Application of Newton-GS Iterative Method with Second-Order Quadrature Scheme in Solving Nonlinear Fredholm Integral Equations

Labiyana Hanif Ali, Jumat Sulaiman, Azali Saudi, and Xu Ming Ming

Abstract In this study, we discuss the application of Newton-GS iterative method with quadrature schemes in solving nonlinear Fredholm integral equations. This study proposes the application of Newton-Gauss–Seidel (NGS) iteration with the second-order quadrature scheme in getting the approximate solution of nonlinear Fredholm integral equations of the second kind (NFIE-2) in comparison with the first-order quadrature scheme. The main idea of this study is to apply the second-order quadrature scheme to discretize the NFIE-2 to form a system of nonlinear integral equations. Then we convert the nonlinear system into the corresponding linear system by imposing the Newton approach. By having this large-scale and sparse linear system, the numerical implementation of Newton-Jacobi (NJ) and NGS iteration approaches alone with first- and second-quadrature schemes have recorded their number of iterations, computational time, and maximum absolute error. As a result of these measured parameters, the comparative study can be performed to gauge the effectiveness of NGS with second-order quadrature scheme when compared with the numerical results of first-order quadrature scheme and NJ iteration. Based on numerical experiments, it can be important to highlight that the implementation of NGS iteration with second-order quadrature scheme has significantly improved the accuracy of its approximate results.

Keywords Nonlinear Fredholm integral equations · Quadrature scheme · Simpson's 1/3 rule · Gauss–Seidel iteration · Newton linearization

L. H. Ali (✉) · J. Sulaiman · X. M. Ming
Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Sabah, Malaysia

J. Sulaiman
e-mail: jumat@ums.edu.my

X. M. Ming
e-mail: xmmzg@sina.com

A. Saudi
Faculty of Computing and Informatics, Universiti Malaysia Sabah, Sabah, Malaysia
e-mail: azali@ums.edu.my

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1 Introduction

Academically, the problem of NFIE-2 has been solved using several analytical and numerical methods such as generalized extrapolation method [1], collocation method [2–4], multi projection method [5], Homotopy perturbation method [6], Adomian decomposition method [7], parameter continuation method [8], successive approximation method [9], etc. However, due to its important application in science and engineering, the studies of solving this problem is continued until now. Recently, the researchers have proposed several methods including Nyström method [10], Nyström-quasilinearization method [11], and trigonometric basis functions [12] to solve the problem of NFIE-2.

Inspired by the accuracy of the data through implementation of higher order quadrature scheme, this study is focusing on the implementation of the second-order quadrature scheme to solve the NFIE-2 in the following form

$$u(t) = f(t) + \int_a^b k(t, x, u(x))dx, \quad x \in [a, b], \tag{1}$$

where k is continuous on interval $[a, b]$, $f(t)$ is known function and $u(t)$ is the unknown function [13]. Family of quadrature schemes is one of numerical integration schemes which used widely in numerical studies due to its useful properties. A system of linear or nonlinear equations can be generated from a single approximation equation through discretization process using a particular quadrature scheme. Many studies have been conducted with implementing these quadrature schemes to discretize the problem of differential and integral equations such as in [14–18]. The implementation of different type or order of quadrature schemes have influenced the accuracy of the approximate solutions.

Since one of the advantages of the second-order quadrature rule, in comparison with first-order quadrature rule is that it provides more accurate approximation equations by considering more grid points on interval $[a, b]$, subsequently, the implementation of the composite Simpson's 1/3 (CS1) scheme is expected to be resulting more accurate solutions in terms of maximum absolute error. Prior to progressively utilizing these quadrature schemes, we design the methodology of solving nonlinear Fredholm integral equations using the NGS and NJ iterative methods with the composite Simpson's 1/3 (CS1) scheme namely NGS-CS1 and NJ-CS1. As for comparative effect, we also establish the formulation and implementation of NGS and NJ iterative methods with the composite Trapezoidal (CT) scheme namely NGS-CT and NJ-CT. Moreover, we just use NJ-CT as a control method to test the efficiency of other three proposed iterative methods particularly NGS-CS1, NGS-CT and NJ-CS1 in solving NFIE-2.

This paper is organized as follows: In Sect. 2, we will discuss the methodology of this study by implementing the second-order quadrature scheme mainly on the composite Simpson's 1/3 (CS1) scheme towards NFIE-2 to form a system of nonlinear integral equations. Then, we discuss the Newton-GS iteration on solving the corresponding nonlinear system. In Sect. 3, we present some numerical examples from the previous studies to illustrate the effectiveness of the proposed approach. Then, we discuss the numerical findings of this studies. Finally, we make some conclusions and suggestion for further studies on solving NFIE-2 in Sect. 4.

