Harish Garg *Editor*

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Theory and Applications

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ISBN 978-981-16-1988-5 ISBN 978-981-16-1989-2 (eBook)
https://doi.org/10.1007/978-981-16-1989-2

Mathematics Subject Classification: 94D05, 68Uxx, 68Txx, 68-XX

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

With the complexity of the socio-economic environment, today's decision-making is one of the most notable ventures, where the mission is to decide the best alternative under the numerous known or unknown criteria. In cognition of things, people may not possess a precise or sufficient level of knowledge of the problem domain and hence they usually face uncertainties in their preferences over the objects. To address it, a theory of fuzzy set, introduced by Lotfi A. Zadeh in 1965, had enclosed a lot of ground with excellent achievements in almost all branches of science. Since its appearance, many application and extensions of it have been developed which found in both theoretical and practical studies from engineering area to arts and humanities, and from life sciences to physical sciences.

In this book, a new extension of the fuzzy sets, entitled as Pythagorean fuzzy sets, is introduced by eminent researchers with several applications. In this set, the performance of the cognitive in terms of fuzzy environment is considered with the help of degrees of membership and non-membership.

This book consists of three parts. The first part involves five chapters presenting contribution on the information measures of Pythagorean fuzzy sets, such as correlation coefficients, divergence measure, similarity measures and isomorphism ranking methods between different sets. The second part contains seven chapters. These chapters include Pythagorean fuzzy decision-making methods and different applications to the real-life problems. Finally, the last part which contains four chapters on the theory of the extension of Pythagorean fuzzy sets and their applications to the decision-making process.

Chapter "A Survey on Recent Applications of Pythagorean Fuzzy Sets: A State-of-the-Art Between 2013 and 2020" in the first part of the book is to conduct a deep survey on the recent applications of the Pythagorean fuzzy sets. This chapter presents a comprehensive literature review to classify, analyse and interprets the existing research to identify the research trends for the applications of the PFSs is presented. Also, the insights regarding the future research directions, challenges and limitations are given. This literature review also analyzes the chronological development of the extensions of the fuzzy set. Chapter "Some New Weighted Correlation Coefficients Between Pythagorean Fuzzy Sets and Their Applications" defines the correlation and weighted correlation coefficients between the pairs of the Pythagorean fuzzy sets. By utilizing these correlation coefficients, it presents an approach to solve the medical diagnosis and pattern recognition problems. Chapter "Parametric Directed Divergence Measure for Pythagorean Fuzzy Set and Their Applications to Multi-criteria Decision-Making" proposes novel parametric directed divergence measures of order α and degree β to solve the decision making problems. A problem related to investment plan is taken to demonstrate it. Chapter "Some Trigonometric Similarity Measures Based on the Choquet Integral for Pythagorean Fuzzy Sets and Applications to Pattern Recognition" introduces several trigonometric similarity measures based on the Choquet integral for Pythagorean fuzzy sets by using the trigonometric functions cosine and cotangent. An application related to pattern recognition and medical diagnoses are discussed with the proposed similarity measures. Chapter "Isomorphic Operators and Ranking Methods for Pythagorean and Intuitionistic Fuzzy Sets" describes the isomorphism between three pairs of fuzzy sets, namely intuitionistic fuzzy sets and Pythagorean fuzzy sets, interval-valued intuitionistic fuzzy sets and interval-valued Pythagorean fuzzy sets, dual hesitant fuzzy sets and dual hesitant Pythagorean fuzzy sets from three aspects: operational laws, aggregation operators and ranking methods.

In the second part of the book, Chapter "A Risk Prioritization Method Based on Interval-Valued Pythagorean Fuzzy TOPSIS and Its Application for Prioritization of the Risks Emerged at Hospitals During the Covid-19 Pandemic" presents a risk prioritization approach by using extended techniques for order preferences by similarity to ideal solution (TOPSIS) under interval-valued Pythagorean fuzzy environment. The approach is applied for the case of prioritizing the risks that emerged at hospitals during the Covid-19 pandemic. Chapter "Assessment of Agriculture Crop Selection Using Pythagorean Fuzzy CRITIC–VIKOR Decision-Making Framework" presents a new hybrid Pythagorean fuzzy model with CRITIC and VIKOR methods named as PF-CRITIC-VIKOR and employs to solve the Kharif crop selection problem. In this model, the criteria weights are computed by the CRITIC approach and the preference order of Kharif crops is evaluated by VIKOR model, which provides easy mathematical steps with accurate and consistent results for assessing the crops. In addition, entropy measures are utilized to assess to compute the experts' importance degrees. Chapter "Choquet Integral Under Pythagorean Fuzzy Environment and Their Application in Decision Making" introduces Pythagorean fuzzy Choquet integral operators, which not only consider the importance of elements or their ordered positions but also consider the interaction among the criteria or ordered positions in criteria of decision making process. A case of sustainable solid waste management problem of between the major cities in Malaysia is presented to illustrate the application of the proposed aggregation operators. Chapter "On Developing Pythagorean Fuzzy Dombi Geometric Bonferroni Mean Operators with Their Application to Multicriteria Decision Making" introduces Pythagorean fuzzy Dombi geometric Bonferroni mean and Pythagorean fuzzy weighted Dombi geometric Bonferroni mean operators. Based on these aggregation operators, it presents an approach for multi-criteria decision-making problems under the Pythagorean fuzzy environment. Chapter "Schweizer–Sklar Muirhead Mean Aggregation Operators Based on Pythagorean Fuzzy Sets and Their Application in Multi-criteria Decision-Making" is on the exploration of the Schweizer–Sklar (SS) operations based on

Pythagorean fuzzy set and studied their score function, accuracy function. Based on these SS operations, Muirhead mean (MM) operators namely, Pythagorean fuzzy Muirhead mean (PFMM) and Pythagorean fuzzy weighted Muirhead mean (PFWMM) are defined for the Pythagorean fuzzy numbers to aggregate the opinions of different decision makers. Later on, based on this PFWMM operator, a decision-making algorithm is introduced to solve the multi-attribute decision-making algorithms. In the Chapter "Pythagorean Fuzzy MCDM Method Based on CODAS", COmbinative Distance-based ASsessment (CODAS) method is extended to its Pythagorean CODAS version for handling the impreciseness and vagueness in decision making process. Chapter "A Novel Pythagorean Fuzzy MULTIMOORA Applied to the Evaluation of Energy Storage Technologies" is on to the evaluation of energy storage technologies. As a result of the industrialization and the growing population, energy demand has been increasing in the world. In this chapter, a conventional multi-objective optimization by ratio analysis plus the full multiplicative form (MULTIMOORA) method extend into its Pythagorean fuzzy version. The proposed method adopts the aggregation approach in which distances are utilized on a fuzzy basis. A practical example that considers the evaluation of energy storage technologies is provided to illustrate the technique.

The third and last part of the book deals on the theory of the extension of Pythagorean fuzzy sets such as Hesitant fuzzy set, Linguistic fuzzy set, soft set and their applications to the solve the decision-making problems. In this part, Chapter "Application of Linear Programming in Diet Problem Under Pythagorean Fuzzy Environment" deals with Pythagorean fuzzy linear programming (PFLP) in which the associated cost and variables are treated as Pythagorean fuzzy numbers. With the aid of the score functions of the Pythagorean fuzzy numbers, a PFLP model is converted into its proportional crisp linear programming. The utility of the method is tested by solving some linear programming problems related to diet problems. Chapter "Maclaurin Symmetric Mean-Based Archimedean Aggregation Operators for Aggregating Hesitant Pythagorean Fuzzy Elements and Their Applications to Multicriteria Decision Making" in this part deals with hesitant Pythagorean fuzzy information, an extension of the Pythagorean fuzzy set, to solve the decision-making problems. In this chapter, weighted Maclaurin symmetric mean (MSM) with Archimedean *t*-conorms and *t*-norms (At-CNs & *t*-CNs) aggregation operators are defined to aggregate the hesitant Pythagorean fuzzy information. Based on the proposed operators, it presents an approach for multi-criteria decision-making problems under the hesitant Pythagorean fuzzy environment. Chapter "Extensions of Linguistic Pythagorean Fuzzy Sets and Their Applications in Multi-attribute Group Decision-Making" extends the linguistic Pythagorean fuzzy sets to dual hesitant linguistic Pythagorean fuzzy sets (DHLPFSs) and probabilistic DHLPFSs (PDHLPFSs) in which each element is represented with a linguistic term. The basic operational laws, ranking method and aggregation operators of DHLPFSs and PDHLPFSs are stated. Based on these, multi-attribute group decision making algorithms are established. The last chapter (Chapter "Pythagorean Fuzzy Soft Sets-Based MADM") describes Pythagorean fuzzy soft sets (PFSSs) with their properties. In this chapter, some notions related to PFSS along with their algebraic properties are defined. The four algorithms, that is, choice value method, PFS-TOPSIS, VIKOR and method of similarity measures, for modeling uncertainties in MADM problems based upon PFSSs are established together with several numerical example.

We hope that this book will provide a useful resource of ideas, techniques, and methods for the research on the theory and applications of Pythagorean fuzzy sets. We are grateful to the referees for their valuable and highly appreciated works contributed to select the high quality of chapters published in this book. We would like to also thank the Springer Nature and its team for supporting throughout its publishing.

Patiala, India Harish Garg

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About the Editor

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Pythagorean Fuzzy Information Measures

A Survey on Recent Applications of Pythagorean Fuzzy Sets: A State-of-the-Art Between 2013 and 2020

Muhammet Deveci, Levent Eriskin, and Mumtaz Karatas

Acronyms

AHP	Analytic Hierarchy Process
ARAS	Fuzzy Additive Ratio Assessment
CoCoSo	Combined Compromise Solution
CODAS	Combinative Distance-Based Assessment
COPRAS	Complex Proportional Assessment
CRITIC	Criteria Importance Through Inter-criteria Correlation
DEA	Data Envelopment Analysis
DM	Decision-Maker
DNMA	Double Normalization-Based Multiple Aggregation
EDAS	Evaluation Based on Distance from Average Solution
ELECTRE	ELimination Et Choix Traduisant la Realité
FS	Fuzzy Sets
GRA	Gray Relational Analysis
IFS	Intuitionistic Fuzzy Set
MABAC	Multi-attributive Border Approximation Area Comparison
MAIRCA	Multi-Attribute Ideal Real Comparative Analysis
MCDM	Multi-Criteria Decision-Making
MOORA	Multi-Objective Optimization on the basis of Ratio Analysis
PROMETHEE	Preference Ranking Organization METHod for Enrichment of Evaluations

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_1

PFS	Pythagorean Fuzzy Set
PFN	Pythagorean Fuzzy Number
QUALIFLEX	QUALItative FLEXible
TODIM	An acronym in Portuguese of interactive and multi-criteria decision-making
TOPSIS	Technique for Order of Preference by Similarity to Ideal Solution
VIKOR	Višekriterijumsko Kompromisno Rangiranje
WASPAS	Weighted Aggregated Sum Product Assessment
WDBA	Weighted Distance-Based Approximation

1 Introduction

The theory of fuzzy sets (FSs) known as type-1 fuzzy sets, which characterize the uncertainties by membership functions, was introduced by Zadeh [1]. Due to its potential to address uncertainty, it has achieved a great success in various fields [2]. Several extensions of fuzzy sets in the literature have been proposed by various researchers such as type-2 fuzzy sets [3], interval type-2 fuzzy sets [4], intuitionistic fuzzy sets [5], neutrosophic sets [6], hesitant fuzzy sets [7], Pythagorean fuzzy sets [8], picture fuzzy sets [9], q-rung Orthopair fuzzy sets [10], and so on. These sets have been successfully applied in most of the decision-making problems under uncertain environment such as personnel selection [11], supplier selection [12], evaluation of airline service quality [13], health technology assessment [14, 15], factory site selection [16], energy storage method selection [17], and offshore wind farm site selection [18] problems.

Since FSs can only express the vagueness, they do not have the ability to handle hesitation inherent in human thinking [19, 20]. In order to define the hesitations more clearly, intuitionistic fuzzy sets (IFSs) were developed by Atanassov [5], which are important generalization of the fuzzy sets. This approach uses the degree of membership and non-membership to model vagueness and imprecision while the sum of the two membership degrees must be less than or equal to 1. The main contribution of the IFSs is their ability to deal with the hesitancy that may exist due to imprecise information [2]. However, if the sum of *(membership)+(non-membership)* is >1, the IFSs fail to overcome this situation. Therefore, Pythagorean fuzzy sets (PFSs) were proposed to address this shortcoming of IFS.

Yager and Abbasov [8] pioneered the PFSs to extend the IFSs which are represented by the degree of membership and non-membership. PFSs are successful extensions of the IFSs and a new tool to cope with uncertainty regarding the degree of memberships. The sum of the two degrees can be less or more than 1, however, the sum of the squares of two degrees is ≤ 1. PFSs are very successful in dealing with vagueness and imprecision involving human thoughts and subjective judgments [21]. When PFSs are compared to IFSs, we observe that they provide more flexibility

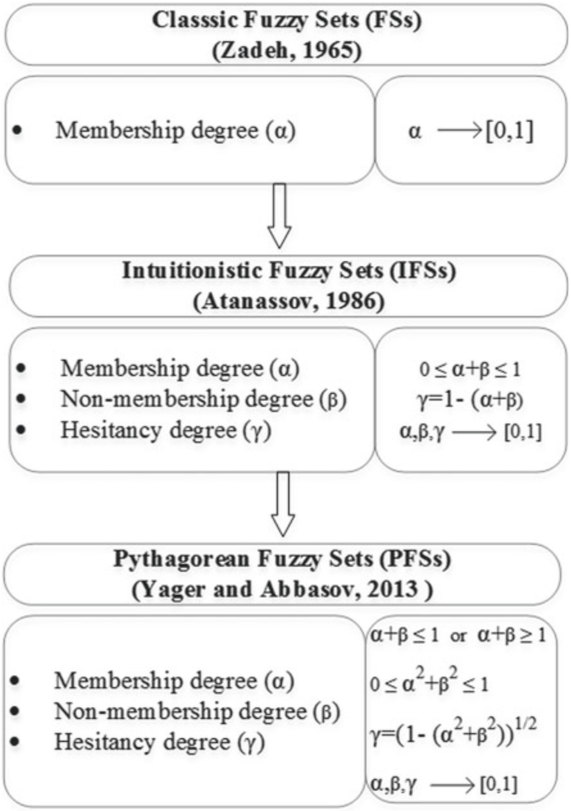

Fig. 1 The development of PFSs and the extensions of classical fuzzy sets

and power to express the uncertainty, since the space of PFSs membership degrees is larger than the space of IFSs (see Fig. 2) [22]. For example, a decision-maker (DM) may provide his/her evaluation for the degree of membership of the element $\hat{x} \in \hat{X}$ with 0.7 and provide his/her evaluation for the degree of non-membership of the element $\hat{x}$ with 0.6. Since the sum of these two values (1.3) is greater than 1, PFSs are preferred for modeling the membership degrees because IFSs cannot meet this condition. Having these properties, PFSs have attracted the attention of many researchers and been applied to many real-life multi-criteria decision-making (MCDM) problems in recent years [23, 24]. The main properties and historical development processes of the FSs, IFSs, and PFSs are illustrated in Fig. 1.

Since PFSs are extensions of the IFSs, they naturally involve the metric space of the IFSs. Moreover, PFSs not only have advantages of the IFSs, but also provide a wider search space to reflect the agreement, disagreement, and hesitancy in decision-making [22].

Having all these properties, PFSs have enormous potential for modeling uncertainty inherent in most of the real-life MCDM problems. To the best knowledge of the authors, there exists no comprehensive review regarding the theory and application areas, the PFSs that will help researchers extract quick and meaningful information. To that end, this study attempts to present a comprehensive survey of application areas and methods that are based on PFSs. In particular, upon collecting a number of quantitative data, such as the year, number of citations, origin country of the articles included in our work, we focus on the areas to which they are applied as well as the MCDM methods and tools they are implemented to. With this multi-dimensional survey approach, we seek to provide a deeper understanding and awareness of how previous research has incorporated the PFSs in different problem domains. This also enables us to examine the most common advantages and challenges observed in PFSs-based applications.

The rest of this paper is structured as follows. Section 2 introduces the basic concepts, aggregation operators, and distance measures of the PFSs. The results of the comprehensive survey are presented in Sect. 3. Finally, Sect. 4 concludes and provides insights regarding advantages, challenges, limitations, and future research directions of the PFSs.

2 Basic Concepts and Operators of Pythagorean Fuzzy Sets

In this section, some basic concepts, operators, and distance measures about PFSs are reviewed.

2.1 Basic Concept of the Pythagorean Fuzzy Sets

First introduced by [5], IFSs are one of the extensions of the classical fuzzy sets to address uncertainty.

Definition 1 Let a set $\hat{F}$ in $\hat{X} = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\}$ be a finite universe of discourse. IFSs F can be defined as follows:

$$\hat{F} = \left\{ \langle \hat{x} : \alpha_{\hat{F}}(\hat{x}), \beta_{\hat{F}}(\hat{x}) \rangle | \hat{x} \in \hat{X} \right\}, \tag{1}$$

where $\alpha_{\hat{F}}, \beta_{\hat{F}} : \hat{X} \to [0, 1]$ denote, respectively, the degree of membership and the degree of non-membership of the element $\hat{x} \in \hat{X}$, and $0 \le \alpha_{\hat{F}}(\hat{x}) + \beta_{\hat{F}}(\hat{x}) \le 1$. The pair $(\alpha_{\hat{F}}(\hat{x}), \beta_{\hat{F}}(\hat{x}))$ can be called as intuitionistic fuzzy number (IFN) and each IFN can be simply expressed as $\hat{\theta} = \left(\alpha_{\hat{\theta}}, \beta_{\hat{\theta}}\right)$, where $\alpha_{\hat{\theta}}, \beta_{\hat{\theta}} : \hat{X} \to [0, 1]$ and $\alpha_{\hat{\theta}} + \beta_{\hat{\theta}} \le 1$.

The degree of hesitation $\gamma_{\hat{F}}(\hat{x})$ of the element $\hat{x}$ to $\hat{F}$ can be defined as follows: $\gamma_{\hat{F}}(\hat{x}) = 1 - (\alpha_{\hat{F}}(\hat{x}) + \beta_{\hat{F}}(\hat{x}))$. $\gamma_{\hat{F}} : \hat{X} \to [0, 1]$ and if $\gamma_{\hat{F}}(\hat{x}) = 0$, the IFS A is close to a fuzzy set.

IFSs consist of three membership degrees that include membership, non-membership, and hesitancy degrees. However, in some instances, when the sum of $\alpha_{\hat{F}}(\hat{x}) + \beta_{\hat{F}}(\hat{x})$ is greater than >1, the requirement of IFSs is not met. Obviously, a new extension of IFSs is needed because it cannot address this situation. Hence, the PFSs have been proposed by [8] as an extention to the IFSs. The PFSs are a new tool to handle vagueness regarding the degree of membership [25].

Definition 2 Let $\hat{X}$ be a nonempty set. A PFS $\hat{P}$ can be defined by [26] as follows:

$$\hat{P} = \left\{ \langle \hat{x} : \alpha_{\hat{P}}(\hat{x}), \beta_{\hat{P}}(\hat{x}) \rangle | \hat{x} \in \hat{X} \right\}, \tag{2}$$

where $\alpha_{\hat{P}}, \beta_{\hat{P}} : \hat{X} \to [0, 1]$ denote the degree of membership and the degree of non-membership of the element $\hat{x} \in \hat{X}$ to $\hat{P}$, respectively. The following condition must be satisfied for every $\hat{x} \in \hat{X}$:

$$0 \le \alpha_{\hat{P}}(\hat{x})^2 + \beta_{\hat{P}}(\hat{x})^2 \le 1. \tag{3}$$

The degree of hesitation $\gamma_{\hat{P}}(\hat{x}) : \hat{X} \to [0, 1]$ of $\hat{x}$ to $\hat{P}$ can be defined as follows:

$$\gamma_{\hat{P}}(\hat{x}) = \sqrt{1 - (\alpha_{\hat{P}}(\hat{x})^2 + \beta_{\hat{P}}(\hat{x})^2)}. \tag{4}$$

If the value of $\gamma_{\hat{P}}(\hat{x})$ is small, then the information about $\hat{P}$ is more precise [27].

Pythagorean fuzzy number (PFN) can be also expressed as $\hat{P} = \left(\alpha_{\hat{P}}, \beta_{\hat{P}}\right)$ and each PFN can be simply expressed as $\hat{\delta} = \left(\alpha_{\hat{\delta}}, \beta_{\hat{\delta}}\right)$ by [28].

$$\gamma_{\hat{\delta}}(\hat{x}) = \sqrt{1 - (\alpha_{\hat{\delta}}(\hat{x})^2 + \beta_{\hat{\delta}}(\hat{x})^2)} \quad \text{and} \quad 0 \le \alpha_{\hat{\delta}}(\hat{x})^2 + \beta_{\hat{\delta}}(\hat{x})^2 \le 1, \tag{5}$$

where $\alpha_{\hat{\delta}}, \beta_{\hat{\delta}} : \hat{X} \to [0, 1]$.

The geometric interpretations of the space of a Pythagorean and intuitionistic membership grades are depicted in Fig. 2 (adopted by [8, 28]). The main difference between PFN and IFN is that they have different constraints, and if an element $\hat{x}$ in $\hat{P}$ is IFN, then it must also be a PFN. However, not all PFNs are IFNs.

In many real-world situations, DMs prefer to use PFNs instead of PFSs to state their evaluation values for alternatives in terms of evaluation criteria [29]. For instance, the evaluation value of Alternative A_i $\hat{\delta}_{ij} = \hat{P}(0.9, 0.3)$ expressed by the DM demonstrates the membership degree as PFN. It should be noted that the alternative A_i is a great alternative in regards to Criterion C_j as 0.9, and meanwhile alternative A_i is not great alternative as 0.4, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ state the alternatives and criteria, respectively.

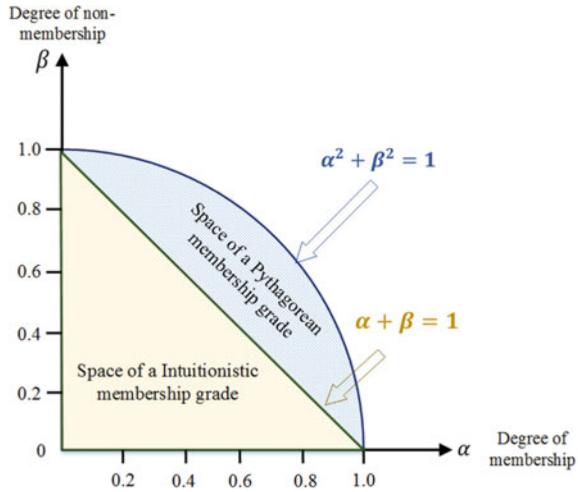

Fig. 2 Comparison of spaces of Pythagorean and Intuitionistic membership grades

2.2 Principal Operations

Let $\hat{\delta}_1 = \left(\alpha_{\hat{\delta}_1}, \beta_{\hat{\delta}_1}\right)$ and $\hat{\delta}_2 = \left(\alpha_{\hat{\delta}_2}, \beta_{\hat{\delta}_2}\right)$ be two PFNs in the set. Their basic operations for PFNs can be expressed as follows [8, 26, 28, 30, 31]:

Definition 3 Three basic operations are initially defined by [26] as follows:

1. $\hat{\delta}_1 \cap \hat{\delta}_2 = P(max\{\alpha_{\hat{\delta}_1}, \alpha_{\hat{\delta}_2}\}, min\{\beta_{\hat{\delta}_1}, \beta_{\hat{\delta}_2}\})$.
2. $\hat{\delta}_1 \cup \hat{\delta}_2 = P(min\{\alpha_{\hat{\delta}_1}, \alpha_{\hat{\delta}_2}\}, max\{\beta_{\hat{\delta}_1}, \beta_{\hat{\delta}_2}\})$.
3. $\hat{\delta}^\eta = P(\alpha_{\hat{\delta}}, \beta_{\hat{\delta}})$.

Definition 4 Four operations for PFNs are recreated by [28] as follows:

1. $\hat{\delta}_1 \oplus \hat{\delta}_2 = \left(\sqrt{\alpha^2_{\hat{\delta}_1} + \alpha^2_{[1.]\delta_2} - \alpha^2_{\hat{\delta}_1} \cdot \alpha^2_{\hat{\delta}_2}}, \beta_{\hat{\delta}_1} \cdot \beta_{\hat{\delta}_2}\right)$.
2. $\hat{\delta}_1 \otimes \hat{\delta}_2 = \left(\alpha_{\hat{\delta}_1} \cdot \alpha_{\hat{\delta}_2}, \sqrt{\beta^2_{\hat{\delta}_1} + \beta^2_{\hat{\delta}_2} - \beta^2_{\hat{\delta}_1} \cdot \beta^2_{\hat{\delta}_2}}\right)$.
3. $\rho\hat{\delta} = \left(\sqrt{1 - (1 - \alpha^2_{\hat{\delta}})^\rho}, \beta^\rho_{\hat{\delta}}\right), \rho > 0$.
4. $\hat{\delta}^\rho = \left(\alpha^\rho_{\hat{\delta}}, \sqrt{1 - (1 - \beta^2_{\hat{\delta}})^\rho}\right), \rho > 0$.

Definition 5 The corresponding operations are also described by [25] as follows:

1. $\hat{\delta}_1 \ominus \hat{\delta}_2 = \left(\sqrt{\frac{\alpha^2_{\hat{\delta}_1} - \alpha^2_{\hat{\delta}_2}}{1 - \alpha^2_{\hat{\delta}_2}}}, \frac{\beta_{\hat{\delta}_1}}{\beta_{\hat{\delta}_2}}\right)$, if $\alpha_{\hat{\delta}_2} \le \alpha_{\hat{\delta}_1}$, $\beta_{\hat{\delta}_1} \le min\left\{\beta_{\hat{\delta}_2}, \frac{\beta_{\hat{\delta}_2}\gamma_{\hat{\delta}_1}}{\gamma_{\hat{\delta}_2}}\right\}$.
2. $\hat{\delta}_1 \oslash \hat{\delta}_2 = \left(\frac{\alpha_{\hat{\delta}_1}}{\alpha_{\hat{\delta}_2}}, \sqrt{\frac{\beta^2_{\hat{\delta}_1} - \beta^2_{\hat{\delta}_2}}{1 - \beta^2_{\hat{\delta}_2}}}\right)$, if $\beta_{\hat{\delta}_2} \le \beta_{\hat{\delta}_1}$, $\alpha_{\hat{\delta}_1} \le min\left\{\alpha_{\hat{\delta}_2}, \frac{\alpha_{\hat{\delta}_2}\gamma_{\hat{\delta}_1}}{\gamma_{\hat{\delta}_2}}\right\}$.

2.3 Score and Accuracy Functions

In this section, a variety of score functions are presented from the literature to emphasize the significance of PFNs.

Definition 6 [28] Let $\hat{\delta}$ be a PFN. The score function of $\hat{\delta}$ is described as follows:

$$Score(\hat{\delta}) = (\alpha_{\hat{\delta}})^2 - (\beta_{\hat{\delta}})^2, \tag{6}$$

where $-1 \leq Score(\hat{\delta}) \leq 1$.

Based on the defined score function of PFNs, the comparison laws of the PFNs are expressed as follows:

Definition 7 [28] Let $\hat{\delta}_1 = \left(\alpha_{\hat{\delta}_1}, \beta_{\hat{\delta}_1}\right)$ and $\hat{\delta}_2 = \left(\alpha_{\hat{\delta}_2}, \beta_{\hat{\delta}_2}\right)$ be two PFNs. $Score(\hat{\delta}_1)$ and $Score(\hat{\delta}_2)$ are expressed as follows:

1. If $Score(\hat{\delta}_1) > Score(\hat{\delta}_2)$, then $\hat{\delta}_1 > \hat{\delta}_2$.
2. If $Score(\hat{\delta}_1) < Score(\hat{\delta}_2)$, then $\hat{\delta}_1 < \hat{\delta}_2$.
3. If $Score(\hat{\delta}_1) = Score(\hat{\delta}_2)$, then $\hat{\delta}_1 \sim \hat{\delta}_2$.

Example 1 Let $\hat{\delta}_1 = \left(\sqrt{7}/5, 3/8\right)$ and $\hat{\delta}_2 = \left(\sqrt{2}/3, 2/7\right)$ be two PFNs. We can calculate the score values of $Score(\hat{\delta}_1)$ and $Score(\hat{\delta}_2)$ according to Definition 7.

$Score(\hat{\delta}_1)=(\sqrt{7}/5)^2 - (3/8)^2=1/7$, and $Score(\hat{\delta}_2) = (\sqrt{2}/3)^2 - (2/7)^2 = 1/7$. The results show that $Score(\hat{\delta}_1) = Score(\hat{\delta}_2)$, then $\hat{\delta}_1 \sim \hat{\delta}_2$.

Definition 8 To solve this problem for the equality case, the accuracy function has been proposed by [25]. The accuracy function is expressed as follow:

$$Accuracy(\hat{\delta}) = (\alpha_{\hat{\delta}})^2 + (\beta_{\hat{\delta}})^2, \tag{7}$$

where $Accuracy(\hat{\delta}) \in [0, 1]$.

1. If $Score(\hat{\delta}_1) > Score(\hat{\delta}_2)$, then $\hat{\delta}_1 > \hat{\delta}_2$.
2. If $Score(\hat{\delta}_1) = Score(\hat{\delta}_2)$, then
   (a) If $Accuracy(\hat{\delta}_1) > Accuracy(\hat{\delta}_2)$, then $\hat{\delta}_1 > \hat{\delta}_2$.
   (b) If $Accuracy(\hat{\delta}_1) = Accuracy(\hat{\delta}_2)$, then $\hat{\delta}_1 \sim \hat{\delta}_2$.

2.4 Distance Measures

This subsection examines some widely used distance measures such as Hamming distance, Euclidean distance, and Taxicab distance. The distance between PFNs ($\hat{\delta}_1$ and $\hat{\delta}_2$) can be calculated with different measures as follows [21]:

Definition 9 [28] The Hamming distance measure is described as follow:

$$\phi^{ZX}(\hat{\delta}_1, \hat{\delta}_2) = \frac{1}{2}\Big(\Big|(\alpha_{\hat{\delta}_1})^2 - (\alpha_{\hat{\delta}_2})^2\Big| + \Big|(\beta_{\hat{\delta}_1})^2 - (\beta_{\hat{\delta}_2})^2\Big| + \Big|(\gamma_{\hat{\delta}_1})^2 - (\gamma_{\hat{\delta}_2})^2\Big|\Big). \quad (8)$$

Definition 10 [32] The Euclidean distance measure is described as follow:

$$\phi^{RXG}(\hat{\delta}_1, \hat{\delta}_2) = \Big\{\frac{1}{2}\Big[\Big((\alpha_{\hat{\delta}_1})^2 - (\alpha_{\hat{\delta}_2})^2\Big)^2 + \Big((\beta_{\hat{\delta}_1})^2 - (\beta_{\hat{\delta}_2})^2\Big)^2 + \Big((\gamma_{\hat{\delta}_1})^2 - (\gamma_{\hat{\delta}_2})^2\Big)^2\Big]\Big\}^{\frac{1}{2}}. \quad (9)$$

Definition 11 [33] The Taxicab distance measure is described as follow:

$$\phi^{B}(\hat{\delta}_1, \hat{\delta}_2) = \frac{1}{2}\Big(\big|\alpha_{\hat{\delta}_1} - \alpha_{\hat{\delta}_2}\big| + \big|\beta_{\hat{\delta}_1} - \beta_{\hat{\delta}_2}\big| + \big|\gamma_{\hat{\delta}_1} - \gamma_{\hat{\delta}_2}\big|\Big). \quad (10)$$

Definition 12 [34] The Generalized distance measure is described as follow:

$$\phi^{C}_{\tau}(\hat{\delta}_1, \hat{\delta}_2) = \Big[\frac{1}{2}\Big(\Big|(\alpha_{\hat{\delta}_1})^2 - (\alpha_{\hat{\delta}_2})^2\Big|^{\tau} + \Big|(\beta_{\hat{\delta}_1})^2 - (\beta_{\hat{\delta}_2})^2\Big|^{\tau} + \Big|(\gamma_{\hat{\delta}_1})^2 - (\gamma_{\hat{\delta}_2})^2\Big|^{\tau}\Big)\Big]^{\frac{1}{\tau}}, \quad (11)$$

where τ is a distance parameter that satisfies $\tau \geq 1$. It degenerates to Hamming distance as Eq. (8) and Euclidean distance as Eq. (9) when $\tau = 1$ and $\tau = 2$, respectively.

Definition 13 [35] Another generalized distance measure is described as follows:

$$\phi^{LZ}_{\tau}(\hat{\delta}_1, \hat{\delta}_2) = \Big[\frac{1}{4}\Big(\big|\alpha_{\hat{\delta}_1} - \alpha_{\hat{\delta}_2}\big|^{\tau} + \big|\beta_{\hat{\delta}_1} - \beta_{\hat{\delta}_2}\big|^{\tau} + \big|\gamma_{\hat{\delta}_1} - \gamma_{\hat{\delta}_2}\big|^{\tau} + \big|\psi_{\hat{\delta}_1} - \psi_{\hat{\delta}_2}\big|^{\tau}\Big)\Big]^{\frac{1}{\tau}}, \quad (12)$$

where $\psi_{\hat{\delta}_1}$ and $\psi_{\hat{\delta}_2}$ denote on a range of to 1 how fully the strengths $\gamma_{\hat{\delta}_1}$ and $\gamma_{\hat{\delta}_2}$, respectively.

2.5 *Pythagorean Fuzzy Aggregation Operators*

Some main aggregation operators are presented in this section. Before moving further for a discussion of Pythagorean fuzzy aggregation operators, we would like to emphasize the importance of the aggregation concept in the decision-making process. In this regard, Fig. 3 shows a typical MCDM process for a ranking problematic. For a ranking problematic involving multiple criteria to consider, the first step is to determine the evaluation criteria. Then individual field experts evaluate these criteria based on their own expertise. Essentially, all these evaluations contribute to the overall information content of the problem. One significant task at this stage is to aggregate all these information content provided by the individual experts. The way we aggregate the information content may have dramatic impact on the results of the decision, hence, choosing the proper aggregation operator that hinders information loss is of utmost importance. Finally, the aggregated information is mapped into

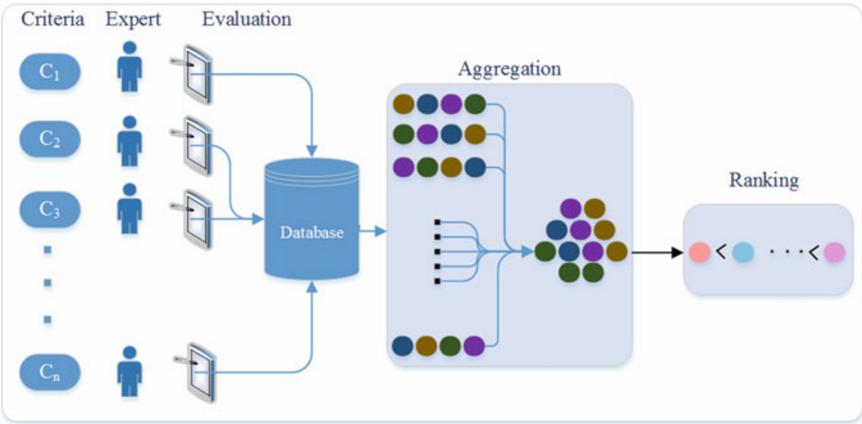

Fig. 3 Multi-criteria decision-making process for a ranking problematic

a score for each alternative which is used for ranking. Aforementioned decision-making process reveals that aggregation operators are important components of the MCDM methodologies which deserve further attention.

Let $\hat{P}_i = (\hat{\alpha}_i, \hat{\beta}_i)(i = 1, 2, \ldots, m)$ be PFNs and $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ be the weight vector of $\hat{P}_i$, and $\sum_i^m \omega_i = 1$.

Definition 14 [26] An $O - PFWA$ (original Pythagorean fuzzy weighted averaging) is defined by

$$O - PFWA(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m) = \left(\sum_{i=1}^{m} \omega_i \hat{\alpha}_i, \sum_{i=1}^{m} \omega_i \hat{\beta}_i \right). \tag{13}$$

If $\omega = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m})$, the $O - PFWA$ operator degenerates into the O-Pythagorean fuzzy average (O-PFA) operator:

$$O - PFWA(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m) = \left(\frac{\sum_{i=1}^{m} \hat{\alpha}_i}{m}, \frac{\sum_{i=1}^{m} \hat{\beta}_i}{m} \right). \tag{14}$$

Definition 15 [26] An $O - PFWG$ (orginal Pythagorean fuzzy weighted geometric) is defined by

$$O - PFWG(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m) = \left(\prod_{i=1}^{m} \hat{\alpha}_i^{\omega_i}, \prod_{i=1}^{m} \hat{\beta}_i^{\omega_i} \right). \tag{15}$$

If $\omega = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m})$, the $O - PFWG$ operator degenerates into the O-Pythagorean fuzzy geometric (O-PFA) operator:

$$O - PFWG(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m) = \left(\prod_{i=1}^{m} \hat{\alpha}_i^{\frac{1}{m}}, \prod_{i=1}^{m} \hat{\beta}_i^{\frac{1}{m}} \right). \tag{16}$$

Definition 16 [36] An $PFWA$ (Pythagorean fuzzy weighted averaging) operator is presented by

$$PFWA(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m) = \left(\sqrt{1 - \prod_{i=1}^{m} (1 - (\hat{\alpha}_i)^2)^{\omega_i}}, \prod_{i=1}^{m} (\hat{\beta}_i)^{\omega_i} \right). \tag{17}$$

If $\omega = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m})$, the PFWA operator degenerates into the PFA operator:

$$PFA(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m) = \left(\sqrt{1 - \prod_{i=1}^{m} (1 - (\hat{\alpha}_i)^2)^{\frac{1}{m}}}, \prod_{i=1}^{m} (\hat{\beta}_i)^{\frac{1}{m}} \right). \tag{18}$$

Definition 17 [36] An $PFWG$ (Pythagorean fuzzy weighted geometric) operator is defined by

$$PFWA(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m) = \left(\prod_{i=1}^{m} (\hat{\alpha}_i)^{\omega_i}, \sqrt{1 - \prod_{i=1}^{m} (1 - (\hat{\beta}_i)^2)^{\omega_i}}. \right) \tag{19}$$

If $\omega = (\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m})$, the PFWA operator degenerates into the PFA operator:

$$PFWA(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_m) = \left(\prod_{i=1}^{m} (\hat{\alpha}_i)^{\frac{1}{m}}, \sqrt{1 - \prod_{i=1}^{m} (1 - (\hat{\beta}_i)^2)^{\frac{1}{m}}} \right). \tag{20}$$

Apart from these aforementioned operators, there are other operators such as novel neutrality operations-based Pythagorean fuzzy geometric aggregation [37] and new logarithmic operational laws and their aggregation operators [38] in the literature.

3 Literature Review

3.1 Survey Methodology

Our survey mainly focused on journal papers, conference papers, and book chapters addressing PFSs. The search was conducted on two prominent resource libraries for scientific literature which cover most of the PFS applications, namely, ScienceDirect and Scopus. Since it was the work of Yager [26] that paved the way for the

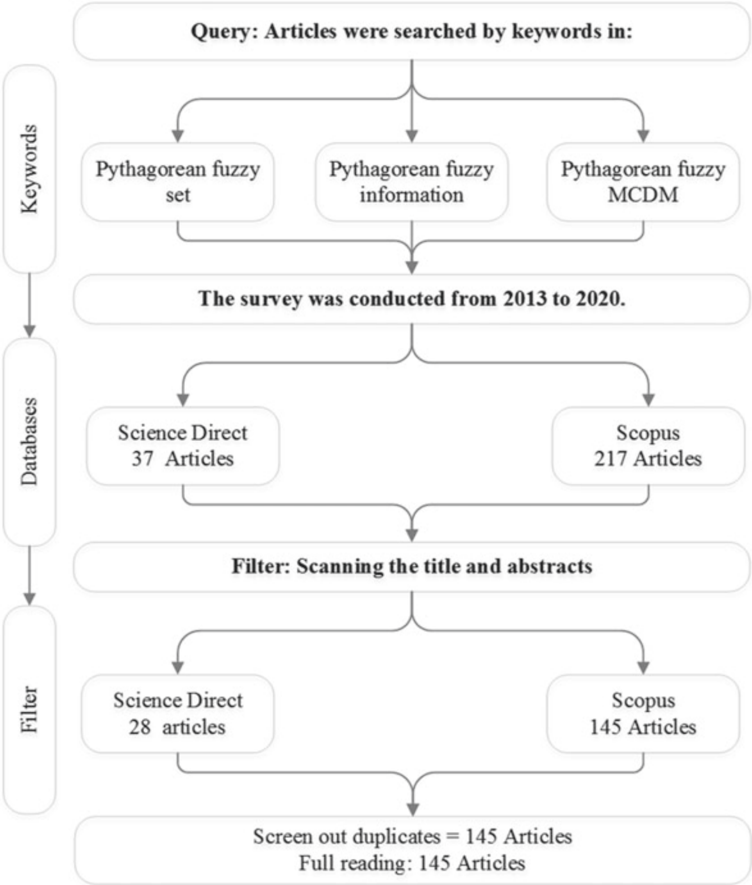

Fig. 4 Flowchart of the literature review on PFSs

PFSs, the search was performed starting from year 2013 until 2020. The keywords "*pythagorean fuzzy information*," "*pythagorean fuzzy MCDM*," and "*pythagorean fuzzy set*" steered the search by examining the titles, abstracts, and keywords of the papers. Among total of 145 papers identified, 136 of the papers are detected with the keyword "*pythagorean fuzzy set*", 46 papers are with the keyword "*pythagorean fuzzy information*", and 1 paper with the keyword "*pythagorean fuzzy MCDM*". Flowchart of the survey methodology is given in Fig. 4.

Figure 5 displays the ratio of studies reached with these keywords. All of the detected papers are found in Scopus database while ScienceDirect database yielded only 28 of the papers.

In an effort to provide as much information as possible from the existing research, we examined the identified papers in various dimensions. These dimensions that are believed to elicit relevant information are given as follows:

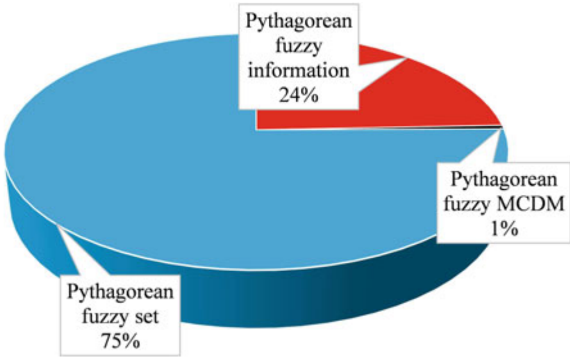

Fig. 5 Percentage of studies with respect to the three keywords used in this study

- **Year**: Year of the publication.
- **Country**: The country where the study was made. If not reported, country of the first author is considered.
- **Citations**: The number of times the publication has been cited.
- **Journal**: The name of the journal where the paper was published.
- **Application area**: The area to which the proposed approach was applied.
- **Methods and tools used**: Other MCDM tools that were used in combination with PFSs.

One of the aims of this survey is to reveal application areas of PFSs and future research directions, hence, we generally focused on and discussed application-oriented papers rather than the theoretical ones.

3.2 Survey Results

In this section, we give quantitative information regarding reviewed papers according to identified review dimensions, firstly. Then, we discuss the literature based on the application area and methods and tools dimensions.

Figure 6 shows the share of each country in the literature of PFSs. The figure shows the number of studies with respect to the origin of country. The results reveal that, China, the origin country of more than half of the 145 studies included in this study, is leading the studies incorporating the PFS concept. China is followed by the origin countries Turkey, Pakistan, Taiwan, and India with the number of studies varying between 9 and 18 out of the 145 studies. Figure 7, on the other hand, draws the attention to the yearly number of PFS related studies published between 2015 and 2020. The bars and solid line in the figure represent the individual yearly published paper numbers and cumulative number of papers, respectively. According to the figure, following [26]'s study in 2013, there is a growing interest in PFS implementations especially after 2017. In particular, 38 and 63 papers have been published in 2018 and 2019, respectively, and 25 papers appear to be published in the first half of 2020.

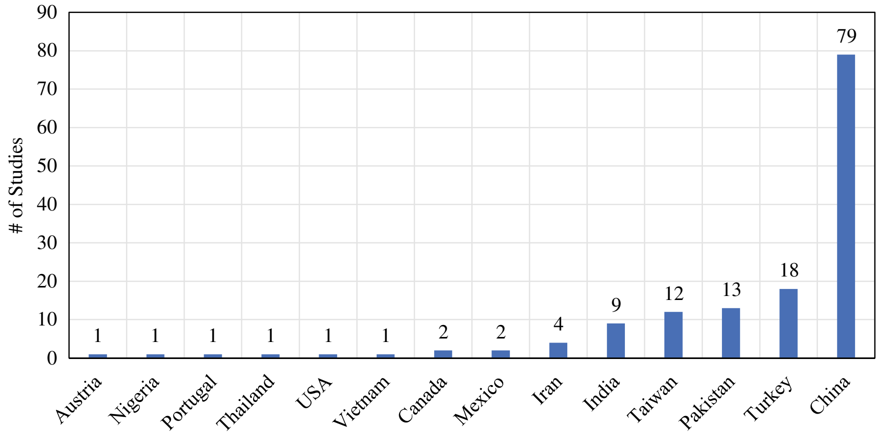

Fig. 6 Number of studies with respect to countries

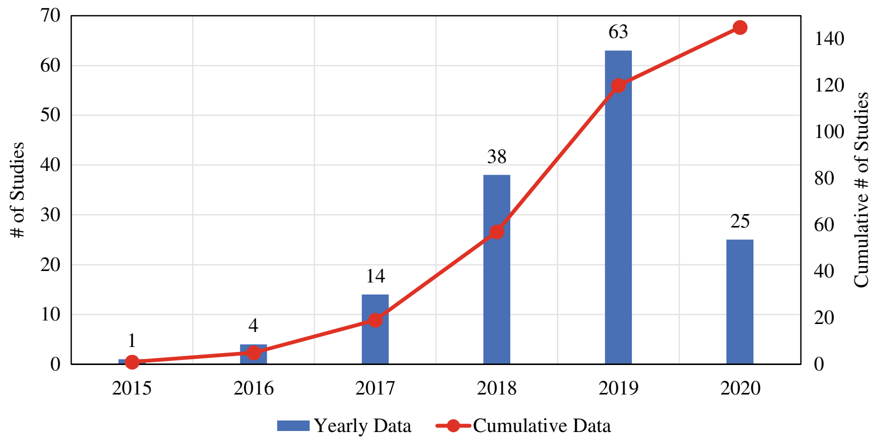

Fig. 7 Yearly and cumulative yearly number of studies.

Table 1 reports the breakdown of yearly number of papers published with respect to the origin country. Once again, the table reveals the growing interest in PFS implementations starting after 2017. It is noteworthy that China is constantly improving its contribution to the body of knowledge related to the PFS implementations.

Figure 8 depicts the frequency of citations for the 145 studies included in this paper. The solid line shows the cumulative percentage values. According to the results, more than half of these studies are cited by less than or equal to 10 papers, and approximately a third of them have been cited for between 11 and 50 times. Although the concept of PFSs is relatively new to the literature, the number of citations collected by studies which incorporate PFSs shows that there is increasing attention and awareness around the world of the capability of PFSs.

Table 1 The number of yearly PFSs-based studies with respect to different countries

Country	2015	2016	2017	2018	2019	2020	Sum
Austria	–	–	–	1	–	–	**1**
Canada	–	1	1	–	–	–	**2**
China	1	3	11	17	35	12	**79**
India	–	–	1	2	4	2	**9**
Iran	–	–	1	–	2	1	**4**
Mexico	–	–	–	1	1	–	**2**
Nigeria	–	–	–	–	1	–	**1**
Pakistan	–	–	–	4	4	5	**13**
Portugal	–	–	–	–	1	–	**1**
Taiwan	–	–	–	7	4	1	**12**
Thailand	–	–	–	–	–	1	**1**
Turkey	–	–	–	6	9	3	**18**
USA	–	–	–	–	1	–	**1**
Vietnam	–	–	–	–	1	–	**1**
Sum	**1**	**4**	**14**	**38**	**63**	**25**	**145**

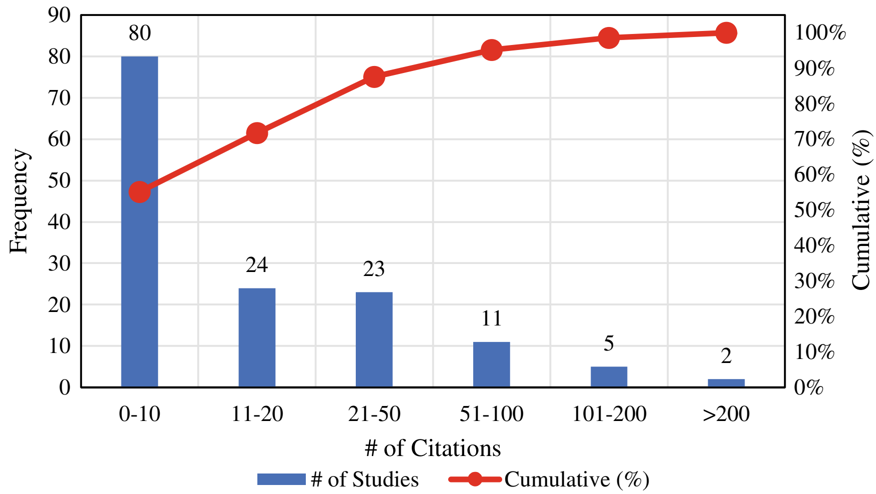

Fig. 8 Histogram for the number of citation

In Table 2 the papers are categorized with respect to the journals they are published in for each year between 2015 and 2020. Note that the journals are listed in decreasing order of the number of articles published (given in the last column). Among the 53 journals listed, IEEE Access, Mathematics, and Soft Computing are the top three journals with 14, 11, and 10 articles published, respectively, in the last three years. Applied Soft Computing, Complexity, and Symmetry are the next three journals with 7 published articles in the PFS domain. We also observe that most of the journals published articles concerning PFSs are peer-reviewed journals in engineering, computational science, information, and mathematics domains. We also note that 4 of the studies out of 145 are conference papers; hence, we do not list them in Table 2.

As emphasized previously, the concept of membership and non-membership degree provides an effective means for modeling vagueness and imprecision as addressed by the IFSs. As a successful extension to the IFSs, the PFSs provide a representation on a larger body of membership and non-standard membership grades which enables DMs to take uncertainty into consideration more flexibly [39]. Being superior to other types of fuzzy extensions, the PFSs are better options for modeling real-life phenomenons. In this regard, many real-life MCDM problems from a variety of fields have been addressed by the PFSs.

Table 3 shows the number of yearly studies published with respect to different application areas. The table reveals that MCDM problems pertaining to supply chain management, investment/capital management, risk management, and project selection areas have been frequently addressed by the PFSs-based MCDM approaches. Additionally, the interest in supply chain management applications is growing recently. When we examine the country origins of these studies with respect to the application areas, we observe that researchers from China mainly focused on supply chain management applications, as seen in Table 4. Turkey, as the second leading country in terms of number of publications, generally publishes on the risk assessment area. Practitioners from other countries, on the other hand, show interest on a variety of application areas without focusing on a particular one.

Classification of papers with respect to application areas and years is presented in Table 5. As remarked, the application area where the PFSs are utilized mostly is the supply chain management. A sustainable supply chain management is important for a company to improve long-term performance while meeting economic, social, and environmental objectives. Selecting the most proper supplier in this regard has become a critical decision problematic for a company since it directly affects organization's success. Having multiple criteria to consider most of which involve uncertain information, the PFSs are utilized for choosing a sustainable supplier [27, 40–44], green supplier [45–48], and partner [49–51] extensively.

Risk assessment is another field where PFSs are commonly applied. Risk assessment enables DMs to identify and analyze potential future events that may impact on assets, projects, environments, or individuals. Even though the results of the assessment can be expressed in terms of qualitative or quantitative fashion, particularly quantitative measures help DMs to identify the amount of tolerability of the risks. Inherent uncertainty in the future events make fuzzy approaches useful alter-

Table 2 The number of yearly PFSs-based studies published in different journals

Journal	2015	2016	2017	2018	2019	2020	Sum
IEEE Access	–	–	–	3	9	2	**14**
Mathematics	–	–	–	6	3	2	**11**
Soft Computing	–	–	–	1	4	5	**10**
Applied Soft Computing	–	1	2	1	3	–	**7**
Complexity	–	–	1	4	1	1	**7**
Symmetry	–	–	–	2	3	2	**7**
International Journal of Fuzzy Systems	–	–	1	1	3	1	**6**
Computers and Industrial Engineering	–	–	–	1	3	1	**5**
Expert Systems With Applications	–	–	–	1	1	2	**4**
IEEE Transactions on Fuzzy Systems	–	1	1	1	–	1	**4**
Information	–	–	–	2	1	1	**4**
Neural Computing and Applications	–	–	–	–	4	–	**4**
Economic Research-Ekonomska Istraživanja	–	–	–	–	2	1	**3**
International Journal of Computational Intelligence Systems	–	–	–	2	1	–	**3**
Applied Intelligence	–	–	–	–	2	–	**2**
Archives of Control Sciences	–	–	1	–	1	–	**2**
Artificial Intelligence Review	–	–	1	–	1	–	**2**
Cognitive Computation	–	–	–	–	2	–	**2**
Computational and Applied Mathematics	–	–	–	–	2	–	**2**
Human and Ecological Risk Assessment	–	–	–	1	1	–	**2**
Information Sciences	–	1	–	1	–	–	**2**
Journal of Ambient Intelligence and Humanized Computing	–	–	–	–	1	1	**2**
Journal of Cleaner Production	–	–	–	1	1	–	**2**
Journal of Experimental and Theoretical Artificial Intelligence	–	–	–	1	1	–	**2**
Journal of Intelligent Systems	–	–	–	1	–	1	**2**
Knowledge and Information Systems	–	–	1	–	1	–	**2**
Safety Science	–	–	–	2	–	–	**2**
Mathematical Problems in Engineering	–	–	–	1	–	–	**1**
Bulletin of the Brazilian Mathematical Society	–	–	–	–	1	–	**1**
Computational and Mathematical Organization Theory	–	–	1	–	–	–	**1**
Discrete Dynamics in Nature and Society	1	–	–	–	–	–	**1**
Engineering Applications of Artificial Intelligence	–	–	–	–	1	–	**1**
EURO Journal on Decision Processes	–	–	–	–	–	1	**1**
IEEE Transactions on Engineering Management	–	–	–	–	1	–	**1**
IEEE/CAA Journal of Automatica Sinica	–	–	–	–	1	–	**1**
Information Fusion	–	–	–	1	–	–	**1**
International Journal of Approximate Reasoning	–	–	–	–	–	1	**1**
International Journal of Hydrogen Energy	–	–	–	–	1	–	**1**
International Journal of Information Technology and Decision-Making	–	1	–	–	–	–	**1**
International Journal of Occupational Safety and Ergonomics	–	–	–	1	–	–	**1**
International Journal of Uncertainty	–	–	–	1	–	–	**1**
International Journal of Intelligent Systems	–	–	–	1	–	–	**1**
Inzinerine Ekonomika-Engineering Economics	–	–	–	–	–	1	**1**
Iranian Journal of Fuzzy Systems	–	–	–	–	1	–	**1**
Journal of Applied Mathematics and Computing	–	–	–	–	–	1	**1**
Journal of Failure Analysis and Prevention	–	–	–	–	1	–	**1**
Journal of Mathematics and Computer Science	–	–	–	–	–	1	**1**
Journal of Natural Gas Science and Engineering	–	–	–	–	1	–	**1**
Journal of Safety Research	–	–	–	–	1	–	**1**
Journal of the Operational Research Society	–	–	–	–	1	–	**1**
New Mathematics and Natural Computation	–	–	–	1	–	–	**1**
Scientia Iranica	–	–	–	1	–	–	**1**
Sustainability	–	–	–	–	1	–	**1**
Sum	**1**	**4**	**9**	**39**	**62**	**26**	**141**

Table 3 The number of yearly PFSs-based studies published with respect to different application areas

Application Area	2015	2016	2017	2018	2019	2020	Sum
Supply Chain Management	–	–	4	6	12	2	**24**
Investment/Capital Management	–	–	1	10	4	3	**18**
Risk Assessment	1	1	1	7	7	–	**17**
Project Selection	–	–	1	4	6	1	**12**
System/Alternative Evaluation	–	–	1	2	4	4	**11**
Product Selection	–	–	3	3	1	1	**8**
Healthcare Management	–	–	–	–	5	3	**8**
Location Selection	–	–	–	1	3	3	**7**
Company Selection	–	1	2	1	3	–	**7**
Information System Management	–	–	1	2	3	1	**7**
Construction Management	–	–	–	1	2	1	**4**
Employee Selection		1	1	–	–	2	**4**
Pattern Recognition	–	–	–	–	4	–	**4**
Emergency Management	–	–	–	1	1	1	**3**
Environment Management	–	–	–	1	2	–	**3**
Military Planning	–	–	–	–	2	1	**3**
Technology Management	–	–	–	–	3	–	**3**
Logistics Management	–	–	–	1	–	1	**2**
Sum	**1**	**4**	**14**	**39**	**62**	**25**	**145**

natives for quantifying these risks. For instance, technological innovation projects play a vital role for high-tech firms to obtain competitive advantage against their rival firms. Hence, evaluating risks associated with potential projects is important for a company [52]. Occupational health and safety risks [53–56], safety risks in gas pipeline construction and mining projects [57–59], personal credit default risks [60], assessment of commercial banks' credit risks [61] are some other examples of PFS applications.

Companies quite often face investment decisions for handling the financial and other assets. In accordance with these decisions, short or long-term strategies for acquiring and disposing of portfolio holdings are determined. In some cases, governments or private companies need to make strategic decisions regarding which technology to invest. All these decisions require numerous factors or criteria to consider in the presence of uncertainty of the future. Having the ability to model uncertain environments very successfully, the fuzzy set theory is the prominent option for addressing investment/capital management decisions. As seen from Table 5, PFSs have frequently been applied to investment/capital management problems such as renewable energy investments [62], financing decision on aggressive/conservative policies of working capital management [63, 64], evaluating Internet companies for investment [22], determining multinational company's future investment group strategies [65,

Table 4 The number of yearly PFSs-based studies published with respect to different application areas and countries

Application Area	Austria	Canada	China	India	Iran	Mexico	Nigeria	Pakistan	Portugal	Taiwan	Thailand	Turkey	USA	Vietnam	Sum
Supply Chain Management	–	–	19	1	–	1	–	2	–	–	–	1	–	–	**24**
Investment/Capital Management	–	–	7	4	1	–	–	2	–	3	–	1	–	–	**18**
Risk Assessment	–	–	8	–	–	–	–	–	–	1	–	8	–	–	**17**
Project Selection	–	–	5	1	3	–	–	1	–	2	–	–	–	–	**12**
System/Alternative Evaluation	–	1	3	–	–	–	–	3	1	1	–	2	–	–	**11**
Product Selection	–	–	6	1	–	–	–	–	–	1	–	–	–	–	**8**
Healthcare Management	–	–	5	1	–	–	–	1	–	1	–	–	–	–	**8**
Location Selection	–	–	4	1	–	–	–	–	–	–	–	2	–	–	**7**
Company Selection	–	–	6	–	–	–	–	1	–	–	–	–	–	–	**7**
Information System Management	–	–	4	–	–	–	–	3	–	–	–	–	–	–	**7**
Construction Management	–	–	–	–	–	–	–	–	–	4	–	–	–	–	**4**
Employee Selection	–	–	2	–	–	–	–	1	–	–	–	1	–	–	**4**
Pattern Recognition	–	–	2	–	–	–	1	–	–	–	–	–	–	1	**4**
Emergency Management	–	–	3	–	–	–	–	–	–	–	–	–	–	–	**3**
Environment Management	–	–	2	–	–	–	–	1	–	–	–	–	–	–	**3**
Military Planning	–	–	3	–	–	–	–	–	–	–	–	–	–	–	**3**
Technology Management	–	–	1	1	–	–	–	–	–	–	–	1	–	–	**3**
Logistics Management	–	–	1	–	–	1	–	–	–	–	–	–	–	–	**2**
Sum	**0**	**1**	**81**	**10**	**4**	**2**	**1**	**15**	**1**	**13**	**0**	**16**	**0**	**1**	**145**

Table 5 Application areas of PFSs-based studies

Application area	# of Studies	Year	Papers
Company Selection	7	2016	[30]
		2017	[31, 72]
		2018	[73]
		2019	[74–76]
Construction Management	4	2018	[52]
		2019	[77, 78]
		2020	[21]
Emergency Management	3	2018	[79]
		2019	[80]
		2020	[81]
Employee Selection	4	2016	[32]
		2017	[82]
		2020	[83, 84]
Environment Management	3	2018	[19]
		2019	[85, 86]
Healthcare Management	8	2019	[87–91]
		2020	[92–94]
Information System Management	7	2017	[31]
		2018	[95, 96]
		2019	[75, 97, 98]
		2020	[99]
Investment/Capital Management	18	2017	[22]
		2018	[34, 62–64, 66, 67, 70, 100, 101],
		2019	[68, 76, 102, 103],
		2020	[65, 69, 71, 104]
Location Selection	7	2018	[105]
		2019	[39, 106, 107]
		2020	[108–110]
Logistics Management	2	2018	[111]
		2020	[112]
Military Planning	3	2019	[107, 113]
		2020	[114]
Pattern Recognition	4	2019	[91, 115–117]

(continued)

66], investment decision on Research and Development (R&D) projects [34], evaluating manufacturing companies to invest in [67–69], personal investment decisions [70], market expansion [71].

