Pythagorean Fuzzy Soft Sets-Based MADM

Khalid Naeem and Muhammad Riaz

1 Introduction

The fuzzy morphological methodology delivers promising yields in quite a lot of areas, whose narrative is pretty qualitative. The inspiration for the use of words or sentences in preference to numbers is that philological descriptions or cataloging are frequently fewer absolute than arithmetical or algebraic ones. Problems that are equipped with unreliable conditions commonly occur in taking decisions, nonetheless are challenging due to perplexing condition of modeling and handling that arises with such uncertainties. To tackle multifarious and complicated problems in day-to-day life situations, the modus operandi customarily utilized as discussed in literature of classical mathematics is not of assistance each time because of the presence of uncertainties and indistinctness. There are abundant procedures that can be believed as mathematical models for coping with imprecision, inexactness, and uncertainties. Inauspiciously, all these simulations are fitted out with technical hitches and complications. To get control on these sorts of insufficiencies, Zadeh [64] brought together the notion of fuzzy sets (FSs). An FS is a substantial mathematical model for stamping an assembling of articles having unintelligible boundary. Atanassov [3–5] moved one step ahead by proposing intuitionistic fuzzy sets (IFSs). Atanassov [6] presented geometrical version of the components of intuitionistic fuzzy objects. Yager [60], by altering the condition on parameters, unveiled Pythagorean fuzzy subsets. Yager and Abbasov [61] studied Pythagorean membership grades. Later, Yager [62] employed these grades in decision-making. Molodtsov [41] patented the perception of a novel sort of model for sorting out uncertainties, traditionally acknowledged as soft sets (SSs).

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FSs, SSs, and their further expansions are resilient mathematical models for solving many real-world problems. The researchers have coined various mathematical models to deal with real-world problems. Çağman et al. [7] explored fuzzy soft sets with applications. Feng et al. [15] presented an adjustable approach to fuzzy soft set-based decision-making. Majumdar and Samanta [39] presented generalized fuzzy soft sets. Feng et al. [16] promote the study of SSs pooled with FSs and rough sets. Davvaz and Sadrabadi [11] presented usage of IFSs in medicine. Maji et al. [38] acquainted with the notion of intuitionistic fuzzy soft sets (IFSSs). Feng et al. [17] presented an additional outlook on generalized IFSSs and associated multi-attribute decision-making methods. Li and Cui [36] studied topological structure of IFSSs. Osmanoglu and Tokat [48] also presented IFS topology independently. Garg and Arora [20] devised a nonlinear-programming methodology for multi-attribute decision-making problem with interval-valued IFSSs information. Guleria and Bajaj [25] used matrices to represent Pythagorean fuzzy soft sets. Naz et al. [45] extended the notion of PFSs to PF graphs. Akram and Naz [2] presented energy of PF graphs with applications. Peng and Yang [49] deliberated some results for PFSs. Peng et al. [50] familiarized some PF information measures with their useful implementations. Peng et al. [51] globalized PFSs to corresponding SSs and solidified their uses. Peng and Selvachandran summed up the notions of Pythagorean fuzzy sets in [52]. Riaz and Naeem [55, 56] obtained some indispensable philosophies of SSs organized with soft σ-algebra and put on show some employments of soft mappings. Fei et al. [14] discussed Pythagorean fuzzy decision-making using soft likelihood functions. Fei and Deng [13], recently, studied multi-criteria decision-making in Pythagorean fuzzy environment.

The decision-taking techniques of TOPSIS and VIKOR have been deliberated by voluminous researchers including Hwang and Yoon [28], Adeel et al. [1], Eraslan and Karaaslan [12], Naeem et al. [44], Liu et al. [37], Kumar and Garg [33], Riaz et al. [54, 57], Li and Nan [34], Opricovic and Tzeng [46, 47], Mohd and Abdullah [40], Naeem et al. [43], Kalkan et al. [30], and Zhang and Xu [65]. Garg and Arora [18] presented generalized IFS power aggregation operator along with its practical usage. Garg and Arora [19] explored dual hesitant fuzzy soft aggregation operators with applications. Garg and Arora [23] explored TOPSIS method based on correlation coefficient for solving decision-making problems with IFSS information. Garg [22] presented, for the purpose of multiple attribute group decision analysis, novel neutrality operations based-Pythagorean fuzzy geometric aggregation operators. Li et al. [35] studied some novel Pythagorean hybrid weighted aggregation operators using Pythagorean fuzzy numbers along with their applications to decision-making. Recently, Garg [21] unveiled Pythagorean fuzzy aggregation operators based upon neutrality operations and rendered its utilizations in the process of multiple attribute group decision-making. Garg and Arora [24] presented Maclaurin symmetric mean aggregation operators based on t-norm operations for the dual hesitant fuzzy soft set.

The notion of similarity measure is indispensably significant in nearly every arena of science and technology. It is ordinarily forged for testing the validity of an object, situation, or document. Similarity measure serves as a substantial tool to decide the level of alikeness between two or more data sets. The similarity measures established by means of the notions of FSs, SSs, IFSs, and PFSs are broadly and efficiently applied in medical diagnosis, pattern recognition, signal detection, image processing, security verification systems, artificial intelligence, machine learning, etc. Similarity measures on various models are explored by Hong and Kim [26], Kharal [32], Kamaci [31], Hung and Yang [27], Hyung et al. [29], Ye [63], Chen [8, 9], Chen et al. [10], Wang et al. [58], and Muthukumar and Krishnan [42]. In recent times, Peng and Garg [53] made public multiparametric similarity measures on PFSs with applications to pattern recognition.

The goal of this chapter is to study Pythagorean fuzzy soft sets (PFSSs) and their practical implementations. We make use of different techniques including choice value method PFS-TOPSIS, VIKOR, and similarity measures for modeling uncertainties in decision-making problems. PFSSs offer a plenty of uses in decision-taking problems of daily life situations ranging from micro to high-level decisions. The chapter is organized as follows: Sect. 2 gives access to essential operations and fundamental characteristics of PFSSs. We devote Sect. 3 for an application of multi-criteria group decision-making (MCGDM) utilizing PFS matrices. In the very next section, we propose PFS-TOPSIS algorithm accompanied by its application in choosing appropriate persons for key ministries in a government. In Sect. 5, we propose PFS-VIKOR and utilize it on the selection of brand ambassadors for a multi-national company. In Sect. 6, we devise a similarity measure (SM) and weighted similarity measure for PFSSs. Based on this SM, we present an application in life sciences. In conclusion, we summarize our work in Sect. 7.

For better understanding of this unit, the reader is suggested to see [5, 7, 38, 41, 60–62, 64] for preliminary notions.

2 Structure of Pythagorean Fuzzy Soft Sets

Peng et al. [51] floated the notion of *Pythagorean fuzzy soft set* (PFSSs) and presented some of their applications. Later, Guleria and Bajaj [25] made use of matrices to represent PFSSs. The matrices used are known as *Pythagorean fuzzy soft matrices* (PFS matrices).

In this segment, we study some fundamental concepts, basic properties, and algebraic operations on PFSSs. X will represent the universe of discourse and E the aggregate of attributes with $A, A_1, A_2, A_3 \subseteq E$, in this section.

Definition 2.1 A *Pythagorean fuzzy soft set* (PFSS) on X is a family of the form

$$
\begin{aligned}
(\curlywedge, A) &= \Big\{\big(e, \{\partial, \sigma_{\curlywedge_A}(\partial), \varrho_{\curlywedge_A}(\partial)\}\big) : e \in A, \partial \in X\Big\}\\
&= \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\curlywedge_A}(\partial), \varrho_{\curlywedge_A}(\partial))}\right\}\right) : e \in A, \partial \in X\right\}\\
&= \left\{\left(e, \left\{\frac{(\sigma_{\curlywedge_A}(\partial), \varrho_{\curlywedge_A}(\partial))}{\partial}\right\}\right) : e \in A, \partial \in X\right\},
\end{aligned}
$$

Table 1 Tabulatory representation of PFSS $\curlywedge_A$

$\curlywedge_A$	e_1	e_2	$\cdots$	e_n
∂_1	$(\sigma_{11}, \varrho_{11})$	$(\sigma_{12}, \varrho_{12})$	$\cdots$	$(\sigma_{1n}, \varrho_{1n})$
∂_2	$(\sigma_{21}, \varrho_{21})$	$(\sigma_{22}, \varrho_{22})$	$\cdots$	$(\sigma_{2n}, \varrho_{2n})$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
∂_m	$(\sigma_{m1}, \varrho_{m1})$	$(\sigma_{m2}, \varrho_{m2})$	$\cdots$	$(\sigma_{mn}, \varrho_{mn})$

where $\sigma_{\curlywedge_A}$ and $\varrho_{\curlywedge_A}$ are mappings dragging members of X to [0, 1], obeying the requirement

$$0 \leq \sigma^2_{\curlywedge_A}(\partial) + \varrho^2_{\curlywedge_A}(\partial) \leq 1$$

.

If we write $\sigma_{ij} = \sigma_{\curlywedge_A}(e_j)(\partial_i)$ and $\varrho_{ij} = \varrho_{\curlywedge_A}(e_j)(\partial_i), i = 1, \ldots, m; j = 1, \ldots, n,$ then the PFSS $\curlywedge_A$ may be expressed in tabular array as in Table 1.

The matrix representing PFSS $\curlywedge_A$ is termed as *Pythagorean fuzzy soft matrix* (PFS matrix), and has form

$$\begin{aligned} \curlywedge_A &= [(\sigma_{ij}, \varrho_{ij})]_{m\times n} \\ &= \begin{pmatrix} (\sigma_{11}, \varrho_{11}) & (\sigma_{12}, \varrho_{12}) & \cdots & (\sigma_{1n}, \varrho_{1n}) \\ (\sigma_{21}, \varrho_{21}) & (\sigma_{22}, \varrho_{22}) & \cdots & (\sigma_{2n}, \varrho_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\sigma_{m1}, \varrho_{m1}) & (\sigma_{m2}, \varrho_{m2}) & \cdots & (\sigma_{mn}, \varrho_{mn}) \end{pmatrix} \end{aligned}$$

Example 2.2 Let $X = \{\partial_i : i = 1, \cdots, 4\}$ and $E = \{e_i : i = 1, 2, \ldots, 5\}$. Take $A = \{e_2, e_5\}$. Then,

$$\curlywedge_A = \Big\{\Big(e_2, \Big\{\Big(\frac{\partial_1}{(0.13, 0.91)}\Big), \Big(\frac{\partial_3}{(0.25, 0.62)}\Big), \Big(\frac{\partial_4}{(0.24, 0.89)}\Big)\Big\}\Big), \Big(e_5, \Big\{\Big(\frac{\partial_1}{(0.71, 0.29)}\Big), \Big(\frac{\partial_2}{(0.41, 0.06)}\Big)\Big\}\Big)\Big\}$$

is a PFSS over X. The tabular representation of $\curlywedge_A$ is given in Table 2.

Table 2 Tabular representation of $\curlywedge_A$

$\curlywedge_A$	e_1	e_2	e_3	e_4	e_5
∂_1	(0,1)	(0.13,0.91)	(0,1)	(0,1)	(0.71,0.29)
∂_2	(0,1)	(0,1)	(0,1)	(0,1)	(0.41,0.06)
∂_3	(0,1)	(0.25,0.62)	(0,1)	(0,1)	(0,1)
∂_4	(0,1)	(0.24,0.89)	(0,1)	(0,1)	(0,1)

The corresponding PFS matrix is

$$\begin{aligned}\angle_A &= [\sigma_{ij}, \varrho_{ij}]_{4\times 5}\\ &= \begin{pmatrix} (0,1) & (0.13, 0.91) & (0,1) & (0,1) & (0.71, 0.29)\\ (0,1) & (0,1) & (0,1) & (0,1) & (0.41, 0.06)\\ (0,1) & (0.25, 0.62) & (0,1) & (0,1) & (0,1)\\ (0,1) & (0.24, 0.89) & (0,1) & (0,1) & (0,1)\end{pmatrix}\end{aligned}$$

Definition 2.3 A PFSS $\angle^{(1)}_{A_1}$ is called *PFS subset* of $\angle^{(2)}_{A_2}$, i.e., $\angle^{(1)}_{A_1}\tilde{\subseteq}\angle^{(2)}_{A_2}$, if

(i) $A_1 \subseteq A_2$, and
(ii) $\angle^{(1)}(e)$ is PFS subset of $\angle^{(2)}(e)$, for all $e \in A_1$.

It is remarkable to notice that $\angle_A \tilde{\subseteq} G_B$ by no means requires that each member of $\angle^{(1)}_A$ must also present in $\angle^{(1)}_B$, contrary to classical set theory.

Definition 2.4 The *union* of two PFSSs $(\angle_1, A_1)$ and $(\angle_2, A_2)$ defined over X is given as $(\angle, A_1 \cup A_2) = (\angle_1, A_1)\,\tilde{\cup}\,(\angle_2, A_2)$, and for all $e \in A$,

$$\angle(e) = \begin{cases} \angle_1(e), & \text{if } e \in A_1 \ \&\ e \notin A_2\\ \angle_2(e), & \text{if } e \in A_2 \ \&\ e \notin A_1\\ \angle_1(e) \cup \angle_2(e), & \text{if } e \in A_1 \cap A_2,\end{cases}$$

where $\angle_1(e) \cup \angle_2(e)$ is the union of two PFSSs.

