A Total Variation Regularization Method for Inverse Source Problem with Uniform Noise

Huan Pan and You-Wei Wen

Abstract The problem of inverse source problem is considered in this paper. The main aim of this problem is to determine the source density function from the state function which is corrupted by uniform noise. Under the framework of maximum a posteriori estimator, the problem can be converted into an optimization problem where the objective function is composed of an L_∞ norm and a total variation (TV) regularization term. By introducing an auxiliary variable, the optimization problem is further converted into a minimax problem. Then first order primal-dual method is applied to find the saddle point of the minimax problem. Numerical examples are given to demonstrate that our proposed method outperforms the other testing methods.

Keywords Inverse problem · Uniform noise · Total variation · L_∞-norm constraint · Linear systems.

1 Introduction

In this paper, we consider the numerical solution of an elliptic inverse source problem [16, 17]. Inverse source problems arise in many areas of mathematical physics, and applications in recent year are rapidly expanding to such areas as geophysics, chemistry, medicine, engineering and mathematical imaging [5, 25]. The phenomena in these applications are generally described by partial differential equations. An inverse source problem for an elliptic partial differential equations on the domain $\Omega \in R^2$ with homogeneous Dirichlet boundary condition is given as follows [10]:

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H. Pan · Y.-W. Wen (✉)
Key Laboratory of Computing and Stochastic Mathematics (LCSM) (Ministry of Education of China), School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, Hunan, P.R. China

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$$\begin{cases} -\nabla \cdot (a(x)\nabla u) + \langle b(x), \nabla u\rangle_{L^2(\Omega)} + c(x)u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $a(x)$ and $c(x)$ are two given bounded and positive functions in Ω, $b(x)$ denotes the velocity of flow, $u(x)$ represents state function, and $f(x)$ is the source density function. If the coefficients $a(x), c(x)$ and the source function $f(x)$ are directly given, we need to estimate the state function $u(x)$, the problem in (1) is called a forward source problem. However, in any physical and engineering problems such as pollutant detection and imaging science, we can acquire the state function $u(x)$ at the boundary of the reconstruction region, i.e., the measurement data $u(x)$ is available, but we need to estimate the source function $f(x)$. It is an inverse source problem [15]. The main aim of the inverse source problem is to determine f from the state function u.

We shall focus our attention to find a numerical solution of the inverse source problem (1) in this paper. The discrete model of (1) can be represented by using vectors and matrices. With the lexicographical ordering of $\mathbf{u}$ and $\mathbf{f}$, their relationship can be expressed as follows:

$$K\mathbf{u} = \mathbf{f}.$$

Here K is the matrix generated by the elliptic partial differential equations. Assume that the size of $\mathbf{u}$ is $N \times M$, $\mathbf{u}_{ij}$ denotes the $((i-1)N + j$-th component of $\mathbf{u}$. If the solution $\mathbf{u}$ is obtained, the source $\mathbf{f}$ can be computed directly by the matrix and the vector product. The solution $\mathbf{u}$ is generally associated with the boundary value which is an observation with errors, this is that $\mathbf{u}$ is corrupted by the noise $\mathbf{n}$ and the observation $\mathbf{u}_\delta$ is given by $\mathbf{u}_\delta = \mathbf{u} + \mathbf{n}$. Hence we obtain $\mathbf{f} = K(\mathbf{u}_\delta - \mathbf{n})$. Since the observation data is corrupted by the measurement errors (noise), the source $\mathbf{f}$ can not be calculated by the product of the matrix and the vector. The observation data $\mathbf{u}_\delta$ can be rewritten as

$$\mathbf{u}_\delta = K^{-1}\mathbf{f} + \mathbf{n}.$$

In mathematics, the inverse source problem is ill-conditioned in the sense of Hadamard [13, 20], namely, small perturbation (quantization errors) in the measurement data may lead to the lack of stability of numerical inversions. The ill-conditioning can be alleviated to stabilize the solution by incorporating the priori source information, and the solution $\mathbf{f}$ can be formulated as a minimizer of the following minimization problem

$$\min_{\mathbf{f}} \psi(\mathbf{u}_\delta, \mathbf{f}) + \lambda\phi(\mathbf{f}).$$

Here the function $\psi(\mathbf{u}_\delta, \mathbf{f})$ is the data-fitting term to represent the distribution of the measurement error $\mathbf{n}$, the function $\phi(\mathbf{f})$ is the regularization term to represent the prior knowledge of $\mathbf{f}$, and λ is a regularization parameter.

