

# One-Dimensional Flows of a Polytropic Gas: Lie Group Classification, Conservation Laws, Invariant and Conservative Difference Schemes



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**Abstract** The chapter considers one-dimensional flows of a polytropic gas in the Lagrangian coordinates in three cases: plain one-dimensional flows, radially symmetric flows, and spherically symmetric flows. The one-dimensional flow of a polytropic gas is described by one second-order partial differential equation in the Lagrangian variables. The Lie group classification of this PDE is performed. Its variational structure allows to construct conservation laws with the help of Noether's theorem. These conservation laws are also recalculated for the gas dynamics variables in the Eulerian and mass Lagrangian coordinates. Additionally, invariant and conservative difference schemes are provided.

## 1 Introduction

Symmetries of the differential equations of mathematical physics are their fundamental features. They reflect the geometric structure of solutions and physical principles of the considered models. We recall that the Lie group symmetries yield a number of useful properties of differential equations (see [1–6]):

- A group action transforms the complete set of solutions into itself; so it is possible to obtain new solutions from a given one;

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- There exists a standard procedure to obtain the whole set of invariants of a symmetry group of transformations; it yields the forms of invariant solutions in which they could be found (symmetry reduction of PDEs);
- For ODEs the known symmetry yields the reduction of the order;
- The invariance of ODEs and PDEs is a necessary condition for the application of Noether's theorem to variational problems to obtain conservation laws (first integrals for ODEs).

The symmetry properties of the gas dynamics equations were studied both in the Eulerian coordinates [1, 7] and in the Lagrangian coordinates [8–10]. Extensive group analysis of the one-dimensional gas dynamics equations in the mass Lagrangian coordinates was given in [8–10]. Here, it should be also mentioned that nonlocal conservation laws of the one-dimensional gas dynamics equations in the mass Lagrangian coordinates were found in [11]. The authors of [12, 13] analyzed the Euler–Lagrange equations corresponding to the one-dimensional gas dynamics equations in the mass Lagrangian coordinates: extensions of the known conservation laws were derived. These conservation laws correspond to special forms of the entropy. The group nature of these conservation laws is given in the present chapter.

As mentioned above, besides assisting with the construction of exact solutions, the knowledge of an admitted Lie group allows one to derive conservation laws. Conservation laws provide information on the basic properties of solutions of differential equations. They are also needed in the analyses of stability and global behavior of solutions. Noether's theorem [14] is the tool that relates symmetries and conservation laws. However, an application of Noether's theorem depends on the following condition: the differential equations under consideration need to be presented as the Euler–Lagrange equations with an appropriate Lagrangian, i.e., Noether's theorem requires variational structure. There are also other approaches to find conservation laws, which try to avoid this requirement [15–18].

The application of symmetries to difference and discrete equations is a more recent field of research [19–22]. One of its directions is the discretization of differential equations with the preservation of the Lie point symmetries. It is relevant to the construction of numerical schemes which inherit qualitative properties of the underlying differential equations. This approach was a base for a series of publications [19, 23–27], which are summarized in the book [22]. The method is based on finding finite-difference invariants which correspond to the chosen mesh stencil and using them to construct invariant difference equations and meshes. Recently, this approach was applied to shallow water systems, wave equations, and the Green–Naghdi system [28–30].

The recent paper [31] was devoted to the Lie group classification, conservation laws, and invariant difference schemes of plain one-dimensional flows of a polytropic gas. Here, we extend these results to radially symmetric flows in two-dimensional space and spherically symmetric flows in three-dimensional space. We refer to all such flows as one-dimensional flows. The results of [31] stand as a particular case in this chapter.

There are two distinct ways to model phenomena in gas dynamics (see, e.g., [32–34]). The typical approach uses the Eulerian coordinates, where flow quantities (at each instant of time) are described in fixed points. Alternatively, the Lagrangian description is used: the particles are identified by the positions which they occupy at some initial time. In the Lagrangian description, there are also two ways to analyze the processes occurring in a gas. One of them uses a system of first-order PDEs for the gas dynamics variables. The other approach uses a scalar second-order PDE to which this system can be reduced. The latter way allows one to use variational approach for the analysis of the gas dynamics equations.

The purpose of the chapter is to present an overview of the authors’ results concerning the analysis of the gas dynamics equations of a polytopic gas. It is devoted to symmetries, conservation laws, and construction of numerical schemes, which preserve qualitative properties of the gas dynamics equations.

The article is organized as follows. In the forthcoming section, we recall Noether’s theorem. Section 3 describes the gas dynamics equations, their reduction to a single second-order PDE, and the Lie point symmetries of this PDE. In Sects. 4 and 5, we consider the general case and the three special cases of the Lie group classification. Invariance and conservative properties of difference schemes are discussed in Sect. 6. Finally, Sect. 7 presents concluding remarks.

## 2 Symmetries and Noether’s Theorem

We briefly remind Noether’s theorem [14], which will be used to find conservation laws with the help of symmetries. In the general case, we have several independent variables and dependent variables, which are denoted as  $x = (x^1, x^2, \dots, x^n)$  and  $u = (u^1, u^2, \dots, u^m)$ , respectively. All derivatives of order  $k$  are denoted as  $u_k$ .

A point symmetry operator has the form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^k \frac{\partial}{\partial u^k} + \eta_i^k \frac{\partial}{\partial u_i^k} + \dots, \tag{1}$$

where we assume that  $\xi^i = \xi^i(x, u)$ ,  $\eta^k = \eta^k(x, u)$ , and that the operator is prolonged to all derivatives  $u_{i_1 \dots i_l}^k$  we need to consider. We denote the considered function as  $F(x, u, u_1, \dots, u_k)$ . It involves derivatives up to some finite order  $k$ .

Noether’s theorem is based on the identity [2, 14]

$$XF + FD_i \xi^i = (\eta^k - \xi^i u_i^k) \frac{\delta F}{\delta u^k} + D_i(N^i F), \tag{2}$$

where

$$\frac{\delta}{\delta u^k} = \sum_{s=0}^{\infty} (-1)^s D_{i_1} \dots D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \dots i_s}^k}, \quad k = 1, 2, \dots, m, \tag{3}$$

are variational operators, and

$$N^i = \xi^i + \sum_{s=0}^{\infty} D_{i_1} \dots D_{i_s} (\eta^k - \xi^i u_i^k) \frac{\delta}{\delta u_{i_1 i_2 \dots i_s}^k}, \quad i = 1, 2, \dots, n. \quad (4)$$

The higher variational operators  $\frac{\delta}{\delta u_{i_1 i_2 \dots i_s}^k}$  are obtained from the variational operators (3) by replacing  $u^k$  with the corresponding derivatives  $u_{i_1 i_2 \dots i_s}^k$ .

**Theorem 1** (E. Noether) *Let the Lagrangian function  $L(x, u, u_1, \dots, u_k)$  satisfy equation*

$$XL + LD_i \xi^i = D_i B^i \quad (5)$$

with a vector  $\mathbf{B} = (B^1, B^2, \dots, B^n)$  and a group generator

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^k(x, u) \frac{\partial}{\partial u^k},$$

then the generator  $X$  is an admitted symmetry of the system of the Euler–Lagrange equations

$$\frac{\delta L}{\delta u^k} = 0, \quad k = 1, 2, \dots, m, \quad (6)$$

and the vector

$$(N^1 L - B^1, N^2 L - B^2, \dots, N^n L - B^n) \quad (7)$$

is a conserved vector.

In the case  $\mathbf{B} = (B^1, B^2, \dots, B^n) = \mathbf{0}$ , we call the symmetry  $X$  variational; otherwise, we say that the symmetry  $X$  is divergent.

It is well known that variational and divergent symmetries are also symmetries of the Euler–Lagrange equations [1–3]. For the Lie point symmetries, i.e., symmetries with coefficients  $\xi^i = \xi^i(x, u)$ ,  $\eta^k = \eta^k(x, u)$ , and first-order Lagrangians  $L = L(x, u, u_1)$ , it easily follows from the identities [35]

$$\frac{\delta}{\delta u^j} (XL + LD_i \xi^i - D_i B^i) = X \left( \frac{\delta L}{\delta u^j} \right) + \left( \frac{\partial \eta^k}{\partial u^j} - \frac{\partial \xi^i}{\partial u^j} u_i^k + \delta_{kj} D_i \xi^i \right) \frac{\delta L}{\delta u^k},$$

$$j = 1, 2, \dots, m,$$

where  $\delta_{kj}$  is the Kronecker symbol.

### 3 Equations of Gas Dynamics for One-Dimensional Flows

We consider three types of gas flows, namely, flows in one-dimensional space, radially symmetric flows in two-dimensional space, and spherically symmetric flows in three-dimensional space. We will refer to these flows as *one-dimensional* flows.

The gas is assumed to be polytropic [34, 36–38]. For a polytropic gas, the pressure  $p$  and the density  $\rho$  are related as

$$p = S\rho^\gamma, \quad (8)$$

where variable  $S$  is the function of the entropy  $\tilde{S}$

$$S = e^{(\tilde{S}-\tilde{S}_0)/c_v}, \quad c_v = \frac{R}{\gamma - 1}.$$

Here,  $R$  is the gas constant,  $c_v$  is the specific heat capacity at constant volume, and  $\tilde{S}_0$  is constant. The adiabatic constant is given as

$$\gamma = 1 + \frac{R}{c_v} > 1.$$

We will also need the equation of state for the polytropic gas, i.e., equation for the specific internal energy

$$\varepsilon = \varepsilon(\rho, p). \quad (9)$$

It has the form

$$\varepsilon = \frac{p}{(\gamma - 1)\rho}. \quad (10)$$

The gas dynamics equations will be given in the Eulerian and Lagrangian coordinates. Eventually, they will be reduced to one scalar PDE of the second order, which will be analyzed for the admitted Lie point symmetries.

#### 3.1 Eulerian Coordinates

In the Eulerian coordinates  $(t, r)$ , the gas dynamics equation can be written as (see, e.g., [32–34])

$$\rho_t + u\rho_r + \frac{\rho}{r^n}(r^n u)_r = 0, \quad (11a)$$

$$u_t + uu_r + \frac{1}{\rho}p_r = 0, \quad (11b)$$

$$S_t + uS_r = 0. \quad (11c)$$

Here, we distinguish the case  $n = 0$  with coordinate  $-\infty < r < \infty$  and velocity  $u$  from the cases  $n = 1, 2$  with radial distance from the origin  $0 < r < \infty$  and the radial velocity  $u$ .

We have  $n = 0, 1, 2$  for the plain one-dimensional flows, the radially symmetric two-dimensional flows, and the spherically symmetric three-dimensional flows, respectively. Note that for these cases  $n = d - 1$ , where  $d = 1, 2, 3$  is the space dimension.

