifold to its components in the coordinate system chosen. As such this is *not* a tensor quantity or a fully coordinate quantity but hangs between the two. The *covariant derivative* of a contravariant vector, V^a, is $V^a_{;b} = V^a_{,b} + \Gamma^a_{bc} V^c$ and of a covariant vector W_a is $W_{a;b} = W_{a,b} - W_c \Gamma^c_{ab}$. The difference of the second derivative obtained by going first in one direction and then in another, or vice versa, gives a measure of the curvature of the space, measured by the *Riemann–Christoffel curvature tensor*

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd} \Gamma^a_{ce} - \Gamma^e_{bc} \Gamma^a_{de} . \tag{14}$$

The trace of the curvature tensor is the Ricci tensor, $R_{bd} = R^a{}_{bad}$, whose trace $R = g^{bd} R_{bd}$ is the Ricci scalar. The Ricci tensor is symmetric and the curvature tensor is symmetric in the first and second pairs of indices and symmetric under the interchange of the two pairs of indices. Further, the skew part for any three of the indices is zero.

This reduces the number of linearly independent components. In m-dimensions, for $R^{a}{}_{bcd}$, there are $m^2(m^2-1)$, for R_{bd}, there are $m(m+1)$, and for R obviously just one.

As in flat spaces, so in curved spaces the "straightest" available path between two points is also the shortest. Such curves are called *geodesics* and satisfy the *geodesic equation*

$$y^{i\,\prime\prime} + \Gamma^{i}_{jk} y^{j\,\prime} y^{k\,\prime} = 0\,. \tag{15}$$

The above procedure relies on the differentiation of the basis vector, so how can it tell us about the curvature of the manifold? The point is that we define the vector on the manifold as a mapping of a point to a nearby point on the manifold, without reference to the coordinate system. Now we map the vector to the coordinate system and differentiate there using the covariant derivative as explained above and then map the quantity back to the manifold. This defines the derivative operator on the manifold without the coordinate system being involved, though an arbitrary coordinate system was used to be able to get the definition. The covariant and partial derivatives are identical for scalars and only differ for vectors or tensors. One is still left with the differentiation of the basis vector being "carried back" to the manifold. To eliminate this extra term, we can use one vector to move another on the manifold, which is called the *Lie derivative* of the vector moved, say $\mathbf{p}$ relative to the one along which it is moved, say $\mathbf{t}$. Going back to the covariant derivative, this is $[\mathscr{L}_{\mathbf{t}}\mathbf{p}]^a = t^b p^a_{;b} - p^b t^a_{;b}$. Since the Christoffel symbol is symmetric in the lower indices the two Christoffel symbols cancel and only the partial derivatives, $t^b p^a_{,b} - p^b t^a_{,b}$, are left in the expression. What has happened is that the derivatives of the basis vectors have cancelled out and so the worrying term is no longer present in the Lie derivative. We can transport one vector along the geodesic given by the other on the manifold by using the Taylor series, to obtain the other vector at the new point. Thus, if the geodesic lies between points P and Q on the manifold, and $\mathbf{t} : P \to Q$, then $\mathbf{p}|_Q$ is given in terms of $\mathbf{p}|_P$ by

$$\mathbf{p}|_Q = \exp[\mathscr{L}_{\mathbf{t}}]\mathbf{p}|_P\,. \tag{16}$$

Requiring that $\mathbf{p}$ be left invariant as it goes along $\mathbf{t}$, amounts to requiring that $\mathscr{L}_{\mathbf{t}}\mathbf{p}$ be zero. Consequently, the metric tensor, and hence the geometry, will be left invariant under Lie transport will if the Lie derivative of the metric tensor is zero. Such a direction is called an *isometry*, and is a generator of the symmetry implicit in the geometry.

Using the above definitions, in component form the equations for an isometric direction, k^a become,

$$g_{ab,c}k^c = g_{ac}k^c{}_{,b} + g_{bc}k^c{}_{,a}\,. \tag{17}$$

These are called *Killing's equations* and a vector satisfying this equation is called a *Killing vector*, or an isometry. Notice that (15) depends on the metric coefficients, which are functions of the dependent variable but do not depend explicitly on the independent variable. Thus the geodesic equations possess a translational symmetry along the independent variable.

It was noted by Aminova and Aminov [15], and independently, but later, by [16] that there is a direct connection between the symmetries of a system of geodesic equations and the isometries of the underlying manifold on which the solutions live. Aminova and Aminov further noted that projecting the geodesic equations down along x, one obtains a cubically semilinear system of second-order ODEs. We independently arrived at the same point [17]. We further noted that these are similar in the case of scalar ODE to the original Lie linearizable ODE. The projected equations then have the coefficients given as functions of the Christoffel symbols for the higher dimensional system. It turns out that the Lie conditions correspond exactly to the requirement that the curvature tensor constructed from those Christoffel symbols be zero, i.e., *the space is flat*. Further, there is a redundancy in the Christoffel symbols due to the freedom of choice of coordinates. When we project from two down to one-dimensional systems (i.e., the scalar equation) the redundancy is of two. These are the two arbitrary functions that Lie needed for his equations. As such, we have obtained the Lie linearization purely from Geometry. *This* is the sense in which Geometry explains what is special about order two. The requirement of flatness is natural as the shortest path between two points in a flat space is a straight line. We know the metric tensor in Cartesian coordinates and the equation of the straight line. It is possible to find the coordinate transformation that converts a flat metric locally to any given metric [18]. The coordinate transformation to get the metric tensor required to give the coefficients yields the linearizing transformation and one obtains the solution of the linearizable system in terms of the the original variables as a nonlinear superposition for the general solution. A more complete analysis of this linearization followed later [19]. A code was developed to determine if the system of second-order ODEs corresponds to a system of geodesics, and if it does to determine the metric tensor [20]. Put together, we could, in principle, feed in any system of the Lie type, check if it is a projected system of geodesics and then obtain the solution. It is this power of Geometry that we use at much of the base for the complex methods.

3 Complex Symmetry Analysis

The maximal Lie algebra for m-dimensional system of second-order ODEs is $sl(m+2)$, which is $sl(m+2, \mathbb{R})$ for real and $sl(m+2, \mathbb{C})$ for complex variables, with $[(m+2)^2 - 1]$ real or complex generators. Now, to obtain ODEs after splitting, the independent variable must be real and the dependent variables complex. In that case, to each generator containing the dependent variables in the complex ODE, there will be two after splitting. Thus, for $m = 1$ there should be 8 generators in the real case and 16 in the complex. However, splitting the scalar ODE into its real and imaginary parts yields a two-dimensional real system, which has an $sl(4, \mathbb{R})$ with 15 generators. This demonstrates that going from the real to the complex is non-trivial. The complication arises due to the fact that the complex ODE is not just the two-dimensional real system written explicitly but also the implicit Cauchy–Riemann conditions, which are two first-order equations. Thus the complex system regarded as a higher dimensional real system is constrained. This causes the reduction of symmetry of the complex system. Complex symmetry analysis follows up on the non-trivial consequences emerging from the constraints [21, 22].