In this paper, we assume that the measurement data $u(x)$ is corrupted by a uniform distribution noise. This is that $\mathbf{n}_i$ (the i-th entry of $\mathbf{n}$) are the independent identically distributed samples with uniform distribution $U(-c, c)$, here c denotes the noise level. According the distribution function of $\mathbf{n}$, we can derive the data-fitting term by $\psi(\mathbf{u}_\delta, \mathbf{f}) = \left\| K^{-1}\mathbf{f} - \mathbf{u}_\delta \right\|_\infty$, see [10, 27]. In the literatures [2, 13, 18, 19], a Tikhonov-type function was used to represent the prior knowledge in the inverse problems.

Numerical difficulty is caused due to non-differentiability of the L_∞-norm in the data-fitting term. In [10, 27], the minimization problem was reformulated into a constrained one. In [10], a Moreau-Yosida approximation for L_∞-norm constraint was considered, and the authors then applied a semi-smooth Newton method to solve for the resulting optimality condition. In [27], the L_∞-norm constraint was handled by active set constraints arising from the optimality conditions, and then an efficient semi-smooth Newton method was applied to find a solution.

In this paper, we consider that the source function is a piecewise continuous function and apply the total variation (TV) function [24] to represent its prior knowledge. The TV regularization has been widely used in many problems such as image denoising [1, 24], image restoration [3, 6], image segmentation [7, 8] and so on. However, to best of our knowledge, there are few papers using the TV function as a regularization term in the inverse source problem. We remark that both the data-fitting term and the regularization term considered in this paper are non-differentiable, we develop different numerical scheme to find a minimizer.

The remainder of the paper is structured as follows. In Sect. 2, we review the inverse source problem and propose total variation regularization method to find its solution. In Sect. 3, we transform the inverse source problem into an equivalent minimax problem and then apply first order primal-dual algorithm to solve it. In Sect. 4, Appling our proposed approach to address given numerical examples of the Inverse Source Problem. Finally, the Sect. 5 concludes this paper.

2 Total Variation Regularization for Inverse Source Problem

In this section, we consider a total variation (TV) regularization approach for inverse source problem. The minimization problem can be written as

$$\min_{\mathbf{f}} \left\| K^{-1}\mathbf{f} - \mathbf{u}_\delta \right\|_\infty + \lambda \left\| \nabla \mathbf{f} \right\|_1 . \quad (2)$$

Here $\|\nabla \mathbf{f}\|_1$ denotes the TV norm of $\mathbf{f}$. The TV norm is defined by $\phi(\mathbf{f}) = \|\nabla \mathbf{f}\|_1$, here

$$(\nabla \mathbf{f})_{i,j} = ((\nabla_x \mathbf{f})_{i,j}, (\nabla_y \mathbf{f})_{i,j})$$

with

$$(\nabla_x \mathbf{f})_{i,j} = \begin{cases} \mathbf{f}_{i+1,j} - \mathbf{f}_{i,j}, & \text{if } i < N, \\ 0, & \text{if } i = N, \end{cases} \quad \left(\nabla_y \mathbf{f}\right)_{i,j} = \begin{cases} \mathbf{f}_{i,j+1} - \mathbf{f}_{i,j}, & \text{if } j < M, \\ 0, & \text{if } j = M. \end{cases}$$

We remark that the data-fitting term in (2) is derived by the assumption of uniform noise in the observation data. Considering an independent $U(-\delta, \delta)$ random variable X, where δ stands for the noise level. Since $\mathbf{n}_i$ (the i-th entry of $\mathbf{n}$) are the independent identically distributed samples with uniform distribution, the likelihood function is given by

$$\prod_{i=1}^{L} f_X(\mathbf{n}_i | \mathbf{u}_\delta, \delta) \propto \mathcal{I}(\mathbf{n}_1, \ldots, \mathbf{n}_L \in [-\delta, \delta]),$$

where the indicator function $\mathcal{I}(S)$ equals to 1 if S happens and 0 otherwise. If at least one $\mathbf{n}_i$ (i.e., $(\mathbf{u}_\delta - K^{-1}\mathbf{f})_i$) falls outside of the interval $[-\delta, \delta]$, the likelihood will be equal to 0. Therefore, the solution of (2) should be any $\mathbf{u}$ that satisfies $\left\|\mathbf{u}_\delta - K^{-1}\mathbf{f}\right\|_\infty \le \delta$. Therefore, the minimization problem in (2) can be rewritten as

$$\min_{\mathbf{f}} \|\nabla \mathbf{f}\|_1 \quad \text{s.t.} \quad \left\|K^{-1}\mathbf{f} - \mathbf{u}_\delta\right\|_\infty \le \delta. \tag{3}$$