We also use other representations of Eq. (11c)

$$p_t + up_r + \frac{\gamma p}{r^n} (r^n u)_r = 0 \quad (12)$$

or

$$\varepsilon_t + u\varepsilon_r + \frac{p}{r^n \rho} (r^n u)_r = 0. \quad (13)$$

### 3.2 Lagrangian Coordinates

As well known [32–34], the mass Lagrangian coordinate  $s$  and the Eulerian coordinate  $r$  of the particle  $s$  are related by the formulas

$$u = \varphi_t, \quad \rho = \frac{1}{\varphi^n \varphi_s}, \quad (14)$$

where  $r = \varphi(t, s)$  defines the motion of a particle  $s$ .

In the Eulerian coordinates  $(t, r)$ , we can introduce the mass Lagrangian coordinate  $s$  as a potential by the system

$$s_r = r^n \rho, \quad s_t = -r^n \rho u, \quad (15)$$

which is equivalent to the 1-form

$$ds = r^n \rho dr - r^n \rho u dt.$$

Here, we rely on the possibility to rewrite the Eq. (11a) as the conservation law

$$(r^n \rho)_t + (r^n \rho u)_r = 0,$$

representing conservation of mass.

In the mass Lagrangian coordinates  $(t, s)$ , which we will call the Lagrangian coordinates, the time derivative stands for the differentiation along the pathlines. It is called the material derivative. Total derivatives in the Lagrangian coordinates  $D_t^L$  and  $D_s$  are related to those in the Eulerian coordinates  $D_t^E$  and  $D_r$  as

$$D_t^L = D_t^E + uD_r, \quad D_s = \frac{1}{r^n \rho} D_r. \tag{16}$$

We rewrite the gas dynamics equations (11) in the Lagrangian coordinates  $(t, s)$  as

$$\rho_t + \rho^2 (r^n u)_s = 0, \tag{17a}$$

$$u_t + r^n p_s = 0, \tag{17b}$$

$$S_t = 0. \tag{17c}$$

We remark that here the gas dynamics variables  $\rho, u, p,$  and  $S$  are functions of the Lagrangian coordinates  $t$  and  $s$  while in the system (11), they are functions of the Eulerian coordinates  $t$  and  $r$ .

The Eulerian spatial coordinate  $\varphi = r$  is a dependent variable in the mass Lagrangian coordinates. Equations (14) can be rewritten in the form

$$r_t = u, \quad r_s = \frac{1}{r^n \rho}. \tag{18}$$

It is also possible to use the 1-form

$$dr = \frac{ds}{r^n \rho} + u dt.$$

Notice that as for Eq. (11c), we also use other representations of Eq. (17c)

$$p_t + \gamma \rho p (r^n u)_s = 0 \tag{19}$$

or

$$\varepsilon_t + p (r^n u)_s = 0. \tag{20}$$

Equation (17c) can be solved

$$S = S(s), \tag{21}$$

where  $S(s)$  is an arbitrary function.

Using these results, it is possible to rewrite the last remaining Eq. (17b), as a partial differential equation of the second order

$$\varphi_{tt} + \varphi^{n(1-\gamma)} \varphi_s^{-\gamma} (S' - n\gamma S \frac{\varphi_s}{\varphi} - \gamma S \frac{\varphi_{ss}}{\varphi_s}) = 0. \tag{22}$$

This PDE is called the *gas dynamics equation* in the Lagrangian coordinates [34, 36].

PDE (22) has a variational formulation, namely, it is the Euler–Lagrange equation

$$\frac{\delta L}{\delta \varphi} = \frac{\partial L}{\partial \varphi} - D_t^L \left( \frac{\partial L}{\partial \varphi_t} \right) - D_s \left( \frac{\partial L}{\partial \varphi_s} \right) = 0 \quad (23)$$

for the Lagrangian

$$L = \frac{1}{2} \varphi_t^2 - \frac{S(s)}{\gamma - 1} \varphi^{(1-\gamma)n} \varphi_s^{1-\gamma}. \quad (24)$$

### 3.3 Conservation Laws

We specify Noether’s theorem, given in Sect. 2, for PDE (22). We consider the Lie point symmetries of the form

$$X = \xi^t(t, s, \varphi) \frac{\partial}{\partial t} + \xi^s(t, s, \varphi) \frac{\partial}{\partial s} + \eta^\varphi(t, s, \varphi) \frac{\partial}{\partial \varphi}. \quad (25)$$

Such symmetries of the PDE (22) can be used to compute conservation laws if they are also variational or divergence symmetries of the Lagrangian (24). We require that they satisfy the condition of the elementary action invariance [2]

$$XL + L(D_t^L \xi^t + D_s \xi^s) = D_t^L B_1 + D_s B_2 \quad (26)$$

for some functions  $B_1(t, s, \varphi)$  and  $B_2(t, s, \varphi)$ . If this condition holds with  $B_1 = B_2 = 0$ , then the symmetry (25) is called variational. We refer to both variational and divergent symmetries as the Noether symmetries.

Given a variational or divergent symmetry, we can find the corresponding conservation law

$$D_t^L [T^t] + D_s [T^s] = 0, \quad (27)$$

where the densities are given by the formulas

$$\begin{aligned} T^t &= \xi^t L + (\eta^\varphi - \xi^t \varphi_t - \xi^s \varphi_s) \frac{\partial L}{\partial \varphi_t} - B_1, \\ T^s &= \xi^s L + (\eta^\varphi - \xi^t \varphi_t - \xi^s \varphi_s) \frac{\partial L}{\partial \varphi_s} - B_2. \end{aligned} \quad (28)$$

Conservation laws (27) can be rewritten for the Eulerian coordinates as

$$D_t^E [{}^e T^t] + D_r [{}^e T^r] = 0. \quad (29)$$

The relation between two above forms of conservation laws can be proved by direct verification



$$D_t^L T^t + D_s T^s = \varphi_s (D_t^E (r^n \rho T^t) + D_r (r^n \rho u T^t + T^s)).$$

Therefore, if we have densities  $T^t$  and  $T^s$  of a conservation law in the Lagrangian coordinates, we can find the corresponding densities in the Eulerian coordinates as

$${}^e T^t = r^n \rho T^t, \quad {}^e T^r = r^n \rho u T^t + T^s. \tag{30}$$

### 3.4 Equivalence Transformations

PDE (22) contains an arbitrary function  $S(s)$ . Thus, we need the group classification with respect to it. The generators of the equivalence Lie group has the form

$$X^e = \xi^t \frac{\partial}{\partial t} + \xi^s \frac{\partial}{\partial s} + \eta^\varphi \frac{\partial}{\partial \varphi} + \eta^S \frac{\partial}{\partial S}, \tag{31}$$

where generators coefficients  $\xi^t, \xi^s, \eta^\varphi$  and  $\eta^S$  depend on  $(t, s, \varphi, S)$ .

Computation gives the generators of the equivalence group. There are five generators

$$\begin{aligned} X_1^e &= \frac{\partial}{\partial t}, & X_2^e &= \frac{\partial}{\partial s}, & X_3^e &= t \frac{\partial}{\partial t} - 2S \frac{\partial}{\partial S}, \\ X_4^e &= s \frac{\partial}{\partial s} + (1 - \gamma)S \frac{\partial}{\partial S}, & X_5^e &= \varphi \frac{\partial}{\partial \varphi} + ((n + 1)\gamma - n + 1)S \frac{\partial}{\partial S} \end{aligned} \tag{32}$$

for the general case. For  $n = 0$ , there are two additional equivalence transformations given by

$$X_{*,n}^e = \frac{\partial}{\partial \varphi} \quad \text{and} \quad X_{**,n}^e = t \frac{\partial}{\partial \varphi}. \tag{33}$$

For the special values of the adiabatic exponent  $\gamma_* = \frac{n+3}{n+1}$ , we obtain one additional generator

$$X_{*,\gamma}^e = t^2 \frac{\partial}{\partial t} + t\varphi \frac{\partial}{\partial \varphi}. \tag{34}$$

### 3.5 Group Classification of the Gas Dynamics Equation

The Lie algebra of the admitted transformations is given by the generators

$$X = \sum_{i=1}^8 k_i Y_i, \tag{35}$$

where

$$\begin{aligned}
 Y_1 &= \frac{\partial}{\partial t}, & Y_2 &= \frac{\partial}{\partial s}, & Y_3 &= \frac{\partial}{\partial \varphi}, \\
 Y_4 &= t \frac{\partial}{\partial t}, & Y_5 &= s \frac{\partial}{\partial s}, & Y_6 &= \varphi \frac{\partial}{\partial \varphi}, \\
 Y_7 &= t \frac{\partial}{\partial \varphi}, & Y_8 &= t^2 \frac{\partial}{\partial t} + t\varphi \frac{\partial}{\partial \varphi}.
 \end{aligned} \tag{36}$$

The coefficients  $k_i$  satisfy the system

$$(k_5 s + k_2) S_s = (-2k_4 + (1 - \gamma)k_5 + ((n + 1)\gamma - n + 1)k_6) S, \tag{37a}$$

$$((n + 1)\gamma - n - 3)k_8 = 0, \tag{37b}$$

$$nk_3 = 0, \tag{37c}$$

$$nk_7 = 0. \tag{37d}$$

For the general case, we get two admitted symmetries

$$\begin{aligned}
 X_1 &= Y_1 = \frac{\partial}{\partial t}, & X_2 &= ((n + 1)\gamma - n + 1)Y_4 + 2Y_6 \\
 & & &= ((n + 1)\gamma - n + 1)t \frac{\partial}{\partial t} + 2\varphi \frac{\partial}{\partial \varphi}.
 \end{aligned} \tag{38}$$

For  $n = 0$ , there are two additional symmetries

$$X_{*,n} = Y_3 = \frac{\partial}{\partial \varphi} \quad \text{and} \quad X_{**,n} = Y_7 = t \frac{\partial}{\partial \varphi}. \tag{39}$$

For the special values  $\gamma_* = \frac{n+3}{n+1}$ , there is one additional symmetry

$$X_{*,\gamma} = Y_8 = t^2 \frac{\partial}{\partial t} + t\varphi \frac{\partial}{\partial \varphi}. \tag{40}$$

The condition (37a) is the classifying equation for function  $S(s)$ . It can be rewritten as

$$(\alpha_1 s + \alpha_0) S_s = \beta S \tag{41}$$

for some constants  $\alpha_0$ ,  $\alpha_1$ , and  $\beta$ . This classifying equation was studied in [39]. It was shown that one need to consider four cases of the entropy function  $S(s)$ , the general case and three special cases:

- arbitrary  $S(s)$ ;
- $S(s) = A_0$ ,  $A_0 = \text{const}$ ;
- $S(s) = A_0 s^q$ ,  $q \neq 0$ ,  $A_0 = \text{const}$ ;
- $S(s) = A_0 e^{qs}$ ,  $q \neq 0$ ,  $A_0 = \text{const}$ .