To be more concrete, if the real independent variable is x and the complex dependent variable for a scalar ODE is $w = (u + \iota v)$, the complex translation operator, $\mathbf{W}_1$ splits as

$$\mathbf{W}_1 = \partial/\partial w = \frac{1}{2}(\partial/\partial u - \iota\partial/\partial v) = \mathbf{U}_1 + \iota\mathbf{V}_1 \ , \tag{18}$$

so that $\mathbf{W}_1 w = 1$ and $\mathbf{W}_1 \overline{w} = 0$. Hence the complex translation splits into two real orthogonal translations. Now there are no rotations for a single complex variable but there is a scaling symmetry $\mathbf{W}_2$,

$$\mathbf{W}_2 = w\partial/\partial w = \frac{1}{2}[(u\partial/\partial u + v\partial/\partial v) + \iota(v\partial/\partial u - u\partial/\partial v)] = \mathbf{U}_2 + \iota\mathbf{V}_2 \ . \tag{19}$$

Thus we get a real scaling, $\mathbf{U}_2$ *and* a rotation in two-dimensional, $\mathbf{V}_2$. In the context of our focus on "why complex methods are so effective" notice that, by definition, translations leave vector lengths invariant while scalings change lengths. The odd feature is that the complex scaling yields a rotation under the splitting, apart from the expected scaling. How did the complex scaling "know" that a real rotation was needed and had to be coded into the complex scaling?

The natural next step is to go to two complex dimensional systems, with the complex translation and rotation symmetry generators

$$\mathbf{W}_1 = \partial/\partial w_1 \ , \ \mathbf{W}_2 = \partial/\partial w_2 \ , \ \mathbf{R} = w_2\partial/\partial w_1 - w_1\partial/\partial w_2 \ , \tag{20}$$

which split into

$$\frac{1}{2}(\partial/\partial u_1 - \iota\partial/\partial v_1) = \mathbf{U}_1 - \iota\mathbf{V}_1 \ , \tag{21}$$

$$\frac{1}{2}(\partial/\partial u_2 - \iota\partial/\partial v_2) = \mathbf{U}_2 - \iota\mathbf{V}_2 \ , \tag{22}$$

$$\begin{aligned}&\frac{1}{2}[(u_2\partial/\partial u_1 - u_1\partial/\partial u_2) + (v_2\partial/\partial v_1 - v_1\partial/\partial v_2)\\ &+\iota\{(u_1\partial/\partial v_2 + v_1\partial/\partial u_2) - (u_2\partial/\partial v_1 + v_2\partial/\partial v_1)\}]\\ &= (\mathbf{R}_1 + \mathbf{R}_2) + \iota(\mathbf{L}_1 - \mathbf{L}_2) \ ,\end{aligned} \tag{23}$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the expected rotations and $\mathbf{L}_1$ and $\mathbf{L}_2$ are two "Lorentz transformations," i.e., rotations through an imaginary angle. Notice that the rotations were arbitrarily identified. Instead of rotating between two u's and two v's, we could have "mixed" them to get rotations between the u's and v's, or broken the two Lorentz transformations differently to obtain two more cross rotations. Is *this* at the heart of the "unexpected effectiveness of the complex methods?"

What has happened is that we actually have the sixteen "quasi-scalings," which we can write, using the notation $k_1 = u_1, k_2 = u_2, k_3 = v_1, k_4 = v_2$, as $k_i\partial/\partial k_j$. To see the full significance of this point, let us proceed to the m complex dimensional system split into $2m$ real variables, k_i. Write the $2m$ translations as $\mathbf{X}_i$ and the $4m^2$ quasi-scalings as $\mathbf{Y}_{ij}$. Then the Lie algebra satisfied by these generators is:

$$[\mathbf{X}_i, \mathbf{X}_j] = 0\ ,\ [\mathbf{X}_i, \mathbf{Y}_{jk}] = \delta_{ij}\mathbf{X}_k\ ,\ [\mathbf{Y}_{ij}, \mathbf{Y}_{kl}] = \delta_{ik}\mathbf{Y}_{jl} - \delta_{jl}\mathbf{Y}_{ik}\ . \tag{24}$$

Thus we have $2m$ translations, $\mathbf{X}_i$, $m(2m-1)$ rotations $\mathbf{R}_{ij}$ given by the commutators of the quasi-scalings, $2m$ genuine scalings $\mathbf{S}_i = k_i\partial/\partial k_i$ and the remaining $m(2m-1)$ are proper "partial scaling." Somehow the Cauchy–Riemann (CR) conditions constrain the symmetries so that the quasi-scalings provide no new generators, and we are left with just the geometrically expected symmetries. The question remains, "How do the CR conditions get rid of the extra symmetries?"

For application to DEs, the prolongation of the generators proceeds in the usual way and the CR conditions do not need any further prolongation. Symmetry methods are used by enumerating all possible algebras of a given dimension. One-dimensional algebras are not in general sufficient for "group methods" to work. As such, we need at least a two-dimensional algebra. In the simpler cases of lower order and lower-dimensional systems, there are few higher dimensional algebras available and the classification problem is easy. For lower dimensions one gets whole classes of possible ODEs associated with each algebra of the given dimension. Thus, for the scalar ODE for two-dimensional algebras, there are four possible algebras, each with an ODE associated with it. On splitting the complex scalar ODE there is a much richer structure as one gets a two-dimensional, three three-dimensional and three four-dimensional algebras, each with its associated class of systems of two ODEs.

For every complex scalar ODE there is a system of two ODEs. However, the converse is obviously not true. Consequently there must be some compatibility conditions that the system satisfies for the correspondence to hold. The way they are obtained is to take the general relevant order complex scalar ODE and split it. The general form of the system corresponding to the general complex ODE is, thereby, obtained. What general form? This depends on the class of systems that is to be converted. The symmetries of the complex scalar ODE and the corresponding real system have been shown to be inequivalent [23, 24]. The procedure can, equally well, be applied to systems of complex ODEs being converted to systems of real ODEs of twice the dimension. A serious problem arises of being able to apply complex methods to odd-dimensional systems. The method used was to introduce an algebraic constraint, but that changes the system. Another method that could be explored would be to adjoin a real DE to a complex system, but it would entail additional complications, and may not be workable or worthwhile.