Table 5 (continued)

Application Area	# of Studies	Year	Papers
Product Selection	8	2017	[31, 118, 119]
		2018	[120–122]
		2019	[123]
		2020	[124]
Project Selection	12	2017	[125]
		2018	[121, 126–128]
		2019	[129–134]
		2020	[135]
Risk Assessment	17	2015	[136]
		2016	[29]
		2017	[137]
		2018	[52–56, 58, 138]
		2019	[57, 59–61, 139–141]
Supply Chain Management	24	2017	[40, 46, 142, 143]
		2018	[41, 47, 48, 51, 144, 145]
		2019	[27, 42, 43, 43–45, 49, 50, 76, 146–148]
		2020	[91, 149]
System/Alternative Evaluation	11	2017	[150]
		2018	[34, 51]
		2019	[31, 61, 151, 152]
		2020	[94, 153–155]
Technology Management	3	2019	[23, 156, 157]

Project selection problems have also drawn attention of PFS practitioners. Pertaining decision-making process needs considering many aspects of the project such as budget, schedule, safety, reliability, and feasibility [129]. R&D and high-tech projects, for instance, require substantial budgets and span long periods of time. Due to these features, failure of these projects may harm governmental development policies or reputations of companies as well as causing social costs to the society. When the problem at hand is to keep a project portfolio which corresponds to periodic and continuous process of evaluating a set of projects, the problem gets more complex and analytical techniques dealing with potential uncertainties are required. In order to address this decision problem, several MCDM approaches utilizing PFSs have been proposed in the literature. Remarkable examples involve PFSs-based WASPAS and MOORA framework for high-technology project evaluation and project portfolio selection [129], determining the best rail project for China railway construction and high-speed rail operating system among the projects of different countries [130], selecting competitive projects from the oil, gas, and petrochemical markets [131], ranking China-Pakistan Economic Corridor (CPEC) projects [127], R&D project selection [128], selecting the best energy project [125, 132].

In some cases, companies or people need to determine the best product that fits their requirements. For instance, a person might need to decide which luxury car to buy, however, price of the luxury car and uncertainty associated with the evaluation criteria make the decision-making process a demanding one [120]. Examples of such products span from Internet wealth management assets [124] to batch of SSDs for computer systems [121]. Fuzzy practitioners addressed several product selections and evaluation problems with PFSs.

Healthcare management deals with management and administration of healthcare systems, hospitals, and hospital networks. As a common application, evaluation of healthcare service quality of hospitals requires considering tangible and intangible criteria such as hospital hygiene, appearance, adequacy of equipment, and facilities. As an another example, selecting the best location of a new healthcare facility also necessitate considering multiple criteria such as demand, cost, distance to the nearest transportation. Presumable hesitancy of decision-makers in evaluating these alternatives with respect to given criteria makes these decision problems a good application area for PFSs. Among PFS applications to healthcare management problems; evaluation and selection of adequate medical service institutions for enhancing the effectiveness of hospital-based post-acute care [87], medical diagnosis by evaluating the observed symptoms [90, 92], ranking physicians according to patient ratings [93], priority ranking of various rehabilitation treatment measures for hospitalized patients [88], ranking countries in terms of healthcare systems [89] are representative examples to mention.

The main aim of the location selection problems is to determine the best location for a facility. These problems generally involve multiple tangible and intangible criteria some of which conflict with each other. For instance, when selecting an appropriate landfill site location, some of the criteria dictate a location that is far away from the city center in order to decrease social costs, while the criteria associated with transportation costs imply a closer one [39]. Due to superiority of the PFSs to model uncertainty more flexibly, they are applied to problems such as hydrogen production facility location selection [106], scenic spot for vacation [109], and optimal siting of electric vehicle charging stations [105] in the literature.

Information system management consists of managing a variety of tasks pertaining to information systems and overseeing network security or Internet operations. The systems under consideration might be Enterprise Resource Planning (ERP) systems or softwares that will expedite the business processes or increase the productivity. Considering the remarkable contributions of these systems to an organization, selecting the best one turns out to be a significant decision problem for the information system managers. Several PFSs-based MCDM approaches have been proposed in the literature to help confused information system managers to deal with this problem. Among real-life applications; electronic health record system selection [98], ERP system selection [75, 96], software selection [95, 97, 99] can be given.

In the business world, another decision problem encountered by the managers is evaluating and selecting the most appropriate company in accordance with some purpose. The companies under consideration might be candidates to corporate with or to invest in. Among company selection examples utilizing PFSs; evaluating airline

Table 6 The number of yearly studies published with respect to different methods

Method	2015	2016	2017	2018	2019	2020	Sum
TOPSIS	–	1	2	6	12	3	**24**
VIKOR	–	–	–	3	5	1	**9**
AHP	–	–	–	4	4	1	**9**
TODIM	–	1	1	–	3	–	**5**
MOORA	–	–	–	1	3	–	**4**
WASPAS	–	–	–	–	2	1	**3**
ELECTRE	–	–	–	1	1	1	**3**
COPRAS	–	–	1	–	1	–	**2**
PROMETHEE	–	–	–	1	1	–	**2**
CoCoSo	–	–	–	–	1	–	**1**
CODAS	–	–	–	–	–	1	**1**
CRITIC	–	–	–	–	1	–	**1**
DEA	–	–	–	–	1	–	**1**
DNMA	–	–	–	–	–	1	**1**
EDAS	–	–	1	–	–	–	**1**
GRA	–	–	–	–	**1**	–	**1**
QUALIFLEX	–	1	–	–	–	–	**1**
WDBA	–	–	–	–	1	–	**1**
Sum	**0**	**3**	**5**	**16**	**37**	**9**	**70**

companies to determine the best one [30, 72, 73], evaluating emerging technology companies [74], ranking companies from different sectors for investment decisions [76] can be mentioned.

Another field that the PFS-based MCDM approaches are utilized is pattern recognition. Having applications in statistical learning, signal processing, computer graphics, and image processing, pattern recognition corresponds to the detection of patterns and regularities associated with the data. As shown by many studies, PFSs are good alternatives for representing patterns in the data. Once the patterns are represented with the fuzzy sets, the recognition process is performed with the proposed similarity measures. Examples of patterns considered in the proposed studies are building materials [116] and symptoms for medical diagnosis [91, 115, 117].

As seen from Tables 3-5, application areas other than the explicitly discussed ones where PFSs are frequently utilized are; construction management, emergency management, environment management, logistics management, technology management, and military planning. On the other hand, we need to remark that this list is not an exhaustive one. We believe that prospective PFSs-based studies will add other application areas to the current list in the near future.

In our review, we also investigate the most commonly used tools and methods which incorporate the PFSs. Table 6 reports the number of yearly studies published with respect to different methods. The table reveals that PFSs have been intensively applied with many well-known MCDM methods such as TOPSIS, VIKOR, AHP, PROMETHEE, ELECTRE, WASPAS, TODIM, and MOORA. The results show that PFSs are mostly implemented with the three MCDM tools TOPSIS, VIKOR, and AHP, followed by other approaches such as TODIM, MOORA, ELECTRE, CORPAS, and PROMETHEE.

The TOPSIS methodology is used to solve MCDM problems by computing the shortest distance from the positive ideal solution as well as the farthest distance from the negative ideal solution [158]. In [30], the authors propose a Pythagorean fuzzy ordered weighted averaging weighted average distance (PFOWAWAD) operator which basically utilizes both the ordered weighted averaging operator and the weighted average. Next, they develop a hybrid methodology which uses both the PFOWAWAD and the TOPSIS and show its performance on the problem of assessing the service quality of different airlines. As another TOPSIS application example, [125] implement hesitant Pythagorean fuzzy sets (HPFSs) with TOPSIS in the context of energy project selection. Considering a risk assessment application for workplace safety, [53] propose an MCDM-based risk assessment methodology which employs Pythagorean and trapezoidal fuzzy sets via fuzzy AHP and fuzzy TOPSIS. Similarly, [58] extend the TOPSIS method with the PFSs for risk assessment and prioritizing hazards. Wan et al. [19] extend the TOPSIS approach for determining DMs' weights based on PFNs for a haze management application. Yang et al. [127] implement a Pythagorean fuzzy TOPSIS (PF-TOPSIS) based approach which is based on entropy measures in the context of assessing China-Pakistan Economic Corridor projects. Similarly, [113] considers the entropy measure of PFSs for evaluating uncertainties and determining attribute weights in TOPSIS. Liang et al. [128] first develop a model of Pythagorean fuzzy decision-theoretic rough sets (PFDTRSs) and then integrates with TOPSIS. In another application, [74] adopt a PF-TOPSIS to assess emerging technology commercialization. Some other important studies which implement TOPSIS with PFSs include [27, 50, 81, 88, 106, 107, 107, 123, 124, 130, 133, 139, 154].

There also exist a few studies which use PF-TOPSIS to compare their proposed MCDM approaches. For example, [46] proposes a Pythagorean fuzzy mathematical programming approach which formulates a bi-objective mathematical model that seeks to minimize the two inconsistency based on Pythagorean fuzzy-positive ideal solution (PFPIS) and Pythagorean fuzzy-negative ideal solution (PFNIS), respectively. Then the authors present a comparison of their approach with Pythagorean fuzzy TOPSIS and Pythagorean fuzzy TODIM techniques for a green supplier selection application. In [105], use a generalized Pythagorean fuzzy ordered weighted standardized distance (GPFOWSD) operator with the VIKOR method for selecting sites for electric vehicle charging stations, and compare their approach with TOPSIS with PFSs. Building on fuzzy TODIM and the intuitionistic fuzzy TODIM approaches, [32] proposes an extension to the TODIM approach to handle MCDM problems by using PFSs. TODIM, an interactive MCDM tool, is capable of characterizing the DMs' psychological behaviors and attitudes in MCDM problems which involve uncertainty and risk. The authors compare the performance of TODIM (with

PFSs) with those of Pythagorean fuzzy TOPSIS, fuzzy TODIM, and the intuitionistic fuzzy TODIM approaches on a case study on selecting the governor of a bank.

The VIKOR methodogy, introduced by Opricovic [159], is another well-known MCDM tool used in decision problems which involve non-commensurable and generally conflicting assessment criteria. Among articles included in our study, [34] is a good example which integrates PFSs and VIKOR and demonstrates the performance of Pythagorean fuzzy VIKOR (PF-VIKOR) the on a number of real-world applications. Considering a renewable energy technology selection problem, [23] implement a PF-VIKOR approach which uses Pythagorean fuzzy-entropy and Pythagorean fuzzy-divergence measures. In another study [56] implement a combined Pythagorean fuzzy AHP and fuzzy VIKOR (PF-AHP-FVIKOR) approach for risk assessment purposes. Chen [87], on the other hand considers the remoteness and remoteness-based multiple criteria ranking indices within PF-VIKOR in the context of hospital selection. Mete et al. [139] develop a decision-support system that is based on the PF-VIKOR method and apply it on a pipeline project risk assessment problem. Using the same PF-VIKOR approach, [59] present a safety risk assessment application in mine industry. Reference [141] extend the Hamy mean operator (a tool used to handle interaction between aggregated arguments) and propose the Pythagorean uncertain linguistic variable Hamy mean (PULVHM) operator and integrate it with the VIKOR on an investment project selection problem. Zhang et al. [124] develop a Pythagorean fuzzy double normalization-based multiple aggregation (PF-DNMA) method for selecting Internet financial products and compare it with fuzzy TOPSIS and VIKOR methods.

AHP, a popular MCMD technique introduced by Saaty [160], is mostly used to obtain the relative importance of a set of alternatives in MCDM problems by pairwise comparisons and matrix operations. Razi and Karatas [161] summarize the main steps of AHP as defining the problem and objective, constructing decision tree, establishing priorities by pairwise comparisons, synthesizing these comparisons to generate alternative weights. As one of the first PFS implementations in AHP, [39, 54] develop the Pythagorean fuzzy AHP (PF-AHP) method and demonstrate it in a landfill site selection problem. Ilbahar et al. [55] propose an integrated methodology called the Pythagorean Fuzzy Proportional Risk Assessment (PFPRA) approach which utilizes Fine Kinney, PF-AHP and a fuzzy inference system for risk assessment. The authors compare the performance of their proposed model with that of the Pythagorean Fuzzy Failure Modes and Effects Analysis (PFFMEA). In another study, [49] proposes an integrated assessment procedure which uses PF-AHP and complex proportional assessment for group decision-making processes in the context of supply chain partner selection. Considering knowledge management problems, [151] developed a 2-tuple IFNs, PFSs, and Bayesian network mechanism with fuzzy AHP. In [57], the authors propose a failure mode and effect analysis (FMEA) based hybrid AHP-MOORA method with PFSs for evaluating risks in natural gas pipeline projects. As another hybrid application example, [154] propose a PF-AHP and PF-TOPSIS solution methodology for evaluating hospital service quality.

ELECTRE method introduced by [162] is also based on pairwise comparisons of multiple criteria and seeks to provide as much as possible set of actions by eliminating outranked alternatives. Chen [63] develop a risk attitudinal assignment model which involves PFSs and interval-valued Pythagorean fuzzy sets (IVPFSs). Next the authors extend the ELECTRE method and propose the IVPF-ELECTRE and implement it to financial decision-making problems. Akram et al. [86] propose the Pythagorean fuzzy ELECTRE (PF-ELECTRE) approach for multi-criteria group decision-making problems and apply it to two real-world examples observed in health safety and environment management domains. Chen [21] implement the Chebyshev distance measures for PFSs integrated to ELECTRE method.

Being a utility theory-based MCDM approach, WASPAS is a unique combination of the weighted sum model (WSM) and weighted product model (WPM) [163]. Mohagheghi et al. [131] implement the interval-valued Pythagorean fuzzy sets (IVPFSs) with WASPAS on a project portfolio selection problem. In a similar study, [93] develop a WASPAS method under PFSs for a physician selection problem. Mohagheghi and Mousavi [129] propose a decision-making process based on the WASPAS and MOORA methods with PFSs on a project assessment and project selection problem.

The MOORA method [164] is another MCDM tool used for comparing discrete alternatives especially observed in well-being economy. Among studies which employ this approach, spsciteperez2018moora proposes an integrated method that utilizes MOORA multi-objective optimization and PFSs. The MOORA under the PFS environment (PF-MOORA) is shown to be a potentially effective application in a variety of domains. The authors also report that the comparison results obtained from the PF-MOORA in terms of rankings and selection of the best alternative are consistent with those obtained from the PF-TOPSIS. Liang [98] consider the multiplicative form of MOORA and propose the implementation of PFSs with MULTIMOORA to solve MCDM problems in the context of hospital electronic health record selection problem.

The PROMETHEE approach introduced by Brans [165] and later extended by Brans and Vincke [166] is an outranking technique for ranking a set of discrete alternatives in the presence of multiple conflicting criteria. References [52, 77] propose an extension to this approach and develop the Pythagorean fuzzy PROMETHEE (PF-PROMETHEE) which is based on the PFSs for multiple criteria analysis. The technique is tested on a bridge construction method selection problem. See Table 7 for a detailed breakdown of the methods and tools used with respect to years and individual studies.

4 Conclusion and Future Outlook

As we discussed in Sect. 3.2, PFSs-based MCDM approaches have been applied to a variety of fields. Common features of the decision problems pertaining to these fields are; the problems are complex that requires considering multiple criteria simultaneously and they inherently involve vagueness and imprecision, hence, cannot be solved with conventional MCDM approaches which do not address uncertainty. All these

Table 7 Methods and tools used

Methods used	# of Studies	Year	Papers
AHP	9	2018	[53–56]
		2019	[39, 49, 57, 151]
		2020	[154]
CoCoSo	1	2019	[31]
CODAS	1	2020	[65]
COPRAS	2	2017	[31]
		2019	[49]
CRITIC	1	2019	[31]
DEA	1	2019	[45]
DNMA	1	2020	[124]
EDAS	1	2017	[31]
ELECTRE	3	2018	[63]
		2019	[86]
		2020	[21]
GRA	1	2019	[61]
MOORA	4	2018	[111]
		2019	[57, 98, 129]
PROMETHEE	2	2018	[52]
		2019	[77]
QUALIFLEX	1	2016	[29]
TODIM	5	2016	[32]
		2017	[46]
		2019	[60, 61, 152]
TOPSIS	24	2016	[30]
		2017	[125]
		2018	[19, 46, 53, 58, 105, 127, 128]
		2019	[27, 50, 74, 88, 106, 107, 107, 113, 123, 130, 133, 139]
		2020	[81, 124, 154]
VIKOR	9	2018	[34, 56, 105]
		2019	[23, 59, 87, 139, 141],
		2020	[124]
WASPAS	3	2019	[129, 131]
		2020	[93]
WDBA	1	2019	[80]

features make PFSs-based MCDM approaches to sound and reliable methodologies for solving these problems.

In line with this assessment, we observe that PFSs are mostly applied to supply chain management, investment/capital management, risk assessment, project selection, and system/alternative selection problems which constitute 56% of the studies discussed. Among application areas that present rare usage of PFSs are emergency management, environment management, military planning, technology management, and logistics management. Conversely, we believe that these areas also have significant potential for PFS applications due to vagueness and imprecision involved in pertaining problems. In military planning, for instance, the future threat environment is always uncertain and relevant problems are complex ones having multiple criteria to consider. In this regard, PFSs are good options for modeling uncertainty associated with the military planning problems.

Many different PFSs-based MCDM approaches such as Multi-attributive Border Approximation Area Comparison (MABAC), Additive Ratio Assessment Method (ARAS), and Multi-Attribute Ideal Real Comparative Analysis (MAIRCA) can be considered in application areas. In recent years, many researchers have been discovering some new extensions of the Pythagorean fuzzy sets. The new operator is also being developed and combined with PFSs to achieve great success in MCDM problems. It is possible to reflect the new advancements, such as operations and distance measures with better PFSs-based MCDM approaches for future studies which may follow the proposed or similar methods. Additionally, new extentions to the PFSs are probable. Another potential future work regarding the PFSs is to develop a novel hybrid PFSs-based MCDM approach for real-life military decision-making problems.

As an important advantage, PFSs involve more information in terms of both membership and non-membership degrees than the classical hesitant fuzzy sets. Therefore, PFSs-based MCDM methods and tools are capable of solving decision problems with multiple and conflicting criteria in a more efficient way. Additionally, our review revealed that PFSs have been successfully implemented with a variety of MCDM tools by many researchers. Among them TOPSIS, VIKOR, AHP, TODIM, MOORA, WASPAS, and ELECTRE are the most commonly employed techniques. In our review of literature we have also observed that PF-based MCDM tools tend to be more practicable and suitable in real-world problems compared to the classical approaches. In other words, proposed MCDM tools provide a better representation of the problem of interest and DM preferences under Pythagorean fuzzy environment. Due to the increasing complexity of real DM problems, it is always a challenge to represent attribute values accurately and appropriately. Therefore, PFS can be useful tool for handling uncertainty such as vagueness.

The majority real-world decision problems are complex. This introduces a design issue with respect to the application of fuzzy theory to such problems, since as the complexity of a problem increases, the number of inputs also increases along with the uncertainty level. Hence, experts often struggle in adapting membership functions that capture various aspects of the problem fully and properly. Meta-heuristics maybe be used to search and optimize the best parameter settings for the PFSs membership functions.

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Some New Weighted Correlation Coefficients Between Pythagorean Fuzzy Sets and Their Applications

P. A. Ejegwa and C. Jana

1 Introduction

Many real-life problems are enmeshed with uncertainties hence making decision-making a herculean task. To address such common challenges, Zadeh [64] introduced fuzzy sets to resolve/curb the embedded uncertainties in decision-making. Some decision-making problems could not be controlled with a fuzzy approach because fuzzy set only considered membership grade whereas, many real-life problems have the component of both membership grade and non-membership grade with the possibility of hesitation. Such cases can best be addressed by IFSs [1, 2]. IFS is described with membership grade μ, non-membership grade ν and hesitation margin π in such a way that their sum is one and $\mu + \nu$ is less than or equal to one. Due to the usefulness of IFS, it has been applied to tackle pattern recognition problems [43, 57], career determination/appointment processes [7, 18, 19, 23] and other MCDM problems discussed in [3–5, 21, 22, 24, 46, 51, 52]. Some improved similarity and distance measures based on the set pair analysis theory with applications have been studied [39, 40].

The idea of IFS though vital, cannot be suitable in a condition where a decision-maker wants to take decision in a multi-criteria problem when $\mu + \nu$ is greater than one. Suppose $\mu = \frac{1}{2}$ and $\nu = \frac{3}{5}$, clearly IFS cannot model such a situation. This provoked Atanassov [2] to propose intuitionistic fuzzy set of second type or Pythagorean fuzzy sets (PFSs) [58, 61] to generalize IFSs such that $\mu + \nu$ is also greater than one and $\mu^2 + \nu^2 + \pi^2 = 1$. PFS is a special case of IFS with additional conditions and thus has more ability to restraint hesitations more appropriate with higher degree of accuracy. The concept of PFSs have been sufficiently explored by different authors so far [8, 13, 60]. Some new generalized Pythagorean fuzzy information and aggregation operators using Einstein operations have been studied in [26, 31] with application

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_2

to decision-making. Garg [32] studied some methods for strategic decision-making with immediate probabilities in Pythagorean fuzzy environment, and the idea of linguistic PFSs has been studied with application to multi-attribute decision-making problems [34]. The notion of interval-valued PFSs has been explicated with regards to score function and exponential operational laws with applications [28, 29, 33]. Many applications of PFSs have been discussed in pattern recognitions [10, 12, 15], TOPSIS method applications [27, 66], MCDM problems using different approaches [9, 11, 35, 58, 59, 61, 62, 67] and other applicative areas [6, 20, 29, 36, 65]. Several measuring tools have been employed to measure the similarity and dissimilarity indexes between PFSs with applications to MCDM problems as discussed in [8, 11, 12, 15, 20].

The concept of correlation coefficient which is a vital tool for measuring interdependency, similarity, and interrelationship between two variables was first studied in statistics by Karl Pearson in 1895 to measure the interrelation between two variables or data. By way of extension, numerous professions like engineering and sciences among others have applied the tool to address their peculiar challenges. To equip correlation coefficient to better handle fuzzy data, the idea was encapsulated into intuitionistic fuzzy context and applied to many MCDM problems. The first work on the correlation coefficient between IFSs (CCIFSs) was carried out by Gerstenkorn and Manko [42]. Hung [44] used a statistical approach to develop CCIFSs by capturing only the membership and non-membership functions of IFSs, and CCIFSs was proposed based on centroid method in [45]. Mitchell [48] studied a new CCIFSs based on integral function. Park et al. [49] and Szmidt and Kacprzyk [53] extended the method in [44] by incorporating the hesitation margin of IFS. Liu et al. [47] introduced a new CCIFSs with the application. Garg and Kumar [38] proposed novel CCIFSs based on set pair analysis and applied the approach to solve some MCDM problems. The concept of correlation coefficient and its applications have been extended to complex intuitionistic fuzzy and intuitionistic multiplicative environments, respectively [30, 41]. TOPSIS method based on correlation coefficient was proposed in [37] to solve decision-making problems with intuitionistic fuzzy soft set information. Several other methods of CCIFSs have been studied and applied to decision-making problems [14, 54, 56, 62, 63].

Garg [25] initiated the study of correlation coefficient between Pythagorean fuzzy sets (CCPFSs) by proposing two novel correlation coefficient techniques to determine the interdependency between PFSs, and applied the techniques to MCDM problems. Thao [55] extended the work on CCIFSs in [54] to CCPFSs and applied the approach to solve some MCDM problems. Singh and Ganie [50] proposed some CCPFSs procedures with applications, but the procedures do not incorporate all the orthodox parameters of PFSs. Ejegwa [16] proposed a triparametric CCPFSs method which generalized one of the CCPFSs techniques studied in [25], and applied the method to decision-making problems. Though one cannot doubt the important of distance and similarity measures as viable soft computing tools, the preference for correlation coefficient measure in information measure theory is because of its considerations of both similarity (which is the dual of distance) and interrelationship/interdependence indexes between PFSs.

In the computation of CCPFSs, the idea of weights of the elements of sets upon which PFSs are built are often ignored, which many times lead to misleading results. Thus, Garg [25] proposed some weighted correlation coefficients between PFSs (WCCPFSs). From the work of Garg [25], we are enthused to provide improved methods of computing WCCPFSs for the enhancement of efficient application. In this chapter, some new WCCPFSs methods are proposed which are provable to be more reliable with better performance indexes than the existing ones. The objectives of the work are to

(i) explore the WCCPFSs methods studied in [25] and propose some new WCCPFSs methods to enhance accuracy and reliability in measuring CCPFSs.
(ii) mathematically corroborate the proposed WCCPFSs methods with the axiomatic conditions for CCPFSs, and numerically verify the authenticity of the proposed methods over the existing ones.
(iii) establish the applications of the proposed methods in some MCDM problems.

The rest of the chapter is delineated as follow; Sect. 2 briefly revises some basic notions of PFSs and Sect. 3 discusses some CCPFSs methods studied in [16, 25] with numerical verifications. Section 4 discusses existing WCCPFSs methods, introduces new WCCPFSs methods and numerically verifies their authenticity. Section 5 demonstrates the application of the new WCCPFSs methods in pattern recognition and medical diagnosis problems, all represented in Pythagorean fuzzy values. Section 6 concludes the chapter and gives some areas for future research.

2 Basic Notions of Pythagorean Fuzzy Sets

Definition 2.1 [1] An intuitionistic fuzzy set of X denoted by A (where X is a non-empty set) is an object having the form

$$\mathsf{A} = \{\langle \frac{\mu_{\mathsf{A}}(x), \nu_{\mathsf{A}}(x)}{x} \rangle \mid x \in X\}, \tag{1}$$

where the functions $\mu_{\mathsf{A}}(x),\ \nu_{\mathsf{A}}(x) : X \to [0, 1]$ define the degrees of membership and non-membership of the element $x \in X$ such that

$$0 \leq \mu_{\mathsf{A}}(x) + \nu_{\mathsf{A}}(x) \leq 1.$$

For any intuitionistic fuzzy set A of X, $\pi_{\mathsf{A}}(x) = 1 - \mu_{\mathsf{A}}(x) - \nu_{\mathsf{A}}(x)$ is the intuitionistic fuzzy set index or hesitation margin of A.

Definition 2.2 [58] A Pythagorean fuzzy set of X denoted by A (where X is a non-empty set) is the set of ordered pairs defined by

$$\mathsf{A} = \{\langle \frac{\mu_{\mathsf{A}}(x), \nu_{\mathsf{A}}(x)}{x} \rangle \mid x \in X\}, \tag{2}$$

where the functions $\mu_{\mathsf{A}}(x)$, $\nu_{\mathsf{A}}(x)\colon X \to [0,1]$ define the degrees of membership and non-membership of the element $x \in X$ to A such that $0 \le (\mu_{\mathsf{A}}(x))^2 + (\nu_{\mathsf{A}}(x))^2 \le 1$. Assuming $(\mu_{\mathsf{A}}(x))^2 + (\nu_{\mathsf{A}}(x))^2 \le 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by $\pi_{\mathsf{A}}(x) = \sqrt{1 - [(\mu_{\mathsf{A}}(x))^2 + (\nu_{\mathsf{A}}(x))^2]}$ and $\pi_{\mathsf{A}}(x) \in [0,1]$.

Definition 2.3 [61] Suppose A and B are PFSs of X, then

(i) $\overline{\mathsf{A}} = \{\langle \frac{\nu_{\mathsf{A}}(x), \mu_{\mathsf{A}}(x)}{x} \rangle | x \in X\}$.
(ii) $\mathsf{A} \cup \mathsf{B} = \{\langle \max(\frac{\mu_{\mathsf{A}}(x), \mu_{\mathsf{B}}(x)}{x}), \min(\frac{\nu_{\mathsf{A}}(x), \nu_{\mathsf{B}}(x)}{x}) \rangle | x \in X\}$.
(iii) $\mathsf{A} \cap \mathsf{B} = \{\langle \min(\frac{\mu_{\mathsf{A}}(x), \mu_{\mathsf{B}}(x)}{x}), \max(\frac{\nu_{\mathsf{A}}(x), \nu_{\mathsf{B}}(x)}{x}) \rangle | x \in X\}$.

It follows that, $\mathsf{A} = \mathsf{B}$ iff $\mu_{\mathsf{A}}(x) = \mu_{\mathsf{B}}(x)$, $\nu_{\mathsf{A}}(x) = \nu_{\mathsf{B}}(x)$ $\forall x \in X$, and $\mathsf{A} \subseteq \mathsf{B}$ iff $\mu_{\mathsf{A}}(x) \le \mu_{\mathsf{B}}(x)$, $\nu_{\mathsf{A}}(x) \ge \nu_{\mathsf{B}}(x)$ $\forall x \in X$. We say $\mathsf{A} \subset \mathsf{B}$ iff $\mathsf{A} \subseteq \mathsf{B}$ and $\mathsf{A} \neq \mathsf{B}$.

Remark 2.4 Suppose A, B and C are PFSs of X. By Definition 2.3, the following properties hold:

(i)
$$\overline{\overline{\mathsf{A}}} = \mathsf{A}$$
(ii)
$$\mathsf{A} \cap \mathsf{A} = \mathsf{A}$$
$$\mathsf{A} \cup \mathsf{A} = \mathsf{A}$$
(iii)
$$\mathsf{A} \cap \mathsf{B} = \mathsf{B} \cap \mathsf{A}$$
$$\mathsf{A} \cup \mathsf{B} = \mathsf{B} \cup \mathsf{A}$$
(iv)
$$\mathsf{A} \cap (\mathsf{B} \cap \mathsf{C}) = (\mathsf{A} \cap \mathsf{B}) \cap \mathsf{C}$$
$$\mathsf{A} \cup (\mathsf{B} \cup \mathsf{C}) = (\mathsf{A} \cup \mathsf{B}) \cup \mathsf{C}$$
(v)
$$\mathsf{A} \cap (\mathsf{B} \cup \mathsf{C}) = (\mathsf{A} \cap \mathsf{B}) \cup (\mathsf{A} \cap \mathsf{C})$$
$$\mathsf{A} \cup (\mathsf{B} \cap \mathsf{C}) = (\mathsf{A} \cup \mathsf{B}) \cap (\mathsf{A} \cup \mathsf{C})$$
(vi)
$$\overline{(\mathsf{A} \cap \mathsf{B})} = \overline{\mathsf{A}} \cup \overline{\mathsf{B}}$$
$$\overline{(\mathsf{A} \cup \mathsf{B})} = \overline{\mathsf{A}} \cap \overline{\mathsf{B}}.$$

Definition 2.5 [12] Pythagorean fuzzy pairs (PFPs) or Pythagorean fuzzy values (PFVs) is characterized by the form $\langle a, b \rangle$ such that $a^2 + b^2 \le 1$ where $a, b \in [0,1]$. PFPs are used for the assessment of objects for which the components (a and b) are interpreted as membership degree and non-membership degree or degree of validity and degree of non-validity, respectively.

3 Correlation Coefficients Between PFSs

Correlation coefficient in the Pythagorean fuzzy environment was pioneered by the work of Garg [25]. The concept of CCPFSs is very valuable in solving MCDM problems. What follows is the axiomatic definition of CCPFSs.

Definition 3.1 [16] Suppose A and B are PFSs of X. Then, the CCPFSs for A and B denoted by $\mathcal{K}(\mathsf{A}, \mathsf{B})$ is a measuring function $\mathcal{K}$: $PFS \times PFS \to [0, 1]$ which satisfies the following conditions;

(i) $\mathcal{K}(\mathsf{A}, \mathsf{B}) \in [0, 1]$,
(ii) $\mathcal{K}(\mathsf{A}, \mathsf{B}) = \mathcal{K}(\mathsf{B}, \mathsf{A})$,
(iii) $\mathcal{K}(\mathsf{A}, \mathsf{B}) = 1$ if and only if $\mathsf{A} = \mathsf{B}$.

Now, we recall the existing CCPFSs methods in [16, 25] as follows:

3.1 Some Existing/New CCPFSs Methods

Assume A and B are PFSs of $X = \{x_i\}$ for $i = 1, \ldots, n$. Then, the CCPFSs for A and B as in [25] are as follows:

$$\mathcal{K}_1(\mathsf{A}, \mathsf{B}) = \frac{C(\mathsf{A}, \mathsf{B})}{\max[C(\mathsf{A}, \mathsf{A}), C(\mathsf{B}, \mathsf{B})]} \tag{3}$$

and

$$\mathcal{K}_2(\mathsf{A}, \mathsf{B}) = \frac{C(\mathsf{A}, \mathsf{B})}{\sqrt{C(\mathsf{A}, \mathsf{A})C(\mathsf{B}, \mathsf{B})}}, \tag{4}$$

where

$$\left.\begin{aligned} C(\mathsf{A}, \mathsf{A}) &= \sum_{i=1}^{n}[\mu_{\mathsf{A}}^4(x_i) + \nu_{\mathsf{A}}^4(x_i) + \pi_{\mathsf{A}}^4(x_i)] \\ C(\mathsf{B}, \mathsf{B}) &= \sum_{i=1}^{n}[\mu_{\mathsf{B}}^4(x_i) + \nu_{\mathsf{B}}^4(x_i) + \pi_{\mathsf{B}}^4(x_i)] \end{aligned}\right\}, \tag{5}$$

$$C(\mathsf{A}, \mathsf{B}) = \sum_{i=1}^{n}[\mu_{\mathsf{A}}^2(x_i)\mu_{\mathsf{B}}^2(x_i) + \nu_{\mathsf{A}}^2(x_i)\nu_{\mathsf{B}}^2(x_i) + \pi_{\mathsf{A}}^2(x_i)\pi_{\mathsf{B}}^2(x_i)]. \tag{6}$$

Ejegwa [16] generalized Eq. (3) as follows:

$$\mathcal{K}(\mathsf{A},\mathsf{B}) = \frac{C(\mathsf{A},\mathsf{B})}{\max[C(\mathsf{A},\mathsf{A}), C(\mathsf{B},\mathsf{B})]}, \tag{7}$$

where

$$\left.\begin{aligned} C(\mathsf{A},\mathsf{A}) &= \sum_{i=1}^{n}[\mu_{\mathsf{A}}^{k}(x_i) + \nu_{\mathsf{A}}^{k}(x_i) + \pi_{\mathsf{A}}^{k}(x_i)] \\ C(\mathsf{B},\mathsf{B}) &= \sum_{i=1}^{n}[\mu_{\mathsf{B}}^{k}(x_i) + \nu_{\mathsf{B}}^{k}(x_i) + \pi_{\mathsf{B}}^{k}(x_i)] \end{aligned}\right\}, \tag{8}$$

and

$$C(\mathsf{A},\mathsf{B}) = \sum_{i=1}^{n}[\mu_{\mathsf{A}}^{\frac{k}{2}}(x_i)\mu_{\mathsf{B}}^{\frac{k}{2}}(x_i) + \nu_{\mathsf{A}}^{\frac{k}{2}}(x_i)\nu_{\mathsf{B}}^{\frac{k}{2}}(x_i) + \pi_{\mathsf{A}}^{\frac{k}{2}}(x_i)\pi_{\mathsf{B}}^{\frac{k}{2}}(x_i)], \tag{9}$$

for $k = 1, \ldots, 4$.

In particular, for $k = 3$, we have

$$\mathcal{K}_3(\mathsf{A},\mathsf{B}) = \frac{C(\mathsf{A},\mathsf{B})}{\max[C(\mathsf{A},\mathsf{A}), C(\mathsf{B},\mathsf{B})]}, \tag{10}$$

where

$$\left.\begin{aligned} C(\mathsf{A},\mathsf{A}) &= \sum_{i=1}^{n}[\mu_{\mathsf{A}}^{3}(x_i) + \nu_{\mathsf{A}}^{3}(x_i) + \pi_{\mathsf{A}}^{3}(x_i)] \\ C(\mathsf{B},\mathsf{B}) &= \sum_{i=1}^{n}[\mu_{\mathsf{B}}^{3}(x_i) + \nu_{\mathsf{B}}^{3}(x_i) + \pi_{\mathsf{B}}^{3}(x_i)] \end{aligned}\right\}, \tag{11}$$

and

$$C(\mathsf{A},\mathsf{B}) = \sum_{i=1}^{n}[\sqrt{(\mu_{\mathsf{A}}(x_i)\mu_{\mathsf{B}}(x_i))^3} + \sqrt{(\nu_{\mathsf{A}}(x_i)\nu_{\mathsf{B}}(x_i))^3} + \sqrt{(\pi_{\mathsf{A}}(x_i)\pi_{\mathsf{B}}(x_i))^3}]. \tag{12}$$

By modifying Eq. (10), we obtain the following new CCPFSs methods as follows:

$$\mathcal{K}_4(\mathsf{A},\mathsf{B}) = \frac{C(\mathsf{A},\mathsf{B})}{\text{Aver}[C(\mathsf{A},\mathsf{A}), C(\mathsf{B},\mathsf{B})]} \tag{13}$$

and

$$\mathcal{K}_5(\mathsf{A},\mathsf{B}) = \frac{C(\mathsf{A},\mathsf{B})}{\sqrt{C(\mathsf{A},\mathsf{A})C(\mathsf{B},\mathsf{B})}}, \tag{14}$$

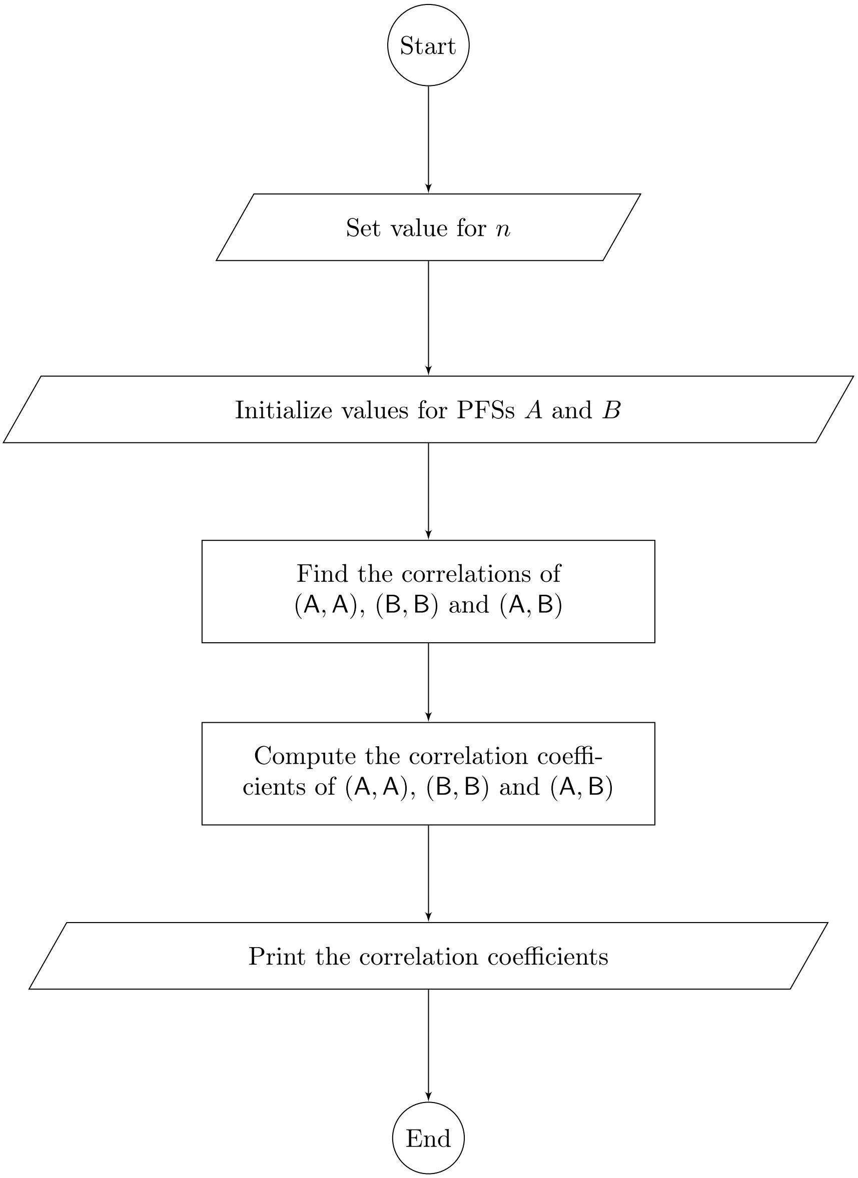
where $C(\mathsf{A},\mathsf{A})$, $C(\mathsf{B},\mathsf{B})$ and $C(\mathsf{A},\mathsf{B})$ are equivalent to Eqs. (11) and (12). Certainly, $\mathcal{K}_3(\mathsf{A},\mathsf{B}) \in [0,1]$, $\mathcal{K}_4(\mathsf{A},\mathsf{B}) \in [0,1]$ and $\mathcal{K}_5(\mathsf{A},\mathsf{B}) \in [0,1]$, respectively.

Proposition 3.2 *The CCPFSs* $\mathcal{K}_4(\mathsf{A},\mathsf{B})$ *and* $\mathcal{K}_5(\mathsf{A},\mathsf{B})$ *are equal if and only if* $C(\mathsf{A},\mathsf{A}) = C(\mathsf{B},\mathsf{B})$.

Proof Straightforward. □

Remark 3.3 If $\mathcal{K}_4(\mathsf{A}, \mathsf{B}) = \mathcal{K}_5(\mathsf{A}, \mathsf{B})$ and $C(\mathsf{A}, \mathsf{A}) \neq C(\mathsf{B}, \mathsf{B})$, then it must be as a result of approximation in the computational processes.

3.1.1 Flowchart for the New CCPFSs Methods



3.2 Numerical Illustrations for Computing CCPFSs

Here, we give examples of PFSs and apply the CCPFSs methods to find the interrelationship between the PFSs. Assume that A, B, and C are PFSs of $X = \{a, b, c\}$ such that

$$\mathsf{A} = \{\langle\frac{0.3, 0.6, 0.7416}{a}\rangle, \langle\frac{0.5, 0.3, 0.8124}{b}\rangle, \langle\frac{0.4, 0.5, 0.7681}{a}\rangle\},$$

$$\mathsf{B} = \{\langle\frac{0.3, 0.6, 0.7416}{a}\rangle, \langle\frac{0.5, 0.3162, 0.8062}{b}\rangle, \langle\frac{0.3873, 0.5, 0.7746}{a}\rangle\}$$

and

$$\mathsf{C} = \{\langle\frac{0.1, 0.1, 0.9899}{a}\rangle, \langle\frac{1, 0, 0}{b}\rangle, \langle\frac{0, 1, 0}{a}\rangle\}.$$

Now, we find the correlation coefficients between (A, C), and (B, C), respectively, using Eqs. (3), (4), (10), (13), and (14).

By using Eqs. (3) and (4), we obtain

$$\begin{aligned} C(\mathsf{A}, \mathsf{C}) &= \sum_{i=1}^{3}[(0.3^2 \times 0.1^2) + (0.6^2 \times 0.1^2) + (0.7416^2 \times 0.9899^2) \\ &+ (0.5^2 \times 1^2) + (0.3^2 \times 0^2) + (0.8124^2 \times 0^2) \\ &+ (0.3873^2 \times 0^2) + (0.5^2 \times 1^2) + (0.7746^2 \times 0^2)] \\ &= 1.0434 \end{aligned}$$

$$\begin{aligned} C(\mathsf{B}, \mathsf{C}) &= \sum_{i=1}^{3}[(0.3^2 \times 0.1^2) + (0.6^2 \times 0.1^2) + (0.7416^2 \times 0.9899^2) \\ &+ (0.5^2 \times 1^2) + (0.3162^2 \times 0^2) + (0.8062^2 \times 0^2) \\ &+ (0.4^2 \times 0^2) + (0.5^2 \times 1^2) + (0.7681^2 \times 0^2)] \\ &= 1.0434 \end{aligned}$$

$$\begin{aligned} C(\mathsf{A}, \mathsf{A}) &= \sum_{i=1}^{3}[0.3^4 + 0.6^4 + 0.7416^4 + 0.5^4 + 0.3^4 \\ &+ 0.8124^4 + 0.4^4 + 0.5^4 + 0.7681^4] \\ &= 1.3825 \end{aligned}$$

$$\begin{aligned} C(\mathsf{B}, \mathsf{B}) &= \sum_{i=1}^{3}[0.3^4 + 0.6^4 + 0.7416^4 + 0.5^4 + 0.3162^4 \\ &+ 0.8062^4 + 0.3873^4 + 0.5^4 + 0.7746^4] \\ &= 1.3801 \end{aligned}$$

$$\begin{aligned} C(\mathsf{C}, \mathsf{C}) &= \sum_{i=1}^{3}[0.1^4 + 0.1^4 + 0.9899^4 + 1^4 + 0^4 \\ &+ 0^4 + 0^4 + 1^4 + 0^4] \\ &= 2.9604. \end{aligned}$$

Hence,

$$\left.\begin{aligned} \mathcal{K}_1(\mathsf{A}, \mathsf{C}) &= \frac{1.0434}{\max[1.3825, 2.9604]} = 0.3525, \\ \mathcal{K}_1(\mathsf{B}, \mathsf{C}) &= \frac{1.0434}{\max[1.3801, 2.9604]} = 0.3525, \end{aligned}\right\}$$

$$\left.\begin{aligned} \mathcal{K}_2(\mathsf{A}, \mathsf{C}) &= \frac{1.0434}{\sqrt{1.3825 \times 2.9604}} = 0.5158, \\ \mathcal{K}_2(\mathsf{B}, \mathsf{C}) &= \frac{1.0434}{\sqrt{1.3801 \times 2.9604}} = 0.5162. \end{aligned}\right\}$$

By using Eqs. (10), (13), and (14), we have

$$\begin{aligned} C(\mathsf{A}, \mathsf{C}) &= \sum_{i=1}^{3}[\sqrt{(0.3 \times 0.1)^3} + \sqrt{(0.6 \times 0.1)^3} + \sqrt{(0.7416 \times 0.9899)^3} \\ &+ \sqrt{(0.5 \times 1)^3} + \sqrt{(0.3 \times 0)^3} + \sqrt{(0.8124 \times 0)^3} \\ &+ \sqrt{(0.3873 \times 0)^3} + \sqrt{(0.5 \times 1)^3} + \sqrt{(0.7746 \times 0)^3}] \\ &= 1.3560 \end{aligned}$$

$$\begin{aligned} C(\mathsf{B}, \mathsf{C}) &= \sum_{i=1}^{3}[\sqrt{(0.3 \times 0.1)^3} + \sqrt{(0.6 \times 0.1)^3} + \sqrt{(0.7416 \times 0.9899)^3} \\ &+ \sqrt{(0.5 \times 1)^3} + \sqrt{(0.3162 \times 0)^3} + \sqrt{(0.8062 \times 0)^3} \\ &+ \sqrt{(0.4 \times 0)^3} + \sqrt{(0.5 \times 1)^3} + \sqrt{(0.7681 \times 0)^3}] \\ &= 1.3560 \end{aligned}$$

$$\begin{aligned} C(\mathsf{A}, \mathsf{A}) &= \sum_{i=1}^{3}[0.3^3 + 0.6^3 + 0.7416^3 + 0.5^3 + 0.3^3 \\ &+ 0.8124^3 + 0.4^3 + 0.5^3 + 0.7681^3] \\ &= 1.9812 \end{aligned}$$

$$\begin{aligned} C(\mathsf{B}, \mathsf{B}) &= \sum_{i=1}^{3}[0.3^3 + 0.6^3 + 0.7416^3 + 0.5^3 + 0.3162^3 \\ &+ 0.8062^3 + 0.3873^3 + 0.5^3 + 0.7746^3] \\ &= 1.9793 \end{aligned}$$

$$\begin{aligned} C(\mathsf{C}, \mathsf{C}) &= \sum_{i=1}^{3}[0.1^3 + 0.1^3 + 0.9899^3 + 1^3 + 0^3 \\ &+ 0^3 + 0^3 + 1^3 + 0^3] \\ &= 2.9720. \end{aligned}$$

Hence,

$$\left.\begin{aligned} \mathcal{K}_3(\mathsf{A}, \mathsf{C}) &= \frac{1.3560}{\max[1.9812, 2.9720]} = 0.4563, \\ \mathcal{K}_3(\mathsf{B}, \mathsf{C}) &= \frac{1.3560}{\max[1.9793, 2.9720]} = 0.4563, \end{aligned}\right\}$$

$$\left.\begin{aligned} \mathcal{K}_4(\mathsf{A}, \mathsf{C}) &= \frac{1.3560}{\mathrm{Aver}[1.9812, 2.9720]} = 0.5475, \\ \mathcal{K}_4(\mathsf{B}, \mathsf{C}) &= \frac{1.3560}{\mathrm{Aver}[1.9793, 2.9720]} = 0.5477, \end{aligned}\right\}$$

$$\left.\begin{aligned} \mathcal{K}_5(\mathsf{A}, \mathsf{C}) &= \frac{1.3560}{\sqrt{1.9812 \times 2.9720}} = 0.5588, \\ \mathcal{K}_5(\mathsf{B}, \mathsf{C}) &= \frac{1.3560}{\sqrt{1.9793 \times 2.9720}} = 0.5591. \end{aligned}\right\}$$

3.2.1 Comparison of the New Methods of Computing CCPFSs with the Existing Methods

Table 1 contains the computational results for easy analysis.

From Table 1, we infer that the (i) CCPFSs methods via maximum approach in [16, 25] cannot determine the interrelationship between almost two equal PFSs with respect to an unrelated PFS, (ii) new CCPFSs methods are very reliable and can determine the interrelationship between almost two equal PFSs with respect to an

Table 1 CCPFSs outputs

CCPFSs	(A, C)	(B, C)
$\mathcal{K}_1$	0.3525	0.3525
$\mathcal{K}_2$	0.5158	0.5162
$\mathcal{K}_3$	0.4563	0.4563
$\mathcal{K}_4$	**0.5475**	**0.5477**
$\mathcal{K}_5$	**0.5588**	**0.5591**

unrelated PFS. Again, the new CCPFSs methods have better performance indexes when compare to the ones in [16, 25]. From the computations,we conclude that (B, C) are more related to each other than (A, C) because

$$\mathcal{K}_i(\mathsf{B}, \mathsf{C}) > \mathcal{K}_i(\mathsf{A}, \mathsf{C}) \ \forall i = 1, 2, 3, 4, 5.$$

4 Some Existing/New WCCPFSs Methods

In many applicative areas, different elements of sets have different weights. In order to have a reliable interdependence index between PFSs, the impact of the weights must be put into consideration. Suppose A and B are PFSs of $X = \{x_i\}$ for $i = 1, \ldots, n$ such that the weights of the elements of X is a set $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ with $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$.

4.1 Some Existing WCCPFSs Methods

We recall some WCCPFSs methods proposed by Garg [25] as follows:

$$\tilde{\mathcal{K}}_1(\mathsf{A}, \mathsf{B}) = \frac{C_\alpha(\mathsf{A}, \mathsf{B})}{\max[C_\alpha(\mathsf{A}, \mathsf{A}), C_\alpha(\mathsf{B}, \mathsf{B})]} \tag{15}$$

and

$$\tilde{\mathcal{K}}_2(\mathsf{A}, \mathsf{B}) = \frac{C_\alpha(\mathsf{A}, \mathsf{B})}{\sqrt{C_\alpha(\mathsf{A}, \mathsf{A}) C_\alpha(\mathsf{B}, \mathsf{B})}}, \tag{16}$$

where

$$\left.\begin{aligned} C_\alpha(\mathsf{A}, \mathsf{A}) &= \sum_{i=1}^{n} \alpha_i[\mu_{\mathsf{A}}^4(x_i) + \nu_{\mathsf{A}}^4(x_i) + \pi_{\mathsf{A}}^4(x_i)] \\ C_\alpha(\mathsf{B}, \mathsf{B}) &= \sum_{i=1}^{n} \alpha_i[\mu_{\mathsf{B}}^4(x_i) + \nu_{\mathsf{B}}^4(x_i) + \pi_{\mathsf{B}}^4(x_i)] \end{aligned}\right\}, \tag{17}$$

, and

$$C_\alpha(\mathsf{A},\mathsf{B}) = \sum_{i=1}^{n} \alpha_i[\mu^2_{\mathsf{A}}(x_i)\mu^2_{\mathsf{B}}(x_i) + \nu^2_{\mathsf{A}}(x_i)\nu^2_{\mathsf{B}}(x_i) + \pi^2_{\mathsf{A}}(x_i)\pi^2_{\mathsf{B}}(x_i)]. \tag{18}$$

4.2 New Methods of Computing WCCPFSs

By modifying Eqs. (10), (13) and (14), we have the following new WCCPFSs A and B:

$$\tilde{\mathcal{K}}_3(\mathsf{A},\mathsf{B}) = \frac{C_\alpha(\mathsf{A},\mathsf{B})}{\max[C_\alpha(\mathsf{A},\mathsf{A}), C_\alpha(\mathsf{B},\mathsf{B})]}, \tag{19}$$

$$\tilde{\mathcal{K}}_4(\mathsf{A},\mathsf{B}) = \frac{C_\alpha(\mathsf{A},\mathsf{B})}{\mathrm{Aver}[C_\alpha(\mathsf{A},\mathsf{A}), C_\alpha(\mathsf{B},\mathsf{B})]} \tag{20}$$

and

$$\tilde{\mathcal{K}}_5(\mathsf{A},\mathsf{B}) = \frac{C_\alpha(\mathsf{A},\mathsf{B})}{\sqrt{C_\alpha(\mathsf{A},\mathsf{A})C_\alpha(\mathsf{B},\mathsf{B})}}, \tag{21}$$

where

$$\left.\begin{aligned} C_\alpha(\mathsf{A},\mathsf{A}) &= \sum_{i=1}^{n} \alpha_i[\mu^3_{\mathsf{A}}(x_i) + \nu^3_{\mathsf{A}}(x_i) + \pi^3_{\mathsf{A}}(x_i)] \\ C_\alpha(\mathsf{B},\mathsf{B}) &= \sum_{i=1}^{n} \alpha_i[\mu^3_{\mathsf{B}}(x_i) + \nu^3_{\mathsf{B}}(x_i) + \pi^3_{\mathsf{B}}(x_i)] \end{aligned}\right\}, \tag{22}$$

and

$$C_\alpha(\mathsf{A},\mathsf{B}) = \sum_{i=1}^{n} \alpha_i[\sqrt{(\mu_{\mathsf{A}}(x_i)\mu_{\mathsf{B}}(x_i))^3} + \sqrt{(\nu_{\mathsf{A}}(x_i)\nu_{\mathsf{B}}(x_i))^3} + \sqrt{(\pi_{\mathsf{A}}(x_i)\pi_{\mathsf{B}}(x_i))^3}]. \tag{23}$$

Proposition 4.1 *The WCCPFSs* $\tilde{\mathcal{K}}_4(\mathsf{A},\mathsf{B})$ *and* $\tilde{\mathcal{K}}_5(\mathsf{A},\mathsf{B})$ *are equal if and only if* $C_\alpha(\mathsf{A},\mathsf{A}) = C_\alpha(\mathsf{B},\mathsf{B})$.

Proof Straightforward. □

Proposition 4.2 *The WCCPFSs* $\tilde{\mathcal{K}}_3(\mathsf{A},\mathsf{B})$, $\tilde{\mathcal{K}}_4(\mathsf{A},\mathsf{B})$ *and* $\tilde{\mathcal{K}}_5(\mathsf{A},\mathsf{B})$ *are CCPFSs.*

Proof We are to prove that $\tilde{\mathcal{K}}_3(\mathsf{A},\mathsf{B})$, $\tilde{\mathcal{K}}_4(\mathsf{A},\mathsf{B})$ and $\tilde{\mathcal{K}}_5(\mathsf{A},\mathsf{B})$ are CCPFSs. First, we show that $\tilde{\mathcal{K}}_3(\mathsf{A},\mathsf{B})$ is a CCPFS. Thus, we verify that $\tilde{\mathcal{K}}_3(\mathsf{A},\mathsf{B})$ satisfies the conditions in Definition 3.1.

Clearly, $\tilde{\mathcal{K}}_3(\mathsf{A},\mathsf{B}) \in [0, 1]$ implies $0 \leq \tilde{\mathcal{K}}_3(\mathsf{A},\mathsf{B}) \leq 1$. Certainly, $\tilde{\mathcal{K}}_3(\mathsf{A},\mathsf{B}) \geq 0$ since $C_\alpha(\mathsf{A},\mathsf{B}) \geq 0$ and $[C_\alpha(\mathsf{A},\mathsf{A}), C_\alpha(\mathsf{B},\mathsf{B})] \geq 0$. Now, we prove that $\tilde{\mathcal{K}}_3(\mathsf{A},\mathsf{B}) \leq 1$. Assume we have the following:

$$\sum_{i=1}^{n} \mu_{\mathsf{A}}^{3}(x_i) = \omega_1, \quad \sum_{i=1}^{n} \mu_{\mathsf{B}}^{3}(x_i) = \omega_2,$$

$$\sum_{i=1}^{n} \nu_{\mathsf{A}}^{3}(x_i) = \omega_3, \quad \sum_{i=1}^{n} \nu_{\mathsf{B}}^{3}(x_i) = \omega_4,$$

$$\sum_{i=1}^{n} \pi_{\mathsf{A}}^{3}(x_i) = \omega_5, \quad \sum_{i=1}^{n} \pi_{\mathsf{B}}^{3}(x_i) = \omega_6.$$

But, $\tilde{\mathcal{K}}_3(\mathsf{A}, \mathsf{B}) = \dfrac{C_\alpha(\mathsf{A}, \mathsf{B})}{\max[C_\alpha(\mathsf{A}, \mathsf{A}), C_\alpha(\mathsf{B}, \mathsf{B})]}$. By Cauchy–Schwarz's inequality, we get

$$\begin{aligned}
\tilde{\mathcal{K}}_3(\mathsf{A}, \mathsf{B}) &= \frac{\sum_{i=1}^{n} \alpha_i [\mu_{\mathsf{A}}^{\frac{3}{2}}(x_i)\mu_{\mathsf{B}}^{\frac{3}{2}}(x_i) + \nu_{\mathsf{A}}^{\frac{3}{2}}(x_i)\nu_{\mathsf{B}}^{\frac{3}{2}}(x_i) + \pi_{\mathsf{A}}^{\frac{3}{2}}(x_i)\pi_{\mathsf{B}}^{\frac{3}{2}}(x_i)]}{\max[\sum_{i=1}^{n} \alpha_i (\mu_{\mathsf{A}}^{3}(x_i) + \nu_{\mathsf{A}}^{3}(x_i) + \pi_{\mathsf{A}}^{3}(x_i)), \sum_{i=1}^{n} \alpha_i (\mu_{\mathsf{B}}^{3}(x_i) + \nu_{\mathsf{B}}^{3}(x_i) + \pi_{\mathsf{B}}^{3}(x_i))]} \\
&= \frac{\sum_{i=1}^{n} \alpha_i [\mu_{\mathsf{A}}^{\frac{3}{2}}(x_i)\mu_{\mathsf{B}}^{\frac{3}{2}}(x_i)] + \sum_{i=1}^{n} \alpha_i [\nu_{\mathsf{A}}^{\frac{3}{2}}(x_i)\nu_{\mathsf{B}}^{\frac{3}{2}}(x_i)] + \sum_{i=1}^{n} \alpha_i [\pi_{\mathsf{A}}^{\frac{3}{2}}(x_i)\pi_{\mathsf{B}}^{\frac{3}{2}}(x_i)]}{\max[\alpha_i (\sum_{i=1}^{n} \mu_{\mathsf{A}}^{3}(x_i) + \sum_{i=1}^{n} \nu_{\mathsf{A}}^{3}(x_i) + \sum_{i=1}^{n} \pi_{\mathsf{A}}^{3}(x_i)), \alpha_i (\sum_{i=1}^{n} \mu_{\mathsf{B}}^{3}(x_i) + \sum_{i=1}^{n} \nu_{\mathsf{B}}^{3}(x_i) + \sum_{i=1}^{n} \pi_{\mathsf{B}}^{3}(x_i))]} \\
&\le \frac{\alpha_i [\sum_{i=1}^{n} \mu_{\mathsf{A}}^{3}(x_i) \sum_{i=1}^{n} \mu_{\mathsf{B}}^{3}(x_i)]^{\frac{1}{2}} + \alpha_i [\sum_{i=1}^{n} \nu_{\mathsf{A}}^{3}(x_i) \sum_{i=1}^{n} \nu_{\mathsf{B}}^{3}(x_i)]^{\frac{1}{2}} + \alpha_i [\sum_{i=1}^{n} \pi_{\mathsf{A}}^{3}(x_i) \sum_{i=1}^{n} \pi_{\mathsf{B}}^{3}(x_i)]^{\frac{1}{2}}}{\max[\alpha_i (\sum_{i=1}^{n} \mu_{\mathsf{A}}^{3}(x_i) + \sum_{i=1}^{n} \nu_{\mathsf{A}}^{3}(x_i) + \sum_{i=1}^{n} \pi_{\mathsf{A}}^{3}(x_i)), \alpha_i (\sum_{i=1}^{n} \mu_{\mathsf{B}}^{3}(x_i) + \sum_{i=1}^{n} \nu_{\mathsf{B}}^{3}(x_i) + \sum_{i=1}^{n} \pi_{\mathsf{B}}^{3}(x_i))} \\
&= \frac{\alpha_i [(\omega_1\omega_2)^{\frac{1}{2}} + (\omega_3\omega_4)^{\frac{1}{2}} + (\omega_5\omega_6)^{\frac{1}{2}}]}{\max[\alpha_i (\omega_1 + \omega_3 + \omega_5), \alpha_i (\omega_2 + \omega_4 + \omega_6)]}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\tilde{\mathcal{K}}_3(\mathsf{A}, \mathsf{B}) - 1 &\le \frac{\alpha_i [(\omega_1\omega_2)^{\frac{1}{2}} + (\omega_3\omega_4)^{\frac{1}{2}} + (\omega_5\omega_6)^{\frac{1}{2}}]}{\max[\alpha_i (\omega_1 + \omega_3 + \omega_5), \alpha_i (\omega_2 + \omega_4 + \omega_6)]} - 1 \\
&= \frac{\alpha_i [(\omega_1\omega_2)^{\frac{1}{2}} + (\omega_3\omega_4)^{\frac{1}{2}} + (\omega_5\omega_6)^{\frac{1}{2}}] - \max[\alpha_i (\omega_1 + \omega_3 + \omega_5), \alpha_i (\omega_2 + \omega_4 + \omega_6)]}{\max[\alpha_i (\omega_1 + \omega_3 + \omega_5), \alpha_i (\omega_2 + \omega_4 + \omega_6)]} \\
&= -\frac{\{\max[\alpha_i (\omega_1 + \omega_3 + \omega_5), \alpha_i (\omega_2 + \omega_4 + \omega_6)] - \alpha_i [(\omega_1\omega_2)^{\frac{1}{2}} + (\omega_3\omega_4)^{\frac{1}{2}} + (\omega_5\omega_6)^{\frac{1}{2}}]\}}{\max[\alpha_i (\omega_1 + \omega_3 + \omega_5), \alpha_i (\omega_2 + \omega_4 + \omega_6)]} \\
&\le 0.
\end{aligned}$$

So $\tilde{\mathcal{K}}_3(\mathsf{A}, \mathsf{B}) \le 1$. Hence, $\tilde{\mathcal{K}}_3(\mathsf{A}, \mathsf{B}) \in [0, 1]$.