2 Methodology

The methodology of this study is design by two main parts: in part one, we discretize the nonlinear Fredholm integral equations using first- and second-order quadrature schemes to form nonlinear integral system. In part two, using NJ and NGS iterative methods, firstly, we need to do the linearization process over the generated nonlinear system using Newton's method to get the corresponding linear system. Later, the linear system can be solved iteratively by using NGS-CS1, NGS-CT, NJ-CS1 and NJ-CT iterative methods respectively to get their numerical solutions.

2.1 Discretization of NFIE-2 Using First- and Second-Order Quadrature Schemes

The formulation of integration function in solving NFIE-2 in (1) using quadrature scheme can be defined as follows

$$\int_a^b f(x)dx = \sum_{j=0}^{n} A_j f\left(x_j\right) + \epsilon_n(f) \tag{2}$$

where t_j, $(j = 0, 1, \ldots, n)$ are abscissas of the partition points of the integration interval on interval $[a, b]$, A_j, $j = 0, 1, 2, \ldots, n$ are the numerical coefficients and $\epsilon_n(y)$ is the truncation error. Constant A_j, $j = 0, 1, 2, \ldots, n$ for Trapezium rule is defined as [16, 17]

$$A_j = \begin{cases} \frac{1}{2}h, & j = 0, n \\ h, & otherwise \end{cases} \tag{3}$$

whereas the value of A_j, $j = 0, 1, 2, \ldots, n$ for Simpson's 1/3 is denoted in the following expression

$$A_j = \begin{cases} \frac{1}{3}h, & j = 0, n \\ \frac{4}{3}h, & j = 1, 3, 5, \ldots, n-1 \\ \frac{2}{3}h, & otherwise \end{cases} \tag{4}$$

where

$$h = \frac{b - a}{n}. \quad (5)$$

The differences between Trapezoidal and Simpson's 1/3 rule can be illustrated as follows.

Figure 1 shows the comparison of finite grid networks between Trapezoidal rule and Simpson's 1/3 rule on interval $[a, b]$. For illustration, let the total number of subintervals to be n. The implementation of Trapezoidal rule will consider as much $(n + 1)$ node points while the Simpson's 1/3 will also consider $(n + 1)$ total node point. This means Simpson's 1/3 rule consider more node points on interval $[a, b]$ resulting the neighboring point between the node points to be increased. Thus, it will

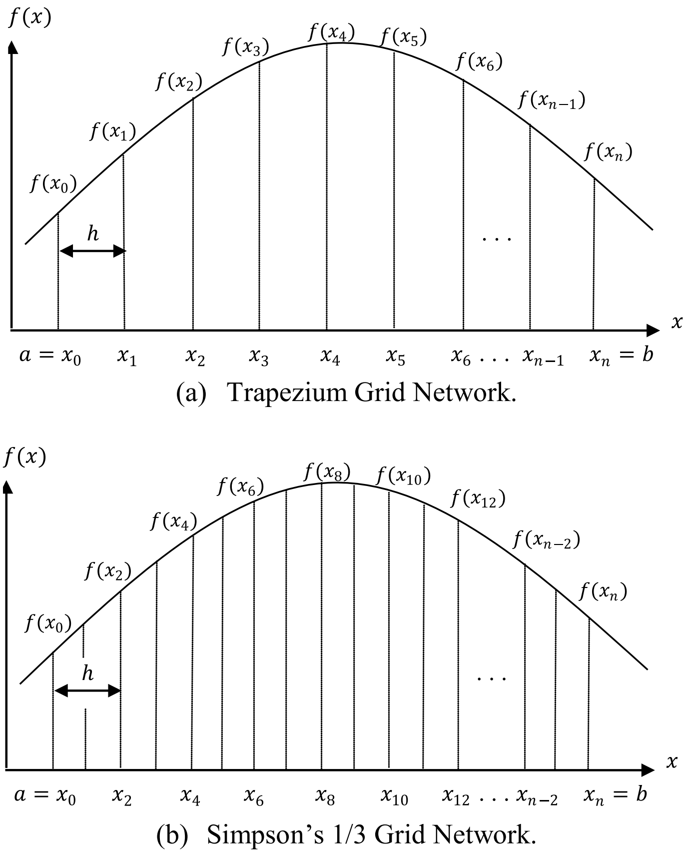

Fig. 1 The comparison of finite grid networks for **a** Trapezium and **b** Simpson's 1/3 on interval $[a, b]$

help to reduce the neighboring distance between the node points and provide more accurate approximate solution of the problem.