4 Complex Linearization

Every first-order complex scalar ODE

$$y'(x) = \omega(x, y) , \tag{25}$$

is linearizable [1, 4], where $y = f + \iota g$ and $\omega = \phi + \iota\gamma$. Thus the system

$$f'(x) = \phi(x, f(x), g(x)) , \; g'(x) = \gamma(x, f(x), g(x)) , \tag{26}$$

is linearizable, provided the CR equations

$$\frac{\partial\phi}{\partial f} = \frac{\partial\gamma}{\partial g} , \; \frac{\partial\phi}{\partial g} = -\frac{\partial\gamma}{\partial f} , \tag{27}$$

hold. We see that not every two-dimensional system of first-order ODEs is linearizable, but only those that satisfy the CR-equations as the linearization constraint equations. Notice that here the role of the CR-equations as a pair of first-order integrability conditions is obvious.

We now come to the general semilinear second-order complex scalar ODE,

$$y''(x) = \omega(x, y; y') . \tag{28}$$

Since it is not true that all second-order ODEs are linearizable, when we split the ODE, the CR-equations do not give linearization conditions, but only compatibility conditions,

$$f''(x) = \phi(x, f, g; f', g') , \; g''(x) = \gamma(x, f, g; f', g') \tag{29}$$

$$\frac{\partial\phi}{\partial f} = \frac{\partial\gamma}{\partial g} , \; \frac{\partial\phi}{\partial g} = -\frac{\partial\gamma}{\partial f} , \; \frac{\partial\phi}{\partial f'} = \frac{\partial\gamma}{\partial g'} , \; \frac{\partial\phi}{\partial g'} = -\frac{\partial\gamma}{\partial f'} . \tag{30}$$

For linearization there are further requirements that must be met. Lie's linearizable scalar second-order ODE, given by (8), can be split into its real and imaginary parts, bearing in mind that the four coefficients are also complex. The resulting two-dimensional system must be of the form

$$\begin{aligned} f'' + (a_1 f'^3 - 3a_2 f'^2 g' - 3a_1 f' g'^2 + a_2 g'^3) \\ +(b_1 f'^2 - 2b_2 f' g' - b_1 g'^2) + (c_1 f' - c_2 g') + h_1 = 0, \\ g''(x) + (a_2 f'^3 + 3a_1 f'^2 g' - 3a_2 f' g'^2 - a_1 g'^3) \\ +(b_2 f'^2 + 2b_1 f' g' - b_2 g'^2) + (c_2 f' + c_1 g') + h_2 = 0, \end{aligned} \tag{31}$$

subject not only to the CR-equations (30) written out explicitly for ϕ and γ but also to the CR-equations for each of the four complex coefficients, a, b, c, h. This is a system of two second-order real ODEs involving eight real functions to bring the system into the Lie form, which must satisfy a system of four first-order constraints that ensure integrability. We call such a system "complex linearizable."

It was proved by Goringe and Leach [12] that for a system of two second-order ODEs linear with constant coefficients there are 7, 8 or 15 symmetries. When extended to the fully general case 5 and 6 generators were added [13]. It may be recalled that the geometrically linearizable system has a Lie algebra of $sl(m+2, \mathbb{R})$, which yields 15 generators for $m = 2$. As such, only in the maximal symmetry case can we use the power of geometry at present to directly obtain the linearizing transformation and hence the solution. What is more disturbing is that the geometrical arguments for linearization are somehow bypassed in general – we are getting straight lines in a curved space. *How can that be?* Could it be that the space corresponding to the system of geodesics for 5, 6, 7 and 8 generators is like a higher dimensional cylinder, with some flat sections? It would be worth exploring this possibility.

The general second-order linear complex scalar equation

$$y''(z) + A(z)y'(z) + B(z)y(z) = 0\,, \tag{32}$$

where z is a complex variable, can be transformed to the form,

$$y''(w) + \alpha(w)y(w) = 0\,, \tag{33}$$

by re-scaling the dependent variable by a position dependent function or, equivalently, by transforming the independent variable, z, appropriately to an independent variable, w, to get rid of the first derivative term. This can then be split to

$$f''(x) + \alpha_1(x)f(x) - \alpha_2(x)g(x) = 0\,,\ g''(x) + \alpha_2(x)f(x) + \alpha_1(x)g(x) = 0\,, \tag{34}$$

where $\alpha = \alpha_1 + \iota\alpha_2$. When this was applied to the free particle equation (with $\alpha = 0$) [23, 25], the 15 generator Lie algebra case was recovered, which is amenable to geometric linearization. For the constant and the variable cases, the 7 and 6 generator algebras were also obtained. Though the system is not geometrically linearizable, the complex equation is and hence its power can be used to solve the scalar ODE and then convert to the system to get the solution of the system.

It is wonderful that that two more of the five classes of linearizable systems can be accessed by complex linearization, making them amenable to the geometric procedure that more-or-less writes down the solution for us, but now the question arises: "where did the other two go?" If the complex method works, "why does it work partially and not fully?" The answer may lie in a step that was glossed over. The scalar ODE was first transformed with the complex independent variable to obtain the simpler form (33) and then it was restricted to the real form. This procedure will not commute in general. If the independent variable is first restricted and then used, the reduction will not occur. There seems to be no good reason to take the reduced form (33) instead of the complete homogeneous liner form. Throwing away the first derivative term in the system may "throw the baby out with the bath-water." Can one not apply the Lie linearization procedure to the full (homogeneous) linear form to obtain the two-dimensional system? Perhaps that would provide the missing cases of 5 and 8 symmetry generators.

Notice that the second-order complex scalar ODE has eight complex symmetries to be linearizable but needs only two to be solvable by Lie's method. On the other hand the two-dimensional system needs at least five real symmetries to be linearizable and four to be solvable. Thus the minimum number of real symmetries required in both cases is four. We see that starting with linearizable second-order complex scalar ODEs, we can end up with two-dimensional systems with fewer symmetries. Is it possible to get a system with only four symmetries that is solvable? In that case, by the easy linearization of a complex ODE the more complicated process of solving the associated system can be bypassed. It was found that this could be done [23, 25–27]. In fact, not only could non-linearizable systems be solved by linearization (of a complex scalar ODE) but one could go further and find complex linearizable scalar ODEs corresponding to systems with *less* symmetries. Thus systems *not solvable* by symmetry methods in the usual way, could be solved by complex linearization. How much lower can one go? It turned out that there is an example with *no* symmetry. We cite the examples of four, one and zero here:

(a) Four symmetry case

$$f'' - f'^{\,3} + 3f'g'^{\,2} = 0\,,\;\; g'' - 3f'^{\,2}g' + g'^{\,3} = 0\,, \tag{35}$$

with the solution

$$\begin{aligned} f(x) &= c_1 \pm (\sqrt{(a-x)^2+b^2} + a - x)^{1/2},\\ g(x) &= c_2 \pm (\sqrt{(a-x)^2+b^2} - a + x)^{1/2}; \end{aligned} \tag{36}$$

(b) One symmetry case

$$\begin{aligned} &f'' - xff'^{\,3} + 3xgf'^{\,2}g' + 3xff'g'^{\,2} - xgg'^{\,3} = 0\,,\\ &g'' - xgf'^{\,3} - 3xff'^{\,2}g' + 3xgf'g'^{\,2} + xfg'^{\,3} = 0\,, \end{aligned} \tag{37}$$

with the implicit solution

$$\begin{aligned} &\mathscr{R}[c_1 Ai(-f-\iota g) + c_2 Bi(-f-\iota g)] = x\,,\\ &\mathscr{I}[c_1 Ai(-f-\iota g) + c_2 Bi(-f-\iota g)] = 0\,, \end{aligned} \tag{38}$$

where $\mathscr{R}$, $\mathscr{I}$ are the real and imaginary parts of the arguments and Ai, Bi are the two Airy functions;

(c) No symmetry case

$$\begin{aligned} &f'' + (f^2 - g^2 - x^2)(f'^{\,3} - 3f'g'^{\,2}) - 2fg(3f'^{\,2}g' - g'^{\,3}) = 0\,,\\ &g'' + (f^2 - g^2 - x^2)(3f'^{\,2}g' - g'^{\,3}) + 2fg(f'^{\,3} - 3f'g'^{\,2}) = 0\,, \end{aligned} \tag{39}$$

which corresponds to the complex scalar ODE

$$y'' - xy^2 y'^3 = 0, \tag{40}$$

which is linearizable to $Y'' = 0$, yielding the solution directly.

5 Complex Noether Symmetries and Integrals

Noether's theorem [28] forms a basis of the use of symmetries in Mechanics and through it in all of Physics. It essentially generalizes Hamilton's principle of least action, which can be reformulated as saying that if there is time-translational invariance, energy will be conserved. The action, S, is a functional of a Lagrangian function, $\mathscr{L}[t, q^i(t), \dot{q}^i(t)]$, where q^i are the coordinates of a system of particles in the higher dimension and t is the time. If there is no explicit dependence on the time the action is minimized and a quantity associated with the Lagrangian, called the Hamiltonian, $\mathscr{H}$ is a conserved quantity. More generally, the theorem says that for every continuous symmetry, there is a conserved quantity. It was further generalized to extend to a continuum of "particles," i.e., a *field*, and thence to relativistic fields and further to quantum fields [29]. Hamilton's original method, used also by Noether, is to use the calculus of variations and require that the variation of the action be zero. This provides the necessary conditions for minimization. The sufficient condition, that the second variation be positive, is generally ignored or glossed over, but should be used to avoid getting spurious solutions.

Noether symmetries, as opposed to the usual symmetries, yield *double reduction* of the DEs for which they apply [30, 31], serving like two of the symmetries. Thus, if there is time translational invariance in an ODE (as for the time independent Schrödinger equation or the steady state heat equation) one can replace the derivative operator by a constant. Further, the energy conservation yields an invariant combination of the generalized coordinates and their derivatives, getting rid of another variable. Formally, $\mathbf{X}^{[1]}$, given by (3) for a single independent variable, is a *Noether symmetry* if there exists an appropriate (*gauge*) function, G, such that

$$\mathbf{X}^{[1]}\mathscr{L} + \mathscr{L}\frac{dA}{dx} = \frac{dG}{dx}, \tag{41}$$

where d/dx is the total derivative. This can be extended to PDEs by using several independent variables, x^a and the corresponding total derivatives with respect to each independent variable as well as introducing a vector gauge function, G^a.

An obvious problem of extending the variational principle to the complex domain arises: functionals map the space of functions into the *reals*, $\mathbb{R}$. Obviously the Lagrangian must be real for the action to be real, so that a minimum can be defined on it. This problem was "swept under the rug" at the time in [21, 32]. In defining distributions for complex arguments, the problem of defining functionals is addressed

[33–35] but not the problem of defining a minimum for a complex action. What is required is that the variations of the real and imaginary parts be separately zero and the minimum for both together requires that the magnitude of the action, $|S|$, be minimum. It is worth mentioning that on purely physical considerations Bender and Boettcher had also proposed complex Hamiltonians [36].

Let us now proceed with the complex Lagrangian [21, 32]. Let $\mathscr{L} = \mathscr{L}_1 + \iota\mathscr{L}_2$. Then the Euler–Lagrange equation splits into the pair of coupled equations:

$$\begin{aligned}
\frac{\partial\mathscr{L}_1}{\partial f} + \frac{\partial\mathscr{L}_2}{\partial g} - \frac{d}{dx}\Big(\frac{\partial\mathscr{L}_1}{\partial f'} + \frac{\partial\mathscr{L}_2}{\partial g'}\Big) = 0\ ,\\
\frac{\partial\mathscr{L}_2}{\partial f} - \frac{\partial\mathscr{L}_1}{\partial g} - \frac{d}{dx}\Big(\frac{\partial\mathscr{L}_2}{\partial f'} - \frac{\partial\mathscr{L}_1}{\partial g'}\Big) = 0\ ,
\end{aligned} \tag{42}$$

which is not a pair of Euler–Lagrange equations. These were called "Euler–Lagrange-like" equations, but perhaps a better name would have been "*complex-EL*" equations.

The Noether operators, $\mathbf{X}_1^{[1]}$, $\mathbf{X}_2^{[1]}$ corresponding to the Lagrangians $\mathscr{L}_1$, $\mathscr{L}_2$

$$\begin{aligned}
\mathbf{X}_1^{[1]} &= \xi_1\partial_x + \frac{1}{2}(\eta_1\partial_f + \eta_2\partial_g + \eta_1'\partial_{f'} + \eta_2'\partial_{g'})\ ,\\
\mathbf{X}_2^{[1]} &= \xi_2\partial_x + \frac{1}{2}(\eta_2\partial_f - \eta_1\partial_g + \eta_2'\partial_{f'} - \eta_1'\partial_{f'})\ ,
\end{aligned} \tag{43}$$

must satisfy the equation

$$\begin{aligned}
\mathbf{X_1}^{[1]}\mathscr{L}_1 - \mathbf{X_2}^{[1]}\mathscr{L}_2 + (D\xi_1)\mathscr{L}_1 - (D\xi_2)\mathscr{L}_2 = DG^1\ ,\\
\mathbf{X_1}^{[1]}\mathscr{L}_2 + \mathbf{X_2}^{[1]}\mathscr{L}_1 + (D\xi_1)\mathscr{L}_2 + (D\xi_2)\mathscr{L}_1 = DG^2\ ,
\end{aligned} \tag{44}$$