The same four cases were obtained for plain one-dimensional flows in [31]. Let us note that the equivalence transformations can be used to simplify these cases to  $A_0 = 1$ .

### 4 Arbitrary Entropy $S(s)$

Equation (17a) can be rewritten in the form of a conservation law as

$$\left[ \frac{1}{\rho} \right]_t - [r^n u]_s = 0.$$

Thus, conservation of mass is included into the equations of the gas dynamics system (17). In the Eulerian coordinates, it has the form

$$[r^n \rho]_t + [r^n \rho u]_r = 0.$$

Equation (17c) gives the conservation of the entropy along pathlines as the conservation law

$$S_t = 0.$$

Let us examine the symmetries of the kernel of admitted Lie algebras (38), (39), and (40) for being variational or divergent symmetries, which provide conservation laws.

#### 4.1 General Case $n \neq 0, \gamma \neq \frac{n+3}{n+1}$

In the general cases, the admitted symmetries (38) provide us with one variational symmetry

$$Z_1 = X_1 = \frac{\partial}{\partial t}. \tag{42}$$

It leads to the conservation of energy with densities

$$T_1^t = \frac{\varphi_t^2}{2} + \frac{S}{\gamma - 1} \varphi^{n(1-\gamma)} \varphi_s^{1-\gamma}, \quad T_1^s = S \varphi^{n(1-\gamma)} \varphi_t \varphi_s^{-\gamma}. \tag{43}$$

For the gas dynamics variables, this conservation law gets rewritten as

$$T_1^t = \frac{u^2}{2} + \frac{S}{\gamma - 1} \rho^{\gamma-1}, \quad T_1^s = r^n S \rho^\gamma u. \tag{44}$$

In the Eulerian coordinates, it has the densities

$${}^e T_1^t = r^n \left( \frac{\rho u^2}{2} + \frac{S}{\gamma - 1} \rho^\gamma \right), \quad {}^e T_1^r = r^n \left( \frac{\rho u^2}{2} + \frac{\gamma S}{\gamma - 1} \rho^\gamma \right) u.$$

#### 4.2 Case $n = 0$ , $\gamma \neq \frac{n+3}{n+1}$

We get one more variational symmetry

$$Z_{*,n} = X_{*,n} = \frac{\partial}{\partial \varphi}, \quad (45)$$

and one divergent symmetry

$$Z_{**,n} = X_{**,n} = t \frac{\partial}{\partial \varphi} \quad \text{with} \quad (B_1, B_2) = (\varphi, 0). \quad (46)$$

These symmetries provide conservation laws

$$T_{*,n}^t = \varphi_t, \quad T_{*,n}^s = S \varphi_s^{-\gamma}; \quad (47)$$

$$T_{**,n}^t = \varphi - \varphi_t t, \quad T_{**,n}^s = -t S \varphi_s^{-\gamma}, \quad (48)$$

representing the conservation of momentum and the motion of the center of mass, respectively.

In gas dynamics variables, we can rewrite these conservation laws as

$$T_{*,n}^t = u, \quad T_{*,n}^s = S \rho^\gamma; \quad (49)$$

$$T_{**,n}^t = r - tu, \quad T_{**,n}^s = -t S \rho^\gamma. \quad (50)$$

Notice that the conserved vector  $(T_{**,n}^t, T_{**,n}^s)$  contains the function  $\varphi \equiv r$ . In the Eulerian coordinates, we get

$${}^e T_{*,n}^t = \rho u, \quad {}^e T_{*,n}^r = \rho u^2 + S \rho^\gamma;$$

$${}^e T_{**,n}^t = \rho(r - tu), \quad {}^e T_{**,n}^r = \rho u(r - tu) - t S \rho^\gamma.$$

### 4.3 Case $n \neq 0$ , $\gamma_* = \frac{n+3}{n+1}$

For  $\gamma = \gamma_*$ , the symmetries (38) and (40) lead to two variational symmetries: (42) and

$$Z_{*,\gamma} = \frac{1}{2}X_2 = 2t \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial \varphi} \tag{51}$$

and one divergence symmetry

$$Z_{**,\gamma} = X_{*,\gamma} = t^2 \frac{\partial}{\partial t} + t\varphi \frac{\partial}{\partial \varphi} \quad \text{with} \quad (B_1, B_2) = \left( \frac{\varphi^2}{2}, 0 \right). \tag{52}$$

In addition to the conservation of energy, given in point Sect. 4.1, there are conservation laws with densities

$$T_{*,\gamma}^t = 2t \left( \frac{\varphi_t^2}{2} + \frac{S}{\gamma-1} \varphi^{n(1-\gamma)} \varphi_s^{1-\gamma} \right) - \varphi \varphi_t, \tag{53}$$

$$T_{*,\gamma}^s = (2t\varphi_t - \varphi) S \varphi^{n(1-\gamma)} \varphi_s^{-\gamma};$$

$$T_{**,\gamma}^t = t^2 \left( \frac{\varphi_t^2}{2} + \frac{S}{\gamma-1} \varphi^{n(1-\gamma)} \varphi_s^{1-\gamma} \right) - t\varphi \varphi_t + \frac{\varphi^2}{2}, \tag{54}$$

$$T_{**,\gamma}^s = (t^2\varphi_t - t\varphi) S \varphi^{n(1-\gamma)} \varphi_s^{-\gamma}.$$

We can rewrite these conservation laws for the gas dynamics variables

$$T_{*,\gamma}^t = 2t \left( \frac{u^2}{2} + \frac{S}{\gamma-1} \rho^{\gamma-1} \right) - ru, \quad T_{*,\gamma}^s = r^n (2tu - r) S \rho^\gamma; \tag{55}$$

$$T_{**,\gamma}^t = t^2 \left( \frac{u^2}{2} + \frac{S}{\gamma-1} \rho^{\gamma-1} \right) - tr u + \frac{r^2}{2}, \quad T_{**,\gamma}^s = r^n (t^2 u - tr) S \rho^\gamma \tag{56}$$

as well as in the Eulerian coordinates

$${}^e T_{*,\gamma}^t = r^n \left( 2t \left( \frac{\rho u^2}{2} + \frac{S}{\gamma-1} \rho^\gamma \right) - r \rho u \right),$$

$${}^e T_{*,\gamma}^r = r^n \left( 2t \left( \frac{\rho u^2}{2} + \frac{\gamma S}{\gamma-1} \rho^\gamma \right) u - r(\rho u^2 + S \rho^\gamma) \right);$$

$${}^e T_{**,\gamma}^t = r^n \left( t^2 \left( \frac{\rho u^2}{2} + \frac{S}{\gamma-1} \rho^\gamma \right) - tr \rho u + \frac{r^2 \rho}{2} \right),$$

$${}^e T_{**,\gamma}^r = r^n \left( t^2 \left( \frac{\rho u^2}{2} + \frac{\gamma S}{\gamma-1} \rho^\gamma \right) u - tr(\rho u^2 + S \rho^\gamma) + \frac{r^2 \rho u}{2} \right).$$

#### 4.4 Case $n = 0$ , $\gamma_* = 3$

In this case, the conservation law of the general case get extended by both the conservation laws given in point Sect.4.2 and by the conservation laws given in point Sect. 4.3.

### 5 Special Cases of Entropy

Group classification of the PDE (22) gives three special cases of the entropy function. They are examined in this section. These cases inherit the symmetries and conservation laws of the arbitrary entropy  $S(s)$ , given in the preceding section. We present only additional symmetries and conservation laws.

#### 5.1 Isentropic Case $S(s) = A_0$

In the Eulerian coordinates, this case is presented as

$$S(r) = A_0 \quad \text{or} \quad S_r = 0. \quad (57)$$

For all cases (the case of general  $n$  and  $\gamma$ , the case  $n = 0$ , and the case of special values  $\gamma = \gamma_*$ ), there are two additional symmetries

$$X_3 = Y_2 = \frac{\partial}{\partial s}, \quad X_4 = (\gamma - 1)Y_4 - 2Y_5 = (\gamma - 1)t \frac{\partial}{\partial t} - 2s \frac{\partial}{\partial s}. \quad (58)$$

##### 5.1.1 General Case $n \neq 0$ , $\gamma \neq \frac{n+3}{n+1}$

In the general case, there are two additional variational symmetries

$$\begin{aligned} Z_2 &= X_3 = \frac{\partial}{\partial s}, \\ Z_3 &= \frac{\gamma + 1}{2} X_2 + \frac{n + 3 - (n + 1)\gamma}{2} X_4 \\ &= ((n + 3)\gamma - n - 1)t \frac{\partial}{\partial t} + ((n + 1)\gamma - n - 3)s \frac{\partial}{\partial s} + (\gamma + 1)\varphi \frac{\partial}{\partial \varphi}. \end{aligned} \quad (59)$$

The conservation laws of this case consist of the conservation law given in point Sect. 4.1 (for arbitrary  $S(s)$ ) and the two additional ones, given by densities

$$T_2^t = \varphi_s \varphi_t, \quad T_2^s = -\frac{\varphi_t^2}{2} + \frac{\gamma S}{\gamma - 1} \varphi^{n(1-\gamma)} \varphi_s^{1-\gamma}; \quad (60)$$

$$\begin{aligned}
 T_3^t &= ((n+3)\gamma - n - 1)t\left(\frac{\varphi_t^2}{2} + \frac{S}{\gamma-1}\varphi^{n(1-\gamma)}\varphi_s^{1-\gamma}\right) + ((n+1)\gamma - n - 3)s\varphi_s\varphi_t \\
 &\quad - (\gamma+1)\varphi\varphi_t, \\
 T_3^s &= ((n+3)\gamma - n - 1)tS\varphi^{n(1-\gamma)}\varphi_t\varphi_s^{-\gamma} + ((n+1)\gamma - n - 3)s \\
 &\quad \left(-\frac{\varphi_t^2}{2} + \frac{\gamma S}{\gamma-1}\varphi^{n(1-\gamma)}\varphi_s^{1-\gamma}\right) - (\gamma+1)S\varphi^{n(1-\gamma)+1}\varphi_s^{-\gamma}.
 \end{aligned} \tag{61}$$

If rewritten for the gas dynamics variables, they take the form

$$T_2^t = \frac{u}{r^n \rho}, \quad T_2^s = -\frac{u^2}{2} + \frac{\gamma S}{\gamma-1}\rho^{\gamma-1}; \tag{62}$$

$$\begin{aligned}
 T_3^t &= ((n+3)\gamma - n - 1)t\left(\frac{u^2}{2} + \frac{S}{\gamma-1}\rho^{\gamma-1}\right) + ((n+1)\gamma - n - 3)s\frac{u}{r^n \rho} - (\gamma+1)ru, \\
 T_3^s &= ((n+3)\gamma - n - 1)tSr^n\rho^\gamma u + ((n+1)\gamma - n - 3)s\left(-\frac{u^2}{2} + \frac{\gamma S}{\gamma-1}\rho^{\gamma-1}\right) \\
 &\quad - (\gamma+1)Sr^{n+1}\rho^\gamma.
 \end{aligned} \tag{63}$$

In the Eulerian coordinates, these conservation laws have densities

$$\begin{aligned}
 {}^e T_2^t &= u, \quad {}^e T_2^s = \frac{u^2}{2} + \frac{\gamma S}{\gamma-1}\rho^{\gamma-1}; \\
 T_3^t &= ((n+3)\gamma - n - 1)tr^n\left(\frac{\rho u^2}{2} + \frac{S}{\gamma-1}\rho^\gamma\right) + ((n+1)\gamma - n - 3)su - (\gamma+1)r^{n+1}\rho u, \\
 T_3^s &= ((n+3)\gamma - n - 1)tr^n\left(\frac{\rho u^2}{2} + \frac{\gamma S}{\gamma-1}\rho^\gamma\right)u \\
 &\quad + ((n+1)\gamma - n - 3)s\left(\frac{u^2}{2} + \frac{\gamma S}{\gamma-1}\rho^{\gamma-1}\right) - (\gamma+1)r^{n+1}(\rho u^2 + S\rho^\gamma),
 \end{aligned}$$

where  $S(s)$ , and  $s$  is defined by system (15).

