Parametric Directed Divergence Measure for Pythagorean Fuzzy Set and Their Applications to Multi-criteria Decision-Making

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1 Introduction

In literature so far, the classical information theory has been widely used and represents the vagueness in the data in classical measure theory but the measures are valid for precisely given data. Even, due to the various constraints in day-to-day life, decision-makers may give their judgements under the uncertain and imprecise situation. Thus, there is always a degree of hesitancy between the preferences of the decision-making so that the analysis conducted under such circumstances is not ideal and hence does not tell the exact information to the system analyst. To cope up with impreciseness, vagueness, and the uncertainty in the data, the intuitionistic fuzzy sets (IFSs) [1] are successful extension of the fuzzy set (FS) [2]. Over the last several years, the IFS has received much attention by introducing the various kinds of information measures, aggregation operators and employed them to solve the decision-making problems under the different environment [3–10]. But, there is some limitation in the studies of IFSs as it is valid only for the environments where the degree's sum is less than one. However, this condition is ruled out in many situations. For instance, if a person gives their preference in the form of membership and non-membership degrees toward a particular object as 0.8 and 0.5, then the situation is not handled with IFSs. In order to resolve it, Yager [11, 12] proposed the Pythagorean fuzzy (PF) sets (PFSs) by relaxing this sum condition to its square sum less than one, i.e., corresponding to the above-considered example, we see that $(0.8)^2 + (0.5)^2 \leq 1$ and hence PFSs are an extension of the existing IFSs. After their pioneer work, Yager and Abbasov [13] studied the relationship between the Pythagorean numbers and the complex numbers. Later on, several aggregation operators under the PFS environment have been investigated by researchers [14, 15] using different norm operations. Zhang and Xu [16] extended the TOPSIS approach from

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IF to the PF environment. Garg [6] presented a confidence level-based averaging and geometric aggregation operators, by incorporating the confidence level of the decision-makers (DMs) to the analysis. In the continuation, several authors introduced different types of aggregation operators under PFSs [17–19] to solve many decision-making problems.

Now, the degree of distance/similarity/divergence measures has been focused by the authors and received attention in the last four decades for solving the decision-making, pattern recognition, medical diagnosis problems. However, the prime task for decision-maker (DM) is to rank the alternatives to get the best [20–23]. For this, researchers have made efforts to enrich the concept of information measures in Pythagorean fuzzy environments [24]. Zhang and Xu [16] suggested a distance measure to solve a realistic problem under PFS. While Yang et al. [25] pointed out an unreasonable case of proof in [16]. Wei and Wei [26] presented some similarity measures between PFSs which are actually based on the cosine function. Li et al. [27] introduced the Hamming distance measure, the Euclidean distance measure, and the Minkowski distance measure between PFSs, with their detailed properties. Zhang [28] explored a novel similarity measure for PFSs, to deal the selection problem of photovoltaic cells. Zeng et al. [29] considered five parameters for distance and similarity measures of Pythagorean fuzzy sets and applied them in the selection of China's Internet stocks. Peng et al. [30] presented similarity measure, distance measure, entropy, and inclusion measure for PFSs, put forward transformation relationships, and successfully applied them in pattern recognition, clustering analysis, and medical diagnosis [31].

Thus, the observation from the above studies is that all the measures do not incorporate the idea of decision-maker's preferences into the measure and also these measures do not follow the linear order. That's why, there is always a trouble in getting the exact nature of the alternative. Therefore, we present a parametric directed divergence measure order α and degree β for Pythagorean fuzzy set (PFS). Through this proposed measure, the decision-maker can make more reliable and flexible decisions for different values of parameters α and β. Several properties have been investigated based on this measure with a numerical example to demonstrate the performance of measure. Finally, concrete conclusion has been presented.

2 Basic Concepts

In this section, some basic definitions of IFSs and PFSs have been presented on the universal set X.

2.1 Intuitionistic Fuzzy Set [1]

Definition 2.1 An IFS (intuitionistic fuzzy set) is defined as

$$\bar{A} = \left\{ \left\langle x, \mu_{\bar{A}}(x), \nu_{\bar{A}}(x) \right\rangle : x \in X \right\},$$

where $\mu_{\bar{A}}(x)$ and $\nu_{\bar{A}}(x)$ represent the membership and non-membership degrees such that $0 \leq \mu_{\bar{A}}(x) + \nu_{\bar{A}}(x) \leq 1$ with $\mu_{\bar{A}}(x), \nu_{\bar{A}}(x) \in [0, 1]$..