for some gauge functions G^1, G^2, where $D = d/dx$. It might seem that the arbitrariness of the gauge functions allows infinitely many solutions and hence the "must satisfy" says nothing. This is not the case. In the scalar case one is requiring that the left side of the equation be an exact differential. For the coupled system, one is demanding that both left sides be total differentials, albeit of different "potentials." The resulting invariants are:

$$\begin{aligned}
I_1 &= \xi\mathscr{L}_1 - \xi_2\mathscr{L}_2 + \frac{1}{2}(\eta_1 - f'\xi_1 + g'\xi_2)\Big(\frac{\partial\mathscr{L}_1}{\partial f'} + \frac{\mathscr{L}_2}{\partial g'}\Big)\\
&\quad - \frac{1}{2}(\eta_2 - f'\xi_2 - g'\xi_1)\Big(\frac{\partial\mathscr{L}_2}{\partial f'} - \frac{\mathscr{L}_1}{\partial g'}\Big) - B_1\ ,\\
I_2 &= \xi\mathscr{L}_2 + \xi_2\mathscr{L}_1 + \frac{1}{2}(\eta_1 - f'\xi_1 + g'\xi_2)\Big(\frac{\partial\mathscr{L}_2}{\partial f'} - \frac{\mathscr{L}_1}{\partial g'}\Big)\\
&\quad + \frac{1}{2}(\eta_2 - f'\xi_2 - g'\xi_1)\Big(\frac{\partial\mathscr{L}_1}{\partial f'} + \frac{\mathscr{L}_2}{\partial g'}\Big) - B_2\ .
\end{aligned} \tag{45}$$

The invariants of complex scalar second-order ODEs are often easier to obtain than those of a two-dimensional real system [37]. The question arises, as with complex linearization so with invariants, are they found for systems that could not be obtained for the two-dimensional system? With the question in mind of why the complex method is providing results that the real system did not, it is necessary to pursue the matter further.

In the simplest case, $y'' = xy'$, the complex method merely reproduces the results for the real system, albeit more simply. In the case of the complex simple harmonic oscillator it correctly gives a coupled system of harmonic oscillators [38] and provides the expression for the energy transferring back and forth between the two. As was put there, *one sees the energy in the field by putting on complex glasses*. It is found that new invariants arise for the complex Lagrangian in some cases. Unfortunately it was expressed in [37] in a way that misleadingly suggests that there are two Lagrangians for real two-dimensional systems arising from a variational principle. We present an example.

Example Consider the system of two second-order semi-linear ODEs

$$\begin{aligned} f'' + 3ff' - 3gg' + f^3 - 3fg^2 = 0\,, \\ g'' + 3fg' + 3gf' + 3f^2g - g^3 = 0\,. \end{aligned} \tag{46}$$

It is not clear that it has any Lagrangian. If there is no ordinary conservation law arising from a variational principle, one can still get a conserved quantity (the generalization of the Hamiltonian) from what are called *partial Lagrangians* [39] or there may be nothing like a Lagrangian. It would be worth exploring which of the alternatives applies in this example and in general. This system corresponds to the complex ODE

$$y'' + 3yy' + y^3 = 0\,, \tag{47}$$

which has five infinitesimal Noether symmetry generators and the corresponding five invariants which split into ten real invariants for the system. It is noted in [37] that the two parts of the Lagrangian are equivalent Lagrangians for the system, which yields only one invariant. It would be interesting to explore if this spew of invariants is related to the spew of infinitesimal generators spawned by the split translation generator of the complex line.

6 Iterative Splitting of a Complex Scalar ODE

The idea of iterative splitting [40] is a strange one: (a) start with a complex (say) scalar ODE and obtain the split two-dimensional system of ODEs; (b) now get hit on the head and develop amnesia, so you forget where the split system came from and use the splitting procedure on it to get a four-dimensional system of ODEs; or (c) get a four-dimensional system of PDEs if you forgot that you intended to restrict yourself to ODEs. Why is the idea strange? To see this we have to get into what was not discussed before: the range of functions to which the procedures are applicable.

When we proceed for the splitting, we assume that the dependent variables are *complex analytic* functions and that the functions in the split system are *real analytic* functions. Now the ratio of the cardinality of the set of all complex analytic functions to the set of all complex functions is a second infinitesimal. Similarly, for the real analytic to all real functions. However, it does not appear that in going complex we have restricted our space "any more" than we have done for the real. In fact one feels that we have somehow made it "more general." This vague feeling lulls us into a false sense of security, as we see when we require the CR-equations. When we repeat the step of splitting we have required that the two dependent variables be *complex analytic functions themselves*. This obviously significantly restricts the space of permissible functions after the split. This will appear in the emergence of a second set of CR-equations. Obviously, there will be infinitely many functions that satisfy the requirement but the restriction on the space of permissible functions will make a big difference for what can be used in DEs. In our amnesia we have wandered into a cave with a narrow opening containing a magic lamp. To get out of the cave we may have to leave our magic lamp of splitting behind. The trick will be to bring a more constrained genie out without the lamp. Let us be more concrete. For complex symmetry analysis for ODEs, we need that f and g be n times differentiable functions of x and w for a complex analytic function $(f, g, f', g', \ldots, f^{(n)}, g^{(n)})$ and for PDEs that $y = f + \iota g$ be a complex analytic function of $z = u + \iota v$ and w be a complex analytic function of $(z, y, y', \ldots, y^{(n)})$.