### 5.1.2 Special Cases

For all special cases, namely case  $n = 0$ ,  $\gamma \neq \frac{n+3}{n+1}$ ; case  $n \neq 0$ ,  $\gamma_* = \frac{n+3}{n+1}$ ; and case  $n = 0$ ,  $\gamma_* = 3$ , we get conservation laws of the arbitrary entropy  $S(s)$ , which were described in Sect. 4, supplemented by the conservation laws given in point Sect. 5.1.1.

Note that

$$Z_3|_{\gamma=\gamma_*} = 2\frac{n+2}{n+1}Z_{*,\gamma},$$

in other words for  $\gamma = \gamma_*$ , only the conservation law corresponding to  $Z_2$  in point Sect. 5.1.1 is new.

## 5.2 Entropy Case $S(s) = A_0 s^q$

In the Eulerian coordinates, this entropy case is described by the differential constraint

$$S_{rr} = \left( \frac{\rho_r}{\rho} + \frac{n}{r} \right) S_r + \frac{q-1}{q} \frac{S_r^2}{S}. \quad (64)$$

For such  $S(s)$ , there is one additional symmetry

$$X_3 = (\gamma + q - 1)Y_4 - 2Y_5 = (\gamma + q - 1)t \frac{\partial}{\partial t} - 2s \frac{\partial}{\partial s}. \quad (65)$$

### 5.2.1 General Case $n \neq 0, \gamma \neq \frac{n+3}{n+1}$

For the general case, there is one additional variational symmetry

$$\begin{aligned} Z_2 &= \frac{\gamma + q + 1}{2} X_2 + \frac{n + 3 - (n + 1)\gamma}{2} X_3 \\ &= ((n + 3)\gamma + 2q - n - 1)t \frac{\partial}{\partial t} + ((n + 1)\gamma - n - 3)s \frac{\partial}{\partial s} \\ &\quad + (\gamma + q + 1)\varphi \frac{\partial}{\partial \varphi}. \end{aligned} \quad (66)$$

Thus, in addition to the conservation of energy given in Sect. 4.1, we obtain the conservation law

$$\begin{aligned} T_2^t &= ((n + 3)\gamma + 2q - n - 1)t \left( \frac{\varphi_t^2}{2} + \frac{S}{\gamma - 1} \varphi^{n(1-\gamma)} \varphi_s^{1-\gamma} \right) \\ &\quad + ((n + 1)\gamma - n - 3)s \varphi_s \varphi_t - (\gamma + q + 1)\varphi \varphi_t, \\ T_2^s &= ((n + 3)\gamma + 2q - n - 1)t S \varphi^{n(1-\gamma)} \varphi_t \varphi_s^{-\gamma} + ((n + 1)\gamma - n - 3)s \\ &\quad \left( -\frac{\varphi_t^2}{2} + \frac{\gamma S}{\gamma - 1} \varphi^{n(1-\gamma)} \varphi_s^{1-\gamma} \right) - (\gamma + q + 1)S \varphi^{n(1-\gamma)+1} \varphi_s^{-\gamma}. \end{aligned} \quad (67)$$

For the gas dynamics variables, it takes the form



$$\begin{aligned}
 T_2^t &= ((n+3)\gamma + 2q - n - 1)t\left(\frac{u^2}{2} + \frac{S}{\gamma - 1}\rho^{\gamma-1}\right) + ((n+1)\gamma - n - 3)s\frac{u}{r^n\rho} \\
 &\quad - (\gamma + q + 1)ru, \\
 T_2^s &= ((n+3)\gamma + 2q - n - 1)tSr^n\rho^\gamma u + ((n+1)\gamma - n - 3)s\left(-\frac{u^2}{2} + \frac{\gamma S}{\gamma - 1}\rho^{\gamma-1}\right) \\
 &\quad - (\gamma + q + 1)Sr^{n+1}\rho^\gamma.
 \end{aligned}
 \tag{68}$$

To rewrite this conservation laws in the Eulerian coordinates, we use the relation

$$s = qr^n\rho\frac{S}{S_r} \tag{69}$$

to present the Lagrangian coordinate  $s$ . This relation allows to write down the densities of the conservation law as follows:

$$\begin{aligned}
 {}^eT_2^t &= ((n+3)\gamma + 2q - n - 1)tr^n\left(\frac{\rho u^2}{2} + \frac{S}{\gamma - 1}\rho^\gamma\right) + ((n+1)\gamma - n - 3)qr^n\rho u\frac{S}{S_r} \\
 &\quad - (\gamma + q + 1)r^{n+1}\rho u, \\
 {}^eT_2^r &= ((n+3)\gamma + 2q - n - 1)tr^n\left(\frac{\rho u^2}{2} + \frac{\gamma S}{\gamma - 1}\rho^\gamma\right)u + ((n+1)\gamma - n - 3)qr^n\rho\frac{S}{S_r} \\
 &\quad \left(\frac{u^2}{2} + \frac{\gamma S}{\gamma - 1}\rho^{\gamma-1}\right) - (\gamma + q + 1)r^{n+1}(\rho u^2 + S\rho^\gamma).
 \end{aligned}$$

### 5.2.2 Case $n = 0, \gamma \neq \frac{n+3}{n+1}$

For  $n = 0$ , the additional Noether symmetries are the same as in the general case. Therefore, we get conservation laws given in points Sects. 4.1, 4.2, and 5.2.1.

### 5.2.3 Case $n \neq 0, \gamma_* = \frac{n+3}{n+1}$

The special case of  $\gamma_*$  splits for values of  $q$ . For general  $q$ , we get the same Noether symmetries as in the case of arbitrary  $S(s)$ . Therefore, we obtain the same conservation laws as given in points Sects. 4.1 and 4.3.

For the particular case  $q_* = -2\frac{n+2}{n+1}$ , there is one additional variational symmetry

$$Z_{*,q} = -\frac{1}{2}X_3 = t\frac{\partial}{\partial t} + s\frac{\partial}{\partial s}. \tag{70}$$

It provides with the following conservation law:

$$\begin{aligned} T_{*,q}^t &= t \left( \frac{\varphi_t^2}{2} + \frac{S}{\gamma-1} \varphi^{n(1-\gamma)} \varphi_s^{1-\gamma} \right) + s \varphi_s \varphi_t, \\ T_{*,q}^s &= t S \varphi^{n(1-\gamma)} \varphi_t \varphi_s^{-\gamma} + s \left( -\frac{\varphi_t^2}{2} + \frac{\gamma S}{\gamma-1} \varphi^{n(1-\gamma)} \varphi_s^{1-\gamma} \right). \end{aligned} \quad (71)$$

It is also possible to present this conservation laws for the gas dynamics variables

$$\begin{aligned} T_{*,q}^t &= t \left( \frac{u^2}{2} + \frac{S}{\gamma-1} \rho^{\gamma-1} \right) + s \frac{u}{r^n \rho}, \\ T_{*,q}^s &= t S r^n \rho^\gamma u + s \left( -\frac{u^2}{2} + \frac{\gamma S}{\gamma-1} \rho^{\gamma-1} \right). \end{aligned} \quad (72)$$

To rewrite these densities in the Eulerian coordinates, we employ the relation (69) and obtain densities

$$\begin{aligned} {}^e T_{*,q}^t &= r^n \left( t \left( \frac{\rho u^2}{2} + \frac{S}{\gamma-1} \rho^\gamma \right) + q \rho u \frac{S}{S_r} \right), \\ {}^e T_{*,q}^r &= r^n \left( t \rho u + q \rho \frac{S}{S_r} \right) \left( \frac{u^2}{2} + \frac{\gamma S}{\gamma-1} \rho^{\gamma-1} \right). \end{aligned}$$

#### 5.2.4 Case $n = 0, \gamma_* = 3$

We get the same conservation laws as described in the previous point. Note that  $n = 0$  leads to  $q_* = -4$ .

### 5.3 Entropy Case $S(s) = A_0 e^{qs}$

Let us note that this special case can be given in the Eulerian coordinates by the differential constraint

$$S_r = q r^n \rho S. \quad (73)$$

For all cases of Sect. 4, there is one additional symmetry

$$X_3 = -2Y_2 + qY_4 = qt \frac{\partial}{\partial t} - 2 \frac{\partial}{\partial s}. \quad (74)$$

### 5.3.1 General Case $n \neq 0, \gamma \neq \frac{n+3}{n+1}$

For the general case, there is one additional variational symmetry

$$\begin{aligned} Z_2 &= \frac{q}{2} X_2 + \frac{n+3-(n+1)\gamma}{2} X_3 \\ &= 2qt \frac{\partial}{\partial t} + ((n+1)\gamma - n - 3) \frac{\partial}{\partial s} + q\varphi \frac{\partial}{\partial \varphi}. \end{aligned} \quad (75)$$

The supplementary conservation law has densities

$$\begin{aligned} T_2^t &= 2qt \left( \frac{\varphi_t^2}{2} + \frac{S}{\gamma-1} \varphi^{n(1-\gamma)} \varphi_s^{1-\gamma} \right) + ((n+1)\gamma - n - 3) \varphi_s \varphi_t - q\varphi \varphi_t, \\ T_2^s &= 2qt S \varphi^{n(1-\gamma)} \varphi_t \varphi_s^{-\gamma} + ((n+1)\gamma - n - 3) \left( -\frac{\varphi_t^2}{2} + \frac{\gamma S}{\gamma-1} \varphi^{n(1-\gamma)} \varphi_s^{1-\gamma} \right) \\ &\quad - qS \varphi^{n(1-\gamma)+1} \varphi_s^{-\gamma}. \end{aligned} \quad (76)$$