Definition 2.5 The *intersection* of two PFSSs $(\angle_1, A_1)$ and $(\angle_2, A_2)$ is another PFSS $(\angle, A_1 \cap A_2) = (\angle_1, A_1)\,\tilde{\cap}\,(\angle_2, A_2)$, where $\angle(e) = \angle_1(e) \cap \angle_2(e)$ for all $e \in A_1 \cap A_2$.

Definition 2.6 The *difference* of two PFSSs $(\angle_1, A_1)$ and $(\angle_2, A_2)$ over X is defined as

$$(\angle_1, A_1)\,\tilde{\setminus}\,(\angle_2, A_2) = \left\{\left(e, \left\{\partial, \min\{\sigma_{\angle_1(e)}(\partial), \varrho_{\angle_2(e)}(\partial)\}, \max\{\varrho_{\angle_1(e)}(\partial), \sigma_{\angle_2(e)}(\partial)\}\right\}\right) : \partial \in X, e \in E\right\}.$$

Definition 2.7 The *complement* of a PFSS $(\angle, A)$ is a mapping $\angle^c : A \to PF^X$ given by $\angle^c(e) = [\angle(e)]^c$, for all $e \in A$. It is represented as $(\angle, A)^c$ or sometimes by $(\angle^c, A)$. Thus, if

$$\angle(e) = \{(\partial, \sigma_{\angle(e)}(\partial), \varrho_{\angle(e)}(\partial)) : \partial \in X\}$$

then

$$\angle^c(e) = \{(\partial, \varrho_{\angle(e)}(\partial), \sigma_{\angle(e)}(\partial)) : \partial \in X\}$$

for all $e \in A$.

Definition 2.8 A PFSS defined over X is termed as *null PFSS* if it is in the form

$$\Phi = \left\{\left(e, \left\{\frac{\partial}{(0, 1)}\right\}\right) : e \in E, \partial \in X\right\}.$$

Definition 2.9 A PFSS defined over X is termed as *absolute PFSS* if it is in the form

$$\breve{X} = \left\{\left(e, \left\{\frac{\partial}{(1, 0)}\right\}\right) : e \in E, \partial \in X\right\}.$$

Definition 2.10 If

$$\leftthreetimes^{(1)}_{A_1} = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes^{(1)}_{A_1}}(\partial), \varrho_{\leftthreetimes^{(1)}_{A_1}}(\partial))}\right\}\right) : e \in A_1, \partial \in X\right\}$$

and

$$\leftthreetimes^{(2)}_{A_2} = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes^{(2)}_{A_2}}(\partial), \varrho_{\leftthreetimes^{(2)}_{A_2}}(\partial))}\right\}\right) : e \in A_2, \partial \in X\right\}$$

are two PFSSs, then

$$\leftthreetimes^{(1)}_{A_1} \tilde{\oplus} \leftthreetimes^{(2)}_{A_2} = \left\{\left(e, \left\{\frac{\partial}{(\sqrt{(\sigma_{\leftthreetimes^{(1)}_{A_1}}(\partial))^2 + (\sigma_{\leftthreetimes^{(2)}_{A_2}}(\partial))^2 - (\sigma_{\leftthreetimes^{(1)}_{A_1}}(\partial)\sigma_{\leftthreetimes^{(2)}_{A_2}}(\partial))^2}, \varrho_{\leftthreetimes^{(1)}_{A_1}}(\partial)\varrho_{\leftthreetimes^{(2)}_{A_2}}(\partial))}\right\}\right) : e \in E, \partial \in X\right\}$$

and

$$\leftthreetimes^{(1)}_{A_1} \tilde{\otimes} \leftthreetimes^{(2)}_{A_2} = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes^{(1)}_{A_1}}(\partial)\sigma_{\leftthreetimes^{(2)}_{A_2}}(\partial), \sqrt{(\varrho_{\leftthreetimes^{(1)}_{A_1}}(\partial))^2 + (\varrho_{\leftthreetimes^{(2)}_{A_2}}(\partial))^2 - (\varrho_{\leftthreetimes^{(1)}_{A_1}}(\partial)\varrho_{\leftthreetimes^{(2)}_{A_2}}(\partial))^2})}\right\}\right) : e \in E, \partial \in X\right\}$$

Definition 2.11 The *necessity operator* on the PFSS

$$\leftthreetimes_A = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes_A}(\partial), \varrho_{\leftthreetimes_A}(\partial))}\right\}\right) : e \in A, \partial \in X\right\}$$

is defined as

$$\tilde{\Box}\leftthreetimes_A = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes_A}(\partial), \sqrt{1 - \sigma^2_{\leftthreetimes_A}(\partial)})}\right\}\right) : e \in A, \partial \in X\right\}.$$

Definition 2.12 The *possibility operator* on the PFSS

$$\leftthreetimes_A = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes_A}(\partial), \varrho_{\leftthreetimes_A}(\partial))}\right\}\right) : e \in A, \partial \in X\right\}$$

is defined as

$$\widetilde{\diamond}\curlywedge_A = \left\{\left(e, \left\{\frac{\partial}{(\sqrt{1-\varrho^2_{\curlywedge_A}(\partial)}, \varrho_{\curlywedge_A}(\partial))}\right\}\right) : e \in A, \partial \in X\right\}$$

Remark The modal operators presented in Definition 2.11 and 2.12 transform any PFSS to the corresponding FSS.

We elaborate the notions presented above with the help of following example.

Example 2.13 Take $X = \{\partial_1, \cdots, \partial_4\}$ and $E = \{e_1, e_2, \cdots, e_6\}$. Assume that $A_1 = \{e_2, e_4\}$, $A_2 = \{e_1, e_4, e_5\}$ and $A_3 = \{e_2, e_4, e_6\}$. Consider the PFSSs

$$\curlywedge^{(1)}_{A_1} = \begin{pmatrix} (0,1) & (0.27,0.78) & (0,1) & (0.39,0.48) & (0,1) & (0,1) \\ (0,1) & (0.11,0.04) & (0,1) & (0.73,0.54) & (0,1) & (0,1) \\ (0,1) & (0.56,0.60) & (0,1) & (0.59,0.51) & (0,1) & (0,1) \\ (0,1) & (0.62,0.62) & (0,1) & (0.37,0.56) & (0,1) & (0,1) \end{pmatrix},$$

$$\curlywedge^{(2)}_{A_2} = \begin{pmatrix} (0.56,0.27) & (0,1) & (0,1) & (0.45,0.58) & (0.33,0.78) & (0,1) \\ (0.11,0.85) & (0,1) & (0,1) & (0.09,0.28) & (0.42,0.51) & (0,1) \\ (0.76,0.49) & (0,1) & (0,1) & (0.62,0.67) & (0.92,0.21) & (0,1) \\ (0.54,0.71) & (0,1) & (0,1) & (0.54,0.82) & (0.87,0.48) & (0,1) \end{pmatrix}$$

and

$$\curlywedge^{(3)}_{A_3} = \begin{pmatrix} (0,1) & (0.31,0.54) & (0,1) & (0.39,0.01) & (0,1) & (0.22,0.87) \\ (0,1) & (0.25,0.04) & (0,1) & (0.76,0.21) & (0,1) & (0.53,0.16) \\ (0,1) & (0.49,0.32) & (0,1) & (0.62,0.37) & (0,1) & (0.42,0.19) \\ (0,1) & (0.63,0.45) & (0,1) & (0.46,0.54) & (0,1) & (0.88,0.32) \end{pmatrix}.$$

It may be observed that $\curlywedge^{(1)}_{A_1}\widetilde{\subseteq}\curlywedge^{(3)}_{A_3}$, whereas neither $\curlywedge^{(1)}_{A_1}\widetilde{\subseteq}\curlywedge^{(2)}_{A_2}$ nor $\curlywedge^{(2)}_{A_2}\widetilde{\subseteq}\curlywedge^{(3)}_{A_3}$. Moreover,

$$\curlywedge^{(1)}_{A_1}\widetilde{\cup}\curlywedge^{(2)}_{A_2} = \begin{pmatrix} (0.56,0.27) & (0.27,0.78) & (0,1) & (0.45,0.48) & (0.33,0.78) & (0,1) \\ (0.11,0.85) & (0.11,0.04) & (0,1) & (0.73,0.28) & (0.42,0.51) & (0,1) \\ (0.76,0.49) & (0.56,0.60) & (0,1) & (0.62,0.51) & (0.92,0.21) & (0,1) \\ (0.54,0.71) & (0.62,0.62) & (0,1) & (0.54,0.56) & (0.87,0.48) & (0,1) \end{pmatrix},$$

$$\curlywedge^{(1)}_{A_1}\widetilde{\cap}\curlywedge^{(2)}_{A_2} = \begin{pmatrix} (0,1) & (0,1) & (0,1) & (0.39,0.58) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.09,0.54) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.59,0.67) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.37,0.82) & (0,1) & (0,1) \end{pmatrix},$$

$$\left(\curlywedge_{A_1}^{(1)}\right)^c = \begin{pmatrix} (1,0) & (0.78,0.27) & (1,0) & (0.48,0.39) & (1,0) & (1,0) \\ (1,0) & (0.04,0.11) & (1,0) & (0.54,0.73) & (1,0) & (1,0) \\ (1,0) & (0.60,0.56) & (1,0) & (0.51,0.59) & (1,0) & (1,0) \\ (1,0) & (0.62,0.62) & (1,0) & (0.56,0.37) & (1,0) & (1,0) \end{pmatrix},$$

$$\curlywedge_{A_1}^{(1)} \widetilde{\setminus} \curlywedge_{A_2}^{(2)} = \begin{pmatrix} (0,1) & (0,1) & (0,1) & (0.39,0.48) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.28,0.54) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.59,0.62) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.37,0.56) & (0,1) & (0,1) \end{pmatrix},$$

$$\widetilde{\Box} \curlywedge_{A_1}^{(1)} = \begin{pmatrix} (0,1) & (0.27,0.96) & (0,1) & (0.39,0.92) & (0,1) & (0,1) \\ (0,1) & (0.11,0.99) & (0,1) & (0.73,0.68) & (0,1) & (0,1) \\ (0,1) & (0.56,0.83) & (0,1) & (0.59,0.81) & (0,1) & (0,1) \\ (0,1) & (0.62,0.78) & (0,1) & (0.37,0.93) & (0,1) & (0,1) \end{pmatrix},$$

$$\widetilde{\Diamond} \curlywedge_{A_1}^{(1)} = \begin{pmatrix} (0,1) & (0.62,0.78) & (0,1) & (0.88,0.48) & (0,1) & (0,1) \\ (0,1) & (0.99,0.04) & (0,1) & (0.84,0.54) & (0,1) & (0,1) \\ (0,1) & (0.80,0.60) & (0,1) & (0.86,0.51) & (0,1) & (0,1) \\ (0,1) & (0.78,0.62) & (0,1) & (0.83,0.56) & (0,1) & (0,1) \end{pmatrix},$$

$$\curlywedge_{A_1}^{(1)} \widetilde{\oplus} \curlywedge_{A_2}^{(2)} = \begin{pmatrix} (0.56,0.27) & (0.27,0.78) & (0,1) & (0.57,0.28) & (0.33,0.78) & (0,1) \\ (0.11,0.85) & (0.11,0.04) & (0,1) & (0.73,0.15) & (0.42,0.51) & (0,1) \\ (0.76,0.49) & (0.56,0.60) & (0,1) & (0.77,0.34) & (0.92,0.21) & (0,1) \\ (0.54,0.71) & (0.62,0.62) & (0,1) & (0.62,0.46) & (0.87,0.48) & (0,1) \end{pmatrix}$$

and

$$\curlywedge_{A_1}^{(1)} \widetilde{\otimes} \curlywedge_{A_2}^{(2)} = \begin{pmatrix} (0,1) & (0,1) & (0,1) & (0.18,0.70) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.06,0.59) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.36,0.77) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.20,0.88) & (0,1) & (0,1) \end{pmatrix}.$$

Proposition 2.14 *Every PFSS $\curlywedge_A$ may be sandwiched between Φ and $\breve{X}$, i.e., $\Phi \widetilde{\subseteq} \curlywedge_A \widetilde{\subseteq} \breve{X}$.*

Proposition 2.15 *If $\curlywedge_{A_1}^{(1)}$, $\curlywedge_{A_2}^{(2)}$ and $\curlywedge_{A_3}^{(3)}$ are three PFSSs over X, then*

(i) $\curlywedge_{A_1}^{(1)} \widetilde{\cap} \curlywedge_{A_1}^{(1)} = \curlywedge_{A_1}^{(1)}$.
(ii) $\curlywedge_{A_1}^{(1)} \widetilde{\cup} \curlywedge_{A_1}^{(1)} = \curlywedge_{A_1}^{(1)}$.
(iii) $\curlywedge_{A_1}^{(1)} \widetilde{\cap} \curlywedge_{A_2}^{(2)} = \curlywedge_{A_2}^{(2)} \widetilde{\cap} \curlywedge_{A_1}^{(1)}$.