In fact, the minimization problem in (2) and (3) are mathematically equivalent. Given a regularization parameter λ in (2), there exists a δ such that the solution of (2) is also the solution of (3). In contrast, given a δ in (3), there also exists a regularization parameter λ in (2) such that the solution of (3) is also the solution of (2), moreover, $1/\lambda$ is the Lagrangian multiplier corresponding the L_∞-norm inequality constraint. It is very important to choose a suitable regularization parameter λ in (2), because λ balances the data-fitting term and the regularization term and avoids to over-fitting or under-fitting the data. Compare to tune the regularization parameter λ, it is more easier to choose the noise level δ because δ is the noise level in the observation data. When δ is not available, it can be estimated by the method of moments [27]. In this paper, we will focus on the numerical scheme to solve (3). Although many methods have been proposed in the literature to find the minimizer of TV-based optimization problem, it is non-trivial to find the minimizer of (3) because both the TV norm and the L_∞ norm are non-differentiable, also the minimizer should satisfy the inequality constraint. In the next section, we will consider the numerical scheme to find a minimizer of (3).

3 Primal-Dual Approach

In this section, we find the minimizer of the inverse source problem (3) by transforming it into a minimax problems. Then we solve it by a primal-dual method [9, 11, 14, 22, 23, 26, 30, 31]. We will apply Chambolle-Pock first order primal-dual algorithm in [9] to seek the saddle point of our minimax problem. We therefore give a brief introduction of the method here.

3.1 Chambolle-Pock's First-Order Primal-Dual Algorithm

In [9], Chambolle and Pock considered solving the minimax problem:

$$\min_{\mathbf{v}} \max_{\mathbf{z}} \Phi(\mathbf{v}) + \langle \mathbf{v}, H\mathbf{z} \rangle - \Psi(\mathbf{z}). \tag{4}$$

Here Φ, Ψ are propoer, convex and lower semi-continuous functions, and H is a linear operator with induced norm $\|H\|$. They proposed to solve the problem by a first-order primal-dual algorithm as follows:

$$\begin{cases} \mathbf{v}^{(k+1)} = \underset{\mathbf{v}}{\operatorname{argmin}} \Phi(\mathbf{v}) + \langle \mathbf{v}, H\mathbf{z} \rangle + \frac{1}{2t} \left\| \mathbf{v} - \mathbf{v}^{(k)} \right\|_2^2, \\ \widehat{\mathbf{v}}^{(k+1)} = \mathbf{v}^{(k+1)} + \mu(\mathbf{v}^{(k+1)} - \mathbf{v}^{(k)}), \\ \mathbf{z}^{(k+1)} = \underset{\mathbf{z}}{\operatorname{argmax}} \left\langle \widehat{\mathbf{v}}^{(k+1)}, H\mathbf{z} \right\rangle - \Psi(\mathbf{z}) - \frac{1}{2s} \left\| \mathbf{z} - \mathbf{z}^{(k)} \right\|_2^2. \end{cases} \tag{5}$$

The parameters $s, t > 0$ are step sizes of the primal and dual variables respectively, and μ is the combination parameter. In the iterative procedure, proximal-point iterations are applied to the sub-differentials of the $\mathbf{v}$ and $\mathbf{z}$ subproblems in (5) with the primal variable and the dual variable fixed alternately.

3.2 Minimax Problem

Let us describe the notations that we will use in the followings. For $\boldsymbol{\xi} \in \mathbb{R}^{NM} \times \mathbb{R}^{NM}$, $\boldsymbol{\xi}_{i,j} = (\boldsymbol{\xi}_{i,j,1}, \boldsymbol{\xi}_{i,j,2}) \in \mathbb{R}^2$ denotes the $(i + (j-1)n)$-th component of $\boldsymbol{\xi}$. Define the inner product $\langle \boldsymbol{\xi}, \mathbf{q} \rangle = \sum_{i,j} \boldsymbol{\xi}_{i,j} \mathbf{q}_{i,j}$ for $\boldsymbol{\xi}, \mathbf{q} \in \mathbb{R}^{nm} \times \mathbb{R}^{nm}$. Define $\|\boldsymbol{\xi}\|_\infty = \max_{i,j} |\boldsymbol{\xi}_{i,j}|$. Define $\operatorname{div} = -\nabla^T$ as the discrete version of the divergence operator, where ∇^T is the adjoint of ∇, i.e.,