One might have thought of generalizing the complexification of the DE to the quaternions, $q = 1 + ai + bj + ck$, subject to the requirements that $i^2 = j^2 = k^2 = -1$ and $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$. It is easily verified that the requirements that $dq/dq = 1$ and $dq^2/dq = 2q$ are incompatible. Thus we cannot bound up the steps from one to four in a single leap and need to look elsewhere for a generalization. Why generalize? Apart from the search for simpler ways to get more powerful results, one wants to obtain insights into the working of the first step by going beyond. As mentioned earlier, the idea is to complexify twice over. In view of the important role of the CR-equations for double splitting, it is worthwhile to state them explicitly for the first splitting. For the initial complex scalar ODE (28), taking $z = x$ and proceeding with the split $y = f + \iota g$, we obtain a pair of ODEs as given before. Now we must also write

$$\omega(x; f, g; f', g') = w^r(x; f, g; f', g') + \iota w^i(x; f, g; f', g') \,. \tag{48}$$

Then the CR-equations are:

$$w^r_f = w^i_g \,, w^r_g = -w^i_f \,; w^r_{f'} = f^i_{g'} \,; w^r_{g'} = -w^i_{f'} \,. \tag{49}$$

It is easier to obtain four-dimensional systems of ODEs or PDEs by double splitting than a three-dimensional system because the number of equations would naturally be even. One can retain one of the functions of the split to be real and the other to be complex, so as to get the desired three-dimensional system, but the number of functions still remains even. To circumvent this problem, retain f as it is but split $g = h + \iota k$ in (28) and take all real terms that do not contain g or its derivative in one term and the rest in a second, complex, term

$$\omega(x, y, y') = w(x; f; f') + W(x; f, g; f', g') . \tag{50}$$

Now split W to write

$$W(x; f, g; f', g') = U(x; f, h, k; f', h', k') + \iota\, V(x; f, h, k; f', h', k') , \tag{51}$$

so that we obtain the three-dimensional second-order system:

$$\begin{aligned} f'' &= w(x; f; f') , \\ h'' &= U(x; f, h, k; f', h', k') , \\ k'' &= V(x; f, h, k; f', h', k') , \end{aligned} \tag{52}$$

subject to the CR-equations

$$U_h = V_k \ , \ U_k = -V_h \ ; \ U_{h'} = V_{k'} \ , \ U_{k'} = -V_{h'} . \tag{53}$$

Notice that the system of three coupled ODEs does not seem very general, as the first of (53) is independent of the other two dependent variables. However, it is not entirely clear how much of a restriction this is. We could try to take linear combinations of the three dependent variables so that in one equation we eliminate the other two. The problem is reminiscent of finding the Jordan canonical form and may need the symmetry structure of the system to be examined for the purpose. Incidentally, the second split given in [40] causes confusion by using ιW instead of W in (50) but is entirely equivalent to the one presented here.

We present an illustrative example here:

Example The system of generalized Emden–Fowler ODEs:

$$f'' = -2x^{-5}hk \ , \ h'' = -2sx^{-5}fk \ , \ k'' = 2x^{-5}fh \ , \tag{54}$$

corresponds to the completely integrable [41] scalar Emden–Fowler ODE

$$y'' = x^{-5}y^2 \ , \tag{55}$$

subject to the algebraic constraint

$$f^2 + h^2 = k^2 \,, \tag{56}$$

which has the symmetry generators

$$\mathbf{X}_1 = x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} \,, \; \mathbf{X}_2 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \tag{57}$$

and is completely integrable. Double-splitting these symmetry generators yields eight Lie-like operators and *no* symmetries of the system. These are

$$Y_1 = x\partial_x + \frac{3}{2}f\partial_f + \frac{3}{4}h\partial_h + \frac{3}{4}k\partial_k \,, \; Y_2 = \frac{3}{4}k\partial_h - \frac{3}{4}h\partial_k \,,$$

$$Y_3 = \frac{3}{2}k\partial_f + \frac{3}{4}f\partial_k \,, \; Y_4 = \frac{3}{2}h\partial_f - \frac{3}{4}f\partial_h,$$

$$Y_5 = x^2\partial_x + \frac{1}{2}xf\partial_f + \frac{1}{4}xh\partial_h + \frac{1}{4}xk\partial_k \,, \; Y_6 = \frac{1}{4}xk\partial_h - \frac{1}{4}xh\partial_k \,,$$

$$Y_7 = \frac{1}{2}xk\partial_f + \frac{1}{4}xf\partial_k \,, \; Y_8 = \frac{1}{2}xh\partial_f - \frac{1}{4}xf\partial_h \,.$$

Our system (54) is completely integrable despite having too few symmetries. Cases with no symmetry were already seen above [26], but the first example of a two-dimensional integrable system with no symmetry was given in [42]. Here we have an integrable system of *three* ODEs with only two symmetries. The first integral of (55) is given by

$$I = \frac{1}{2}x^2y'^2 + \frac{1}{2}y^2 - \frac{1}{3}x^{-3}y^3 - xyy' \,,$$

which has the symmetry X_2 [43]. The invariant obtained from it, $v = y/x$ reduces the equation to a simple quadrature as

$$x^2v' = \pm\sqrt{c + \frac{2}{3}v^3},$$

yielding the solution of (54). It is worth noting that the Lie-like operators have proliferated on double splitting and are likely to increase still more for further splits. The Lie symmetries seem lost in the abundance of Lie-like operators.

Put $f(x) = k(x) + \iota\, l(x)$ and $g(x) = m(x) + \iota\, n(x)$ in (48) to obtain the four-dimensional system of ODEs by double-splitting,

$$\begin{aligned} w^r(x; f, g; f', g') &= u^r(x; \mathbf{k}; \mathbf{k}') + \iota v^r(x; \mathbf{k}; \mathbf{k}') \,, \\ w^i(x; f, g; f', g') &= u^i(x; \mathbf{k}; \mathbf{k}') + \iota v^i(x; \mathbf{k}; \mathbf{k}') \,, \end{aligned} \tag{58}$$

where $\mathbf{k} := (k, l, m, n)$, yielding the system of four ODEs

$$k''(x) = u^r(x; \mathbf{k}; \mathbf{k}') \ , \ l''(x) = u^i(x; \mathbf{k}; \mathbf{k}') \ , \\ m''(x) = v^r(x; \mathbf{k}; \mathbf{k}') \ , \ n''(x) = v^i(x; \mathbf{k}; \mathbf{k}') \ , \tag{59}$$

subject to the CR-conditions

$$u^r_k + v^r_l = u^i_m + v^i_n \ , \ u^r_l - v^r_k = u^i_n - v^i_m \ , \\ u^r_m + v^r_n = -u^i_n - v^i_l \ , \ u^r_n - v^r_m = -u^i_l + v^i_k \ , \\ u^r_{k'} + v^r_{l'} = u^i_{m'} + v^i_{n'} \ , \ u^r_{l'} - v^r_{k'} = u^i_{n'} - v^i_{m'} \ , \\ u^r_{m'} + v^r_{n'} = -u^i_{k'} - v^i_{l'} \ , \ u^r_{n'} - v^r_{m'} = -u^i_{l'} + v^i_{n'} \ . \tag{60}$$

The prolonged symmetry generator can now be written as

$$\mathbf{X} = \xi(x, \mathbf{k})\frac{\partial}{\partial x} + \underline{\eta}(x, \mathbf{k}).\nabla_{\mathbf{k}} + \underline{\eta}^{[1]}(x; \mathbf{k}, \mathbf{k}').\nabla_{\mathbf{k}'} \ . \tag{61}$$

Writing this equation out in detail makes it too unwieldy to convey much wisdom.