In general, we can form the following nonlinear approximation equations based on the implementation of quadrature scheme in integration part of (1) for $i, j = 0, 1, 2, \ldots, n$, as follows

$$\begin{aligned} u_i - A_j hk(t, x_0, u_0) - A_j k(t, x_1, u_1) - A_j k(t, x_2, u_2) - \cdots \\ - A_j hk(t, x_n, u_n) = f_i \end{aligned} \tag{6}$$

The nonlinear approximation (6) can be easily specified by replacing the A_j based on the given values in (3) and (4) for first- and second-order quadrature schemes. The nonlinear function of (6) can be defined as

$$\begin{aligned} G_i(u_0, u_1, u_2, \ldots, u_n) = u_i - A_j hk(t, x_0, u_0) - A_j hk(t, x_1, u_1) \\ - A_j hk(t, x_2, u_2) - \cdots - A_j hk(t, x_n, u_n) - f_i \end{aligned} \tag{7}$$

Thus, we can form the following nonlinear system of NFIE-2 in the following form

$$G_i(u_0, u_1, u_2, \ldots, u_n) = 0, \quad i = 0, 1, 2, 3, \ldots, n \tag{8}$$

2.2 Formulation of NGS Iteration with Quadrature Schemes

In the second part, we will discuss the formulation of NGS iteration with quadrature schemes to solve the generated nonlinear system in Part A. Using Newton's method, we can represent the corresponding nonlinear system (8) into a linear system as follows [19]

$$J\left(\underline{u}^{(k)}\right)\Delta u^{(k)} = -G\left(\underline{u}^{(k)}\right) \tag{9}$$

where

$$J\left(\underline{u}^{(k)}\right) = \begin{bmatrix} \frac{dG_0}{du_0} & \frac{dG_0}{du_1} & \frac{dG_0}{du_2} & \cdots & \frac{dG_0}{du_n} \\ \frac{dG_1}{du_0} & \frac{dG_1}{du_1} & \frac{dG_1}{du_2} & \cdots & \frac{dG_1}{du_n} \\ \frac{dG_2}{du_0} & \frac{dG_2}{du_1} & \frac{dG_2}{du_2} & \cdots & \frac{dG_2}{du_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{dG_n}{du_0} & \frac{dG_n}{du_1} & \frac{dG_n}{du_2} & \cdots & \frac{dG_n}{du_n} \end{bmatrix}_{(n+1)\times(n+1)},$$

$$\Delta u^{(k)} = \left[\Delta\, u_0 \;\; \Delta u_1 \;\; \Delta u_2 \;\; \cdots \;\; \Delta u_n\right]^T,$$

and $\underline{u}^{(k)}$ is determined by $u_i^{(k+1)} = u_i^{(k)} + \Delta u_i$.

To solve the linear system of (9), Jacobian matrix, $J\left(\underline{u}^{(k)}\right)$ needs to be decompose into $J\left(\underline{u}^{(k)}\right) = D - L - U$, where D is diagonal matrix, L is strictly lower matrix, and U is strictly upper matrix so we can formulate the formulation of Gauss–Seidel iteration to solve NFIE-2 in (9) as follows [20]

$$\Delta\underline{u}^{(k+1)} = (D - L)^{-1} U \Delta\underline{u}^{(k)} + (D - L)^{-1} \underline{f} \tag{10}$$

Algorithm 1 Implementation of NGS-CS1 Iteration.

i. Let $\nabla\underline{u}^{(k)} = 0$, $k = 0$ and $\varepsilon = 10^{-10}$.
ii. Set $q = 0$ and compute matrix $J\left(\underline{u}^{(k)}\right)$ and $G\left(\underline{u}^{(k)}\right)$.
iii. Compute the current value, $\Delta u_i^{(k+1)}$
   a. For $i = 0, 1, 2, \ldots, n$ and $j = 0, 1, 2, \ldots, n$, calculate

$$\Delta u_i^{(k+1)} \leftarrow \frac{1}{A_{i,i}} \begin{pmatrix} f_i - \frac{1}{A_{i,i}} \left(\sum_{j=0}^{i-1} A_{i,j} \Delta u_j^{(k+1)} \right) \\ -\frac{1}{A_{i,i}} \left(\sum_{j=i+1}^{n} A_{i,j} \Delta u_j^{(k)} \right) \end{pmatrix}.$$

   b. Conduct the convergence test, $\left|\Delta u_i^{(k+1)} - \Delta u_i^{(k)}\right| \leq \varepsilon$. If satisfied, continue to step iv, otherwise repeat step iii(a).
iv. Conduct the convergence test, $\left|G\left(\underline{u}^{(k+1)}\right)\right| \leq \varepsilon$. If satisfied, display the approximate solution, and otherwise repeat step iii.
v. Display the output.
vi. Stop.