For the gas dynamics variables, we get

$$\begin{aligned} T_2^t &= 2qt \left( \frac{u^2}{2} + \frac{S}{\gamma-1} \rho^{\gamma-1} \right) + ((n+1)\gamma - n - 3) \frac{u}{r^n \rho} - qru, \\ T_2^s &= 2qt S r^n \rho^\gamma u + ((n+1)\gamma - n - 3) \left( -\frac{u^2}{2} + \frac{\gamma S}{\gamma-1} \rho^{\gamma-1} \right) - qS r^{n+1} \rho^\gamma. \end{aligned} \quad (77)$$

Finally, we rewrite these densities in the Eulerian coordinates

$$\begin{aligned} {}^e T_2^t &= 2qtr^n \left( \frac{\rho u^2}{2} + \frac{S}{\gamma-1} \rho^\gamma \right) + ((n+1)\gamma - n - 3)u - qr^{n+1} \rho u, \\ {}^e T_2^r &= 2qtr^n \left( \frac{\rho u^2}{2} + \frac{\gamma S}{\gamma-1} \rho^\gamma \right) u + ((n+1)\gamma - n - 3) \left( \frac{u^2}{2} + \frac{\gamma S}{\gamma-1} \rho^{\gamma-1} \right) \\ &\quad - qr^{n+1} (\rho u^2 + S \rho^\gamma). \end{aligned}$$

### 5.3.2 Special Cases

For all special cases, we get the same additional conservation law as in the general case of  $n$  and  $\gamma$ . We remark that because of

$$Z_2|_{\gamma=\gamma_*} = qZ_{*,\gamma}$$

the corresponding conservation law, given in point Sect. 5.3.1, is not new for the special values  $\gamma = \gamma_*$ .

## 5.4 Discussion

The complete Lie group classification of the gas dynamics equation in the Lagrangian coordinates (22) allows us to find all conservation laws which can be found using Noether's theorem and admitted symmetries. The group classification has three cases of the entropy for which there exist additional symmetries. In the Eulerian coordinates, these three cases are defined by differential constraints of first or second order. Notice that the overdetermined systems which consist of the gas dynamics equations and one of the considered differential constraints are involutive. The authors of [12, 13] also found conservation laws corresponding to special forms of the entropy. Here, the symmetry nature of these conservation laws is explained.

In contrast to [11], the conservation laws, obtained in this chapter, are local. It should be also noted that these conservation laws are naturally derived: their counterparts in the Lagrangian coordinates were derived directly using Noether's theorem without any additional assumptions. In contrast to the two-dimensional Lagrangian gas dynamics, the special cases of the entropy in the Lagrangian coordinates are given explicitly. In the two-dimensional case [40], the entropy is arbitrary, but the admitted symmetry operators contain functions satisfying quasilinear partial differential equations.

In a conservative form, the one-dimensional gas dynamics equations (11) are [33]

$$[r^n \rho]_t + [r^n \rho u]_r = 0, \quad (78a)$$

$$[r^n \rho u]_t + [r^n (\rho u^2 + p)]_r = nr^{n-1} p, \quad (78b)$$

$$[r^n (\rho \varepsilon + \frac{\rho u^2}{2})]_t + [r^n (\rho \varepsilon + \frac{\rho u^2}{2} + p)u]_r = 0, \quad (78c)$$

where  $[\dots]_t$  and  $[\dots]_r$  denote total derivatives with respect to time  $t$  and the Eulerian coordinate  $r$ . One notes that the equation corresponding to the conservation law of momentum is not homogeneous. However, most methods for constructing conservation laws can only construct homogeneous conservation laws.

Consider inhomogeneous conservation laws of the one-dimensional gas dynamics equations

$$D_t [f^t] + D_r [f^r] = f, \quad (79)$$

where  $D_t$  and  $D_r$  are the total derivatives and the functions  $f^t$ ,  $f^r$ , and  $f$  depend on  $(t, r, \rho, u, p)$ . The method which is used to derive such conservation laws consists of obtaining an overdetermined system of partial differential equations for the functions  $f^t$ ,  $f^r$ , and  $f$  and finding its general solution. The overdetermined system is derived by substituting the main derivatives  $\rho_t$ ,  $u_t$ , and  $p_t$  found from the gas dynamics equations into (79) and splitting it with respect to the parametric derivatives.

Calculations show that the general solution of this system provides the conservation laws

$$[\rho F]_t + [\rho u F]_r = \rho (F_t + u (F_r - \frac{n}{r} F)), \quad (80)$$

$$[h\rho u]_t + [h(\rho u^2 + p)]_r = h_t \rho u + h_r (\rho u^2 + p) - \frac{n}{r} h \rho u^2, \quad (81)$$

$$\begin{aligned} & [h(\rho \frac{u^2}{2} + \frac{p}{\gamma-1})]_t + [h(\rho \frac{u^2}{2} + \frac{\gamma p}{\gamma-1})u]_r \\ & = h_t (\rho \frac{u^2}{2} + \frac{p}{\gamma-1}) + (h_r - \frac{n}{r} h) (\rho \frac{u^2}{2} + \frac{\gamma p}{\gamma-1})u, \end{aligned} \quad (82)$$

where  $h(t, r)$  and  $F(t, r, p\rho^{-\gamma})$  are arbitrary functions.

Equation (80) becomes a homogeneous conservation law if and only if

$$F(t, r, z) = r^n g(z), \quad z = \frac{p}{\rho^\gamma},$$

which for  $g \equiv 1$  gives Eq. (78a).

Equation (81) can be a homogeneous conservation law only if  $n = 0$ . Notice that for  $h = r^n$ , this equation becomes (78b). Equation (82) provides a homogeneous conservation law; only for  $h = r^n$ , it gives Eq. (78c).

It should be also noted here that if the overdetermined system defined above is extended by the condition  $f = 0$ , then one obtains all possible homogeneous zero-order conservation laws of the one-dimensional gas dynamics equations.

## 6 Difference Models Preserving Symmetries

The first problem in discretization of differential equations is the choice of a difference mesh. The peculiarity of our approach is that we add mesh equation(s) into the difference model:

$$F_i(z) = 0, \quad i = 1, \dots, I; \quad (83a)$$

$$\Omega_j(z) = 0, \quad j = 1, \dots, J. \quad (83b)$$

Here, the first set of equations approximates the underlying differential system and the second set of equations describes the difference mesh;  $z$  is a set of difference variables needed for approximation. As it was shown in [19, 22], the invariance of the mesh structure is a necessary condition for the invariance of the difference model. The mesh equations can be presented with the help of difference invariants or, alternatively, one can check the invariance of any chosen mesh by means of a certain criterium (see [19, 22]).

Symmetries of difference schemes allow one to construct difference counterparts of the differential conservation laws. The latter provides the absence of fake sources of energy, momentum, etc. in difference models which play an important role in solutions with high gradients. Moreover, the presence of (local) difference conservation

laws gives a possibility to apply the difference counterpart of the Gauss–Ostrogradskii theorem [41] that leads to global conservation properties of the numerical solutions.

For discretization of the gas dynamics system (11a), (11b), (12), which is given in the Eulerian coordinates, the simplest choice seems to be an orthogonal mesh in  $(t, r)$  plane. As it will be shown below, this mesh is not invariant with respect to symmetries which we aim to preserve in the difference models. This noninvariance destroys the invariance of difference equations considered on such a mesh. We will choose another coordinate system in which one can preserve mesh geometry and, hence, the invariance of the whole difference model.

## 6.1 The Gas Dynamics Equations

In Sect. 3, we considered entropy as one of the dependent variables. Since the entropy is conserved along pathlines only for smooth solutions, it is appropriate to choose another form of the gas dynamics equations for numerical modeling.

### 6.1.1 Eulerian Coordinates

We start with the equations for the gas dynamics variables  $\rho$ ,  $u$ , and  $p$ :

$$\rho_t + u\rho_r + \frac{\rho}{r^n}(r^n u)_r = 0, \quad (84a)$$

$$u_t + uu_r + \frac{1}{\rho}p_r = 0, \quad (84b)$$

$$p_t + up_r + \frac{\gamma p}{r^n}(r^n u)_r = 0, \quad (84c)$$

which admits four symmetries for any  $n$  and  $\gamma$

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_2 &= t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}, \\ X_3 &= 2t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} - u \frac{\partial}{\partial u} + 2\rho \frac{\partial}{\partial \rho}, & X_4 &= \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}. \end{aligned} \quad (85)$$

For  $n = 0$ , there are two additional symmetries

$$X_{*,n} = \frac{\partial}{\partial r} \quad \text{and} \quad X_{**,n} = t \frac{\partial}{\partial r} + \frac{\partial}{\partial u}. \quad (86)$$

For the special values  $\gamma_* = \frac{n+3}{n+1}$ , there is one additional symmetry

$$X_{*,\gamma} = t^2 \frac{\partial}{\partial t} + tr \frac{\partial}{\partial r} + (r - tu) \frac{\partial}{\partial u} - (n+1)t\rho \frac{\partial}{\partial \rho} - (n+3)tp \frac{\partial}{\partial p}. \quad (87)$$

### 6.1.2 Conservation Laws

System (84) possesses the following conservation laws.

#### 1. General case of $n$ and $\gamma$

In the general case, we get

- Conservation of mass

$$[r^n \rho]_t + [r^n \rho u]_r = 0; \quad (88)$$

- Conservation of energy

$$[r^n (\rho \varepsilon + \frac{\rho u^2}{2})]_t + [r^n (\rho \varepsilon + \frac{\rho u^2}{2} + p)u]_r = 0; \quad (89)$$

- Conservation law

$$[r^n \rho F(\frac{p}{\rho^\gamma})]_t + [r^n \rho u F(\frac{p}{\rho^\gamma})]_r = 0, \quad (90)$$

where  $F$  is a differentiable function. It holds due to the conservation of mass (88) and conservation of entropy along the pathlines, given by

$$(\frac{p}{\rho^\gamma})_t + u(\frac{p}{\rho^\gamma})_r = 0.$$

#### 2. Case $n = 0$

For the particular case  $n = 0$  (plain one-dimensional flows), we obtain two additional conservation laws:

- Momentum

$$[\rho u]_t + [\rho u^2 + p]_r = 0; \quad (91)$$

- Motion of the center of mass

$$[\rho(r - tu)]_t + [\rho u(r - tu) - tp]_r = 0. \quad (92)$$

#### 3. Special values of $\gamma_* = \frac{n+3}{n+1}$

For  $\gamma = \gamma_*$ , there are two additional conservation laws

$$\begin{aligned} & \{r^n [2t(\rho\varepsilon + \frac{\rho u^2}{2}) - r\rho u]\}_t \\ & + \{r^n [2t(\rho\varepsilon + \frac{\rho u^2}{2} + p)u - r(\rho u^2 + p)]\}_r = 0 \end{aligned} \quad (93)$$

and

$$\begin{aligned} & \{r^n [t^2(\rho\varepsilon + \frac{\rho u^2}{2}) - tr\rho u + \frac{r^2}{2}\rho]\}_t \\ & + \{r^n [t^2(\rho\varepsilon + \frac{\rho u^2}{2} + p)u - tr(\rho u^2 + p) + \frac{r^2}{2}\rho u]\}_r = 0. \end{aligned} \quad (94)$$

One can find conservation laws by direct computation or by an appropriate reduction of the three-dimensional conservation laws. Conservation laws of three-dimensional gas dynamics were obtained in [42] (see also [2]) with the help of a variational formulation and Noether's theorem (it requires some assumptions) and by direct computation in [43]. Among the 13 conservation laws of the three-dimensional case, all but one can be integrated over discontinuities [43]. The only conservation law which cannot be integrated over discontinuities gets reduced to (90) in the case of one-dimensional flows. It cannot be integrated over discontinuities because the entropy is not conserved for the discontinuous solutions [37, 44]. In [2, 42], one can find a symmetry interpretation of the conservation laws, i.e., the correspondence between the conservation laws and the Lie point symmetries of the three-dimensional gas dynamics equations.