Let us now come to the system of four PDEs. This is the most straightforward of the various possibilities considered. At the first step we regard both the independent and the dependent variables of (28) as complex, so that we take z instead of x there and do the usual split with $z = s + \iota t$, so that *both the independent and dependent variables are split*. This gives a system of two second-order PDEs for two functions of two variables. This is the standard complex symmetry analysis talked of earlier for PDEs. The double split repeats the process and yields a system of four PDEs of four variables. Due to the number of variables involved in the double split it becomes impossible to follow our notation above here and we copy the equations as given in [40], including the CR-conditions and the prolonged generator.

$$\begin{aligned} & w_{ss} - w_{tt} + 2x_{st} - w_{uu} + w_{vv} - 2x_{uv} + 2y_{su} - 2y_{tv} \\ & +2z_{sv} + 2z_{tv} = 4g(\mathbf{s}; \mathbf{w}, \nabla_{\mathbf{s}}\mathbf{w}); \\ & x_{ss} - x_{tt} - 2w_{st} - x_{uu} + x_{vv} + 2w_{uv} + 2z_{su} - 2z_{tv} \\ & -2y_{sv} - 2y_{tv} = 4h(\mathbf{s}; \mathbf{w}, \nabla_{\mathbf{s}}\mathbf{w}); \\ & y_{ss} - y_{tt} + 2z_{st} - y_{uu} + y_{vv} - 2z_{uv} + 2w_{su} - 2w_{tv} \\ & +2x_{sv} + 2x_{tv} = 4k(\mathbf{s}; \mathbf{w}, \nabla_{\mathbf{s}}\mathbf{w}); \\ & z_{ss} - z_{tt} - 2y_{st} - z_{uu} + z_{vv} + 2y_{uv} + 2x_{su} - 2x_{tv} \\ & -2w_{sv} - 2w_{tv} = 4l(\mathbf{s}; \mathbf{w}, \nabla_{\mathbf{s}}\mathbf{w}); \end{aligned} \tag{62}$$

subject to the CR-conditions

$$
\begin{aligned}
w_s + x_t &= y_u + z_v \ , \ w_t - x_s = y_v - z_u \ , \\
w_u + x_v &= -y_s - z_t \ , \ w_v - x_u = -y_t + z_s \ ; \\
g_s + h_t &= k_u + l_v \ , \ g_t - h_s = k_v - l_u \ , \\
g_u + h_v &= -k_s - l_t \ , \ g_v - h_u = -k_t + l_s \ ; \\
g_w + h_x &= k_y + l_z \ , \ g_x - h_w = k_z - l_y \ , \\
g_y + h_z &= -k_w - l_x \ , \ g_z - h_y = -k_x + l_w \ .
\end{aligned} \tag{63}
$$

The derivatives in the rest of the CR-conditions can be written in more familiar form using the variables

$$
\begin{aligned}
\alpha &= w_s + x_t + y_u + z_v \ , \ \beta = w_t - x_s + y_v - z_u \ ; \\
\gamma &= w_u + x_v - y_s - z_t \ , \ \delta = w_v - x_u - y_t + z_s \ ,
\end{aligned} \tag{64}
$$

so that the rest of the CR-conditions are

$$
\begin{aligned}
g_\alpha - h_\beta &= k_\gamma - l_\delta \ , \ g_\beta + h_\alpha = k_\delta + l_\gamma \ ; \\
g_\gamma - h_\delta &= -k_\alpha + l_\beta \ , \ g_\delta + h_\gamma = -k_\beta - l_\alpha \ .
\end{aligned} \tag{65}
$$

The prolonged symmetry generator for the system is

$$
\mathbf{X}^{[1]} = \underline{\xi}(\mathbf{s}, \mathbf{g}).\nabla_{\mathbf{s}} + \underline{\eta}(\mathbf{s}, \mathbf{g}).\nabla_{\mathbf{g}} + \underline{\eta}^{[\mathbf{1}]}(\mathbf{s}, \mathbf{g}, \nabla_{\mathbf{s}}\mathbf{g}).\nabla_{\nabla_{\mathbf{s}}\mathbf{g}} \ . \tag{66}
$$

We again rely on an example to illustrate our system.

Example The free-particle system of equations is given by (62), with the right side set equal to zero. The CR-conditions are trivial. There are now 32 Lie-like operators, of which only 24 are symmetry generators. As before the local projective symmetries (eight in all) are lost, but the dilations are *not* lost. The generic problem of PDEs of an infinity of symmetry generators, persists but the 24 symmetries *do* form a Lie algebra, $\mathscr{A}_{24}$ say, which serves as a "core" for the system of PDEs, in that one could write the full Lie algebra $\mathscr{A} = \mathscr{A}_\infty \bigoplus \mathscr{A}_{24}$ and "throw away" the infinite dimensional algebra $\mathscr{A}_\infty$, to be left with a solution with 24 arbitrary constants.

Notice that for the three-dimensional system of ODEs, one could reverse the order of taking the split into two and one, to get a "dual" system. This would not be so simple for the split into a four-dimensional system, and there would be no obvious "dual." One could do the split first and then two "singles" after that; a "single," a split and a "single"; or two "singles" and a split. All would yield four ODEs for functions of one variable, but they would all be different. The first and third would, in some sense, be "duals." The same applies for the PDEs. In fact, the complex method for ODEs is not unique. Instead of first restricting the complex independent variable to be real, we could have first split and then restricted. The results would not be the same.

7 Discussion and Conclusion

In this chapter we reviewed the developments in Lie symmetry analysis that made explicit use of the complex analyticity of the solutions of complex differential equations. It might be recalled that Lie, himself, had assumed that the functions were complex analytic, but had not made the requirement explicit. So long as one remains entirely in the complex domain nothing new *can* emerge from the discussion. It is only when one splits the dependent and independent variables into their real and imaginary parts that the new features arise. In that case, a complex scalar ODE yields a pair of PDEs for two real functions of two real variables. As could be expected, the really new features arise when the independent variable is restricted to the real domain, which is needed to obtain ODEs. The areas where we particularly explored the consequences of the complex methods were linearization and Noether invariants. This is not to say that there are no consequences for more general situations, or that they would turn out to be less interesting, novel or useful. It is simply that these were the easiest to tackle, and hence provided a quick check on whether anything new would arise. In fact, there is reason to expect, as we shall discuss shortly, that the more general cases will lead to even more unexpected results. After all, if the complex linearizable ODE leads to the solution of ODEs not amenable to solution by symmetry methods, how many more may become solvable if the complex ODE is solvable, even if it is not linearizable?