(iv) $\curlywedge_{A_1}^{(1)} \widetilde{\cup} \curlywedge_{A_2}^{(2)} = \curlywedge_{A_2}^{(2)} \widetilde{\cup} \curlywedge_{A_1}^{(1)}$.
(v) $\curlywedge_{A_1}^{(1)} \widetilde{\cap} (\curlywedge_{A_2}^{(2)} \widetilde{\cap} \curlywedge_{A_3}^{(3)}) = (\curlywedge_{A_1}^{(1)} \widetilde{\cap} \curlywedge_{A_2}^{(2)}) \widetilde{\cap} \curlywedge_{A_3}^{(3)}$.
(vi) $\curlywedge_{A_1}^{(1)} \widetilde{\cup} (\curlywedge_{A_2}^{(2)} \widetilde{\cup} \curlywedge_{A_3}^{(3)}) = (\curlywedge_{A_1}^{(1)} \widetilde{\cup} \curlywedge_{A_2}^{(2)}) \widetilde{\cup} \curlywedge_{A_3}^{(3)}$.
(vii) $\curlywedge_{A_1}^{(1)} \widetilde{\cup} (\curlywedge_{A_2}^{(2)} \widetilde{\cap} \curlywedge_{A_3}^{(3)}) = (\curlywedge_{A_1}^{(1)} \widetilde{\cup} \curlywedge_{A_2}^{(2)}) \widetilde{\cap} (\curlywedge_{A_1}^{(1)} \widetilde{\cup} \curlywedge_{A_3}^{(3)})$.
(viii) $\curlywedge_{A_1}^{(1)} \widetilde{\cap} (\curlywedge_{A_2}^{(2)} \widetilde{\cup} \curlywedge_{A_3}^{(3)}) = (\curlywedge_{A_1}^{(1)} \widetilde{\cap} \curlywedge_{A_2}^{(2)}) \widetilde{\cup} (\curlywedge_{A_1}^{(1)} \widetilde{\cap} \curlywedge_{A_3}^{(3)})$.

Proposition 2.16 *If $\curlywedge_{A_1}^{(1)}$ and $\curlywedge_{A_2}^{(2)}$ are two PFSSs over X, then*

(i) $\curlywedge_{A_1}^{(1)} \widetilde{\cap} \curlywedge_{A_2}^{(2)} \widetilde{\subseteq} \curlywedge_{A_1}^{(1)} \widetilde{\subseteq} \curlywedge_{A_1}^{(1)} \widetilde{\cup} \curlywedge_{A_2}^{(2)}$
(ii) $\curlywedge_{A_1}^{(1)} \widetilde{\cap} \curlywedge_{A_2}^{(2)} \widetilde{\subseteq} \curlywedge_{A_2}^{(2)} \widetilde{\subseteq} \curlywedge_{A_1}^{(1)} \widetilde{\cup} \curlywedge_{A_2}^{(2)}$.

The above propositions are easy consequences of definition.

Remark Consider the PFSSs $\curlywedge_{A_1}^{(1)}$ and $\curlywedge_{A_2}^{(2)}$ given in Example 2.13. We have

$$\left(\curlywedge_{A_1}^{(1)} \widetilde{\cup} \curlywedge_{A_2}^{(2)}\right)^c = \begin{pmatrix} (0.27, 0.56) & (0.78, 0.27) & (1, 0) & (0.48, 0.45) & (0.78, 0.33) & (1, 0) \\ (0.85, 0.11) & (0.04, 0.11) & (1, 0) & (0.28, 0.73) & (0.51, 0.42) & (1, 0) \\ (0.49, 0.76) & (0.60, 0.56) & (1, 0) & (0.51, 0.62) & (0.21, 0.92) & (1, 0) \\ (0.71, 0.54) & (0.62, 0.62) & (1, 0) & (0.56, 0.54) & (0.48, 0.87) & (1, 0) \end{pmatrix} \tag{1}$$

$$\left(\curlywedge_{A_1}^{(1)}\right)^c = \begin{pmatrix} (1, 0) & (0.78, 0.27) & (1, 0) & (0.48, 0.39) & (1, 0) & (1, 0) \\ (1, 0) & (0.04, 0.11) & (1, 0) & (0.54, 0.73) & (1, 0) & (1, 0) \\ (1, 0) & (0.60, 0.56) & (1, 0) & (0.51, 0.59) & (1, 0) & (1, 0) \\ (1, 0) & (0.62, 0.62) & (1, 0) & (0.56, 0.37) & (1, 0) & (1, 0) \end{pmatrix},$$

$$(\curlywedge_{A_2}^{(2)})^c = \begin{pmatrix} (0.27, 0.56) & (1, 0) & (1, 0) & (0.58, 0.45) & (0.78, 0.33) & (1, 0) \\ (0.85, 0.11) & (1, 0) & (1, 0) & (0.28, 0.09) & (0.51, 0.42) & (1, 0) \\ (0.49, 0.76) & (1, 0) & (1, 0) & (0.67, 0.62) & (0.21, 0.92) & (1, 0) \\ (0.71, 0.54) & (1, 0) & (1, 0) & (0.82, 0.54) & (0.48, 0.87) & (1, 0) \end{pmatrix}$$

Since $A_1 \cap A_2 = \{e_4\}$, so

$$\left(\curlywedge_{A_1}^{(1)}\right)^c \widetilde{\cap} \left(\curlywedge_{A_2}^{(2)}\right)^c = \begin{pmatrix} (0, 1) & (0, 1) & (0, 1) & (0.48, 0.45) & (0, 1) & (0, 1) \\ (0, 1) & (0, 1) & (0, 1) & (0.28, 0.73) & (0, 1) & (0, 1) \\ (0, 1) & (0, 1) & (0, 1) & (0.51, 0.62) & (0, 1) & (0, 1) \\ (0, 1) & (0, 1) & (0, 1) & (0.56, 0.54) & (0, 1) & (0, 1) \end{pmatrix} \tag{2}$$

From (1) & (2), we conclude that De Morgan's laws do not make sense in PFSS theory.

Theorem 2.17 *If $(\curlywedge_1, A_1)$ and $(\curlywedge_2, A_2)$ are two PFSSs over X, then*

(a) $\Big((\curlywedge_1, A_1) \widetilde{\cup} (\curlywedge_2, A_2)\Big)^c \neq (\curlywedge_1, A_1)^c \widetilde{\cap} (\curlywedge_2, A_2)^c$, *and*

(b) $\left((\curlywedge_1, A_1)\widetilde{\cap}(\curlywedge_2, A_2)\right)^c \neq (\curlywedge_1, A_1)^c\widetilde{\cup}(\curlywedge_2, A_2)^c$.

Remark Consider again the PFSS $\curlywedge^{(1)}_{A_1}$ given in Example 2.13. We have

$$\curlywedge^{(1)}_{A_1} = \begin{pmatrix} (0,1) & (0.27, 0.78) & (0,1) & (0.39, 0.48) & (0,1) & (0,1) \\ (0,1) & (0.11, 0.04) & (0,1) & (0.73, 0.54) & (0,1) & (0,1) \\ (0,1) & (0.56, 0.60) & (0,1) & (0.59, 0.51) & (0,1) & (0,1) \\ (0,1) & (0.62, 0.62) & (0,1) & (0.37, 0.56) & (0,1) & (0,1) \end{pmatrix}$$

$$\therefore (\curlywedge^{(1)}_{A_1})^c = \begin{pmatrix} (1,0) & (0.78, 0.27) & (1,0) & (0.48, 0.39) & (1,0) & (1,0) \\ (1,0) & (0.04, 0.11) & (1,0) & (0.54, 0.73) & (1,0) & (1,0) \\ (1,0) & (0.60, 0.56) & (1,0) & (0.51, 0.59) & (1,0) & (1,0) \\ (1,0) & (0.62, 0.62) & (1,0) & (0.56, 0.37) & (1,0) & (1,0) \end{pmatrix}$$

Now,

$$\curlywedge^{(1)}_{A_1}\widetilde{\cup}(\curlywedge^{(1)}_{A_1})^c = \begin{pmatrix} (1,0) & (0.78, 0.27) & (1,0) & (0.48, 0.39) & (1,0) & (1,0) \\ (1,0) & (0.11, 0.04) & (1,0) & (0.73, 0.54) & (1,0) & (1,0) \\ (1,0) & (0.60, 0.56) & (1,0) & (0.59, 0.51) & (1,0) & (1,0) \\ (1,0) & (0.62, 0.62) & (1,0) & (0.56, 0.37) & (1,0) & (1,0) \end{pmatrix} \neq \breve{X}$$

and

$$\curlywedge^{(1)}_{A_1}\widetilde{\cap}(\curlywedge^{(1)}_{A_1})^c = \begin{pmatrix} (0,1) & (0.27, 0.78) & (0,1) & (0.39, 0.48) & (0,1) & (0,1) \\ (0,1) & (0.04, 0.11) & (0,1) & (0.54, 0.73) & (0,1) & (0,1) \\ (0,1) & (0.56, 0.60) & (0,1) & (0.51, 0.59) & (0,1) & (0,1) \\ (0,1) & (0.62, 0.62) & (0,1) & (0.37, 0.56) & (0,1) & (0,1) \end{pmatrix} \neq \Phi$$

These observations lead to the following theorem.

Theorem 2.18 *If* $(\curlywedge, A)$ *is any PFSS over* X*, then*

(1) $\curlywedge_A\widetilde{\cup}\curlywedge^c_A \neq \breve{X}$*, and*
(2) $\curlywedge_A\widetilde{\cap}\curlywedge^c_A \neq \Phi$.

Definition 2.19 We know that

$$\curlywedge^{(1)}_{A_1}\widetilde{\otimes}\curlywedge^{(2)}_{A_2} = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\curlywedge^{(1)}_{A_1}}(\partial)\sigma_{\curlywedge^{(2)}_{A_2}}(\partial), \sqrt{(\varrho_{\curlywedge^{(1)}_{A_1}}(\partial))^2+(\varrho_{\curlywedge^{(2)}_{A_2}}(\partial))^2-(\varrho_{\curlywedge^{(1)}_{A_1}}(\partial)\varrho_{\curlywedge^{(2)}_{A_2}}(\partial))^2})}\right\}\right) : e \in E, \partial \in X\right\}.$$

If we substitute $\curlywedge^{(1)}_{A_1} = \curlywedge^{(2)}_{A_2} = \curlywedge_A$, then

$$\curlywedge_A \widetilde{\otimes} \curlywedge_A = \left\{ \left(e, \left\{ \frac{\partial}{(\sigma^2_{\curlywedge_A}(\partial), \sqrt{2\varrho^2_{\curlywedge_A}(\partial) - \varrho^4_{\curlywedge_A}(\partial)})} \right\} \right) : e \in E, \partial \in X \right\}.$$

That is,

$$(\curlywedge_A)^2 = \left\{ \left(e, \left\{ \frac{\partial}{\left(\sigma^2_{\curlywedge_A}(\partial), \sqrt{1 - (1 - \varrho^2_{\curlywedge_A})^2}\right)} \right\} \right) : e \in E, \partial \in X \right\}.$$

In general, if k is any non-negative real number, then

$$(\curlywedge_A)^k = \left\{ \left(e, \left\{ \frac{\partial}{\left(\sigma^k_{\curlywedge_A}(\partial), \sqrt{1 - (1 - \varrho^2_{\curlywedge_A})^k}\right)} \right\} \right) : e \in E, \partial \in X \right\}.$$

In particular, for $k = \frac{1}{2}$, we have

$$(\curlywedge_A)^{\frac{1}{2}} = \left\{ \left(e, \left\{ \frac{\partial}{\left(\sqrt{\sigma_{\curlywedge_A}(\partial)}, \sqrt{1 - \sqrt{1 - \varrho^2_{\curlywedge_A}}}\right)} \right\} \right) : e \in E, \partial \in X \right\}$$

$(\curlywedge_A)^2$ is called *concentration* of $\curlywedge_A$, denoted as $con(\curlywedge_A)$ whereas $(\curlywedge_A)^{\frac{1}{2}}$ is entitled as *dilation* of $\curlywedge_A$, denoted as $dil(\curlywedge_A)$.

Example 2.20 For PFSS $\curlywedge^{(1)}_{A_1} = \curlywedge_A$ given in Example 2.13, concentration and dilation are

$$con(\curlywedge_A) = \begin{pmatrix} (0,1) & (0.07, 0.92) & (0,1) & (0.15, 0.64) & (0,1) & (0,1) \\ (0,1) & (0.01, 0.06) & (0,1) & (0.53, 0.71) & (0,1) & (0,1) \\ (0,1) & (0.31, 0.77) & (0,1) & (0.35, 0.67) & (0,1) & (0,1) \\ (0,1) & (0.38, 0.79) & (0,1) & (0.14, 0.73) & (0,1) & (0,1) \end{pmatrix}$$

and

$$dil(\curlywedge_A) = \begin{pmatrix} (0,1) & (0.52, 0.61) & (0,1) & (0.62, 0.35) & (0,1) & (0,1) \\ (0,1) & (0.33, 0.03) & (0,1) & (0.85, 0.40) & (0,1) & (0,1) \\ (0,1) & (0.75, 0.45) & (0,1) & (0.77, 0.37) & (0,1) & (0,1) \\ (0,1) & (0.79, 0.46) & (0,1) & (0.61, 0.41) & (0,1) & (0,1) \end{pmatrix}$$

respectively.