### 6.1.3 Lagrangian Coordinates

We rewrite the gas dynamics equations (84) in the Lagrangian coordinates  $(t, s)$  as

$$\left(\frac{1}{\rho}\right)_t = (r^n u)_s, \quad (95a)$$

$$u_t + r^n p_s = 0, \quad (95b)$$

$$\varepsilon_t = -p(r^n u)_s, \quad (95c)$$

$$r_t = u. \quad (95d)$$

Note that in the Lagrangian coordinates, variable  $r$  is dependent. It is given by Eq. (95d), which is included in the system of the gas dynamics equations, and the relation

$$r_s = \frac{1}{r^n \rho}. \quad (96)$$



From Eq. (95), it is easy to see that

$$\varepsilon_t = -p \left( \frac{1}{\rho} \right)_t. \quad (97)$$

This relation is important for the balance between the specific internal energy and the specific kinetic energy.

We rewrite symmetries (85) and additional symmetries (86) and (87) in the Lagrangian coordinates. There are four symmetries in the general case

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, & X_2 &= t \frac{\partial}{\partial t} + (n+1)s \frac{\partial}{\partial s} + r \frac{\partial}{\partial r}, \\ X_3 &= 2t \frac{\partial}{\partial t} + (n+3)s \frac{\partial}{\partial s} - u \frac{\partial}{\partial u} + 2\rho \frac{\partial}{\partial \rho} + r \frac{\partial}{\partial r}, \\ X_4 &= s \frac{\partial}{\partial s} + \rho \frac{\partial}{\partial \rho} + p \frac{\partial}{\partial p}; \end{aligned} \quad (98)$$

two additional symmetries for the particular case  $n = 0$

$$X_{*,n} = \frac{\partial}{\partial r} \quad \text{and} \quad X_{**,n} = t \frac{\partial}{\partial r} + \frac{\partial}{\partial u}; \quad (99)$$

and one additional symmetry for the special case  $\gamma_*$

$$X_{*,\gamma} = t^2 \frac{\partial}{\partial t} + (r - tu) \frac{\partial}{\partial u} - (n+1)t\rho \frac{\partial}{\partial \rho} - (n+3)tp \frac{\partial}{\partial p} + tr \frac{\partial}{\partial r}. \quad (100)$$

We also include the translation symmetry for the mass Lagrange coordinate, which is given by the generator

$$X_0 = \frac{\partial}{\partial s}. \quad (101)$$

#### 6.1.4 Conservation Laws

Let us rewrite the conservation laws for the Lagrangian coordinates. We obtain

##### 1. General case of $n$ and $\gamma$

There hold

- Conservation of mass

$$\left[ \frac{1}{\rho} \right]_t - [r^n u]_s = 0; \quad (102)$$

- Conservation of energy

$$\left[\varepsilon + \frac{1}{2}u^2\right]_t + [r^n pu]_s = 0; \quad (103)$$

- Conservation of entropy along pathlines

$$\left[\frac{p}{\rho^\gamma}\right]_t = 0. \quad (104)$$

## 2. Case $n = 0$

There are additional

- Conservation of momentum

$$[u]_t + [p]_s = 0; \quad (105)$$

- Motion of the center of mass

$$[r - tu]_t - [tp]_s = 0. \quad (106)$$

## 3. Special values of $\gamma_* = \frac{n+3}{n+1}$

For  $\gamma = \gamma_*$ , there are two additional conservation laws

$$\left[2t\left(\varepsilon + \frac{1}{2}u^2\right) - ru\right]_t + [r^n p(2tu - r)]_s = 0 \quad (107)$$

and

$$\left[t^2\left(\varepsilon + \frac{1}{2}u^2\right) - tru + \frac{r^2}{2}\right]_t + [r^n p(t^2u - tr)]_s = 0. \quad (108)$$

## 6.2 The Numerical Schemes

In this section, we consider numerical schemes and their symmetries. Besides, our goal is to construct schemes that have difference conservation laws analogous to the conservation laws of the underlying differential system. We restrict ourselves by the homogenous conservation laws.

### 6.2.1 Invariance and Eulerian Coordinates for $n = 0$

For discretization of the gas dynamics system (84), which is given in the Eulerian coordinates, the simplest choice seems to be an orthogonal mesh in  $(t, r)$  plane. However, this mesh is not invariant that destroys invariance of difference equations considered on such mesh. Indeed, as it was shown in [19, 22], the necessary condition for a mesh to preserve its orthogonality under a group transformation generated by

the operator

$$X = \xi^t \frac{\partial}{\partial t} + \xi^r \frac{\partial}{\partial r} + \dots \quad (109)$$

is the following:

$$D_{+h}(\xi^t) = -D_{+\tau}(\xi^x), \quad (110)$$

where  $D_{+h}$  and  $D_{+\tau}$  are the operators of difference differentiation in  $r$  and  $t$  directions, respectively.

System (84) admits the 6-parameter Lie symmetry group of point transformations that corresponds to the Lie algebra of infinitesimal operators (85) and (86). In the special case  $\gamma_* = 3$ , there is one more symmetry (87).

It is easy to see that the Galilean transformation given by the operator  $X_{**,n}$  does not satisfy the criterion (110). The same is true for  $X_{*,\gamma}$ . It means that one should look for an invariant moving mesh in the Eulerian coordinates.

To obtain an invariant moving mesh, we chose the following difference stencil with two time layers:

- independent variables:

$$t = t_j, \quad \hat{t} = t_{j+1}; \quad r = r_i^j, \quad r_+ = r_{i+1}^j, \quad \hat{r} = r_i^{j+1}, \quad \hat{r}_+ = r_{i+1}^{j+1};$$

- dependent variables in the nodes of the mesh (the same notation as for  $r$ ):

$$u, \quad u_+, \quad \hat{u}, \quad \hat{u}_+; \quad \rho, \quad \rho_-, \quad \hat{\rho}, \quad \hat{\rho}_-; \quad p, \quad p_-, \quad \hat{p}, \quad \hat{p}_-.$$

Then, we find the finite-difference invariants for symmetries (85) and (86) as solutions of the system of linear equations

$$X_i I(t, \hat{t}, r, r_+, \hat{r}, \hat{r}_+, \dots, p, p_-, \hat{p}, \hat{p}_-) = 0 \quad (111)$$

for the considered symmetries. Here, we assume that the operator is prolonged for all variables of the stencil [22]. There are 12 functionally independent invariants

$$\begin{aligned} & \frac{\hat{h}_+}{h_+}, \quad \frac{\tau}{h_+} \sqrt{\frac{p}{\rho}}, \quad \sqrt{\frac{\rho}{p}} \left( \frac{\hat{r} - r}{\tau} - u \right), \\ & \sqrt{\frac{\rho}{p}} (u_+ - u), \quad \sqrt{\frac{\rho}{p}} (\hat{u} - u), \quad \sqrt{\frac{\rho}{p}} (\hat{u}_+ - \hat{u}), \\ & \frac{p_+}{p}, \quad \frac{\hat{p}}{p}, \quad \frac{\hat{p}_+}{\hat{p}}, \quad \frac{\hat{\rho}}{\rho}, \quad \frac{\hat{\rho}_+}{\hat{\rho}}, \quad \frac{\rho_+}{\rho}, \end{aligned}$$

where  $\tau = \hat{t} - t$ ,  $h_+ = r_+ - r$ , and  $\hat{h}_+ = \hat{r}_+ - \hat{r}$ .

Notice that the only one difference invariant contains the value  $\hat{r}$ . This invariant suggests, for example, an invariant moving mesh given by

$$\sqrt{\frac{\rho}{p}} \left( \frac{\hat{r} - r}{\tau} - u \right) = 0$$

or, equivalently,

$$\frac{\hat{r} - r}{\tau} = u. \quad (112)$$

In the continuous limit, it corresponds to the evolution of the spacial variable  $r$  given as

$$\frac{dr}{dt} = u. \quad (113)$$

Thus, we arrive at choosing the mass Lagrangian coordinates with the operator of differentiation with respect to  $t$

$$D_t^L = D_t^E + u D_r.$$

### 6.2.2 Notations

We introduce the mesh for the mass Lagrangian coordinate  $s$ :

$$h^s = s_{i+1} - s_i \quad \text{and} \quad h_-^s = s_i - s_{i-1}. \quad (114)$$

Generally, the spacing can be nonuniform. For simplicity, we use a uniform mesh  $h^s = h_-^s$ .

For time, we consider the mesh with points  $t_j$ . Since we consider the schemes with two time layers, we denote the time step as  $\tau$ . Of course, we can consider nonuniform time meshes with steplengths  $\tau_j = t_{j+1} - t_j$ .

Now the operators have the form

$$X = \xi^t \frac{\partial}{\partial t} + \xi^s \frac{\partial}{\partial s} + \dots \quad (115)$$

and the criterium of invariant orthogonality

$$D_{+h_s}(\xi^t) = -D_{+\tau}(\xi^s) \quad (116)$$

holds for all considered symmetries (98), (99), (100), and (101). Here,  $D_{+h_s}$  and  $D_{+\tau}$  are the operators of difference differentiation in  $s$  and  $t$  directions, respectively.