2.2 Hesitant Fuzzy Set [32, 33]

Definition 2.2 A HFS (hesitant fuzzy set) is defined as a function $HFS : X \to [0, 1]$ and is given by

$$\bar{A} = \left\{ \left\langle x, h_{\bar{A}}(x) \right\rangle : x \in X \right\},$$

where $h_{\bar{A}}(x)$ represents HFE (hesitant fuzzy element).

2.3 Pythagorean Fuzzy Set [11, 16]

Definition 2.3 A PFS (Pythagorean fuzzy set) is defined as a set of ordered pairs given by

$$\bar{A} = \left\{ \left\langle x, \mu_{\bar{A}}(x), \nu_{\bar{A}}(x) \right\rangle : x \in X \right\},$$

where $\mu_{\bar{A}}(x)$ and $\nu_{\bar{A}}(x)$ represent the membership and non-membership degrees such that $(\mu_{\bar{A}}(x))^2 + (\nu_{\bar{A}}(x))^2 \leq 1$ with $\mu_{\bar{A}}(x), \nu_{\bar{A}}(x) \in [0, 1]$. For convenience, the pair of these membership functions is called a Pythagorean fuzzy number (PFN) and it is denoted as $\alpha = <\mu_\alpha, \nu_\alpha>$.

3 Proposed Parametric Directed Divergence Measure for Pythagorean Fuzzy Set (PFS)

In this section, we have proposed a flexible and generalized parametric divergence measure of order α and degree β denoted as class of (α, β), under the environment of PFSs. Some desirable properties of this measure are also being studied.

3.1 Parametric Divergence Measure for PFSs

Definition 3.1 Let A and B be the two PFSs defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, then a parametric directed divergence measure for PFSs based on parameters α and β is denoted as $D_\alpha^\beta(A|B)$ and defined as

$$D_\alpha^\beta(A|B) = \frac{\alpha}{n(2-\beta)} \sum \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}}{\mu_A^{\frac{2\alpha}{(2-\beta)}} + \mu_B^{\frac{2\alpha}{(2-\beta)}}} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}}{\nu_A^{\frac{2\alpha}{(2-\beta)}} + \nu_B^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}}{\pi_A^{\frac{2\alpha}{(2-\beta)}} + \pi_B^{\frac{2\alpha}{(2-\beta)}}} \right) \end{array} \right],$$

where μ, ν, and π are the membership, non-membership and hesitancy functions, respectively, and it is valid for $\alpha,\ \beta > 0$ and except $\beta \neq 2$.

It is clearly seen from the definition that the $D_\alpha^\beta(A|B)$ is not symmetric, so to imbue the measure with symmetry, a parametric symmetric divergence measure for PFSs has been defined as follows.

3.2 Parametric Symmetric Divergence Measure for PFSs

Definition 3.2 Let A and B be the two PFSs defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, then a parametric directed divergence measure for PFSs based on parameters α and β is denoted as $D_\alpha^\beta(A;, B)$ and defined as

$$D_\alpha^\beta(A;B) = D_\alpha^\beta(A\,|B) + D_\alpha^\beta(B|A) \quad \Rightarrow$$

$$D_\alpha^\beta(A;B) = \frac{\alpha}{n(2-\beta)} \sum \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}}{\mu_A^{\frac{2\alpha}{(2-\beta)}} + \mu_B^{\frac{2\alpha}{(2-\beta)}}} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}}{\nu_A^{\frac{2\alpha}{(2-\beta)}} + \nu_B^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}}{\pi_A^{\frac{2\alpha}{(2-\beta)}} + \pi_B^{\frac{2\alpha}{(2-\beta)}}} \right) \end{array} \right] + \frac{\alpha}{n(2-\beta)} \sum \left[\begin{array}{r} \mu_B^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}}{\mu_A^{\frac{2\alpha}{(2-\beta)}} + \mu_B^{\frac{2\alpha}{(2-\beta)}}} \right) + \nu_B^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}}{\nu_A^{\frac{2\alpha}{(2-\beta)}} + \nu_B^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_B^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}}{\pi_A^{\frac{2\alpha}{(2-\beta)}} + \pi_B^{\frac{2\alpha}{(2-\beta)}}} \right) \end{array} \right]$$

From the definition of $D_\alpha^\beta(A \mid B)$ and $D_\alpha^\beta(A;\, B)$, it has been observed that

$$D_\alpha^\beta(A \mid B) \geq 0, \quad D_\alpha^\beta(A;\, B) \geq 0,$$

and $A = B \Rightarrow D_\alpha^\beta(A \mid B) = D_\alpha^\beta(A; B)$.