We observe that in concentration of the PFSS, the value of membership function is reduced and that of non-membership function exceeds the corresponding original values. On the other hand, in case of dilation of the PFSS, the value of membership function exceeds and that of non-membership function reduces as compared to the corresponding original values.

Keeping in mind this observation, we may link phonetic terms like "*very*", "*moderate*", "*highly*", and "*not*" with the PFSS $\curlywedge_A$ by giving different non-negative real values to k. For Example,

$$k = \frac{1}{2} \Rightarrow \text{"}very\text{"}$$
$$k = \frac{3}{4} \Rightarrow \text{"}moderate\text{"}$$
$$k = \frac{1}{5} \Rightarrow \text{"}highly\text{"}$$
$$k = 4 \Rightarrow \text{"}not\text{"}$$

For conceiving these notions effectively, consider the following example.

Example 2.21 Choose $X = \{\text{Angelica, Smith, Adina, Paul}\}$ as the class of students and $E = \{e_1, \cdots, e_5\}$ as the collection of attributes, where

$$\begin{aligned} e_1 &= \text{Sharp in Mathematics} \\ e_2 &= \text{Sharp in Physics} \\ e_3 &= \text{Sharp in Chemistry} \\ e_4 &= \text{Obedient} \\ e_5 &= \text{Active in physical games} \end{aligned}$$

Assume that the PFSS representing members of X and the value of trait e_j in the form of PFNs is

$$\curlywedge_A = \begin{pmatrix} (0.83, 0.28) & (0.54, 0.21) & (0.37, 0.64) & (0.59, 0.16) & (0.86, 0.11) \\ (0.31, 0.26) & (0.56, 0.57) & (0.43, 0.32) & (0.74, 0.25) & (0.13, 0.05) \\ (0.52, 0.27) & (0.64, 0.12) & (0.45, 0.57) & (0.61, 0.35) & (0.29, 0.51) \\ (0.48, 0.59) & (0.35, 0.21) & (0.57, 0.13) & (0.21, 0.21) & (0.88, 0.41) \end{pmatrix}$$

The entry at (1, 1) position, i.e., (0.83, 0.28) shows that Angelica's tendency towards sharpness in mathematics is 83% whereas against it is 28%.

Now,

$$very(\curlywedge_A) = \begin{pmatrix} (0.91, 0.20) & (0.73, 0.15) & (0.61, 0.48) & (0.77, 0.11) & (0.93, 0.08) \\ (0.56, 0.19) & (0.75, 0.42) & (0.66, 0.23) & (0.86, 0.18) & (0.36, 0.04) \\ (0.72, 0.19) & (0.80, 0.09) & (0.67, 0.42) & (0.78, 0.25) & (0.54, 0.37) \\ (0.69, 0.44) & (0.59, 0.15) & (0.75, 0.09) & (0.46, 0.15) & (0.94, 0.30) \end{pmatrix},$$

$$moderate(\curlywedge_A) = \begin{pmatrix} (0.87, 0.24) & (0.63, 0.18) & (0.47, 0.57) & (0.67, 0.14) & (0.89, 0.10) \\ (0.42, 0.23) & (0.65, 0.51) & (0.53, 0.28) & (0.80, 0.22) & (0.22, 0.04) \\ (0.61, 0.23) & (0.72, 0.10) & (0.55, 0.51) & (0.69, 0.31) & (0.40, 0.45) \\ (0.58, 0.52) & (0.46, 0.18) & (0.66, 0.11) & (0.31, 0.18) & (0.91, 0.36) \end{pmatrix},$$

$$highly(\nless_A) = \begin{pmatrix} (0.96, 0.13) & (0.88, 0.09) & (0.82, 0.32) & (0.90, 0.07) & (0.97, 0.05) \\ (0.79, 0.12) & (0.89, 0.27) & (0.84, 0.15) & (0.94, 0.11) & (0.66, 0.02) \\ (0.88, 0.12) & (0.91, 0.05) & (0.85, 0.27) & (0.91, 0.16) & (0.78, 0.24) \\ (0.86, 0.29) & (0.81, 0.09) & (0.89, 0.06) & (0.73, 0.09) & (0.97, 0.19) \end{pmatrix}$$

and

$$not(\nless_A) = \begin{pmatrix} (0.47, 0.53) & (0.09, 0.41) & (0.02, 0.94) & (0.12, 0.31) & (0.55, 0.22) \\ (0.01, 0.49) & (0.10, 0.89) & (0.03, 0.59) & (0.30, 0.48) & (0.00, 0.10) \\ (0.07, 0.51) & (0.17, 0.24) & (0.04, 0.89) & (0.14, 0.64) & (0.01, 0.84) \\ (0.05, 0.91) & (0.02, 0.41) & (0.11, 0.26) & (0.00, 0.41) & (0.60, 0.72) \end{pmatrix}.$$

Definition 2.22 A PFSS $(\nless, A)$ is termed as a *Pythagorean fuzzy soft point* (PFS point), denoted as $\vartheta_{\nless}$, if for the element $\vartheta \in A$ we have

(i) $\nless(\vartheta) \neq \Phi$, and
(ii) $\nless(\vartheta') = \check{X}$, for all $\vartheta' \in A - \{\vartheta\}$.

Definition 2.23 A PFS point $\vartheta_{\nless}\widetilde{\in}(\nless, A)$ is said to be in PFSS $(\nless_1, A_1)$, i.e., $\vartheta_{\nless}\widetilde{\in}(\nless_1, A_1)$ if $\vartheta \in A_1 \Rightarrow \nless(\vartheta) \subseteq \nless_1(\vartheta)$.

Example 2.24 Let $X = \{i, n, k\}$ and $E = \{\vartheta_1, \vartheta_2\}$, then

$$\vartheta_{\nless_1} = \{(\vartheta_1, \{(i, 0.42, 0.57), (k, 0.43, 0.42)\})\},$$

and

$$\vartheta_{\nless_2} = \{(\vartheta_2, \{(n, 0.37, 0.56), (k, 0.68, 0.29)\})\}$$

are two distinct PFS points contained in the PFSS

$$\nless_E = \big\{(\vartheta_1, \{(i, 0.42, 0.57), (k, 0.43, 0.42)\}), (\vartheta_2, \{(n, 0.37, 0.56), (k, 0.68, 0.29)\})\big\}.$$

Notice that $\nless_E = \vartheta_{\nless_1}\widetilde{\cup}\vartheta_{\nless_2}$, i.e., a PFS is union of its PFS points.

3 Multi-criteria Group Decision-Making Using Pythagorean Fuzzy Soft Information

There are lots of expressions that we casually use in daily life that have fuzzy structure. Usually we use numerical or, sometimes, verbal expressions to explain an event, refer to something, evaluating expertise of someone, and in many other situations include fuzziness. It is customary to use lingual expressions. These expressions generally do not express cast-iron certainty when deciding on a situation or elucidating some event. For example, the words poor, middle class, lower middle class, upper middle class, upper class and rich, etc. are used according to the income of an individual. We use the word "fast" to express a speed of 80 km/h while traveling on a rough road but call it "slow" while moving on motorways. These examples illustrate how human brain works and decides in ambiguous and uncertain situations, and how it recognizes, assesses, and commands events.

Science and technology have made tremendous developments with the advent of FSs. The mathematics of FSs has gained a large number of practical implementations in both theoretical and applied studies ranging from life sciences to artificial intelligence, and from physical sciences to engineering and humanities.

Often, we face problems in daily life situations which are not precise and clear. This issue leads us to different sorts of decision-making mechanisms. We endeavor to reach at some flawless and intellectual decision employing these mechanisms. For that reason, it is the need of the hour to have improved mathematical models and techniques for handling uncertainty and imprecision.

Shrewdly choice making is an energetic portion of trade, financial matters, social sciences, and other real-world issues. It marks out from day-by-day moo level operational appraisals at low-ranking administration level to long-term key arranging confronted by senior members of any organization. Conclusions that are delivered at any level can cause genuine or awful results, but is there an unequivocal format that choice producers ought to embrace in arrange to guarantee victory, or ought to supersede the standard plans of tackling a problem?

The choice producers ought to contract numerous components into consideration before reaching a unanimous and consistent choice. So it is basic to discover all these components are taken before the assurance is finalized. In parliamentary law to guarantee that all the vital realities and figures are scrutinized, it is irreplaceable to arrange the choice making advancement with an ordered demeanor.

Above and beyond other colossal applications, the science of mathematics helps us too in coming to conclusions on logical evidence. PFSSs take a broad view of both of IFSs and SSs in the sense that all intuitionistic fuzzy numbers used to express membership and non-membership degrees are part of Pythagorean fuzzy numbers. In daily decision-taking problems, PFSSs cover a greater membership space than the IFSS. As a consequence, PFSSs are more capable than IFSSs to model imprecision and uncertainty in choice making problems.

In this segment, we present an algorithm for handling multiple criteria group decision-making problem using choice value method under the umbrella of PFSSs supported by an illustration.

Algorithm 1:

Step 1: Input $X = \{\partial_i : i = 1, 2, \cdots, m\}$ as an aggregate of choices and $E = \{e_i : i = 1, 2, \cdots, n\}$ as a collection of attributes.

Step 2: Construct the PFS matrix with the assistance of experts.

Step 3: Compute relative importance, i.e., weight w_i of each attribute such that $\Sigma_{i=1}^{n} w_i = 1$.

Step 4: Compute the matrix of choice values using $\mathrm{C} = \curlywedge_E \times W^t$.

Step 5: Compute the score value s for each alternative using $s_j = n_{\sigma_j} - n_{\varrho_j}$, where n_{σ_j} denotes number of times σ_j goes beyond or equals other values of σ_k, $k \neq j$.

Step 6: The alternative for which score value is highest is the requisite choice.

Table 3 Tabular representation of $\curlywedge_E$

$\curlywedge_E$	e_1	e_2	e_3	e_4	e_5
∂_1	(0.42,0.56)	(0.37,0.54)	(0.59,0.11)	(0.23,0.59)	(0.11,0.92)
∂_2	(0.34,0.13)	(0.52,0.41)	(0.54,0.11)	(0.33,0.02)	(0.22,0.14)
∂_3	(0.89,0.24)	(0.77,0.31)	(0.56,0.15)	(0.50,0.13)	(0.28,0.13)
∂_4	(0.43,0.44)	(0.56,0.67)	(0.83,0.29)	(0.47,0.58)	(0.37,0.09)
∂_5	(0.56,0.67)	(0.49,0.52)	(0.57,0.38)	(0.21,0.34)	(0.38,0.36)
∂_6	(0.79,0.34)	(0.44,0.43)	(0.56,0.58)	(0.91,0.39)	(0.33,0.39)
∂_7	(0.54,0.24)	(0.51,0.42)	(0.55,0.55)	(0.11,0.09)	(0.39,0.56)

As a case study, we employ Algorithm 1 in stock exchange investment problem using hypothetical information.

Example 3.1 Suppose that an investor wishes to invest some money in some business with least risk. Let $X = \{\partial_i : i = 1, \cdots, 7\}$ be the collection of choices under consideration. For the purpose of reducing the risk factor, he decides to invest his capital in the ratio 3:2 in accordance with the top ranked two businesses. After getting advice from his four financial advisors, he chooses the set of attributes as $E = \{e_i : i = 1, \cdots, 5\}$, where

$$\begin{aligned}
e_1 &= \text{Standing reputation of the business}\\
e_2 &= \text{Impact on market}\\
e_3 &= \text{Prospects}\\
e_4 &= \text{Product Viability}\\
e_5 &= \text{Investment Safety}
\end{aligned}$$

Studying the history and trends of these businesses, the members of the technical team of the investor arranges the gathered information in the form of Table 3 of the PFS-set $\curlywedge_E$.