We split the dependent variables into kinematic and thermodynamic. The kinematic variables  $u$  and  $r$  are prescribed to the nodes. For example, for  $u$ , we have

$$u = u_i^j, \quad u_+ = u_{i+1}^j, \quad \hat{u} = u_i^{j+1}, \quad \hat{u}_+ = u_{i+1}^{j+1}.$$

The thermodynamic variables  $\rho$  and  $p$  are taken in the midpoints as

$$\begin{aligned} \rho_- &= \rho_{i-1/2}^j, & \rho &= \rho_{i+1/2}^j, & \rho_+ &= \rho_{i+3/2}^j, \\ \hat{\rho}_- &= \rho_{i-1/2}^{j+1}, & \hat{\rho} &= \rho_{i+1/2}^{j+1}, & \hat{\rho}_+ &= \rho_{i+3/2}^{j+1}. \end{aligned}$$

To describe the scheme, we need the time and spatial derivatives

$$u_t = \frac{\hat{u} - u}{\tau}, \quad u_s = \frac{u_{i+1}^j - u_i^j}{s_{i+1} - s_i} = \frac{u_+ - u}{h^s}, \quad p_{\bar{s}} = \frac{p_{i+1/2}^j - p_{i-1/2}^j}{\frac{1}{2}(s_{i+1} - s_{i-1})} = \frac{p - p_-}{h^s}$$

and weighted values defined as

$$y^{(\alpha)} = \alpha \hat{y} + (1 - \alpha)y, \quad 0 \leq \alpha \leq 1.$$

### 6.2.3 The Samarskii–Popov Scheme

In [45] (see also [32]), the authors introduced a conservative scheme for plain one-dimensional flows ( $n = 0$ ). It was generalized to the other one-dimensional flows ( $n = 1, 2$ ) in [32]. This scheme is a discretization of Eq. (95)

$$\left(\frac{1}{\rho}\right)_t = (Ru^{(0.5)})_s, \tag{117a}$$

$$u_t = -Rp_s^{(\alpha)}, \tag{117b}$$

$$\varepsilon_t = -p^{(\alpha)}(Ru^{(0.5)})_s, \tag{117c}$$

$$r_t = u^{(0.5)}, \tag{117d}$$

where  $R$  is a discretization of  $r^n$  chosen as

$$R = \frac{\hat{r}^{n+1} - r^{n+1}}{(n+1)(\hat{r} - r)} = \begin{cases} 1, & n = 0; \\ \frac{\hat{r} + r}{2}, & n = 1; \\ \frac{\hat{r}^2 + \hat{r}r + r^2}{3}, & n = 2. \end{cases}$$

Scheme (117) has four equations for five variables  $\rho$ ,  $u$ ,  $\varepsilon$ ,  $r$ , and  $p$ . It should be supplemented by a discrete equation of state, a discrete analog of (9). For example, it can be taken in the same form that means

$$\varepsilon_{i+1/2}^j = \varepsilon(\rho_{i+1/2}^j, p_{i+1/2}^j). \tag{118}$$

### 6.2.4 Properties of the Samarskii–Popov Scheme

For a polytropic gas scheme (117), (118) is invariant with respect to the symmetries (98) and (101) corresponding to the general case. For  $n = 0$ , it is also invariant to symmetries (99). The scheme is not invariant for the additional symmetry (100), which exists for the special values  $\gamma_*$ .

Let us review important properties of the scheme. It possesses many qualitative properties of the underlying differential equations. For any equation of state  $\varepsilon = \varepsilon(\rho, p)$ , i.e., not only for the polytropic gas (10), this scheme has the following conservation laws:

- Conservation of mass

$$\left[ \frac{1}{\rho} \right]_t - [Ru^{(0.5)}]_s = 0; \quad (119)$$

- Conservation of energy

$$\left[ \varepsilon + \frac{u^2 + u_+^2}{4} \right]_t + [Rp_*^{(\alpha)} u^{(0.5)}]_s = 0, \quad (120)$$

where

$$p_*^{(\alpha)} = (p_*)_i^{(\alpha)} = \frac{p_{i-1/2}^{(\alpha)} + p_{i+1/2}^{(\alpha)}}{2}.$$

For  $n = 0$ , there are two additional conservation laws:

- Conservation of momentum

$$[u]_t + [p^{(\alpha)}]_s = 0; \quad (121)$$

- Motion of the center of mass

$$[r - tu]_t - [t^{(0.5)} p^{(\alpha)}]_s = 0. \quad (122)$$

These conservation laws correspond to (102), (103), (105), and (106). There are no discrete conservation laws corresponding to (107) and (108), which hold for the special values of  $\gamma_*$ .

**Remark 1** Modifying the equation of state (10), it is possible to achieve conservation of the conservation laws (107) and (108), which hold for  $\gamma_* = \frac{n+3}{n+1}$ , under discretization. We refer to [46] for the case  $n = 0$  and to [47] for the generalization to  $n = 1, 2$ .

Scheme (117) consists of four equations for five variables  $\rho, u, p, \varepsilon$ , and  $r$ . We will not impose the discrete equation of state (118). The freedom to choose a discretization of the equation of state will be used to impose one additional conservation law. Let us look for an equation of state which gives us the following difference analog of the additional conservation law (107):

$$\left[ 2t \left( \varepsilon + \frac{\langle u^2 \rangle}{2} \right) - \langle ru \rangle \right]_t + [Rp_*^{(\alpha)}(2t^{(0.5)}u^{(0.5)} - r^{(0.5)})]_s = 0, \quad (123)$$

where we use a special notation for the average value of two function values taken in the neighboring nodes of the same time layer

$$\langle f(u, r) \rangle = \frac{f(u, r) + f(u_+, r_+)}{2}.$$

It leads to the following specific internal energy equation:

$$\varepsilon^{(0.5)} = \frac{p^{(\alpha)}}{\gamma - 1} \left( \frac{1}{\rho} \right)^{(0.5)} - \frac{\tau^2}{8} \langle (u_t)^2 \rangle + \frac{1}{2} p^{(\alpha)} [r^{(0.5)}R - (r^{n+1})^{(0.5)}]_s. \quad (124)$$

We will take it as the *discrete equation of state*, which approximates (10).

In this case, we also get a difference analog of the second additional conservation law (108) as

$$\left[ t^2 \left( \varepsilon + \frac{\langle u^2 \rangle}{2} \right) - t \langle ru \rangle + \frac{\langle r^2 \rangle}{2} + \frac{\tau^2}{8} \langle u^2 \rangle \right]_t + [Rp_*^{(\alpha)}((t^2)^{(0.5)}u^{(0.5)} - t^{(0.5)}r^{(0.5)})]_s = 0. \quad (125)$$

Note that it has a correcting term  $\frac{\tau^2}{8} \langle u^2 \rangle$ , which disappears in the continuous limit.

Thus, we obtained difference scheme (117) supplemented by discrete state equation (124). In this scheme, the pressure values  $p$  and  $\hat{p}$  appear only as a weighted value  $p^{(\alpha)}$ , i.e.,  $\alpha$  has no longer meaning of a parameter. We can consider this value as the pressure in the midpoint of the cell  $(t_{j+1/2}, s_{i+1/2})$ , i.e., for  $\alpha = 0.5$ .

The scheme holds a discrete counterpart of the relation (97), namely

$$\varepsilon_t = -p^{(\alpha)} \left( \frac{1}{\rho} \right)_t. \quad (126)$$

This is an important supplement to the conservation of total energy (120), which provides the balance of the specific internal energy and the specific kinematic energy.

In the case of a polytropic gas, the equations of gas dynamics hold the conservation of entropy (104) along pathlines (for smooth solutions). There is no such property for the scheme (117). However, the scheme holds the relation

$$\frac{\Delta p}{p^{(\alpha)}} = \gamma \frac{\Delta \rho}{\rho^{(\alpha)}}, \quad \Delta p = \hat{p} - p, \quad \Delta \rho = \hat{\rho} - \rho \quad (127)$$

that approximates (104) presented with the help of differentials

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho}.$$

### 6.3 Invariance of Difference Schemes

In this point, we show how to construct invariant schemes with the help of finite-difference invariants. Scheme (117) can be expressed in terms of invariants for the general case of  $\gamma$ . Its modification described in Remark 1 possesses the additional conservation laws which hold for the special values  $\gamma_*$ . However, it is not invariant with respect to the additional symmetry  $X_{*,\gamma}$  which exists for these special values. For the special values  $\gamma_*$ , invariant schemes are constructed. The case  $n = 0$  was reported in [31].

#### 6.3.1 General Case $n \neq 0, \gamma \neq \frac{n+3}{n+1}$

We chose an orthogonal mesh in the Lagrangian coordinates and a stencil with the following variables:

- independent variables:

$$t = t_j, \quad \hat{t} = t_{j+1}, \quad s = s_i, \quad s_+ = s_{i+1}, \quad s_- = s_{i-1};$$

- kinematic variables in the nodes:

$$u = u_i^j, \quad u_+ = u_{i+1}^j, \quad \hat{u} = u_i^{j+1}, \quad \hat{u}_+ = u_{i+1}^{j+1}, \quad r, \quad r_+, \quad \hat{r}, \quad \hat{r}_+;$$

- thermodynamic variables in the midpoints:

$$\rho = \rho_i^j, \quad \rho_- = \rho_{i-1}^j, \quad \hat{\rho} = \rho_i^{j+1}, \quad \hat{\rho}_- = \rho_{i-1}^{j+1}, \quad p, \quad p_-, \quad \hat{p}, \quad \hat{p}_-.$$

For these 21 stencil variables, we find  $16 = 21 - 5$  invariants of the symmetries (98) and (101):

$$\begin{aligned} I_1 &= \frac{h_-^s}{h^s}, \quad I_2 = \frac{\rho r^{n+1}}{h^s}, \quad I_3 = \frac{\tau}{h^s} r^n \sqrt{\rho p}, \quad I_4 = \frac{\tau u}{r}, \\ I_5 &= \frac{u_+}{u}, \quad I_6 = \frac{\hat{u}}{u}, \quad I_7 = \frac{\hat{u}_+}{\hat{u}}, \quad I_8 = \frac{r_+}{r}, \quad I_9 = \frac{\hat{r}}{r}, \quad I_{10} = \frac{\hat{r}_+}{\hat{r}}, \\ I_{11} &= \frac{\rho_-}{\rho}, \quad I_{12} = \frac{\hat{\rho}}{\rho}, \quad I_{13} = \frac{\hat{\rho}_-}{\hat{\rho}}, \quad I_{14} = \frac{p_-}{p}, \quad I_{15} = \frac{\hat{p}}{p}, \quad I_{16} = \frac{\hat{p}_-}{\hat{p}}. \end{aligned}$$