$$(\operatorname{div} \boldsymbol{\xi})_{i,j} = \begin{cases} \xi^x_{i,j} & i = 1 \\ \xi^x_{i,j} - \xi^x_{i-1,j} & 1 < i < N \\ -\xi^x_{i-1,j} & i = N \end{cases} + \begin{cases} \xi^y_{i,j} & j = 1, \\ \xi^y_{i,j} - \xi^y_{i,j-1} & 1 < j < N, \\ -\xi^y_{i,j-1} & j = N. \end{cases}$$

We represent the TV norm using the dual form, i.e.,

$$\|\nabla \mathbf{f}\|_1 = \max_{\|\boldsymbol{\xi}\|_\infty \le 1} \langle \operatorname{div} \boldsymbol{\xi}, \mathbf{f} \rangle. \tag{6}$$

Using the dual formulation, the minimization problem (3) can be written as the following minimax problem:

$$\min_{\left\|K^{-1}\mathbf{f} - \mathbf{u}_\delta\right\|_\infty \le \delta} \max_{\|\boldsymbol{\xi}\|_\infty \le 1} \langle \mathbf{f}, \operatorname{div} \boldsymbol{\xi} \rangle. \tag{7}$$

Introducing the auxiliary varible $\mathbf{r} = \mathbf{u}_\delta - K^{-1}\mathbf{f}$, we obtain $\mathbf{f} - K(\mathbf{u}_\delta - \mathbf{r}) = 0$. We consider Lagrangian function for the resulting equation

$$\mathcal{L}(\mathbf{f}, \mathbf{r}, \boldsymbol{\xi}, \mathbf{y}) \equiv \langle \mathbf{f}, \mathrm{div}\boldsymbol{\xi} \rangle + \langle \mathbf{y}, \mathbf{f} - K(\mathbf{u}_\delta - \mathbf{r}) \rangle . \tag{8}$$

Here $\mathbf{y}$ is the Lagrange multiplier associated with the equality constraint $\mathbf{f} - K(\mathbf{u}_\delta - \mathbf{r}) = 0$. Hence, we have

$$\max_{\|\boldsymbol{\xi}\|_\infty \le 1, \mathbf{y}} \mathcal{L}(\mathbf{f}, \mathbf{r}, \boldsymbol{\xi}, \mathbf{y}) = \begin{cases} \|\nabla \mathbf{f}\|_1 , & \text{if } \ \mathbf{f} - K(\mathbf{u}_\delta - \mathbf{r}) = 0, \\ \infty, & \text{otherwise.} \end{cases}$$

Also we have

$$\min_{\mathbf{f}} \mathcal{L}(\mathbf{f}, \mathbf{r}, \boldsymbol{\xi}, \mathbf{y}) = \begin{cases} \langle \mathrm{div}\boldsymbol{\xi}, K(\mathbf{u}_\delta - \mathbf{r}) \rangle , & \text{if } \ \mathrm{div}\boldsymbol{\xi} + \mathbf{y} = 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

According to [4, Proposition 5.5.4], we know that the minimum and the maximum in (8) can be swapped and there exists a saddle point of $\mathcal{L}$. We obtain

$$\min_{\|\mathbf{r}\|_\infty \le \delta, \mathbf{f}} \max_{\|\boldsymbol{\xi}\|_\infty \le 1, \mathbf{y}} \mathcal{L}(\mathbf{f}, \mathbf{r}, \boldsymbol{\xi}, \mathbf{y}) = \max_{\|\boldsymbol{\xi}\|_\infty \le 1} \min_{\|\mathbf{r}\|_\infty \le \delta} \langle \mathrm{div}\boldsymbol{\xi}, K(\mathbf{u}_\delta - \mathbf{r}) \rangle .$$

Thus we have the following theorem.