This information may be put in the form of PFS matrix as

$$\curlywedge_E = \begin{pmatrix}
(0.42, 0.56) & (0.37, 0.54) & (0.59, 0.11) & (0.23, 0.59) & (0.11, 0.92)\\
(0.34, 0.13) & (0.52, 0.41) & (0.54, 0.11) & (0.33, 0.02) & (0.22, 0.14)\\
(0.89, 0.24) & (0.77, 0.31) & (0.56, 0.15) & (0.50, 0.13) & (0.28, 0.13)\\
(0.43, 0.44) & (0.56, 0.67) & (0.83, 0.29) & (0.47, 0.58) & (0.37, 0.09)\\
(0.56, 0.67) & (0.49, 0.52) & (0.57, 0.38) & (0.21, 0.34) & (0.38, 0.36)\\
(0.79, 0.34) & (0.44, 0.43) & (0.56, 0.58) & (0.91, 0.39) & (0.33, 0.39)\\
(0.54, 0.24) & (0.51, 0.42) & (0.55, 0.55) & (0.11, 0.09) & (0.39, 0.56)
\end{pmatrix}$$

Assume that the four financial advisors provide the relative importance, i.e., weights, which are fuzzified, to each attribute and are given in the form of following matrix:

$$M = \begin{pmatrix} 0.54 & 0.38 & 0.59 & 0.89 & 0.76 \\ 0.37 & 0.47 & 0.48 & 0.94 & 0.88 \\ 0.82 & 0.46 & 0.76 & 0.23 & 0.79 \\ 0.18 & 0.32 & 0.57 & 0.46 & 0.69 \end{pmatrix}$$

After normalizing the entries of M, the normalized matrix appears to be

$$\widehat{M} = \begin{pmatrix} 0.507 & 0.461 & 0.485 & 0.639 & 0.485 \\ 0.348 & 0.570 & 0.394 & 0.675 & 0.562 \\ 0.770 & 0.558 & 0.625 & 0.165 & 0.504 \\ 0.169 & 0.388 & 0.468 & 0.330 & 0.441 \end{pmatrix}$$

Thus, the weighted values for the attributes are

$$W(e_1) = 0.188,\ W(e_2) = 0.207,\ W(e_3) = 0.207,\ W(e_4) = 0.190,\ W(e_5) = 0.209.$$

Hence, the weight vector is

$$W = \begin{pmatrix} 0.188 & 0.207 & 0.207 & 0.190 & 0.209 \end{pmatrix}$$

Thus, the PF-matrix for choice values is

$$\begin{aligned} \mathrm{C} &= \curlywedge_E \times W^t \\ &= \begin{pmatrix} (0.42, 0.56) & (0.37, 0.54) & (0.59, 0.11) & (0.23, 0.59) & (0.11, 0.92) \\ (0.34, 0.13) & (0.52, 0.41) & (0.54, 0.11) & (0.33, 0.02) & (0.22, 0.14) \\ (0.89, 0.24) & (0.77, 0.31) & (0.56, 0.15) & (0.50, 0.13) & (0.28, 0.13) \\ (0.43, 0.44) & (0.56, 0.67) & (0.83, 0.29) & (0.47, 0.58) & (0.37, 0.09) \\ (0.56, 0.67) & (0.49, 0.52) & (0.57, 0.38) & (0.21, 0.34) & (0.38, 0.36) \\ (0.79, 0.34) & (0.44, 0.43) & (0.56, 0.58) & (0.91, 0.39) & (0.33, 0.39) \\ (0.54, 0.24) & (0.51, 0.42) & (0.55, 0.55) & (0.11, 0.09) & (0.39, 0.56) \end{pmatrix} \begin{pmatrix} 0.188 \\ 0.207 \\ 0.207 \\ 0.190 \\ 0.209 \end{pmatrix} \\ &= \begin{pmatrix} (0.3444, 0.5442) \\ (0.3920, 0.1651) \\ (0.5962, 0.1922) \\ (0.5352, 0.4105) \\ (0.4440, 0.4521) \\ (0.5974, 0.4286) \\ (0.4234, 0.3801) \end{pmatrix} \end{aligned}$$

The values of the score function along with ranking are given in Table 4. Table 4 demonstrates that

$$\partial_3 \succ \partial_6 \succ \partial_2 = \partial_4 \succ \partial_7 \succ \partial_5 \succ \partial_1$$

This ranking is depicted in Fig. 1.

Table 4 Values of score function and ranking

X	s	Ranking
∂_1	$0 - 6 = -6$	6
∂_2	$1 - 0 = 1$	3
∂_3	$5 - 1 = 4$	1
∂_4	$4 - 3 = 1$	3
∂_5	$3 - 5 = -2$	5
∂_6	$6 - 4 = 2$	2
∂_7	$2 - 2 = 0$	4

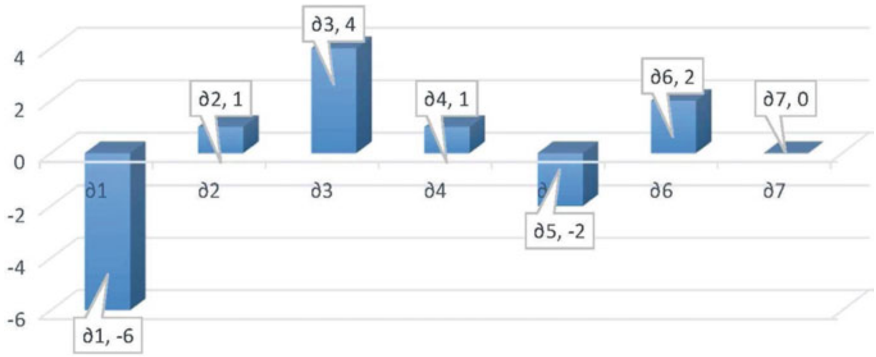


Fig. 1 Ranking of companies

In view of this ranking, it may be concluded that the investor should invest 60% of the capital on ∂_3, and 40% on ∂_6.

3.1 Comparison Analysis

We compare the results of our proposed Algorithm 1 with that of some existing methods. The results obtained are shown in Table 5.

The results portrayed in Table 5 approve the validity of the proposed technique.

Table 5 Comparison of results of suggested algorithm 1 with some existing techniques

Method	Ranking of choices
Algorithm 1 (Suggested)	$\partial_3 \succ \partial_6 \succ \partial_2 = \partial_4$
Guleria and Bajaj (Case-I) [25]	$\partial_3 \succ \partial_6 \succ \partial_2 \succ \partial_4$
Guleria and Bajaj (Case-II) [25]	$\partial_3 \succ \partial_6 \succ \partial_2 \succ \partial_4$
Peng et al. [51]	$\partial_3 \succ \partial_6 \succ \partial_4 \succ \partial_2$

Table 6 Phonetic labels for assessing alternatives

Linguistic Terms	Fuzzy Weights
Very Necessary (VN)	(0.80, 1]
Mandatory (M)	(0.50, 0.80]
More or Less Required (MLR)	(0.20, 0.50]
Average Requirement (AR)	(0.10, 0.20]
Of No Use (ONU)	[0, 0.10]

4 TOPSIS Approach for Choice Making with Pythagorean Fuzzy Soft Sets

In this section, we study the utilization of PFSSs in intelligent decision-taking. For this purpose, we first extend TOPSIS to PFSS. The proposed version will be called PFS-TOPSIS. Afterwards, we shall consider a problem of choosing suitable candidates for key ministries of a country, where PFSSs may be used.

We launch by illuminating the offered modus operandi a step at a time. The suggested PFS-TOPSIS is generality of fuzzy soft TOPSIS suggested in [12] by Eraslan and Karaaslan.

Algorithm 2:

Step 1: Recognizing the problem: Suppose that $DM = \{\mathcal{D}_i : i = 1, \cdots, n\}$ is team of decision experts, $C = \{\ddot{c}_i = 1, \cdots, l\}$ is the assemblage of choices and $Q = \{q_j : j = 1, \cdots, m\}$ is family of attributes.

Step 2: Picking the phonetic terms as given in Table 6, prepare weighted parameter matrix as $[w_{ij}]_{n\times m}$, where w_{ij} is the weight allocated by the decision expert $\mathcal{D}_i$ to the attribute q_j.

Step 3: Normalize the weighted matrix to get $\hat{N} = [\hat{n}_{ij}]_{n\times m}$, where $\hat{n}_{ij} = \frac{w_{ij}}{\sqrt{\Sigma_{i=1}^{n} w_{ij}^2}}$ and obtaining the weight vector $\mathcal{W} = (\mathfrak{w}_1, \mathfrak{w}_2, \cdots, \mathfrak{w}_m)$, where $\mathfrak{w}_j = \frac{\sum_{i=1}^{n} \hat{n}_{ij}}{m \sum_{k=1}^{m} \hat{n}_{ik}}$.

Step 4: Construct PFS matrix

$$D_i = [v^i_{jk}]_{l\times m} = \begin{pmatrix} v^i_{11} & v^i_{12} & \cdots & v^i_{1m} \\ v^i_{21} & v^i_{22} & \cdots & v^i_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ v^i_{j1} & v^i_{j2} & \cdots & v^i_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ v^i_{l1} & v^i_{l2} & \cdots & v^i_{lm}, \end{pmatrix}$$

where v^i_{jk} is a PFS-element, provided by ith decision expert. Then obtain the aggregated matrix

$$D = \frac{D_1 + D_2 + \cdots + D_n}{n} = [\dot{v}_{jk}]_{l\times m}.$$

Step 5: Achieve the weighted PFS matrix

$$D_w = [\ddot{r}_{jk}]_{l\times m} = \begin{pmatrix} \ddot{r}_{11} & \ddot{r}_{12} & \cdots & \ddot{r}_{1m} \\ \ddot{r}_{21} & \ddot{r}_{22} & \cdots & \ddot{r}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \ddot{r}_{j1} & \ddot{r}_{j2} & \cdots & \ddot{r}_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ \ddot{r}_{l1} & \ddot{r}_{l2} & \cdots & \ddot{r}_{lm}, \end{pmatrix}$$

where $\ddot{r}_{jk} = \mathfrak{w}_k \times \dot{v}_{jk}$.

Step 6: Track the PFS-valued positive ideal solution (PFSV-PIS) and PFS-valued negative ideal solution (PFSV-NIS). For this purpose, we employ in order

$$\begin{aligned} \text{PFSV-PIS} &= \{\ddot{r}_1^+, \ddot{r}_2^+, \cdots, \ddot{r}_m^+\} \\ &= \left\{(\vee_k \ddot{r}_{jk}, \wedge_k \ddot{r}_{jk}); k = 1, \cdots, m\right\} \\ &= \left\{(\sigma_k^+, \varrho_k^+) : k = 1, \cdots, m\right\} \end{aligned}$$

and

$$\begin{aligned} \text{PFSV-NIS} &= \{\ddot{r}_1^-, \ddot{r}_2^-, \cdots, \ddot{r}_m^-\} \\ &= \left\{(\wedge_k \ddot{r}_{jk}, \vee_k \ddot{r}_{jk}); k = 1, \cdots, m\right\} \\ &= \left\{(\sigma_k^-, \varrho_k^-) : k = 1, \cdots, m\right\}, \end{aligned}$$

where $\vee$ stands for PFS union and $\wedge$ represents PFS intersection.

Step 7: Compute distances of each alternative from PFSV-PIS and PFSV-NIS, respectively, utilizing

$$\mathfrak{z}_j^+ = \sqrt{\Sigma_{k=1}^m \left\{\left(\sigma_{jk} - \sigma_k^+\right)^2 + \left(\varrho_{jk} - \varrho_k^+\right)^2\right\}}$$

and

$$\mathfrak{z}_j^- = \sqrt{\Sigma_{k=1}^m \left\{\left(\sigma_{jk} - \sigma_k^-\right)^2 + \left(\varrho_{jk} - \varrho_k^-\right)^2\right\}}.$$

Step 8: Attain the closeness coefficient of each alternative with ideal solution by making use of

$$\mathcal{C}_j^* = \frac{\mathfrak{z}_j^-}{\mathfrak{z}_j^+ + \mathfrak{z}_j^-} \in [0, 1].$$

Step 9: Arrange the ranking of choices in decreasing (or increasing) for obtaining the priority order of the choices.

As an illustration of Algorithm 2, we discuss a state managerial problem following the procedural steps given in Algorithm 2.

Example 4.1 Suppose that a political party clean sweeps in general elections in a country. The party has got chance for the first time to make national government and wishes to prove that it is the best. The party chairman wants to deliver to the people of the country his best. The party wants to fill the positions of key ministries by choosing ministers, who should also be competent, well educated/trained and meritorious in their respective fields. The party's top leadership constitutes a committee of experts to help him solve this riddle on scientific grounds. They also decide that no member should be given more than one ministry. Assume that

$$C = \{c_1, c_2, \ldots, c_6\}$$

is the set of candidates who are to be deputed in different key ministries (ministries of foreign affairs, defence, finance, and information & broadcasting in order). Further suppose that

$$Q = \{q_1, q_2, \ldots, q_5\}$$

is the set of qualification/merit mandatory for filling a position. The committee interviews each candidate carefully to see who is appropriate for which ministry.