The scheme (117) is invariant with respect to the considered symmetries and can be expressed in terms of the invariants as

$$\frac{1}{I_{12}} - 1 = I_2 I_4 \left( \frac{(I_9 I_{10})^{n+1} - I_8^{n+1}}{(n+1)(I_9 I_{10} - I_8)} \frac{I_6 I_7 + I_5}{2} - \frac{I_9^{n+1} - 1}{(n+1)(I_9 - 1)} \frac{I_6 + 1}{2} \right), \quad (128a)$$

$$I_6 - 1 = -\frac{I_3^2}{I_2 I_4} \frac{I_9^{n+1} - 1}{(n+1)(I_9 - 1)} (\alpha I_{15}(1 - I_{16}) + (1 - \alpha)(1 - I_{14})), \quad (128b)$$

$$\frac{1}{\gamma - 1} \left( \frac{I_{15}}{I_{12}} - 1 \right) = -I_2 I_4 (\alpha I_{15} + (1 - \alpha)) \quad (128c)$$

$$\begin{aligned} & \times \left( \frac{(I_9 I_{10})^{n+1} - I_8^{n+1}}{(n+1)(I_9 I_{10} - I_8)} \frac{I_6 I_7 + I_5}{2} - \frac{I_9^{n+1} - 1}{(n+1)(I_9 - 1)} \frac{I_6 + 1}{2} \right), \\ & I_9 - 1 = \frac{1}{2} I_4 (1 + I_7). \end{aligned} \quad (128d)$$

### 6.3.2 Special Case $n = 0$ , $\gamma \neq \frac{n+3}{n+1}$

In the space of 21 stencil variables, there are 14 invariants for 7 symmetries (98), (99), (101):

$$\begin{aligned} I_1 &= \frac{h^s}{h^s}, \quad I_2 = \frac{\tau}{h^s} \sqrt{\rho p}, \quad I_3 = \sqrt{\frac{\rho}{p}} \left( \frac{\hat{r} - r}{\tau} - u \right), \quad I_4 = \sqrt{\frac{\rho}{p}} \left( \frac{\hat{r} - r}{\tau} - \hat{u} \right), \\ I_5 &= \sqrt{\frac{\rho}{p}} (u_+ - u), \quad I_6 = \sqrt{\frac{\rho}{p}} (\hat{u}_+ - \hat{u}), \quad I_7 = \frac{\rho(r_+ - r)}{h^s}, \quad I_8 = \frac{\hat{\rho}(\hat{r}_+ - \hat{r})}{h^s}, \\ I_9 &= \frac{\rho_-}{\rho}, \quad I_{10} = \frac{\hat{\rho}}{\rho}, \quad I_{11} = \frac{\hat{\rho}_-}{\hat{\rho}}, \quad I_{12} = \frac{p_-}{p}, \quad I_{13} = \frac{\hat{p}}{p}, \quad I_{14} = \frac{\hat{p}_-}{\hat{p}}. \end{aligned}$$

One can find the scheme (117) for  $n = 0$  approximating the gas dynamics system (95) with the help of these invariants as

$$\frac{1}{I_{10}} - 1 = I_2 \frac{I_5 + I_6}{2}, \quad (129a)$$

$$I_3 - I_4 = -I_2 (\alpha (I_{13} - I_{13} I_{14}) + (1 - \alpha)(1 - I_{12})), \quad (129b)$$

$$\frac{1}{\gamma - 1} \left( \frac{I_{13}}{I_{10}} - 1 \right) = -I_2 (\alpha I_{13} + (1 - \alpha)) \frac{I_5 + I_6}{2}, \quad (129c)$$

$$I_3 + I_4 = 0. \quad (129d)$$

### 6.3.3 Special Case $n \neq 0$ , $\gamma_* = \frac{n+3}{n+1}$

We use the same mesh and stencil as for the general case of  $\gamma$ . Due to the additional symmetry (100), we get one invariant less. We obtain the following finite-difference invariants:

$$J_1 = \frac{h^s}{h^s}, \quad J_2 = \frac{\rho r^{n+1}}{h_s}, \quad J_3 = \frac{\hat{\rho} \hat{r}^{n+1}}{h^s}, \quad J_4 = \frac{\tau r^n}{h_s} \rho^{\frac{1}{2} - \frac{1}{n+1}} \hat{\rho}^{\frac{1}{n+1}} p^{\frac{1}{2}}, \quad J_5 = \frac{\hat{p}}{p} \left( \frac{\rho}{\hat{\rho}} \right)^{\frac{n+3}{n+1}},$$

$$J_6 = \frac{r + \tau u}{\hat{r}} \quad J_7 = \frac{r_+ + \tau u_+}{\hat{r}_+}, \quad J_8 = \frac{\hat{r} - \tau \hat{u}}{r}, \quad J_9 = \frac{\hat{r}_+ - \tau \hat{u}_+}{r_+},$$

$$J_{10} = \frac{r_+}{r} \quad J_{11} = \frac{\hat{r}_+}{\hat{r}}, \quad J_{12} = \frac{\rho_-}{\rho}, \quad J_{13} = \frac{\hat{\rho}_-}{\hat{\rho}}, \quad J_{14} = \frac{p_-}{p}, \quad J_{15} = \frac{\hat{p}_-}{\hat{p}}.$$

Using these invariants, we suggest an invariant scheme

$$\hat{\rho}(\hat{r}_+^{n+1} - \hat{r}^{n+1}) = \rho(r_+^{n+1} - r^{n+1}), \quad (130a)$$

$$\frac{\hat{u} - u}{\tau} = - \left( \frac{\hat{\rho}}{\rho} \right)^{\frac{2}{n+1}} r^n \frac{p - p_-}{h^s}, \quad (130b)$$

$$\frac{\hat{p}}{\hat{\rho}^{\frac{n+3}{n+1}}} = \frac{p}{\rho^{\frac{n+3}{n+1}}}, \quad (130c)$$

$$\frac{\hat{r} - r}{\tau} = u, \quad (130d)$$

which allows explicit computations. It is expressed in terms of the invariants as

$$J_3(J_{11}^{n+1} - 1) = J_2(J_{10}^{n+1} - 1), \quad (131a)$$

$$J_8 - 1 = \frac{J_4^2}{J_2} (1 - J_{14}), \quad (131b)$$

$$J_5 = 1, \quad (131c)$$

$$J_6 = 1. \quad (131d)$$

In addition to the invariance, the scheme (130) possesses conservation of mass, given by Eq. (130a), and conservation of the entropy along pathlines, given by (130c).

We remark that the conservation of mass property can be rewritten as

$$\frac{1}{\tau} \left( \frac{1}{\hat{\rho}} - \frac{1}{\rho} \right) = \frac{R_+ u_+ - R u}{h^s} \quad \text{or} \quad \left[ \frac{1}{\rho} \right]_t - [R u]_s = 0 \quad (132)$$

with

$$h_s = \hat{\rho} \frac{\hat{r}_+^{n+1} - \hat{r}^{n+1}}{n+1} = \rho \frac{r_+^{n+1} - r^{n+1}}{n+1}.$$

### 6.3.4 Special Case $n = 0, \gamma_* = 3$

In comparison to the case  $n = 0, \gamma \neq 3$ , we have one more symmetry, namely (100). Therefore, we get one invariant less. There are 13 invariants:

$$\begin{aligned}
 J_1 &= \frac{h^s}{h^s}, & J_2 &= \frac{\tau}{h^s}(\rho p \hat{\rho} \hat{p})^{\frac{1}{4}}, & J_3 &= \sqrt{\frac{\rho}{p}}\left(\frac{\hat{r}-r}{\tau} - u\right), & J_4 &= \sqrt{\frac{\hat{\rho}}{\hat{p}}}\left(\frac{\hat{r}-r}{\tau} - \hat{u}\right), \\
 J_5 &= \sqrt{\frac{\rho}{p}}\left(\frac{h_+}{\tau} + u_+ - u\right), & J_6 &= \sqrt{\frac{\hat{\rho}}{\hat{p}}}\left(-\frac{\hat{h}_+}{\tau} + \hat{u}_+ - \hat{u}\right), & J_7 &= \frac{\rho(r_+ - r)}{h^s}, \\
 J_8 &= \frac{\hat{\rho}(\hat{r}_+ - \hat{r})}{h^s}, & J_9 &= \frac{\hat{p}}{p}\left(\frac{\rho}{\hat{\rho}}\right)^3, & J_{10} &= \frac{\rho_-}{\rho}, & J_{11} &= \frac{\hat{\rho}_-}{\hat{\rho}}, & J_{12} &= \frac{p_-}{p}, & J_{13} &= \frac{\hat{p}_-}{\hat{p}}.
 \end{aligned}$$

There are many possibilities to approximate the gas dynamics system (95a), (95b), (104), (95d) with the help of these invariants. We propose the following explicit invariant scheme:

$$\hat{\rho}(\hat{r}_+ - \hat{r}) = \rho(r_+ - r), \quad (133a)$$

$$\frac{\hat{u} - u}{\tau} = -\left(\frac{\hat{\rho}}{\rho}\right)^2 \frac{p_- - p}{h^s}, \quad (133b)$$

$$\frac{\hat{p}}{\hat{\rho}^3} = \frac{p}{\rho^3}, \quad (133c)$$

$$\frac{\hat{r} - r}{\tau} = u. \quad (133d)$$

In term of the invariants, this scheme is written as

$$J_7 = J_8, \quad (134a)$$

$$J_4 = J_2 J_9^{-3/4} (1 - J_{12}), \quad (134b)$$

$$J_9 = 1, \quad (134c)$$

$$J_3 = 0. \quad (134d)$$

The scheme conserves the entropy, or  $S$ , along the pathlines and possesses conservation of mass (133a). Note that the first equation can be rewritten as

$$\frac{1}{\tau} \left( \frac{1}{\hat{\rho}} - \frac{1}{\rho} \right) = \frac{u_+ - u}{h^s} \quad \text{or} \quad \left[ \frac{1}{\rho} \right]_t - [u]_s = 0 \quad (135)$$

with

$$h^s = \hat{\rho}(\hat{r}_+ - \hat{r}) = \rho(r_+ - r).$$

We remark that implicit invariant schemes are also possible.

## 7 Conclusion

In the chapter, we examined one-dimensional flows of a polytropic gas and their Lie point symmetry properties. By the one-dimensional flows we mean plain one-dimensional flows, the gas dynamics flows with radial symmetry, and the gas dynamics flows with spherical symmetry. There was performed the Lie group classification of the gas dynamics equations reduced to a single second-order PDE in the Lagrangian coordinates. The entropy function was a parameter of the classification. Four cases were identified. In the general case, there are conservation laws of mass and energy. For the special cases, there were found additional conservation laws. The conservation laws obtained for the second-order PDE were later rewritten for the gas dynamics variables. They were also transformed from the Lagrangian coordinates to the Eulerian ones.

Difference models were discussed for different cases of  $n$  and  $\gamma$ . It is shown that the Samarskii–Popov scheme is invariant for the symmetries of the general case of  $\gamma$ , but not for the additional symmetry of the special case  $\gamma_* = \frac{n+3}{n+1}$ . This scheme possesses conservation of mass and energy, for  $n = 0$  also conservation of momentum and motion of the center of mass. It does not have conservation of the entropy along the pathlines. For the special values  $\gamma_*$ , we suggest invariant schemes, which have conservation of mass and conservation of the entropy along the pathlines.

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