3 Numerical Experiments and Discussion

This study considers five large mesh size which are 512, 1024, 2048, 4096, and 8192. Using three main parameters, number of iteration (Iter), computational time (Time) and maximum absolute error (Err), we compared the data obtained for four suggested iterative methods, NGS-CS1, NGS-CT, NJ-CS1 and NJ-CT on three numerical examples as stated here.

Example 1 Consider the following NFIE-2 problem [21]

$$u(t) = 1 - \frac{5}{12}t + \int_0^1 tx[u(x)]^2 dx \tag{11}$$

where the exact solution for this problem is $u(t) = 1 + \frac{1}{3}t$.

Example 2 Consider the following NFIE-2 problem [22]

$$u(t) = -\frac{t}{9} - \frac{t^2}{8} + t^3 + \int_0^1 \left(t^2x + tx^2\right)u^2(x)dx \tag{12}$$

where the exact solution for this problem is $u(t) = t^3$.

Example 3 Consider the following NFIE-2 problem [23]

$$u(t) = t + \frac{\cos\left(e^{(1)} + t\right) - \cos(1 + t)}{20} + \int_0^1 \frac{\sin\left(e^{(x)} + t\right)}{20} e^{(u(x))}dx \tag{13}$$

where the exact solution for this problem is $u(t) = t$.

Tables 1, 2 and 3 show the implementation of NJ and NGS using Trapezium and Simpson's 1/3 does not bring much different in terms of number of iteration and iteration time. But in terms of maximum absolute error, the approximate solutions for all proposed problems recorded more accurate approximate error using Simpson's 1/3 rule compared to Trapezium rule. Figures 2, 3 and 4 demonstrates th3e comparison of Newton-iterative methods in terms of maximum absolute error graphically for Examples 1, 2, and , respectively. The figures show NGS-CS1 iterative method have provided more accurate solutions compared to methods with Trapezium rule. This is

Table 1 Numerical results of NJ and NGS iterative methods using Trapezium and Simpson's 1/3 for Example 1

		Mesh size				
		512	1024	2048	4096	8192
Iter	NJ-CT	329	330	330	330	330
	NGS-CT	183	183	183	184	184
	NJ-CS1	329	330	330	330	330
	NGS-CS1	183	183	183	184	184
Time	NJ-CT	1.66	6.65	26.56	106.12	423.31
	NGS-CT	0.94	3.76	14.97	60.23	240.11
	NJ-CS1	1.68	6.62	26.45	105.7	422.29
	NGS-CS1	0.96	3.75	14.96	60.15	241.12
Err	NJ-CT	3.18E−06	7.94E−07	1.98E−07	4.93E−08	1.20E−08
	NGS-CT	3.18E−06	7.95E−07	1.99E−07	4.95E−08	1.23E−08
	NJ-CS1	3.89E−10	3.92E−10	3.94E−10	3.95E−10	3.95E−10
	NGS-CS1	1.23E−10	1.24E−10	1.25E−10	1.25E−10	1.25E−10

Table 2 Numerical results of NJ and NGS iterative methods using Trapezium and Simpson's 1/3 for Example 2

		Mesh size				
		512	1024	2048	4096	8192
Iter	NJ-CT	105	105	105	105	105
	NGS-CT	64	64	64	64	64
	NJ-CS1	105	105	105	105	105
	NGS-CS1	64	64	64	64	64
Time	NJ-CT	0.55	2.19	8.78	35.05	138.82
	NGS-CT	0.35	1.41	5.54	22.00	87.79
	NJ-CS1	0.58	2.19	8.70	34.75	138.96
	NGS-CS1	0.36	1.39	5.48	21.91	87.93
Err	NJ-CT	1.13E−05	2.82E−06	7.04E−07	1.76E−07	4.40E−08
	NGS-CT	1.13E−05	2.82E−06	7.04E−07	1.76E−07	4.40E−08
	NJ-CS1	4.47E−11	5.57E−11	6.28E−11	6.36E−11	6.39E−11
	NGS-CS1	9.81E−11	6.99E−12	1.14E−11	1.18E−11	1.19E−11