Picking the weights from Table 6, the experts provide the following weighted parameter matrix

$$\mathcal{P} = \begin{pmatrix} \text{VN} & \text{MLR} & \text{MLR} & \text{ONU} & \text{VN} \\ \text{M} & \text{AR} & \text{AR} & \text{AR} & \text{VN} \\ \text{M} & \text{M} & \text{VN} & \text{M} & \text{M} \\ \text{MLR} & \text{AR} & \text{MLR} & \text{AR} & \text{VN} \end{pmatrix} = \begin{pmatrix} 0.90 & 0.40 & 0.30 & 0.10 & 0.90 \\ 0.70 & 0.15 & 0.20 & 0.15 & 0.85 \\ 0.60 & 0.70 & 0.90 & 0.80 & 0.75 \\ 0.40 & 0.15 & 0.40 & 0.15 & 0.90 \end{pmatrix}$$

The normalized weighted matrix is

$$\hat{N} = \begin{pmatrix} 0.667 & 0.480 & 0.286 & 0.120 & 0.528 \\ 0.519 & 0.180 & 0.191 & 0.180 & 0.499 \\ 0.445 & 0.840 & 0.858 & 0.960 & 0.440 \\ 0.296 & 0.180 & 0.381 & 0.180 & 0.528 \end{pmatrix}$$

and hence the weight vector is $\mathcal{W} = (0.220, 0.192, 0.196, 0.164, 0.228)$.

Assume that the four experts provide the following PFS matrices in which the PFN at (i, j)th position demarcated grades of candidates row-wise and the attribute column-wise.

$$D_1 = \begin{pmatrix} (0.57, 0.39) & (0.49, 0.74) & (0.77, 0.38) & (0.54, 0.21) & (0.12, 0.48) \\ (0.66, 0.51) & (0.54, 0.54) & (0.32, 0.13) & (0.99, 0.13) & (0.54, 0.07) \\ (0.15, 0.68) & (0.19, 0.32) & (0.76, 0.41) & (0.45, 0.15) & (0.11, 0.49) \\ (0.67, 0.74) & (0.09, 0.83) & (0.59, 0.31) & (0.84, 0.16) & (0.37, 0.21) \\ (0.59, 0.17) & (0.33, 0.67) & (0.34, 0.68) & (0.52, 0.19) & (0.58, 0.61) \\ (0.27, 0.54) & (0.49, 0.46) & (0.48, 0.59) & (0.55, 0.54) & (0.38, 0.01) \end{pmatrix}$$

$$D_2 = \begin{pmatrix} (0.34, 0.52) & (0.58, 0.21) & (0.47, 0.21) & (0.70, 0.31) & (0.11, 0.34) \\ (0.47, 0.33) & (0.39, 0.32) & (0.56, 0.20) & (0.38, 0.11) & (0.26, 0.18) \\ (0.59, 0.17) & (0.33, 0.17) & (0.19, 0.28) & (0.59, 0.06) & (0.78, 0.16) \\ (0.44, 0.17) & (0.38, 0.23) & (0.58, 0.27) & (0.71, 0.24) & (0.54, 0.02) \\ (0.32, 0.28) & (0.56, 0.11) & (0.44, 0.37) & (0.49, 0.29) & (0.55, 0.55) \\ (0.34, 0.47) & (0.52, 0.37) & (0.11, 0.18) & (0.47, 0.13) & (0.47, 0.27) \end{pmatrix}$$

$$D_3 = \begin{pmatrix} (0.11, 0.58) & (0.37, 0.22) & (0.56, 0.11) & (0.21, 0.69) & (0.79, 0.32) \\ (0.13, 0.67) & (0.46, 0.13) & (0.36, 0.54) & (0.56, 0.27) & (0.46, 0.61) \\ (0.59, 0.13) & (0.25, 0.11) & (0.62, 0.33) & (0.47, 0.28) & (0.28, 0.47) \\ (0.11, 0.49) & (0.23, 0.05) & (0.50, 0.28) & (0.34, 0.48) & (0.61, 0.54) \\ (0.17, 0.29) & (0.82, 0.34) & (0.56, 0.51) & (0.50, 0.28) & (0.49, 0.12) \\ (0.33, 0.69) & (0.57, 0.61) & (0.48, 0.57) & (0.33, 0.02) & (0.46, 0.31) \end{pmatrix}$$

$$D_4 = \begin{pmatrix} (0.40, 0.59) & (0.41, 0.32) & (0.49, 0.12) & (0.35, 0.65) & (0.39, 0.12) \\ (0.25, 0.17) & (0.38, 0.10) & (0.85, 0.26) & (0.44, 0.57) & (0.92, 0.14) \\ (0.38, 0.51) & (0.36, 0.11) & (0.52, 0.29) & (0.48, 0.38) & (0.52, 0.35) \\ (0.56, 0.11) & (0.73, 0.16) & (0.35, 0.27) & (0.58, 0.62) & (0.62, 0.63) \\ (0.11, 0.01) & (0.33, 0.37) & (0.28, 0.38) & (0.47, 0.32) & (0.71, 0.19) \\ (0.58, 0.17) & (0.44, 0.15) & (0.56, 0.16) & (0.33, 0.21) & (0.88, 0.26) \end{pmatrix}$$

Thus, the aggregated matrix is

$$D = \begin{pmatrix} (0.355, 0.520) & (0.463, 0.373) & (0.573, 0.205) & (0.450, 0.465) & (0.353, 0.315) \\ (0.378, 0.420) & (0.443, 0.273) & (0.523, 0.283) & (0.593, 0.270) & (0.545, 0.250) \\ (0.428, 0.373) & (0.373, 0.178) & (0.523, 0.328) & (0.498, 0.218) & (0.423, 0.368) \\ (0.445, 0.378) & (0.358, 0.318) & (0.505, 0.283) & (0.618, 0.375) & (0.535, 0.350) \\ (0.298, 0.188) & (0.510, 0.373) & (0.405, 0.485) & (0.495, 0.270) & (0.583, 0.368) \\ (0.380, 0.468) & (0.505, 0.398) & (0.408, 0.375) & (0.420, 0.225) & (0.548, 0.213) \end{pmatrix}$$

and hence the weighted PFS matrix is

Table 7 Distance & closeness coefficient of each candidate

Candidate	$\mathfrak{Z}_j^+$	$\mathfrak{Z}_j^-$	C_j^*
c_1	0.1137	0.0714	0.3857
c_2	0.0620	0.0843	0.5762
c_3	0.0774	0.0830	0.5175
c_4	0.0741	0.0782	0.5135
c_5	0.0909	0.0934	0.5068
c_6	0.0953	0.0725	0.4321

$$D_w = \begin{pmatrix} (0.078, 0.114) & (0.089, 0.072) & (0.112, 0.040) & (0.074, 0.076) & (0.080, 0.072) \\ (0.083, 0.092) & (0.085, 0.052) & (0.103, 0.055) & (0.097, 0.044) & (0.124, 0.057) \\ (0.094, 0.082) & (0.072, 0.034) & (0.103, 0.064) & (0.082, 0.036) & (0.096, 0.084) \\ (0.098, 0.083) & (0.069, 0.061) & (0.099, 0.055) & (0.101, 0.062) & (0.122, 0.080) \\ (0.066, 0.041) & (0.098, 0.072) & (0.079, 0.095) & (0.081, 0.044) & (0.133, 0.084) \\ (0.084, 0.103) & (0.097, 0.076) & (0.080, 0.074) & (0.069, 0.037) & (0.125, 0.049) \end{pmatrix}$$

The positive and negative ideal solutions are

$$\begin{aligned} \text{PFSV-PIS} &= \{\ddot{r}_1^+, \ddot{r}_2^+, \cdots, \ddot{r}_5^+\} \\ &= \{(0.098, 0.041), (0.098, 0.034), (0.112, 0.040), (0.101, 0.036), (0.133, 0.049)\} \end{aligned}$$

and

$$\begin{aligned} \text{PFSV-NIS} &= \{\ddot{r}_1^-, \ddot{r}_2^-, \cdots, \ddot{r}_5^-\} \\ &= \{(0.066, 0.114), (0.069, 0.076), (0.079, 0.095), (0.069, 0.076), (0.096, 0.084)\} \end{aligned}$$

respectively.

The distance of each candidate from PFSV-PIS and PFSV-NIS accompanied by their relative closeness coefficients are displayed in Table 7.

Hence, the ranking preference is

$$c_2 \succ c_3 \succ c_4 \succ c_5 \succ c_6 \succ c_1$$

. This preference order is depicted in Fig. 2.

The above priority order advocates that the ministry of foreign affairs should be given to c_2, defence to c_3, finance to c_4 and the ministry of information & broadcasting to c_5.

Fig. 2 Ranking of candidates

5 Multiple Criteria Group Decision-Making Using PFS-VIKOR Method

The word VIKOR is abbreviated version of "Vlse Kriterijumska Optimizacija Kompromisno Resenje" from Serbian language to mean manifold-criteria analysis (or optimization) and middle ground way out. This technique was devised by Serafim Opricovic to handle choice making problems having dissenting and non-commensurable principles, with the assumption that finding the middle grounds is apt for resolving any clash. The team of experts rummages around for a solution that neighbors the superlative idyllic solution, and the choices are evaluated following all recognized rules. VIKOR has transpired as a widely held multi-criteria decision-making technique mainly because of its computational straightforwardness and scrupulousness of solution.

We elucidate the suggested technique bit by bit as below. First six steps of PFS-VIKOR are the same as of PFS-TOPSIS given in Algorithm 2, so we skip them.

Algorithm 3 (PFS-VIKOR):

Step 7: Use the formulae

$$S_i = \Sigma_{j=1}^{m} \mathfrak{w}_j \left(\frac{d\left(\ddot{r}_j^+, \ddot{r}_{ij}\right)}{d\left(\ddot{r}_j^+, \ddot{r}_j^-\right)} \right)$$

$$R_i = \max_{j=1}^{m} \mathfrak{w}_j \left(\frac{d\left(\ddot{r}_j^+, \ddot{r}_{ij}\right)}{d\left(\ddot{r}_j^+, \ddot{r}_j^-\right)} \right)$$

$$Q_i = \kappa \left(\frac{S_i - S^-}{S^+ - S^-} \right) + (1 - \kappa) \left(\frac{R_i - R^-}{R^+ - R^-} \right),$$

where $S^+ = \max_i S_i$, $S^- = \min_i S_i$, $R^+ = \max_i R_i$, and $R^- = \min_i R_i$, to get the values of group utility S_i, individual regret R_i, and compromise Q_i. The real number κ is termed as coefficient of decision mechanism. The role of the coefficient κ is that if compromise solution is to be selected by majority, we choose $\kappa > 0.5$; for consensus we use $\kappa = 0.5$, and $\kappa < 0.5$ represents veto. $\mathfrak{w}_j$ represents the weight of the jth criteria, which expresses its relative importance.

Step 8: Arrange S_i, R_i, and Q_i in ascending array. The choice $\ddot{r}_\alpha$ would be announced middle ground solution if it has the minimum value of Q_i and further gratifies the following two necessities in chorus:

(a) If $\ddot{r}_{\alpha_1}$ and $\ddot{r}_{\alpha_2}$ are two best choices regarding Q_i, then

$$Q(\ddot{r}_{\alpha_2}) - Q(\ddot{r}_{\alpha_1}) \geq \frac{1}{n-1}$$

n being the number of attributes.

(b) The choice $\ddot{r}_{\alpha_1}$ must be best ranked by at least one of R_i and S_i.

There will exist multiple compromise solutions otherwise, which may be located as under:

(i) $\ddot{r}_{\alpha_1}$ and α_2 will be the compromise solutions in case merely (a) is gratified.

(ii) $\ddot{r}_{\alpha_1}, \ddot{r}_{\alpha_2}, \cdots, \ddot{r}_{\alpha_u}$ would be the compromise solutions in case (a) is not fulfilled, where $\ddot{r}_{\alpha_u}$ may be found employing

$$Q(\ddot{r}_{\alpha_u}) - Q(\ddot{r}_{\alpha_1}) \geq \frac{1}{n-1}.$$

Example 5.1 Assume that a multi-national company wants to choose some brand ambassadors for advertisement of its products. The CEO of that company constitutes a committee of four experts to give recommendations about the selection of ambassadors. The number of ambassadors may vary from one to any reasonable number. The CEO needs a unanimous decision about their selection. The committee decides to work on scientific grounds. Assume that

$$C = \{a_1, a_2, \cdots, a_6\}$$

is the set of persons under consideration as embassador. Further suppose that

$$Q = \{q_1, q_2, \cdots, q_5\}$$

is the set of qualities under consideration for the selection of any individual. The committee ponders on the personalities and the effectiveness of those individuals on the mob.

Picking the weights from Table 6, the experts provide the following weighted parameter matrix

$$\mathcal{P} = \begin{pmatrix} \text{VN} & \text{MLR} & \text{MLR} & \text{ONU} & \text{VN} \\ \text{M} & \text{AR} & \text{AR} & \text{AR} & \text{VN} \\ \text{M} & \text{M} & \text{VN} & \text{M} & \text{M} \\ \text{MLR} & \text{AR} & \text{MLR} & \text{AR} & \text{VN} \end{pmatrix}$$

$$= \begin{pmatrix} 0.90 & 0.40 & 0.30 & 0.10 & 0.90 \\ 0.70 & 0.15 & 0.20 & 0.15 & 0.85 \\ 0.60 & 0.70 & 0.90 & 0.80 & 0.75 \\ 0.40 & 0.15 & 0.40 & 0.15 & 0.90 \end{pmatrix}$$

The normalized weighted matrix is

$$\hat{N} = \begin{pmatrix} 0.667 & 0.480 & 0.286 & 0.120 & 0.528 \\ 0.519 & 0.180 & 0.191 & 0.180 & 0.499 \\ 0.445 & 0.840 & 0.858 & 0.960 & 0.440 \\ 0.296 & 0.180 & 0.381 & 0.180 & 0.528 \end{pmatrix}$$

and hence the weight vector is $\mathcal{W} = (0.220, 0.192, 0.196, 0.164, 0.228)$.