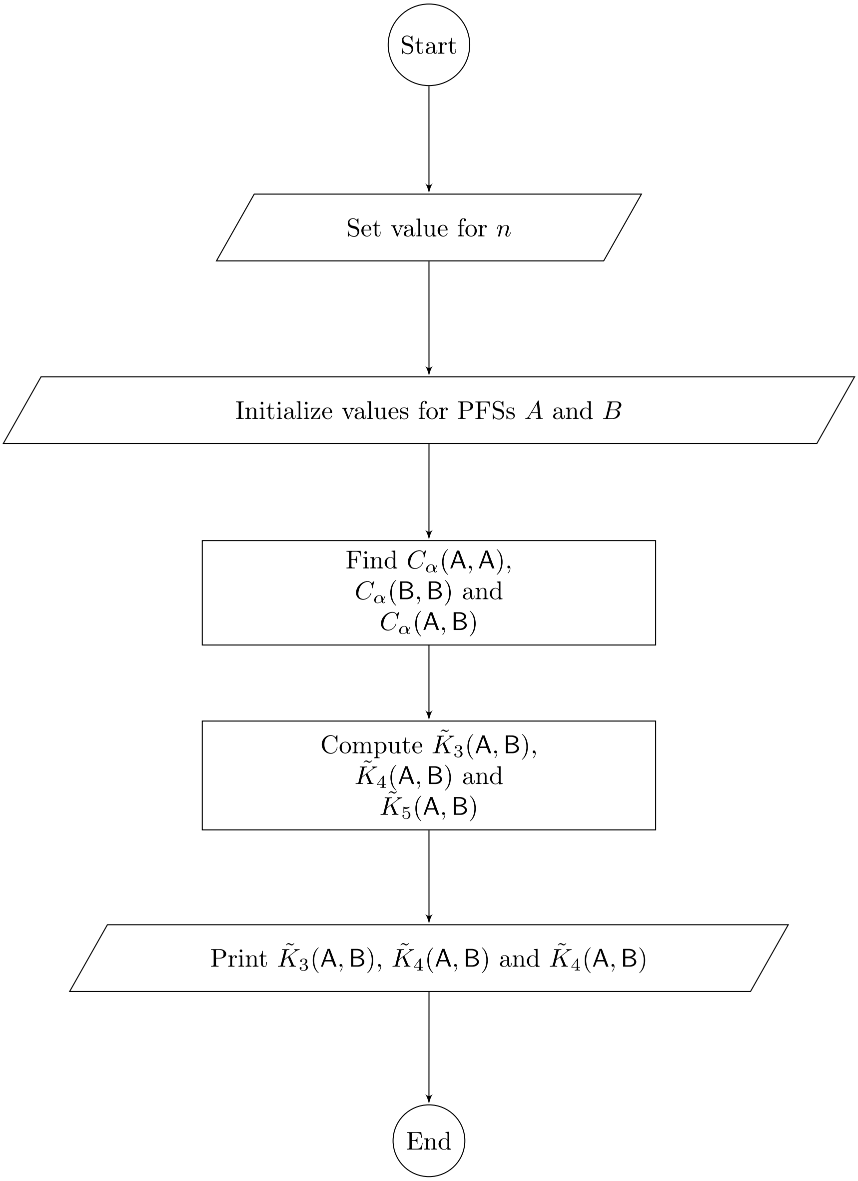
Certainly, $\tilde{\mathcal{K}}_3(\mathsf{A}, \mathsf{B}) = \tilde{\mathcal{K}}_3(\mathsf{B}, \mathsf{A})$, so we omit details. Also, we show that $\tilde{\mathcal{K}}_3(\mathsf{A}, \mathsf{B}) = 1 \Leftrightarrow \mathsf{A} = \mathsf{B}$. Suppose $\mathsf{A} = \mathsf{B}$, then we obtain

$$\tilde{\mathcal{K}}_3(\mathsf{A}, \mathsf{B}) = \frac{C_\alpha(\mathsf{A}, \mathsf{A})}{\max[C_\alpha(\mathsf{A}, \mathsf{A}), C_\alpha(\mathsf{A}, \mathsf{A})]} = \frac{C_\alpha(\mathsf{A}, \mathsf{A})}{C_\alpha(\mathsf{A}, \mathsf{A})} = 1.$$

The converse is straightforward. Therefore, $\tilde{\mathcal{K}}_3(\mathsf{A}, \mathsf{B})$ is a CCPFS. The proofs for $\tilde{\mathcal{K}}_4(\mathsf{A}, \mathsf{B})$ and $\tilde{\mathcal{K}}_5(\mathsf{A}, \mathsf{B})$ are similar.

□

4.2.1 Flowchart for the New WCCPFSs Methods



4.3 Numerical Verifications of the WCCPFSs Methods

By using the information in Subsection 3.2, and putting into consideration the effect of the weights of the elements of $X = \{a, b, c\}$, we compute the interdependence indexes of A and B. Assume $\alpha = \{0.4, 0.32, 0.28\}$, and using Eqs. (15) and (16), we obtain

$$\begin{aligned} C_\alpha(\mathsf{A}, \mathsf{C}) = \sum_{i=1}^{3} & [0.4((0.3^2 \times 0.1^2) + (0.6^2 \times 0.1^2) + (0.7416^2 \times 0.9899^2)) \\ & + 0.32((0.5^2 \times 1^2) + (0.3^2 \times 0^2) + (0.8124^2 \times 0^2)) \\ & + 0.28((0.3873^2 \times 0^2) + (0.5^2 \times 1^2) + (0.7746^2 \times 0^2))] \\ & = 0.3674 \end{aligned}$$

$$\begin{aligned} C_\alpha(\mathsf{B}, \mathsf{C}) = \sum_{i=1}^{3} & [0.4((0.3^2 \times 0.1^2) + (0.6^2 \times 0.1^2) + (0.7416^2 \times 0.9899^2)) \\ & + 0.32((0.5^2 \times 1^2) + (0.3162^2 \times 0^2) + (0.8062^2 \times 0^2)) \\ & + 0.28((0.4^2 \times 0^2) + (0.5^2 \times 1^2) + (0.7681^2 \times 0^2))] \\ & = 0.3674 \end{aligned}$$

$$\begin{aligned} C_\alpha(\mathsf{A}, \mathsf{A}) = \sum_{i=1}^{3} & [0.4(0.3^4 + 0.6^4 + 0.7416^4) + 0.32(0.5^4 + 0.3^4 + 0.8124^4) \\ & + 0.28(+0.4^4 + 0.5^4 + 0.7681^4)] \\ & = 0.4602 \end{aligned}$$

$$\begin{aligned} C_\alpha(\mathsf{B}, \mathsf{B}) = \sum_{i=1}^{3} & [0.4(0.3^4 + 0.6^4 + 0.7416^4) + 0.32(+0.5^4 + 0.3162^4 + 0.8062^4) \\ & + 0.28(0.3873^4 + 0.5^4 + 0.7746^4)] \\ & = 0.4591 \end{aligned}$$

$$\begin{aligned} C_\alpha(\mathsf{C}, \mathsf{C}) = \sum_{i=1}^{3} & [0.4(0.1^4 + 0.1^4 + 0.9899^4) + 0.32(1^4 + 0^4 + 0^4) \\ & + 0.28(0^4 + 1^4 + 0^4)] \\ & = 0.9841. \end{aligned}$$

Hence,

$$\left.\begin{aligned}\tilde{\mathcal{K}}_1(\mathsf{A},\mathsf{C}) &= \frac{0.3674}{\max[0.4602,\,0.9841]} = 0.3733,\\ \tilde{\mathcal{K}}_1(\mathsf{B},\mathsf{C}) &= \frac{0.3674}{\max[0.4591,\,0.9841]} = 0.3733,\end{aligned}\right\}$$

$$\left.\begin{aligned}\tilde{\mathcal{K}}_2(\mathsf{A},\mathsf{C}) &= \frac{0.3674}{\sqrt{0.4602\times 0.9841}} = 0.5459,\\ \tilde{\mathcal{K}}_2(\mathsf{B},\mathsf{C}) &= \frac{0.3674}{\sqrt{0.4591\times 0.9841}} = 0.5466.\end{aligned}\right\}$$

By using Eqs. (19), (20) and (21), we have

$$\begin{aligned}C_\alpha(\mathsf{A},\mathsf{C}) &= \sum_{i=1}^{3}[0.4(\sqrt{(0.3\times 0.1)^3}+\sqrt{(0.6\times 0.1)^3}+\sqrt{(0.7416\times 0.9899)^3})\\ &+0.32(\sqrt{(0.5\times 1)^3}+\sqrt{(0.3\times 0)^3}+\sqrt{(0.8124\times 0)^3})\\ &+0.28(\sqrt{(0.3873\times 0)^3}+\sqrt{(0.5\times 1)^3}+\sqrt{(0.7746\times 0)^3})]\\ &=0.4717\end{aligned}$$

$$\begin{aligned}C_\alpha(\mathsf{B},\mathsf{C}) &= \sum_{i=1}^{3}[0.4(\sqrt{(0.3\times 0.1)^3}+\sqrt{(0.6\times 0.1)^3}+\sqrt{(0.7416\times 0.9899)^3})\\ &+0.32(\sqrt{(0.5\times 1)^3}+\sqrt{(0.3162\times 0)^3}+\sqrt{(0.8062\times 0)^3})\\ &+0.28(\sqrt{(0.4\times 0)^3}+\sqrt{(0.5\times 1)^3}+\sqrt{(0.7681\times 0)^3})]\\ &=0.4717\end{aligned}$$

$$\begin{aligned}C_\alpha(\mathsf{A},\mathsf{A}) &= \sum_{i=1}^{3}[0.4(0.3^3+0.6^3+0.7416^3)+0.32(0.5^3+0.3^3+0.8124^3)\\ &+0.28(+0.4^3+0.5^3+0.7681^3)]\\ &=0.6604\end{aligned}$$

$$\begin{aligned}C_\alpha(\mathsf{B},\mathsf{B}) &= \sum_{i=1}^{3}[0.4(0.3^3+0.6^3+0.7416^3)+0.32(+0.5^3+0.3162^3+0.8062^3)\\ &+0.28(0.3873^3+0.5^3+0.7746^3)]\\ &=0.6595\end{aligned}$$

$$
\begin{aligned}
C_\alpha(\mathsf{C}, \mathsf{C}) &= \sum_{i=1}^{3}[0.4(0.1^3 + 0.1^3 + 0.9899^3) + 0.32(1^3 + 0^4 + 0^3) \\
&+ 0.28(0^3 + 1^3 + 0^3)] \\
&= 0.9888.
\end{aligned}
$$

Hence,

$$
\left.
\begin{aligned}
\tilde{\mathcal{K}}_3(\mathsf{A}, \mathsf{C}) &= \frac{0.4717}{\max[0.6604, 0.9888]} = 0.4770, \\
\tilde{\mathcal{K}}_3(\mathsf{B}, \mathsf{C}) &= \frac{0.4717}{\max[0.6595, 0.9888]} = 0.4770,
\end{aligned}
\right\}
$$

$$
\left.
\begin{aligned}
\tilde{\mathcal{K}}_4(\mathsf{A}, \mathsf{C}) &= \frac{0.4717}{\text{Aver}[0.6604, 0.9888]} = 0.5720, \\
\tilde{\mathcal{K}}_4(\mathsf{B}, \mathsf{C}) &= \frac{0.4717}{\text{Aver}[0.6595, 0.9888]} = 0.5723,
\end{aligned}
\right\}
$$

$$
\left.
\begin{aligned}
\mathcal{K}_5(\mathsf{A}, \mathsf{C}) &= \frac{0.4717}{\sqrt{0.6604 \times 0.9888}} = 0.5837, \\
\mathcal{K}_5(\mathsf{B}, \mathsf{C}) &= \frac{0.4717}{\sqrt{0.6595 \times 0.9888}} = 0.5841.
\end{aligned}
\right\}
$$

4.3.1 Comparison of the New Methods of Computing WCCPFSs with the Existing Methods

Table 2 contains the computational results for easy analysis.

By comparing Tables 1 and 2, it is not superfluous to say that WCCPFSs give a better measure of interrelationship. This bespeaks the impact of weights on measuring correlation coefficient. From Table 2, we surmise that the (i) WCCPFSs techniques via maximum method in [25] and $\tilde{\mathcal{K}}_3$ cannot determine the interrelationship between almost two equal PFSs with respect to an unrelated PFS, (ii) new WCCPFSs techniques are more reasonable and accurate and can determine the interrelationship between almost two equal PFSs with respect to an unrelated PFS. Again, the new

Table 2 WCCPFSs outputs

WCCPFSs	(A, C)	(B, C)
$\tilde{\mathcal{K}}_1$	0.3733	0.3733
$\tilde{\mathcal{K}}_2$	0.5459	0.5466
$\tilde{\mathcal{K}}_3$	**0.4770**	**0.4770**
$\tilde{\mathcal{K}}_4$	**0.5720**	**0.5723**
$\tilde{\mathcal{K}}_5$	**0.5837**	**0.5841**

WCCPFSs techniques have better performance indexes in contrast to the ones in [25]. From the computations, we conclude that (B, C) are more related to each other than (A, C).

5 Determination of Pattern Recognition and Medical Diagnostic Problem via WCCPFSs

In this section, we apply the WCCPFSs methods discussed so far to problems of pattern recognition and medical diagnosis to ascertain the more efficient approach and agreement of decision via the WCCPFSs techniques.

5.1 Applicative Example in Pattern Recognition

Pattern recognition is the process of identifying patterns by using machine learning procedure. Pattern recognition has a lot to do with artificial intelligence and machine learning. The idea of pattern recognition is important because of its application potential in neural networks, software engineering, computer vision, etc. Assume there are three pattern C_i, represented in Pythagorean fuzzy values in $X = \{x_i\}$, for $i = 1, \ldots, 3$ and $\alpha = \{0.4, 0.3, 0.3\}$. If there is an unknown pattern P represented in Pythagorean fuzzy values in $X = \{x_i\}$. The Pythagorean fuzzy representations of these patterns are in Table 3.

Table 3 Pythagorean fuzzy representations of patterns

Feature space			
PFS	x_1	x_2	x_3
μ_{C_1}	1.0000	0.8000	0.7000
ν_{C_1}	0.0000	0.0000	0.1000
π_{C_1}	0.0000	0.6000	0.7071
μ_{C_2}	0.8000	1.0000	0.9000
ν_{C_2}	0.1000	0.0000	0.1000
π_{C_2}	0.5916	0.0000	0.4243
μ_{C_3}	0.6000	0.8000	1.0000
ν_{C_3}	0.2000	0.0000	0.0000
π_{C_3}	0.7746	0.6000	0.0000
μ_{P}	0.5000	0.6000	0.8000
ν_{P}	0.3000	0.2000	0.1000
π_{P}	0.8124	0.7746	0.5916

To enable us to classify P into any of $\mathsf{C}_i, i = 1, 2, 3$, we deploy the WCCPFSs in [25] and the proposed WCCPFSs as follows:

Using Eqs. (15) and (16), we obtain

$$C_\alpha(\mathsf{C}_1, \mathsf{P}) = 0.3805,\ C_\alpha(\mathsf{C}_2, \mathsf{P}) = 0.4392,\ C_\alpha(\mathsf{C}_3, \mathsf{P}) = 0.5218,$$

$$C_\alpha(\mathsf{C}_1, \mathsf{C}_1) = 0.7088,\ C_\alpha(\mathsf{C}_2, \mathsf{C}_2) = 0.7195,\ C_\alpha(\mathsf{C}_3, \mathsf{C}_3) = 0.6582,\ C_\alpha(\mathsf{P}, \mathsf{P}) = 0.5095.$$

Hence,

$$\tilde{\mathcal{K}}_1(\mathsf{C}_1, \mathsf{P}) = 0.5368,\ \tilde{\mathcal{K}}_1(\mathsf{C}_2, \mathsf{P}) = 0.6104,\ \tilde{\mathcal{K}}_1(\mathsf{C}_3, \mathsf{P}) = 0.7928.$$

$$\tilde{\mathcal{K}}_2(\mathsf{C}_1, \mathsf{P}) = 0.6332,\ \tilde{\mathcal{K}}_2(\mathsf{C}_2, \mathsf{P}) = 0.7254,\ \tilde{\mathcal{K}}_2(\mathsf{C}_3, \mathsf{P}) = 0.9011.$$

Using Eqs. (19), (20) and (21), we have

$$C_\alpha(\mathsf{C}_1, \mathsf{P}) = 0.5434,\ C_\alpha(\mathsf{C}_2, \mathsf{P}) = 0.5973,\ C_\alpha(\mathsf{C}_3, \mathsf{P}) = 0.6808,$$

$$C_\alpha(\mathsf{C}_1, \mathsf{C}_1) = 0.8277,\ C_\alpha(\mathsf{C}_2, \mathsf{C}_2) = 0.8299,\ C_\alpha(\mathsf{C}_3, \mathsf{C}_3) = 0.7939,\ C_\alpha(\mathsf{P}, \mathsf{P}) = 0.6312.$$

Hence,

$$\tilde{\mathcal{K}}_3(\mathsf{C}_1, \mathsf{P}) = 0.6565,\ \tilde{\mathcal{K}}_3(\mathsf{C}_2, \mathsf{P}) = 0.7197,\ \tilde{\mathcal{K}}_3(\mathsf{C}_3, \mathsf{P}) = 0.8575.$$

$$\tilde{\mathcal{K}}_4(\mathsf{C}_1, \mathsf{P}) = 0.7449,\ \tilde{\mathcal{K}}_4(\mathsf{C}_2, \mathsf{P}) = 0.8175,\ \tilde{\mathcal{K}}_4(\mathsf{C}_3, \mathsf{P}) = 0.9554.$$

$$\tilde{\mathcal{K}}_5(\mathsf{C}_1, \mathsf{P}) = 0.7518,\ \tilde{\mathcal{K}}_5(\mathsf{C}_2, \mathsf{P}) = 0.8253,\ \tilde{\mathcal{K}}_5(\mathsf{C}_3, \mathsf{P}) = 0.9617.$$

Table 4 presents the results for glance analysis.

From Table 4, P is suitable to be classified with C_3 because $\tilde{\mathcal{K}}_i(\mathsf{C}_3, \mathsf{P}) > \tilde{\mathcal{K}}_i(\mathsf{C}_2, \mathsf{P}) > \tilde{\mathcal{K}}_i(\mathsf{C}_1, \mathsf{P})\ \forall\ i = 1, \ldots, 5$.

Table 4 WCCPFSs outputs

WCCPFSs	$(\mathsf{C}_1, \mathsf{P})$	$(\mathsf{C}_2, \mathsf{P})$	$(\mathsf{C}_3, \mathsf{P})$
$\tilde{\mathcal{K}}_1$	0.5368	0.6104	0.7928
$\tilde{\mathcal{K}}_2$	0.6332	0.7254	0.9011
$\tilde{\mathcal{K}}_3$	**0.6565**	**0.7197**	**0.8575**
$\tilde{\mathcal{K}}_4$	**0.7449**	**0.8175**	**0.9554**
$\tilde{\mathcal{K}}_5$	**0.7518**	**0.8253**	**0.9617**

5.2 Applicative Example in Medical Diagnosis

Medical diagnosis is a delicate exercise because failure to make the right decision may lead to the death of the patient. Diagnosis of diseases is challenging due to embedded fuzziness in the processes. Here, we present a scenario of a mathematical approach of diagnosing a patient medical status via WCCPFSs methods, where the symptoms or clinical manifestations of the diseases are represented in Pythagorean fuzzy values by using hypothetical cases.

Suppose we have a set of diseases $\mathsf{D} = \{\mathsf{D}_1, \mathsf{D}_2, \mathsf{D}_3, \mathsf{D}_4, \mathsf{D}_5\}$ represented in Pythagorean fuzzy values, where D_1 = viral fever, D_2 = malaria, D_3 = typhoid fever, D_4 = peptic ulcer, D_5 = chest problem, and a set of symptoms

$$\mathsf{S} = \{\mathsf{s}_1, \mathsf{s}_2, \mathsf{s}_3, \mathsf{s}_4, \mathsf{s}_5\}$$

for s_1 = temperature, s_2 = headache, s_3 = stomach pain, s_4 = cough, s_5 = chest pain, which are the clinical manifestations of D_i, $i = 1, \ldots, 5$. From the knowledge of the clinical manifestations, the weight of the symptoms as $\alpha = \{0.3, 0.25, 0.1, 0.25, 0.1\}$.

Assume a patient P with a manifest symptoms S is also capture in Pythagorean fuzzy values. Table 5 contains Pythagorean fuzzy information of D_i, $i = 1, \ldots, 5$ and P with respect to S.

Now, we find which of the diseases D_i has the greatest interrelationship with the patient P with respect to the clinical manifestations S by deploying Eqs. (15), (16), (19), (20), and (21).

By using Eqs. (15) and (16), we have

$$C_\alpha(\mathsf{D}_1, \mathsf{P}) = 0.4609,\ C_\alpha(\mathsf{D}_2, \mathsf{P}) = 0.4740,\ C_\alpha(\mathsf{D}_3, \mathsf{P}) = 0.4213,$$

$$C_\alpha(\mathsf{D}_4, \mathsf{P}) = 0.3252,\ C_\alpha(\mathsf{D}_5, \mathsf{P}) = 0.2289,\ C_\alpha(\mathsf{D}_1, \mathsf{D}_1) = 0.5930,$$

$$C_\alpha(\mathsf{D}_2, \mathsf{D}_2) = 0.5203,\ C_\alpha(\mathsf{D}_3, \mathsf{D}_3) = 0.5761,\ C_\alpha(\mathsf{D}_4, \mathsf{D}_4) = 0.5297,$$

Table 5 Pythagorean fuzzy representations of diagnostic process

Clinical manifestations					
PFS	s_1	s_2	s_3	s_4	s_5
μ_{D_1}	0.4000	0.3000	0.1000	0.4000	0.1000
ν_{D_1}	0.0000	0.5000	0.7000	0.3000	0.7000
π_{D_1}	0.9165	0.8124	0.7071	0.8660	0.7071
μ_{D_2}	0.7000	0.2000	0.0000	0.7000	0.1000
ν_{D_2}	0.0000	0.6000	0.9000	0.0000	0.8000
π_{D_2}	0.7141	0.7746	0.4359	0.7141	0.5916
μ_{D_3}	0.3000	0.6000	0.2000	0.2000	0.1000
ν_{D_3}	0.3000	0.1000	0.7000	0.6000	0.9000
π_{D_3}	0.9055	0.7937	0.6856	0.7746	0.4243
μ_{D_4}	0.1000	0.2000	0.8000	0.2000	0.2000
ν_{D_4}	0.7000	0.4000	0.0000	0.7000	0.7000
π_{D_4}	0.7071	0.8944	0.6000	0.6856	0.6856
μ_{D_5}	0.1000	0.0000	0.2000	0.2000	0.8000
ν_{D_5}	0.8000	0.8000	0.8000	0.8000	0.1000
π_{D_5}	0.5916	0.6000	0.5657	0.5657	0.5916
μ_{P}	0.8000	0.6000	0.2000	0.6000	0.1000
ν_{P}	0.1000	0.1000	0.8000	0.1000	0.6000
π_{P}	0.5916	0.7937	0.5657	0.7937	0.7937

$$C_\alpha(\mathsf{D}_5, \mathsf{D}_5) = 0.5274,\ C_\alpha(\mathsf{P}, \mathsf{P}) = 0.4859.$$

Hence,

$$\tilde{\mathcal{K}}_1(\mathsf{D}_1, \mathsf{P}) = 0.7772,\ \tilde{\mathcal{K}}_1(\mathsf{D}_2, \mathsf{P}) = 0.9110,\ \tilde{\mathcal{K}}_1(\mathsf{D}_3, \mathsf{P}) = 0.7313,$$

$$\tilde{\mathcal{K}}_1(\mathsf{D}_4, \mathsf{P}) = 0.6139,\ \tilde{\mathcal{K}}_1(\mathsf{D}_5, \mathsf{P}) = 0.4340.$$

$$\tilde{\mathcal{K}}_2(\mathsf{D}_1, \mathsf{P}) = 0.8586,\ \tilde{\mathcal{K}}_2(\mathsf{D}_2, \mathsf{P}) = 0.9427,\ \tilde{\mathcal{K}}_2(\mathsf{D}_3, \mathsf{P}) = 0.7963,$$

$$\tilde{\mathcal{K}}_2(\mathsf{D}_4, \mathsf{P}) = 0.6410,\ \tilde{\mathcal{K}}_2(\mathsf{D}_5, \mathsf{P}) = 0.4522.$$

Using Eqs. (19), (20) and (21), we obtain

$$C_\alpha(\mathsf{D}_1, \mathsf{P}) = 0.6354,\ C_\alpha(\mathsf{D}_2, \mathsf{P}) = 0.6840,\ C_\alpha(\mathsf{D}_3, \mathsf{P}) = 0.5945,$$

Table 6 WCCPFSs outputs

WCCPFSs	$(\mathsf{D}_1, \mathsf{P})$	$(\mathsf{D}_2, \mathsf{P})$	$(\mathsf{D}_3, \mathsf{P})$	$(\mathsf{D}_4, \mathsf{P})$	$(\mathsf{D}_5, \mathsf{P})$
$\tilde{\mathcal{K}}_1$	0.7772	0.9110	0.7313	0.6139	0.4340
$\tilde{\mathcal{K}}_2$	0.8586	0.9427	0.7963	0.6410	0.4522
$\tilde{\mathcal{K}}_3$	**0.8508**	**0.9549**	**0.8052**	**0.6486**	**0.5037**
$\tilde{\mathcal{K}}_4$	**0.8685**	**0.9562**	**0.8174**	**0.6493**	**0.5040**
$\tilde{\mathcal{K}}_5$	**0.8688**	**0.9562**	**0.8175**	**0.6494**	**0.5040**

$$C_\alpha(\mathsf{D}_4, \mathsf{P}) = 0.4646,\ C_\alpha(\mathsf{D}_5, \mathsf{P}) = 0.3608,\ C_\alpha(\mathsf{D}_1, \mathsf{D}_1) = 0.7468,$$

$$C_\alpha(\mathsf{D}_2, \mathsf{D}_2) = 0.7143,\ C_\alpha(\mathsf{D}_3, \mathsf{D}_3) = 0.7383,\ C_\alpha(\mathsf{D}_4, \mathsf{D}_4) = 0.7146,$$

$$C_\alpha(\mathsf{D}_5, \mathsf{D}_5) = 0.7154,\ C_\alpha(\mathsf{P}, \mathsf{P}) = 0.7163.$$

Hence,

$$\tilde{\mathcal{K}}_3(\mathsf{D}_1, \mathsf{P}) = 0.8508,\ \tilde{\mathcal{K}}_3(\mathsf{D}_2, \mathsf{P}) = 0.9549,\ \tilde{\mathcal{K}}_3(\mathsf{D}_3, \mathsf{P}) = 0.8052,$$

$$\tilde{\mathcal{K}}_3(\mathsf{D}_4, \mathsf{P}) = 0.6486,\ \tilde{\mathcal{K}}_3(\mathsf{D}_5, \mathsf{P}) = 0.5037.$$

$$\tilde{\mathcal{K}}_4(\mathsf{D}_1, \mathsf{P}) = 0.8685,\ \tilde{\mathcal{K}}_4(\mathsf{D}_2, \mathsf{P}) = 0.9562,\ \tilde{\mathcal{K}}_4(\mathsf{D}_3, \mathsf{P}) = 0.8174,$$

$$\tilde{\mathcal{K}}_4(\mathsf{D}_4, \mathsf{P}) = 0.6493,\ \tilde{\mathcal{K}}_4(\mathsf{D}_5, \mathsf{P}) = 0.5040.$$

$$\tilde{\mathcal{K}}_5(\mathsf{D}_1, \mathsf{P}) = 0.8688,\ \tilde{\mathcal{K}}_5(\mathsf{D}_2, \mathsf{P}) = 0.9562,\ \tilde{\mathcal{K}}_5(\mathsf{D}_3, \mathsf{P}) = 0.8175,$$

$$\tilde{\mathcal{K}}_5(\mathsf{D}_4, \mathsf{P}) = 0.6494,\ \tilde{\mathcal{K}}_5(\mathsf{D}_5, \mathsf{P}) = 0.5040.$$

Table 6 presents the results for glance analysis.

From Table 6, it is inferred that the patient is suffering from malaria since

$$\tilde{\mathcal{K}}_i(\mathsf{D}_2, \mathsf{P}) > \tilde{\mathcal{K}}_i(\mathsf{D}_1, \mathsf{P}) > \tilde{\mathcal{K}}_i(\mathsf{D}_3, \mathsf{P}) > \tilde{\mathcal{K}}_i(\mathsf{D}_4, \mathsf{P}) > \tilde{\mathcal{K}}_i(\mathsf{D}_5, \mathsf{P})$$

for $i = 1, \ldots, 5$.

6 Conclusion

In this chapter, we have studied some techniques of calculating CCPFSs and WCCPFSs, respectively. It is found that the approach of WCCPFSs is more reliable than CCPFSs. By juxtaposing the existing methods of computing WCCPFSs and the novel ones, it is proven that the novel methods of calculating WCCPFSs are more accurate and efficient. Some MCDM problems were considered via the existing and the novel WCCPFSs methods to demonstrate applicability. The novel methods of computing WCCPFSs could be applied to more MCDM problems via object-oriented approach in cases of larger population. Extending the concept of weights on elements of PFSs to other existing correlation coefficients in Pythagorean fuzzy domain [16, 50] could be of great interest in other different applicative areas through clustering algorithm.

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Parametric Directed Divergence Measure for Pythagorean Fuzzy Set and Their Applications to Multi-criteria Decision-Making

Nikunj Agarwal

1 Introduction

In literature so far, the classical information theory has been widely used and represents the vagueness in the data in classical measure theory but the measures are valid for precisely given data. Even, due to the various constraints in day-to-day life, decision-makers may give their judgements under the uncertain and imprecise situation. Thus, there is always a degree of hesitancy between the preferences of the decision-making so that the analysis conducted under such circumstances is not ideal and hence does not tell the exact information to the system analyst. To cope up with impreciseness, vagueness, and the uncertainty in the data, the intuitionistic fuzzy sets (IFSs) [1] are successful extension of the fuzzy set (FS) [2]. Over the last several years, the IFS has received much attention by introducing the various kinds of information measures, aggregation operators and employed them to solve the decision-making problems under the different environment [3–10]. But, there is some limitation in the studies of IFSs as it is valid only for the environments where the degree's sum is less than one. However, this condition is ruled out in many situations. For instance, if a person gives their preference in the form of membership and non-membership degrees toward a particular object as 0.8 and 0.5, then the situation is not handled with IFSs. In order to resolve it, Yager [11, 12] proposed the Pythagorean fuzzy (PF) sets (PFSs) by relaxing this sum condition to its square sum less than one, i.e., corresponding to the above-considered example, we see that $(0.8)^2 + (0.5)^2 \leq 1$ and hence PFSs are an extension of the existing IFSs. After their pioneer work, Yager and Abbasov [13] studied the relationship between the Pythagorean numbers and the complex numbers. Later on, several aggregation operators under the PFS environment have been investigated by researchers [14, 15] using different norm operations. Zhang and Xu [16] extended the TOPSIS approach from

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_3

IF to the PF environment. Garg [6] presented a confidence level-based averaging and geometric aggregation operators, by incorporating the confidence level of the decision-makers (DMs) to the analysis. In the continuation, several authors introduced different types of aggregation operators under PFSs [17–19] to solve many decision-making problems.

Now, the degree of distance/similarity/divergence measures has been focused by the authors and received attention in the last four decades for solving the decision-making, pattern recognition, medical diagnosis problems. However, the prime task for decision-maker (DM) is to rank the alternatives to get the best [20–23]. For this, researchers have made efforts to enrich the concept of information measures in Pythagorean fuzzy environments [24]. Zhang and Xu [16] suggested a distance measure to solve a realistic problem under PFS. While Yang et al. [25] pointed out an unreasonable case of proof in [16]. Wei and Wei [26] presented some similarity measures between PFSs which are actually based on the cosine function. Li et al. [27] introduced the Hamming distance measure, the Euclidean distance measure, and the Minkowski distance measure between PFSs, with their detailed properties. Zhang [28] explored a novel similarity measure for PFSs, to deal the selection problem of photovoltaic cells. Zeng et al. [29] considered five parameters for distance and similarity measures of Pythagorean fuzzy sets and applied them in the selection of China's Internet stocks. Peng et al. [30] presented similarity measure, distance measure, entropy, and inclusion measure for PFSs, put forward transformation relationships, and successfully applied them in pattern recognition, clustering analysis, and medical diagnosis [31].

Thus, the observation from the above studies is that all the measures do not incorporate the idea of decision-maker's preferences into the measure and also these measures do not follow the linear order. That's why, there is always a trouble in getting the exact nature of the alternative. Therefore, we present a parametric directed divergence measure order α and degree β for Pythagorean fuzzy set (PFS). Through this proposed measure, the decision-maker can make more reliable and flexible decisions for different values of parameters α and β. Several properties have been investigated based on this measure with a numerical example to demonstrate the performance of measure. Finally, concrete conclusion has been presented.

2 Basic Concepts

In this section, some basic definitions of IFSs and PFSs have been presented on the universal set X.

2.1 Intuitionistic Fuzzy Set [1]

Definition 2.1 An IFS (intuitionistic fuzzy set) is defined as

$$\bar{A} = \left\{ \left\langle x, \mu_{\bar{A}}(x), \nu_{\bar{A}}(x) \right\rangle : x \in X \right\},$$

where $\mu_{\bar{A}}(x)$ and $\nu_{\bar{A}}(x)$ represent the membership and non-membership degrees such that $0 \le \mu_{\bar{A}}(x) + \nu_{\bar{A}}(x) \le 1$ with $\mu_{\bar{A}}(x), \nu_{\bar{A}}(x) \in [0, 1]$..

2.2 Hesitant Fuzzy Set [32, 33]

Definition 2.2 A HFS (hesitant fuzzy set) is defined as a function $HFS : X \to [0, 1]$ and is given by

$$\bar{A} = \left\{ \left\langle x, h_{\bar{A}}(x) \right\rangle : x \in X \right\},$$

where $h_{\bar{A}}(x)$ represents HFE (hesitant fuzzy element).

2.3 Pythagorean Fuzzy Set [11, 16]

Definition 2.3 A PFS (Pythagorean fuzzy set) is defined as a set of ordered pairs given by

$$\bar{A} = \left\{ \left\langle x, \mu_{\bar{A}}(x), \nu_{\bar{A}}(x) \right\rangle : x \in X \right\},$$

where $\mu_{\bar{A}}(x)$ and $\nu_{\bar{A}}(x)$ represent the membership and non-membership degrees such that $(\mu_{\bar{A}}(x))^2 + (\nu_{\bar{A}}(x))^2 \le 1$ with $\mu_{\bar{A}}(x), \nu_{\bar{A}}(x) \in [0, 1]$. For convenience, the pair of these membership functions is called a Pythagorean fuzzy number (PFN) and it is denoted as $\alpha = <\mu_\alpha, \nu_\alpha>$.

3 Proposed Parametric Directed Divergence Measure for Pythagorean Fuzzy Set (PFS)

In this section, we have proposed a flexible and generalized parametric divergence measure of order α and degree β denoted as class of (α, β), under the environment of PFSs. Some desirable properties of this measure are also being studied.

3.1 Parametric Divergence Measure for PFSs

Definition 3.1 Let A and B be the two PFSs defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, then a parametric directed divergence measure for PFSs based on parameters α and β is denoted as $D_\alpha^\beta(A|B)$ and defined as

$$D_\alpha^\beta(A|B) = \frac{\alpha}{n(2-\beta)} \sum \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}}{\mu_A^{\frac{2\alpha}{(2-\beta)}} + \mu_B^{\frac{2\alpha}{(2-\beta)}}} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}}{\nu_A^{\frac{2\alpha}{(2-\beta)}} + \nu_B^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}}{\pi_A^{\frac{2\alpha}{(2-\beta)}} + \pi_B^{\frac{2\alpha}{(2-\beta)}}} \right) \end{array} \right],$$

where μ, ν, and π are the membership, non-membership and hesitancy functions, respectively, and it is valid for $\alpha,\ \beta > 0$ and except $\beta \neq 2$.

It is clearly seen from the definition that the $D_\alpha^\beta(A|B)$ is not symmetric, so to imbue the measure with symmetry, a parametric symmetric divergence measure for PFSs has been defined as follows.

3.2 Parametric Symmetric Divergence Measure for PFSs

Definition 3.2 Let A and B be the two PFSs defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, then a parametric directed divergence measure for PFSs based on parameters α and β is denoted as $D_\alpha^\beta(A;, B)$ and defined as

$$D_\alpha^\beta(A;B) = D_\alpha^\beta(A\,|B) + D_\alpha^\beta(B|A) \quad \Rightarrow$$

$$D_\alpha^\beta(A;B) = \frac{\alpha}{n(2-\beta)} \sum \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}}{\mu_A^{\frac{2\alpha}{(2-\beta)}} + \mu_B^{\frac{2\alpha}{(2-\beta)}}} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}}{\nu_A^{\frac{2\alpha}{(2-\beta)}} + \nu_B^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}}{\pi_A^{\frac{2\alpha}{(2-\beta)}} + \pi_B^{\frac{2\alpha}{(2-\beta)}}} \right) \end{array} \right].$$

$$+ \frac{\alpha}{n(2-\beta)} \sum \left[\begin{array}{r} \mu_B^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}}{\mu_A^{\frac{2\alpha}{(2-\beta)}} + \mu_B^{\frac{2\alpha}{(2-\beta)}}} \right) + \nu_B^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}}{\nu_A^{\frac{2\alpha}{(2-\beta)}} + \nu_B^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_B^{\frac{2\alpha}{(2-\beta)}} \log\left(\dfrac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}}{\pi_A^{\frac{2\alpha}{(2-\beta)}} + \pi_B^{\frac{2\alpha}{(2-\beta)}}} \right) \end{array} \right]$$

From the definition of $D_\alpha^\beta(A \mid B)$ and $D_\alpha^\beta(A;\, B)$, it has been observed that

$$D_\alpha^\beta(A \mid B) \geq 0, \quad D_\alpha^\beta(A;\, B) \geq 0,$$

and $A = B \Rightarrow D_\alpha^\beta(A \mid B) = D_\alpha^\beta(A; B)$.

Divide the universe X into two parts X_1 and X_2, where

$$\begin{aligned} X_1 &= \{x_i : x_i \in X, A(x_i) \subseteq B(x)\}, \text{ i.e.,} \\ \mu_A(x_i) \le \mu_B(x_i), &\ \nu_A(x_i) \ge \nu_B(x_i)\ \forall x_i \in X_1, \\ X_2 &= \{x_i : x_i \in X, A(x_i) \supseteq B(x)\}, \text{ i.e.,} \\ \mu_A(x_i) \ge \mu_B(x_i), &\ \nu_A(x_i) \le \nu_B(x_i)\ \forall x_i \in X_2, \end{aligned}.$$

Now, we propose some properties based on the above considerations.

3.3 Some Properties of Parametric Symmetric Divergence Measure for PFSs

Property 3.3.1 Let A and B be the two PFSs defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, such that they satisfy for any $x_i \in X$ either $A \subseteq B$ or $B \subseteq A$,

$$D_\alpha^\beta(A \cup B;\ A \cap B) = D_\alpha^\beta(A; B).$$

Proof It is clear that

$$D_\alpha^\beta(A \cup B;\ A \cap B) = D_\alpha^\beta(A \cup B | A \cap B) + D_\alpha^\beta(A \cap B | A \cup B).$$

On the other hand,

$$\begin{aligned} &D_\alpha^\beta(A \cup B | A \cap B) \\ &= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^n \left[\begin{array}{r} \mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}} + \nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \\ &= \frac{\alpha}{n(2-\beta)} \left[\begin{array}{l} \sum_{x\in X_1} \left[\begin{array}{r} \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_B^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_B^{\frac{2\alpha}{(2-\beta)}} + \nu_A^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] + \\ \sum_{x\in X_2} \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}} \log\left(\frac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}} + \nu_B^{\frac{2\alpha}{(2-\beta)}}} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \end{array} \right] \end{aligned} \quad (1)$$

and

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A\cap B|\,A\cup B)\\
&=\frac{\alpha}{n\,(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{r}\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\,+\,\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}\,\log\left(\dfrac{2\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}\,+\,\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}}\right)\\
+\,\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&=\frac{\alpha}{n\,(2-\beta)}\left[\begin{array}{l}\displaystyle\sum_{x\in X_1}\left[\begin{array}{r}\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\,+\,\nu_{A}^{\frac{2\alpha}{(2-\beta)}}\,\log\left(\dfrac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}\,+\,\nu_{B}^{\frac{2\alpha}{(2-\beta)}}}\right)\\
+\,\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]+\\
\displaystyle\sum_{x\in X_2}\left[\begin{array}{r}\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\,+\,\nu_{B}^{\frac{2\alpha}{(2-\beta)}}\,\log\left(\dfrac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}\,+\,\nu_{A}^{\frac{2\alpha}{(2-\beta)}}}\right)\\
+\,\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\end{array}\right].
\end{aligned}
\tag{2}
$$

Then, by adding Eqs. (1) and (2), we get

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A\cup B|\,A\cap B)\,+\,D_{\alpha}^{\beta}(A\cap B|\,A\cup B)\\
&=\frac{\alpha}{n\,(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{r}\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\,+\,\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\,\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right].\\
&+\;\frac{\alpha}{n\,(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{r}\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\,+\,\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\nu_{B}^{\frac{2\alpha}{(2-\beta)}(x_i)}}\right)\\
+\,\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,\log\left(\dfrac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\,+\,\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&=D_{\alpha}^{\beta}(A\,|B)\,+\,D_{\alpha}^{\beta}(B|\,A)=D_{\alpha}^{\beta}(A;\;B)
\end{aligned}
$$

Thus, the results hold.

Property 3.3.2 For any two PFSs A and B defined on universal set $X = \{x_1,\ x_2,\ \ldots,x_n\}$, we have

(1) $D_{\alpha}^{\beta}(A;A\cup B)=D_{\alpha}^{\beta}(B;A\cap B)$

(2) $D_{\alpha}^{\beta}(A;A\cap B)=D_{\alpha}^{\beta}(B;A\cup B)$

(3) $D_\alpha^\beta(A; A \cup B) \ + \ D_\alpha^\beta(A; A \cap B) = D_\alpha^\beta(A; B)$

(4) $D_\alpha^\beta(B; A \cup B) \ + \ D_\alpha^\beta(B; A \cap B) = D_\alpha^\beta(A; B)$.

Proof All can be proved similarly. So, we prove only the first one, i.e.,

$$\begin{aligned} D_\alpha^\beta(A; A \cup B) &= D_\alpha^\beta(B; A \cap B) \\ &\Rightarrow D_\alpha^\beta(A\,|A \cup B) \ + \ D_\alpha^\beta(A \cup B|A). \\ &= D_\alpha^\beta(B|A \cap B) \ + \ D_\alpha^\beta(A \cap B|B) \end{aligned} \tag{3}$$

We consider the LHS. By the definition of divergence,

$$\begin{aligned} &D_\alpha^\beta(A\,|A \cup B) \\ &= \frac{\alpha}{n\,(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \\ &= \frac{\alpha}{n\,(2-\beta)} \sum_{x \in X_1} \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \\ &= \frac{\alpha}{n\,(2-\beta)} \sum_{x \in X_{2_1}} \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \\ &\Theta \log(1) = 0 \\ &= \frac{\alpha}{n\,(2-\beta)} \sum_{x \in X_1} \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right] \end{aligned}$$

and

$$D_{\alpha}^{\beta}(A \cup B|A)$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right]$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{x \in X_1} \left[\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right].$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{x \in X_{2_1}} \left[\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right]$$

$$\Theta \log(1) = 0$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{x \in X_1} \left[\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right]$$

Similarly, RHS

$$D_{\alpha}^{\beta}(B|A \cap B)$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{x \in X_1} \left[\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right]$$

and

$$D_{\alpha}^{\beta}(A \cap B|B)$$

$$= \frac{\alpha}{n(2-\beta)} \sum_{x \in X_1} \left[\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right].$$

By using Eq. (3), the expressions are same from both sides. This proves the result.

Property 3.3.3 For any two PFSs A and B defined on universal set $X = \{x_1,\ x_2,\ \ldots, x_n\}$, we have

(1) $D_\alpha^\beta(A; C)\ +\ D_\alpha^\beta(B; C) - D_\alpha^\beta(A \cup B; C) \geq 0$

(2) $D_\alpha^\beta(A; C)\ +\ D_\alpha^\beta(B; C) - D_\alpha^\beta(A \cap B; C) \geq 0.$

Proof Both can be proved similarly. So, we prove only the first one, i.e.,

$$D_\alpha^\beta(A; C) = D_\alpha^\beta(A\,|C)\ +\ D_\alpha^\beta(C|A)$$

$$= \frac{\alpha}{n\,(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{r} \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right)\ +\ \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ +\ \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right]$$

$$+\ \frac{\alpha}{n\,(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{r} \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right)\ +\ \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ +\ \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right]$$

and

$$D_\alpha^\beta(B; C) = D_\alpha^\beta(B\,|C)\ +\ D_\alpha^\beta(C|B)$$

$$= \frac{\alpha}{n\,(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{r} \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right)\ +\ \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ +\ \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right].$$

$$+\ \frac{\alpha}{n\,(2-\beta)} \sum_{i=1}^{n} \left[\begin{array}{r} \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right)\ +\ \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ +\ \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\dfrac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\ +\ \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \end{array} \right]$$

Now,

$$
\begin{aligned}
&D_\alpha^\beta(A\cup B;C)\\
&=\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{l}
\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
\qquad+\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)
\end{array}\right]\\
&+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{l}
\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
\qquad+\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)
\end{array}\right]\\
&=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\left[\begin{array}{l}
\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)
\end{array}\right].\\
&+\frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}\left[\begin{array}{l}
\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)
\end{array}\right]
\end{aligned}
$$

Then,

$$
\begin{aligned}
&D_\alpha^\beta(A;C)+D_\alpha^\beta(B;C)-D_\alpha^\beta(A\cup B;C)=\\
&=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}\left[\begin{array}{l}
\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)
\end{array}\right].\\
&+\frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\left[\begin{array}{l}
\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{C}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)
\end{array}\right]
\end{aligned}
$$

Since all the membership and non-membership lies between [0, 1]. This completes the proof.

Property 3.3.4 For any two PFSs A and B defined on universal set $X = \{x_1, x_2, \ldots, x_n\}$, we have

$$D_\alpha^\beta(A \cap B; C) + D_\alpha^\beta(A \cup B; C) = D_\alpha^\beta(A; C) + D_\alpha^\beta(B; C)$$

Proof

$$\begin{aligned} &D_\alpha^\beta(A \cap B; C) \\ &= \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right. \\ &\qquad \left. + \pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right] \\ &+ \frac{\alpha}{n(2-\beta)} \sum_{i=1}^{n} \left[\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right. \\ &\qquad \left. + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_{A\cap B}^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right] \\ &= \frac{\alpha}{n(2-\beta)} \sum_{x\in X_1} \left[\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right. \\ &\qquad + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ &\qquad \left. + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right] \\ &+ \frac{\alpha}{n(2-\beta)} \sum_{x\in X_2} \left[\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right. \\ &\qquad + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \\ &\qquad \left. + \nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) + \pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) \log\left(\frac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i) + \pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)} \right) \right] \end{aligned}$$

Now,

$$
\begin{aligned}
&D_\alpha^\beta(A\cup B;C)\\
&=\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{r}\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&\quad+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{array}{r}\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_{A\cup B}^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]\\
&=\frac{\alpha}{n(2-\beta)}\sum_{x\in X_1}\left[\begin{array}{l}\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right].\\
&\quad+\frac{\alpha}{n(2-\beta)}\sum_{x\in X_2}\left[\begin{array}{l}\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
+\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\dfrac{2\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_C^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{array}\right]
\end{aligned}
$$

By adding all of the above equations, we get the required result and this completes the proof.

Property 3.3.5 For any two PFSs A and B defined on universal set $X = \{x_1,\ x_2,\ \ldots, x_n\}$, we have

(1) $D_\alpha^\beta(A\,;B) = D_\alpha^\beta(A^c;\,B^c)$

(2) $D_\alpha^\beta(A\,;B^c) = D_\alpha^\beta(A^c;\,B)$

(3) $D_\alpha^\beta(A\,;B)\ +\ D_\alpha^\beta(A^c\,;B) = D_\alpha^\beta(A^c;\,B^c)\ +\ D_\alpha^\beta(A\,;B^c)$.

Proof Clearly, first and second parts are similar and the third one can be proved by adding these two. So, we prove only (1).

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A;B)\\
&= \frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{aligned}&\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
&+\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{aligned}\right]\\
&+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{aligned}&\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
&+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{aligned}\right].
\end{aligned}
$$

$$
\begin{aligned}
&D_{\alpha}^{\beta}(A^{c};B^{c})=\\
&= \frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{aligned}&\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
&+\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{aligned}\right]\\
&+\frac{\alpha}{n(2-\beta)}\sum_{i=1}^{n}\left[\begin{aligned}&\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\nu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)+\nu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)+\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\mu^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\mu_B^{\frac{2\alpha}{(2-\beta)}}(x_i)+\mu_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\\
&+\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)\log\left(\frac{2\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)}{\pi_B^{\frac{2\alpha}{(2-\beta)}}(x_i)+\pi_A^{\frac{2\alpha}{(2-\beta)}}(x_i)}\right)\end{aligned}\right]
\end{aligned}
$$

Then (1) holds.

4 Decision-Making Method Based on Proposed Parametric Directed Divergence Measure for Pythagorean Fuzzy Set (PFS)

In this section, we shall investigate the decision-making problem based on Proposed Parametric Directed Divergence Measure D_{α}^{β} in which the attribute values are evaluated by the expert which give their preferences in terms of Pythagorean fuzzy numbers PFNs. Assume that a set of "m" alternatives $A = \{A_1, A_2, \ldots, A_m\}$ to be considered under the set of "n" criterion $G = \{G_1, G_2, \ldots G_n\}$. Experts have evaluated these "m" alternatives under each criterion and give their rating value in the form of IFNs. Then, we have the following steps for computing the best alternative(s) based on the proposed measure.

Step 1: ***Construction of decision-making matrix***: Suppose $D_{m\times n}(x_{ij}) = \langle \mu_{ij}, \nu_{ij}\rangle$ be the intuitionistic fuzzy decision matrix, where μ_{ij} represents the degree that the alternative A_i satisfies the criteria G_j and ν_{ij} indicates the degree that the alternative A_i does not satisfy the criteria G_j given by the decision-maker such that $\mu_{ij}, \nu_{ij} \in [0, 1]$ with $\mu_{ij} + \nu_{ij} \leq 1, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n$. So, the intuitionistic fuzzy decision matrix is constructed as follows:

$$D_{m\times n}(x_{ij}) = \begin{bmatrix} \langle \mu_{11}, \nu_{11}\rangle & \langle \mu_{12}, \nu_{12}\rangle & \cdots & \langle \mu_{1n}, \nu_{1n}\rangle \\ \langle \mu_{21}, \nu_{21}\rangle & \langle \mu_{22}, \nu_{22}\rangle & \cdots & \langle \mu_{2n}, \nu_{2n}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mu_{m1}, \nu_{m1}\rangle & \langle \mu_{m2}, \nu_{m2}\rangle & \cdots & \langle \mu_{mn}, \nu_{mn}\rangle \end{bmatrix}.$$

Step 2: ***Compute the ideal alternatives***: Ideal alternative is denoted as A^* and given as

$$A^* = \left\{\langle \mu_1^*,\ \nu_1^*\rangle,\ \langle \mu_2^*,\ \nu_2^*\rangle,\ \ldots,\ \langle \mu_n^*,\ \nu_n^*\rangle\right\},$$

where $\mu_j^* = \max_{i=1}^{m}(\mu_{ij})$ and $\nu_j^* = \min_{i=1}^{m}(\nu_{ij})$.

Step 3: ***Evaluation of proposed Symmetric Divergence Measure***: Now we calculate $D_\alpha^\beta(A_i;\ A^*)$, $i = 1, 2, \ldots, m$ by the given formula:

$$\begin{aligned} &D_\alpha^\beta(A_i; A^*) \\ &= \frac{\alpha}{n(2-\beta)}\sum_{j=1}^{n}\left[\begin{aligned} &\mu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})\log\left(\frac{2\mu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\mu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \mu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right) + \nu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})\log\left(\frac{2\nu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\nu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \nu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right) \\ &\qquad + \pi_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})\log\left(\frac{2\pi_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\pi_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \pi_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right)\end{aligned}\right] \\ &+ \frac{\alpha}{n(2-\beta)}\sum_{j=1}^{n}\left[\begin{aligned} &\mu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})\log\left(\frac{2\mu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\mu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \mu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right) + \nu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})\log\left(\frac{2\nu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\nu_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \nu_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right) \\ &\qquad + \pi_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})\log\left(\frac{2\pi_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}{\pi_{j^*}^{\frac{2\alpha}{(2-\beta)}}(x_{ij}) + \pi_{ij}^{\frac{2\alpha}{(2-\beta)}}(x_{ij})}\right)\end{aligned}\right]. \end{aligned}$$

Step 4: ***Ranking the alternative***: Rank all the alternative according to indexing as obtained from $k = \arg \min_{1\le i\le m}\{D_\alpha^\beta(A_i; A^*)\}$.

5 Illustrative Example

In this section, one illustrative example from the field of decision-making has been taken for demonstrating the proposed approach.

Example: Decision-Making Problem. Consider the field of investment, where a person wants to invest some sort of money. As in these days, more and more companies have attracted the customers by reducing price and giving some other kind of benefits, so it is difficult for the investor to choose the best market for investment. In order to avoid the risk factor in the market and to make the decision more clear, they constitute a committee to invest the money in five major companies, namely, retail, food, computer, petrochemical, and a car company, respectively, denoted by $A_1,\ A_2,\ A_3,\ A_4,\ A_5$. Experts have been hired who gave their preferences of each alternative under the set of four major analyses, namely, the growth (G_1), the risk (G_2), the social-political impact (G_3) and the environmental impact (G_4). The rating value of each alternative $A_i = (i = 1, 2, \ldots, 5)$ under each factor has been assessed in terms of PFNs $\alpha_{ij} = \langle \mu_{ij}, \nu_{ij} \rangle_{5\times 4}$ and is summarized as follows.

$$D = \begin{bmatrix} <0.5,\ 0.4> & <0.6,\ 0.3> & <0.3,\ 0.6> & <0.2,\ 0.7> \\ <0.7,\ 0.3> & <0.7,\ 0.2> & <0.7,\ 0.2> & <0.4,\ 0.5> \\ <0.6,\ 0.4> & <0.5,\ 0.4> & <0.5,\ 0.3> & <0.6,\ 0.3> \\ <0.8,\ 0.1> & <0.6,\ 0.3> & <0.3,\ 0.4> & <0.2,\ 0.6> \\ <0.6,\ 0.2> & <0.4,\ 0.3> & <0.7,\ 0.1> & <0.5,\ 0.3> \end{bmatrix}.$$

By using these normalized data, the ideal value for all the criteria is given by

$$A^* = \{<0.8,\ 0.1>,\ <0.7,\ 0.2>,\ <0.7,\ 0.1>,\ <0.6,\ 0.3>\}.$$

Thus, based on it the directed divergence measure from ideal alternative to each alternative is computed by taking $\alpha = 1,\ \beta = 0.5$ and their corresponding measures are summarized as follows:

$$\begin{array}{l} D_\alpha^\beta(A_1; A^*) = 0.1289; \quad D_\alpha^\beta(A_2; A^*) = 0.0228; \quad D_\alpha^\beta(A_3; A^*) = 0.0468; \\ D_\alpha^\beta(A_4; A^*) = 0.0742; \quad D_\alpha^\beta(A_5; A^*) = 0.0258 \end{array}.$$

So, the ranking order of these alternatives is

$$A_2 \succ A_5 \succ A_3 \succ A_4 \succ A_1.$$

Hence, Food company is the best one for investment point of view.

6 Conclusions

Here, a parametric directed divergence measure of order α and degree β under the environment of Pythagorean fuzzy sets (PFSs) has been explored. We also discussed some desirable properties of this measure. For demonstration, a decision-making problem (investment problem) has been solved by using this technique. The parameters of this measure provide the flexibility to the decision-makers and that thing makes it more generalized. Thus, we conclude that the proposed divergence measure is suitable to solve several real-life problems and can be found as an alternative one among the various approaches to solve the decision-making problems. In future, we will be dealing with some more complicated problems or more realistic problems in the field of fuzzy cluster analysis, medical diagnosis, etc.

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Some Trigonometric Similarity Measures Based on the Choquet Integral for Pythagorean Fuzzy Sets and Applications to Pattern Recognition

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1 Introduction

The notion of the fuzzy set was presented by Zadeh [35] via a membership function and it was expanded to the notion of intuitionistic fuzzy set (IFS) by Atanassov [1] via a membership function with a non-membership function such that the sum of these functions is less than or equal to one. However, data in real-world problems cannot always be represented by a fuzzy set or an IFS. For instance, if a decision-maker or expert uses the intuitionistic fuzzy environment to give their preferences with the membership degree 0.7 and the non-membership degree 0.4, then we see that the sum of these degrees is equal to 1.1 which is larger than 1 and so this case cannot be characterized with an IFS. Thus, these were expanded to some more useful notions to solve real-world problems. With this motivation, Yager [30] presented the notion of Pythagorean fuzzy set (PFS) that is represented by a membership function with a non-membership function such that the sum of squares of these functions is less than or equal to 1. For instance, in the example above, we obtain that $0.7^2 + 0.4^2 \leq 1$. Therefore, a PFS is more useful than an IFS as well as a fuzzy set while handling real-life applications with imprecision and uncertainty.