Theorem 1 *Define* $\mathcal{Q}(\mathbf{r}, \boldsymbol{\xi}) = \langle \mathrm{div}\boldsymbol{\xi}, K(\mathbf{u}_\delta - \mathbf{r}) \rangle$*, then we have*

$$\min_{\left\|K^{-1}\mathbf{f} - \mathbf{u}_\delta\right\|_\infty \le \delta} \|\nabla \mathbf{f}\|_1 = \max_{\|\boldsymbol{\xi}\|_\infty \le 1} \min_{\|\mathbf{r}\|_\infty \le \delta} \mathcal{Q}(\mathbf{r}, \boldsymbol{\xi}).$$

Moreover, the minimum in the left-hand side above is attained at $\mathbf{f}^* = K(\mathbf{u}_\delta - \mathbf{r}^*)$*, here* $(\mathbf{r}^*, \boldsymbol{\xi}^*)$ *is the saddle point of the function* $\mathcal{Q}(\mathbf{r}, \boldsymbol{\xi})$*.*

Now we apply Chambolle-Pock's first-order primal-dual method (5) to compute the saddle point of $\mathcal{Q}(\mathbf{r}, \boldsymbol{\xi})$, the iterative scheme is given as follows:

$$\mathbf{r}^{k+1} = \operatorname*{argmin}_{\|\mathbf{r}\|_\infty \le \delta} \mathcal{Q}(\mathbf{r}, \boldsymbol{\xi}^k) + \frac{1}{2s} \left\|\mathbf{r} - \mathbf{r}^k\right\|_2^2 \tag{9}$$

$$\widehat{\mathbf{r}}^{k+1} = \mathbf{r}^{k+1} + \theta(\mathbf{r}^{k+1} - \mathbf{r}^k) \tag{10}$$

$$\boldsymbol{\xi}^{k+1} = \operatorname*{argmax}_{\|\boldsymbol{\xi}\|_\infty \le 1} \mathcal{Q}(\widehat{\mathbf{r}}^{k+1}, \boldsymbol{\xi}) - \frac{1}{2t} \left\|\boldsymbol{\xi} - \boldsymbol{\xi}^k\right\|_2^2 \tag{11}$$

3.3 Subproblem for $\mathbf{r}$

The minimization of (9) reduces to

$$\mathbf{r}^{k+1} = \underset{\mathbf{r}}{\operatorname{argmin}} \left\langle \operatorname{div}\boldsymbol{\xi}^k, K(\mathbf{u}^\delta - \mathbf{r}) \right\rangle + \frac{1}{2s} \left\| \mathbf{r} - \mathbf{r}^k \right\|_2^2 \tag{12}$$

$$= \underset{\mathbf{r}}{\operatorname{argmin}} \left\| \mathbf{r} - (\mathbf{r}^k - sK^T \operatorname{div}\boldsymbol{\xi}^k) \right\|_2^2 \tag{13}$$

We first introduce the concept of the projection operator.

$$\mathcal{P}(\mathbf{w}) = \underset{\mathbf{r}\in\Omega}{\operatorname{argmin}} \|\mathbf{r} - \mathbf{w}\|_2^2 . \tag{14}$$

In general, the projection onto a general convex set is difficult and computationally expensive. As the L_∞-constraints can be formulated as the bounded constraints, the corresponding closed-form solution is given by

$$[\mathcal{P}(\mathbf{w})]_i = \begin{cases} \delta, & \mathbf{w}_i \geq \delta. \\ \mathbf{w}_i, & |\mathbf{w}_i| < \delta. \\ -\delta, & \mathbf{w}_i \leq -\delta. \end{cases}$$

By using a suitable projection operator, we can view $\mathbf{r}^{k+1}$ as the projection of $\left(\mathbf{r}^{k+1} - sK^T \operatorname{div}\boldsymbol{\xi}^k\right)$ on Ω. Thus we obtain

$$\mathbf{r}^{k+1} = \mathcal{P}\left(\mathbf{r}^{k+1} - sK^T \operatorname{div}\boldsymbol{\xi}^k\right). \tag{15}$$

3.4 Subproblem for $\boldsymbol{\xi}$

We change the maximization problem for $\boldsymbol{\xi}$ in (11) to a minimization one and obtain:

$$\boldsymbol{\xi}^{k+1} = \underset{\boldsymbol{\xi}}{\operatorname{argmax}} \left\langle \operatorname{div}\boldsymbol{\xi}, K(\mathbf{u}^\delta - \widehat{\mathbf{r}}^{k+1}) \right\rangle - \frac{1}{2t} \left\| \boldsymbol{\xi} - \boldsymbol{\xi}^k \right\|_2^2 \tag{16}$$

$$= \underset{\|\boldsymbol{\xi}\|_\infty \leq 1}{\operatorname{argmin}} -\left\langle \operatorname{div}\boldsymbol{\xi}, K(\mathbf{u}^\delta - \widehat{\mathbf{r}}^{k+1}) \right\rangle + \frac{1}{2t} \left\| \boldsymbol{\xi} - \boldsymbol{\xi}^k \right\|_2^2 \tag{17}$$