Divide the universe X into two parts X_1 and X_2, where

$$\begin{aligned} X_1 &= \{x_i : x_i \in X, A(x_i) \subseteq B(x)\}, \text{ i.e.,} \\ \mu_A(x_i) &\le \mu_B(x_i),\ \nu_A(x_i) \ge \nu_B(x_i)\ \forall\, x_i \in X_1, \\ X_2 &= \{x_i : x_i \in X, A(x_i) \supseteq B(x)\}, \text{ i.e.,} \\ \mu_A(x_i) &\ge \mu_B(x_i),\ \nu_A(x_i) \le \nu_B(x_i)\ \forall\, x_i \in X_2, \end{aligned}.$$

Now, we propose some properties based on the above considerations.

3.3 Some Properties of Parametric Symmetric Divergence Measure for PFSs

Property 3.3.1 Let A and B be the two PFSs defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, such that they satisfy for any $x_i \in X$ either $A \subseteq B$ or $B \subseteq A$,

$$D_\alpha^\beta(A \cup B;\ A \cap B) = D_\alpha^\beta(A; B).$$

Proof It is clear that

$$D_\alpha^\beta(A \cup B;\ A \cap B) = D_\alpha^\beta(A \cup B| A \cap B) + D_\alpha^\beta(A \cap B|A \cup B).$$

On the other hand,

$$\begin{aligned}
&D_\alpha^\beta(A \cup B| A \cap B) \\
&= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{r} \mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}} + \nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \\
&= \frac{\alpha}{n(2-\beta)} \left[\begin{array}{l} \displaystyle\sum_{x\in X_1} \left[\begin{array}{r} \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_B^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_B^{\frac{2\alpha}{(2-\beta)}} + \nu_A^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] + \\ \displaystyle\sum_{x\in X_2} \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}} + \nu_B^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \end{array} \right]
\end{aligned} \tag{1}$$

and

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A\cap B|\, A\cup B)\\
&=\frac{\alpha}{n\,(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{r}\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\;+\;\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}\,\log\left(\dfrac{2\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}\;+\;\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}}\right)\\
+\,\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&=\frac{\alpha}{n\,(2-\beta)}\left[\begin{array}{l}\displaystyle\sum_{x\in X_1}\left[\begin{array}{r}\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\;+\;\nu_{A}^{\frac{2\alpha}{(2-\beta)}}\,\log\left(\dfrac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}\;+\;\nu_{B}^{\frac{2\alpha}{(2-\beta)}}}\right)\\
+\,\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]+\\
\displaystyle\sum_{x\in X_2}\left[\begin{array}{r}\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\;+\;\nu_{B}^{\frac{2\alpha}{(2-\beta)}}\,\log\left(\dfrac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}\;+\;\nu_{A}^{\frac{2\alpha}{(2-\beta)}}}\right)\\
+\,\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\end{array}\right].
\end{aligned}
\tag{2}
$$

Then, by adding Eqs. (1) and (2), we get

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A\cup B|\, A\cap B)\;+\;D_{\alpha}^{\beta}(A\cap B|\, A\cup B)\\
&=\frac{\alpha}{n\,(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{r}\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\;+\;\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\,\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right].\\
&+\;\frac{\alpha}{n\,(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{r}\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\;+\;\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\nu_{B}^{\frac{2\alpha}{(2-\beta)}(x_i)}}\right)\\
+\,\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\;+\;\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&=D_{\alpha}^{\beta}(A\,|B)\;+\;D_{\alpha}^{\beta}(B|\,A)=D_{\alpha}^{\beta}(A;\;B)
\end{aligned}
$$

Thus, the results hold.

Property 3.3.2 For any two PFSs A and B defined on universal set $X = \{x_1,\ x_2,\ \ldots,x_n\}$, we have

(1) $D_{\alpha}^{\beta}(A;A\cup B)=D_{\alpha}^{\beta}(B;A\cap B)$

(2) $D_{\alpha}^{\beta}(A;A\cap B)=D_{\alpha}^{\beta}(B;A\cup B)$

(3) $D_\alpha^\beta(A; A\cup B) + D_\alpha^\beta(A; A\cap B) = D_\alpha^\beta(A; B)$

(4) $D_\alpha^\beta(B; A\cup B) + D_\alpha^\beta(B; A\cap B) = D_\alpha^\beta(A; B)$.