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_4

Some expansions of PFSs such as interval-valued Pythagorean fuzzy sets [21], complex Pythagorean fuzzy sets [25], and Pythagorean fuzzy linguistic sets [9, 13, 22] were improved and they were applied to some extensive fields. One of these extensive areas of application is the notion of similarity measure for PFSs, which is an effective tool to find the degree of similarity between two objects. There are several versions of similarity measures for PFSs that satisfy some certain conditions and they have various applications to some different fields such as pattern recognition and medical diagnosis. For example, Peng and Garg [23] presented the concept of multiparametric similarity measures for PFSs with multiple parameters. Nyugen et al. [20] introduced the concept of exponential similarity measures by considering the exponential function. Firozja et al. [4] proposed new similarity measures by using triangular conorms. Another version of similarity measures is a cosine similarity measure [28, 33]. There exist various studies that introduce versions of cosine similarity measures for PFSs and that give applications of these similarity measures on pattern recognition, medical diagnosis, decision-making, and face recognition systems [5, 28, 31].

The process of combining several numerical values into a single representative one is called aggregation, and a numerical function performing this process is called an aggregation function. This concept has various application areas such as artificial intelligence, operations research, economics and finance, pattern recognition and image processing, data fusion, multicriteria decision-making, automated reasoning, etc. (see, e.g., [15]). The arithmetic mean and the weighted mean are the most well-known aggregation operators. Moreover, a cosine similarity measure uses the arithmetic mean or the weighted arithmetic mean to aggregate the cosine values of the angle among conjugate components of the vector representations of two PFSs. Various researchers have studied and enhanced the theory of PFSs using aggregation operators. For example, Zeng et al. [36] developed some induced ordered weighted aggregation operator for PFSs. Wei and Lu [29] presented a power aggregation operator for PFS to solve decision-making problems. Recently, Garg [10, 12] presented some neutrality operation-based Pythagorean fuzzy geometric aggregation operators and then defined some novel Pythagorean fuzzy weighted, ordered weighted, and hybrid neutral averaging aggregation operators for Pythagorean fuzzy information, which can neutrally treat the membership and non-membership degrees. Moreover, Garg [8, 11] developed some new probabilistic aggregation operators with Pythagorean fuzzy information by using an ordered weighted average operator. All of these existing aggregation operators ignore the interaction between the elements. In this paper, to overcome this deficiency, we use the notion of Choquet integral with respect to a fuzzy measure that considers the interaction between elements.

Choquet integral is a non-linear continuous aggregation operator that uses non-additive measures. In 1953, Choquet [3] presented the notion of fuzzy measure (or capacity or non-additive measure) and Choquet integral. The notion of fuzzy measure permits assignment of "weights"on subsets of the universal set and it has been used in a wide range of fields as a common disciplinary method such as decision-making, pattern recognition, and medical diagnosis [7, 14, 16, 26, 27]. Actually, the Choquet integral is an extension of Lebesgue integral and a non-additive extension of the weighted arithmetic mean. Although a fuzzy integral has more complicated structure due to the lack of additivity in contrast to the additive integrals such as Lebesgue integral, use of a fuzzy measure and a fuzzy integral is more effective in the aggregation. In [19], it is shown that the Choquet integral performs significantly more orders than the weighted arithmetic mean and that the difference gets larger when the number of the elements of the set gets larger. Moreover, it has been proved in [18] that when the number of the element of the finite set increases, the probability of getting more optimal ranking in the Choquet integral increases compared to the weighted arithmetic mean. Actually, fuzzy measures and fuzzy integrals let us to take the preferences into account that are not contained in the weights in the weighted arithmetic mean [24]. In the literature, there are some studies that consider the Choquet integral as an aggregation function for some fuzzy sets (see, e.g., [6, 17]) and there are some studies that consider fuzzy measure theory to introduce a similarity measure for IFSs (see, e.g., [2, 32]). In this chapter, we use the Choquet integral to present some trigonometric similarity measures for PFSs instead of weighted arithmetic mean.

This chapter presents a synthesis which is an innovative tool by considering the Choquet integral to define a similarity measure for PFSs via some trigonometric functions inspired by the definition of the weighted cosine similarity measure for PFSs. The remainder of this chapter is organized as follows: in Sect. 2, the notion of PFS is recalled and the existing similarity measures for PFS are given. In Sect. 3, we recall the notions of fuzzy measure and the Choquet integral. Then we propose ten trigonometric similarity measures for PFSs via the Choquet integral. In Sect. 4, we compare the proposed similarity measures with some existing similarity measures for PFSs and to express the effectiveness of proposed similarity measures, we apply them on some pattern recognition and medical diagnosis problems.

2 Preliminaries

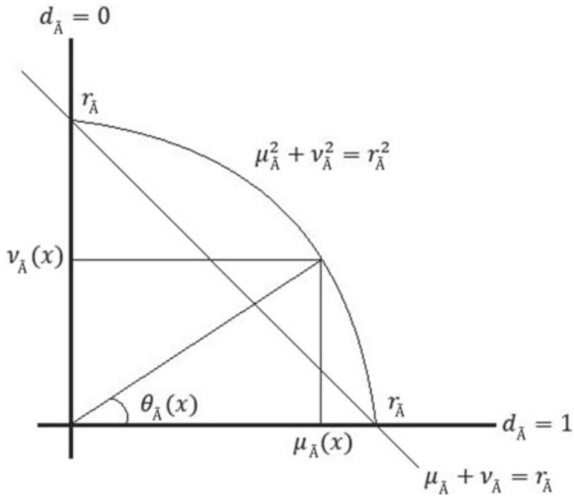
The notion of PFS was defined by Yager [30] to model real-life problems including imprecision, uncertainty, and vagueness situations more precisely with higher accuracy. In this section, we start with recalling the notion of PFS and some existing trigonometric similarity measures for PFSs.

Definition 1 Let $\mathbb{U} = \{\xi_1, \xi_2, \ldots, \xi_n\}$ be a finite set. A PFS $\tilde{A}$ in $\mathbb{U}$ is given with

$$\tilde{A} = \left\{ < \xi, \mu_{\tilde{A}}(\xi), \nu_{\tilde{A}}(\xi) > |\xi \in \mathbb{U} \right\}, \tag{1}$$

where $\mu_{\tilde{A}}$ and $\nu_{\tilde{A}}$ are functions from $\mathbb{U}$ to [0, 1] with the condition $\mu^2_{\tilde{A}}(\xi) + \nu^2_{\tilde{A}}(\xi) \leq 1$, for any $\xi \in X$. The numbers $\mu_{\tilde{A}}(\xi)$ and $\nu_{\tilde{A}}(\xi)$ indicate the membership degree and the non-membership degree of the element ξ to the set $\tilde{A}$, respectively [30].

Fig. 1 Comparison of IFSs and PFSs



For any PFS $\tilde{A}$ of $\mathbb{U}$, $\pi_{\tilde{A}}(\xi) = (1 - \mu_{\tilde{A}}^2(\xi) - \nu_{\tilde{A}}^2(\xi))^{1/2}$ is called the hesitancy degree of ξ to $\tilde{A}$, for each $\xi \in \mathbb{U}$ [30]. It is obvious that $0 \le \pi_{\tilde{A}}(\xi) \le 1$.

If $(r_{\tilde{A}}(\xi), \theta_{\tilde{A}}(\xi))$ is the polar coordinates of $(\mu_{\tilde{A}}(\xi), \nu_{\tilde{A}}(\xi))$ for a point $\xi \in \mathbb{U}$, then the function $d_{\tilde{A}} : X \to [0, 1]$ can be considered the direction of commitment at point ξ where $d_{\tilde{A}}(\xi) := (1 - \theta_{\tilde{A}}(\xi))\frac{\pi}{2}$ [30]. The function $d_{\tilde{A}}$ scales the first quadrant between zero and one, i.e., if $\theta_{\tilde{A}}(\xi) = \frac{\pi}{2}$ then $\mu_{\tilde{A}}(\xi) = 0$ and $\nu_{\tilde{A}}(\xi) = r_{\tilde{A}}(\xi)$ which means the direction $d_{\tilde{A}}(\xi) = 0$ and if $\theta_{\tilde{A}}(\xi) = 0$ then $\mu_{\tilde{A}}(\xi) = r_{\tilde{A}}(\xi)$ and $\nu_{\tilde{A}}(\xi) = 0$ which means the direction $d_{\tilde{A}}(\xi) = 1$. Therefore, a PFS $\tilde{A}$ can be expressed by either $(\mu_{\tilde{A}}, \nu_{\tilde{A}})$ or $(r_{\tilde{A}}, d_{\tilde{A}})$ (see Fig. 1).

Now, we examine some similarity measures for PFSs. Peng and Garg [23] proposed two similarity measures for PFSs by using relation of similarity measures with distance measures.

Let $\tilde{A} = \left\{< \xi, \mu_{\tilde{A}}(\xi), \nu_{\tilde{A}}(\xi) > | \xi \in \mathbb{U}\right\}$ and $\tilde{B} = \left\{< \xi, \mu_{\tilde{B}}(\xi), \nu_{\tilde{B}}(\xi) > | \xi \in \mathbb{U}\right\}$ be two PFSs in a finite set $\mathbb{U} = \{\xi_1, \xi_2, \ldots, \xi_n\}$. Two similarity measures between $\tilde{A}$ and $\tilde{B}$ are given with

$$S_1(\tilde{A}, \tilde{B}) := 1 - \left\{ \frac{1}{2nt_k^p} \sum_{i=1}^{n} \left(\begin{array}{l} \left|(t_k - 1)(\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)) - (\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i))\right|^p \\ + \left|(t_k - k)(\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)) - k(\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i))\right|^p \end{array} \right) \right\}^{\frac{1}{p}} \tag{2}$$

and

$$S_2(\tilde{A}, \tilde{B}) := 1 - \left\{ \frac{1}{nt_k^p} \sum_{i=1}^{n} \max \begin{pmatrix} \left|(t_k - 1)(\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)) - (\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i))\right|^p, \\ \left|(t_k - k)(\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)) - k(\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i))\right|^p \end{pmatrix} \right\}^{\frac{1}{p}}, \tag{3}$$

where $k \geq 0$, $p \geq 1$, and t_k are parameters such that $t_k \geq k + 1$.

Following similarity measures are the weighted versions of the similarity measures recalled in (2)–(3):

$$S_1^{\omega}(\tilde{A}, \tilde{B}) := 1 - \left\{ \frac{1}{2t_k^p} \sum_{i=1}^{n} \omega_i \begin{pmatrix} \left|(t_k - 1)(\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)) - (\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i))\right|^p \\ + \left|(t_k - k)(\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)) - k(\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i))\right|^p \end{pmatrix} \right\}^{\frac{1}{p}} \tag{4}$$

and

$$S_2^{\omega}(\tilde{A}, \tilde{B}) := 1 - \left\{ \frac{1}{t_k^p} \sum_{i=1}^{n} \omega_i \max \begin{pmatrix} \left|(t_k - 1)(\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)) - (\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i))\right|^p, \\ \left|(t_k - k)(\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)) - k(\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i))\right|^p \end{pmatrix} \right\}^{\frac{1}{p}}, \tag{5}$$

where $\omega_i \in [0, 1]$ for each $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \omega_i = 1$.

Effectiveness of these similarity measures was illustrated with some case studies of pattern recognition in [23] and the decision-maker tried to explain the effect of variables on the samples using the 1–8 scale for k and the 1–9 scale for p and t_k. That is, by limiting the parameters, the decision-maker studied the examples under special choices. Moreover, when $t_k = 1, k = 0$ and $\tilde{A}$ and $\tilde{B}$ two PFSs such that $\nu_{\tilde{A}}^2(\xi_i) = \nu_{\tilde{B}}^2(\xi_i)$ for any $i = 1, 2, \ldots, n$ and we obtain that $S_1(\tilde{A}, \tilde{B}) = S_2(\tilde{A}, \tilde{B}) = 1$ for any $p \geq 1$. As a result, the μ membership function and p parameter lose their importance and information carried by μ becomes insignificant and so it is neglected. Moreover, a contradiction emerges when the similarity of different PFSs is equal to one. The main reason of this contradiction is the use of relation of similarity measure with distance measure to obtain similarity measure for PFSs.

Nyugen et al. [20] proposed three weighted exponential similarity measures for PFSs by using exponential function:

$$SM_0(\tilde{A}, \tilde{B}) := \sum_{i=1}^{n} \omega_i \; S_i^{\mu}(\tilde{A}, \tilde{B}) \; S_i^{\nu}(\tilde{A}, \tilde{B}), \tag{6}$$

$$SM_1(\tilde{A}, \tilde{B}) := \sum_{i=1}^{n} \omega_i \left(\frac{S_i^{\mu}(\tilde{A}, \tilde{B}) + S_i^{\nu}(\tilde{A}, \tilde{B})}{2} \right), \tag{7}$$

and

$$SM_p(\tilde{A}, \tilde{B}) := \sum_{i=1}^{n} \omega_i \left(\frac{\left((S_i^{\mu}(\tilde{A}, \tilde{B}))^p + (S_i^{\nu}(\tilde{A}, \tilde{B}))^p \right)^{\dfrac{1}{p}}}{2} \right), \text{ for all } p \in \{1, 2, \ldots\} \tag{8}$$

where $S_i^{\mu}(\tilde{A}, \tilde{B}) := e^{-\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right|}$ and $S_i^{\nu}(\tilde{A}, \tilde{B}) := e^{-\left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right|}$ and $\omega_i \in (0, 1]$ for each $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \omega_i = 1$.

These similarity measures were applied to some pattern recognition problems. Since the proposed similarity measures are given with the help of the exponential function, as the number of elements of the universal set and p variable increase, the effort of calculating the similarity increases.

Firozja et al. [4] presented similarity measures for PFSs via the notion of S-norm. They applied them some pattern recognition and medical diagnosis problems to show the effectiveness of proposed similarity measures. Based on S-norm three weighted similarity measures between $\tilde{A}$ and $\tilde{B}$ are given with

$$\mathbb{S}_1 := \sum_{i=1}^{n} \omega_i \sqrt{1 - \left\{ (\mu_{\tilde{A}}(\xi_i) - \mu_{\tilde{B}}(\xi_i))^2 \vee (\nu_{\tilde{A}}(\xi_i) - \nu_{\tilde{B}}(\xi_i))^2 \right\}}, \tag{9}$$

$$\mathbb{S}_2 := \sum_{i=1}^{n} \omega_i \sqrt{1 - \left\{ (\mu_{\tilde{A}}(\xi_i) - \mu_{\tilde{B}}(\xi_i))^2 + (\nu_{\tilde{A}}(\xi_i) - \nu_{\tilde{B}}(\xi_i))^2 \wedge 1 \right\}}, \tag{10}$$

$$\mathbb{S}_3 := \sum_{i=1}^{n} \omega_i \sqrt{1 - \left\{ (\mu_{\tilde{A}}(\xi_i) - \mu_{\tilde{B}}(\xi_i))^2 + (\nu_{\tilde{A}}(\xi_i) - \nu_{\tilde{B}}(\xi_i))^2 - (\mu_{\tilde{A}}(\xi_i) - \mu_{\tilde{B}}(\xi_i))^2 (\nu_{\tilde{A}}(\xi_i) - \nu_{\tilde{B}}(\xi_i))^2 \right\}}, \tag{11}$$

where "$\vee$" and "$\wedge$" denote the maximum operator and minimum operator, respectively. Moreover, $\omega_i \in [0, 1]$ for each $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \omega_i = 1$.

However, the proposed similarity measures do not evaluate the Pythagorean fuzzy information carried by μ and ν well enough in some cases. In [4], the distance between components is more important than the information carried by fuzzy values. As a result, the decision-making process becomes difficult because the similarities of two different Pythagorean fuzzy sets or values are equal.

Wei and Wei [28] proposed two similarity measures via cosine function among PFS $\tilde{A}$ and PFS $\tilde{B}$ by using the arithmetic mean:

$$PFC^1(\tilde{A}, \tilde{B}) := \frac{1}{n}\sum_{i=1}^{n}\frac{\mu_{\tilde{A}}^2(\xi_i)\mu_{\tilde{B}}^2(\xi_i) + \nu_{\tilde{A}}^2(\xi_i)\nu_{\tilde{B}}^2(\xi_i)}{\sqrt{\mu_{\tilde{A}}^4(\xi_i) + \nu_{\tilde{A}}^4(\xi_i)}\sqrt{\mu_{\tilde{B}}^4(\xi_i) + \nu_{\tilde{B}}^4(\xi_i)}}, \tag{12}$$

$$PFC^2(\tilde{A}, \tilde{B}) := \frac{1}{n}\sum_{i=1}^{n}\frac{\mu_{\tilde{A}}^2(\xi_i)\mu_{\tilde{B}}^2(\xi_i) + \nu_{\tilde{A}}^2(\xi_i)\nu_{\tilde{B}}^2(\xi_i) + \pi_{\tilde{A}}^2(\xi_i)\pi_{\tilde{B}}^2(\xi_i)}{\sqrt{\mu_{\tilde{A}}^4(\xi_i) + \nu_{\tilde{A}}^4(\xi_i) + \pi_{\tilde{A}}^4(\xi_i)}\sqrt{\mu_{\tilde{B}}^4(\xi_i) + \nu_{\tilde{B}}^4(\xi_i) + \pi_{\tilde{B}}^4(\xi_i)}}. \tag{13}$$

They also proposed four more similarity measures via cosine function among PFS $\tilde{A}$ and PFS $\tilde{B}$ by using the arithmetic mean:

$$PFCS^1(\tilde{A}, \tilde{B}) := \frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi}{2}\left(\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| \vee \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right|\right)\right], \tag{14}$$

$$PFCS^2(\tilde{A}, \tilde{B}) := \frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi}{4}\left(\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| + \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right|\right)\right], \tag{15}$$

$$PFCS^3(\tilde{A}, \tilde{B}) := \frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi}{2}\left(\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| \vee \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right| \vee \left|\pi_{\tilde{A}}^2(\xi_i) - \pi_{\tilde{B}}^2(\xi_i)\right|\right)\right], \tag{16}$$

$$PFCS^4(\tilde{A}, \tilde{B}) := \frac{1}{n}\sum_{i=1}^{n}\cos\left[\frac{\pi}{4}\left(\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| + \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right| + \left|\pi_{\tilde{A}}^2(\xi_i) - \pi_{\tilde{B}}^2(\xi_i)\right|\right)\right]. \tag{17}$$

Moreover, they proposed four similarity measures via cotangent function among PFS $\tilde{A}$ and PFS $\tilde{B}$ by using the arithmetic mean:

$$PFCT^1(\tilde{A}, \tilde{B}) := \frac{1}{n}\sum_{i=1}^{n}\cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| \vee \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right|\right)\right], \tag{18}$$

$$PFCT^2(\tilde{A}, \tilde{B}) := \frac{1}{n}\sum_{i=1}^{n}\cot\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| + \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right|\right)\right], \tag{19}$$

$$PFCT^3(\tilde{A},\tilde{B}) := \frac{1}{n}\sum_{i=1}^{n}\cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right|\vee\left|\nu^2_{\tilde{A}}(\xi_i)-\nu^2_{\tilde{B}}(\xi_i)\right|\vee\left|\pi^2_{\tilde{A}}(\xi_i)-\pi^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{20}$$

$$PFCT^4(\tilde{A},\tilde{B}) := \frac{1}{n}\sum_{i=1}^{n}\cot\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right|+\left|\nu^2_{\tilde{A}}(\xi_i)-\nu^2_{\tilde{B}}(\xi_i)\right|+\left|\pi^2_{\tilde{A}}(\xi_i)-\pi^2_{\tilde{B}}(\xi_i)\right|\right)\right]. \tag{21}$$

Following similarity measures [28] are the weighted versions of the similarity measures recalled in (12)–(21):

$$WPFC^1(\tilde{A},\tilde{B}) = \sum_{i=1}^{n}\omega_i\frac{\mu^2_{\tilde{A}}(\xi_i)\mu^2_{\tilde{B}}(\xi_i)+\nu^2_{\tilde{A}}(\xi_i)\nu^2_{\tilde{B}}(\xi_i)}{\sqrt{\mu^4_{\tilde{A}}(\xi_i)+\nu^4_{\tilde{A}}(\xi_i)}\sqrt{\mu^4_{\tilde{B}}(\xi_i)+\nu^4_{\tilde{B}}(\xi_i)}}, \tag{22}$$

$$WPFC^2(\tilde{A},\tilde{B}) := \sum_{i=1}^{n}\omega_i\frac{\mu^2_{\tilde{A}}(\xi_i)\mu^2_{\tilde{B}}(\xi_i)+\nu^2_{\tilde{A}}(\xi_i)\nu^2_{\tilde{B}}(\xi_i)+\pi^2_{\tilde{A}}(\xi_i)\pi^2_{\tilde{B}}(\xi_i)}{\sqrt{\mu^4_{\tilde{A}}(\xi_i)+\nu^4_{\tilde{A}}(\xi_i)+\pi^4_{\tilde{A}}(\xi_i)}\sqrt{\mu^4_{\tilde{B}}(\xi_i)+\nu^4_{\tilde{B}}(\xi_i)+\pi^4_{\tilde{B}}(\xi_i)}}, \tag{23}$$

$$WPFCS^1(\tilde{A},\tilde{B}) := \sum_{i=1}^{n}\omega_i\cos\left[\frac{\pi}{2}\left(\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right|\vee\left|\nu^2_{\tilde{A}}(\xi_i)-\nu^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{24}$$

$$WPFCS^2(\tilde{A},\tilde{B}) := \sum_{i=1}^{n}\omega_i\cos\left[\frac{\pi}{4}\left(\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right|+\left|\nu^2_{\tilde{A}}(\xi_i)-\nu^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{25}$$

$$WPFCS^3(\tilde{A},\tilde{B}) := \sum_{i=1}^{n}\omega_i\cos\left[\frac{\pi}{2}\left(\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right|\vee\left|\nu^2_{\tilde{A}}(\xi_i)-\nu^2_{\tilde{B}}(\xi_i)\right|\vee\left|\pi^2_{\tilde{A}}(\xi_i)-\pi^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{26}$$

$$WPFCS^4(\tilde{A},\tilde{B}) := \sum_{i=1}^{n}\omega_i\cos\left[\frac{\pi}{4}\left(\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right|+\left|\nu^2_{\tilde{A}}(\xi_i)-\nu^2_{\tilde{B}}(\xi_i)\right|+\left|\pi^2_{\tilde{A}}(\xi_i)-\pi^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{27}$$

$$WPFCT^1(\tilde{A}, \tilde{B}) := \sum_{i=1}^{n} \omega_i \cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| \vee \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right|\right)\right], \tag{28}$$

$$WPFCT^2(\tilde{A}, \tilde{B}) := \sum_{i=1}^{n} \omega_i \cot\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| + \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right|\right)\right], \tag{29}$$

$$WPFCT^3(\tilde{A}, \tilde{B}) := \sum_{i=1}^{n} \omega_i \cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| \vee \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right| \vee \left|\pi_{\tilde{A}}^2(\xi_i) - \pi_{\tilde{B}}^2(\xi_i)\right|\right)\right], \tag{30}$$

$$WPFCT^4(\tilde{A}, \tilde{B}) := \sum_{i=1}^{n} \omega_i \cot\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| + \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right| + \left|\pi_{\tilde{A}}^2(\xi_i) - \pi_{\tilde{B}}^2(\xi_i)\right|\right)\right], \tag{31}$$

where $0 \leq \omega_1, \omega_2, \ldots, \omega_n \leq 1$ with $\sum_{i=1}^{n} \omega_i = 1$.

3 Trigonometric Similarity Measures Defined with the Choquet Integral For PFSs

As we mentioned before, the notion of the Choquet integral is a non-additive extension of the notion of weighted arithmetic mean. In this study, we consider the Choquet integral and cosine and cotangent functions to construct ten new similarity measures motivating from the trigonometric similarity measures (22)–(31) defined by [28].

First of all, let us recall some basic notions of fuzzy measure theory that are used in this section.

Definition 2 Let $\mathbb{U} \neq \varnothing$ be a finite set and let $P(\mathbb{U})$ be the family of all subsets of $\mathbb{U}$. If

(i) $\sigma(\varnothing) = 0$,
(ii) $\sigma(\mathbb{U}) = 1$,
(iii) $A \subseteq B$ implies $\sigma(A) \leq \sigma(B)$ (monotonicity), then the set function $\sigma : P(\mathbb{U}) \to [0, 1]$ is called a fuzzy measure on $\mathbb{U}$ [3].

Note that, a fuzzy measure need not to be additive.

Definition 3 Let $\mathbb{U} = \{\xi_1, \xi_2, \dots, \xi_n\}$ be a finite set and let σ be a fuzzy measure on $\mathbb{U}$. The Choquet integral [3] of a function $f : \mathbb{U} \to [0, 1]$ with respect to σ is defined by

$$(C)\int_{\mathbb{U}} f\, d\sigma := \sum_{k=1}^{n} \left(f(\xi_{(k)}) - f(\xi_{(k-1)})\right) \sigma(E_{(k)}), \tag{32}$$

where the sequence $\left\{\xi_{(k)}\right\}_{k=0}^{n}$ is a new permutation of the sequence $\{\xi_k\}_{k=0}^{n}$ such that $0 := f(\xi_{(0)}) \le f(\xi_{(1)}) \le f(\xi_{(2)}) \le \cdots \le f(\xi_{(n)})$ and $E_{(k)} := \left\{\xi_{(k)}, \xi_{(k+1)}, \dots, \xi_{(n)}\right\}$.

If σ is an additive measure then the Choquet integral reduces to weighted arithmetic mean.

Throughout this section we let $\mathbb{U} = \{\xi_1, \xi_2, \dots, \xi_n\}$ be a finite set, $\tilde{A}$ and $\tilde{B}$ be two PFSs in $\mathbb{U}$ and σ be a fuzzy measure on $\mathbb{U}$.

Definition 4 A Choquet cosine similarity measure among PFS $\tilde{A}$ and PFS $\tilde{B}$ is given with

$$W_{PFC^1}^{(C,\sigma)}(\tilde{A}, \tilde{B}) := (C)\int_{\mathbb{U}} f_{\tilde{A},\tilde{B}}(\xi)\, d\sigma(\xi), \tag{33}$$

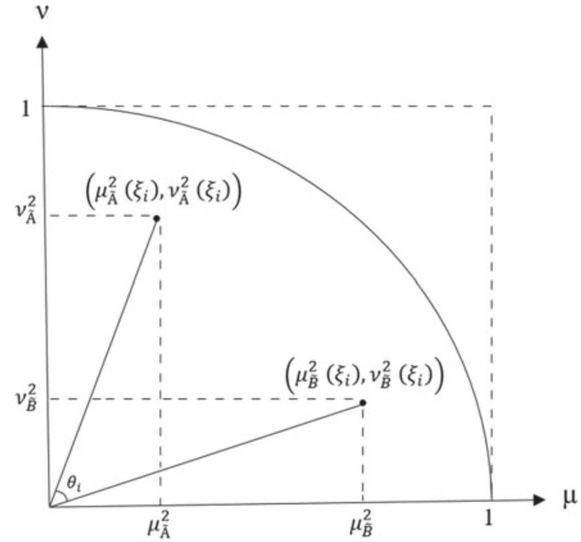
where

$$f_{\tilde{A},\tilde{B}}(\xi_i) := \frac{\mu_{\tilde{A}}^2(\xi_i)\mu_{\tilde{B}}^2(\xi_i) + \nu_{\tilde{A}}^2(\xi_i)\nu_{\tilde{B}}^2(\xi_i)}{\sqrt{\mu_{\tilde{A}}^4(\xi_i) + \nu_{\tilde{A}}^4(\xi_i)}\sqrt{\mu_{\tilde{B}}^4(\xi_i) + \nu_{\tilde{B}}^4(\xi_i)}}, \tag{34}$$

for $i = 1, 2, \dots, n$.

In Fig. 2, we see that $\cos\theta_i = f_{\tilde{A},\tilde{B}}(\xi_i)$ for any $i = 1, 2, \dots, n$.

Fig. 2 The angle between vector representations



Proposition 1 *The cosine similarity measure* $W^{(C,\sigma)}_{PFC^1}$ *satisfies the following properties:*
(P₁) $0 \leq W^{(C,\sigma)}_{PFC^1}(\tilde{A}, \tilde{B}) \leq 1$

(P₂) $W^{(C,\sigma)}_{PFC^1}(\tilde{A}, \tilde{B}) = W^{(C,\sigma)}_{PFC^1}(\tilde{B}, \tilde{A})$

(P₃) *If* $\tilde{A} = \tilde{B}$ *then* $W^{(C,\sigma)}_{PFC^1}(\tilde{A}, \tilde{B}) = 1$.

Proof **(P_1)** Since $f_{\tilde{A},\tilde{B}}(\xi_i) \in [0, 1]$ for any $i = 1, 2, \ldots, n$ and the Choquet integral is monotone we have $0 \leq W^{(C,\sigma)}_{PFC^1}(\tilde{A}, \tilde{B}) \leq 1$ immediately.
(P_2) It is trivial since $f_{\tilde{A},\tilde{B}}(\xi_i) = f_{\tilde{B},\tilde{A}}(\xi_i)$ for any $i = 1, 2, \ldots, n$
(P_3) If $\tilde{A} = \tilde{B}$ then we have $\mu_{\tilde{A}}(\xi_i) = \mu_{\tilde{B}}(\xi_i)$ and $\nu_{\tilde{A}}(\xi_i) = \nu_{\tilde{B}}(\xi_i)$, for $i = 1, 2, \ldots, n$, which yields that $f_{\tilde{A},\tilde{B}}(\xi_i) = 1$. Hence, we obtain $W^{(C,\sigma)}_{tPFC^1}(\tilde{A}, \tilde{B}) = 1$. □

Considering the functions μ, ν, and π we propose another similarity measure via cosine function and the Choquet integral.

Definition 5 A Choquet cosine similarity measure among PFS $\tilde{A}$ and PFS $\tilde{B}$ is given with

$$W^{(C,\sigma)}_{PFC^2}(\tilde{A}, \tilde{B}) := (C) \int_{\mathbb{U}} g_{\tilde{A},\tilde{B}}(\xi)\, d\sigma(\xi), \tag{35}$$

where

$$g_{\tilde{A},\tilde{B}}(\xi_i) := \frac{\mu^2_{\tilde{A}}(\xi_i)\mu^2_{\tilde{B}}(\xi_i) + \nu^2_{\tilde{A}}(\xi_i)\nu^2_{\tilde{B}}(\xi_i) + \pi^2_{\tilde{A}}(\xi_i)\pi^2_{\tilde{B}}(\xi_i)}{\sqrt{\mu^4_{\tilde{A}}(\xi_i) + \nu^4_{\tilde{A}}(\xi_i) + \pi^4_{\tilde{A}}(\xi_i)}\sqrt{\mu^4_{\tilde{B}}(\xi_i) + \nu^4_{\tilde{B}}(\xi_i) + \pi^4_{\tilde{B}}(\xi_i)}}, \tag{36}$$

for $i = 1, 2, \ldots, n$.

Remark 1 Similar to the proof of Proposition 1 it can be proved that $W^{(C,\sigma)}_{PFC^2}$ satisfies the conditions $P_1 - P_3$.

Definition 6 Considering the cosine function and the Choquet integral two new similarity measures among PFS $\tilde{A}$ and PFS $\tilde{B}$ are given with

$$W^{(C,\sigma)}_{PFCS^1}(\tilde{A}, \tilde{B}) := (C) \int_{\mathbb{U}} h^{(1)}_{\tilde{A},\tilde{B}}(\xi)\, d\sigma(\xi), \tag{37}$$

$$W^{(C,\sigma)}_{PFCS^2}(\tilde{A}, \tilde{B}) := (C) \int_{\mathbb{U}} h^{(2)}_{\tilde{A},\tilde{B}}(\xi)\, d\sigma(\xi), \tag{38}$$

where

$$h^{(1)}_{\tilde{A},\tilde{B}}(\xi_i) := \cos\left[\frac{\pi}{2}\left(\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{B}}(\xi_i)\right| \vee \left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{39}$$

and

$$h^{(2)}_{\tilde{A},\tilde{B}}(\xi_i) := \cos\left[\frac{\pi}{4}\left(\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{B}}(\xi_i)\right| + \left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{40}$$

for $i = 1, 2, \ldots, n$, respectively.

Proposition 2 *The cosine similarity measure $W^{(C,\sigma)}_{PFCS^k}$ satisfies P_1, P_2 and the following properties:*
($\mathbf{P_3'}$) *$\tilde{A} = \tilde{B}$ if and only if $W^{(C,\sigma)}_{PFCS^k}(\tilde{A}, \tilde{B}) = 1$.*

($\mathbf{P_4}$) *If $\tilde{C}$ is an PFS in $\mathbb{U}$ and $\tilde{A} \subset \tilde{B} \subset \tilde{C}$ then $W^{(C,\sigma)}_{PFCS^k}(\tilde{A}, \tilde{C}) \le W^{(C,\sigma)}_{PFCS^k}(\tilde{A}, \tilde{B})$ and $W^{(C,\sigma)}_{PFCS^k}(\tilde{A}, \tilde{C}) \le W^{(C,\sigma)}_{PFCS^k}(\tilde{B}, \tilde{C})$.*

Proof P_1 and P_2 can be proved similar to Proposition 1.
($\mathbf{P_3'}$) For any two PFSs $\tilde{A}$ and $\tilde{B}$ in $\mathbb{U}$, if $\tilde{A}=\tilde{B}$, this implies $\mu^2_{\tilde{A}}(\xi_i)=\mu^2_{\tilde{B}}(\xi_i)$ and $\nu^2_{\tilde{A}}(\xi_i)=\nu^2_{\tilde{B}}(\xi_i)$, for $i=1, 2, \ldots, n$. Thus, $\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right|=0$ and $\left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{B}}(\xi_i)\right|=0$. Therefore, we have $h^{(k)}_{\tilde{A},\tilde{B}}(\xi_i) = 1$ and so $W^{(C,\sigma)}_{PFC^k}(\tilde{A}, \tilde{B}) = 1$ for $k = 1, 2$.
Conversely, let $W^{(C,\sigma)}_{PFCS^k}(\tilde{A}, \tilde{B}) = 1$ for $k = 1, 2$. Then, since $\cos 0 = 1$ we have $h^{(k)}_{\tilde{A},\tilde{B}}(\xi_i) = 1$ which yields that $\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{B}}(\xi_i)\right| = 0$ and $\left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{B}}(\xi_i)\right| = 0$, $i = 1, 2, \ldots, n$. Therefore, we obtain $\mu^2_{\tilde{A}}(\xi_i) = \mu^2_{\tilde{B}}(\xi_i)$ and $\nu^2_{\tilde{A}}(\xi_i) = \nu^2_{\tilde{B}}(\xi_i)$, for $i = 1, 2, \ldots, n$. Hence, $\tilde{A} = \tilde{B}$.
($\mathbf{P_4}$) If $\tilde{A} \subset \tilde{B} \subset \tilde{C}$ then $\mu_{\tilde{A}}(\xi_i) \le \mu_{\tilde{B}}(\xi_i) \le \mu_{\tilde{C}}(\xi_i)$ and $\nu_{\tilde{A}}(\xi_i) \ge \nu_{\tilde{B}}(\xi_i) \ge \nu_{\tilde{C}}(\xi_i)$, for $i = 1, 2, \ldots, n$. Then, $\mu^2_{\tilde{A}}(\xi_i) \le \mu^2_{\tilde{B}}(\xi_i) \le \mu^2_{\tilde{C}}(\xi_i)$ and $\nu^2_{\tilde{A}}(\xi_i) \ge \nu^2_{\tilde{B}}(\xi_i) \ge \nu^2_{\tilde{C}}(\xi_i)$, for $i = 1, 2, \ldots, n$. Thus, we have

$$\begin{aligned}
\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{B}}(\xi_i)\right| &\le \left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{C}}(\xi_i)\right| \\
\left|\mu^2_{\tilde{B}}(\xi_i) - \mu^2_{\tilde{C}}(\xi_i)\right| &\le \left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{C}}(\xi_i)\right| \\
\left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{B}}(\xi_i)\right| &\le \left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{C}}(\xi_i)\right| \\
\left|\nu^2_{\tilde{B}}(\xi_i) - \nu^2_{\tilde{C}}(\xi_i)\right| &\le \left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{C}}(\xi_i)\right|.
\end{aligned}$$

So, we obtain $h^{(k)}_{\tilde{A},\tilde{C}}(\xi_i) \le h^{(k)}_{\tilde{A},\tilde{B}}(\xi_i)$ and $h^{(k)}_{\tilde{A},\tilde{C}}(\xi_i) \le h^{(k)}_{\tilde{B},\tilde{C}}(\xi_i)$ which yields that $W^{(C,\sigma)}_{PFCS^k}(\tilde{A}, \tilde{C}) \le W^{(C,\sigma)}_{PFCS^k}(\tilde{A}, \tilde{B})$ and $W^{(C,\sigma)}_{PFCS^k}(\tilde{A}, \tilde{C}) \le W^{(C,\sigma)}_{PFCS^k}(\tilde{B}, \tilde{C})$, for $k = 1, 2$. Hence, the proof is completed.

□

Definition 7 Considering the cosine function and the Choquet integral two similarity measures among PFS $\tilde{A}$ and PFS $\tilde{B}$ are given with

$$W^{(C,\sigma)}_{PFCS^3}(\tilde{A}, \tilde{B}) := (C)\int_{\mathbb{U}} p^{(1)}_{\tilde{A},\tilde{B}}(\xi)\, d\sigma(\xi), \tag{41}$$

$$W^{(C,\sigma)}_{PFCS^4}(\tilde{A},\tilde{B}) := (C)\int\limits_{\mathbb{U}} p^{(2)}_{\tilde{A},\tilde{B}}(\xi)\,d\sigma(\xi), \tag{42}$$

where

$$p^{(1)}_{\tilde{A},\tilde{B}}(\xi_i) := \cos\left[\frac{\pi}{2}\left(\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right| \vee \left|\nu^2_{\tilde{A}}(\xi_i)-\nu^2_{\tilde{B}}(\xi_i)\right| \vee \left|\pi^2_{\tilde{A}}(\xi_i)-\pi^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{43}$$

and

$$p^{(2)}_{\tilde{A},\tilde{B}}(\xi_i) := \cos\left[\frac{\pi}{4}\left(\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right| + \left|\nu^2_{\tilde{A}}(\xi_i)-\nu^2_{\tilde{B}}(\xi_i)\right| + \left|\pi^2_{\tilde{A}}(\xi_i)-\pi^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{44}$$

for $i = 1, 2, \ldots, n$, respectively.

Remark 2 Similar to the proof of Proposition 2 it can be proved that $W^{(C,\sigma)}_{PFCS^3}$ and $W^{(C,\sigma)}_{PFCS^4}$ satisfy the conditions P_1, P_2, P'_3 and P_4.

Definition 8 Considering the cotangent function and the Choquet integral two similarity measures among PFS $\tilde{A}$ and PFS $\tilde{B}$ are given with

$$W^{(C,\sigma)}_{PFCT^1}(\tilde{A},\tilde{B}) := (C)\int\limits_{\mathbb{U}} r^{(1)}_{\tilde{A},\tilde{B}}(\xi)\,d\sigma(\xi), \tag{45}$$

$$W^{(C,\sigma)}_{PFCT^2}(\tilde{A},\tilde{B}) := (C)\int\limits_{\mathbb{U}} r^{(2)}_{\tilde{A},\tilde{B}}(\xi)\,d\sigma(\xi), \tag{46}$$

where

$$r^{(1)}_{\tilde{A},\tilde{B}}(\xi_i) := \cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right| \vee \left|\nu^2_{\tilde{A}}(\xi_i)-\nu^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{47}$$

and

$$r^{(2)}_{\tilde{A},\tilde{B}}(\xi_i) := \cot\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\left|\mu^2_{\tilde{A}}(\xi_i)-\mu^2_{\tilde{B}}(\xi_i)\right| + \left|\nu^2_{\tilde{A}}(\xi_i)-\nu^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{48}$$

for $i = 1, 2, \ldots, n$, respectively.

Definition 9 Considering the cotangent function and the Choquet integral two new similarity measures among PFS $\tilde{A}$ and PFS $\tilde{B}$ are given with

$$W^{(C,\sigma)}_{PFCT^3}(\tilde{A},\tilde{B}) := (C)\int\limits_{\mathbb{U}} q^{(1)}_{\tilde{A},\tilde{B}}(\xi)\,d\sigma(\xi), \tag{49}$$

$$W^{(C,\sigma)}_{PFCT^4}(\tilde{A},\tilde{B}) := (C)\int\limits_{\mathbb{U}} q^{(2)}_{\tilde{A},\tilde{B}}(\xi)\,d\sigma(\xi), \tag{50}$$

where

$$q^{(1)}_{\tilde{A},\tilde{B}}(\xi_i) := \cot\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{B}}(\xi_i)\right| \vee \left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{B}}(\xi_i)\right| \vee \left|\pi^2_{\tilde{A}}(\xi_i) - \pi^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{51}$$

and

$$q^{(2)}_{\tilde{A},\tilde{B}}(\xi_i) := \cot\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{B}}(\xi_i)\right| + \left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{B}}(\xi_i)\right| + \left|\pi^2_{\tilde{A}}(\xi_i) - \pi^2_{\tilde{B}}(\xi_i)\right|\right)\right], \tag{52}$$

for $i = 1, 2, \ldots, n$, respectively.

Proposition 3 *The cotangent similarity measure $W^{(C,\sigma)}_{PFCT^k}(\tilde{A}, \tilde{B})$, $(k = 1, 2, 3, 4)$ satisfies P_1, P_2, P'_3 and P_4.*

Proof P_1 and P_2 can be proved similar to Proposition 1.
($\mathbf{P'_3}$) For any two PFSs $\tilde{A}$ and $\tilde{B}$ in $\mathbb{U}$, if $\tilde{A} = \tilde{B}$, then we have $\mu^2_{\tilde{A}}(\xi_i) = \mu^2_{\tilde{B}}(\xi_i)$ and $\nu^2_{\tilde{A}}(\xi_i) = \nu^2_{\tilde{B}}(\xi_i)$, for $i = 1, 2, \ldots, n$. Thus, we obtain $\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{B}}(\xi_i)\right| = 0$ and $\left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{B}}(\xi_i)\right| = 0$ which implies $q^{(k)}_{\tilde{A},\tilde{B}}(\xi_i) = 1$ and so $W^{(C,\sigma)}_{PFCT^k}(\tilde{A}, \tilde{B}) = 1$ for $k = 1, 2, 3, 4$. Conversely, if $W^{(C,\sigma)}_{PFCT^k}(\tilde{A}, \tilde{B}) = 1$ for $k = 1, 2, 3, 4$, then since $\cot\dfrac{\pi}{4} = 1$ we have $q^{(k)}_{\tilde{A},\tilde{B}}(\xi_i) = 1$ and so $\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{B}}(\xi_i)\right| = 0$ and $\left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{B}}(\xi_i)\right| = 0$, $i = 1, 2, \ldots, n$. Thus, we obtain $\mu^2_{\tilde{A}}(\xi_i) = \mu^2_{\tilde{B}}(\xi_i)$ and $\nu^2_{\tilde{A}}(\xi_i) = \nu^2_{\tilde{B}}(\xi_i)$, for $i = 1, 2, \ldots, n$ which yields that $\tilde{A} = \tilde{B}$.
($\mathbf{P_4}$) If $\tilde{A} \subset \tilde{B} \subset \tilde{C}$ then $\mu_{\tilde{A}}(\xi_i) \leq \mu_{\tilde{B}}(\xi_i) \leq \mu_{\tilde{C}}(\xi_i)$ and $\nu_{\tilde{A}}(\xi_i) \geq \nu_{\tilde{B}}(\xi_i) \geq \nu_{\tilde{C}}(\xi_i)$, for $i = 1, 2, \ldots, n$. Then, $\mu^2_{\tilde{A}}(\xi_i) \leq \mu^2_{\tilde{B}}(\xi_i) \leq \mu^2_{\tilde{C}}(\xi_i)$ and $\nu^2_{\tilde{A}}(\xi_i) \geq \nu^2_{\tilde{B}}(\xi_i) \geq \nu^2_{\tilde{C}}(\xi_i)$, for $i = 1, 2, \ldots, n$.
Case 1 : Let $\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{C}}(\xi_i)\right| \geq \left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{C}}(\xi_i)\right|$. Then from the assumption of non-membership functions, we have

$$\begin{aligned}
\left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{B}}(\xi_i)\right| &\leq \left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{C}}(\xi_i)\right| \leq \left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{C}}(\xi_i)\right| \\
\left|\nu^2_{\tilde{B}}(\xi_i) - \nu^2_{\tilde{C}}(\xi_i)\right| &\leq \left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{C}}(\xi_i)\right| \leq \left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{C}}(\xi_i)\right|.
\end{aligned}$$

On the other hand, from the assumption of the membership functions, we have

$$\begin{aligned}
\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{B}}(\xi_i)\right| &\leq \left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{C}}(\xi_i)\right| \\
\left|\mu^2_{\tilde{B}}(\xi_i) - \mu^2_{\tilde{C}}(\xi_i)\right| &\leq \left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{C}}(\xi_i)\right|
\end{aligned}$$

Case 2 : Let $\left|\mu^2_{\tilde{A}}(\xi_i) - \mu^2_{\tilde{C}}(\xi_i)\right| \leq \left|\nu^2_{\tilde{A}}(\xi_i) - \nu^2_{\tilde{C}}(\xi_i)\right|$. Then from the assumption of membership functions, we have

$$\left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{B}}^2(\xi_i)\right| \le \left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{C}}^2(\xi_i)\right| \le \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{C}}^2(\xi_i)\right|$$
$$\left|\mu_{\tilde{B}}^2(\xi_i) - \mu_{\tilde{C}}^2(\xi_i)\right| \le \left|\mu_{\tilde{A}}^2(\xi_i) - \mu_{\tilde{C}}^2(\xi_i)\right| \le \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{C}}^2(\xi_i)\right|.$$

Furthermore, from the assumption of the non-membership functions, we have

$$\left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{B}}^2(\xi_i)\right| \le \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{C}}^2(\xi_i)\right|$$
$$\left|\nu_{\tilde{B}}^2(\xi_i) - \nu_{\tilde{C}}^2(\xi_i)\right| \le \left|\nu_{\tilde{A}}^2(\xi_i) - \nu_{\tilde{C}}^2(\xi_i)\right|.$$

Both in Case 1 and Case 2, we have $q_{\tilde{A},\tilde{C}}^{(k)}(\xi_i) \le q_{\tilde{A},\tilde{B}}^{(k)}(\xi_i)$ and $q_{\tilde{A},\tilde{C}}^{(k)}(\xi_i) \le q_{\tilde{B},\tilde{C}}^{(k)}(\xi_i)$ which implies that $W_{PFCT^k}^{(C,\sigma)}(\tilde{A},\tilde{C}) \le W_{PFCT^k}^{(C,\sigma)}(\tilde{A},\tilde{B})$ and $W_{PFCT^k}^{(C,\sigma)}(\tilde{A},\tilde{C}) \le W_{PFCT^k}^{(C,\sigma)}(\tilde{B},\tilde{C})$, for $k = 1, 2, 3, 4$. Thus, the proof is completed. □

Remark 3 If we consider additive measures instead of fuzzy measures, then the similarity measures proposed in Definitions 4–9 are reduced to (22)–(31), respectively. On the other hand, considering the weights as measures of singletons we conclude that trigonometric similarity measures (22)–(31) are considered as trigonometric similarity measures based on the Choquet integral given in Definitions 4–9, respectively.

4 Applications

In this section, to show the effectiveness of the proposed Choquet similarity measures, we give some applications on pattern recognition and medical diagnosis problems.

4.1 Pattern Recognition Problem

A pattern recognition problem investigates how an object is coherent with a given pattern. We apply the proposed Choquet similarity measures to show their outperforming and suitability in pattern recognition. We consider a pattern recognition problem which is studied in [5, 28]. Wei and Wei [28] obtained that pattern $\tilde{A}$ belongs to class $\tilde{A}_3$, using similarity measures between (12) and (21) (Table 1 of [28]).

Example 1 Let $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ be three patterns which are represented by using following PFSs of a finite set $\mathbb{U} = \{\xi_1, \xi_2, \xi_3\}$:

Table 1 Fuzzy measure

$\sigma(\varnothing) = 0$	$\sigma(\{\xi_1\}) = 0.5$	$\sigma(\{\xi_2\}) = 0.3$
$\sigma(\{\xi_3\}) = 0.2$	$\sigma(\{\xi_1, \xi_2\}) = 0.7$	$\sigma(\{\xi_1, \xi_3\}) = 0.8$
$\sigma(\{\xi_2, \xi_3\}) = 0.6$	$\sigma(\{\xi_1, \xi_2, \xi_3\}) = 1$	

$$\begin{aligned}
\tilde{A}_1 &= \{\langle \xi_1, 1, 0\rangle, \langle \xi_2, 0.8, 0\rangle, \langle \xi_3, 0.7, 0.1\rangle\} \\
\tilde{A}_2 &= \{\langle \xi_1, 0.8, 0.1\rangle, \langle \xi_2, 1, 0\rangle, \langle \xi_3, 0.9, 0.1\rangle\} \\
\tilde{A}_3 &= \{\langle \xi_1, 0.6, 0.2\rangle, \langle \xi_2, 0.8, 0\rangle, \langle \xi_3, 1, 0\rangle\}
\end{aligned}$$

Let $\tilde{A} = \{\langle \xi_1, 0.5, 0.3\rangle, \langle \xi_2, 0.6, 0.2\rangle, \langle \xi_3, 0.8, 0.1\rangle\}$ be a pattern that needs to be classified in one of three classes $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$.

We use a hypothetical fuzzy measure. For this purpose, we use the hypothetical weights used in [28] as the fuzzy measures of singletons and we create the remaining measures using the monotonicity property of the fuzzy measure as follows (Table 1).

From the recognition principle of maximum degree of similarity between PFSs, the process of assigning the pattern $\tilde{A}$ to $\tilde{A}_i$ is described by

$$k = \arg \max_{1 \leq i \leq 3} \left\{ W_{PFS}(C, \sigma)(\tilde{A}_i, \tilde{A}). \right\} \tag{53}$$

The rankings obtained by ten similarity measures are visualized in Fig. 3 and comparison of the results is given in Table 2. The numerical result presented in Tables 2 and Eq. (53) shows that $k = \tilde{A}_3$ for each similarity measure. Namely, $\tilde{A}$ pattern belongs to class $\tilde{A}_3$ with respect to each trigonometric similarity measure. When the results are compared to the results in [5, 28], we see that they are in agreement (see Table 2).

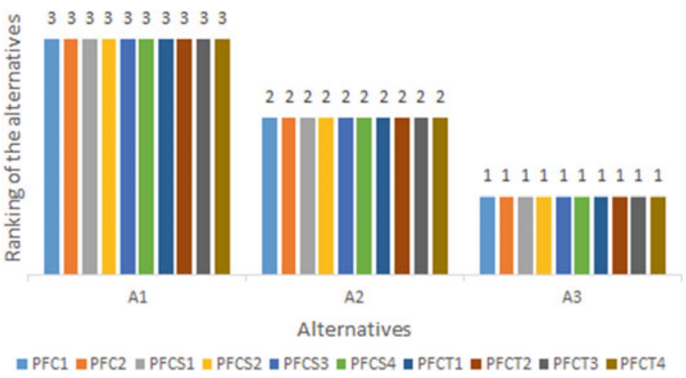


Fig. 3 Visualization of the rankings of the alternatives according to ten trigonometric similarity measures

Table 2 Comparison of classification result of Example 1

Similarity measure	Similarity scores		
	$(\tilde{A}_1, \tilde{A})$	$(\tilde{A}_2, \tilde{A})$	$(\tilde{A}_3, \tilde{A})$
$WPFC^1$	0.9686	0.9712	**0.9844**
$WPFC^2$	0.6273	0.7237	**0.9266**
$WPFCS^1$	0.6573	0.7627	**0.9329**
$WPFCS^2$	0.8843	0.9228	**0.9782**
$WPFCS^3$	0.6573	0.7627	**0.9329**
$WPFCS^4$	0.6573	0.7627	**0.9329**
$WPFCT^1$	0.4475	0.4995	**0.7206**
$WPFCT^2$	0.6554	0.6887	**0.8225**
$WPFCT^3$	0.4475	0.4995	**0.7206**
$WPFCT^4$	0.4475	0.4995	**0.7206**
$W_{PFC^1}^{(C,\sigma)}$	0.9738	0.9759	**0.9864**
$W_{PFC^2}^{(C,\sigma)}$	0.6788	0.7478	**0.9275**
$W_{PFCS^1}^{(C,\sigma)}$	0.7094	0.7909	**0.9268**
$W_{PFCS^2}^{(C,\sigma)}$	0.9020	0.9299	**0.9771**
$W_{PFCS^3}^{(C,\sigma)}$	0.7094	0.7909	**0.9268**
$W_{PFCS^4}^{(C,\sigma)}$	0.7094	0.7909	**0.9268**
$W_{PFCT^1}^{(C,\sigma)}$	0.4910	0.5223	**0.7120**
$W_{PFCT^2}^{(C,\sigma)}$	0.6838	0.7001	**0.8193**
$W_{PFCT^3}^{(C,\sigma)}$	0.4910	0.5223	**0.7120**
$W_{PFCT^4}^{(C,\sigma)}$	0.4910	0.5223	**0.7120**

4.2 *Medical Diagnosis Problem*

A medical diagnosis aims to determine which disease explains the symptoms of a patient. In this process, patterns of symptoms are compared with patterns of disease. Now, we apply proposed Choquet similarity measures to show their outperforming and suitability in medical diagnosis problems. We consider a medical diagnosis problem that was studied in [28]. Wei and Wei [28] obtained unknown class Φ belonging to class Ψ_2 according to similarity measures between (13) and (21) except for (12) (see Table 3 of [28]).

Example 2 Let us consider a set of diagnoses and symptoms as follows:
Ψ={Ψ_1(Viral fever), Ψ_2(Malaria), Ψ_3(Typhoid), Ψ_4(Stomach problem), Ψ_5(Chest Problem)}
Z={ζ_1(Temperature), ζ_2(Headache), ζ_3(Stomach pain), ζ_4(Cough), ζ_5(Chest pain)}.

Assume that a patient that has all the symptoms is represented by the following PFS:

$\Phi(\text{Patient}) = \{\langle \zeta_1, 0.8, 0.1\rangle, \langle \zeta_2, 0.6, 0.1\rangle, \langle \zeta_3, 0.2, 0.8\rangle, \langle \zeta_4, 0.6, 0.1\rangle, \langle \zeta_5, 0.1, 0.6\rangle\}$.

Table 3 Comparison of classification result of Example 2

Similarity measure	Similarity scores				
	(Ψ_1, Φ)	(Ψ_2, Φ)	(Ψ_3, Φ)	(Ψ_4, Φ)	(Ψ_5, Φ)
$WPFC^1$	0.8237	0.7840	**0.8283**	0.3512	0.2360
$WPFC^2$	0.8865	**0.8904**	0.8116	0.6629	0.5205
$WPFCS^1$	0.9191	**0.9250**	0.8599	0.7627	0.6392
$WPFCS^2$	0.9623	**0.9554**	0.9449	0.8115	0.7502
$WPFCS^3$	0.9151	**0.9244**	0.8599	0.7601	0.6392
$WPFCS^4$	0.9151	**0.9244**	0.8599	0.7601	0.6392
$WPFCT^1$	**0.6965**	0.6917	0.6623	0.5193	0.4393
$WPFCT^2$	**0.7861**	0.7802	0.7778	0.5844	0.5210
$WPFCT^3$	0.6876	**0.6898**	0.6623	0.5096	0.4393
$WPFCT^4$	0.6876	**0.6898**	0.6623	0.5096	0.4393
$W_{PFC^1}^{(C,\sigma)}$	0.9162	**0.9560**	0.7280	0.3345	0.2324
$W_{PFC^2}^{(C,\sigma)}$	0.8755	**0.8890**	0.7846	0.5988	0.5192
$W_{PFCS^1}^{(C,\sigma)}$	0.9125	**0.9260**	0.8400	0.7211	0.6392
$W_{PFCS^2}^{(C,\sigma)}$	0.9556	**0.9757**	0.9318	0.7924	0.7485
$W_{PFCS^3}^{(C,\sigma)}$	0.9088	**0.9256**	0.8400	0.7184	0.6392
$W_{PFCS^4}^{(C,\sigma)}$	0.8785	**0.9256**	0.8736	0.7184	0.6392
$W_{PFCT^1}^{(C,\sigma)}$	0.6824	**0.6860**	0.6359	0.4858	0.4392
$W_{PFCT^2}^{(C,\sigma)}$	0.7672	**0.8166**	0.7510	0.5479	0.5196
$W_{PFCT^3}^{(C,\sigma)}$	0.6747	**0.6847**	0.6359	0.4762	0.4392
$W_{PFCT^4}^{(C,\sigma)}$	0.6322	**0.6847**	0.6714	0.4762	0.4392

Moreover, assume that each diagnosis $\Psi_i (i = 1, 2, 3, 4, 5)$ is given as PFSs:

$$\begin{aligned}
\Psi_1(\text{Viral fever}) &= \{\langle\zeta_1, 0.4, 0.0\rangle, \langle\zeta_2, 0.3, 0.5\rangle, \langle\zeta_3, 0.1, 0.7\rangle, \langle\zeta_4, 0.4, 0.3\rangle, \langle\zeta_5, 0.1, 0.7\rangle\}\\
\Psi_2(\text{Malaria}) &= \{\langle\zeta_1, 0.7, 0.0\rangle, \langle\zeta_2, 0.2, 0.6\rangle, \langle\zeta_3, 0.0, 0.9\rangle, \langle\zeta_4, 0.7, 0.0\rangle, \langle\zeta_5, 0.1, 0.8\rangle\}\\
\Psi_3(\text{Typhoid}) &= \{\langle\zeta_1, 0.3, 0.3\rangle, \langle\zeta_2, 0.6, 0.1\rangle, \langle\zeta_3, 0.2, 0.7\rangle, \langle\zeta_4, 0.2, 0.6\rangle, \langle\zeta_5, 0.1, 0.9\rangle\}\\
\Psi_4(\text{Stomach problem}) &= \{\langle\zeta_1, 0.1, 0.7\rangle, \langle\zeta_2, 0.2, 0.4\rangle, \langle\zeta_3, 0.8, 0.0\rangle, \langle\zeta_4, 0.2, 0.7\rangle, \langle\zeta_5, 0.2, 0.7\rangle\}\\
\Psi_5(\text{Chest Problem}) &= \{\langle\zeta_1, 0.1, 0.8\rangle, \langle\zeta_2, 0.0, 0.8\rangle, \langle\zeta_3, 0.2, 0.8\rangle, \langle\zeta_4, 0.2, 0.8\rangle, \langle\zeta_5, 0.8, 0.1\rangle\}.
\end{aligned}$$

Our aim is to classify Φ into one of the diagnosis $\Psi_i (i = 1, 2, 3, 4, 5)$. First of all we construct a fuzzy measure. As in the pattern recognition problem, we use a hypothetical fuzzy measure for this example by taking into account the hypothetical weights given in [28] as fuzzy measures of singletons (see Table 4).

When we consider Table 3, it is seen that the results are in agreement with the results in [28] all except for $WPFC^1(\Psi_i, \Phi)$, $WPFCT^1(\Psi_i, \Phi)$ and $WPFCT^2(\Psi_i, \Phi)$, whereas they are fully compatible with the results in [5]. From (53), we see that $k = \Psi_2$ for each similarity measure. Namely, Φ patient has viral fever. Therefore, the ten similarity measures proposed are more sensitive and consistent than the ones suggested by Wei and Wei [28].