Thus

$$\boldsymbol{\xi}^{k+1} = \mathcal{P}_{\mathcal{A}}(\boldsymbol{\xi}^k - t\nabla K(\mathbf{u}^\delta - \widehat{\mathbf{r}}^{k+1}))$$

where $\mathcal{A} = \left\{\boldsymbol{\xi} : \|\boldsymbol{\xi}\|_\infty \leq 1\right\}$, the gradient projection of $(\boldsymbol{\xi}^k - t\nabla \mathbf{f}^{k+1})$ onto the set $\mathcal{A}$. In the following, we derive a formula for the gradient projection operator

$$\mathcal{P}_{\mathcal{A}}(\mathbf{q}) = \operatorname*{argmin}_{\mathbf{p}\in\mathcal{A}} \|\mathbf{p}-\mathbf{q}\|_2^2$$

For any $\mathbf{q}$, by the definition of the set $\mathcal{A}$, the Lagrangian function is

$$\|\mathbf{p}-\mathbf{q}\|_2^2 + \sum_{i,j} t_{i,j}\left(|p_{i,j}|^2 - 1\right),$$

where $t_{i,j} \geq 0$ is the Lagrangian multiplier associated with the constraint $|p_{i,j}|^2 \leq 1$. Its complementarity conditions implies that for the optimal $t_{i,j}$, either $t_{i,j} = 0$ with $|p_{i,j}|, |q_{i,j}| < 1$, or $t_{i,j} > 0$ with $|p_{i,j}| = 1$ and $|q_{i,j}| \geq 1$. In the former case, we have $p_{i,j} = q_{i,j}$. In the latter case, the KKT conditions yields $p_{i,j} - q_{i,j} + t_{i,j}p_{i,j} = 0$ for all i, j. Therefore, we have $t_{i,j} = |q_{i,j}| - 1$, and thus $p_{i,j} = q_{i,j}/|q_{i,j}|$. Hence, we obtain

$$(\mathcal{P}_{\mathcal{A}}(\mathbf{q}))_{i,j} = \frac{1}{\max(1, |q_{i,j}|)} q_{i,j}. \tag{18}$$

4 Numerical Results

In this section, three numerical experiments are implemented to demonstrate the effectiveness of the proposed method, that is to consider the inverse source problem (1) with domain $\Omega = [0, 1]^2$. We investigate the influence of noise level on the numerical results, specifically, set $\delta = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$. In order to be able to stabilize the convergence of approximate solution by our proposed method, choosing primal variation step size $s = 5 \times 10^{-8}$ and dual variation step size $t = 5 \times 10^{-2}$. We show the exact solution $\mathbf{f}^*$ with size 256×256 in Fig. 1a, the exact data $\mathbf{u}^*$ in Fig. 1b and observation data $\mathbf{u}_\delta$ with noise level $\delta = 0.1, 0.4, 0.9$ respectively in Fig. 2.

In the following experiments, we compare our algorithm(TV) with semi-smooth Newton mehthod (SSN) [27], Primal-Dual method (PD) [21] and Forward Backward

(a) exact solution $\mathbf{f}^*$ (b) exact data $\mathbf{u}^*$

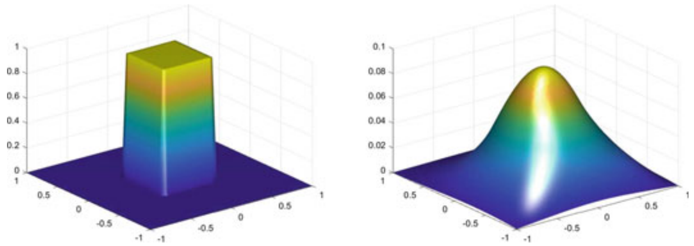
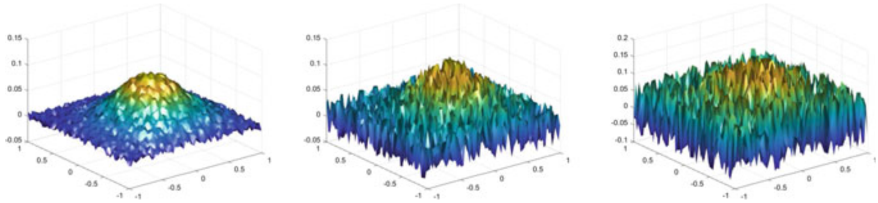