Proof All can be proved similarly. So, we prove only the first one, i.e.,

$$\begin{aligned} D_\alpha^\beta(A; A\cup B) &= D_\alpha^\beta(B; A\cap B) \\ &\Rightarrow D_\alpha^\beta(A\,|A\cup B) + D_\alpha^\beta(A\cup B|A). \\ &= D_\alpha^\beta(B|A\cap B) + D_\alpha^\beta(A\cap B|B) \end{aligned} \tag{3}$$

We consider the LHS. By the definition of divergence,

$$\begin{aligned} &D_\alpha^\beta(A\,|A\cup B) \\ &= \frac{\alpha}{n\,(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right) \\ +\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right] \\ &= \frac{\alpha}{n\,(2-\beta)}\sum_{x\in X_1}\left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right) \\ +\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right] \\ &= \frac{\alpha}{n\,(2-\beta)}\sum_{x\in X_{2_1}}\left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right) \\ +\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right] \\ &\Theta\ \log(1) = 0 \\ &= \frac{\alpha}{n\,(2-\beta)}\sum_{x\in X_1}\left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right) \\ +\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right] \end{aligned}$$

and

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A\cup B|A)\\
&=\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{r}\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\left[\begin{array}{r}\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right].\\
&=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_{2_1}}\left[\begin{array}{r}\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&\Theta\ \log(1)=0\\
&=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\left[\begin{array}{r}\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]
\end{aligned}
$$

Similarly, RHS

$$
\begin{aligned}
&D_{\alpha}^{\beta}(B|A\cap B)\\
&=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\left[\begin{array}{r}\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A\cap B|B)\\
&=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\left[\begin{array}{r}\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right].
\end{aligned}
$$

By using Eq. (3), the expressions are same from both sides. This proves the result.

Property 3.3.3 For any two PFSs A and B defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, we have

(1) $D_\alpha^\beta(A; C) + D_\alpha^\beta(B; C) - D_\alpha^\beta(A \cup B; C) \geq 0$

(2) $D_\alpha^\beta(A; C) + D_\alpha^\beta(B; C) - D_\alpha^\beta(A \cap B; C) \geq 0.$

Proof Both can be proved similarly. So, we prove only the first one, i.e.,

$$D_\alpha^\beta(A; C) = D_\alpha^\beta(A|C) + D_\alpha^\beta(C|A)$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{l} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ \qquad + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right]$$

$$+ \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{l} \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ \qquad + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right]$$

and

$$D_\alpha^\beta(B; C) = D_\alpha^\beta(B|C) + D_\alpha^\beta(C|B)$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{l} \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ \qquad + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right].$$

$$+ \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{l} \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ \qquad + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right]$$

Now,

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A \cup B; C) \\
&= \frac{\alpha}{n\,(2-\beta)} \sum_{i=1}^{n} \left[\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right] \\
&+ \frac{\alpha}{n\,(2-\beta)} \sum_{i=1}^{n} \left[\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right] \\
&= \frac{\alpha}{n\,(2-\beta)} \sum_{x \in X_1} \left[\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right. \\
&\quad + \pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\
&\quad \left. + \nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right]. \\
&+ \frac{\alpha}{n\,(2-\beta)} \sum_{x \in X_2} \left[\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right. \\
&\quad + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\
&\quad \left. + \nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right]
\end{aligned}
$$

Then,

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A; C) + D_{\alpha}^{\beta}(B; C) - D_{\alpha}^{\beta}(A \cup B; C) = \\
&= \frac{\alpha}{n\,(2-\beta)} \sum_{x \in X_2} \left[\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right. \\
&\quad + \pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\
&\quad \left. + \nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right]. \\
&+ \frac{\alpha}{n\,(2-\beta)} \sum_{x \in X_1} \left[\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right. \\
&\quad + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\
&\quad \left. + \nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right]
\end{aligned}
$$

Since all the membership and non-membership lies between [0, 1]. This completes the proof.