Table 4 Fuzzy measure

$\sigma(\emptyset) = 0$	$\sigma(\{\zeta_1\}) = 0.15$	$\sigma(\{\zeta_2\}) = 0.25$
$\sigma(\{\zeta_3\}) = 0.20$	$\sigma(\{\zeta_4\}) = 0.15$	$\sigma(\{\zeta_5\}) = 0.25$
$\sigma(\{\zeta_1, \zeta_2\}) = 0.35$	$\sigma(\{\zeta_1, \zeta_3\}) = 0.30$	$\sigma(\{\zeta_1, \zeta_4\}) = 0.25$
$\sigma(\{\zeta_1, \zeta_5\}) = 0.39$	$\sigma(\{\zeta_2, \zeta_3\}) = 0.40$	$\sigma(\{\zeta_2, \zeta_4\}) = 0.31$
$\sigma(\{\zeta_2, \zeta_5\}) = 0.45$	$\sigma(\{\zeta_3, \zeta_4\}) = 0.30$	$\sigma(\{\zeta_3, \zeta_5\}) = 0.42$
$\sigma(\{\zeta_4, \zeta_5\}) = 0.35$	$\sigma(\{\zeta_1, \zeta_2, \zeta_3\}) = 0.45$	$\sigma(\{\zeta_1, \zeta_2, \zeta_4\}) = 0.40$
$\sigma(\{\zeta_1, \zeta_2, \zeta_5\}) = 0.50$	$\sigma(\{\zeta_1, \zeta_3, \zeta_4\}) = 0.35$	$\sigma(\{\zeta_1, \zeta_3, \zeta_5\}) = 0.50$
$\sigma(\{\zeta_1, \zeta_4, \zeta_5\}) = 0.60$	$\sigma(\{\zeta_2, \zeta_3, \zeta_4\}) = 0.45$	$\sigma(\{\zeta_2, \zeta_3, \zeta_5\}) = 0.55$
$\sigma(\{\zeta_2, \zeta_4, \zeta_5\}) = 0.46$	$\sigma(\{\zeta_3, \zeta_4, \zeta_5\}) = 0.45$	$\sigma(\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}) = 0.7$
$\sigma(\{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}) = 0.75$	$\sigma(\{\zeta_1, \zeta_2, \zeta_4, \zeta_5\}) = 0.80$	$\sigma(\{\zeta_1, \zeta_3, \zeta_4, \zeta_5\}) = 0.95$
$\sigma(\{\zeta_2, \zeta_3, \zeta_4, \zeta_5\}) = 0.85$	$\sigma(\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}) = 1$	

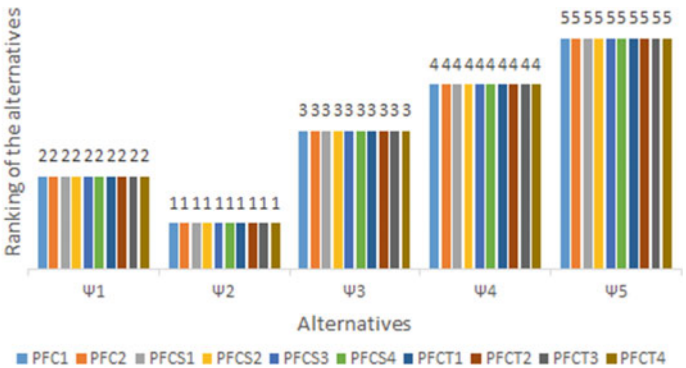


Fig. 4 Visualization of the rankings of the alternatives according to ten trigonometric similarity measures

The rankings obtained by ten similarity measures are also visualized in Fig. 4.

Now, we give the comparison of the proposed similarity measures with the similarity measures (6)–(11). The numerical results presented in Table 5 show that the proposed Choquet cosine similarity measure consistent with [4] and [20]. Namely, pattern $\tilde{A}$ belongs to class $\tilde{A}_3$. However, as the decision-maker and the fuzzy environment change, the weights vary, and this increases the sensitivity. For example, when we consider Table 6, patient P has typhoid in [4] while it has malaria with respect to proposed Choquet integral model. The reason of this change is that when solving the medical diagnosis problem in this model, the weights of symptoms and their interaction with each other are taken into account with the help of fuzzy measure. The similarity measures given in Tables 5 and 6 are defined similar to Definitions 4–9 with the help of Choquet integral.

Table 5 Comparison of classification result of Example 1

Similarity measure	Similarity scores		
	$(\tilde{A}_1, \tilde{A})$	$(\tilde{A}_2, \tilde{A})$	$(\tilde{A}_3, \tilde{A})$
$\mathbb{S}_1$ [4]	0.92594	0.95092	**0.98739**
$\mathbb{S}_2$ [4]	0.89294	0.93369	**0.97766**
$\mathbb{S}_3$ [4]	0.90006	0.93573	**0.97797**
SM_0 [20]	0.60585	0.63322	**0.78205**
SM_1 [20]	0.79017	0.80766	**0.88802**
SM_2 [20]	0.57250	0.58144	**0.63113**
$W_{\mathbb{S}_1}^{(C,\sigma)}$	0.93726	0.95462	**0.98730**
$W_{\mathbb{S}_2}^{(C,\sigma)}$	0.90758	0.93748	**0.98165**
$W_{\mathbb{S}_3}^{(C,\sigma)}$	0.91342	0.93932	**0.98189**
$W_{SM_0}^{(C,\sigma)}$	0.63526	0.64504	**0.77848**
$W_{SM_1}^{(C,\sigma)}$	0.80658	0.81316	**0.88653**
$W_{SM_2}^{(C,\sigma)}$	0.58214	0.58378	**0.63054**

Table 6 Comparison of classification results of Example 2

Similarity measure	Similarity scores				
	(Ψ_1, Φ)	(Ψ_2, Φ)	(Ψ_3, Φ)	(Ψ_4, Φ)	(Ψ_5, Φ)
$\mathbb{S}_1$[4]	0.9635	0.9536	**0.9768**	0.6962	0.8856
$\mathbb{S}_2$ [4]	0.9471	0.9253	**0.9733**	0.2746	0.7180
$\mathbb{S}_3$ [4]	0.9493	0.9318	**0.9742**	0.5922	0.8193
$W_{\mathbb{S}_1}^{(C,\sigma)}$	0.9513	**0.9778**	0.9320	0.8028	0.7712
$W_{\mathbb{S}_2}^{(C,\sigma)}$	0.9445	**0.9706**	0.9028	0.5592	0.5117
$W_{\mathbb{S}_3}^{(C,\sigma)}$	0.9454	**0.9712**	0.9104	0.7199	0.6632

5 Conclusion

In this paper, we propose new similarity measures based on the Choquet integral for PFSs. Moreover, we apply this measures to pattern recognition and medical diagnosis problems and then we compare our results with some existing results. We see that our results are consistent with the literature while some of them are incompatible. The main reason of this difference is the sensitivity of the Choquet integral. The proposed Choquet integral model has a wide range of applications. In the future, we shall expand the proposed Choquet integral model with some different information measures and fuzzy environments and we shall apply them to decision-making, risk analysis, and many other fields under uncertain environments [9, 13, 34].

Acknowledgments The authors are grateful to the reviewers for carefully reading the chapter and for offering substantial comments and suggestions which enabled them to improve the presentation.

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Isomorphic Operators and Ranking Methods for Pythagorean and Intuitionistic Fuzzy Sets

Yi Yang and Zhen-Song Chen

1 Introduction

After decades of research, the theories and methods of intuitionistic fuzzy sets (IFSs) [1] have formed a relatively perfect decision-making system and been applied to multi-attribute decision-making (MADM) problems in various fields. The aggregation operators and the ranking methods of fuzzy sets are the key components of the decision-making method. In the process of solving MADM problems, fuzzy information aggregation operators and sorting methods play a key role. Operators are mainly used to aggregate multiple attribute evaluation information to obtain the comprehensive evaluation values of the alternatives. Sorting methods are mainly applied to get the ranking of comprehensive evaluation value to obtain the optimal alternative.

Because intuitionistic fuzzy sets have the advantage of considering both membership degree and nonmembership degree, in recent years, scholars have cross-fusing other types of fuzzy sets, such as interval-valued fuzzy sets (IVFSs) [2] and hesitation fuzzy sets (HFSs) [4], with intuitionistic fuzzy sets to derive interval-valued intuitionistic fuzzy sets (IVIFSs) [3] and dual hesitation fuzzy sets (DHFSs) [5]. These new fuzzy sets integrate the advantages of the combiner, which makes the related operation rules, aggregation operators, and ranking methods get close attention of scholars. A series of fuzzy multi-attribute decision-making (FMADM) methods are developed and applied. In 2014, the concept of Pythagorean fuzzy sets (PFSs) [6] is introduced, which expands the value range of membership degree and nonmembership degree

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_5

from triangle region to quarter circle region. In other words, intuitionistic fuzzy sets belong to the category of PFSs. Following the development of intuitionistic fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy sets are extended to Pythagorean fuzzy environment, and interval-valued Pythagorean fuzzy sets (IVPFSs) [7, 8] and dual hesitant Pythagorean fuzzy sets (DHPFSs) [9] are derived. The aggregation operators and sorting methods of these extended fuzzy sets have been a hot topic in recent years.

The aggregation operators of fuzzy sets are generally based on operational laws, mainly including four types of operations such as addition, multiplication, scalar multiplication, and exponentiation. Dual Archimedean t-norm and s-norm [10, 11] are the core tools to define these laws, especially Algebraic t-norm/s-norm, Einstein t-norm/s-norm, Hamming t-norm/s-norm, and frank t-norm/s-norm are the four most commonly used tools. On this basis, a series of Archimedes fuzzy aggregation operators are generated. For example, Pythagorean fuzzy interactive Hamacher power aggregation operators [32] and generalized Pythagorean fuzzy Einstein geometric aggregation operators [33] are developed to construct multiple attribute group decision-making methods. In terms of sorting methods, the score function and accuracy function of fuzzy numbers are generally used to construct sorting methods to distinguish different fuzzy numbers. The Pythagorean fuzzy sets and its extended sets mainly follow the relevant theories and methods of intuitionistic fuzzy sets in terms of aggregation operators and ranking methods. It is mainly reflected in the operators based on Archimedean t-norm and s-norm, and the ranking methods are based on score function and accuracy function. However, the current researches on Pythagorean fuzzy sets mainly focus on how to extend the theory and method of intuitionistic fuzzy sets to Pythagorean fuzzy decision-making environment, and pay less attention to the internal relationship between the two kinds of fuzzy sets. Therefore, from the perspective of isomorphism, this study will reveal the internal relationship and transformation relationship between the relevant theoretical methods of two types of fuzzy sets, and provides method support for solving the two kinds of fuzzy multi-attribute decision-making problems.

The purpose of this study is to reveal the substantial relationship between three kinds of intuitionistic fuzzy sets and three kinds of Pythagorean fuzzy sets, including IFSs and PFSs, IVIFSs and IVPFSs, and DHFSs and DHPFSs. The isomorphism between three pairs of fuzzy sets is studied from three aspects: operational laws, aggregation operators, and ranking methods.

2 Preliminaries

Some basic concepts of several kinds of fuzzy sets and the related concepts of t-norm and s-norm are reviewed.

2.1 Related Definitions of Intuitionistic Fuzzy Sets and Pythagorean Fuzzy Sets

Atanassov [1] generalizes Zadeh's fuzzy set theory [12] with the concept of intuitionistic fuzzy sets (IFSs) as defined below:

Definition 1 ([1]) Let X be a universe of discourse. An IFS I in X is given by

$$I = \{\langle x, \mu_I(x), v_I(x)\rangle | x \in X\}, \tag{1}$$

where the function $\mu_I : X \to [0, 1]$ defines the degree of membership and $v_I : X \to [0, 1]$ defines the degree of nonmembership of the element $x \in X$ to I, respectively, and for every $x \in X$, it holds that $\mu_I(x) + v_I(x) \leq 1$.

Pythagorean fuzzy set (PFS) is a generalization of intuitionistic fuzzy sets (IFSs) and its definition is given as follows:

Definition 2 ([6]) Let X be a universe of discourse. A PFS P in X is given by

$$P = \{\langle x, \mu_P(x), v_P(x)\rangle | x \in X\}, \tag{2}$$

where the function $\mu_P : X \to [0, 1]$ defines the degree of membership and $v_P : X \to [0, 1]$ defines the degree of nonmembership of the element $x \in X$ to P, respectively, and for every $x \in X$, it holds that $(\mu_P(x))^2 + (v_P(x))^2 \leq 1$.The degree of indeterminacy $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (v_P(x))^2}$.

For simplicity, we called $(\mu_P(x), v_P(x))$ a Pythagorean fuzzy number (PFN) denoted by $P = (\mu_P, v_P)$, where $\mu_P, v_P \in [0, 1]$,$\pi_P = \sqrt{1 - \mu_P^2 - v_P^2}$, and $\mu_P^2 + v_P^2 \leq 1$.

Interval-valued intuitionistic fuzzy set (IVIFS) is a combination of interval-valued fuzzy sets and intuitionistic fuzzy sets and its definition is given as follows:

Definition 3 ([3]) We let K denote a finite universal set. An IVIFS B on K is provided as

$$B = \{\langle q, \mu_B(q), v_B(q)\rangle | q \in K\}, \tag{3}$$

where $\mu_B : K \to [0, 1]$ is the interval membership function, $v_B : K \to [0, 1]$ is the interval nonmembership function of B, satisfying $\mu_B = \left[\mu_B^-(q), \mu_B^+(q)\right]$, $v_B = \left[v_B^-(q), v_B^+(q)\right]$, $0 \leq \mu_B^-(q) \leq \mu_B^+(q) \leq 1$, $0 \leq v_B^-(q) \leq v_B^+(q) \leq 1$, $\mu_B^+(q) + v_B^+(q) \leq 1$ for any $q \in K$. Especially, $\pi_B = \left[1 - \mu_B^+(q) - v_B^+(q), 1 - \mu_B^-(q) - v_B^-(q)\right]$ is the indeterminacy degree of

$q \in K$. The pair $\beta = \left(\left[\mu_B^-(q), \mu_B^+(q)\right], \left[v_B^-(q), v_B^+(q)\right]\right)$ is called IVIFN and simply expressed as $\beta = \left(\left[\mu^-, \mu^+\right], \left[v^-, v^+\right]\right)$, where $\left[\mu^-, \mu^+\right], \left[v^-, v^+\right] \subseteq [0, 1]$ and $\mu^+ + v^+ \leq 1$.

According to Definition 1, if $\mu^- = \mu^+$ and $v^- = v^+$, then IVIFN β reduces to an intuitionistic fuzzy number (IFN) [1].

Interval-valued Pythagorean fuzzy set (IVPFS) is a combination of interval-valued fuzzy sets and Pythagorean fuzzy sets and its definition is given as follows:

Definition 4 ([7, 8]) Given a finite universal set K, an interval-valued PFS (IVPFS) P on K is given by

$$P = \{\langle y, \mu_P(y), v_P(y)\rangle | y \in K\}, \tag{4}$$

where $\mu_P(y) : K \to \varepsilon([0, 1])$ is the membership degree, $v_P(y) : K \to \varepsilon([0, 1])$ is the nonmembership degree, and $\sup\left(\mu_P^2(y)\right) + \sup\left(v_P^2(y)\right) \leq 1$. And $\varepsilon([0, 1])$ is the set of all closed intervals in the unit interval, and we call the two-tuples $(\mu_P(y), v_P(y))$ as interval-valued Pythagorean fuzzy number (IVPFN). Let $\mu_P(y) = \left[a^-, a^+\right]$ and $v_P(y) = \left[b^-, b^+\right]$, then the IVPFN can be expressed as $\beta = \left(\left[a^-, a^+\right], \left[b^-, b^+\right]\right)$, and $\left(b^+\right)^2 + \left(a^+\right)^2 \leq 1$.

According to Definition 2, if $a^- = a^+$ and $b^- = b^+$, then IVPFN β reduces to an Pythagorean fuzzy number (PFN) [6].

Considering the advantages of IFSs and hesitant fuzzy sets, the concept of dual hesitant fuzzy sets (DHFSs) is defined, which are composed of membership hesitant fuzzy sets and nonmembership hesitant fuzzy sets.

Definition 5 Let X be a universe of discourse. A DHFS δ in X is given by

$$\delta = \{\langle x, p_I(x), q_I(x)\rangle | x \in X\} \tag{5}$$

in which $p_I(x)$ and $q_I(x)$ are two sets of some values in [0, 1], denoting the possible membership degrees and nonmembership degrees of the element $x \in X$ to δ, respectively, with the conditions: $\max_{\gamma \in p_I}\{\gamma\} + \max_{\eta \in q_I}\{\eta\} \leq 1$, where $\gamma \in p_I(x)$ and $\eta \in q_I(x)$ for all $x \in X$. For convenience, the pair $\delta = (p_I(x), q_I(x))$ is called a hesitant Pythagorean fuzzy number (HPFN) denoted by $\delta = (p, q)$, with the conditions: $\gamma \in p, \eta \in q\gamma, \eta \in [0, 1]$, and $\max_{\gamma \in p}\{\gamma\} + \max_{\eta \in q}\{\eta\} \leq 1$.

Considering the advantages of PFSs and hesitant fuzzy sets, the concept of dual hesitant fuzzy sets (DHFSs) is defined, which are composed of membership hesitant fuzzy sets and nonmembership hesitant fuzzy sets.

Definition 6 Let X be a universe of discourse. A DHPFS ψ in X is given by

$$\psi = \{\langle x, a_P(x), b_P(x)\rangle | x \in X\} \tag{6}$$

in which $a_P(x)$ and $b_P(x)$ are two sets of some values in [0, 1], denoting the possible membership degrees and nonmembership degrees of the element $x \in X$ to ψ, respectively, with the conditions: $\max_{\tau\in a_P}\{\tau^2\} + \max_{\varsigma\in b_P}\{\varsigma^2\} \leq 1$, where $\tau \in a_P(x)$ and $\varsigma \in b_P(x)$ for all $x \in X$. For convenience, the pair $\psi = (a_P(x), b_P(x))$ is called a hesitant Pythagorean fuzzy number (HPFN) denoted by $\psi = (a, b)$, with the conditions: $\tau \in a$, $\varsigma \in b\tau$, $\varsigma \in [0, 1]$, and $\max_{\tau\in a}\{\tau^2\} + \max_{\varsigma\in q}\{\varsigma^2\} \leq 1$.

Figure 1 describes the relationship between various fuzzy sets. From Fig. 1, we have

(1) An intuitionistic fuzzy number (IFN) is also a Pythagorean fuzzy number (PFN), but not all PFNs are IFNs;
(2) A dual hesitant fuzzy number (DHFN) is also a dual hesitant Pythagorean fuzzy number (DHPFN), but not all DHPFNs are DHFNs;
(3) An interval-valued intuitionistic fuzzy number (IVIFN) is also an interval-valued Pythagorean fuzzy number (IVPFN), but not all IVPFNs are IVIFNs.

These results mean we can use the PFNs (DHPFNs/IVPFNs) with less limitation than IFNs (DHIFNs/IVIFNs) in some special situation.

2.2 T-Norm and Its Dual T-Conorm

A triple (T, S, N), where T is a t-norm, S a t-conorm and N a fuzzy complement is called a dual triple if T and S are dual with respect to N. The dual triple (T, S, N) is a utility tool to define the generalized operational laws for a variety of different fuzzy environments.

The concepts of t-norm and t-conorm are given as follows:

Definition 7 ([10]). A function $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if it satisfies the following four conditions:

(1) (Neutral element): $T(1, x) = x$, for all x.
(2) (Commutativity): $T(y, x) = T(x, y)$, for all x and y.
(3) (Associativity): $T(T(x, y), z) = T(x, T(y, z))$, for all x, y, and z.
(4) (Monotonicity): If $x \leq x'$ and $y \leq y'$, then $T(x, y) \leq T(x', y')$.

Definition 8 ([10]) A function $S : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-conorm if it satisfies the following four conditions:

(1) (Neutral element): $S(0, x) = x$, for all x.
(2) (Commutativity): $S(y, x) = S(x, y)$, for all x and y.

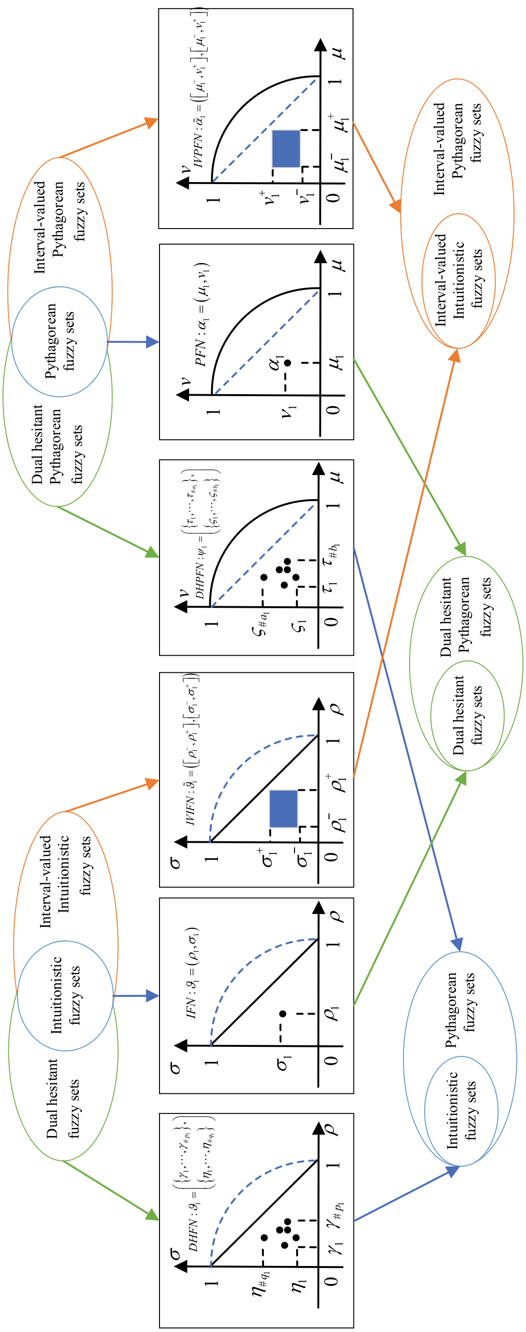

Fig. 1 Inclusion relations of six kinds of fuzzy sets

(3) (Associativity): $S(S(x, y), z) = S(x, S(y, z))$, for all x, y, and z.
(4) (Monotonicity): if $x \le x'$ and $y \le y'$, then $S(x, y) \le S\big(x', y'\big)$.

Continuous Archimedean t-norms are very useful because they can be represented via additive generator. Now, the definitions of Archimedean t-norm and t-conorm are given as follows:

Definition 9 ([10]) A t-norm is called Archimedean t-norm if for each $(a, b) \in]0, 1[^2$ there is an $n \in \{1, 2, \cdots\}$ with $T\overbrace{(a, \cdots, a)}^{n-times} < b$.

Definition 10 ([10]) A t-conorm is called Archimedean t-conorm if for each $(a, b) \in]0, 1[^2$ there is an $n \in \{1, 2, \cdots\}$ with $S\overbrace{(a, \cdots, a)}^{n-times} > b$.

Any continuous Archimedean t-norm and t-conorm can be represented with the help of a continuous additive generator.

Proposition 1 ([10]) *A continuous Archimedean t-norm T is expressed via its additive generator $g : [0, 1] \to [0, \infty]$, which verifies $g(1) = 0$ as $T(x, y) = g^{-1}(g(x) + g(y))$, where g is a continuous strictly decreasing function and $g^{-1}(t) = \sup\{z \in [0, 1]|g(z) > t, t \in [0, \infty]\}$ is the pseudo-inverse of g.*

Proposition 2 ([10]) *A continuous Archimedean t-conorm S is expressed via its additive generator $h : [0, 1] \to [0, \infty]$, which verifies $h(0) = 0$ as $S(x, y) = h^{-1}(h(x) + h(y))$, where h is a continuous strictly increasing function and $h^{-1}(t) = \sup\{z \in [0, 1]|h(z) < t, t \in [0, \infty]\}$ is the pseudo-inverse of h.*

Definition 11 ([10]) A fuzzy complement is a mapping denoted by $N : [0, 1] \to [0, 1]$ that satisfies the following:

(1) Boundary conditions: $N(0) = 1$ and $N(1) = 0$.
(2) Monotonicity: for all $a, b \in [0, 1]$, if $a \le b$, then $N(a) \ge N(b)$,
(3) Continuity
(4) Involution: $N(N(a)) = a$ for all $a \in [0, 1]$.

Yager class of fuzzy complements [13, 14] is defined by $N(a) = (1 - a^p)^{1/p}$, where $p \in (0, \infty)$. When $p = 1$, this function becomes the classical fuzzy complements $N_I(a) = 1 - a$. while $p = 2$ will make this function become the Pythagorean complement [6] $N_P(a) = \sqrt{1 - a^2}$.

Klir and Yuan [11] first put forward the concept of dual triple (T, S, N), which denote that T and S are dual with respect to N, and let any such triple be called a dual triple.

Definition 12 ([11]) A t-norm T and a t-conorm S are dual with respect to a fuzzy complement N iff $T(x, y) = N(S(N(x), N(y)))$ and $S(x, y) = N(T(N(x), N(y)))$.

Theorem 1 ([11]). *Given a t-norm T and an involutive fuzzy complement N, the binary operation S on $[0, 1]$ defined by $S(x, y) = N(T(N(x), N(y)))$ for all x, $y \in [0, 1]$ is a t-conorm such that (T, S, N) is a dual triple.*

Proposition 3 ([11]) *Let T be a t-norm, S its dual t-conorm, and $g : [0, 1] \to [0, \infty]$ an additive generator of T. The function $h : [0, 1] \to [0, \infty]$ defined by $h(t) = g(N(t))$ is an additive generator of S.*

2.3 Four Types of Dual Archimedean T-Norm and S-Norm

It is found that the traditional triple (T, S, N_I) is mainly used to define the operational laws and aggregation operators of intuitionistic fuzzy sets, dual hesitant fuzzy sets, and interval-valued intuitionistic fuzzy sets. For convenience, this study denotes such triples as (T_I, S_I, N_I). Triple (T, S, N_P) is mainly used in the operation laws and aggregation operators of Pythagorean fuzzy sets, dual hesitant Pythagorean fuzzy sets, and interval-valued Pythagorean fuzzy sets. For convenience, this study denotes such triples as (T_P, S_P, N_P).

This subsection introduces four classic types of dual T-norm and S-norm, whose specific forms are different in the intuitionistic fuzzy environment and the Pythagorean fuzzy environment. But there is a certain transformational relationship. See Table 1 for details.

Table 1 Generators of operations for IFSs and PFSs

	Generators for IFSs/References	Generators for PFSs/References	Relation
Algebraic operations	$g_I(t) = -\log t$, [15]	$g_P(t) = -\log t^2$, [16]	$g_P(t) = g_I(\phi(t))$
Einstein operations	$g_I(t) = \log\left(\frac{2-t}{t}\right)$, [15]	$g_P(t) = \log\left(\frac{2-t^2}{t^2}\right)$, [16]	$g_P(t) = g_I(\phi(t))$
Hamacher operations	$g_I(t) = \log\left(\frac{\tau+(1-\tau)t}{t}\right)$, [15]	$g_P(t) = \log\left(\frac{\tau+(1-\tau)t^2}{t^2}\right)$, [16]	$g_P(t) = g_I(\phi(t))$
Frank operations	$g_I(t) = \ln\frac{\gamma-1}{\gamma^t-1}, \gamma > 1$, [15]	$g_P(t) = \ln\frac{\gamma-1}{\gamma^{t^2}-1}, \gamma > 1$, [17]	$g_P(t) = g_I(\phi(t))$

Theorem 2 *If the additive generators of S_I and S_P satisfy $g_P(t) = g_I(\phi(t))$, then*

$$S_P(x, y) = \phi^{-1}(S_I(\phi(x), \phi(y))),\;\; T_P(x, y) = \phi^{-1}(T_I(\phi(x), \phi(y))),$$

where T_I and T_P are the dual t-norm of S_I and S_P, respectively, and $\phi(t) = t^2$ is an automorphism on [0, 1].

Theorem 2 reveals the transformation between intuitionistic fuzzy triple (T_I, S_I, N_I) and Pythagorean fuzzy triple (T_P, S_P, N_P).

3 Operations Isomorphism

Operations are mainly used to construct aggregation operators. As mentioned in the previous section, the operational laws of the three types of fuzzy sets are all based on t-norm and s-norm. From the isomorphism perspective, this section will reveal the correlation between the operational laws of the three types of intuitionistic fuzzy sets and Pythagorean fuzzy sets.

3.1 Operations Isomorphism Between IFSs and PFSs

Firstly, the relationship between the intuitionistic fuzzy operations and the Pythagorean fuzzy operations is analyzed.

Dual Archimedean t-norm and s-norm are used to define the generalized intuitionistic fuzzy operations, which can be reduced to special operations such as Einstein, Hamacher, and Frank operation.

Definition 13 ([15]). For three IFNs $\vartheta = (\rho, \sigma)$ and $\vartheta_i = (\rho_i, \sigma_i)(i = 1, 2)$, then we have

(1) $$\begin{aligned}\vartheta_1 \oplus \vartheta_2 &= (S_I(\rho_1, \rho_2), T_I(\sigma_1, \sigma_2)) \\ &= \left(h_I^{-1}(h_I(\rho_1) + h_I(\rho_2)), g_I^{-1}(g_I(\sigma_1) + g_I(\sigma_2))\right)\end{aligned};$$

(2) $$\begin{aligned}\vartheta_1 \otimes \vartheta_2 &= (T_I(\rho_1, \rho_2), S_I(\sigma_1, \sigma_2)) \\ &= \left(g_I^{-1}(g_I(\rho_1) + g_I(\rho_2)), h_I^{-1}(h_I(\sigma_1) + h_I(\sigma_2))\right)\end{aligned};$$

(3) $\lambda\vartheta = \left(h_I^{-1}(\lambda h_I(\rho)), g_I^{-1}(\lambda g_I(\sigma))\right), \lambda > 0;$

(4) $\vartheta^\lambda = \left(g_I^{-1}(\lambda g_I(\rho)), h_I^{-1}(\lambda h_I(\sigma))\right), \lambda > 0;$

where T_I and S_I are dual with respect to N_I and $N_I(a) = 1 - a$. g_I and h_I are the additive generators of T_I and S_I, respectively.

By referring to the intuitionistic fuzzy generalized operations, some new dual Archimedean t-norm and s-norm suitable for Pythagorean fuzzy environment are proposed and used to construct Pythagorean fuzzy generalized operations.

Definition 14 ([16]). For three PFNs $\alpha = (\mu_\alpha, \nu_\alpha)$ and $\alpha_i = \left(\mu_{\alpha_i}, \nu_{\alpha_i}\right)(i = 1, 2)$, then we have

(1) $$\begin{aligned}\alpha_1 \oplus \alpha_2 &= (S_P(\mu_1, \mu_2), T_P(\nu_1, \nu_2)) \\ &= \left(h_p^{-1}(h_P(\mu_1) + h_P(\mu_2)), g_p^{-1}(g_P(\nu_1) + g_P(\nu_2))\right)\end{aligned};$$

(2) $$\begin{aligned}\alpha_1 \otimes \alpha_2 &= (T_P(\mu_1, \mu_2), S_P(\nu_1, \nu_2)) \\ &= \left(g_p^{-1}(g_P(\mu_1) + g_P(\mu_2)), h_p^{-1}(h_P(\nu_1) + h_P(\nu_2))\right)\end{aligned};$$

(3) $\lambda\alpha = \left(h_p^{-1}(\lambda h_P(\mu)), g_p^{-1}(\lambda g_P(\nu))\right), \lambda > 0;$

(4) $\alpha^\lambda = \left(g_p^{-1}(\lambda g_P(\mu)), h_p^{-1}(\lambda h_P(\nu))\right), \lambda > 0;$

where T_P and S_P are dual with respect to N_P and $N_P(a) = \sqrt{1 - a^2}$. g_P and h_P are the additive generators of T_P and S_P, respectively.

For convenience, in this study, the set of all the intuitionistic fuzzy sets is denoted as A_I, and the set of all the Pythagorean fuzzy sets is denoted as A_P.

Definition 15 Let $\alpha_i = (\mu_i, \nu_i)_P \in A_P$, a mapping $\wp_{P\to I} : A_P \to A_I$ is defined as follows:

$$\wp_{P\to I}(\alpha_i) = (\phi(\mu_i), \phi(\nu_i))_I \in A_I,$$

where $\varphi(x) = x^2$ is an automorphism on [0, 1].

Definition 15 reveals that IFSs and PFSs have a mathematical isomorphism $\wp_{P\to I}$.

Theorem 3 *Let $\alpha_i = \langle\mu_i, \nu_i\rangle_P \in A_P$, $\wp_{P\to I}(\alpha_i) = (\phi(\mu_i), \phi(\nu_i))_I \in A_I$, and the operations of $\alpha_i = \langle\mu_i, \nu_i\rangle_P$ on A_P and the operations of $\wp_{P\to I}(\alpha_i) = (\phi(\mu_i), \phi(\nu_i))_I$ on A_I are defined based on the t-norm and s-norm, then we have*

(1) $\wp_{P\to I}(\alpha_1 \oplus \alpha_2) = \wp_{P\to I}(\alpha_1) \oplus \wp_{P\to I}(\alpha_2);$
(2) $\wp_{P\to I}(\alpha_1 \otimes \alpha_2) = \wp_{P\to I}(\alpha_1) \otimes \wp_{P\to I}(\alpha_2);$
(3) $\wp_{P\to I}(\lambda\alpha_1) = \lambda\wp_{P\to I}(\alpha_1);$
(4) $\wp_{P\to I}\left(\alpha_1^\lambda\right) = \wp_{P\to I}(\alpha_1)^\lambda.$

From Theorem 3, $\wp_{P\to I}$ can be proved to be an isomorphic mapping from A_P to A_I. Theorem 3 reveals that IFSs and PFSs do not only have a mathematical isomorphism, but also have a operations isomorphism. The later is a substantial relationship between IFSs and PFSs not reported up to now in literature.

3.2 Operations Isomorphism Between IVIFSs and IVPFSs

Atanassov [18] proposed the interval-valued intuitionistic fuzzy Algebraic operations, and several classical t-norm and s-norm are applied to interval-valued intuitionistic fuzzy environment, and various operations, such as Hamacher, Frank and Einstein operations, are proposed [19–21]. In this study, the uniform generalized interval-valued intuitionistic operations are provided as follow.

Definition 16 For three IVIFNs $\tilde{\vartheta} = \left(\left[\rho^{-}, \rho^{+}\right], \left[\sigma^{-}, \sigma^{+}\right]\right)$ and $\tilde{\vartheta}_i = \left(\left[\rho_i^{-}, \rho_i^{+}\right], \left[\sigma_i^{-}, \sigma_i^{+}\right]\right)(i = 1, 2)$, then

(1) $\tilde{\vartheta}_1 \oplus \tilde{\vartheta}_2 = \left(\left[S_I\left(\rho_1^{-}, \rho_2^{-}\right), S_I\left(\rho_1^{+}, \rho_2^{+}\right)\right], \left[T_I\left(\sigma_1^{-}, \sigma_2^{-}\right), T_I\left(\sigma_1^{+}, \sigma_2^{+}\right)\right]\right);$

(2) $\tilde{\vartheta}_1 \otimes \tilde{\vartheta}_2 = \left(\left[T_I\left(\rho_1^{-}, \rho_2^{-}\right), T_I\left(\rho_1^{+}, \rho_2^{+}\right)\right], \left[S_I\left(\sigma_1^{-}, \sigma_2^{-}\right), S_I\left(\sigma_1^{+}, \sigma_2^{+}\right)\right]\right);$

(3) $\lambda\tilde{\vartheta} = \left(\left[h_I^{-1}(\lambda h_I(\rho^{-})), h_I^{-1}(\lambda h_I(\rho^{+}))\right], \left[g_I^{-1}(\lambda g_I(\sigma^{-})), g_I^{-1}(\lambda g_I(\sigma^{+}))\right]\right), \lambda > 0;$

(4) $\tilde{\vartheta}^{\lambda} = \left(\left[g_I^{-1}(\lambda g_I(\rho^{-})), g_I^{-1}(\lambda g_I(\rho^{+}))\right], \left[h_I^{-1}(\lambda h_I(\sigma^{-})), h_I^{-1}(\lambda h_I(\sigma^{+}))\right]\right), \lambda > 0.$

where $T_I(x, y) = g_I^{-1}(g_I(x) + g_I(y))$ and $S_I(x, y) = h_I^{-1}(h_I(x) + h_I(y))$ are dual with respect to N_I and $N_I(a) = 1 - a$. g_I and h_I are the additive generators of T_I and S_I, respectively.

Similar to interval-valued intuitionistic fuzzy operations, some kinds of interval-valued Pythagorean fuzzy operations are proposed. The uniform generalized interval-valued Pythagorean operations are provided as follow.

Definition 17 ([22]). For three IVPFNs $\tilde{\alpha} = \left(\left[\mu^{-}, \mu^{+}\right], \left[v^{-}, v^{+}\right]\right)$ and $\tilde{\alpha}_i = \left(\left[\mu_i^{-}, \mu_i^{+}\right], \left[v_i^{-}, v_i^{+}\right]\right)(i = 1, 2)$, then

(1) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left(\left[S_P\left(\mu_1^{-}, \mu_2^{-}\right), S_P\left(\mu_1^{+}, \mu_2^{+}\right)\right], \left[T_P\left(v_1^{-}, v_2^{-}\right), T_P\left(v_1^{+}, v_2^{+}\right)\right]\right);$

(2) $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \left(\left[T_P\left(\mu_1^{-}, \mu_2^{-}\right), T_P\left(\mu_1^{+}, \mu_2^{+}\right)\right], \left[S_P\left(v_1^{-}, v_2^{-}\right), S_P\left(v_1^{+}, v_2^{+}\right)\right]\right);$

(3) $\lambda\tilde{\alpha} = \left(\left[h_P^{-1}(\lambda h_P(\mu^{-})), h_P^{-1}(\lambda h_P(\mu^{+}))\right], \left[g_P^{-1}(\lambda g_P(v^{-})), g_P^{-1}(\lambda g_P(v^{+}))\right]\right), \lambda > 0;$

(4) $\tilde{\alpha}^{\lambda} = \left(\left[g_P^{-1}(\lambda g_P(\mu^{-})), g_P^{-1}(\lambda g_P(\mu^{+}))\right], \left[h_P^{-1}(\lambda h_P(v^{-})), h_P^{-1}(\lambda h_P(v^{+}))\right]\right), \lambda > 0.$

where $T_P(x, y) = g_P^{-1}(g_P(x) + g_P(y))$ and $S_P(x, y) = h_P^{-1}(h_P(x) + h_P(y))$ are dual with respect to N_P and $N_P(a) = \sqrt{1 - a^2}$. g_P and h_P are the additive generators of T_P and S_P, respectively.

For convenience, in this study, the set of all the interval-valued intuitionistic fuzzy sets is denoted as $\tilde{A}_I$, and the set of all the interval-valued Pythagorean fuzzy sets is denoted as $\tilde{A}_P$.

Definition 18 Let $\tilde{\alpha}_i = \left(\left[\mu_i^{-}, \mu_i^{+}\right], \left[v_i^{-}, v_i^{+}\right]\right)_P \in A_P$, a mapping $\tilde{\wp}_{P\to I} : \tilde{A}_P \to \tilde{A}_I$ is defined as follows:

$$\tilde{\wp}_{P\to I}(\tilde{\alpha}_i) = \left(\left[\phi\left(\mu_i^{-}\right), \phi\left(\mu_i^{+}\right)\right], \left[\phi\left(v_i^{-}\right), \phi\left(v_i^{+}\right)\right]\right)_I \in \tilde{A}_I,$$

where $\varphi(x) = x^2$ is an automorphism on [0, 1].

Definition 18 reveals that IVIFSs and IVPFSs have a mathematical isomorphism $\tilde{\wp}_{P\to I}$.

Theorem 4 *Let* $\tilde{\alpha}_i = \left(\left[\mu_i^-, \mu_i^+\right], \left[v_i^-, v_i^+\right]\right)_P \in \tilde{A}_P$, $\tilde{\wp}_{P\to I}(\tilde{\alpha}_i) = \left(\left[\phi\left(\mu_i^-\right), \phi\left(\mu_i^+\right)\right], \left[\phi\left(v_i^-\right), \phi\left(v_i^+\right)\right]\right)_I \in \tilde{A}_I$, *and the operations of* $\tilde{\alpha}_i \in \tilde{A}_P$ *on* $\tilde{A}_P$ *and the operations of* $\tilde{\wp}_{P\to I}(\tilde{\alpha}_i) \in \tilde{A}_I$ *on* $\tilde{A}_I$ *are defined based on the t-norm and s-norm, then we have*

(1) $\tilde{\wp}_{P\to I}(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) = \tilde{\wp}_{P\to I}(\tilde{\alpha}_1) \oplus \tilde{\wp}_{P\to I}(\tilde{\alpha}_2)$;
(2) $\tilde{\wp}_{P\to I}(\tilde{\alpha}_1 \otimes \tilde{\alpha}_2) = \tilde{\wp}_{P\to I}(\tilde{\alpha}_1) \otimes \tilde{\wp}_{P\to I}(\tilde{\alpha}_2)$;
(3) $\tilde{\wp}_{P\to I}(\lambda\tilde{\alpha}_1) = \lambda\tilde{\wp}_{P\to I}(\tilde{\alpha}_1)$;
(4) $\tilde{\wp}_{P\to I}\left(\tilde{\alpha}_1^{\lambda}\right) = \tilde{\wp}_{P\to I}(\tilde{\alpha}_1)^{\lambda}$.

From Theorem 4, $\tilde{\wp}_{P\to I}$ can be proved to be an isomorphic mapping from $\tilde{A}_P$ to $\tilde{A}_I$. Theorem 4 reveals that IVIFSs and IVPFSs do not only have a mathematical isomorphism, but also have a operations isomorphism. The later is a substantial relationship between IVIFSs and IVPFSs not reported up to now in literature.

3.3 Operations Isomorphism Between DHFSs and DHPFSs

Inspired by intuitionistic fuzzy Archimedean operations and Pythagorean fuzzy Archimedean operations, some scholars developed dual hesitant fuzzy Archimedean operations and dual hesitant Pythagorean fuzzy Archimedean operations by applying Archimedes t-norm and s-norm [23, 24].

Definition 19 ([23]) For three DHFNs $\delta = (p, q)$ and $\delta_i = (p_i, q_i)(i = 1, 2)$, then we have

(1) $$\begin{aligned}\delta_1 \oplus \delta_2 &= \cup_{\gamma_i\in p_i,\eta_i\in q_i}\{\{S_I(\gamma_1,\gamma_2)\},\{T_I(\eta_1,\eta_2)\}\}\\ &= \cup_{\gamma_i\in p_i,\eta_i\in q_i}\left\{\left\{h_I^{-1}(h_I(\gamma_1)+h_I(\gamma_2))\right\},\left\{g_I^{-1}(g_I(\eta_1)+g_I(\eta_2))\right\}\right\};\end{aligned}$$
(2) $$\begin{aligned}\delta_1 \otimes \delta_2 &= \cup_{\gamma_i\in p_i,\eta_i\in q_i}\{\{T_I(\gamma_1,\gamma_2)\},\{S_I(\eta_1,\eta_2)\}\}\\ &= \cup_{\gamma_i\in p_i,\eta_i\in q_i}\left\{\left\{g_I^{-1}(g_I(\gamma_1)+g_I(\gamma_2))\right\},\left\{h_I^{-1}(h_I(\eta_1)+h_I(\eta_2))\right\}\right\};\end{aligned}$$
(3) $\lambda\delta = \cup_{\gamma\in p,\eta\in q}\left\{\left\{h_I^{-1}(\lambda h_I(\gamma))\right\},\left\{g_I^{-1}(\lambda g_I(\eta))\right\}\right\}, \lambda > 0$;
(4) $\delta^{\lambda} = \cup_{\gamma\in p,\eta\in q}\left\{\left\{g_I^{-1}(\lambda g_I(\gamma))\right\},\left\{h_I^{-1}(\lambda h_I(\eta))\right\}\right\}, \lambda > 0$.

where T_I and S_I are dual with respect to $N_I(a) = 1 - a$. g_I and h_I are the additive generators of T_I and S_I, respectively.

Definition 20 ([24]) For three DHPFNs $\psi = (a, b)$ and $\psi_i = (a_i, b_i)(i = 1, 2)$, then we have

(1) $$\begin{aligned}\psi_1 \oplus \psi_2 &= \cup_{\tau_i\in a_i,\varsigma_i\in b_i}\{\{S_P(\tau_1,\tau_2)\},\{T_P(\varsigma_1,\varsigma_2)\}\}\\ &= \cup_{\tau_i\in a_i,\varsigma_i\in b_i}\left\{\left\{h_P^{-1}(h_P(\tau_1)+h_P(\tau_2))\right\},\left\{g_P^{-1}(g_P(\varsigma_1)+g_P(\varsigma_2))\right\}\right\};\end{aligned}$$
(2) $$\begin{aligned}\psi_1 \otimes \psi_2 &= \cup_{\tau_i\in a_i,\varsigma_i\in b_i}\{\{T_P(\tau_1,\tau_2)\},\{S_P(\varsigma_1,\varsigma_2)\}\}\\ &= \cup_{\tau_i\in a_i,\varsigma_i\in b_i}\left\{\left\{g_P^{-1}(g_P(\tau_1)+g_P(\tau_2))\right\},\left\{h_P^{-1}(h_P(\varsigma_1)+h_P(\varsigma_2))\right\}\right\};\end{aligned}$$

(3) $\lambda\psi = \cup_{\tau\in a,\varsigma\in b}\left\{\left\{h_P^{-1}(\lambda h_P(\tau))\right\}, \left\{g_P^{-1}(\lambda g_P(\varsigma))\right\}\right\}, \lambda > 0$;
(4) $\delta^\lambda = \cup_{\tau\in a,\varsigma\in b}\left\{\left\{g_P^{-1}(\lambda g_P(\tau))\right\}, \left\{h_P^{-1}(\lambda h_P(\varsigma))\right\}\right\}, \lambda > 0$.

where T_P and S_P are dual with respect to $N_P(a) = \sqrt{1-a^2}$. g_P and h_P are the additive generators of T_P and S_P, respectively.

For convenience, in this study, the set of all the dual hesitant fuzzy sets is denoted as $\hat{A}_I$, and the set of all the Pythagorean fuzzy sets is denoted as $\hat{A}_P$.

Definition 21 Let $\psi_i = (a_i, b_i)_P \in \hat{A}_P$, a mapping $\hat{\wp}_{P\to I} : \hat{A}_P \to \hat{A}_I$ is defined as follows

$$\hat{\wp}_{P\to I}(\psi_i) = \left(\hat{\phi}(a_i), \hat{\phi}(b_i)\right)_I \in A_I,$$

where $\varphi(x) = x^2$ is an automorphism on [0, 1], and $\hat{\phi}(a_i) = \cup_{\tau_i\in a_i}\{\phi(\tau_i)\}$ and $\hat{\phi}(b_i) = \cup_{\tau_i\in a_i}\{\phi(\varsigma_i)\}$.

Definition 20 reveals that DHFSs and DHPFSs have a mathematical isomorphism $\tilde{\wp}_{P\to I}$.

Theorem 5 *Let* $\psi_i = (a_i, b_i)_P \in \hat{A}_P$, $\hat{\wp}_{P\to I}(\psi_i) = \left(\hat{\phi}(a_i), \hat{\phi}(b_i)\right)_I \in \hat{A}_I$, *and the operations of* ψ_i *on* $\hat{A}_P$ *and the operations of* $\hat{\wp}_{P\to I}(\psi_i)$ *on* $\hat{A}_I$ *are defined based on the t-norm and s-norm, then we have*

(1) $\hat{\wp}_{P\to I}(\psi_1 \oplus \psi_2) = \hat{\wp}_{P\to I}(\psi_1) \oplus \wp_{P\to I}(\psi_2)$;
(2) $\hat{\wp}_{P\to I}(\psi_1 \otimes \psi_2) = \hat{\wp}_{P\to I}(\psi_1) \otimes \wp_{P\to I}(\psi_2)$;
(3) $\hat{\wp}_{P\to I}(\lambda\psi_1) = \lambda\hat{\wp}_{P\to I}(\psi_1)$;
(4) $\hat{\wp}_{P\to I}\left(\psi_1^\lambda\right) = \hat{\wp}_{P\to I}(\psi_1)^\lambda$.

From Theorem 5, $\hat{\wp}_{P\to I}$ can be proved to be an isomorphic mapping from $\hat{A}_P$ to $\hat{A}_I$. Theorem 5 reveals that DHFSs and DHPFSs do not only have a mathematical isomorphism, but also have an operations isomorphism. The later is a substantial relationship between DHFSs and DHPFSs not reported up to now in literature.

4 Aggregation Operators Isomorphism

In the previous section, the isomorphic relationship between three kinds of Archimedean intuitionistic fuzzy operations and three kinds of Archimedean Pythagorean fuzzy operations is studied. On this basis, in this section, the relationship between three kinds of intuitionistic fuzzy aggregation operators and three kinds of Pythagorean fuzzy aggregation operators will be studied.

4.1 Aggregation Operators Isomorphism Between PFSs and IFSs

By extending four classical Archimedean t-norm and s-norm to the intuitionistic fuzzy environment, the scholars proposed Archimedean intuitionistic fuzzy aggregation operators, which can be used to aggregate intuitionistic fuzzy information and constitute a systematic intuitionistic fuzzy information aggregation system to support the construction of intuitionistic fuzzy decision-making method. It provides a reference for the generalized operators of interval-valued intuitionistic fuzzy sets, Pythagorean fuzzy sets, and so on.

Definition 22 ([16]) Let $\vartheta_i = \left(\rho_{\alpha_i}, \sigma_{\alpha_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs, w_i $(i = 1, 2, \ldots, n)$ is the weights of α_i $(i = 1, 2, \ldots, n)$, and $w_i \in [0, 1] (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_n = 1$. Then,

(1) the Archimedean intuitionistic fuzzy weighted average operator is defined as follows:

$$\text{Arch} - IFWA(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \left(h_I^{-1}\left(\sum_{i=1}^{n} w_i h_I(\rho_i)\right), g_I^{-1}\left(\sum_{i=1}^{n} w_i g_I(\sigma_i)\right)\right),$$

(2) the Archimedean intuitionistic fuzzy weighted geometric operator is defined as follows:

$$\text{Arch} - IFWG(\vartheta_1, \vartheta_2, \ldots, \vartheta_n) = \left(g_I^{-1}\left(\sum_{i=1}^{n} w_i g_I(\rho_i)\right), h_I^{-1}\left(\sum_{i=1}^{n} w_i h_I(\sigma_i)\right)\right).$$

Inspired by the Archimedean intuitionistic fuzzy aggregation operators, the Archimedean fuzzy aggregation operator is proposed based on the Pythagorean fuzzy operational rules in the previous section.

Definition 23 ([15]) Let $\alpha_i = \left(\mu_{\alpha_i}, \nu_{\alpha_i}\right)$ $(i = 1, 2, \ldots, n)$ be a collection of PFNs, w_i $(i = 1, 2, \ldots, n)$ is the weights of α_i $(i = 1, 2, \ldots, n)$, and $w_i \in [0, 1] (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_n = 1$. Then,

(1) the Archimedean Pythagorean fuzzy weighted average operator is defined as follows:

$$\text{Arch} - PFWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(h_P^{-1}\left(\sum_{i=1}^{n} w_i h_p(\mu_i)\right), g_P^{-1}\left(\sum_{i=1}^{n} w_i g_p(\nu_i)\right)\right).$$

(2) the Archimedean Pythagorean fuzzy weighted geometric operator is defined as follows:

$$\text{Arch} - PFWG(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(g_P^{-1}\left(\sum_{i=1}^{n} w_i g_p(\mu_i)\right), h_P^{-1}\left(\sum_{i=1}^{n} w_i h_p(\nu_i)\right)\right).$$

Based on the mapping relation $\wp_{P\to I}$ in Sect. 3.1, the relationship between two kinds of fuzzy aggregation operators is studied

Theorem 6 *Let* $\alpha_i = \langle \mu_i, \nu_i \rangle_P \in A_P (i = 1, 2, \ldots, n)$ *be a collection of PFNs,* $\wp_{P\to I}(\alpha_i) = (\phi(\mu_i), \phi(\nu_i))_I \in A_I$ *be a collection of IFNs, then*

(1) $\wp_{P\to I}(Arch - PFWA(\alpha_1, \alpha_2, \ldots, \alpha_n)) = Arch - IFWA(\wp_{P\to I}(\alpha_1), \wp_{P\to I}(\alpha_2), \ldots, \wp_{P\to I}(\alpha_n))$.

(2) $\wp_{P\to I}(Arch - PFWG(\alpha_1, \alpha_2, \ldots, \alpha_n)) = Arch - IFWG(\wp_{P\to I}(\alpha_1), \wp_{P\to I}(\alpha_2), \ldots, \wp_{P\to I}(\alpha_n))$.

From Theorem 6, the Archimedean Pythagorean and intuitionistic fuzzy aggregation operators are isomorphic with respect to the mapping $\wp_{P\to I}$.

4.2 Aggregation Operators Isomorphism Between IVPFSs and IVIFSs

In recent years, based on the classical Algebraic operations of interval-valued intuitionistic fuzzy sets, scholars extended Eistein, Hamacher, and Frank t-norm and s-norm to the environment of interval-valued intuitionistic fuzzy sets [19–21]. For convenience, these results are expressed in a general form as follows.

Definition 24 Let $\tilde{\vartheta}_i = \left(\left[\rho_i^-, \rho_i^+\right], \left[\sigma_i^-, \sigma_i^+\right]\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IVIFNs, w_i $(i = 1, 2, \ldots n)$ is the weights of $\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$, and $w_i \in [0, 1] (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_n = 1$. Then,

(1) the Archimedean interval-valued intuitionistic fuzzy weighted average operator is defined as follows:

$$\begin{aligned} &\text{Arch-IV}IFWA\left(\tilde{\vartheta}_1, \tilde{\vartheta}_2, \ldots, \tilde{\vartheta}_n\right) \\ &= \left(\left[h_I^{-1}\left(\sum_{i=1}^{n} w_i h_I\left(\rho_i^-\right)\right), h_I^{-1}\left(\sum_{i=1}^{n} w_i h_I\left(\rho_i^+\right)\right)\right],\right. \\ &\quad \left.\left[g_I^{-1}\left(\sum_{i=1}^{n} w_i g_I\left(\sigma_i^-\right)\right), g_I^{-1}\left(\sum_{i=1}^{n} w_i g_I\left(\sigma_i^+\right)\right)\right]\right) \end{aligned}$$

(2) the Archimedean interval-valued intuitionistic fuzzy weighted geometric operator is defined as follows:

$$\text{Arch-IV}IFWG\left(\tilde{\vartheta}_1, \tilde{\vartheta}_2, \ldots, \tilde{\vartheta}_n\right) = \left(\left[g_I^{-1}\left(\sum_{i=1}^{n} w_i g_I\left(\rho_i^-\right)\right), g_I^{-1}\left(\sum_{i=1}^{n} w_i g_I\left(\rho_i^+\right)\right)\right], \left[h_I^{-1}\left(\sum_{i=1}^{n} w_i h_I\left(\sigma_i^-\right)\right), h_I^{-1}\left(\sum_{i=1}^{n} w_i h_I\left(\sigma_i^+\right)\right)\right]\right).$$

The Archimedean interval-valued Pythagorean fuzzy aggregation operators are proposed to aggregate IVPFNs based on the operations.

Definition 25 ([22]) Let $\tilde{\alpha}_i = \left(\left[\mu_i^-, \mu_i^+\right], \left[v_i^-, v_i^+\right]\right)$ $(i = 1, 2, \ldots, n)$ be a collection of IVPFNs, w_i $(i = 1, 2, \ldots, n)$ is the weights of $\tilde{\alpha}_i$ $(i = 1, 2, \ldots, n)$, and $w_i \in [0, 1](i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_n = 1$. Then,

(1) the Archimedean interval-valued Pythagorean fuzzy weighted average operator is defined as follows:

$$\text{Arch-IV}PFWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\left[h_P^{-1}\left(\sum_{i=1}^{n} w_i h_p\left(\mu_i^-\right)\right), h_P^{-1}\left(\sum_{i=1}^{n} w_i h_p\left(\mu_i^+\right)\right)\right], \left[g_P^{-1}\left(\sum_{i=1}^{n} w_i g_p\left(v_i^-\right)\right), g_P^{-1}\left(\sum_{i=1}^{n} w_i g_p\left(v_i^+\right)\right)\right]\right).$$

(2) the Archimedean interval-valued Pythagorean fuzzy weighted geometric operator is defined as follows:

$$\text{Arch-IV}PFWG(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \left(\left[g_P^{-1}\left(\sum_{i=1}^{n} w_i g_p\left(\mu_i^-\right)\right), g_P^{-1}\left(\sum_{i=1}^{n} w_i g_p\left(\mu_i^+\right)\right)\right], \left[h_P^{-1}\left(\sum_{i=1}^{n} w_i h_p\left(v_i^-\right)\right), h_P^{-1}\left(\sum_{i=1}^{n} w_i h_p\left(v_i^+\right)\right)\right]\right).$$

Based on the mapping relation $\tilde{\wp}_{P\to I}$ in Sect. 3.2, the relationship between two kinds of fuzzy aggregation operators is studied.

Theorem 7 *Let* $\tilde{\alpha}_i = \left(\left[\mu_i^-, \mu_i^+\right], \left[v_i^-, v_i^+\right]\right)_P \in \tilde{A}_P(i = 1, 2, \ldots, n)$ *be a collection of IVPFNs,* $\tilde{\wp}_{P\to I}(\tilde{\alpha}_i) = \left(\left[\phi\left(\mu_i^-\right), \phi\left(\mu_i^+\right)\right], \left[\phi\left(v_i^-\right), \phi\left(v_i^+\right)\right]\right)_I \in \tilde{A}_I(i = 1, 2, \ldots, n)$ *be a collection of IVIFNs, then*

(1) $$\tilde{\wp}_{P\to I}(Arch - IVPFWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)) = Arch - IVIFWA(\tilde{\wp}_{P\to I}(\tilde{\alpha}_1), \wp_{P\to I}(\tilde{\alpha}_2), \ldots, \wp_{P\to I}(\tilde{\alpha}_n))$$

(2) $$\begin{aligned}&\tilde{\wp}_{P\to I}(Arch - IVPFWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n))\\ &= Arch - IVIFWA(\tilde{\wp}_{P\to I}(\tilde{\alpha}_1), \wp_{P\to I}(\tilde{\alpha}_2), \ldots, \wp_{P\to I}(\tilde{\alpha}_n))\end{aligned}.$$

From Theorem 7, the Archimedean interval-valued Pythagorean and intuitionistic fuzzy aggregation operators are isomorphic with respect to the mapping $\tilde{\wp}_{P\to I}$.

4.3 Aggregation Operators Isomorphism Between HPFSs and DHFSs

Based on the Archimedean dual hesitant fuzzy operations in the previous section, some series of Archimedean dual hesitant fuzzy power weighted aggregation operators are constructed. In this study, the degenerate operator Archimedes dual hesitant fuzzy weighted aggregation operators are mainly studied.

Definition 26 ([23]) Let $\delta_i = (p_i, q_i)$ $(i = 1, 2, \ldots, n)$ be a collection of IVIFNs, w_i $(i = 1, 2, \ldots, n)$ is the weights of δ_i $(i = 1, 2, \ldots, n)$, and $w_i \in [0, 1](i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_n = 1$. Then,

(1) the Archimedean dual hesitant fuzzy weighted average operator is defined as follows:

$$\begin{aligned}&\text{Arch} - DHFWA(\delta_1, \delta_2, \ldots, \delta_n)\\ &= \cup_{\gamma_i\in p_i,\eta_i\in q_i}\left\{\left\{h_I^{-1}\left(\sum_{i=1}^{n} w_i h_I(\gamma_i)\right)\right\}, \left\{g_I^{-1}\left(\sum_{i=1}^{n} w_i g_I(\eta_i)\right)\right\}\right\}.\end{aligned}$$

(2) the Archimedean dual hesitant fuzzy weighted geometric operator is defined as follows:

$$\begin{aligned}&\text{Arch} - DHFWG(\delta_1, \delta_2, \ldots, \delta_n)\\ &= \cup_{\gamma_i\in p_i,\eta_i\in q_i}\left\{\left\{g_I^{-1}\left(\sum_{i=1}^{n} w_i g_I(\gamma_i)\right)\right\}, \left\{h_I^{-1}\left(\sum_{i=1}^{n} w_i h_I(\eta_i)\right)\right\}\right\}.\end{aligned}$$

The Archimedean dual hesitant Pythagorean fuzzy aggregation operators are proposed to aggregate DHPFNs based on the operations.