Table 3 Numerical results of NJ and NGS iterative methods using Trapezium and Simpson's 1/3 for Example 3

		Mesh size				
		512	1024	2048	4096	8192
Iter	NJ-CT	22	22	22	22	22
	NGS-CT	18	18	18	18	18
	NJ-CS1	22	22	22	22	22
	NGS-CS1	18	18	18	18	18
Time	NJ-CT	1.69	6.73	26.94	108.01	428.01
	NGS-CT	1.42	5.66	22.72	90.89	360.65
	NJ-CS1	1.69	6.76	26.99	107.97	441.91
	NGS-CS1	1.43	5.69	23.01	91.07	363.81
Err	NJ-CT	1.45E−07	3.63E−08	9.08E−09	2.27E−09	5.68E−10
	NGS-CT	1.45E−07	3.63E−08	9.08E−09	2.27E−09	5.68E−10
	NJ-CS1	2.84E−13	3.83E−14	2.82E−14	2.76E−14	2.78E−14
	NGS-CS1	2.95E−13	1.84E−14	5.27E−15	4.80E−15	5.12E−15

due to the implementation of high order quadrature scheme which helps to increase the neighborhood distance between each node points in interval $[a, b]$ which resulting more accurate results. Moreover, when comparing both NJ-CS1 and NGS-CS1, NGS-CS1 iteration recorded more accurate results compared to NJ-CT.

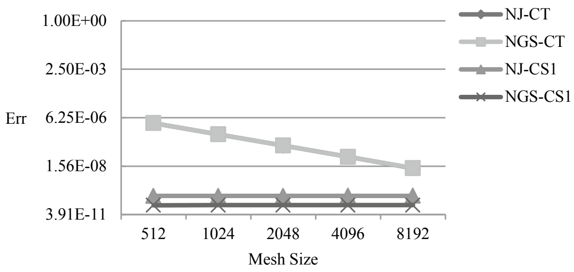

Fig. 2 Plots of maximum absolute error for of NJ and NGS iterative methods using Trapezium and Simpson's 1/3 (Example 1)

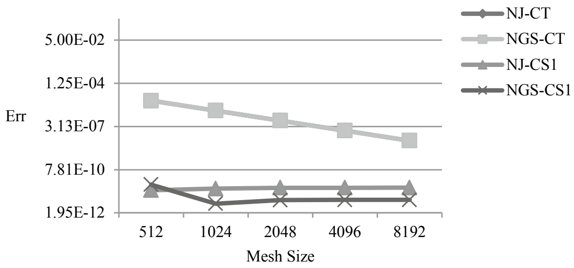

Fig. 3 Plots of maximum absolute error for of NJ and NGS iterative methods using Trapezium and Simpson's 1/3 (Example 2)

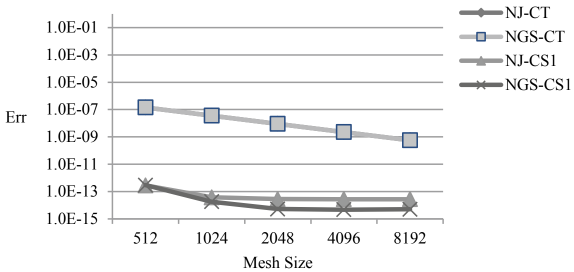

Fig. 4 Plots of maximum absolute error for of NJ and NGS iterative methods using Trapezium and Simpson's 1/3 (Example 3)

4 Conclusions

In this study, we discuss the application of Newton-iterative method with first- and second-order quadrature schemes in getting the approximate solution of NFIE-2. Based on the data obtained, we can conclude that the implementation of second-order quadrature scheme can improve the accuracy of the output compared to the first-order quadrature scheme. This means the approximate solution obtained for NFIE-2 is very close to the exact solutions. Thus, we conclude that Newton-GS iteration with second-order quadrature scheme, NGS-CS1 to be the most efficient method in solving NFIE-2 compared to the rest tested methods in this study. In the future study, this finding had a wide potential to be extended using the combination of the half-sweep iteration concept together with the weighted parameter iteration family, specifically to weighted parameter [16, 17], modified weighted parameter

[24, 25] and Accelerated parameter [26, 27] iteration families to reduce the iteration number and computational time and to improve the accuracy of the approximate solution.

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