Assume that the four experts provide the following PFS matrices in which the PFN at (i, j)th position demarcated grades of candidates row-wise and the attribute column-wise.

$$D_1 = \begin{pmatrix} (0.57, 0.39) & (0.49, 0.74) & (0.77, 0.38) & (0.54, 0.21) & (0.12, 0.48) \\ (0.66, 0.51) & (0.54, 0.54) & (0.32, 0.13) & (0.99, 0.13) & (0.54, 0.07) \\ (0.15, 0.68) & (0.19, 0.32) & (0.76, 0.41) & (0.45, 0.15) & (0.11, 0.49) \\ (0.67, 0.74) & (0.09, 0.83) & (0.59, 0.31) & (0.84, 0.16) & (0.37, 0.21) \\ (0.59, 0.17) & (0.33, 0.67) & (0.34, 0.68) & (0.52, 0.19) & (0.58, 0.61) \\ (0.27, 0.54) & (0.49, 0.46) & (0.48, 0.59) & (0.55, 0.54) & (0.38, 0.01) \end{pmatrix}$$

$$D_2 = \begin{pmatrix} (0.34, 0.52) & (0.58, 0.21) & (0.47, 0.21) & (0.70, 0.31) & (0.11, 0.34) \\ (0.47, 0.33) & (0.39, 0.32) & (0.56, 0.20) & (0.38, 0.11) & (0.26, 0.18) \\ (0.59, 0.17) & (0.33, 0.17) & (0.19, 0.28) & (0.59, 0.06) & (0.78, 0.16) \\ (0.44, 0.17) & (0.38, 0.23) & (0.58, 0.27) & (0.71, 0.24) & (0.54, 0.02) \\ (0.32, 0.28) & (0.56, 0.11) & (0.44, 0.37) & (0.49, 0.29) & (0.55, 0.55) \\ (0.34, 0.47) & (0.52, 0.37) & (0.11, 0.18) & (0.47, 0.13) & (0.47, 0.27) \end{pmatrix}$$

$$D_3 = \begin{pmatrix} (0.11, 0.58) & (0.37, 0.22) & (0.56, 0.11) & (0.21, 0.69) & (0.79, 0.32) \\ (0.13, 0.67) & (0.46, 0.13) & (0.36, 0.54) & (0.56, 0.27) & (0.46, 0.61) \\ (0.59, 0.13) & (0.25, 0.11) & (0.62, 0.33) & (0.47, 0.28) & (0.28, 0.47) \\ (0.11, 0.49) & (0.23, 0.05) & (0.50, 0.28) & (0.34, 0.48) & (0.61, 0.54) \\ (0.17, 0.29) & (0.82, 0.34) & (0.56, 0.51) & (0.50, 0.28) & (0.49, 0.12) \\ (0.33, 0.69) & (0.57, 0.61) & (0.48, 0.57) & (0.33, 0.02) & (0.46, 0.31) \end{pmatrix}$$

$$D_4 = \begin{pmatrix} (0.40, 0.59) & (0.41, 0.32) & (0.49, 0.12) & (0.35, 0.65) & (0.39, 0.12) \\ (0.25, 0.17) & (0.38, 0.10) & (0.85, 0.26) & (0.44, 0.57) & (0.92, 0.14) \\ (0.38, 0.51) & (0.36, 0.11) & (0.52, 0.29) & (0.48, 0.38) & (0.52, 0.35) \\ (0.56, 0.11) & (0.73, 0.16) & (0.35, 0.27) & (0.58, 0.62) & (0.62, 0.63) \\ (0.11, 0.01) & (0.33, 0.37) & (0.28, 0.38) & (0.47, 0.32) & (0.71, 0.19) \\ (0.58, 0.17) & (0.44, 0.15) & (0.56, 0.16) & (0.33, 0.21) & (0.88, 0.26) \end{pmatrix}$$

Thus, the aggregated matrix is

$$D = \begin{pmatrix} (0.355, 0.520) & (0.463, 0.373) & (0.573, 0.205) & (0.450, 0.465) & (0.353, 0.315) \\ (0.378, 0.420) & (0.443, 0.273) & (0.523, 0.283) & (0.593, 0.270) & (0.545, 0.250) \\ (0.428, 0.373) & (0.373, 0.178) & (0.523, 0.328) & (0.498, 0.218) & (0.423, 0.368) \\ (0.445, 0.378) & (0.358, 0.318) & (0.505, 0.283) & (0.618, 0.375) & (0.535, 0.350) \\ (0.298, 0.188) & (0.510, 0.373) & (0.405, 0.485) & (0.495, 0.270) & (0.583, 0.368) \\ (0.380, 0.468) & (0.505, 0.398) & (0.408, 0.375) & (0.420, 0.225) & (0.548, 0.213) \end{pmatrix}$$

and hence the weighted PFS matrix is

$$D_w = \begin{pmatrix} (0.078, 0.114) & (0.089, 0.072) & (0.112, 0.040) & (0.074, 0.076) & (0.080, 0.072) \\ (0.083, 0.092) & (0.085, 0.052) & (0.103, 0.055) & (0.097, 0.044) & (0.124, 0.057) \\ (0.094, 0.082) & (0.072, 0.034) & (0.103, 0.064) & (0.082, 0.036) & (0.096, 0.084) \\ (0.098, 0.083) & (0.069, 0.061) & (0.099, 0.055) & (0.101, 0.062) & (0.122, 0.080) \\ (0.066, 0.041) & (0.098, 0.072) & (0.079, 0.095) & (0.081, 0.044) & (0.133, 0.084) \\ (0.084, 0.103) & (0.097, 0.076) & (0.080, 0.074) & (0.069, 0.037) & (0.125, 0.049) \end{pmatrix}$$

The positive and negative ideal solutions are

$$\begin{aligned} \text{PFSV-PIS} &= \{\ddot{r}_1^+, \ddot{r}_2^+, \cdots, \ddot{r}_5^+\} \\ &= \{(0.098, 0.041), (0.098, 0.034), (0.112, 0.040), (0.101, 0.036), (0.133, 0.049)\} \end{aligned}$$

and

$$\begin{aligned} \text{PFSV-NIS} &= \{\ddot{r}_1^-, \ddot{r}_2^-, \cdots, \ddot{r}_5^-\} \\ &= \{(0.066, 0.114), (0.069, 0.076), (0.079, 0.095), (0.069, 0.076), (0.096, 0.084)\} \end{aligned}$$

respectively.

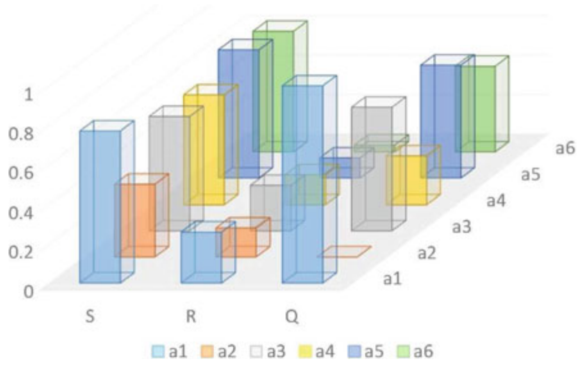
Choosing $\kappa = 0.5$, the values of S_i, R_i, and Q_i for each choice $\ddot{r}_i$ are calculated utilizing

$$\begin{aligned} S_i &= \Sigma_{j=1}^{5} \mathfrak{w}_j \left(\frac{d(\ddot{r}_j^+, \ddot{r}_{ij})}{d(\ddot{r}_j^+, \ddot{r}_j^-)} \right) \\ R_i &= \max_{j=1}^{5} \mathfrak{w}_j \left(\frac{d(\ddot{r}_j^+, \ddot{r}_{ij})}{d(\ddot{r}_j^+, \ddot{r}_j^-)} \right) \\ Q_i &= \kappa \left(\frac{S_i - S^-}{S^+ - S^-} \right) + (1 - \kappa) \left(\frac{R_i - R^-}{R^+ - R^-} \right) \end{aligned}$$

Table 8 Values of S_i, R_i, and Q_i for alternatives

Alternative	S_i	R_i	Q_i
a_1	0.7698	0.2589	1.0000
a_2	0.3663	0.1469	0.0000
a_3	0.5788	0.2280	0.6260
a_4	0.5562	0.1491	0.2457
a_5	0.6531	0.1025	0.5754
a_6	0.6148	0.0358	0.4368

Fig. 3 3D column chart of rankings



and are given in Table 8 below:

The rank of choices is as under:

$$\text{By } Q_i : \quad a_2 \prec a_4 \prec a_6 \prec a_5 \prec a_3 \prec a_1$$

$$\text{By } S_i : \quad a_2 \prec a_4 \prec a_3 \prec a_6 \prec a_5 \prec a_1$$

$$\text{By } R_i : \quad a_6 \prec a_5 \prec a_2 \prec a_4 \prec a_3 \prec a_1$$

Since

$$Q(a_4) - Q(a_2) = 0.2457 \not\geq \frac{1}{4}$$

so (a) is not gratified. Further,

$$Q(a_6) - (a_2) = 0.4368 \geq \frac{1}{4}$$

Thus, the committee recommends that the persons a_2, a_4, and a_6 must be chosen as brand ambassadors. These rankings are depicted in Fig. 3.

6 A Similarity Measure for PFSSs

In this section, we propose a new similarity measure for PFSSs based on cosine similarity measure and Frobenius inner product of matrices and render some of its characteristics.

Definition 6.1 Let $X=\{\partial_i : i=1, \cdots, m\}$ be a crisp set and $E = \{e_j : j = 1, \cdots, n\}$ be the aggregate of attributes. If

$$\measuredangle_1 = \begin{pmatrix} (\sigma_{11}, \varrho_{11})_{\measuredangle_1} & (\sigma_{12}, \varrho_{12})_{\measuredangle_1} & \cdots & (\sigma_{1n}, \varrho_{1n})_{\measuredangle_1} \\ (\sigma_{21}, \varrho_{21})_{\measuredangle_1} & (\sigma_{22}, \varrho_{22})_{\measuredangle_1} & \cdots & (\sigma_{2n}, \varrho_{2n})_{\measuredangle_1} \\ \vdots & \vdots & \ddots & \vdots \\ (\sigma_{m1}, \varrho_{m1})_{\measuredangle_1} & (\sigma_{m2}, \varrho_{m2})_{\measuredangle_1} & \cdots & (\sigma_{mn}, \varrho_{mn})_{\measuredangle_1} \end{pmatrix}$$

and

$$\measuredangle_2 = \begin{pmatrix} (\sigma_{11}, \varrho_{11})_{\measuredangle_2} & (\sigma_{12}, \varrho_{12})_{\measuredangle_2} & \cdots & (\sigma_{1n}, \varrho_{1n})_{\measuredangle_2} \\ (\sigma_{21}, \varrho_{21})_{\measuredangle_2} & (\sigma_{22}, \varrho_{22})_{\measuredangle_2} & \cdots & (\sigma_{2n}, \varrho_{2n})_{\measuredangle_2} \\ \vdots & \vdots & \ddots & \vdots \\ (\sigma_{m1}, \varrho_{m1})_{\measuredangle_2} & (\sigma_{m2}, \varrho_{m2})_{\measuredangle_2} & \cdots & (\sigma_{mn}, \varrho_{mn})_{\measuredangle_2} \end{pmatrix}$$

are PFS matrices of PFSSs $(\measuredangle_1, E)$ and $(\measuredangle_2, E)$, then *similarity measure* between $(\measuredangle_1, E)$ and $(\measuredangle_2, E)$ is given as

$$Sim(\measuredangle_1, \measuredangle_2) = \frac{< \measuredangle_1, \measuredangle_2 >}{\| \measuredangle_1 \| \| \measuredangle_2 \|},$$

where

$$< \measuredangle_1, \measuredangle_2 > = tr(\measuredangle_1^T \measuredangle_2)$$
$$\| \measuredangle_1 \| = \sqrt{< \measuredangle_1, \measuredangle_1 >}.$$

Here $tr(\measuredangle_1^T \measuredangle_2)$ (known as *trace* of the matrix $\measuredangle_1^T \measuredangle_2$) denotes the sum of elements at principal diagonal of the matrix $\measuredangle_1^T \measuredangle_2$. The above definition holds good if hesitation margin ε_{ij} is also taken into account. Moreover, this similarity measure satisfies the following:

(1) $0 \leq Sim(\measuredangle_1, \measuredangle_2) \leq 1$.
(2) $Sim(\measuredangle_1, \measuredangle_2) = 1 \Leftrightarrow \measuredangle_1 = \measuredangle_2$.
(3) $Sim(\measuredangle_1, \measuredangle_2) = Sim(\measuredangle_2, \measuredangle_1)$.
(4) $Sim(\measuredangle, \measuredangle^c) = 1$ iff $\measuredangle$ is a crisp set.
(5) If $(\measuredangle_1, E)\widetilde{\sqsubseteq}(\measuredangle_2, E)\widetilde{\sqsubseteq}(\measuredangle_3, E)$, then $Sim(\measuredangle_1, \measuredangle_3) \leq Sim(\measuredangle_2, \measuredangle_3)$.