Definition 27 ([24]) Let $\psi_i = (a_i, b_i)_P \in \hat{A}_P$ $(i = 1, 2, \ldots, n)$ be a collection of IVPFNs, w_i $(i = 1, 2, \ldots, n)$ is the weights of $\psi_i (i = 1, 2, \ldots, n)$, and $w_i \in [0, 1] (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_n = 1$. Then,

(1) the Archimedean dual hesitant Pythagorean fuzzy weighted average operator is defined as follows:

$$\begin{aligned} &\text{Arch} - \text{DH}PFWA(\psi_1, \psi_2, \ldots, \psi_n) \\ &= \cup_{\tau_i \in a_i, \varsigma_i \in b_i} \left\{ \left\{ h_P^{-1}\left(\sum_{i=1}^{n} w_i h_p(\tau_i) \right) \right\}, \left\{ g_P^{-1}\left(\sum_{i=1}^{n} w_i g_p(\varsigma_i) \right) \right\} \right\}. \end{aligned}$$

(2) the Archimedean dual hesitant Pythagorean fuzzy weighted geometric operator is defined as follows:

$$\begin{aligned} &\text{Arch} - \text{DH}PFWG(\psi_1, \psi_2, \ldots, \psi_n) \\ &= \cup_{\tau_i \in a_i, \varsigma_i \in b_i} \left\{ \left\{ g_P^{-1}\left(\sum_{i=1}^{n} w_i g_p(\tau_i) \right) \right\}, \left\{ h_P^{-1}\left(\sum_{i=1}^{n} w_i h_p(\varsigma_i) \right) \right\} \right\}. \end{aligned}$$

Based on the mapping relation $\hat{\wp}_{P\to I}$ in Sect. 3.3, the relationship between two kinds of fuzzy aggregation operators is studied.

Theorem 8 *Let* $\psi_i = (a_i, b_i)_P \in \hat{A}_P$ $(i = 1, 2, \ldots, n)$ *be a collection of IVPFNs,* $\hat{\wp}_{P\to I}(\psi_i) = \left(\hat{\phi}(a_i), \hat{\phi}(b_i)\right)_I \in \hat{A}_I (i = 1, 2, \ldots, n)$ *be a collection of IVIFNs, then*

(1) $\hat{\wp}_{P\to I}(Arch - DHPFWA(\psi_1, \psi_2, \ldots, \psi_n))$
$= Arch - DHFWA\left(\hat{\wp}_{P\to I}(\psi_1), \hat{\wp}_{P\to I}(\psi_2), \ldots, \hat{\wp}_{P\to I}(\psi_n)\right)$

(2) $$\begin{aligned} &\hat{\wp}_{P\to I}(Arch - DHPFWG(\psi_1, \psi_2, \ldots, \psi_n)) \\ &= Arch - DHFWG\left(\hat{\wp}_{P\to I}(\psi_1), \hat{\wp}_{P\to I}(\psi_2), \ldots, \hat{\wp}_{P\to I}(\psi_n)\right). \end{aligned}$$

From Theorem 8, the Archimedean dual hesitant Pythagorean and dual hesitant fuzzy aggregation operators are isomorphic with respect to the mapping $\hat{\wp}_{P\to I}$.

The effectiveness of isomorphic operators is verified and illustrated by aggregating the evaluation information of Pythagorean fuzzy multi-attribute decision-making problem.

Example 1 Let $X_i (i = 1, 2)$ be two alternatives, $C_j (j = 1, 2, 3, 4)$ be four attributes, and $w_j = 1/4$ is the weight of $C_j (j = 1, 2, 3, 4)$. Suppose the Pythagorean fuzzy decision matrix $D = \left(\alpha_{ij}\right)_{2\times 4} = \left(\left(\mu_{ij}, v_{ij}\right)\right)_{2\times 4}$ is provided in Table 2.

We use the Pythagorean fuzzy weighted average (PFWA) operator [6] to aggregation decision information $\alpha_{ij} (j = 1, 2, 3, 4)$ for alternatives $X_i (i = 1, 2)$, and obtain the comprehensive evaluation information $\alpha_i (i = 1, 2)$:

Table 2 Pythagorean fuzzy decision matrix D

	C_1	C_2	C_3	C_4	PFWA operator
X_1	(0.4, 0.5)	(0.3, 0.6)	(0.4, 0.6)	(0.2, 0.5)	(0.3375, 0.3130)
X_2	(0.5, 0.3)	(0.6, 0.3)	(0.7, 0.2)	(0.4, 0.2)	(0.5715, 0.5384)

Table 3 Intuitionistic fuzzy decision matrix D_I with mapping relation $\wp_{P\to I}$

	C_1	C_2	C_3	C_4	IFWA operator
X_1	(0.16, 0.25)	(0.09, 0.36)	(0.16, 0.36)	(0.04, 0.25)	(0.1139, 0.0980)
X_2	(0.25, 0.09)	(0.36, 0.03)	(0.49, 0.04)	(0.16, 0.04)	(0.3266, 0.2898)

$$\alpha_1 = PFWA(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}) = \left(\sqrt{1-\prod_{j=1}^{4}\left(1-\mu_{1j}^2\right)^{w_j}}, \prod_{j=1}^{4} v_{1j}^{w_j}\right)$$
$$= (0.3375, 0.3130)$$

$$\alpha_2 = PFWA(\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24}) = \left(\sqrt{1-\prod_{j=1}^{4}\left(1-\mu_{2j}^2\right)^{w_j}}, \prod_{j=1}^{4} v_{2j}^{w_j}\right).$$
$$= (0.5715, 0.5384)$$

Based on the mapping relation $\wp_{P\to I}$, we obtain the intuitionistic fuzzy decision matrix $D_I = \left(\alpha'_{ij}\right)_{2\times 4}$, which is shown in Table 3. We use the intuitionistic fuzzy weighted average (IFWA) operator [31] to aggregation decision information $\alpha'_{ij}(j = 1, 2, 3, 4)$ for alternatives $X_i(i = 1, 2)$, and obtain the comprehensive evaluation information $\alpha'_i(i = 1, 2)$.

$$\begin{aligned}\alpha'_1 &= IFWA\left(\alpha'_1, \alpha'_{12}, \alpha'_{13}, \alpha'_{14}\right)\\ &= IFWA\left(\hat{\wp}_{P\to I}(\alpha_{11}), \hat{\wp}_{P\to I}(\alpha_{12}), \hat{\wp}_{P\to I}(\alpha_{13}), \hat{\wp}_{P\to I}(\alpha_{14})\right)\\ &= \left(1-\prod_{j=1}^{4}(1-\rho_{1j})^{w_j}, \prod_{j=1}^{4}\sigma_{1j}^{w_j}\right) = \hat{\wp}_{P\to I}(\alpha_1) = (0.1139, 0.0980)\end{aligned}$$

$$\begin{aligned}\alpha'_2 &= IFWA\left(\alpha'_{21}, \alpha'_{22}, \alpha'_{23}, \alpha'_{24}\right)\\ &= IFWA\left(\hat{\wp}_{P\to I}(\alpha_{21}), \hat{\wp}_{P\to I}(\alpha_{22}), \hat{\wp}_{P\to I}(\alpha_{23}), \hat{\wp}_{P\to I}(\alpha_{24})\right)\\ &= \left(1-\prod_{j=1}^{4}(1-\rho_{1j})^{w_j}, \prod_{j=1}^{4}\sigma_{1j}^{w_j}\right) = \hat{\wp}_{P\to I}(\alpha_2)=(0.3266, 0.2898)\end{aligned}.$$

Example 1 shows that isomorphism operators can transform Pythagorean fuzzy multi-attribute decision-making (FMADM) problem into intuitionistic FMADM problem, and the aggregation result obtained is consistent with that before transformation. Similarly, the hesitant Pythagorean FMADM problem and interval-valued

Pythagorean FMADM problem can be transformed into dual hesitant FMADM problem and interval-valued intuitionistic FMADM problem, respectively.

5 Ranking Methods Isomorphism

For fuzzy multi-attribute decision-making approach, the ranking method of fuzzy numbers is used to sort the comprehensive attribute values to determine the evaluation results. The ranking method of three groups of research objects in this study is mainly based on score function and accuracy function. Based on the isomorphic perspective, this section will study the relationship between three sets of fuzzy sets.

5.1 Ranking Methods Isomorphism Between IFNs and PFNs

The score function and accuracy function of intuitionistic fuzzy sets constitute the sorting method of intuitionistic fuzzy numbers and also lay a foundation for the sorting methods of other fuzzy sets.

Definition 28 ([25]) For any two IFNs, $\vartheta_i = (\rho_i, \sigma_i)(i = 1, 2)$:

(i) if $sco_I(\vartheta_1) < sco_I(\vartheta_2)$, then $\vartheta_1 <_I \vartheta_2$;
(ii) if $sco_I(\vartheta_1) = sco_I(\vartheta_2)$, then

  (a) if $acc_I(\vartheta_1) < acc_I(\vartheta_2)$, then $\vartheta_1 <_I \vartheta_2$;
  (b) if $acc_I(\vartheta_1) = acc_I(\vartheta_2)$, then $\vartheta_1 = \vartheta_2$,

where $sco_I(\vartheta_i) = \rho_i - \sigma_i$ and $acc_I(\vartheta_i) = \rho_i + \sigma_i(i = 1, 2)$ are the score functions and accuracy functions, respectively.

Based on the score function and accuracy function of intuitionistic fuzzy sets, scholars put forward the score function and accuracy function of the Pythagorean fuzzy sets, and then constructed the sorting method for ranking Pythagorean fuzzy numbers.

Definition 29 ([26, 27]) For any two PFNs $\alpha_i = (\mu_i, v_i)(i = 1, 2)$:

(i) if $sco_P(\alpha_1) < sco_P(\alpha_2)$, then $\alpha_1 <_p \alpha_2$;
(ii) if $sco_P(\alpha_1) = sco_P(\alpha_2)$, then

  (a) if $acc_P(\alpha_1) < acc_P(\alpha_2)$, then $\alpha_1 <_p \alpha_2$;
  (b) if $acc_P(\alpha_1) = acc_P(\alpha_2)$, then $\alpha_1 \sim_p \alpha_2$,

where $sco_P(\alpha_i) = \mu_i^2 - v_i^2(i = 1, 2)$ are the score functions of $\alpha_i(i = 1, 2)$, and $acc_P(\alpha_i) = \mu_i^2 + v_i^2(i = 1, 2)$ are the accuracy functions.

Based on the mapping relation $\wp_{P \to I}$ in Sect. 3.1, the relationship between two kinds of fuzzy ranking methods is studied.

Theorem 9 *Let* $\alpha_i = \langle \mu_i, \nu_i \rangle_P \in A_P$, $\wp_{P\to I}(\alpha_i) = (\phi(\mu_i), \phi(\nu_i))_I \in A_I$, *then we have*

(1) $sco_P(\alpha_i) = sco_I(\wp_{P\to I}(\alpha_i))$;
(2) $acc_P(\alpha_i) = acc_I(\wp_{P\to I}(\alpha_i))$.

Theorem 10 *Let* $\alpha_i = \langle \mu_i, \nu_i \rangle_P \in A_P$, $\wp_{P\to I}(\alpha_i) = (\phi(\mu_i), \phi(\nu_i))_I \in A_I$. *Then,*

$\alpha_1 <_p \alpha_2$ *if and only if* $\wp_{P\to I}(\alpha_1) <_I \wp_{P\to I}(\alpha_2)$

and

$\alpha_1 \sim_p \alpha_2$ *if and only if* $\wp_{P\to I}(\alpha_1) \sim_I \wp_{P\to I}(\alpha_2)$.

Theorem 10 reveals IFSs and PFSs have a ranking method isomorphism.

5.2 Ranking Methods Isomorphism Between IVIFNs and IVPFNs

The ranking methods of interval intuitionistic fuzzy numbers and interval Pythagorean fuzzy numbers are mainly defined by using score function and accuracy function. In recent years, scholars have proposed a series of scoring functions and accuracy functions of interval-valued Pythagorean fuzzy numbers (IVPFNs) from different perspectives, and used to construct corresponding decision analysis methods. For example, fuzzy linear programming model [34] and TOPSIS method [35] based on improved score function, decision analysis methods based on improved accuracy function [36, 37]. Considering the generality, this paper mainly analyzes the traditional score function and accuracy function of IVPFNs.

Definition 30 ([28]) For any two IFNs, $\tilde{\vartheta}_i = \left(\left[\rho_i^-, \rho_i^+\right], \left[\sigma_i^-, \sigma_i^+\right]\right)(i = 1, 2)$:

(i) if $Sco_I\left(\tilde{\vartheta}_1\right) < Sco_I\left(\tilde{\vartheta}_2\right)$, then $\tilde{\vartheta}_1 <_{\tilde{I}} \tilde{\vartheta}_2$;
(ii) if $Sco_I\left(\tilde{\vartheta}_1\right) = Sco_I\left(\tilde{\vartheta}_2\right)$, then

 (a) if $Acc_I\left(\tilde{\vartheta}_1\right) < Acc_I\left(\tilde{\vartheta}_2\right)$, then $\tilde{\vartheta}_1 <_{\tilde{I}} \tilde{\vartheta}_2$;
 (b) if $Acc_I\left(\tilde{\vartheta}_1\right) = Acc_I\left(\tilde{\vartheta}_2\right)$, then $\tilde{\vartheta}_1 \sim_{\tilde{I}} \tilde{\vartheta}_2$,

where $Sco_I\left(\tilde{\vartheta}_i\right) = \left(\rho_i^- + \rho_i^+ - \sigma_i^- - \sigma_i^+\right)/2$ and $Acc_I\left(\tilde{\vartheta}_i\right) = \left(\rho_i^- + \rho_i^+ + \sigma_i^- + \sigma_i^+\right)/2(i = 1, 2)$ are the score functions and accuracy functions, respectively.

Definition 31 ([7, 29]). For any two IVPFNs $\tilde{\alpha}_i = \left(\left[\mu_i^-, \mu_i^+\right], \left[\nu_i^-, \nu_i^+\right]\right)(i = 1, 2)$:

(i) if $Sco_P(\tilde{\alpha}_1) < Sco_P(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 <_{\tilde{P}} \tilde{\alpha}_2$;
(ii) if $Sco_P(\tilde{\alpha}_1) = Sco_P(\tilde{\alpha}_2)$, then

(a) if $Acc_P(\tilde{\alpha}_1) < Acc_P(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 <_{\tilde{P}} \tilde{\alpha}_2$;
(b) if $Acc_P(\tilde{\alpha}_1) = Acc_P(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 \sim_{\tilde{P}} \tilde{\alpha}_2$,

where $Sco_P(\tilde{\alpha}_i) = \left(\left(\mu_i^-\right)^2 + \left(\mu_i^+\right)^2 - \left(v_i^-\right)^2 - \left(v_i^+\right)^2\right) \Big/ 2$ and $Acc_P(\tilde{\alpha}_i) = \left(\left(\mu_i^-\right)^2 + \left(\mu_i^+\right)^2 + \left(v_i^-\right)^2 + \left(v_i^+\right)^2\right) \Big/ 2$ are the score functions and accuracy functions, respectively.

Based on the mapping relation $\hat{\wp}_{P \to I}$ in Sect. 3.2, the relationship between two kinds of fuzzy ranking methods is studied.

Theorem 11 *Let* $\tilde{\alpha}_i = \left(\left[\mu_i^-, \mu_i^+\right], \left[v_i^-, v_i^+\right]\right)_P \in \tilde{A}_P$, $\tilde{\wp}_{P \to I}(\tilde{\alpha}_i) = \left(\left[\phi\left(\mu_i^-\right), \phi\left(\mu_i^+\right)\right], \left[\phi\left(v_i^-\right), \phi\left(v_i^+\right)\right]\right)_I \in \tilde{A}_I$, *then*

(1) $Sco_P(\tilde{\alpha}_i) = Sco_I(\tilde{\wp}_{P \to I}(\tilde{\alpha}_i))$;

(2) $Acc_P(\tilde{\alpha}_i) = Acc_I(\tilde{\wp}_{P \to I}(\tilde{\alpha}_i))$.

Theorem 12 *Let* $\tilde{\alpha}_i = \left(\left[\mu_i^-, \mu_i^+\right], \left[v_i^-, v_i^+\right]\right)_P \in \tilde{A}_P$, $\tilde{\wp}_{P \to I}(\tilde{\alpha}_i) = \left(\left[\phi\left(\mu_i^-\right), \phi\left(\mu_i^+\right)\right], \left[\phi\left(v_i^-\right), \phi\left(v_i^+\right)\right]\right)_I \in \tilde{A}_I$. *Then,*

(1) $\tilde{\alpha}_1 <_{\tilde{P}} \tilde{\alpha}_2$ if and only if $\tilde{\wp}_{P \to I}(\tilde{\alpha}_1) <_{\tilde{I}} \tilde{\wp}_{P \to I}(\tilde{\alpha}_2)$;
(2) $\tilde{\alpha}_1 \sim_{\tilde{P}} \tilde{\alpha}_2$ if and only if $\tilde{\wp}_{P \to I}(\tilde{\alpha}_1) \sim_{\tilde{I}} \tilde{\wp}_{P \to I}(\tilde{\alpha}_2)$;

Theorem 12 reveals IVIFSs and IVPFSs have a ranking method isomorphism.

5.3 Ranking Methods Isomorphism Between DHFSs and DHPFSs

The score function and accuracy function of DHFSs and DHPFSs are constructed mainly from the perspective of the expected mean of membership degree values and nonmembership degree values.

Definition 32 ([30]) For any two DHFNs, $\delta_i = (p_i, q_i)$ $(i = 1, 2)$:

(i) if $\hat{S}co_I(\delta_1) < \hat{S}co_I(\delta_2)$, then $\delta_1 <_{\hat{I}} \delta_2$;
(ii) if $\hat{S}co_I(\delta_1) = \hat{S}co_I(\delta_2)$, then

(a) if $\hat{A}cc_I(\delta_1) < \hat{A}cc_I(\delta_2)$, then $\delta_1 <_{\hat{I}} \delta_2$;
(b) if $\hat{A}cc_I(\delta_1) = \hat{A}cc_I(\delta_2)$, then $\delta_1 \sim_{\hat{I}} \delta_2$,

where $\hat{S}co_I(\delta_i) = \frac{1}{\# p_i} \sum_{\gamma_i \in p_i} \gamma_i - \frac{1}{\# q_i} \sum_{\eta_i \in q_i} \eta_i$ and $\hat{A}cc_I(\delta_i) = \frac{1}{\# p_i} \sum_{\gamma_i \in p_i} \gamma_i + \frac{1}{\# q_i} \sum_{\eta_i \in q_i} \eta_i (i = 1, 2)$ are the score functions and accuracy functions, respectively.

Definition 33 ([9]) For any two HPFNs, $\psi_i = (a_i, b_i)$ $(i = 1, 2)$:

(i) if $\hat{S}co_P(\psi_1) < \hat{S}co_P(\psi_2)$, then $\psi_1 <_{\hat{P}} \psi_2$;
(ii) if $\hat{S}co_P(\psi_1) = \hat{S}co_P(\psi_2)$, then

 (a) if $\hat{A}cc_P(\psi_1) < \hat{A}co_P(\psi_2)$, then $\psi_1 <_{\hat{P}} \psi_2$;
 (b) if $\hat{A}cc_P(\psi_1) = \hat{A}co_P(\psi_2)$, then $\psi_1 \sim_{\hat{P}} \psi_2$,

where $\hat{S}co_P(\psi_i) = \frac{1}{\#a_i}\sum_{\tau\in a_i}\tau_i - \frac{1}{\#b_i}\sum_{\varsigma_i\in b_i}\varsigma_i$ and $\hat{A}cc_P(\psi_i) = \frac{1}{\#a_i}\sum_{\tau\in a_i}\tau_i + \frac{1}{\#b_i}\sum_{\varsigma_i\in b_i}\varsigma_i (i = 1, 2)$ are the score functions and accuracy functions, respectively.

Based on the mapping relation $\hat{\wp}_{P\to I}$ in Sect. 3.3, the relationship between two kinds of fuzzy ranking methods is studied.

Theorem 13 *Let $\psi_i = (a_i, b_i)_P \in \hat{A}_P$, $\hat{\wp}_{P\to I}(\psi_i) = \left(\hat{\phi}(a_i), \hat{\phi}(b_i)\right)_I \in \hat{A}_I$, then we have*

(1) $\hat{S}co_P(\psi_i) = \hat{S}co_I\left(\hat{\wp}_{P\to I}(\psi_i)\right)$;

(2) $\hat{A}cc_P(\psi_i) = \hat{A}cc_I\left(\hat{\wp}_{P\to I}(\psi_i)\right)$

Theorem 14 *Let $\tilde{\alpha}_i = \left(\left[\mu_i^-, \mu_i^+\right], \left[v_i^-, v_i^+\right]\right)_P \in \tilde{A}_P$, $\hat{\wp}_{P\to I}(\psi_i) = \left(\hat{\phi}(a_i), \hat{\phi}(b_i)\right)_I \in \hat{A}_I$. Then,*

(1) $\psi_1 <_{\hat{P}} \psi_2$ if and only if $\hat{\wp}_{P\to I}(\psi_1) <_{\tilde{I}} \hat{\wp}_{P\to I}(\psi_2)$
(2) $\psi_1 \sim_{\hat{P}} \psi_2$ if and only if $\hat{\wp}_{P\to I}(\psi_1) \sim_{\tilde{I}} \hat{\wp}_{P\to I}(\psi_2)$.

Theorem 12 reveals DHFSs and DHPFSs have a ranking method isomorphism.

Example 2 (Continuation Example 1) In this example, the evaluation matrix of Tables 1 and 2 in Example 1 is used. Based on the score functions for Pythagorean and intuitionistic fuzzy numbers, we obtain the score matrix for Tables 1 and 2, which is shown in Table 3. Since $sco_P\left(\alpha_{1j}\right) < sco_P\left(\alpha_{2j}\right)$, $j = 1, 2, 3, 4$, then $\alpha_{1j} <_P \alpha_{2j}$, $j = 1, 2, 3, 4$. Since $sco_I\left(\alpha_{1j}^{'}\right) = sco_I\left(\wp_{P\to I}\left(\alpha_{1j}\right)\right) < sco_I\left(\alpha_{2j}^{'}\right) = sco_I\left(\wp_{P\to I}\left(\alpha_{2j}\right)\right)$, $j = 1, 2, 3, 4$, then $\alpha_{1j}^{'} <_P \alpha_{2j}^{'}$, $j = 1, 2, 3, 4$. Table 4 shows that the scoring matrix corresponding to Tables 2 and 3 is the same, and the effectiveness of isomorphic ranking method is verified.

Table 4 Score matrix

	C_1	C_2	C_3	C_4
X_1	−0.09	−0.27	−0.33	−0.12
X_2	0.16	0.27	0.45	0.12

6 Proofs

Proof of Theorem 2

Proof Since

$$g_P(t) = g_I(\phi(t)) \Rightarrow g_P^{-1}(t) = \phi^{-1}\big(g_I^{-1}(t)\big) \Rightarrow g_I^{-1}(t) = \phi\big(g_P^{-1}(t)\big)$$

and

$$h_P(t) = h_I(\phi(t)) \Rightarrow h_P^{-1}(t) = \phi^{-1}\big(h_I^{-1}(t)\big) \Rightarrow h_I^{-1}(t) = \phi\big(h_P^{-1}(t)\big)$$

Then

$$\begin{aligned} S_P(x, y) = h_P^{-1}(h_P(x) + h_P(y)) &= \phi^{-1}\big(h_I^{-1}(h_I(\phi(x)) + h_P(\phi(y)))\big) \\ &= \phi^{-1}(S_I(\phi(x), \phi(y))) \end{aligned}$$

and

$$\begin{aligned} T_P(x, y) = g_P^{-1}(g_P(x) + g_P(y)) &= \phi^{-1}\big(g_I^{-1}(g_I(\phi(x)) + g_P(\phi(y)))\big) \\ &= \phi^{-1}(T_I(\phi(x), \phi(y))). \end{aligned}$$

Theorem 2 is proven.

Proof of Theorem 3

Proof Because $\alpha_i = \langle \mu_i, \nu_i \rangle_P \in A_P$, we have $\mu_i^2 + \nu_i^2 \leq 1$. It means that if α_i is a PFN on A_P, and then $\wp_{P\to I}(\alpha_i)$ is an IFN on A_I.

(1) Following Definition 14 and Theorem 2, we have

$$\alpha_1 \oplus \alpha_2 = (S_P(\mu_1, \mu_2), T_P(\nu_1, \nu_2))$$

and

$$S_P(x, y) = \phi^{-1}(S_I(\phi(x), \phi(y))), \;\; T_P(x, y) = \phi^{-1}(T_I(\phi(x), \phi(y)))$$

then

$$\begin{aligned} \wp_{P\to I}(\alpha_1 \oplus \alpha_2) &= (\phi(S_P(\mu_1, \mu_2)), \phi(T_P(\nu_1, \nu_2))) \\ &= (S_I(\phi(\mu_1), \phi(\mu_2)), T_I(\phi(\nu_1), \phi(\nu_2))) \\ &= \wp_{P\to I}(\alpha_1) \oplus \wp_{P\to I}(\alpha_2) \end{aligned}$$

which completes the proof of (1).

(2) The proof of (2) is similar to (1) and it is hence omitted here.
(3) Following Definition 14, we have $\lambda\alpha = \left(h_p^{-1}(\lambda h_P(\mu)), g_p^{-1}(\lambda g_P(v))\right), \lambda > 0,$

and

$$g_P(t) = g_I(\phi(t)) \Rightarrow g_P^{-1}(t) = \phi^{-1}\left(g_I^{-1}(t)\right) \Rightarrow g_I^{-1}(t) = \phi\left(g_P^{-1}(t)\right),$$

then

$$\begin{aligned}\wp_{P\to I}(\lambda\alpha) &= \left(\phi\left(h_P^{-1}(\lambda h_P(\mu))\right), \phi\left(g_P^{-1}(\lambda g_P(v))\right)\right)\\ &= \left(h_I^{-1}(\lambda h_P(\mu)), g_I^{-1}(\lambda g_P(v))\right)\\ &=\left(h_I^{-1}(\lambda h_I(\phi(\mu))), g_I^{-1}(\lambda g_I(\phi(v)))\right)\\ &= \lambda\wp_{P\to I}(\alpha)\end{aligned},$$

which completes the proof of (3).

(4) The proof of (4) is similar to (3) and it is hence omitted here.

Theorem 3 is proven.
Proof of Theorem 4

Proof (1) Following Definition 17 and Theorem 2, we have

$$\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left(\left[S_P\left(\mu_1^-, \mu_2^-\right), S_P\left(\mu_1^+, \mu_2^+\right)\right], \left[T_P\left(v_1^-, v_2^-\right), T_P\left(v_1^+, v_2^+\right)\right]\right)$$

and

$$S_P(x, y) = \phi^{-1}(S_I(\phi(x), \phi(y))),\ \ T_P(x, y) = \phi^{-1}(T_I(\phi(x), \phi(y))),$$

then

$$\begin{aligned}\tilde{\wp}_{P\to I}(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) &= \left(\left[\phi\left(S_P\left(\mu_1^-, \mu_2^-\right)\right), \phi\left(S_P\left(\mu_1^+, \mu_2^+\right)\right)\right], \left[\phi\left(T_P\left(v_1^-, v_2^-\right)\right), \phi\left(T_P\left(v_1^+, v_2^+\right)\right)\right]\right)\\ &= \begin{pmatrix}\left[\phi(\phi^{-1}(S_I(\phi(\mu_1^-), \phi(\mu_2^-)))), \phi(\phi^{-1}(S_I(\phi(\mu_1^+), \phi(\mu_2^+))))\right],\\ \left[\phi(\phi^{-1}(T_I(\phi(v_1^-), \phi(v_2^-)))), \phi(\phi^{-1}(T_I(\phi(v_1^+), \phi(v_2^+))))\right]\end{pmatrix}\\ &= \left(\left[S_I(\phi(\mu_1^-), \phi(\mu_2^-)), S_I(\phi(\mu_1^+), \phi(\mu_2^+))\right],\right.\\ &\qquad \left.\left[T_I(\phi(v_1^-), \phi(v_2^-)), T_I(\phi(v_1^+), \phi(v_2^+))\right]\right)\\ &= \tilde{\wp}_{P\to I}(\tilde{\alpha}_1) \oplus \tilde{\wp}_{P\to I}(\tilde{\alpha}_2)\end{aligned},$$

which completes the proof of (1).

(2) The proof of (2) is similar to (1) and it is hence omitted here.

(3) Following Definitions 16 and 17, we have $\lambda\tilde{\alpha} = \left(\left[h_P^{-1}\left(\lambda h_P\left(\mu^-\right)\right), h_P^{-1}\left(\lambda h_P\left(\mu^+\right)\right)\right], \left[g_P^{-1}\left(\lambda g_P\left(\nu^-\right)\right), g_P^{-1}\left(\lambda g_P\left(\nu^+\right)\right)\right]\right), \lambda > 0,$ and

$$g_P(t) = g_I(\phi(t)) \Rightarrow g_P^{-1}(t) = \phi^{-1}\left(g_I^{-1}(t)\right) \Rightarrow g_I^{-1}(t) = \phi\left(g_P^{-1}(t)\right)$$

then

$$\begin{aligned}
\tilde{\wp}_{P\to I}(\lambda\tilde{\alpha}) &= \left(\left[\phi\left(h_P^{-1}\left(\lambda h_P\left(\mu^-\right)\right)\right), \phi\left(h_P^{-1}\left(\lambda h_P\left(\mu^+\right)\right)\right)\right],\right.\\
&\quad \left.\left[\phi\left(g_P^{-1}\left(\lambda g_P\left(\nu^-\right)\right)\right), \phi\left(g_P^{-1}\left(\lambda g_P\left(\nu^+\right)\right)\right)\right]\right)\\
&= \begin{pmatrix} \left[\phi\left(\phi^{-1}\left(h_I^{-1}\left(\lambda h_I\left(\phi\left(\mu^-\right)\right)\right)\right)\right), \phi\left(\phi^{-1}\left(h_I^{-1}\left(\lambda h_I\left(\phi\left(\mu^+\right)\right)\right)\right)\right)\right], \\ \left[\phi\left(\phi^{-1}\left(g_I^{-1}\left(\lambda g_I\left(\phi\left(\nu^-\right)\right)\right)\right)\right), \phi\left(\phi^{-1}\left(g_I^{-1}\left(\lambda g_I\left(\phi\left(\nu^+\right)\right)\right)\right)\right)\right] \end{pmatrix}\\
&= \left(\left[h_I^{-1}\left(\lambda h_I\left(\phi\left(\mu^-\right)\right)\right), h_I^{-1}\left(\lambda h_I\left(\phi\left(\mu^+\right)\right)\right)\right],\right.\\
&\quad \left.\left[g_I^{-1}\left(\lambda g_I\left(\phi\left(\nu^-\right)\right)\right), g_I^{-1}\left(\lambda g_I\left(\phi\left(\nu^+\right)\right)\right)\right]\right)\\
&= \lambda\tilde{\wp}_{P\to I}(\tilde{\alpha})
\end{aligned}$$

which completes the proof of (3).

(4) The proof of (4) is similar to (3) and it is hence omitted here.

Theorem 4 is proven.

Proof of Theorem 5

Proof

(1) Following Definition 20 and Theorem 2, we have

$$\psi_1 \oplus \psi_2 = \cup_{\tau_i \in a_i, \varsigma_i \in b_i}\{\{S_P(\tau_1, \tau_2)\}, \{T_P(\varsigma_1, \varsigma_2)\}\}$$

and

$$S_P(x, y) = \phi^{-1}(S_I(\phi(x), \phi(y))),\ \ T_P(x, y) = \phi^{-1}(T_I(\phi(x), \phi(y)))$$

then

$$\begin{aligned}
\hat{\wp}_{P\to I}(\psi_1 \oplus \psi_2) &= \cup_{\tau_i \in a_i, \varsigma_i \in b_i}\{\{\phi(S_P(\tau_1, \tau_2))\}, \{\phi(T_P(\varsigma_1, \varsigma_2))\}\}\\
&= \cup_{\tau_i \in a_i, \varsigma_i \in b_i}\left\{\left\{\phi\left(\phi^{-1}(S_I(\phi(\tau_1), \phi(\tau_2)))\right)\right\},\right.\\
&\qquad\qquad \left.\left\{\phi\left(\phi^{-1}(T_I(\phi(\varsigma_1), \phi(\varsigma_2)))\right)\right\}\right\}\\
&= \cup_{\tau_i \in a_i, \varsigma_i \in b_i}\{\{S_I(\phi(\tau_1), \phi(\tau_2))\}, \{T_I(\phi(\varsigma_1), \phi(\varsigma_2))\}\}\\
&= \wp_{P\to I}(\alpha_1) \oplus \wp_{P\to I}(\alpha_2)
\end{aligned}$$

which completes the proof of (1).

(2) The proof of (2) is similar to (1) and it is hence omitted here.
(3) Following Definitions 19 and 20, we have $\lambda\psi = \cup_{\tau\in a,\varsigma\in b}\left\{\left\{h_P^{-1}(\lambda h_P(\tau))\right\},\left\{g_P^{-1}(\lambda g_P(\varsigma))\right\}\right\}$,

and

$$g_P(t) = g_I(\phi(t)) \Rightarrow g_P^{-1}(t) = \phi^{-1}\left(g_I^{-1}(t)\right) \Rightarrow g_I^{-1}(t) = \phi\left(g_P^{-1}(t)\right)$$

then

$$\begin{aligned}
\hat{\wp}_{P\to I}(\lambda\psi_1) &= \cup_{\tau_1\in a_1,\varsigma_1\in b_1}\left\{\left\{\phi\left(h_P^{-1}(\lambda h_P(\tau_1))\right)\right\},\left\{\phi\left(g_P^{-1}(\lambda g_P(\varsigma_1))\right)\right\}\right\}\\
&= \cup_{\tau_1\in a_1,\varsigma_1\in b_1}\left\{\left\{\phi\left(\phi^{-1}\left(h_I^{-1}(\lambda h_I(\phi(\tau_1)))\right)\right)\right\},\left\{\phi\left(\phi^{-1}\left(g_I^{-1}(\lambda g_I(\phi(\varsigma_1)))\right)\right)\right\}\right\}\\
&= \cup_{\tau_1\in a_1,\varsigma_1\in b_1}\left\{\left\{h_I^{-1}(\lambda h_I(\phi(\tau_1)))\right\},\left\{g_I^{-1}(\lambda g_I(\phi(\varsigma_1)))\right\}\right\}\\
&= \lambda\hat{\wp}_{P\to I}(\psi_1)
\end{aligned}$$

which completes the proof of (3).

(4) The proof of (4) is similar to (3) and it is hence omitted here.

Theorem 5 is proven.

Proof of Theorem 6

Proof

(1) Since

$$g_P(t) = g_I(\phi(t)) \Rightarrow g_P^{-1}(t) = \phi^{-1}\left(g_I^{-1}(t)\right) \Rightarrow g_I^{-1}(t) = \phi\left(g_P^{-1}(t)\right)$$

and

$$h_P(t) = h_I(\phi(t)) \Rightarrow h_P^{-1}(t) = \phi^{-1}\left(h_I^{-1}(t)\right) \Rightarrow h_I^{-1}(t) = \phi\left(h_P^{-1}(t)\right).$$

Then, according to Definition 21, we have

$$
\begin{aligned}
&\wp_{P\to I}(Arch-PFWA(\alpha_1,\alpha_2,\ldots,\alpha_n))\\
&=\wp_{P\to I}\left(h_P^{-1}\left(\sum_{i=1}^{n} w_i h_p(\mu_i)\right), g_P^{-1}\left(\sum_{i=1}^{n} w_i g_p(v_i)\right)\right)\\
&=\left(\phi\left(h_P^{-1}\left(\sum_{i=1}^{n} w_i h_p(\mu_i)\right)\right), \phi\left(g_P^{-1}\left(\sum_{i=1}^{n} w_i g_p(v_i)\right)\right)\right)\\
&=\left(\phi\left(\phi^{-1}\left(h_I^{-1}\left(\sum_{i=1}^{n} w_i h_I(\phi(\mu_i))\right)\right)\right), \phi\left(\phi^{-1}\left(g_I^{-1}\left(\sum_{i=1}^{n} w_i g_I(\phi(v_i))\right)\right)\right)\right).\\
&=\left(h_I^{-1}\left(\sum_{i=1}^{n} w_i h_I(\phi(\mu_i))\right), g_I^{-1}\left(\sum_{i=1}^{n} w_i g_I(\phi(v_i))\right)\right)\\
&=Arch-IFWA(\wp_{P\to I}(\alpha_1),\wp_{P\to I}(\alpha_2),\ldots,\wp_{P\to I}(\alpha_n))
\end{aligned}
$$

(2) The proof of (2) is similar to (1) and it is hence omitted here.

Theorem 6 is proven.

Proof of Theorem 7

Proof

(1) Since

$$g_P(t)=g_I(\phi(t))\Rightarrow g_P^{-1}(t)=\phi^{-1}\big(g_I^{-1}(t)\big)\Rightarrow g_I^{-1}(t)=\phi\big(g_P^{-1}(t)\big)$$

and

$$h_P(t)=h_I(\phi(t))\Rightarrow h_P^{-1}(t)=\phi^{-1}\big(h_I^{-1}(t)\big)\Rightarrow h_I^{-1}(t)=\phi\big(h_P^{-1}(t)\big).$$

Then, according to Definitions 24 and 25, we have

$$
\begin{aligned}
&\wp_{P\to I}(Arch-IVPFWA(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n))\\
&=\left(\left[\phi\left(h_P^{-1}\left(\sum_{i=1}^{n}w_i h_p\left(\mu_i^-\right)\right)\right),\phi\left(h_P^{-1}\left(\sum_{i=1}^{n}w_i h_p\left(\mu_i^+\right)\right)\right)\right],\right.\\
&\left.\left[\phi\left(g_P^{-1}\left(\sum_{i=1}^{n}w_i g_p\left(v_i^-\right)\right)\right),\phi\left(g_P^{-1}\left(\sum_{i=1}^{n}w_i g_p\left(v_i^+\right)\right)\right)\right]\right)\\
&=\left(\begin{array}{l}\left[\phi\left(\phi^{-1}\left(h_I^{-1}\left(\sum_{i=1}^{n}w_i h_I\left(\phi\left(\mu_i^-\right)\right)\right)\right)\right),\phi\left(\phi^{-1}\left(h_I^{-1}\left(\sum_{i=1}^{n}w_i h_I\left(\phi\left(\mu_i^+\right)\right)\right)\right)\right)\right],\\
\left[\phi\left(\phi^{-1}\left(g_I^{-1}\left(\sum_{i=1}^{n}w_i g_I\left(\phi\left(v_i^-\right)\right)\right)\right)\right),\phi\left(\phi^{-1}\left(g_I^{-1}\left(\sum_{i=1}^{n}w_i g_I\left(\phi\left(v_i^+\right)\right)\right)\right)\right)\right]\end{array}\right).\\
&=\left(\left[h_I^{-1}\left(\sum_{i=1}^{n}w_i h_I\left(\phi\left(\mu_i^-\right)\right)\right),h_I^{-1}\left(\sum_{i=1}^{n}w_i h_I\left(\phi\left(\mu_i^+\right)\right)\right)\right],\right.\\
&\left.\left[g_I^{-1}\left(\sum_{i=1}^{n}w_i g_I\left(\phi\left(v_i^-\right)\right)\right),g_I^{-1}\left(\sum_{i=1}^{n}w_i g_I\left(\phi\left(v_i^+\right)\right)\right)\right]\right)\\
&=Arch-IVIFWA(\wp_{P\to I}(\tilde{\alpha}_1),\wp_{P\to I}(\tilde{\alpha}_2),\ldots,\wp_{P\to I}(\tilde{\alpha}_n))
\end{aligned}
$$

(2) The proof of (2) is similar to (1) and it is hence omitted here.

Theorem 7 is proven.

Proof of Theorem 8

Proof

(1) Since $g_P(t)=g_I(\phi(t))\Rightarrow g_P^{-1}(t)=\phi^{-1}\left(g_I^{-1}(t)\right)\Rightarrow g_I^{-1}(t)=\phi\left(g_P^{-1}(t)\right)$ and

$$
h_P(t)=h_I(\phi(t))\Rightarrow h_P^{-1}(t)=\phi^{-1}\left(h_I^{-1}(t)\right)\Rightarrow h_I^{-1}(t)=\phi\left(h_P^{-1}(t)\right).
$$

Then, according to Definitions 26 and 27, we have

$$
\begin{aligned}
&\hat{\wp}_{P\to I}(Arch-HPFWA(\psi_1,\psi_2,\ldots,\psi_n))\\
&=\cup_{\tau_i\in a_i,\varsigma_i\in b_i}\left\{\left\{\phi\left(h_P^{-1}\left(\sum_{i=1}^{n}w_i h_p(\tau_i)\right)\right)\right\},\left\{\phi\left(g_P^{-1}\left(\sum_{i=1}^{n}w_i g_p(\varsigma_i)\right)\right)\right\}\right\}\\
&=\cup_{\tau_i\in a_i,\varsigma_i\in b_i}\left\{\left\{h_I^{-1}\left(\sum_{i=1}^{n}w_i h_I(\phi(\tau_i))\right)\right\},\left\{g_I^{-1}\left(\sum_{i=1}^{n}w_i g_I(\phi(\varsigma_i))\right)\right\}\right\}.\\
&=Arch-IFWG\left(\hat{\wp}_{P\to I}(\psi),\hat{\wp}_{P\to I}(\psi_2),\ldots,\hat{\wp}_{P\to I}(\psi_n)\right)
\end{aligned}
$$

(2) The proof of (2) is similar to (1) and it is hence omitted here.

Theorem 8 is proven.

Proof of Theorem 9

Proof By Definitions 28 and 29, we have

$$sco_P(\alpha_i) = \mu_i^2 - v_i^2 = \phi(\mu_i) - \phi(v_i) = sco_I(\wp_{P\to I}(\alpha_i))$$

$$acc_P(\alpha_i) = \mu_i^2 + v_i^2 = \phi(\mu_i) + \phi(v_i) = acc_I(\wp_{P\to I}(\alpha_i)).$$

Theorem 9 is proven.

Proof of Theorem 10

Proof According to Theorem 9, we have

$$sco_P(\alpha_i) = sco_I(\wp_{P\to I}(\alpha_i)),\ \ acc_P(\alpha_i) = acc_I(\wp_{P\to I}(\alpha_i)).$$

Therefore,

$$\begin{array}{c} sco_P(\alpha_1) < sco_P(\alpha_2) \Leftrightarrow sco_I(\wp_{P\to I}(\alpha_1)) < sco_I(\wp_{P\to I}(\alpha_2)) \\ sco_P(\alpha_1) = sco_P(\alpha_2) \Leftrightarrow sco_I(\wp_{P\to I}(\alpha_1)) = sco_I(\wp_{P\to I}(\alpha_2)) \\ acc_P(\alpha_1) < acc_P(\alpha_2) \Leftrightarrow acc_I(\wp_{P\to I}(\alpha_1)) < acc_I(\wp_{P\to I}(\alpha_2)) \\ acc_P(\alpha_1) = acc_P(\alpha_2) \Leftrightarrow acc_I(\wp_{P\to I}(\alpha_1)) = acc_I(\wp_{P\to I}(\alpha_2)) \end{array}.$$

Thus, we have

Case 1

(i) if $sco_P(\alpha_1) < sco_P(\alpha_2)$, then $\alpha_1 <_p \alpha_2$, and $sco_I(\wp_{P\to I}(\alpha_1)) < sco_I(\wp_{P\to I}(\alpha_2))$, then

$$\wp_{P\to I}(\alpha_1) <_I \wp_{P\to I}(\alpha_2)$$

(ii) if $sco_P(\alpha_1) = sco_P(\alpha_2)$, then $sco_I(\wp_{P\to I}(\alpha_1)) = sco_I(\wp_{P\to I}(\alpha_2))$, and

(a) if $acc_P(\alpha_1) < acc_P(\alpha_2)$ then $\alpha_1 <_p \alpha_2$, and $acc_I(\wp_{P\to I}(\alpha_1)) < acc_I(\wp_{P\to I}(\alpha_2))$, then

$$\wp_{P\to I}(\alpha_1) <_I \wp_{P\to I}(\alpha_2)$$

(b) if $acc_P(\alpha_1) = acc_P(\alpha_2)$ then $\alpha_1 \sim_p \alpha_2$, and $acc_I(\wp_{P\to I}(\alpha_1)) = acc_I(\wp_{P\to I}(\alpha_2))$, then

$$\wp_{P\to I}(\alpha_1) \sim_I \wp_{P\to I}(\alpha_2).$$

Case 2. If $\wp_{P\to I}(\alpha_1) <_I \wp_{P\to I}(\alpha_2)$ then $\alpha_1 <_p \alpha_2$.

The proof of **Case 2** is similar to **Case 1** and it is hence omitted here.

Theorem 10 is proven.

Proof of Theorem 11.

Proof According to Definitions 30 and 31, we have

$$\begin{aligned} Sco_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_i)) &= \frac{\phi(\mu_i^-)+\phi(\mu_i^+)-\phi(v_i^-)-\phi(v_i^+)}{2} \\ &= \frac{(\mu_i^-)^2+(\mu_i^+)^2-(v_i^-)^2-(v_i^+)^2}{2} = Sco_P(\tilde{\alpha}_i), \end{aligned}$$

$$\begin{aligned} Acc_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_i)) &= \frac{\phi(\mu_i^-)+\phi(\mu_i^+)+\phi(v_i^-)+\phi(v_i^+)}{2} \\ &= \frac{(\mu_i^-)^2+(\mu_i^+)^2+(v_i^-)^2+(v_i^+)^2}{2} = Acc_P(\tilde{\alpha}_i). \end{aligned}$$

Theorem 11 is proven.

Proof of Theorem 11.

Proof According to Definitions 30 and 31, we have

$$\begin{aligned} Sco_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_i)) &= \frac{\phi(\mu_i^-)+\phi(\mu_i^+)-\phi(v_i^-)-\phi(v_i^+)}{2} \\ &= \frac{(\mu_i^-)^2+(\mu_i^+)^2-(v_i^-)^2-(v_i^+)^2}{2} = Sco_P(\tilde{\alpha}_i), \end{aligned}$$

$$\begin{aligned} Acc_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_i)) &= \frac{\phi(\mu_i^-)+\phi(\mu_i^+)+\phi(v_i^-)+\phi(v_i^+)}{2} \\ &= \frac{(\mu_i^-)^2+(\mu_i^+)^2+(v_i^-)^2+(v_i^+)^2}{2} = Acc_P(\tilde{\alpha}_i). \end{aligned}$$

Theorem 11 is proven.

Proof of Theorem 12.

Proof According to Theorem 10, we have

$$sco_P(\alpha_i) = sco_I(\wp_{P\to I}(\alpha_i)),\ \ acc_P(\alpha_i) = acc_I(\wp_{P\to I}(\alpha_i)).$$

Therefore,

$$\begin{array}{c} Sco_P(\tilde{\alpha}_1) < Sco_P(\tilde{\alpha}_2) \Leftrightarrow Sco_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_1)) < Sco_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_2)) \\ Sco_P(\tilde{\alpha}_1) = Sco_P(\tilde{\alpha}_2) \Leftrightarrow Sco_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_1)) = Sco_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_2)) \\ Acc_P(\tilde{\alpha}_1) < Acc_P(\tilde{\alpha}_2) \Leftrightarrow Acc_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_1)) < Acc_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_2)) \\ Acc_P(\tilde{\alpha}_1) = Acc_P(\tilde{\alpha}_2) \Leftrightarrow Acc_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_1)) = Acc_I(\tilde{\wp}_{P\to I}(\tilde{\alpha}_2)) \end{array}.$$

Thus, we have
$\tilde{\alpha}_1 <_{\tilde{P}} \tilde{\alpha}_2$ if and only if $\tilde{\wp}_{P\to I}(\tilde{\alpha}_1) <_{\tilde{I}} \tilde{\wp}_{P\to I}(\tilde{\alpha}_2)$.
Theorem 12 is proven.

Proof of Theorem 13.

Proof According to Definitions 32 and 33,

$$\begin{aligned} \hat{S}co_I\left(\hat{\wp}_{P\to I}(\psi_i)\right) &= \frac{1}{\#a_i}\sum\nolimits_{\tau\in a_i}\phi(\tau_i) - \frac{1}{\#b_i}\sum\nolimits_{\varsigma_i\in b_i}\phi(\varsigma_i) \\ &= \frac{1}{\#a_i}\sum\nolimits_{\tau\in a_i}\tau_i^2 - \frac{1}{\#b_i}\sum\nolimits_{\varsigma_i\in b_i}\varsigma_i^2 = \hat{S}co_P(\psi_i) \\ \hat{A}cc_I\left(\hat{\wp}_{P\to I}(\psi_i)\right) &= \frac{1}{\#a_i}\sum\nolimits_{\tau\in a_i}\phi(\tau_i) + \frac{1}{\#b_i}\sum\nolimits_{\varsigma_i\in b_i}\phi(\varsigma_i) \\ &= \frac{1}{\#a_i}\sum\nolimits_{\tau\in a_i}\tau_i^2 + \frac{1}{\#b_i}\sum\nolimits_{\varsigma_i\in b_i}\varsigma_i^2 = \hat{A}cc_P(\psi_i) \end{aligned}.$$

Theorem 13 is proven.

Proof of Theorem 14.

Proof According to Theorem 13, we have

$$\hat{S}co_P(\psi_i) = \hat{S}co_I\left(\hat{\wp}_{P\to I}(\psi_i)\right),\ \ \hat{A}cc_P(\psi_i) = \hat{A}cc_I\left(\hat{\wp}_{P\to I}(\psi_i)\right).$$

Therefore,

$$\begin{array}{c} \hat{S}co_P(\psi_1) < \hat{S}co_P(\psi_2) \Leftrightarrow \hat{S}co_I\left(\hat{\wp}_{P\to I}(\psi_1)\right) < \hat{S}co_I\left(\hat{\wp}_{P\to I}(\psi_2)\right) \\ \hat{S}co_P(\psi_1) = \hat{S}co_P(\psi_2) \Leftrightarrow \hat{S}co_I\left(\hat{\wp}_{P\to I}(\psi_1)\right) = \hat{S}co_I\left(\hat{\wp}_{P\to I}(\psi_2)\right) \\ \hat{A}cc_P(\psi_1) < \hat{A}cc_P(\psi_2) \Leftrightarrow \hat{A}cc_I\left(\hat{\wp}_{P\to I}(\psi_1)\right) < \hat{A}cc_I\left(\hat{\wp}_{P\to I}(\psi_2)\right) \\ \hat{A}cc_P(\psi_1) = \hat{A}cc_P(\psi_2) \Leftrightarrow \hat{A}cc_I\left(\hat{\wp}_{P\to I}(\psi_1)\right) = \hat{A}cc_I\left(\hat{\wp}_{P\to I}(\psi_2)\right) \end{array}.$$

Thus, we have

$\psi_1 <_{\hat{P}} \psi_2$ if and only if $\hat{\wp}_{P\to I}(\psi_1) <_{\tilde{I}} \hat{\wp}_{P\to I}(\psi_2)$.

Theorem 14 is proven.

7 Conclusion

The main results of this paper are shown in Table 5.

The isomorphism operators in Table 5 reveal that the results obtained by the following two patterns are equivalent:

(1) $A\,collection\,of\ PFNs \xrightarrow{operator} comprehensive\ PFN \xrightarrow{isomorphism} comprehensive\ IFN$

(2) $A\,collection\,of\ PFNs \xrightarrow{isomorphism} A\,collection\,of\ IFNs \xrightarrow{operator} comprehensive\ IFN.$

The isomorphic ranking methods in Table 5 reveal that the ranking relationship of two different PFNs remains unchanged after being converted into IFNs. Similarly, the same conclusion can be obtained for IVPFNs (vs. IVIFNs) and DPHFNs (vs. DHFNs).

When dealing with multiple attribute decision-making problems, isomorphism operators and ranking methods can reveal some interesting facts. Let $X_i (i = 1, 2)$ be two alternatives, and their related collections of PFNs are $A_i (i = 1, 2)$. From Table 2, we know that the following two sort results are consistent:

$$
\begin{aligned}
&Two\,collection\,of\ PFNs\left(A\big/B\right)\\
&\Rightarrow \begin{cases} \xrightarrow{operator} Two\ comprehensive\ PFN\left(\alpha_1\big/\alpha_2\right) \\ \xrightarrow{isomorphism} Two\,collection\,of\ IFNs\left(A^{'}\big/B^{'}\right) \\ \xrightarrow{operator} two\ comprehensive\ IFNs\left(\alpha_1^{'}\big/\alpha_2^{'}\right) \end{cases}.\\
&\xrightarrow{ranking} X_1 < X_2
\end{aligned}
$$

The above analysis shows that the Pythagorean fuzzy multi-attribute decision-making problem can be transformed into intuitionistic fuzzy multi-attribute decision-making problem by isomorphism, and the decision-making results are consistent.

In this study, the isomorphic relation between each pair of fuzzy sets is studied from three aspects, such as the operations, aggregation operators, and sorting methods. Aggregation operator and ranking method play an important role in the construction of fuzzy multi-attribute decision-making methods, which are mainly used for information aggregation and ranking. The study of isomorphic operators and isomorphic ranking methods reveals the essential relationship between intuitionistic FMADM method and Pythagorean FMADM method.

Table 5 Results

Isomorphism	Intuitionistic fuzzy sets versus Pythagorean fuzzy sets	
Mathematical isomorphism	Definition 15	$\begin{cases} \alpha_i = (\mu_i, \nu_i)_P \in A_P, \phi(x) = x^2 \\ \wp_{P\to I}(\alpha_i) = (\phi(\mu_i), \phi(\nu_i))_I \in A_I \end{cases}$
Isomorphic operators	Theorem 6	$\begin{cases} \wp_{P\to I}(Arch - PFWA(\alpha_1, \ldots, \alpha_n)) = Arch - IFWA(\wp_{P\to I}(\alpha_1), \ldots, \wp_{P\to I}(\alpha_n)) \\ \wp_{P\to I}(Arch - PFWG(\alpha_1, \ldots, \alpha_n)) = Arch - IFWG(\wp_{P\to I}(\alpha_1), \ldots, \wp_{P\to I}(\alpha_n)) \end{cases}$
Isomorphic ranking methods	Theorem 10	$\begin{cases} \alpha_1 <_p \alpha_2 \, iff \, \wp_{P\to I}(\alpha_1) <_I \wp_{P\to I}(\alpha_2) \\ \alpha_1 \sim_p \alpha_2 \, iff \, \wp_{P\to I}(\alpha_1) \sim_I \wp_{P\to I}(\alpha_2) \end{cases}$
Isomorphism	Interval-valued intuitionistic fuzzy sets versus Interval-valued Pythagorean fuzzy sets	
Mathematical isomorphism	Definition 18	$\begin{cases} \tilde{\alpha}_i = ([\mu_i^-, \mu_i^+], [\nu_i^-, \nu_i^+])_P \in A_P, \phi(x) = x^2 \\ \tilde{\wp}_{P\to I}(\tilde{\alpha}_i) = ([\phi(\mu_i^-), \phi(\mu_i^+)], [\phi(\nu_i^-), \phi(\nu_i^+)])_I \in \tilde{A}_I \end{cases}$
Isomorphic operators	Theorem 7	$\begin{cases} \tilde{\wp}_{P\to I}(Arch - IVPFWA(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)) = Arch - IVIFWA(\tilde{\wp}_{P\to I}(\tilde{\alpha}_1), \ldots, \wp_{P\to I}(\tilde{\alpha}_n)) \\ \tilde{\wp}_{P\to I}(Arch - IVPFWG(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)) = Arch - IVIFWG(\tilde{\wp}_{P\to I}(\tilde{\alpha}_1), \ldots, \wp_{P\to I}(\tilde{\alpha}_n)) \end{cases}$
Isomorphic ranking methods	Theorem 12	$\begin{cases} \tilde{\alpha}_1 <_{\tilde{P}} \tilde{\alpha}_2 \, iff \, \tilde{\wp}_{P\to I}(\tilde{\alpha}_1) <_{\tilde{I}} \tilde{\wp}_{P\to I}(\tilde{\alpha}_2) \\ \tilde{\alpha}_1 \sim_{\tilde{P}} \tilde{\alpha}_2 \, iff \, \tilde{\wp}_{P\to I}(\tilde{\alpha}_1) \sim_{\tilde{I}} \tilde{\wp}_{P\to I}(\tilde{\alpha}_2) \end{cases}$

(continued)

Table 5 (continued)

Isomorphism	Dual hesitant fuzzy sets versus Dual hesitant Pythagorean fuzzy sets	
Mathematical isomorphism	Definition 21	$\begin{cases} \psi_i = (a_i, b_i)_P \in \hat{A}_P, \varphi(x) = x^2 \\ \hat{\wp}_{P\to I}(\psi_i) = \left(\hat{\phi}(a_i), \hat{\phi}(b_i)\right)_I \in A_I, \hat{\phi}(a_i) = \cup_{\tau_i \in a_i}\{\phi(\tau_i)\}, \hat{\phi}(b_i) = \cup_{\tau_i \in a_i}\{\phi(\varsigma_i)\} \end{cases}$
Isomorphic operators	Theorem 8	$\begin{cases} \hat{\wp}_{P\to I}(Arch - DHPFWA(\psi_1, \ldots, \psi_n)) = Arch - DHFWA\left(\hat{\wp}_{P\to I}(\psi_1), \ldots, \hat{\wp}_{P\to I}(\psi_n)\right) \\ \hat{\wp}_{P\to I}(Arch - DHPFWG(\psi_1, \ldots, \psi_n)) = Arch - DHFWG\left(\hat{\wp}_{P\to I}(\psi_1), \ldots, \hat{\wp}_{P\to I}(\psi_n)\right) \end{cases}$
Isomorphic ranking methods	Theorem 14	$\begin{cases} \psi_1 <_{\hat{P}} \psi_2 \, iff \, \hat{\wp}_{P\to I}(\psi_1) <_{\tilde{I}} \hat{\wp}_{P\to I}(\psi_2) \\ \psi_1 \sim_{\hat{P}} \psi_2 \, iff \, \hat{\wp}_{P\to I}(\psi_1) \sim_{\tilde{I}} \hat{\wp}_{P\to I}(\psi_2) \end{cases}$

The advantage of this research is that it can transform the Pythagorean FMADM problem into intuitionistic FMADM problem. Especially in the face of large-scale group decision-making problems, some small groups provide Pythagorean fuzzy decision-making matrix, and some small groups provide intuitionistic fuzzy decision-making matrix. The isomorphic aggregation operators and ranking methods proposed in this paper can effectively deal with such decision situations. The limitations of this paper are mainly reflected in two aspects: first, the limitation of isomorphic operators is that the related operation rules are constructed based on Archimedean t-norm and s-norm; second, the limitation of isomorphic ranking method is that the related score functions and accuracy functions are constructed based on the difference, sum value, and expectation between membership and nonmembership degree. In the future research, we will focus on other types of isomorphism operators of various fuzzy sets, such as Bonferroni operator, Heronian operator, etc.

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Pythagorean Fuzzy Multi-criteria Decision-Making

A Risk Prioritization Method Based on Interval-Valued Pythagorean Fuzzy TOPSIS and Its Application for Prioritization of the Risks Emerged at Hospitals During the Covid-19 Pandemic

Muhammet Gul and Melih Yucesan

1 Introduction

Hospitals stay at a critical point in providing first care to people who are affected in a natural or human-made disaster. Pandemics can lead to an increasing spread of disease, with irregular and suddenly increasing patient demands that can affect the capacity of hospitals and the overall functioning of the health system. At these times, risks arising based on hospital location, building, medical staff, patients, and health care process strongly have a negative impact on the fight against the outbreak. In order to cope with the difficulty of such an epidemic disaster, hospitals must have completed their preparations and taken the required measures against risks before these events occur.

Nowadays, the vast majority of the world is struggling against the epidemic of Covid-19. The world is faced with the demand for infected patients who arrive at the hospitals heavily and irregularly. Hospitals are facilities with complex processes, mostly connected to external support and supply lines [1]. Even in regular times, many hospitals operate at full capacity or close to full capacity. In epidemic conditions, with surge demand, hospitals may find it challenging to carry out their basic functional activities, and capacity may no longer meet this demand [2]. Even a well-prepared hospital for disasters will have a hard time coping with the consequences of a Covid-19 pandemic. It is an effective hospital management policy that will reduce these difficulties to some extent. In a report of WHO, it is highlighted that

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_6

this effective hospital management will help (1) continuity of essential services, (2) well-coordinated implementation of priority action, (3) clear and accurate internal and external communication, (4) swift adaptation to increased demands, (5) effective use of scarce resources, and (6) safe environment for health workers [3].

Considering the possible negative consequences of the Covid-19 pandemic, where the healthcare sector and hospitals are currently in a great struggle, it is clear that making the hospitals prepared and ready for such disaster-based risks should be made quickly and reliably. This reinforces the conclusion that risk assessment studies for determining the hazards that arise in hospitals and eliminating these hazards or reducing their effects to an acceptable level are of great importance to reduce the losses of the epidemic.

During natural disasters and pandemics, necessary measures should be taken to determine the risks in the service process and to minimize these risks in order to increase the service quality of hospitals and to enable the society to access health services more easily. Studies evaluating the risks that may arise in the hospital during the pandemic period are very rare in the literature. In this context, the risks that may arise especially during the pandemic were evaluated and preventive measures were presented.