Property 3.3.4 For any two PFSs A and B defined on universal set $X = \{x_1,\ x_2,\ \ldots, x_n\}$, we have

$$D_\alpha^\beta(A\cap B; C)\ +\ D_\alpha^\beta(A\cup B; C) = D_\alpha^\beta(A; C)\ +\ D_\alpha^\beta(B; C)$$

Proof

$$\begin{aligned}
&D_\alpha^\beta(A\cap B; C)\\
&= \frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
\qquad+\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{l}\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
\qquad+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&= \frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\left[\begin{array}{l}\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&+\frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}\left[\begin{array}{l}\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]
\end{aligned}$$

Now,

$$
\begin{aligned}
&D_\alpha^\beta(A\cup B; C)\\
&= \frac{\alpha}{n\,(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{l} \mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ \qquad + \pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \\
&\quad + \frac{\alpha}{n\,(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{l} \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ \qquad + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \\
&= \frac{\alpha}{n\,(2-\beta)} \sum_{x\in X_1} \left[\begin{array}{l} \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right]. \\
&\quad + \frac{\alpha}{n\,(2-\beta)} \sum_{x\in X_2} \left[\begin{array}{l} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right]
\end{aligned}
$$

By adding all of the above equations, we get the required result and this completes the proof.

Property 3.3.5 For any two PFSs A and B defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, we have

(1) $D_\alpha^\beta(A\,; B) = D_\alpha^\beta(A^c; B^c)$

(2) $D_\alpha^\beta(A\,; B^c) = D_\alpha^\beta(A^c; B)$

(3) $D_\alpha^\beta(A\,; B) + D_\alpha^\beta(A^c\,; B) = D_\alpha^\beta(A^c; B^c) + D_\alpha^\beta(A\,; B^c)$.

Proof Clearly, first and second parts are similar and the third one can be proved by adding these two. So, we prove only (1).

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A;B)\\
&= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{l} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ \qquad + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \\
&+ \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{l} \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ \qquad + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right].
\end{aligned}
$$

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A^c;B^c) =\\
&= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{l} \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ \qquad + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \\
&+ \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{l} \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ \qquad + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right]
\end{aligned}
$$

Then (1) holds.

4 Decision-Making Method Based on Proposed Parametric Directed Divergence Measure for Pythagorean Fuzzy Set (PFS)

In this section, we shall investigate the decision-making problem based on Proposed Parametric Directed Divergence Measure D_{α}^{β} in which the attribute values are evaluated by the expert which give their preferences in terms of Pythagorean fuzzy numbers PFNs. Assume that a set of "m" alternatives $A = \{A_1,\ A_2, \ldots,\ A_m\}$ to be considered under the set of "n" criterion $G = \{G_1,\ G_2, \ldots G_n\}$. Experts have evaluated these "m" alternatives under each criterion and give their rating value in the form of IFNs. Then, we have the following steps for computing the best alternative(s) based on the proposed measure.

Step 1: ***Construction of decision-making matrix***: Suppose $D_{m \times n}(x_{i\,j}) = \langle \mu_{i\,j}, \nu_{i\,j} \rangle$ be the intuitionistic fuzzy decision matrix, where μ_{ij} represents the degree that the alternative A_i satisfies the criteria G_j and ν_{ij} indicates the degree that the alternative A_i does not satisfy the criteria G_j given by the decision-maker such that $\mu_{ij}, \nu_{ij} \in [0, 1]$ with $\mu_{ij} + \nu_{ij} \leq 1$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. So, the intuitionistic fuzzy decision matrix is constructed as follows:

$$D_{m\times n}(x_{ij}) = \begin{bmatrix} \langle\mu_{11},\nu_{11}\rangle & \langle\mu_{12},\nu_{12}\rangle & \cdots & \langle\mu_{1n},\nu_{1n}\rangle \\ \langle\mu_{21},\nu_{21}\rangle & \langle\mu_{22},\nu_{22}\rangle & \cdots & \langle\mu_{2n},\nu_{2n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\mu_{m1},\nu_{m1}\rangle & \langle\mu_{m2},\nu_{m2}\rangle & \cdots & \langle\mu_{mn},\nu_{mn}\rangle \end{bmatrix}.$$

Step 2: ***Compute the ideal alternatives***: Ideal alternative is denoted as A^* and given as

$$A^* = \left\{\langle\mu_1^*,\ \nu_1^*\rangle,\ \langle\mu_2^*,\ \nu_2^*\rangle,\ \ldots,\ \langle\mu_n^*,\ \nu_n^*\rangle\right\},$$

where $\mu_j^* = \max_{i=1}^{m}(\mu_{ij})$ and $\nu_j^* = \min_{i=1}^{m}(\nu_{ij})$.