Fig. 1 Left: the exact solution $\mathbf{f}^*$ with size 256×256; Right: the exact data $\mathbf{u}^*$



$\delta = 0.1$ $\delta = 0.4$ $\delta = 0.9$

Fig. 2 Observed data $\mathbf{u}_\delta$ corrupted by uniform noise levels with the $\delta = 0.1, 0.4, 0.9$ respectively

method (FB) [12]. The Root-Mean-Square-Error (RMSE) is used to quantitatively measure the quality of the estimated solution. It is defined as follows:

$$\text{RMSE} = \frac{1}{\sqrt{L}} \left\| \widehat{\mathbf{f}} - \mathbf{f}^* \right\|_2$$

where $\mathbf{f}^*$ denotes the exact solution and $\widehat{\mathbf{f}}$ denotes the estimated solution. The smaller RMSE is, the better the estimated solution is.

We consider the discrete problem (1) and set $a(x) = 1$ and $c(x) = 0$. Three different functions for $b(x)$ are used in the tests, they are $b(x) = -[2, 0]$, $b(x) = -[0, 1]$ and $b(x) = -[2, 1]$ respectively. In order to quantitatively measure the accuracy of the estimated solutions, we show the RMSE values for different noise level in Table 1. We note that the RMSE of recovery data by four methods gradually increase with the noise level increasing. The RMSE obtained by TV method is smaller than that obtained by other methods.

We show the estimated solutions obtained by different methods in the Figs. 3, 4 and 5 with different noise levels $\delta = 0.1, 0.4, 0.9$, respectively. We can observe that there are some jumps in the estimated solutions obtained by SNN method and FB method. The estimated solutions obtained by PD method look smooth. We remark that the Tikhonov-type regularization function is applied in the these three methods. It is obviously that the estimated solutions obtained by the proposed method are closer to the true solution.

5 Conclusion

In this paper, we study the inverse source problem where observation data are corrupted by uniform noise. The main contribution of this paper is to develop an efficient total variation regularization method for solving the ill-posed inverse source problem of the L_∞-norm data fitting. Numerical examples are given to demonstrate that our proposed method outperforms the other testing methods.

Table 1 RMSE of estimated solution for different noise levels

δ	$\|\mathbf{u}_\delta - \mathbf{u}\|_2$	SSN	PD	FB	TV
		$a = 1, b = -[2, 0]$			
0.1	9.36e-03	1.19e-01	1.25e-01	1.34e-01	2.36e-02
0.3	2.80e-02	1.33e-01	1.36e-01	1.39e-01	3.85e-02
0.5	4.68e-02	1.33e-01	1.41e-01	1.48e-01	6.95e-02
0.7	6.55e-02	1.49e-01	1.46e-01	1.53e-01	9.94e-02
0.9	8.42e-02	1.53e-01	1.50e-01	1.58e-01	1.18e-01
		$a = 1, b = -[0, 1]$			
0.1	9.93e-03	1.19e-01	1.26e-01	1.35e-01	1.82e-02
0.3	2.97e-02	1.32e-01	1.36e-01	1.41e-01	3.34e-02
0.5	4.96e-02	1.34e-01	1.42e-01	1.47e-01	7.16e-02
0.7	6.95e-02	1.50e-01	1.47e-01	1.55e-01	9.73e-02
0.9	8.93e-02	1.52e-01	1.51e-01	1.60e-01	8.86e-02
		$a = 1, b = -[2, 1]$			
0.1	9.24e-03	1.18e-01	1.25e-01	1.27e-01	1.95e-02
0.3	2.77e-02	1.32e-01	1.35e-01	1.37e-01	3.65e-02
0.5	4.62e-02	1.32e-01	1.41e-0	1.43e-01	7.03e-02
0.7	6.47e-02	1.50e-01	1.47e-01	1.50e-01	1.04e-01
0.9	8.32e-02	1.51e-01	1.49e-01	1.52e-01	1.16e-01