Therefore, in this study, an interval-valued Pythagorean fuzzy technique for order preferences by similarity to ideal solution (IVPF-TOPSIS) based risk prioritization approach is proposed. Since the risk assessment studies contain uncertainty due to the subjective nature of human judgments, interval-valued Pythagorean fuzzy sets (IVPFSs) can reflect the fuzziness, ambiguity, and uncertainty well in making decisions. The proposed approach is further applied in prioritizing the risks that emerged at hospitals in times of the Covid-19 pandemic.

The rest of the chapter is presented as follows: literature review of IVPFSs and IVPF-TOPSIS are presented in the next section. The situation analysis of Turkey's covid-19 pandemic is given in Sect. 3. The applied methodology is presented in Sect. 4. The case study is given in Sect. 5. In the last section, the conclusion is presented.

2 Literature Review

The evaluator role of decision-makers in decision-making problems requires the use of fuzzy logic theory in the face of various uncertainties. Subjectivity and uncertainty in the judgments of the evaluators about the evaluation criteria is an important and difficult problem that also occurs in the solution of the problem addressed in this study. The fuzzy set theory (FST) presented by Zadeh [4] has been previously applied to the solution of many decision problems. This theory has been developed over time and transformed into different versions and these versions have been effectively applied to many decision problems. They have also been merged with various multi-criteria decision-making (MCDM) methods ([5–11]). One of these extensions is Pythagorean fuzzy sets. Pythagorean fuzzy sets, first proposed by Yager in 2014, are a new extension of intuitionistic fuzzy sets. These sets inherit the durability feature of intuitionistic fuzzy sets. They do not only depict the imprecise and ambiguous information that intuitionistic fuzzy set can capture, but they can also model the more complex uncertainty in practical situations, which the latter cannot identify. An extended version of Pythagorean fuzzy sets is IVPFSs which this study selects as a methodology to handle ambiguous information [12–15]. IVPFSs allow membership and non-membership degrees to a specific set to have a range value. Therefore, they have wider application potential due to their ability to handle strong uncertainty in the decision-making process and can be used in this research to capture uncertain information in the prioritization of risks that emerged at hospitals in Covid-19 times. When the advantages of the TOPSIS method, which has been successfully applied to many crucial risk prioritization problems, are added to the aforementioned advantage of this special set (IVPFSs), an answer to the question of why the proposed approach is applied to this particular problem can emerged.

In the literature, many scholars have dealt with IVPFSs and applied it to various areas [12–20]. Garg [14], proposes an improved score function for solving MCDM problem with partially known weight information. In another study, new exponential operational laws and their aggregation operators for interval-valued Pythagorean fuzzy multi-criteria decision-making are described [15]. Moreover, a new improved accuracy function [12] and a new score function [13] for IVPFSs are developed. Recently, by Garg, linguistic IVPFSs concept is introduced and applied in MCDM domain [16]. In addition to the theoretical contributions of IVPFSs, there are some studies that implemented IVPFSs with TOPSIS. Yu et al. [11] developed a group decision-making sustainable supplier selection approach using extended TOPSIS under IVPFSs. Sajjad Ali Khan et al. [21] extended the TOPSIS under IVPFSs via Choquet integral. Ho et al. [22] proposed a Pearson-like correlation-based TOPSIS method with IVPFSs and applied it to multiple criteria decision analysis of stroke rehabilitation treatments. Ak and Gul [23] used integration of AHP-TOPSIS under IVPFSs for an information security risk analysis problem. Onar et al. [24] assessed cloud service providers by an IVPF-TOPSIS approach. Yucesan and Gul [25] used IVPF-TOPSIS as an auxiliary MCDM method in hospital service quality evaluation.

Considering the findings obtained from both IVPFSs and interval-valued Pythagoras IVPF-TOPSIS literature, this study contributes to the literature by the following aspects and differs from similar existed ones.

(1) Theoretically, although there are many studies on IVPFSs, there are limited studies on IVPF-TOPSIS. Among these, ones that are adapted to real-life problems consist of the mentioned studies above. Therefore, this method (IVPF-TOPSIS) has been applied to the risk prioritization problem for the first time in the literature to capture the uncertainty information of decision-makers. In addition, the special case of Covid-19 increases its importance in terms of showing the originality of the study and the applicability of such methods in the field of public health.
(2) Secondly, to the problem of prioritizing risks, a GRA-TOPSIS integrated approach based on IVPFSs as in Yu et al. [11] has been applied. The distance

and similarity between the alternatives are evaluated at the same time, which makes the ranking results more solid.

(3) Thirdly, a comparative analysis with other existing IVPF-TOPSIS approaches is performed to provide its validity.

3 Situation Analysis of Turkey's Covid-19 Pandemic

This disease first appeared on December 30, 2019, in Wuhan, China [26]. In nearly 6 months, the infection, which first expanded to Iran and Italy, is spread all over the world. The world is faced with the demand for infected patients who apply to hospital emergency units heavily and irregularly. As of July 4, 2020, it has caused nearly 11,2 million cases and 529,882 deaths in the world [3]. The symptoms of Covid-19 are not specific, and they can range from no symptoms (asymptomatic) to severe pneumonia and death. According to the figures of the report "Report of the WHO-China Joint Mission on Coronavirus Disease 2019 (Covid-19)", typical signs and symptoms are expressed such as fever, dry cough, fatigue, sputum production, shortness of breath, sore throat, and headache[27].

As of June 28, 2020, similar to the fights of countries against this pandemic throughout the world, Turkey continues its struggle. After the first Covid-19 cases in Turkey were notified on March 11, 2020, higher than 3 million tests were performed in total. Between 1 and 28 June, nearly 1,2 million tests were performed. In total, 198,284 laboratory-confirmed Covid-19 cases and 5,097 deaths due to Covid-19 have been reported in Turkey. The total number of hospitalizations was 105,416 and 5,773 patients were hospitalized between 22 and 28 June. The recovery rate and death rate of all confirmed cases were 86.04% and 2.57%, respectively [28]. All these statistical figures show that Turkey is at a better level than the world average. This is related to Turkey's early isolation decisions and as well as the case of being prepared for such an event. Therefore, hospitals are required to prepare for such pandemics.

4 Applied Methodology

Traditionally, it is assumed that the information obtained when making assessments is known as crisp numbers [14, 15]. Moreover, MCDM-based models merged with fuzzy extensions are frequently applied to occupational risk assessment problems in hospitals [29, 30]. However, the use of MCDM in prioritization problems (such as risk prioritization) along the area of public health or the emerging of Covid-19 pandemic is a new platform to make research [31–33]. Therefore, the current study aims to remedy the gap in this respect. This will help hospital decision-makers and national policymakers in making hospitals ready for such outbreaks. The proposed method is presented in Fig. 1. Also, the calculation details are discussed in Sect. 4.

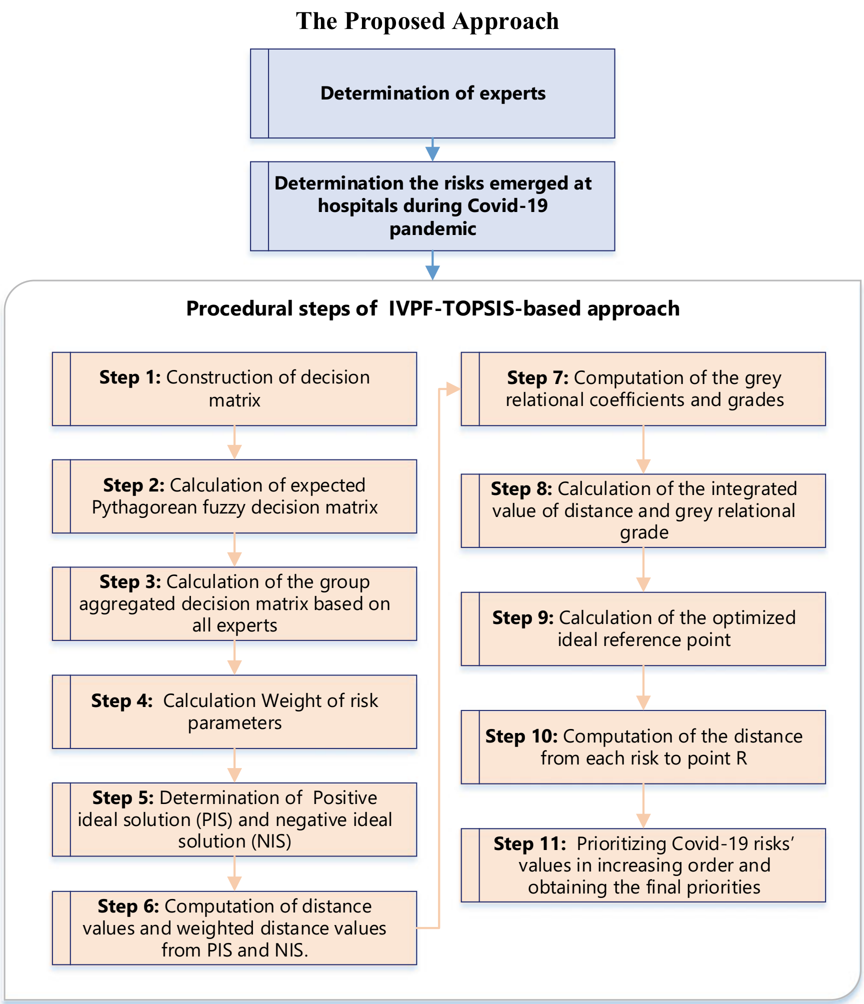


Fig. 1 Flow chart of the proposed approach

In the applied methodology, the uncertain information of decision-makers is considered in IVPFSs. Weights of decision-makers and criteria are determined in the way of Yu et al. [11]. Then, hazards that emerged due to Covid-19 pandemic at hospitals are evaluated by the grey correlation analysis (GRA) and TOPSIS integrated method [11].

4.1 Preliminaries

In this section, we provide some definitions, notions, and formulae regarding IVPFSs.

Definition 1 Let X be a universe of discourse. A Pythagorean fuzzy set P is an object having the form [11–15]:

$$P = \{\langle x,\ \mu_P(x), v_P(x)\rangle | x \in X\} \tag{1}$$

where $\mu_P(x) : X \mapsto [0, 1]$ defines the degree of membership and $v_P(x) : X \mapsto [0, 1][0, 1]$ defines the degree of non-membership of the element $x \in X$ to P, respectively, and, for every $x \in X$, it holds:

$$0 \leq \mu_P(x)^2 + v_P(x)^2 \leq 1 \tag{2}$$

For any P and $x \in X$, $\pi_P(x) = \sqrt{1 - \mu_P^2(x) - v_P^2(x)}$ is called the degree of indeterminacy of x to P.

Definition 2 Let $\beta_1 = P(\mu_{\beta_1}, v_{\beta_1})$ and $\beta_2 = P(\mu_{\beta_2}, v_{\beta_2})$ be two Pythagorean fuzzy numbers in a simple demonstration then the following operations are defined ([34–37]):

$$\beta_1 \oplus \beta_2 = P(\sqrt{\mu_{\beta_1}^2 + \mu_{\beta_2}^2 - \mu_{\beta_1}^2\mu_{\beta_2}^2}, v_{\beta_1} v_{\beta_2}) \tag{3}$$

$$\lambda\beta_1 = P(\sqrt{1 - (1 - \mu_{\beta_1}^2)^\lambda}, (v_{\beta_1})^\lambda), \lambda > 0 \tag{4}$$

$$d(\beta_1, \beta_2) = \sqrt{\left(\mu_{\beta_1} - \mu_{\beta_2}\right)^2 + \left(v_{\beta_1} - v_{\beta_2}\right)^2 + \left(\pi_{\beta_1} - \pi_{\beta_2}\right)^2} \tag{5}$$

Definition 3 An IVPFS $\tilde{P}_i$ of the universe of discourse U is depicted as follows [38]: $\tilde{P}_i = \left\{u, ([\mu_{\tilde{P}_i}^L(u), \mu_{\tilde{P}_i}^U(u)], [v_{P_i}^L(u), v_{P_i}^U(u)] | u \epsilon U\right\}$ where $0 \leq \left(\mu_{\tilde{P}_i}^U(u)\right)^2 + \left(v_{\tilde{P}_i}^U(u)\right)^2 \leq 1$

Definition 4: An intuitionistic entropy of Pythagorean fuzzy set is computed via Eq. (6) [39].

$$e = -\frac{1}{m * ln2}\sum_{i=1}^{m}\left[\mu_{\beta_i}{}^2 ln\left(\mu_{\beta_i}\right)^2 + v_{\beta_i}{}^2 ln\left(v_{\beta_i}\right)^2 - \left(1 - \pi_{\beta_i}{}^2\right) ln\left(1 - \pi_{\beta_i}{}^2\right) - \pi_{\beta_i}{}^2 ln2\right] \tag{6}$$

4.2 Procedural Steps of the Proposed IVPF-TOPSIS-based Approach

First, experts should determine the status of risks in the decision-making process [40]. For an MCDM problem, a decision matrix is required to construct the decision-making process. Since, the problem that this study has dealt with a risk prioritization, we have designed the decision matrix whose elements include the values of all alternatives with respect to each criterion under IVPFSs.

Let $CR = \{cv_1, cv_2, \ldots, cv_m\} m \geq 2$ be a set of alternatives (for this study "*Covid-19 risk*"), $RP = \{rp_1, rp_2, \ldots, rp_n\}$ be a set of criteria set (for this study "*risk parameter*"), $w = \{w_1, w_2, \ldots, w_n\}$ be a set of criteria weights for this study "*risk parameter weight*") that satisfy the conditions of $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$, and $DM = \{dm_1, dm_2, \ldots, dm_h\}$ be a set of decision-makers' weights for this study "*decision-maker (expert) weight*"). Also, while $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ refers to the optimism degree of experts, $\eta = (\eta_1, \eta_2, \ldots, \eta_h)$ shows the relative weights of experts satisfying $\sum_{k=1}^{h} \eta_k = 1$. The procedural steps of our proposed IVPF-TOPSIS-based risk prioritization approach are as follows:

Step 1: This step handles the construction of the decision matrix. In determining the ratings of decision-makers regarding alternatives with respect to the criteria, the IVPFSs-based linguistic scale given in Table 1 is used. Here, k refers to the indices for decision-makers.

Step 2: In this step, expected Pythagorean fuzzy decision matrix of the decision-makers is calculated using the optimism degree λ.

$$\mu_{ij}^{k} = (1 - \lambda_k)\mu_{ij}^{kL} + \lambda_k \mu_{ij}^{kU} \tag{7}$$

$$v_{ij}^{k} = \lambda_k v_{ij}^{kL} + (1 - \lambda_k) v_{ij}^{kU} \tag{8}$$

Step 3: The group aggregated decision matrix based on all experts' expected decision matrices by Eq. (9).

Table 1 The IVPFSs-based linguistic scale [11]

Linguistic term		$[\mu_{ij}^{kL}, \mu_{ij}^{kU}]$		$[v_{ij}^{kL}, v_{ij}^{kU}]$	
Very Good	VG	0.80	0.90	0.00	0.15
Good	G	0.70	0.80	0.15	0.25
Medium Good	MG	0.55	0.70	0.25	0.40
Medium	M	0.45	0.55	0.40	0.55
Medium Poor	MP	0.30	0.45	0.55	0.70
Poor	P	0.20	0.30	0.70	0.80
Very Poor	VP	0.00	0.20	0.80	0.95

$$T=\left\{\sqrt{1-\prod_{k=1}^{h}\left(1-\mu_{ij}^{k^2}\right)^{\eta_k}},\prod_{k=1}^{h}v_{ij}^{k^{\eta_k}}\right\} \tag{9}$$

Step 4: This step is about weight calculation of risk parameters using entropy measure of each parameter as in Eq. (10)

$$ent_j=-\frac{1}{m*ln2}\sum_{i=1}^{m}\left[\mu_{ij}{}^2ln\left(\mu_{ij}\right)^2+v_{ij}{}^2ln\left(v_{ij}\right)^2-\left(1-\pi_{ij}{}^2\right)\ln\left(1-\pi_{ij}{}^2\right)-\pi_{ij}{}^2ln2\right] \tag{10}$$

Also, a divergence is introduced in Eq. (11) and normalized risk parameter weights are computed as in Eq. (12).

$$d_j=1-ent_j, j=1,2,\ldots,n \tag{11}$$

$$w_j=\frac{d_j}{\sum_{j=1}^{n}d_j}, j=1,2,\ldots,n \tag{12}$$

Step 5: Positive ideal solution (PIS) and negative ideal solution (NIS) are determined using Eqs. (13)–(14).

$$PIS=A^+=\left\{\left(\mu_1{}^+,v_1{}^+,\pi_1{}^+\right),\left(\mu_2{}^+,v_2{}^+,\pi_2{}^+\right),\ldots,\left(\mu_n{}^+,v_n{}^+,\pi_n{}^+\right)\right\} \tag{13}$$

$$NIS=A^-=\left\{\left(\mu_1{}^-,v_1{}^-,\pi_1{}^-\right),\left(\mu_2{}^-,v_2{}^-,\pi_2{}^-\right),\ldots,\left(\mu_n{}^-,v_n{}^-,\pi_n{}^-\right)\right\} \tag{14}$$

where $\mu_j{}^+=max\{\mu_{ij}\}$, $v_j{}^+=min\{v_{ij}\}$, $\mu_j{}^-=min\{\mu_{ij}\}$, $v_j{}^-=max\{v_{ij}\}$,

$$\pi_j^+=\sqrt{1-\mu_j^{+2}-v_j^{+2}},\ \pi_j^-=\sqrt{1-\mu_j^{-2}-v_j^{-2}}$$

Step 6: This step concerns with computation of distance values and weighted distance values from PIS and NIS. The required equations are provided in Eqs. (15)–(18).

$$\Delta_{ij}^+=\sqrt{\left\{\left(\mu_{ij}-\mu_j{}^+\right)^2+\left(v_{ij}-v_j{}^+\right)^2+\left(\pi_{ij}-\pi_j{}^+\right)^2\right\}} \tag{15}$$

$$\Delta_{ij}^-=\sqrt{\left\{\left(\mu_{ij}-\mu_j{}^-\right)^2+\left(v_{ij}-v_j{}^-\right)^2+\left(\pi_{ij}-\pi_j{}^-\right)^2\right\}} \tag{16}$$

$$dist_i^+=\sum_{j=1}^{n}w_j\Delta_{ij}^+ \tag{17}$$

$$dist_i^- = \sum_{j=1}^{n} w_j \Delta_{ij}^- \tag{18}$$

Step 7: According to the study of Yu et al. [11], grey relational analysis is merged into TOPSIS. The grey relational coefficients and grades are computed as Eqs. (19)–(22), respectively.

$$\psi_{ij}^+ = \frac{\min_i \min_j \Delta_{ij}^+ + \rho \max_i \max_j \Delta_{ij}^+}{\Delta_{ij}^+ + \rho \max_i \max_j \Delta_{ij}^+} \tag{19}$$

$$\psi_{ij}^- = \frac{\min_i \min_j \Delta_{ij}^- + \rho \max_i \max_j \Delta_{ij}^-}{\Delta_{ij}^- + \rho \max_i \max_j \Delta_{ij}^-} \tag{20}$$

$$\varsigma_i^+ = \frac{1}{n} \sum_{j=1}^{n} w_j \psi_{ij}^+ \tag{21}$$

$$\varsigma_i^- = \frac{1}{n} \sum_{j=1}^{n} w_j \psi_{ij}^- \tag{22}$$

Step 8: In this step, the integrated value of distance and grey relational grade is computed as in Eqs. (23)–(24).

$$int_i{}^+ = \alpha * dist_i{}^- + \beta * \varsigma_i^+, i = 1, 2, \ldots, m \tag{23}$$

$$int_i{}^- = \alpha * dist_i{}^+ + \beta * \varsigma_i^-, i = 1, 2, \ldots, m \tag{24}$$

Here $\alpha + \beta = 1$. Generally, they are set to 0.5. $int_i{}^+$ and $int_i{}^-$ refer to the closeness of the ith Covid-19 risk to PIS and NIS.

Step 9: The optimized ideal reference point is calculated as in Eq. (25).

$$R = \left(r^+, r^-\right) = \left(\max\left(int_i{}^+\right), \min\left(int_i{}^-\right)\right), i = 1, 2, \ldots, m \tag{25}$$

Step 10: In this step, the distance from each Covid-19 risk to point R is computed as in Eq. (26).

$$CC_i = \sqrt{\left(int_i{}^+ - r^+\right)^2 + \left(int_i{}^- - r^-\right)^2} \tag{26}$$

Step 11: The final step is about prioritizing Covid-19 risks' CC_i values in increasing order and obtaining the final priorities.

5 Case Study: Prioritization of the Risks Emerged at Hospitals During the Covid-19 Pandemic

In this section, a case study for the applied methodology is carried out to prioritize the risks that emerged at hospitals during Covid-19 pandemic. For this aim, twenty-one Covid-19 risks are determined benefiting from the experience of the decision-makers who participated in this study at two different hospitals. These hospitals are state hospitals situated in the Black Sea region of Turkey and face a great number of Covid-19 cases till March 2020. The first expert is an academician who studies mathematical modeling of Covid-19 spreading, MCDM, and fuzzy logic modeling. The second and third experts are working in these hospitals as "assistant hospital manager". Regarding the risk list, three different pillars are considered as hospital (indicated as an abbreviation of "Hospital risk: HR"), patient & staff (indicated as an abbreviation of "Patient & staff risk: PSR") and healthcare (indicated as an abbreviation of "Care risk: CR"). Six hospital risks, nine patient & staff risks, and six care risks are identified as in Table 2. The risk parameters are determined as severity, occurrence, and detection which is used as the parameters of a classical failure mode and effect analysis. The twenty-one Covid-19 risks are assessed with respect to these three risk parameters considering the weights of risk parameters, decision-makers, and optimism degree of decision-makers. For this respect, we obtain weights of risk parameters using entropy measures as follows:

$w_j = (0.386, 0.315, 0.299), j = severity, occurrence, detection$. Weights of decision-makers are assumed as $\eta = (\eta_1, \eta_2, \eta_3) = (0.4, 0.3, 0.3)$. Finally, the optimism degrees of decision-makers are considered as $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (0.5, 0.5, 0.5)$ which are all equal.

By completing the requirements for the computational procedural steps of the proposed approach, the following step-by-step operations is performed to find the priorities of the emerged Covid-19-related risks at the hospital sites.

In the first step, following linguistic terms and their corresponding IVPFSs shown in Table 1, the decision matrix assessed by each expert is constructed. Then, the expected decision matrix for each expert is calculated by Eqs. (7) and (8). In the third step of the approach, the group aggregated decision matrix is computed by Eq. (9) and provided in Table 3.

In the fourth step of the proposed approach, we obtained the weight of risk parameters using entropy measure of each parameter using Eqs. (10), (11), and (12). In the fifth step, PIS and NIS values are computed using Eqs. (13) and (14) as in Table 4.

In the sixth step, the weighted distances from each Covid-19 risk to the PIS and the NIS are computed using Eqs. (15)–(18). Then, the grey relational grades are calculated as the seventh step procedures using Eqs. (19)–(22). In the eighth step, the integrated value of distance and grey relational grade is calculated by Eqs. (23)–(24). The computation results are given in Table 5.

In the ninth step, the optimized ideal reference point G is obtained as (0.524, 0.081). Then, the distance from each Covid-19 risk to point G is computed using Eq. (26). The final CC_i results are given in Fig. 2. Considering the ranking of each

Table 2 Descriptions of risks that emerged at hospitals during the Covid-19 pandemic

Risk	Description of the risk at hospitals regarding Covid-19
HR1	The risk associated with isolation of the hospital's admission unit (ED) or the whole hospital
HR2	The risk associated with the ergonomic factors (ventilation, light, noise, etc.) factors of the hospital
HR3	The risk associated with the disinfection of the places/room/medical device
HR4	The risk associated with the disposal of medical waste used in the treatment of Covid-19 patients
HR5	Risks arising from social distance-based placement or layout design in the hospital waiting areas and medical treatment
HR6	Risks arising from the location of the section where the swab sample is taken to make a diagnosis of Covid-19
PSR1	The risk stems from non-utilizing personal protective equipment (PPE)
PSR2	Risk arising from non-following the personal hygiene rules
PSR3	The risk associated with non-following the hospital care rules, appropriate treatment procedures and Covid-19 treatment algorithm
PSR4	The risk associated with non-performing the regular health checks due to the growing literature on Covid-19
PSR5	The risk emerged in the transport of samples of Covid-19 patients
PSR6	Risk of non-documenting the patient records and/or keeping the Covid-19 patients data in an organized way
PSR7	The risk due to lack of communication between the patient, the medical staff, and relatives of patients
PSR8	Risks arising from failure to apply the necessary sterilization to medical equipment contacted by Covid-19 suspects
PSR9	Risks arising from healthcare personnel who are in close contact with Covid-19 patients using the same living space as other staff
CR1	The risk associated with the quantity, lead time, storage, and transfer of medicine under appropriate conditions
CR2	The risk associated with the quantity, lead time, storage, and transfer of PPEs against Covid-19
CR3	Risk arising from the supplying of test kits
CR4	Risk arising from the supplying of respirators
CR5	Risk related to intensive care unit and other inpatients Covid-19 bed capacity
CR6	Risk related to the hospital's food, cleaning, and generator adequacy

Covid-19 risks (by CC_i values in increasing order), the *PSR2* (Risk arising from non-following the personal hygiene rules) is determined as the most crucial risk-related Covid-19 at hospital sites.

As a result of the calculations, PSR2 was determined as the highest risk. Republic of Turkey Ministry of Health has identified the implementation of personal hygiene as an important criterion following the criterion of "usage of masks" in the fight

Table 3 The group aggregated decision matrix

Covid-19 risk	Severity			Occurrence			Detection		
HR1	0.719	0.231	0.656	0.418	0.576	0.703	0.100	0.875	0.474
HR2	0.250	0.750	0.612	0.205	0.798	0.567	0.100	0.875	0.474
HR3	0.418	0.576	0.703	0.516	0.460	0.723	0.250	0.750	0.612
HR4	0.268	0.744	0.613	0.294	0.710	0.640	0.418	0.576	0.703
HR5	0.683	0.268	0.680	0.468	0.516	0.718	0.322	0.696	0.642
HR6	0.826	0.101	0.555	0.205	0.798	0.567	0.581	0.378	0.720
PSR1	0.750	0.200	0.630	0.268	0.744	0.613	0.670	0.281	0.688
PSR2	0.750	0.200	0.630	0.625	0.325	0.710	0.786	0.149	0.600
PSR3	0.797	0.135	0.589	0.217	0.785	0.580	0.205	0.798	0.567
PSR4	0.593	0.364	0.718	0.761	0.172	0.626	0.205	0.798	0.567
PSR5	0.177	0.823	0.540	0.217	0.785	0.580	0.543	0.424	0.725
PSR6	0.161	0.835	0.525	0.294	0.710	0.640	0.375	0.625 ara>	0.685
PSR7	0.205	0.798	0.567	0.643	0.315	0.699	0.500	0.475	0.724
PSR8	0.643	0.315	0.699	0.643	0.315	0.699	0.750	0.200	0.630
PSR9	0.797	0.135	0.589	0.670	0.281	0.688	0.343	0.660	0.668
CR1	0.719	0.231	0.656	0.205	0.798	0.567	0.343	0.660	0.668
CR2	0.719	0.231	0.656	0.161	0.835	0.525	0.418	0.576	0.703
CR3	0.681	0.282	0.676	0.268	0.744	0.613	0.177	0.823	0.540
CR4	0.418	0.576	0.703	0.226	0.791	0.569	0.330	0.685	0.650
CR5	0.719	0.231	0.656	0.258	0.755	0.603	0.268	0.744	0.613
CR6	0.205	0.798	0.567	0.707	0.243	0.664	0.217	0.785	0.580

Table 4 The PIS (A +) and NIS (A−) values

	Severity			Occurrence			Detection		
A+	0.826	0.101	0.555	0.761	0.172	0.626	0.786	0.149	0.600
A−	0.161	0.835	0.525	0.161	0.835	0.525	0.100	0.875	0.474

against Covid-19. Although this risk has a high impact on the spread of the virus, it is very difficult to determine whether personal hygiene rules are followed. For this reason, brochures and instructions reminding the personal hygiene rules were placed in the visible places of the hospital. The second most important risk was identified as PSR8. The detectability of this risk is relatively easy. But if action is not taken, it can cause devastating effects. For this reason, Covid-19 suspects are accepted to carry viruses from the moment they enter the hospital. The tomography and other diagnostic devices used in the diagnosis of the disease are disinfected after each use. The third most important risk has been identified as PSR9 medical staff in Turkey as in other countries were affected by the virus. Some measures

Table 5 The integrated value of distance and grey relational grades

Covid-19 risk	int_i^+	int_i^-
HR1	0.259	0.340
HR2	0.057	0.557
HR3	0.226	0.342
HR4	0.160	0.412
HR5	0.311	0.257
HR6	0.394	0.224
PSR1	0.371	0.208
PSR2	0.524	0.081
PSR3	0.274	0.323
PSR4	0.365	0.238
PSR5	0.143	0.460
PSR6	0.117	0.489
PSR7	0.254	0.330
PSR8	0.460	0.122
PSR9	0.429	0.160
CR1	0.262	0.320
CR2	0.269	0.333
CR3	0.222	0.356
CR4	0.165	0.410
CR5	0.255	0.320
CR6	0.208	0.384

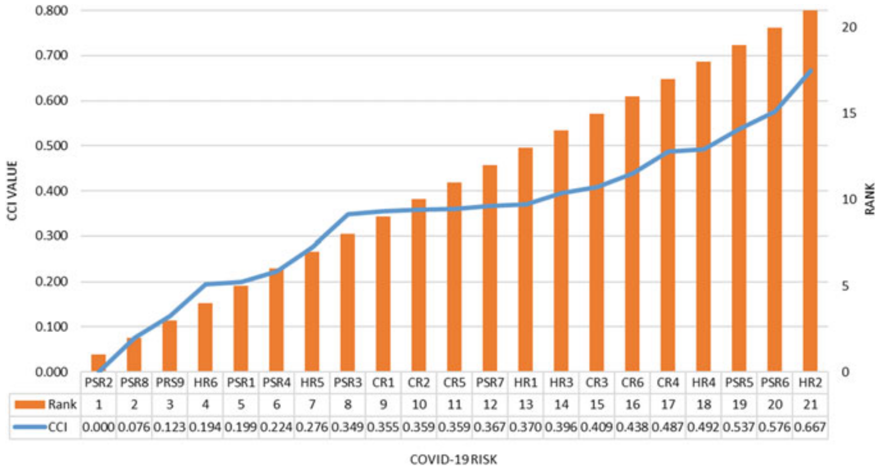


Fig. 2 CCI and priority (rank) values of Covid-19 risks

have been taken to prevent the spread of the virus to medical staff. State-owned guesthouses were made free of charge to healthcare professionals and they were encouraged to reside there. The fourth most important risk has been identified as HR6. Healthcare professionals are closer than 20 cm to take a swab sample, which makes the virus spread possible. In this context, some measures have been taken by the hospitals. Cabins were ordered to prevent contact with medical staff and patients. These cabins are located in areas where there are fewer patients and healthcare professionals. The fifth most important risk has been identified as PSR1. In order to prevent Covid-19 transmission in healthcare professionals, N95 type masks with high protection were provided. Necessary information was provided by announcing the instructions for the use of protective equipment determined by the Ministry of Health to medical staff. The sixth most important risk was identified as PSR4. Patients who needed regular care and treatment during the fight against Covid-19 were adversely affected. For this reason, "clean hospitals "were determined for each region, and there was no Covid-19 patient in those hospitals, and it was served to combat regular diseases. The seventh most important risk has been identified as HR5. In order to provide social isolation, serious changes have been made especially in emergency departments. Covid-19 suspects were separated from the patients who would receive other emergency services and the two groups were prevented from contacting each other.

5.1 Comparative Study

To compare the performance of our approach in this case with other similar IVPF-TOPSIS approaches, we conducted a comparative study. In this study, the results of our current study (final TOPSIS scores and risk rankings) were compared with the analysis results obtained by applying IVPF-TOPSIS of Garg [13] to our case which uses a modified score function. The results obtained are summarized in Table 6.

In applying Garg's IVPF-TOPSIS to our data, we used interval-valued Pythagorean fuzzy weighted average operator (IVPFWAO) of Peng and Yang [41] in aggregating evaluation of each three decision-maker. The weights of decision-makers and weights of three risk parameters are set to the same in the current approach. Figure 3 presents the comparison of the results obtained from the current approach with the results we obtained by applying Garg [13]. Therefore, it was concluded from Fig. 3 that the results calculated by the approach taken from the literature coincide with the results of the proposed approach. The correlation test for final scores and risk rankings also supports this result. The Spearman's Rank Correlation Coefficient (Rho) gives a value of 0.819 (approximately 82%) for both method ranking order results. Also, the Pearson correlation analysis is performed in final IVPF-TOPSIS results of both approaches and value of -0.753 (approximately 75%) is obtained. The negative and higher value shows a strong and inverse relationship between the results. In our proposed approach, a lower final CC_i value is desired unlike in Garg's

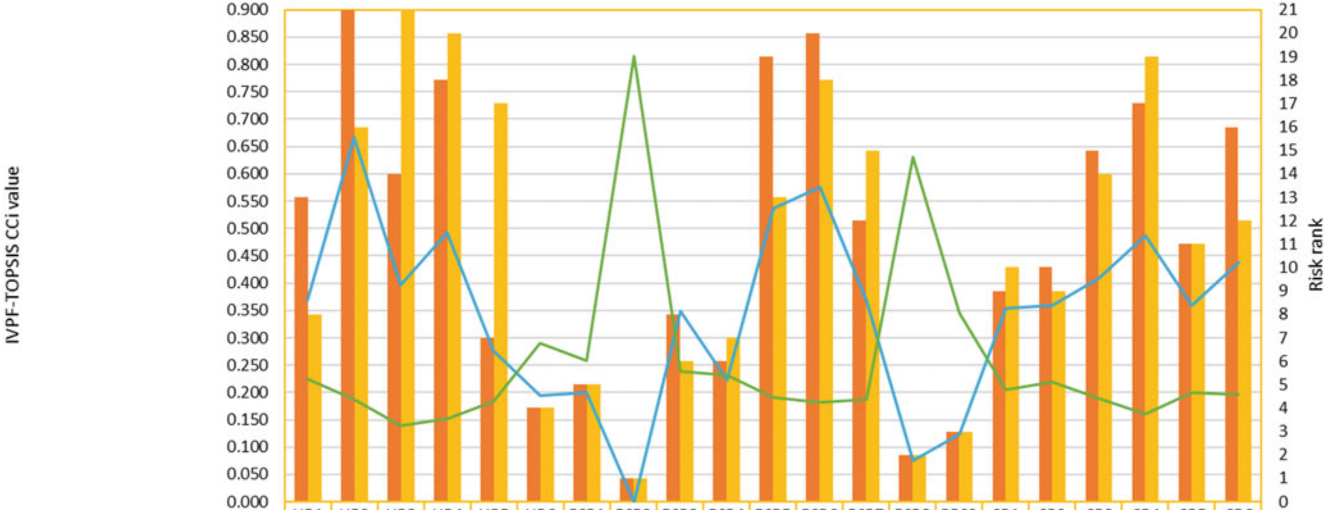
Table 6 Results obtained by applying Garg [13]'s IVPF-TOPSIS to our case

Covid-19 risk	Distance from ideal (d^+)	Distance from anti-ideal (d^-)	Final IVPF-TOPSIS score (CC_i)	Rank
HR1	1.355	0.391	0.224	8
HR2	2.119	0.491	0.188	16
HR3	1.190	0.192	0.139	21
HR4	1.611	0.287	0.151	20
HR5	0.961	0.217	0.184	17
HR6	1.178	0.480	0.289	4
PSR1	1.017	0.355	0.259	5
PSR2	0.086	0.381	0.815	1
PSR3	1.586	0.499	0.239	6
PSR4	1.122	0.338	0.232	7
PSR5	1.901	0.449	0.191	13
PSR6	1.885	0.418	0.182	18
PSR7	1.462	0.339	0.188	15
PSR8	0.140	0.240	0.631	2
PRS9	0.705	0.372	0.346	3
CR1	1.364	0.354	0.206	10
CR2	1.354	0.381	0.220	9
CR3	1.554	0.363	0.189	14
CR4	1.566	0.301	0.161	19
CR5	1.429	0.357	0.200	11
CR6	1.792	0.438	0.197	12

IVPF-TOPSIS CC_i value. In Garg's approach, a higher CC_i is desired. This is why we obtained a negative correlation coefficient.

6 Conclusion

The quality and continuity of hospital services play an important role in reducing the social and economic impacts of the pandemic on countries. In many countries, the health system is not ready for a pandemic. Moreover, the system is locked and there is a serious weakness both in the fight against pandemic and in routine health services. In this context, in order to ensure the continuity of hospital services and to increase the quality of service, the risks that may arise in hospitals, especially during the pandemic period, were identified and preventive actions were taken for high-priority risks.



	HR1	HR2	HR3	HR4	HR5	HR6	PSR1	PSR2	PSR3	PSR4	PSR5	PSR6	PSR7	PSR8	PRS9	CR1	CR2	CR3	CR4	CR5	CR6
Current approach Rank	13	21	14	18	7	4	5	1	8	6	19	20	12	2	3	9	10	15	17	11	16
Garg (2017b)'s approach Rank	8	16	21	20	17	4	5	1	6	7	13	18	15	2	3	10	9	14	19	11	12
Current approach CCi	0.370	0.667	0.396	0.492	0.276	0.194	0.199	0.000	0.349	0.224	0.537	0.576	0.367	0.076	0.123	0.355	0.359	0.409	0.487	0.359	0.438
Garg (2017b)'s approach CCi	0.224	0.188	0.139	0.151	0.184	0.289	0.259	0.815	0.239	0.232	0.191	0.182	0.188	0.631	0.346	0.206	0.220	0.189	0.161	0.200	0.197

Covid-19 risk

Fig. 3 Results of comparative study

The time of the pandemic and the risks in the hospital during regular times differ from each other. In this study, risks that may arise in the hospital are arranged by considering pandemic conditions. In the light of the evaluations, the most important risks were identified, and preventive measures were presented. During the pandemic, it may be necessary to change the infrastructure, revise the processes, differentiate the measures, and reorganize the personnel management. In addition, the interests and concerns of healthcare personnel and patients differ during these periods. When all these factors are considered, while preventing the detected risks improves the quality of health care; It will be a helpful element in the decision-making mechanisms of hospital management. It is not enough to evaluate only one hospital to be able to analyze pandemic situation. The authors will conduct a risk analysis of all hospitals in the region in future studies. In addition, it is planned to integrate the discrete event simulation with the risk analysis for hospitals. Thus, the effects of the preventive measures can be determined and how hospitals can serve in pandemics and natural disasters can be tested with different scenarios. This information will assist decision-makers on which regional hospitals can be declared “pandemic hospitals”.

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Assessment of Agriculture Crop Selection Using Pythagorean Fuzzy CRITIC–VIKOR Decision-Making Framework

Arunodaya Raj Mishra, Pratibha Rani, and Sitesh Bharti

1 Introduction

The selection of an appropriate crop pattern shows an important role in encouraging sustainable farming procedure, optimizing natural resources, and maximizing the economic benefits of any country. In India, there are mainly two seasons for farming: Kharif season and Rabi season. The Kharif season crops like Zea mays, cotton, paddy, jute, sugarcane, Sorghum, etc., are cultivated in the season from April to October. The crop pattern selection process is much more complicated due to various factors that vary from region to region. Many criteria as available environmental resources, water quality, soil quality, and farming process influence the process of crop pattern assessment.

In recent times, lots of studies have been presented regarding agricultural sustainability across the globe. For illustration, Roy and Chan [47] developed a procedure by including agriculture sustainability factors in Bangladesh. Ramırez-Garcıa et al. [39] suggested an multi-criteria decision-making (MCDM) analysis for the selection of covering crop species and cultivars. A method for crop pattern selection was suggested by Sorensen et al. [50]. Pramanik [37] and Bozdag et al. [4] studied location selection for agricultural land in Darjeeling district (India) and Cihanbeyli (Turkey), respectively, by employing AHP- and GIS-based approaches. It has been proven that the fuzzy sets (FSs) theory-based MCDM procedure offers many irreplaceable merits over the classical MCDM methods. The theory of FSs expresses the linguistic assessments and reduces the human inaccuracy in MCDM processes [57]. Rezaei-Moghaddam and Karami [46] designed a framework based on the AHP model

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_7

for selecting the sustainable agricultural progress. Cobuloglu and Buyuktahtakin [7] suggested a new fuzzy-based method for biomass crop assessment. Qureshi et al. [38] evaluated the Indian farming system by a new fuzzy-based method. Deepa et al. [12] proposed a predictive mathematical model to handle and evaluate the agriculture crop in India. Deepa and Ganeshan [10] developed a soft decision system with rough set and VIKOR model to select the agriculture crop in India. Deepa and Ganeshan [11] recommended DRSA based model for agriculture crop classification. Sambasivam et al. [48] applied a combined model with AHP and TOPSIS approaches for assessing an ideal Rabi crop in India. To the best of the authors' awareness, this is the first work which offers crop pattern assessment for Kharif season with sustainable farming procedures under Pythagorean fuzzy sets (PFSs).

In the last two decades, there are various extensions on FSs available in the literature, namely, intuitionistic fuzzy sets (IFSs), PFSs, and others. To avoid the inadequacy of FSs, Atanassov [3] established the concept of IFSs, which are described by belongingness degree (BD) and non-belongingness degree (ND), and satisfies a requirement that the addition of the BD and ND is less than or equal to unity. Mishra [29] introduced a method on the basis of trigonometric entropy and similarity measures aiming for the evaluation of the township development problems with IFSs. Das et al. [9] recommended a robust decision-making framework with intuitionistic fuzzy numbers. Djatna et al. [14] introduced an intuitionistic fuzzy decision tree to classify the different types of stroke disease in BioMed Central data. Zheng and Liu [59] generalized the Triple I model for multiple-rules approximate reasoning on IFSs. Kumari and Mishra [25] evaluated and selected the most suitable green supplier by employing multi-criteria COPRAS approach. So far, several other researchers have studied many theories related to IFSs and implemented in pattern recognition, fuzzy reasoning, fuzzy control, decision-making, etc. [22, 27, 31].

In several applications, there may be a case in which the experts present his/her opinion as $(\frac{\sqrt{3}}{2}, \frac{1}{2})$. Consequently, IFS is unable to deal with this situation because $\frac{\sqrt{3}}{2} + \frac{1}{2} > 1$. To conquer this issue, Yager [55] established the theory of PFSs, described by the BD, ND, and HD, and satisfied a constraint that the square addition of belongingness degree (BD) and non-belongingness degree (ND) is less than or equal to 1. The IFSs and PFSs can express the opinions of decision-makers (DMs) more systematically. Moreover, if one is an intuitionistic fuzzy number (IFN), then it must also be a Pythagorean fuzzy number (PFN), but not all PFNs are the IFNs. Thus, the PFSs are assumed as a more reliable model to solve the complex MCDM problems. Zhang and Xu [58] studied basic operational laws for PFNs. Peng and Yang [34] suggested some elementary operations to solve the MCDM issue for PFSs. Ma and Xu [28] recommended symmetric operators for PFSs with application in MCDM problem. Wu and Wei [53] initiated some Hamacher aggregation operators for PFSs and utilized them for multiple criteria decision analysis. Further, Garg [16] pioneered some new Pythagorean fuzzy (PF) weighted and ordered weighted operators based on confidence levels. Garg [18] suggested an innovative PF-based method with probabilistic information and immediate probabilities philosophy. With the use of PF-entropy and PF-divergence measures-based decision-making framework, the most desirable renewable energy technologies are evaluated and selected by Rani et al. [42]. Further, Garg [17] and Garg [19] introduced some neutral operational laws based on PF-geometric aggregation operators and PF-averaging aggregation operators, respectively, and then employed for multiple criteria decision analysis. In a further study, Wang et al. [51] presented a novel PF-entropy measure and a series of interactive Hamacher power aggregation operators for PFSs. Akram et al. [2] suggested two decision-making methods, namely, ELECTRE-I and TOPSIS with complex PFSs for evaluating the multi-criteria group decision-making problems with complex Pythagorean fuzzy information. Rani et al. [45] suggested a novel PF-WASPAS method for a multi-criteria physician selection problem with uncertain information. Apart from them, various other studies have been presented from different perspectives such as aggregation operators [15, 24, 56], information measures [36, 40, 43, 54], decision-making approaches [1, 20, 41]. Nonetheless, no work has been presented regarding the assessment of Kharif season crops under the environment of PFSs.

The application of MCDM depends on the computing of criteria weight, which is significant for selecting and sorting. The weight determination method is divided into objective weights and subjective weights, depending on whether the weight is computed indirectly from the result or directly by decision-makers. The criteria importance through inter-criteria correlation (CRITIC) method, proposed by Diakoulaki et al. 13, is based on the standard deviation which uses correlation analysis to measure the value of each criterion and achieves relatively objective criteria weights in the MCDM problems. It collects entire preference information contained in the evaluation criteria based on the analysis of the evaluation matrix. Recently, few hybrid methods have been introduced by combining CRITIC and many other MCDM approaches under uncertain environments [23, 35, 52]. However, there is no work in the literature which evaluates the related criteria by PF-CRITIC method in the assessment of Kharif season crops.

MCDM, a part of decision theory, is an act of choosing an ideal choice from a given set of decision variants. Due to widespread changes and development of socio-economic surroundings, practical decision-making problems are becoming more and more complex. During the past few decades, a variety of approaches have been proposed to deal with real-life MCDM problems, where each of them has its own advantages and drawbacks. To tackle the MCDM problems, Opricovic [33] pioneered a compromise programming-based method, named as VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR). The main objective of this method is to present compromise solution(s) that supports the minimum individual regret for the opponent and the maximum group utility for the majority. The VIKOR approach provides the ranking of a set of alternatives and generates compromise solutions with the closeness to the ideal solution similar to TOPSIS approach, but the basic principle of the TOPSIS is that the chosen alternative should be the closest to the ideal solution and farthest to the negative ideal solution. To deal with the uncertainty that arises in MCDM problems, Mishra and Rani [30] studied a collective MCDM framework based on Shapley function, classical VIKOR approach, entropy, and divergence measures with IFSs. With the use PF-VIKOR method, a set of electric vehicle charging stations was ranked by Cui et al. [8]. Chen [6] suggested interval-valued Pythagorean fuzzy-based VIKOR technique for the assessment of pilot hospitals under an uncertain environment. Liang et al. [26] evaluated and ranked the quality of internet banking websites in the Ghanaian banking industry by using a combined methodology based on VIKOR, TODIM, and PFSs. Rani et al. [42] recommended PF-VIKOR-based decision-making method to rank the renewable energy technologies in India. In another study, Naeem et al. [32] developed a new MCDM structure based on TOPSIS and VIKOR methods with Pythagorean fuzzy soft set. Most recent, Rani and Mishra [44] evaluated the eco-industrial thermal plants by employing single-valued neutrosophic set-based VIKOR method. Although, several other authors [21, 41, 49] have developed VIKOR under different fuzzy environments. According to the existing literature, we have found that no one has combined the CRITIC and VIKOR methods with PFSs for the MCDM of the Kharif season crops.

1.1 Motivation and Contributions

Uncertainty is commonly occurred in the Kharif season crops assessment process due to the presence of multiple constraints, lack of knowledge, vague human mind, and inconsistency of the problem, therefore, this process can be considered as complex MCDM problem. In the recent past, PFSs have been verified as one of the flexible and superior tools to cope with the uncertainty and ambiguity that occurred in real-life MCDM problems. Consequently, the present chapter focuses on the environment of PFSs. As far as we know, there is no study in the literature regarding a combined decision-making framework based on the combination of CRITIC and VIKOR approaches with entropy and divergence measures under PFSs environment. Thus, this is the first study which combines the CRITIC and VIKOR approaches with PFSs and named PF-CRITIC–VIKOR. In this methodology, the objective criteria weights estimated by the CRITIC model are more sensible for the MCDM process. Next, the VIKOR method implements a simplistic calculation process with accurate and consistent results for assessing Kharif season crops.

The major contributions of the chapter are explained as follows.

(a) To offer a comprehensive Pythagorean fuzzy modeling structure by covering several factors in the selection of Kharif season crop patterns.
(b) To evaluate the importance of criteria, CRITIC method is applied and VIKOR method is used to assess the Kharif season crop pattern for sustainable farming practices.
(c) New divergence measure is developed to implement VIKOR model and entropy measure based method to compute the decision-makers' weight.

We summarize the present study in the following manner: Section 2 presents basic concepts related to PFSs. Section 3 introduces new divergence and entropy measures for PFSs. Section 4 proposes a novel PF-CRITIC–VIKOR method for the evaluation of MCDM problems within uncertain environment, wherein decision-makers and criteria weights are fully unknown. To reveal the effectiveness of the proposed method, Sect. 5 presents an illustrative case study of Kharif season crops assessment, in which the evaluation values of the alternatives over the criteria are given in terms of PFNs. In addition, comparative and sensitivity analyses are discussed to reveal the potentiality of the proposed method. Section 6 concludes the whole study and presents the future research directions.

2 Preliminaries

This section presents some elementary notions of PFSs, which are used throughout this study.

Definition 1 ([55]) A PFS J on a fixed set Ω is defined as $J = \{\langle u_i, \mu_J(u_i), \nu_J(u_i)\rangle | u_i \in \Omega\}$, where $\mu_J : \Omega \to [0,1]$ and $\nu_J : \Omega \to [0,1]$ denote the BD and ND of an element $u_i \in \Omega$, respectively and for each $u_i \in \Omega$, $0 \le (\mu_J(u_i))^2 + (\nu_J(u_i))^2 \le 1$.

The degree of indeterminacy is denoted by $\pi_J(u_i) = \sqrt{1 - \mu_J^2(u_i) - \nu_J^2(u_i)}$, for every $u_i \in \Omega$. Zhang and Xu [58] defined a PFN as $\eta = J\left(\mu_\eta, \nu_\eta\right)$ which holds $\mu_\eta, \nu_\eta \in [0,1]$ and $0 \le \mu_\eta^2 + \nu_\eta^2 \le 1$.

Definition 2 Consider a PFN $\eta = J\left(\mu_\eta, \nu_\eta\right)$. Then score and accuracy values of η are given by Zhang and Xu [58], Peng and Yang [34]:

$$\mathbb{S}(\eta) = \left(\mu_\eta\right)^2 - \left(\nu_\eta\right)^2, \mathbb{S}(\eta) \in [-1,1] \text{ and } \hbar(\eta) = \left(\mu_\eta\right)^2 + \left(\nu_\eta\right)^2, \hbar(\eta) \in [0,1].$$

As $\mathbb{S}(\eta) \in [-1,1]$, then Wu and Wei [53] presented an improved score value for η, given as

$$\mathbb{S}^*(\eta) = \frac{1}{2}(\mathbb{S}(\eta) + 1), \text{ where } \mathbb{S}(\eta) \in [0,1]. \tag{1}$$

Definition 3 Let $\eta = J\left(\mu_\eta, \nu_\eta\right)$, $\eta_1 = J\left(\mu_{\eta_1}, \nu_{\eta_1}\right)$ and $\eta_2 = J\left(\mu_{\eta_2}, \nu_{\eta_2}\right)$ be the PFNs. Then, the following operations on PFNs as [55].

$$\eta^c = J\left(\nu_\eta, \mu_\eta\right),$$
$$\eta_1 \oplus \eta_2 = J\left(\sqrt{\mu_{\eta_1}^2 + \mu_{\eta_2}^2 - \mu_{\eta_1}^2\mu_{\eta_2}^2}, \nu_{\eta_1}\nu_{\eta_2}\right),$$
$$\eta_1 \otimes \eta_2 = J\left(\mu_{\eta_1}\mu_{\eta_2}, \sqrt{\nu_{\eta_1}^2 + \nu_{\eta_2}^2 - \nu_{\eta_1}^2\nu_{\eta_2}^2}\right),$$
$$\lambda\eta = J\left(\sqrt{1 - \left(1 - \mu_\eta^2\right)^\lambda}, \left(\nu_\eta\right)^\lambda\right), \lambda > 0,$$
$$\eta^\lambda = J\left(\left(\mu_\eta\right)^\lambda, \sqrt{1 - \left(1 - \nu_\eta^2\right)^\lambda}\right), \lambda > 0.$$

Definition 4 Let $J, K \in PFSs(\Omega)$. A real-valued function $D : PFSs(\Omega) \times PFSs(\Omega) \rightarrow \mathbb{R}$ is said to be PF-divergence measure (PF-DM) if satisfies the postulates, given as Rani et al. [42].

(a1). $D(J, K) = D(K, J)$,
(a2). $D(J, K) = 0 \Leftrightarrow J = K$,
(a3). $D(J \cap L, K \cap L) \leq D(J, K)$ for every $L \in PFS(\Omega)$,
(a4). $D(J \cup L, K \cup L) \leq D(J, K)$ for every $L \in PFS(\Omega)$.

Definition 5 A PF-entropy (PFE) $\mathrm{E} : PFS(\Omega) \rightarrow [0, 1]$ is a real-valued mapping which satisfies the following assumptions [36, 42]:

(P1). $0 \leq \mathrm{E}(J) \leq 1$;
(P2). $\mathrm{E}(J) = 0$ iff J is a crisp set;
(P3). $\mathrm{E}(J) = 1$ iff $\mu_J(u_i) = \nu_J(u_i), \forall u_i \in \Omega$;
(P4). $\mathrm{E}(J) = \mathrm{E}(J^c)$;
(P5). For each $u_i \in \Omega$, $\mathrm{E}(J) \leq \mathrm{E}(K)$ if J is less than K,

$$i.e., \ \mu_J(u_i) \leq \mu_K(u_i) \leq \nu_K(u_i) \leq \nu_J(u_i) \text{ or } \nu_J(u_i) \leq \nu_K(u_i) \leq \mu_K(u_i) \leq \mu_J(u_i).$$

3 Proposed Divergence and Entropy Measures

As the divergence and entropy measures have widely been employed in real-world problems, however, very few authors have focused their attention on the development of new PF-DM and PFE. Thus, the aim of the section is to first introduce a new divergence measure within PFSs context. Next, based on the proposed divergence measure, we have introduced a new entropy measure for PFSs. On the basis of these measures, we will develop a scientific decision-making tool in the next section.

Let $J, K \in PFSs(\Omega)$. Then divergence measure for PFSs is defined by

$$D(J, K) = \frac{1}{n}\sum_{i=1}^{n}\left(\left|\mu_J^2(u_i) - \mu_K^2(u_i)\right|.\exp\left(\frac{\left|\mu_J^2(u_i) - \mu_K^2(u_i)\right|}{\frac{1}{2}\left(\mu_J^2(u_i) + \mu_K^2(u_i)\right)}\right)\right.$$
$$+\left|\nu_J^2(u_i) - \nu_K^2(u_i)\right|\exp\left(\frac{\left|\nu_J^2(u_i) - \nu_K^2(u_i)\right|}{\frac{1}{2}\left(\nu_J^2(u_i) + \nu_K^2(u_i)\right)}\right)$$
$$\left.+\left|\pi_J^2(u_i) - \pi_K^2(u_i)\right|\exp\left(\frac{\left|\pi_J^2(u_i) - \pi_K^2(u_i)\right|}{\frac{1}{2}\left(\pi_J^2(u_i) + \pi_K^2(u_i)\right)}\right)\right). \tag{2}$$

Theorem 1 *The function Eq.* (2) *is a valid PF-DM.*

Proof (a1)–(a2). Proofs are omitted here.

(a3). For $J, K, L \in PFSs(\Omega)$, we have

$$\left|\min\left(\mu_J^2(u_i), \mu_L^2(u_i)\right) - \min\left(\mu_K^2(u_i), \mu_L^2(u_i)\right)\right| \le \left|\mu_J^2(u_i) - \mu_K^2(u_i)\right|,$$
$$\left|\min\left(\nu_J^2(u_i), \nu_L^2(u_i)\right) - \min\left(\nu_K^2(u_i), \nu_L^2(u_i)\right)\right| \le \left|\nu_J^2(u_i) - \nu_K^2(u_i)\right|$$
$$\text{and } \left|\min\left(\pi_J^2(u_i), \pi_L^2(u_i)\right) - \min\left(\pi_K^2(u_i), \pi_L^2(u_i)\right)\right| \le \left|\pi_J^2(u_i) - \pi_K^2(u_i)\right|.$$

Therefore, we obtain

$$\begin{aligned}
&\Bigg(\left|\min\left(\mu_J^2(u_i), \mu_L^2(u_i)\right) - \min\left(\mu_K^2(u_i), \mu_L^2(u_i)\right)\right| \\
&\quad \exp\left(\frac{\left|\min\left(\mu_J^2(u_i), \mu_L^2(u_i)\right) - \min\left(\mu_K^2(u_i)\mu_L^2(u_i)\right)\right|}{\frac{1}{2}\left(\min\left(\mu_J^2(u_i), \mu_L^2(u_i)\right) + \min\left(\mu_K^2(u_i), \mu_L^2(u_i)\right)\right)}\right)\Bigg) \\
&\quad \le \left(\left|\mu_J^2(u_i) - \mu_K^2(u_i)\right| \exp\left(\frac{\left|\mu_J^2(u_i) - \mu_K^2(u_i)\right|}{\frac{1}{2}\left(\mu_J^2(u_i) + \mu_K^2(u_i)\right)}\right)\right), \\
&\Bigg(\left|\min\left(\nu_J^2(u_i), \nu_L^2(u_i)\right) - \min\left(\nu_K^2(u_i), \nu_L^2(u_i)\right)\right| \\
&\quad \exp\left(\frac{\left|\min\left(\nu_J^2(u_i), \nu_L^2(u_i)\right) - \min\left(\nu_K^2(u_i), \nu_L^2(u_i)\right)\right|}{\frac{1}{2}\left(\min\left(\nu_J^2(u_i), \nu_L^2(u_i)\right) + \min\left(\nu_K^2(u_i), \nu_L^2(u_i)\right)\right)}\right)\Bigg) \\
&\quad \le \left(\left|\nu_J^2(u_i) - \nu_K^2(u_i)\right| \exp\left(\frac{\left|\nu_J^2(u_i) - \nu_K^2(u_i)\right|}{\frac{1}{2}\left(\nu_J^2(u_i) + \nu_K^2(u_i)\right)}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\Bigg(\left|\min\left(\pi_J^2(u_i), \pi_L^2(u_i)\right) - \min\left(\pi_K^2(u_i), \pi_L^2(u_i)\right)\right| \\
&\quad \exp\left(\frac{\left|\min\left(\pi_J^2(u_i), \pi_L^2(u_i)\right) - \min\left(\pi_K^2(u_i), \pi_L^2(u_i)\right)\right|}{\frac{1}{2}\left(\min\left(\pi_J^2(u_i), \pi_L^2(u_i)\right) + \min\left(\pi_K^2(u_i), \pi_L^2(u_i)\right)\right)}\right)\Bigg) \\
&\quad \le \left(\left|\pi_J^2(u_i) - \pi_K^2(u_i)\right|. \exp\left(\frac{\left|\pi_J^2(u_i) - \pi_K^2(u_i)\right|}{\frac{1}{2}\left(\pi_J^2(u_i) + \pi_K^2(u_i)\right)}\right)\right), \ \forall u_i \in \Omega
\end{aligned}$$

This implies that $D(J \cap L, K \cap L) \le D(J, K)$ for every $L \in PFS(\Omega)$.

(A4). In a similar way, we can prove $D(J \cup L, K \cup L) \le D(J, K)$ for every $L \in PFS(\Omega)$.

Hence, the given measure $D(J, K)$ is a valid PF-divergence measure.

Example 1 To illustrate the effectiveness of the proposed divergence measure, we have compared the performances of the proposed and existing distance and divergence measures, given by Chen [5], Rani et al. [42, 45] on two common datasets, and results are presented in Table 1.

Table 1 Comparison of developed divergence measure with existing ones

Case	J	K	Chen 5 $d_C^1(J, K)$	Chen 5 $d_C^2(J, K)$	Chen 5 $d_C^1(J, K)$ ($\beta = 1$)	Chen 5 $d_C^2(J, K)$ ($\beta = 2$)	Rani et al. 42 $D_{R1}(F, G)$	Rani et al. 45 $D_{R2}(F, G)$	$D(J, K)$
I	$\{\langle v_1, 0.55, 0.45\rangle, \langle v_2, 0.63, 0.55\rangle\}$	$\{\langle v_1, 0.39, 0.50\rangle, \langle v_2, 0.50, 0.59\rangle\}$	**0.1500**	**0.1323**	**0.1500**	**0.1323**	**0.009**	**0.0060**	0.4466
II	$\{\langle v_1, 0.55, 0.45\rangle, \langle v_2, 0.63, 0.55\rangle\}$	$\{\langle v_1, 0.40, 0.51\rangle, \langle v_2, 0.51, 0.60\rangle\}$	0.1400	0.1217	0.1400	0.1217	**0.009**	**0.0060**	0.4066

Note "Bold" denotes counter-intuitive cases results

From Table 1, we can see that $J_1 = J_2$, but $K_1 \neq K_2$ and now, when we compare the outcomes under Case I and Case II, we obtain the distance measures given by Chen [5], Rani et al. [42, 45] generate counter-intuitive results which are highlighted in Table 4. However, the proposed divergence measure is free from counter-intuitive results, which shows its effectiveness over the existing measures.