Example 6.2 Let $X = \{\partial_1, \cdots, \partial_4\}$ be the universe and $E = \{e_i \mid i = 1, 2, 3\}$ be the aggregate of attributes. Consider the PFS matrices

$$\measuredangle_1 = \begin{pmatrix} (0.95, 0.21) & (0.73, 0.46) & (0.53, 0.71) \\ (0.38, 0.82) & (1, 0) & (0.67, 0.52) \\ (0.28, 0.57) & (0.58, 0.31) & (0.62, 0.79) \\ (0, 1) & (0.91, 0.19) & (0.63, 0.74) \end{pmatrix}$$

and

$$\measuredangle_2 = \begin{pmatrix} (0.54, 0.29) & (0.61, 0.67) & (0.76, 0.02) \\ (0.07, 0.53) & (0.56, 0.11) & (0.39, 0.79) \\ (0.58, 0.17) & (0.36, 0.34) & (0.17, 0.58) \\ (0.21, 0.83) & (0.49, 0.48) & (0.21, 0.87) \end{pmatrix}$$

representing PFSSs $(\measuredangle_1, E)$ and $(\measuredangle_2, E)$, respectively. Now,

$$\begin{aligned} < \measuredangle_1, \measuredangle_2 > &= (0.95, 0.21).(0.54, 0.29) + (0.73, 0.46).(0.61, 0.67) + \cdots + (0.63, 0.74).(0.21, 0.87) \\ &= 6.7180, \\ \| \measuredangle_1 \| &= \sqrt{0.95^2 + 0.21^2 + 0.73^2 + \cdots + 0.74^2} \\ &= 3.1089, \\ \| \measuredangle_2 \| &= \sqrt{0.54^2 + 0.29^2 + 0.61^2 + \cdots + 0.87^2} \\ &= 2.4784. \\ \therefore Sim(\measuredangle_1, \measuredangle_2) &= \frac{< \measuredangle_1, \measuredangle_2 >}{\| \measuredangle_1 \| \| \measuredangle_2 \|} \\ &= \frac{6.7180}{3.1089 \times 2.4784} \\ &= 0.8719. \end{aligned}$$

Example 6.3 Let $X = \{\partial_1, \partial_2, \partial_3\}$ and $E = \{e_1, e_2, e_3\}$. Let

$$\measuredangle_1 = \begin{pmatrix} (0.52, 0.73, 0.44) & (0.89, 0.15, 0.43) & (0.62, 0.59, 0.52) \\ (0.46, 0.73, 0.50) & (1, 0, 0) & (0.51, 0.51, 0.69) \\ (0.32, 0.19, 0.93) & (0.64, 0.27, 0.72) & (0.87, 0.03, 0.49) \end{pmatrix}$$

$$\measuredangle_2 = \begin{pmatrix} (0.68, 0.52, 0.52) & (0.31, 0.69, 0.65) & (0.44, 0.02, 0.90) \\ (0.61, 0.50, 0.61) & (0.33, 0.57, 0.75) & (0.81, 0.16, 0.56) \\ (0.52, 0.28, 0.81) & (0.29, 0.22, 0.93) & (0.21, 0.39, 0.90) \end{pmatrix}$$

be the PFS matrices representing the PFSSs $(\measuredangle_1, E)$ and $(\measuredangle_2, E)$, respectively.
Now,

$$\begin{aligned} < \measuredangle_1, \measuredangle_2 > &= (0.52, 0.73, 0.44).(0.68, 0.52, 0.52) + \cdots + (0.74, 0.63, 0.49).(0.35, 0.54, 0.90) \\ &= 7.0581, \\ \| \measuredangle_1 \| &= \sqrt{0.52^2 + 0.73^2 + 0.44^2 + \cdots + 0.49^2} \\ &= 2.9987, \\ \| \measuredangle_2 \| &= \sqrt{0.68^2 + 0.52^2 + 0.31^2 + \cdots + 0.54^2} \\ &= 2.9994 \end{aligned}$$

$$\therefore Sim(\measuredangle_1, \measuredangle_2) = \frac{< \measuredangle_1, \measuredangle_2 >}{\| \measuredangle_1 \| \| \measuredangle_2 \|} = \frac{7.0581}{2.9987 \times 2.9994} = 0.7847.$$

Example 6.4 Let $X = \{\partial_1, \partial_2, \partial_3\}$ and $E = \{e_1, e_2, e_3\}$. Consider the PFS matrices

$$(\measuredangle_1, E) = \begin{pmatrix} (0.27, 0.39) & (0.42, 0.51) & (0.61, 0.43) \\ (0.25, 0.56) & (0.58, 0.49) & (0.92, 0.36) \\ (0.76, 0.23) & (0.46, 0.48) & (0.54, 0.21) \end{pmatrix}$$

$$(\measuredangle_2, E) = \begin{pmatrix} (0.45, 0.21) & (0.26, 0.89) & (0.54, 0.39) \\ (0.29, 0.28) & (0.46, 0.44) & (0.64, 0.31) \\ (0.27, 0.54) & (0.28, 0.33) & (0.89, 0.16) \end{pmatrix}$$

$$(\measuredangle_3, E) = \begin{pmatrix} (0.93, 0.15) & (0.45, 0.59) & (0.33, 0.14) \\ (0.39, 0.28) & (0.51, 0.55) & (0.64, 0.27) \\ (0.71, 0.32) & (0.33, 0.18) & (0.09, 0.56) \end{pmatrix}$$

Then $Sim(\measuredangle_2, \measuredangle_1) = 0.8925 > 0.82$ and $Sim(\measuredangle_1, \measuredangle_3) = 0.8491 > 0.82$ but $Sim(\measuredangle_2, \measuredangle_3) = 0.8027 \ngtr 0.82$. This advocates that the relation of being similar is not transitive.

Definition 6.5 Two PFSSs $(\measuredangle_1, E_1)$ and $(\measuredangle_2, E_2)$ defined over (X, E) are called λ-*similar*, denoted as $(\measuredangle_1, E_1) \approx^{\lambda} (\measuredangle_2, E_2)$, if $Sim(\measuredangle_1, \measuredangle_2) \geq \lambda$ for some $0 < \lambda < 1$.

Proposition 6.6 *The relation of being λ-similar is reflexive and symmetric, but not transitive.*

Corollary 6.7 *The relation of being λ-similar is not an equivalence relation.*

6.1 Weighted Similarity Measure for PFSSs

In this subsection, we present weighted similarity measure between two PFSSs and give some of its peculiar characteristics.

Definition 6.8 Let $\measuredangle_1$ and $\measuredangle_2$ be as given in Definition 6.1. Assume that the weight of e_j is $w_j \in [0, 1]$ for $j = 1, 2, \cdots, n$. The *weighted similarity measure* between $\measuredangle_1$ and $\measuredangle_2$ is given as

$$Sim_W(\measuredangle_1, \measuredangle_2) = \frac{< \measuredangle_1, \measuredangle_2 >}{\| \measuredangle_1 \| \| \measuredangle_2 \|},$$

where

$$< \measuredangle_1, \measuredangle_2 > = \frac{\Sigma_{i,j} w_j (\sigma_{ij}, \varrho_{ij})_{\measuredangle_1} . (\sigma_{ij}, \varrho_{ij})_{\measuredangle_2}}{\Sigma_j w_j}$$

$$\| \measuredangle_1 \| = \sqrt{< \measuredangle_1, \measuredangle_1 >}.$$

This weighted similarity measure satisfies the same properties as given in Definition 6.1.

Example 6.9 Consider the PFSSs given by the PFS matrices

$$\measuredangle_1 = \begin{pmatrix} (0.52, 0.73) & (0.89, 0.15) & (0.62, 0.59) \\ (0.46, 0.73) & (1, 0) & (0.51, 0.51) \\ (0.32, 0.19) & (0.64, 0.27) & (0.87, 0.03) \end{pmatrix}$$

$$\measuredangle_2 = \begin{pmatrix} (0.68, 0.52) & (0.31, 0.69) & (0.44, 0.02) \\ (0.61, 0.50) & (0.33, 0.57) & (0.81, 0.16) \\ (0.52, 0.28) & (0.29, 0.22) & (0.21, 0.39) \end{pmatrix}$$

Assume that the weights of the attributes e_1, e_2, and e_3 are $w_1 = 0.52$, $w_2 = 0.31$, and $w_3 = 0.47$, respectively. Then,

$$\begin{aligned} < \measuredangle_1, \measuredangle_2 > &= 1.7672 \\ \| \measuredangle_1 \| &= 1.7020 \\ \| \measuredangle_2 \| &= 1.3479 \\ \therefore Sim_W(\measuredangle_1, \measuredangle_2) &= 0.7703 \end{aligned}$$

6.2 *Practical Implementation of Proposed Similarity Measure in Life Sciences*

As a model, in this subsection, we employ proposed similarity measure to diagnose whether a person has hepatitis or not. As earlier, we first propose Algorithm 4 before heading towards numerical example where proposed similarity measure may be successfully employed as follows:

Algorithm 4

Step 1: Choose the set $X = \{\eta_1 = \text{hepatitis}, \eta_2 = \text{no hepatitis}\}$.
Step 2: Choose the set of symptoms $E = \{e_1, e_2, \cdots, e_n\}$.
Step 3: Choose a model PFS matrix $(\measuredangle, E)$ with which similarity is to be computed.
Step 4: Choose PFS matrix $(\measuredangle_1, E)$ for the patient.
Step 5: Compute similarity between $(\measuredangle_1, E)$ and $(\measuredangle, E)$.
Step 6: Decide the threshold value $\lambda \in]0, 1[$.
Step 7: The patient is diseased if $Sim(\measuredangle, \measuredangle_1) \geq \lambda$.

Example 6.10 Presume that $X = \{\eta_1 = \text{hepatitis}, \eta_2 = \text{no hepatitis}\}$. Let's choose the set of parameters containing the collection of some detectible symptoms, say, $E = \{e_i : i = 1, 2, \cdots, 5\}$, where

$$\begin{aligned} e_1 &= \text{vomiting}, \\ e_2 &= \text{jaundice}, \\ e_3 &= \text{light/clay-colored stool}, \\ e_4 &= \text{abdominal discomfort, and} \\ e_5 &= \text{dark urine}. \end{aligned}$$

The PFS matrix $(\measuredangle, E)$ over X for hepatitis is given as under, which may be constructed with the aid of clinical/medical experts:

$$(\measuredangle, E) = \begin{pmatrix} (0.62, 0.47) & (0.36, 0.57) \\ (0.89, 0.41) & (0.27, 0.93) \\ (0.58, 0.25) & (0.31, 0.54) \\ (0.51, 0.62) & (0.49, 0.38) \\ (0.63, 0.45) & (0.53, 0.41) \end{pmatrix}$$

The PFS matrix $(\measuredangle_1, E)$ over X for hepatitis based upon an ill person is given as follows:

$$(\measuredangle_1, E) = \begin{pmatrix} (0.11, 0.07) & (0.92, 0.15) \\ (0.14, 0.05) & (0.86, 0.26) \\ (0.08, 0.96) & (0.57, 0.02) \\ (0.36, 0.69) & (0.83, 0.19) \\ (0.46, 0.37) & (0.29, 0.84) \end{pmatrix}$$

Let's decide the threshold value $\lambda = 0.75$. The similarity measure between $(\measuredangle, E)$ and $(\measuredangle_1, E)$ is $Sim(\measuredangle, \measuredangle_1) = 0.6497 < \lambda$, so we conclude that the person does not seem to be victim of hepatitis.

7 Conclusion

We studied some elementary notions of Pythagorean fuzzy soft sets in this chapter. Some fundamental operations and their prime characteristics are also examined with the assistance of elaborative examples. We proposed four algorithms, i.e., choice value method, PFS-TOPSIS, VIKOR, and method of similarity measures, for modeling uncertainties in MADM problems based upon PFSSs. The proposed Algorithms have been efficaciously applied on ranking different alternatives. To comprehend the final rankings, we have made use of statistical charts. The proposed models have tremendous potential for further exploration in theoretical besides application perspective and may be efficiently applied in other hybrid structures of fuzzy sets including Pythagorean m-polar fuzzy sets, Pythagorean m-polar fuzzy soft sets, q-rung fuzzy soft sets, neutrosophic soft sets, and Pythagorean fuzzy parameterized soft sets, etc. with slight amendments. The ideas may be efficiently employed in handling uncertainties in different sectors of real-life situations including business, artificial intelligence, marketing, shortest route problem, image processing, electoral system, pattern recognition, machine learning, medical diagnosis, trade analysis, game theory, forecasting, agri-business analysis, robotics, coding theory, recruitment problems, and many other problems.

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