Step 3: ***Evaluation of proposed Symmetric Divergence Measure***: Now we calculate $D_\alpha^\beta(A_i;\ A^*),\ i = 1, 2, \ldots, m$ by the given formula:

$$\begin{aligned} &D_\alpha^\beta(A_i; A^*) \\ &= \frac{\alpha}{n(2-\beta)} \sum_{j=1}^{n} \left[\begin{aligned} &\mu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) \log\left(\frac{2\mu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\mu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \mu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right) + \nu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) \log\left(\frac{2\nu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\nu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \nu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right) \\ &+ \pi_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) \log\left(\frac{2\pi_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\pi_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \pi_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right) \end{aligned} \right] \\ &+ \frac{\alpha}{n(2-\beta)} \sum_{j=1}^{n} \left[\begin{aligned} &\mu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) \log\left(\frac{2\mu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\mu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \mu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right) + \nu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) \log\left(\frac{2\nu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\nu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \nu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right) \\ &+ \pi_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) \log\left(\frac{2\pi_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\pi_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \pi_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right) \end{aligned} \right]. \end{aligned}$$

Step 4: ***Ranking the alternative***: Rank all the alternative according to indexing as obtained from $k = \arg \min_{1\le i\le m} \{D_\alpha^\beta(A_i;\ A^*)\}$.

5 Illustrative Example

In this section, one illustrative example from the field of decision-making has been taken for demonstrating the proposed approach.

Example: Decision-Making Problem. Consider the field of investment, where a person wants to invest some sort of money. As in these days, more and more companies have attracted the customers by reducing price and giving some other kind of benefits, so it is difficult for the investor to choose the best market for investment. In order to avoid the risk factor in the market and to make the decision more clear, they constitute a committee to invest the money in five major companies, namely, retail, food, computer, petrochemical, and a car company, respectively, denoted by $A_1,\ A_2,\ A_3,\ A_4,\ A_5$. Experts have been hired who gave their preferences of each alternative under the set of four major analyses, namely, the growth (G_1), the risk (G_2), the social-political impact (G_3) and the environmental impact (G_4). The rating value of each alternative $A_i = (i = 1, 2, \ldots, 5)$ under each factor has been assessed in terms of PFNs $\alpha_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle_{5\times 4}$ and is summarized as follows.

$$D = \begin{bmatrix} <0.5,\ 0.4> & <0.6,\ 0.3> & <0.3,\ 0.6> & <0.2,\ 0.7> \\ <0.7,\ 0.3> & <0.7,\ 0.2> & <0.7,\ 0.2> & <0.4,\ 0.5> \\ <0.6,\ 0.4> & <0.5,\ 0.4> & <0.5,\ 0.3> & <0.6,\ 0.3> \\ <0.8,\ 0.1> & <0.6,\ 0.3> & <0.3,\ 0.4> & <0.2,\ 0.6> \\ <0.6,\ 0.2> & <0.4,\ 0.3> & <0.7,\ 0.1> & <0.5,\ 0.3> \end{bmatrix}.$$

By using these normalized data, the ideal value for all the criteria is given by

$$A^* = \{<0.8,\ 0.1>,\ <0.7,\ 0.2>,\ <0.7,\ 0.1>,\ <0.6,\ 0.3>\}.$$

Thus, based on it the directed divergence measure from ideal alternative to each alternative is computed by taking $\alpha = 1,\ \beta = 0.5$ and their corresponding measures are summarized as follows:

$$\begin{array}{l} D_\alpha^\beta(A_1; A^*) = 0.1289; \quad D_\alpha^\beta(A_2; A^*) = 0.0228; \quad D_\alpha^\beta(A_3; A^*) = 0.0468; \\ D_\alpha^\beta(A_4; A^*) = 0.0742; \quad D_\alpha^\beta(A_5; A^*) = 0.0258 \end{array}.$$

So, the ranking order of these alternatives is

$$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1.$$

Hence, Food company is the best one for investment point of view.

6 Conclusions

Here, a parametric directed divergence measure of order α and degree β under the environment of Pythagorean fuzzy sets (PFSs) has been explored. We also discussed some desirable properties of this measure. For demonstration, a decision-making problem (investment problem) has been solved by using this technique. The parameters of this measure provide the flexibility to the decision-makers and that thing makes it more generalized. Thus, we conclude that the proposed divergence measure is suitable to solve several real-life problems and can be found as an alternative one among the various approaches to solve the decision-making problems. In future, we will be dealing with some more complicated problems or more realistic problems in the field of fuzzy cluster analysis, medical diagnosis, etc.

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