In the applications to linearization, we discussed only the complex *scalar* ODE split to get a pair of real ODEs. By geometric methods one obtains the maximum symmetry case of linearizable systems. Using complex methods, two more of the five classes were accessed. Here, we have indicated that it should be possible to access the remaining two classes by *not* using the optimal canonical forms for the complex scalar ODE, as the restriction to the real independent variable (required for obtaining ODEs rather than PDEs by splitting) does not commute with the splitting procedure. This would be worth pursuing. However, the entire discussion is limited to a two-dimensional system. For higher (even) dimensional systems, we can split a higher, say m, dimensional complex system to a $2m$-dimensional system. This has been done for a two-dimensional system split to a four-dimensional system in [44]. It would be important to investigate if all the linearizable classes for the four-dimensional system are obtainable by complex methods, using the same point of avoiding the use of the optimal canonical form. More generally, if the m-d system split to the $2m$-d system covers all linearizable classes of the $2m$-d system. Further, one would need to see whether the double splitting of a scalar ODE yields the same results as the single split of the two-dimensional system. If not, is one of them more restrictive than the other, or is it that both methods give different extensions with some overlap? For odd dimensional systems, we have seen that one can appeal to iterative methods. However, it is worth exploring if direct algebraic constraints could also provide the desired linearization. Again, it would be fascinating to look for the connection between the two methods of enlarging the systems to which it is applied.

Complex methods have also been applied to the linearization of scalar third-order ODEs to deal with a two-dimensional system of third-order ODEs [45, 46] and a classification of two-dimensional linearizable systems of third-order ODEs has been obtained. This is a much harder problem as the ODEs and systems are not apparently connected to geodesic equations and hence to the geometrical methods. A method was developed to reduce the order of the ODE by defining a derivative of the dependent variable as a new dependent variable [47–49], thereby providing a possible connection with a system of geodesic equations, provided it satisfies the required criteria. Going one step further, one could ask if the second-order two-dimensional system could be obtained from a complex scalar ODE, so that the third-order ODE could be treated related to a second-order scalar ODE that could correspond to a geodesic equation? It is by no means clear that this could be done, but it seems very interesting to pursue this line of inquiry further. The procedure mentioned here was also used to reduce fourth-order ODEs to two-dimensional systems of second-order ODEs, albeit there is no classification for them. Of course, the above question would be as interesting for these equations as well. It was noted that the above procedure amounted to using contact transformations for third- or fourth-order ODEs and this provided the first classification for linearization by contact symmetries. It would be most interesting to see what would happen if one used complex methods for the contact transformations. The further ramifications involving iterative splitting may help provide insights into how the various methods, including contact and Lie–B*ä*cklund transformations are interconnected.

So far we have concentrated on reviewing the developments arising from complex methods that were useful but not really discussed the odd features, that turn up when we use the methods. This happens marginally in the first split, where the Lie operators are lost and what were called "Lie-like" which we called "complex-Lie," operators, replace them. Similarly for the Noether symmetries and integrals. This occurs more dramatically when iterative splitting is used. It is worthwhile to pursue this odd feature further. In this chapter we suggested that there may be a connection with the enormous proliferation of symmetry generators for the simple symmetries of the complex line: translation and scaling. It is worth pursuing precisely *how much* the proliferation is. Consider a complex scalar ODE, the translation splits into two and the rotation into four, giving a total of six. Now, at the second iteration, the two translations split into four and the four scaling-type operators split into sixteen, giving a total of twenty. In general, for n iterations we get the total number $N = 2^n + (2^n)^2 = 2^n(2^n + 1)$. Thus for $n = 3,\ N = 72$ and for $n = 4,\ N = 272$. Following the same logic, starting with an m-d complex system of ODEs, there will be initially m translations and m^2 scaling-type generators, which gives the general formula $N = 2^n m(2^n m + 1)$ after n iterations. Hence for a starting two-dimensional system, the effect is simply like increasing the number of iterations by one. Starting with $m = 3$ at two iterations, already $N = 156$. *This* is the extent to which the generators proliferate after n iterations. But the question then is "what difference does this proliferation make?" We will now consider the possibility that this may provide a clue to answer the big question that was mentioned at the start of this chapter: "*Why* does the complex method work?"

Take the case of linearization. Without the use of complex methods the geometric methods only give the *maximal* symmetry case. Complex methods have already provided two of the five linearizable classes for two-dimensional second-order systems and there is good reason to expect that the other two will also be found. To this extent, it is not all that strange that they work. After all, they only give what had been obtained by classical methods, albeit very much more explicitly. However, when complex linearization provides solutions for systems that are not linearizable, one really needs to explain how *that* could ever happen. Even stranger, where the number of symmetry generators of the system are inadequate to solve by symmetry methods, *how on Earth* can the complex methods work their magic there? The answer may lie in the much larger space of these "quasi-symmetry generators." One starts with the inadequately symmetric real system and "lifts" it as a lower-dimensional complex system, where the lifted equation is adequately symmetric and solves it there. However, the linearizing transformation that yields the solution changes the restricted real variable to a complex one, so that it cannot be used to linearize the system. So far it seems reasonable. The question now is, "why is it that there is at least one solution that is retained when one restricts the variable to be real? Why is it not that there is *no* solution for the real system?" An associated question is, "we have brought down one solution, but how do we know that we have not missed other solutions that could have been found?" For a $2m$-d real system of second-order ODEs, we need criteria that tell us precisely how many of the $2m$ complex solutions can be "brought down to the real world." No such criteria are available and they are crucial for using the complex methods to their full potential.

Now let us discuss the odd features of the complex variational principle. It has been noted that the real and imaginary parts of the complex Lagrangian are not the same but are equivalent, in that they satisfy the same Euler–Lagrange equations. Why should that be so? The final quantity that is minimized is the magnitude of the Lagrangian and not its real and imaginary parts. How is it that the two separately "know" that they must satisfy the same equations? Presumably it is because, apart from a constant value, the sum of the squares of the two parts has to become zero and that can only be if each is separately zero. However, the question then is "If they have to satisfy the same equations, why do they differ?" Again, the Noether invariants obtained from complex Lagrangians are many more than would have been expected at first sight. Essentially this must come from the tremendous enlargement of the quasi-symmetry operator space. The same point of obtaining many more invariants for the real system than should be possible, appears for the invariants. As such, the same questions as for linearization need to be answered. Again the explicit criteria are needed.

We have not discussed complex methods for PDEs. It is not that they cannot be used there. They *are* so used. They can be used to obtain systems of PDEs from ODEs. Nor is it that they are not useful. They pick out a "core" finite-dimensional symmetry algebra from an infinite dimensional Lie-algebra. The thing is that it is not so clear what relevance the PDEs would have for the big question asked here.

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