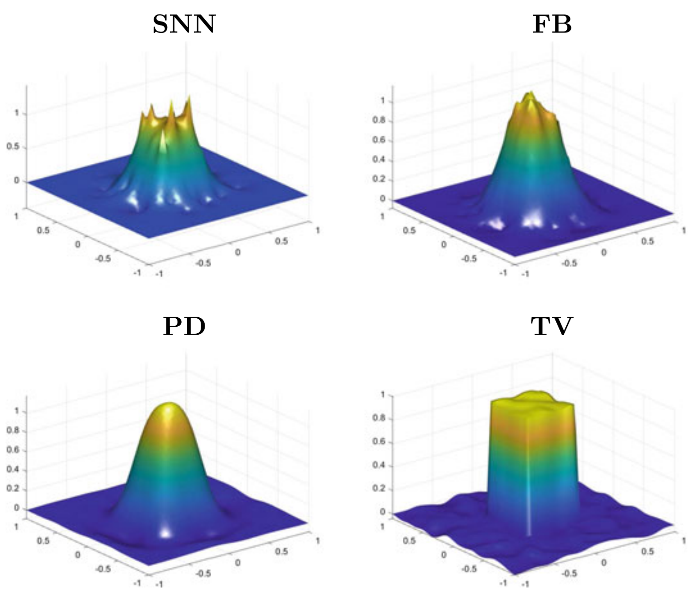


Fig. 3 Root-mean-square-error (RMSE) values obtained by different methods for different noise levels. Here $\delta = 0.1$ and $a = 1, b = -[2, 0]$

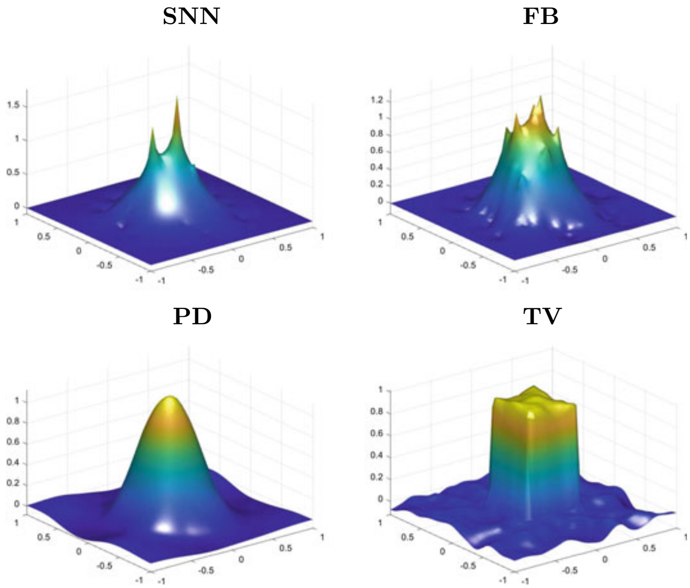


Fig. 4 Four algorithms recovering data graphs with noise levels of $d = 0.4$ based on $a = 1, b = -[0, 1]$

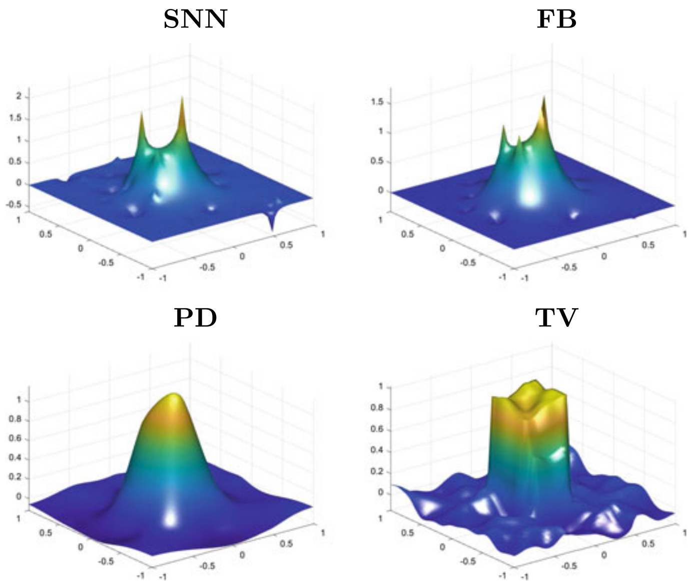


Fig. 5 Four algorithms recovering data graphs with noise levels of $\delta = 0.9$ based on $a = 1, b = -[2, 1]$

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