On the basis of proposed divergence measure, we have developed a relation between PF-DM and PF-entropy measure as follows:

$$\mathrm{E}(J) = 1 - \frac{1}{2\exp(2)} D\big(J, J^c\big), \tag{3}$$

where

$$\mathrm{E}(J) = 1 - \frac{1}{2n\exp(2)} \sum_{i=1}^{n} \left(\left|\mu_J^2(u_i) - \nu_J^2(u_i)\right| \exp\left(\frac{\left|\mu_J^2(u_i) - \nu_J^2(u_i)\right|}{\frac{1}{2}\big(\mu_J^2(u_i) + \nu_J^2(u_i)\big)} \right) \right). \tag{4}$$

Theorem 2 *The function Eq. (4) is a valid PFE.*

Proof: In order to verify this theorem, the expression Eq. (4) must hold the requirements (p1)–(p5) of Definition 5. The proof is obvious; hence, we omitted this proof.

4 Pythagorean Fuzzy-CRITIC–VIKOR (PF-CRITIC–VIKOR) Methodology

This section integrates the CRITIC and the VIKOR approaches under PFSs environment. The CRITIC method is extended with PFSs and employed to assess the attribute weights. In addition, the VIKOR model is extended with PF-DM and PFE to rank the alternatives. Now, the calculation steps for combined PF-CRITIC–VIKOR methodology are presented as follows (Fig. 1):

Step 1*: Initiate the alternative and attribute*

The doctrine of MCDM model is to select the most desirable choice among the p options $C = \{C_1, C_2, \ldots, C_p\}$ under the criterion set $R = \{R_1, R_2, \ldots, R_q\}$. It is assumed that a committee of l experts $A = \{A_1, A_2, \ldots, A_l\}$ is created to obtain the ideal option(s). Suppose $Z = \left(z_{ij}^{(k)}\right)$, $i = 1(1)p$, $j = 1(1)q$ be the PF-decision matrix obtained by decision-makers (DMs), where $z_{ij}^{(k)}$ describes the assessment of an option C_i over attribute R_j in the form of PFNs, expressed by k^{th} DM.

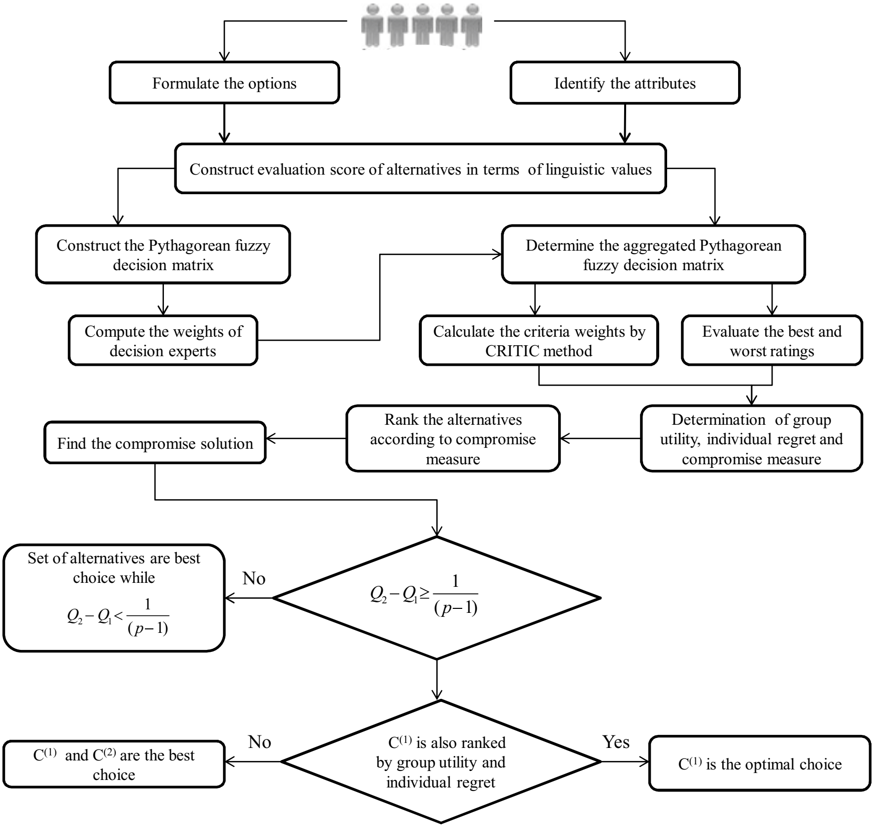

Fig. 1 Flowchart of developed PF-CRITIC–VIKOR method

***Step 2**: Determine the weights of the DMs*

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)^T$ be the weights of l DMs such that $\sum_{k=1}^{l} \lambda_k = 1$. Let the DMs' weights are assumed as Linguistic variables (LVs) that are articulated in PFNs. Let $A_k = (\mu_k, \nu_k)$ be a PFN for the evaluation of the k^{th} DM. Based on PFE, the weight of expert is estimated by

$$\lambda_k = \frac{1 - \mathrm{E}(A_k)}{k - \sum_{j=1}^{\ell} \mathrm{E}(A_k)}. \tag{5}$$

***Step 3**: Build the aggregated PF- decision matrix (APF-DM)*

To construct the APF-DM, combining each individual one into a group decision matrix based on DMs judgments is required. To accomplish this, PF-weighted averaging operator (PFWAO) is employed and then $R = \left(\varepsilon_{ij}\right)_{p \times q}$ where

$$\varepsilon_{ij} = \left(\tilde{\mu}_{ij}, \tilde{\nu}_{ij}\right) = PFWA_{\lambda}\left(z_{ij}^{(1)}, z_{ij}^{(2)}, \ldots, z_{ij}^{(\ell)}\right)$$
$$= \left(\sqrt{1 - \prod_{k=1}^{\ell}\left(1 - \mu_{ijk}^{2}\right)^{\lambda_k}}, \prod_{k=1}^{\ell}\left(\nu_{ijk}\right)^{\lambda_k}\right). \tag{6}$$

***Step 4**: Utilize the CRITIC approach for the evaluation of criteria weights*

Firstly, assume that each criterion has different importance. Let $\omega = \left(\omega_1, \omega_2, \ldots, \omega_q\right)^T$ be the attribute weight vector such that $\omega_j \in [0, 1]$ and $\sum_{j=1}^{q} \omega_j = 1$. This method unites the intensity contrast of every attribute and conflict among the attributes. Intensity contrast of attribute is assessed by the standard deviation (SD) and conflict among the attributes is calculated by using the correlation coefficient (CRC). The steps for CRITIC method under PFSs context are given as

***Step 4-A**:* Estimate the score matrix $S = \left(\varepsilon_{ij}\right)_{p\times q}, i = 1(1)p, j = 1(1)q$, where

$$\varepsilon_{ij} = \tilde{\mu}_{ij}^{2} - \tilde{\nu}_{ij}^{2}, \tag{7}$$

***Step 4-B**:* Construct the standard PF-matrix $\widetilde{S} = \left(\widetilde{\chi}_{ij}\right)_{p\times q}$, where

$$\widetilde{\chi}_{ij} = \begin{cases} \dfrac{\varepsilon_{ij} - \varepsilon_j^-}{\varepsilon_j^+ - \varepsilon_j^-}, & j \in R_b \\ \dfrac{\varepsilon_j^+ - \varepsilon_{ij}}{\varepsilon_j^+ - \varepsilon_j^-}, & j \in R_n \end{cases} \tag{8}$$

wherein $\varepsilon_j^+ = \max\limits_i \varepsilon_{ij}$ and $\varepsilon_j^- = \min\limits_i \varepsilon_{ij}$.

***Step 4-C**:* Compute the attributes SDs with the following expression:

$$\sigma_j = \sqrt{\frac{\sum_{i=1}^{p}\left(\widetilde{\chi}_{ij} - \overline{\chi}_j\right)^2}{p}}, \text{ wherein } \overline{\chi}_j = \sum\nolimits_{i=1}^{p} \widetilde{\chi}_{ij}/p. \tag{9}$$

***Step 4-D**:* Assess the CRC between the criteria pairs:

$$r_{jt} = \frac{\sum_{i=1}^{p}\left(\widetilde{\chi}_{ij} - \overline{\chi}_j\right)\left(\widetilde{\chi}_{it} - \widetilde{\chi}_t\right)}{\sqrt{\sum_{i=1}^{p}\left(\widetilde{\chi}_{ij} - \overline{\chi}_j\right)^2 \sum_{i=1}^{p}\left(\widetilde{\chi}_{it} - \overline{\chi}_t\right)^2}}. \tag{10}$$

***Step 4-E**:* Determine the amount of information of attribute and is presented as

$$c_j = \sigma_j \sum_{t=1}^{q} \left(1 - r_{jt}\right). \tag{11}$$

Step 4-F: Evaluate the objective attribute weight and is defined by

$$\omega_j = \frac{c_j}{\sum_{j=1}^{q} c_j}. \tag{12}$$

Step 5*: Find the best and worst ratings.*

In the proposed approach, the PF-ideal solution (PF-IS) and the PF-anti-ideal solution (PF-AIS) are considered as best and worst values. The formulae for the computation of PF-IS and PF-AIS are given by

$$\phi_j^+ = \begin{cases} \max\limits_i \tilde{\mu}_{ij}, \text{ for benefit criterion } R_j \\ \min\limits_i \tilde{\nu}_{ij}, \text{ for cost criterion } R_j \end{cases} \quad \text{for } j = 1(1)q, \tag{13}$$

$$\phi_j^- = \begin{cases} \min\limits_i \tilde{\mu}_{ij}, \text{ for benefit criterion } R_j \\ \max\limits_i \tilde{\nu}_{ij}, \text{ for cost criterion } R_j \end{cases} \quad \text{for } j = 1(1)q. \tag{14}$$

Step 6*: Calculate the utility score (US), worst group score (WGS), and compromise score (CS) to the PF-IS for each option.*

Corresponding to the proposed PF-DM, we calculate the US S_i and the WGS I_i over each option $C_i (i = 1(1)p)$ and are given by

$$S_i = L_{1,i} = \sum_{j=1}^{n} \omega_j \frac{D\left(\phi_j^+, \varepsilon_{ij}\right)}{D\left(\phi_j^+, \phi_j^-\right)} \tag{15}$$

$$I_i = L_{\infty,i} = \max_{1 \le j \le q} \left(\omega_j \frac{D\left(\phi_j^+, \varepsilon_{ij}\right)}{D\left(\phi_j^+, \phi_j^-\right)} \right). \tag{16}$$

For each option C_i, the CS Q_i is assessed as follows:

$$Q_i = \tau \frac{\left(S_i - S^+\right)}{\left(S^- - S^+\right)} + (1 - \tau) \frac{\left(I_i - I^+\right)}{\left(I^- - I^+\right)}, \tag{17}$$

where $S^+ = \min\limits_i S_i$, $S^- = \max\limits_i S_i$, $I^+ = \min\limits_i I_i$, $I^- = \max\limits_i I_i$, and $\tau \in [0, 1]$ is the coefficient of the decision mechanism.

Step 7*: Rank the options.*

The preference order of the options is assessed by sorting each value of S_i, I_i, and Q_i.

Step 8*: Determine the compromise solution.*

Propose the option C_i as a CS analogous to Q_1 (the least among Q_i values) if

(C1): Option C_i has an acceptable improvement, i.e., $Q_2 - Q_1 \geq \frac{1}{(p-1)}$, where p denotes the number of options.
(C2): The option C_i is stable in the MCDM procedure; that is, it is also the best ranked in S_i or I_i.

If any one condition is not fulfilled, then a group of CSs is proposed, which consists of

(i) Options C_1 and C_2 if only the condition (C2) is not satisfied.
(ii) Options C_1, C_2, C_3,…, C_k if condition (C1) is not pleased; and C_k is assessed by the expression $Q_k - Q_1 < \frac{1}{(p-1)}$.

5 Case Study: Agriculture Crop Selection Problem

The region for the development of decision-making framework is Satna district in Madhya Pradesh (MP), India. Satna district is situated in the northern part of MP, having a geographical region of 7,424 Sq km. The Satna district lies between the latitudes 23°05'N and 25°12'N and longitudes 80°21'E and 81°23'E, which is shown in Fig. 2. The district is customarily and agriculturally rich. Approximately 79.4% of population is rural-based and agriculture practices are the main economic activity. Agriculture is the prime source of income for the people. Satna is considered for the present case study because the crops such as Paddy, Soybean, Blackgram, and Pigeon pea are major economic Kharif crops in this region. To illustrate the application of the developed PF-CRITIC–VIKOR methodology, an empirical study is conducted with agriculture crops in Kharif season, Satna District, Madhya Pradesh, India. For this case study, we have considered four Kharif crops Pigeonpea (C_1), Paddy (C_2), Soybean (C_3) and Blackgram (C_4) in accordance with sustainable agricultural practices. To select the ideal crop, a team of four DMs (A_1, A_2, A_3, A_4) has been formed which comprises of one professor from the agriculture department, one postdoctoral researcher, and two officers of the district agriculture department having at least 5–8 years of experience in the agriculture development. All DMs are proficient in the decision-making process and having durable expertise in several agricultural events. These Kharif season crops are evaluated based on seven criteria, given in Table 2 and Fig. 3. The criteria R_3, R_4, and R_7 are cost-type and rest of all are benefit-type. The procedure of PF-CRITIC–VIKOR methodology for the evaluation of best Kharif season crop is given as follows:

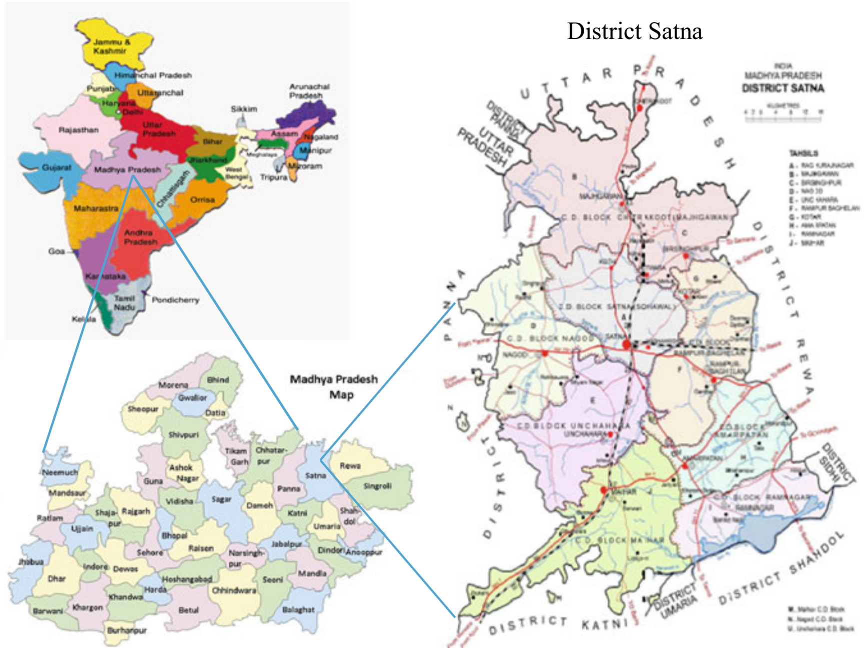

Fig. 2 Study area location map

First, we assume the significance degrees of the DMs in the form of PFNs and are given as {(0.85, 0.25, 0.4637), (0.80, 0.30, 0.5196), (0.70, 0.45, 0.5545), (0.75, 0.40, 0.5268)}. Let $Z = \left(z_{ij}^{(k)}\right)$, $i = 1(1)p$, $j = 1(1)q$ be the PF-decision matrix, articulated by DMs and is depicted in Table 3.

Since DMs weights are first presented in the terms of PFNs, now, the DMs weights are calculated with the use of Eqs. (4)–(5) and given as $\{\lambda_1 = 0.3127, \lambda_2 = 0.2746, \lambda_3 = 0.1866, \lambda_4 = 0.2261\}$.

By applying Eq. (6), the four individual decision opinions are aggregated, and now, the APF-DM $R = \left(\varepsilon_{ij}\right)_{p\times q}$ is offered in Table 4.

To determine the criteria weights, the CRITIC method is extended under PFSs environment. With the use of Eq. (7) and Table 4, first we have calculated the score matrix $S = \left(\varepsilon_{ij}\right)_{p\times q}$. Then, computed the standard PF-matrix $\widetilde{S} = \left(\widetilde{\chi}_{ij}\right)_{p\times q}$ by employing Eq. (8). Further, by Eqs. (9)–(11), the SD, CRC, and amount of information of each criterion are evaluated. At last, the attribute weights are estimated by using Eq. (12) and mentioned in Table 5.

By employing Eqs. (13)–(14), the best and worst ratings of the Kharif crop options are calculated as below:

$$\phi_j^+ = \{(0.749, 0.166, 0.641), (0.705, 0.387, 0.595), (0.456, 0.620, 0.638), \\ (0.614, 0.619, 0.490), (0.741, 0.561, 0.370)$$

Table 2 Detail description of the considered criteria in sustainable agriculture crop selection

Goal	Criteria	Type	Description	Crop option
Agriculture crop selection	Soil quality (R_1)	Benefit	Initial process to enhance soil quality is very expensive, but a constant subsequent of organic farming can contribute to increase the production, and also conserves the soil for cultivation	Pigeonpea (C_1)
	Water quality (R_2)	Benefit	Studies the competence of crop-type in accumulative water quality and dipping water usage	Paddy (C_2)
	Input fertilizers (R_3)	Cost	It considers Nitrogen, Urea, P_2O_5, SSP, K_2O, and MOP	
	Risk (R_4)	Cost	Flood, winter rain, drought, and other natural disasters	Soybean (C_3)
	Facilities/Infrastructure (R_5)	Benefit	Distance from roads, markets, seed, and processing plants	
	Agricultural management policies (R_6)	Benefit	Refers all the procedures from sowing seeds until the harvesting	Blackgram (C_4)
	Environmental impact (R_7)	Cost	Soil erosion, acidification potential, abiotic depletion potential,freshwater and marine water ecotoxicity and polluted water, and others	

$$(0.723, 0.545, 0.425), (0.545, 0.677, 0.495)\} \tag{18}$$

$$\begin{aligned}\phi_j^- = \{&(0.604, 0.304, 0.736), (0.596, 0.400, 0.696), (0.543, 0.620, 0.567),\\ &(0.624, 0.565, 0.540), (0.616, 0.627, 0.478),\\ &(0.680, 0.581, 0.447), (0.590, 0.677, 0.440)\}.\end{aligned} \tag{19}$$

Applying (15)–(17), the values of S_i, I_i and Q_i are estimated and presented in Table 6. According to the given values, the preferences of the agriculture crops are

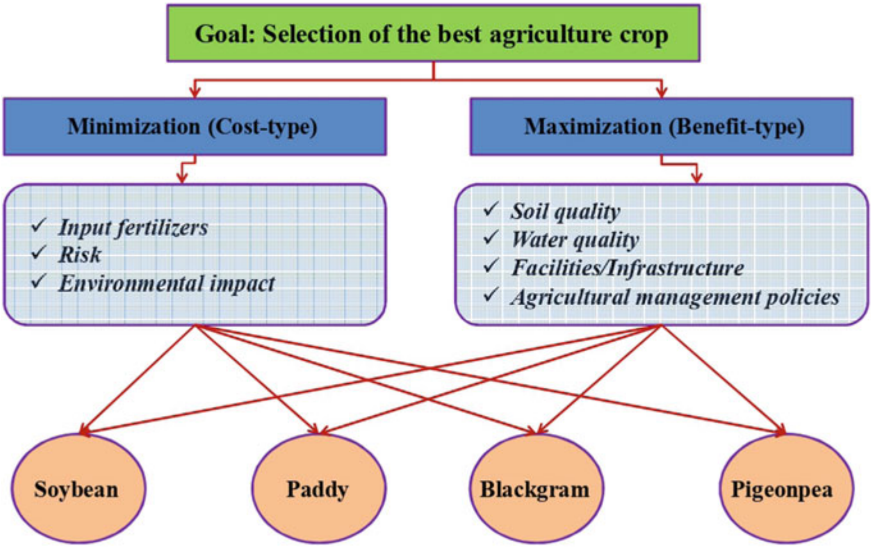

Fig. 3 A comprehensive perspective of the proposed framework

obtained in Table 6. Least value of CS(Q_{i}) determines the best agriculture crop C_2, i.e., Paddy is the desirable choice.

5.1 *Sensitivity Analysis (SA)*

Here, we conduct a SA to measure the impact of change in parameter τ on the final preferences of the crop alternatives. As the τ varies from 0.0 to 0.2, the ranking of the options is $C_2 \succ C_1 \succ C_3 \succ C_4$, from 0.3 to 0.5, the preference ordering is $C_2 \succ C_3 \succ C_1 \succ C_4$, and from 0.6 to 1.0, the preference ordering is $C_2 \succ C_3 \succ C_4 \succ C_1$, i.e., the ranking of given four agriculture crop options is similar in each case. Thus, we can say that the results attained by PF-CRITIC–VIKOR approach are more valuable and well-consistent.

It is clear from Table 7 and Fig. 4 that as the value of τ increases, the compromise measure Q_i of C_2 remains same, while C_3 and C_4 decreases and C_1 increases, i.e., the crop option C_2 is the best choice in each set. And, hence, it is observed that Paddy crop C_2 is supreme than other options.

Table 3 Assessment ratings of agriculture crop selection

Criteria	C_1	C_2	C_3	C_4
R_1	A_1: (0.68, 0.30)	A_1: (0.80, 0.15)	A_1: (0.68, 0.25)	A_1: (0.60, 0.30)
	A_2: (0.60, 0.35)	A_2: (0.70, 0.15)	A_2: (0.65, 0.30)	A_2: (0.55, 0.35)
	A_3: (0.65, 0.30)	A_3: (0.75, 0.20)	A_3: (0.70, 0.25)	A_3: (0.70, 0.20)
	A_4: (0.75, 0.20)	A_4: (0.65, 0.25)	A_4: (0.65, 0.40)	A_4: (0.65, 0.30)
R_2	A_1: (0.60, 0.25)	A_1: (0.62, 0.35)	A_1: (0.70, 0.38)	A_1: (0.62, 0.43)
	A_2: (0.70, 0.20)	A_2: (0.55, 0.45)	A_2: (0.74, 0.40)	A_2: (0.75, 0.38)
	A_3: (0.50, 0.45)	A_3: (0.65, 0.38)	A_3: (0.68, 0.35)	A_3: (0.70, 0.34)
	A_4: (0.45, 0.52)	A_4: (0.58, 0.48)	A_4: (0.65, 0.40)	A_4: (0.65, 0.42)
R_3	A_1: (0.55, 0.65)	A_1: (0.40, 0.65)	A_1: (0.50, 0.55)	A_1: (0.57, 0.60)
	A_2: (0.50, 0.60)	A_2: (0.50, 0.55)	A_2: (0.50, 0.65)	A_2: (0.54, 0.65)
	A_3: (0.58, 0.70)	A_3: (0.55, 0.60)	A_3: (0.55, 0.60)	A_3: (0.50, 0.58)
	A_4: (0.50, 0.70)	A_4: (0.45, 0.70)	A_4: (0.40, 0.55)	A_4: (0.48, 0.64)
R_4	A_1: (0.60, 0.70)	A_1: (0.68, 0.55)	A_1: (0.62, 0.50)	A_1: (0.58, 0.65)
	A_2: (0.65, 0.55)	A_2: (0.60, 0.58)	A_2: (0.65, 0.58)	A_2: (0.52, 0.60)
	A_3: (0.60, 0.50)	A_3: (0.58, 0.62)	A_3: (0.58, 0.67)	A_3: (0.50, 0.60)
	A_4: (0.58, 0.62)	A_4: (0.56, 0.63)	A_4: (0.60, 0.70)	A_4: (0.45, 0.58)
R_5	A_1: (0.60, 0.65)	A_1: (0.72, 0.50)	A_1: (0.73, 0.58)	A_1: (0.77, 0.58)
	A_2: (0.65, 0.60)	A_2: (0.70, 0.55)	A_2: (0.78, 0.50)	A_2: (0.72, 0.67)
	A_3: (0.64, 0.52)	A_3: (0.75, 0.52)	A_3: (0.70, 0.60)	A_3: (0.64, 0.52)
	A_4: (0.57, 0.68)	A_4: (0.68, 0.56)	A_4: (0.70, 0.62)	A_4: (0.68, 0.56)
R_6	A_1: (0.72, 0.55)	A_1: (0.72, 0.58)	A_1: (0.75, 0.52)	A_1: (0.68, 0.56)
	A_2: (0.63, 0.68)	A_2: (0.65, 0.52)	A_2: (0.70, 0.56)	A_2: (0.72, 0.54)
	A_3: (0.67, 0.53)	A_3: (0.64, 0.55)	A_3: (0.72, 0.58)	A_3: (0.70, 0.66)
	A_4: (0.65, 0.52)	A_4: (0.58, 0.50)	A_4: (0.68, 0.57)	A_4: (0.68, 0.58)
R_7	A_1: (0.58, 0.70)	A_1: (0.55, 0.70)	A_1: (0.55, 0.72)	A_1: (0.56, 0.65)
	A_2: (0.60, 0.72)	A_2: (0.60, 0.65)	A_2: (0.62, 0.70)	A_2: (0.55, 0.70)
	A_3: (0.59, 0.68)	A_3: (0.50, 0.68)	A_3: (0.64, 0.69)	A_3: (0.52, 0.66)
	A_4: (0.60, 0.54)	A_4: (0.62, 0.72)	A_4: (0.60, 0.70)	A_4: (0.50, 0.72)

5.2 *Comparative Study*

In the current section, we perform a comparison between the developed framework and the existing model to explore the robustness of the proposed PF-CRITIC–VIKOR model. For this, we have selected an existing method, namely, PF-TOPSIS, given by Zhang and Xu [58].

From Table 4 and Eqs. (13)–(14), the PF-IS and PF-AIS are evaluated. The discrimination measures of each crop alternative with PF-IS and PF-AIS are estimated by Eq. (2). The relative closeness coefficient $W(C_i)$ of each crop option

Table 4 APF-DM for agriculture crop selection

	C_1	C_2	C_3	C_4
R_1	(0.668, 0.296, 0.683)	(0.749, 0.166, 0.641)	(0.668, 0.285, 0.687)	(0.604, 0.304, 0.736)
R_2	(0.612, 0.286, 0.737)	(0.596, 0.400, 0.696)	(0.705, 0.387, 0.595)	(0.679, 0.404, 0.613)
R_3	(0.530, 0.645, 0.550)	(0.456, 0.620, 0.638)	(0.491, 0.584, 0.646)	(0.543, 0.620, 0.567)
R_4	(0.614, 0.619, 0.490)	(0.632, 0.577, 0.517)	(0.624, 0.565, 0.540)	(0.538, 0.619, 0.573)
R_5	(0.616, 0.627, 0.478)	(0.711, 0.526, 0.467)	(0.741, 0.561, 0.370)	(0.733, 0.598, 0.323)
R_6	(0.680, 0.581, 0.447)	(0.675, 0.545, 0.498)	(0.723, 0.545, 0.425)	(0.695, 0.564, 0.446)
R_7	(0.590, 0.677, 0.440)	(0.574, 0.685, 0.448)	(0.589, 0.708, 0.389)	(0.545, 0.677, 0.495)

Table 5 The standard PF-matrix $\widetilde{S} = \left(\widetilde{\chi}_{ij}\right)_{p\times q}$, SD, amount of information and attribute weight

Criteria	C_1	C_2	C_3	C_4	σ_j	c_j	ω_j
R_1	0.333	1.000	0.357	0.000	0.362	2.417	0.1428
R_2	0.643	0.000	1.000	0.671	0.362	2.481	0.1466
R_3	0.524	1.000	0.115	0.000	0.393	2.963	0.1751
R_4	0.468	0.014	0.000	1.000	0.409	3.152	0.1862
R_5	0.000	0.978	1.000	0.782	0.407	1.891	0.1117
R_6	0.000	0.326	1.000	0.393	0.361	1.798	0.1062
R_7	0.000	0.584	0.865	1.000	0.458	2.224	0.1314

Table 6 The values of S_{i}, I_{i} and Q_{i} for the evaluation of agriculture crops

Crop option	S_i	I_{i}	Q_{i}
C_1	0.857	0.216	0.751
C_2	0.576	0.146	0.000
C_3	0.607	0.269	0.494
C_4	0.724	0.286	0.765
Ranking order	$S_2 \succ S_3 \succ S_4 \succ S_1$	$I_2 \succ I_1 \succ I_3 \succ I_4$	$Q_2 \succ Q_3 \succ Q_1 \succ Q_4$

Table 7 Variation of CS over different parameter τ values

τ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
C_1	0.502	0.551	0.601	0.651	0.701	0.751	0.801	0.850	0.900	0.950	1.000
C_2	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
C_3	0.878	0.801	0.725	0.648	0.571	0.494	0.418	0.341	0.264	0.187	0.111
C_4	1.000	0.953	0.906	0.859	0.812	0.765	0.718	0.671	0.624	0.577	0.530

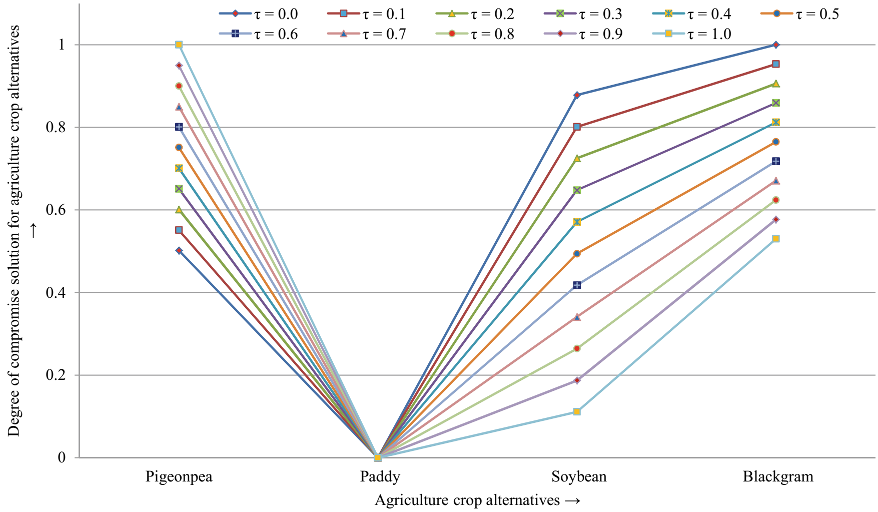

Fig. 4 Variation of CS over parameter τ values for each agriculture crop alternative

is assessed as $W(C_1) = 0.2753$, $W(C_2) = 0.5991$, $W(C_3) = 0.6162$, and $W(C_2) = 0.4431$. Hence, the final preference order of given agriculture crop option is $C_3 \succ C_2 \succ C_4 \succ C_1$. Therefore, the most suitable agriculture crop alternative is C_3, that is, Soybean crop, which is quite different from the proposed method. Further, the rank CRC values with compromise measure are calculated as (0.80, 0.80, 1.00, 0.60). The rank CRC is employed to rank the values for the determination of consistency of the proposed methodology. From Fig. 5, it is found that proposed methodology is

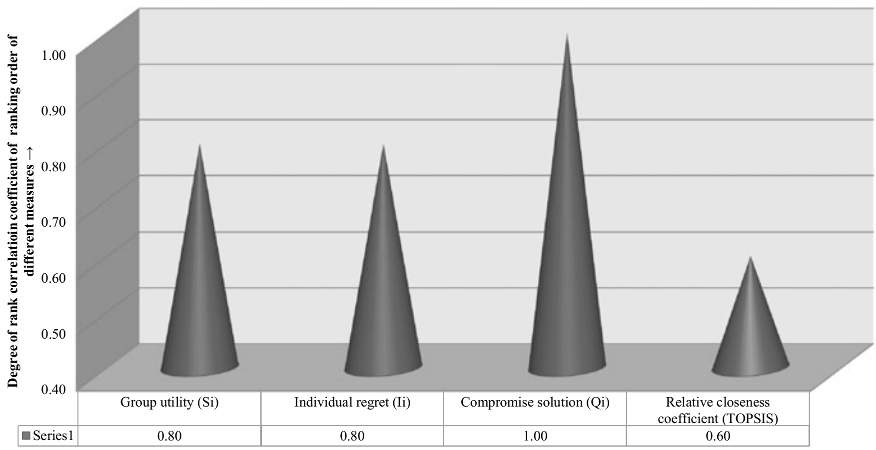

Fig. 5 Correlation design of different measures of VIKOR approach with existing approaches

well consistent than the existing one. Further, the PF-CRITIC–VIKOR framework has the following benefits:

(a) As the DMs weight is taken into account, we have utilized an approach based on PF-entropy which gives the more exact decisions for MCDM problems.
(b) Another benefit of the PF-CRITIC–VIKOR model comparing with PF-TOPSIS [58] model is that it originates the CS(s) which assume not only maximizing US(s) for the majority but minimizing WGS(s) for the opponent as well, while in PF-TOPSIS model, it assumes discrimination from the PF-IS and from the PF-AIS, without considering their relative importance.
(c) The CRITIC is a more straightforward method with less computational effort. Besides, in the case of PCA (Principal component analysis) crisp values of inter-criteria correlation coefficients should be defined, in order to distinguish those criteria considered to be highly correlated. This subjective intervention is avoided by using the CRITIC method, which makes the developed PF-CRITIC–VIKOR approach more sensible, flexible and efficient.

Furthermore, we illustrate a thorough comparison of the proposed approach with further MCDM methods based on different vital parameters applied in the decision-making process (see Table 8). It can be observed that the developed method is definitely a novel contribution as it integrates all major parameters of the MCDM process into comparison with existing methods to solve the MCDM problems under PFS environment.

6 Conclusions

This work presents a novel MCDM framework to rank the Kharif crop alternatives in Satna district. For this, firstly, new divergence and entropy measures have been developed within PFSs context. Secondly, an integrated PF-CRITIC–VIKOR methodology has been proposed with the combination of the CRITIC and the VIKOR methods under PFSs environment. In this methodology, the CRITIC model has been utilized to derive the criteria weights and the proposed entropy measure has been applied to compute the DMs' weights. In addition, the average and worst group scores have been calculated with the help of proposed divergence measure. Additionally, a case study of Kharif crop alternative assessment has been presented to validate the potentiality and strength of the PF-CRITIC–VIKOR methodology. The PF-CRITIC approach determines the weights of the considered seven criteria, which as Risk (0.1862), Input fertilizers (0.1751), Water quality (0.1466), Soil quality (0.1428), Environmental impact (0.1314), Facilities/Infrastructure (0.1117), and Agricultural management policies (0.1062). By employing the PF-CRITIC-VIKOR method, the Kharif crop pattern is obtained as Paddy $\succ$ Soybean $\succ$ Pigeonpea $\succ$ Blackgram. Next, we have demonstrated a sensitivity analysis with parameter values to analyze the stability of proposed PF-CRITIC–VIKOR approach. Afterward, a comparison has been carried out between the proposed and corresponding related model which

Table 8 A comparison of ranking of the alternatives with various existing methods

Authors	Methods	Criteria weights	MCDM model	Expert weights	Ranking order	Optimal choice
Zhang and Xu [58]	Similarity measure-based TOPSIS method	Assumed	Compromising model	Assumed	$C_3 \succ C_2 \succ C_4 \succ C_1$	C_3
Zhang et al. [60]	Similarity measure-based method	Assumed	Compromising model	Not considered	$C_3 \succ C_2 \succ C_4 \succ C_1$	C_3
Rani et al. [42]	Novel divergence and entropy measure-based VIKOR method with PFSs	Entropy and divergence measure-based method	Compromising model	Computed	$C_2 \succ C_3 \succ C_1 \succ C_4$	C_2
Proposed method	Proposed divergence measure and entropy measure-based PF-CRITIC–VIKOR method	CRITIC method	Compromising model	Computed (Entropy measure-based formula)	$C_2 \succ C_3 \succ C_1 \succ C_4$ $(\tau = 0.5)$	C_2

validates the robustness. The final outcomes verify that the proposed model is more effective, reliable, and stable and has less complicated mathematical steps than existing approaches within PFSs context. In addition, it provides a new weight-determining procedure to assess more accurate criteria and decision-weights that improves the permanence of introduced model. As a result, the present PF-CRITIC–VIKOR method will be very useful for agricultural stakeholders in the selection of Kharif crop pattern selection process. However, this method has limitations in dealing with a large number of criteria set.

In future, we will develop new methods like DNMA (Double Normalization-based Multiple Aggregation), GLDS (Gained and Lost Dominance Score), MARCOS (Measurement Alternatives and Ranking according to the Compromise Solution) and others under PFSs environment and employ to choose Kharif season crop selection with adaption and mitigation options resulting from climate change issues for better sustainable agricultural perspectives. In addition, further studies may include subjective and objective weights of the criteria to improve the accuracy of the proposed method.

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Choquet Integral Under Pythagorean Fuzzy Environment and Their Application in Decision Making

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1 Introduction

Recent developments in information processing have heightened the need for a highly efficient information aggregation operators. The Choquet integral is one of the aggregation operators in which it is used to aggregate information and calculate the global score. It was introduced by Choquet [10] as an aggregation operator to solve the interrelationship among criteria of decision problems where ordering of the individual criteria is the ultimate result. Ordering of criteria is more challenging particularly when there exists some uncertainty regarding the criteria. In light of this difficulty, the concept of fuzzy measure is used in Choquet integral where interaction phenomena among the decision criteria can be modeled [40]. The Choquet integral uses the concept of fuzzy measure to indicate the weights or the importance of multiple interdependent criteria in decision making [12]. The main idea of Choquet integral is that the interrelationship between criteria can be modeled through a fuzzy measure where assigning a weight is not only to each criterion but also to each subset of

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_8

criteria [11, 36]. In other words, information about criteria is expressed via a fuzzy measure. It is a non-additive fuzzy integral or a numerical-based approach where interactivity of subjective judgment experts can be eliminated. The Choquet integral encompasses the property of non-additive capacity and corresponds to a large class of aggregation functions [8, 34]. Sub additive or super additive operators are used to integrate functions with respect to the fuzzy measures where many extensions and generalizations of fuzzy sets could be inserted into fuzzy measures.

One of the generalizations of fuzzy sets is intuitionistic fuzzy sets (IFS). Atanassov [5] generalized fuzzy sets to IFS. Element in IFS is expressed by an ordered pair, and each ordered pair is characterized by a membership degree and non-membership degree. The sum of the two degrees must be less than or equal to one [5]. The IFS has been received much attention since its inception and broadly applied as an aggregation operator. In the context of fuzzy measures, instead of using fuzzy sets, Tan and Chen [42] used IFS to propose IFS Choquet integral based on *t*-norms and *t*-conorms. Murofushi and Sugeno [33] used the Choquet integral to propose the interval-valued IFS correlated averaging operator and interval-valued IFS correlated geometric operator to aggregate interval-valued IFS information and applied them to a practical decision making problem. Wang et al. [49] proposed a method based on intuitionistic fuzzy dependent aggregation operators and applied them to supplier selection. Recently, Liang et al. [31] proposed arithmetical average operator and geometric average operator based on IFS to aggregate the intuitionistic fuzzy information. Abdullah et al. [4] propose a combination of interval-valued IFS and Choquet integral to allow a strong interrelationship between criteria of a decision making model. In a case of personnel matching management, very recently Yu and Xu [51] proposed novel IFS Choquet integral aggregation. This aggregation operator is integrated with multi-objective decision-making model to find optimal personnel matching results.

It can be seen that these aggregation operators are built using the dual memberships of IFS where the sum of these two memberships is less than or equal to one. However, in some circumstances, the sum of membership and non-membership degree of criteria could be greater than one. This situation could not be described by the IFS. In order to address this problem, Yager [50] introduced another generalization of fuzzy set which is called as Pythagorean fuzzy set (PFS). Unlike IFS, the sum squares of PFS memberships is less than or equal to one. The PFS has emerged as an effective tool compared to IFS in depicting uncertainty of criteria in decision problems. For example, Chen [9] proposed Chebyshev distance measures in the ELimination Et Choice Translating REality (ELECTRE) method for addressing multiple criteria decision-making problems under uncertainty of PFS. Very recently, the author developed operational laws and their corresponding weighted aggregation operators based on PFS then proposed an algorithm to solve the multiple attribute group decision making problems [18–23]. Some other publications that related to PFS and its applications can be retrieved from Ejegwa [14, 16, 17], Wan Mohd et al. [46], Abdullah and Mohd [3], Wan Mohd and Abdullah [45] and Ejegwa and Awolola [15].

In relation to application of Choquet integral under PFS environment, Khan [30] used the Choquet integral to develop a very comprehensive Pythagorean fuzzy aggregation operator. With no specific application, the author developed a specific aggregation operator and illustrated the proposed method with a group decision making problem. Wang et al. [48] developed the concept of interval-valued hesitant Pythagorean fuzzy sets. To ease the application in selecting project private partner, the authors supported the concept with technique for order preference by similarity to ideal solution and Choquet integral. Khan et al. [29] proposed several aggregation operators based on Pythagorean hesitant fuzzy Choquet integral and applied the developed operators to multi-attribute decision making problem. It can be seen that these literature do not provide any specific real-life applications. It is also noticed that these literature tend to focus on hesitant PFS and various types of aggregator operators instead of real decision making applications. Far too little attention has been paid to introduce PFS in the aggregation operator of Choquet integral. In other words, the two memberships of PFS are not fully utilized in aggregating information through the Choquet integral operator. To bridge this knowledge gap, this chapter proposes PFS-Choquet integral where the concept of fuzzy measures is extended to PFS. The proposed PFS-Choquet is expected to provide better tool in handling uncertain and incomplete information of a case study of sustainable solid waste management (SWM). The rest of this paper is organized as follows. In Sect. 2, we describe the concepts of PFS and Pythagorean fuzzy measures, and some knowledge about the Choquet integral. The proposed PFS-Choquet integral is presented in Sect. 3. In Sect. 4, a case study of SWM is presented to illustrate the proposed method. Finally, conclusion is made in Sect. 5.

2 Preliminaries

This section recalls the definitions of fuzzy measures, PFS, and Choquet integral. These definitions are fully utilized in proposing the PFS-Choquet integral.

Definition 1 (*Yager [50]*) A Pythagorean Fuzzy Sets P in a finite universe of discourse is

$$P = \{< x, \mu_P(x), \nu_P(x) > | x \in X\} >$$

where $\mu_p, \nu_P : X \rightarrow [0, 1][0, 1]$ with the condition that the square sum of its membership degree and non-membership degree is less than or equal to 1.

$$(\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$$

It is clearly seen that sum of squares of its memberships is less than or equal to one.

Definition 2 (*Grabisch [24]; Sugeno [41]*) A fuzzy measure on X is a set function $\mu : 2^x \rightarrow [0, 1]$ satisfying the following axioms.

(1) $\mu(\phi) = 0, \mu(X) = 1$ (boundary conditions)
(2) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ (monotonicity)

Definition 3 (*Murofushi and Sugeno [33]*) A fuzzy density function of a fuzzy measure μ on a finite set X is a function $s : X \rightarrow [0, 1]$ satisfying,

$$s(x) = \mu(\{x\}), x \in X$$

$s(x)$ is called the fuzzy density of singleton x.

Definition 4 (*Tan and Chen [42]*) Let f be a real-valued function on X, and μ be a non-additive measure (fuzzy measure) on X. Then, the Choquet integral of f with respect to non-additive μ is represented as,

$$C_{\mu}(f)\big) = \sum_{i=1}^{n} \mu\big(x_{(i)}\big) - \mu(x_{(i+1)})\Big]f(i) \tag{1}$$

where $(.)$ is finite order of permutation $f(1) \leq \cdots \leq f(n), A(i) = \{i, \ldots, n\}$, and $A_{(n+1)} = \phi$.

It shows that Choquet integral is also called Lebesgue integral up to reordering of the indices. In other words, when the fuzzy measure is additive, it shows that the Choquet integral reduces to Lebesgue integral.

Conditions of fuzzy measures are given in Definition 5.

Definition 5 (*Sugeno [40]*) A fuzzy measure on X is a set function $\mu : P(X) \rightarrow [0, 1]$, satisfying:

i. Boundary condition: $\mu(\phi) = 0$ and $\mu(X) = 1$;
ii. Monotonicity: If $A, B \in P(X)$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

For $A, B \in P(X)$ with $A \cap B \in \varphi$, the fuzzy measure is said to be:

i. an additive measure, if $\mu(A \cup B) = \mu(A) + \mu(B)$;
ii. a super additive measure, if $\mu(A \cup B) > \mu(A) + \mu(B)$;
iii. a sub additive measure if $\mu(A \cup B) < \mu(A) + \mu(B)$.

and,

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B),\ \lambda \in [-1, \infty),\ \forall A, B \in P(X) \text{ and } A \cap B = \varphi \tag{2}$$

This equation is required to find fuzzy measure.

If X is finite, then the parameter λ of a fuzzy measure satisfies

$$\mu(X) = \frac{1}{\lambda}\left(\prod_{i=1}^{n}(1 + \lambda\mu(x_i)) - 1\right), \lambda \neq 0 \tag{3}$$

The parameter λ can be determined with the boundary condition $\mu(x) = 1$, i.e.,

$$\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda\mu(x_i)) \tag{4}$$

Definition 6 (*Choquet [10]*) Let μ be a fuzzy measure on N. The Choquet integral of $x = (x_1, \ldots, x_n) \in [0, 1]^n$ with respect to μ is defined as

$$C_\mu(x_1, \ldots, x_n) = \sum_{i=1}^{n}\left[\mu\left(A_{(i)}\right) - \mu\left(A_{(i+1)}\right)\right]x_i \tag{5}$$

where (.) indicates a finite order of permutation on N such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_n$, and $A_{(i)} = \{(i), \ldots (n)\}, A_{(n+1)} = \phi$

Definition 7 (*Tan and Chen [42]*) Let $x_i = (t_{xi}, f_{xi})$ $(i = 1,2. \ldots..n)$ be a collection of intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. The (discrete) intuitionistic fuzzy Choquet integral of x_i with respect to μ is defined by

$$IF\ C_\mu(x_1, \ldots, x_n) = \sum\nolimits_{i=1}^{n}\left[\mu\left(A_{(i)}\right) - \mu\left(A_{(i+1)}\right)\right]x_i$$

where $(\cdot)$ indicates a permutation on X such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_n$ and $A_{(i)} = \{(i), \ldots (n)\}, A_{(n+1)} = \phi$

These definitions are generally used in the proposed method in which the ultimate decision of Choquet value could be made based on Definition 6. Detailed explanation of the proposed method is presented in the following section.

3 Pythagorean Fuzzy Choquet Integral

It is known that the characteristics of PFSs are closely related to IFSs despite its differences in the condition of dual memberships. Similar to IFSs where arithmetic operations, aggregation operators have been widely discussed in literature, several attempts have been made to understand the algebraic operations of PFS. Peng and Yang [37] for example, proposed division and subtraction operations of PFS and discuss their properties. The properties of aggregation operators (see Definition 5) such as boundedness, idempotency, and monotonicity were also investigated. They developed a Pythagorean fuzzy superiority and inferiority ranking method to solve multi-criteria decision making (MCDM) problem instead of a special aggregation operator. In this paper, we used the knowledge of aggregations operator properties and extended it to propose PFS-Choquet integral. In this paper, the PFS-Choquet integral is defined as follows.

Let $x_i = (t_{xi}, f_{xi})$ $(i = 1,2. \ldots..n)$ be a collection of Pythagorean intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. The (discrete) Pythagorean fuzzy Choquet integral of x_i with respect to μ is defined by

$$PICI_{\mu}(x_1, \ldots, x_n) = \sum\nolimits_{i=1}^{n} \left[\mu\left(A_{(i)}\right) - \mu\left(A_{(i+1)}\right)\right] x_i$$

where $(\cdot)$ indicates a permutation on X such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_n$ and $A_{(i)} = \{(i), \ldots (n)\}, A_{(n+1)} = \phi$.

Differently from the method of Peng and Yang [37] where membership and non-membership are calculated separately at the end of the computational procedures, this proposed method substitutes the separation method with the score functions. This proposed method is not fitted with the separation of membership and non-membership as it is undermined the concept of interrelationship of dual memberships of PFS. In response to this limitation, the score function proposed by Zhang and Xu [53] is substituted to the newly PFS-Choquet integral. In this proposed method, the membership and non-membership are combined to get the score of all criteria of MCDM problems. In addition, linguistic terms used in evaluation are defined in PFS in which the sum of squares of two memberships for linguistic terms is less than or equal to one. The computational procedure of the proposed PFS-Choquet integral method is given as follows.

Step 1: Construct a function X_m where $X = \{x_1, x_2, \cdots, x_\mathrm{n}\}$. Identify input n value (number of evaluation items) and m value (number of inputs).

Step 2: Construct decision matrix using the linguistic variable defined in PFS.

Step 3: Calculated weighted Pythagorean fuzzy set using Eq. (6).

$$\lambda_m P_{i,m} = \left\langle \sqrt{1 - \left(1 - \mu_{i,m}^2\right)^{\lambda}}, \left(v_{i,m}\right)^{\lambda} \right\rangle \tag{6}$$

Step 4: Calculate the aggregation for each input using Eq. 7

$$\lambda_{m_1} P_{i,m_1} \oplus \lambda_{m_2} P_{i,m_2} = \left\langle \sqrt{\mu^2_{\lambda_{m_1} P_{i,m_1}} + \mu^2_{\lambda_{m_2} P_{i,m_2}} - \mu^2_{\lambda_{m_1} Z_{m_1 P_{i,m_1}}} \mu^2_{\lambda_{m_2} P_{i,m_2}}}, v_{\lambda_{m_1} P_{i,m_1}} v_{\lambda_{m_2} P_{i,m_2}} \right\rangle \tag{7}$$

Step 5: Calculate the score function using Eq. (8)

$$s_i = \mu_{P_i}^2 - v_{P_i}^2 \tag{8}$$

Step 6: Obtain relative weight for each criteria using Eq. (9)

$$\frac{\lambda_m P_{i,m}}{\sum_{i=1}^{n} \lambda_m P_{i,m}} \tag{9}$$

The relative weight of criteria is obtained using Eq. (9) in which the defuzzification is already implemented beforehand using the score function equation.

Step 7: Obtain $v(i)$ using Eq. (10)

$$v(i, i+1) = w_{C_i} + w_{C_{i+1}} \tag{10}$$

Step 8: Calculate value of the Pythagorean Choquet integral.

$$PICI_{\mu}(x_1, ..., x_n) = \sum\nolimits_{i=1}^{n} \left[\mu\left(A_{(i)}\right) - \mu\left(A_{(i+1)}\right)\right] x_i \tag{11}$$

where t is a permutation satisfying $x_{t(1)} \leq \ldots \leq x_{t(n)}$.

The ultimate output of the proposed method is the Choquet value in which the strength of aggregated criteria is measured.

4 Application to Sustainable Solid Waste Management

The Choquet integral is one of the aggregation operators that has been used to aggregate information. It has been applied with great success to many different information aggregation cases where vast majority of information in real-world applications is characterized by high level of uncertainties. The success of Choquet integral in information aggregation can be witnessed in economics, insurance, finance, quality of life, and social welfare [27]. It is mainly owing to the non-additive characteristics of Choquet integral in which it can play a key role or provide a capacity for recent advances in decision theory [25]. Moreover, the role of fuzzy measures in dealing uncertainties, and the role of operator integrals in computational aspects are the vital component in Choquet integral that guarantee success in real applications. Several real application research using Choquet integral are supply chain management strategy measurements [47], job-shop scheduling problem [38], selection of optimal supplier in supply chain management [44]. Zhang [52], used Choquet integral for screening geological CO2 storage sites. Very recently, Olawumi and Chan [35] used generalized Choquet fuzzy integral method to determine the importance weights of the sustainability assessment criteria. The sustainable solid waste management problem and methods used in solving the problems are also discussed by many authors. Very recently, Sharma, et al. [39] applied the clustering method k-means algorithms framework for municipal solid waste management. In another recently published work, Tascione et al. [43], used a linear programming model to identify the best scenario of managing solid waste. With about a similar model, Batur et al. [7] used a mixed integer linear programming model for solid waste management by developing a long term system. This paper adds another application of the proposed Choquet integral to the case of sustainable SWM. Specifically, this chapter presents an evaluation of sustainable SWM of two major cities in Malaysia using the proposed method. The two cities are Kuala Lumpur and Johor Bahru. Kuala Lumpur is the capital of Malaysia where it is located at the west coast of Peninsular Malaysia. Johor Bahru is located at the southern part of Peninsular Malaysia and separated by a causeway with Singapore. The evaluation includes the selection of the best city in Malaysia in the context of managing solid waste. The ultimate goal of this application is to obtain the optimized value of Choquet integral in which the better city in managing solid waste could be suggested. Detailed descriptions of the evaluation are given as follows.

4.1 Experts and Criteria

The evaluation begins with the identification of experts and criteria. The evaluation criteria that influence sustainable SWM are retrieved from literature while the weight and priority of the criteria are provided by a group of experts in the field that related to sustainable SWM. This group of four experts comprises several important key personnel at number of sustainable SWM companies and some are academicians that are attached to environmental studies academic program at a public university in Malaysia. Personal communications with experts were conducted to collect linguistic evaluation. These verbal communications mainly aimed at obtaining the weight of importance of criteria of sustainable SWM. Table 1 shows the linguistic terms and rating scale used in this study.

The criteria for this study are retrieved from several literature in sustainable SWM (see Herva and Roca [1, 28] In this study, the selected criteria are Relative Cost (C_1), Environmental Health (C_2), Socio-culture (C3), Public Awareness(C4), Institutional(C5), Technical (C6), Operation & Maintenance Challenges (C7), Population

Table 1 Importance of criteria and PFS rating scale

Linguistic terms of "Importance"	Pythagorean Fuzzy Sets
Very low	$\langle 0, 0\rangle$
Low	$\langle 0.1, 0.9\rangle$
Medium low	$\langle 0.2, 0.9\rangle$
Medium	$\langle 0.4, 0.6\rangle$
Medium high	$\langle 0.5, 0.7\rangle$
High	$\langle 0.7, 0.2\rangle$
Very high	$\langle 0.9, 0.1\rangle$

Size (C8), Human Health (C9), and Consumption Habits (C10). Detailed description of the criteria can be retrieved from Abdullah and Goh [2]. The collected linguistic evaluations with respect to criteria are then computed using the proposed method. Detailed computations are described as follows.

4.2 Computation

Step 1: Construct evaluation matrix.

Based on the defined linguistic in Table 1, the evaluation matrix of Expert 1, Expert 2, Expert 3, Expert 4 are summarized in Table 2.

Step 2: Calculate weighted PFS matrix.

Prior to calculating weighted PFS, the weight of each expert is obtained based on their working experience, knowledge, and also seniority in their company. Table 3 presents the weight and relative weight of experts.

The next step is to calculate the weighted rating. It is calculated using Eq. (6).

The weighted rating of Expert 1, Expert 2, Expert 3, and Expert 4 are summarized in Table 4.

Table 2 Rating of importance of criteria from experts

Criteria	Expert 1	Expert 2	Expert 3	Expert 4
C_1	$\langle 0.7, 0.2\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.2, 0.9\rangle$
C_2	$\langle 0.9, 0.1\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.9, 0.1\rangle$	$\langle 0.9, 0.1\rangle$
C_3	$\langle 0.4, 0.6\rangle$	$\langle 0.2, 0.9\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.5, 0.5\rangle$
C_4	$\langle 0.5, 0.5\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.7, 0.2\rangle$
C_5	$\langle 0.5, 0.5\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.5, 0.5\rangle$
C_6	$\langle 0.2, 0.9\rangle$	$\langle 0.2, 0.9\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.4, 0.6\rangle$
C_7	$\langle 0.2, 0.9\rangle$	$\langle 0.1, 0.9\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.4, 0.6\rangle$
C_8	$\langle 0.7, 0.2\rangle$	$\langle 0.7, 0.2\rangle$	$\langle 0.1, 0.9\rangle$	$\langle 0.2, 0.9\rangle$
C_9	$\langle 0.2, 0.9\rangle$	$\langle 0.4, 0.6\rangle$	$\langle 0.2, 0.9\rangle$	$\langle 0.4, 0.6\rangle$
C_{10}	$\langle 0.4, 0.6\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.5, 0.5\rangle$	$\langle 0.5, 0.5\rangle$

Table 3 Weight score and relative weight of experts

Experts	Weight score	Relative weight, λ_m
1	0.80	0.2540
2	0.75	0.2381
3	0.75	0.2381
4	0.85	0.2698

Table 4 Weighted rating for criteria

Criteria	Expert 1	Expert 2	Expert 3	Expert 4
C_1	$\langle 0.3965, 0.6645\rangle$	$\langle 0.5715, 0.5780\rangle$	$\langle 0.2017, 0.8855\rangle$	$\langle 0.1047, 0.9720\rangle$
C_2	$\langle 0.5866, 0.5572\rangle$	$\langle 0.5715, 0.5780\rangle$	$\langle 0.5715, 0.5780\rangle$	$\langle 0.6010, 0.5372\rangle$
C_3	$\langle 0.2801, 0.8783\rangle$	$\langle 0.0983, 0.9752\rangle$	$\langle 0.2017, 0.8855\rangle$	$\langle 0.2733, 0.8294\rangle$
C_4	$\langle 0.2654, 0.8386\rangle$	$\langle 0.3849, 0.6817\rangle$	$\langle 0.2573, 0.8479\rangle$	$\langle 0.4076, 0.6477\rangle$
C_5	$\langle 0.2654, 0.8386\rangle$	$\langle 0.2573, 0.8479\rangle$	$\langle 0.2017, 0.8855\rangle$	$\langle 0.2733, 0.8294\rangle$
C_6	$\langle 0.1016, 0.9736\rangle$	$\langle 0.0983, 0.9752\rangle$	$\langle 0.2017, 0.8855\rangle$	$\langle 0.2144, 0.8712\rangle$
C_7	$\langle 0.1016, 0.9736\rangle$	$\langle 0.0489, 0.9752\rangle$	$\langle 0.2017, 0.8855\rangle$	$\langle 0.2144, 0.8712\rangle$
C_8	$\langle 0.3965, 0.6645\rangle$	$\langle 0.3849, 0.6817\rangle$	$\langle 0.0489, 0.9752\rangle$	$\langle 0.1047, 0.9720\rangle$
C_9	$\langle 0.2654, 0.8386\rangle$	$\langle 0.2017, 0.8855\rangle$	$\langle 0.0983, 0.9752\rangle$	$\langle 0.2144, 0.8712\rangle$
C_{10}	$\langle 0.5866, 0.5572\rangle$	$\langle 0.3849, 0.6817\rangle$	$\langle 0.2573, 0.8479\rangle$	$\langle 0.2733, 0.8294\rangle$

Step 3: Calculate the aggregated matrix.

The aggregated matrix for Kuala Lumpur, for example, is calculated using Eq. (7).

$$\begin{aligned}\lambda_1 P_1 \oplus \lambda_2 P_1 &= \left\langle \sqrt{0.3965^2 + 0.5715^2 - 0.3965^2 \times 0.5715^2}, 0.6645 \times 0.5780 \right\rangle \\ &= \langle 0.6576, 0.3841\rangle \\ \lambda_1 P_2 \oplus \lambda_2 P_2 &= \left\langle \sqrt{0.5866^2 + 0.5715^2 - 0.5866^2 \times 0.5715^2}, 0.5572 \times 0.5780 \right\rangle \\ &= \langle 0.7472, 0.3221\rangle\end{aligned}$$

The remaining calculation for aggregated rating is computed similarly. Table 5 presents aggregated rating for Kuala Lumpur and Johor Bahru.

Table 5 Aggregated weighted rating of cities

Criteria	Aggregated rating of Kuala Lumpur	Aggregated rating of Johor Bharu
C_1	$\langle 0.6576, 0.3841\rangle$	$\langle 0.2262, 0.8607\rangle$
C_2	$\langle 0.7472, 0.3221\rangle$	$\langle 0.7549, 0.3105\rangle$
C_3	$\langle 0.2293, 0.8566\rangle$	$\langle 0.3351, 0.7344\rangle$
C_4	$\langle 0.4562, 0.5716\rangle$	$\langle 0.4705, 0.5492\rangle$
C_5	$\langle 0.3633, 0.7110\rangle$	$\langle 0.3351, 0.7344\rangle$
C_6	$\langle 0.1410, 0.9495\rangle$	$\langle 0.2911, 0.7715\rangle$
C_7	$\langle 0.1126, 0.9495\rangle$	$\langle 0.2911, 0.7715\rangle$
C_8	$\langle 0.5311, 0.4530\rangle$	$\langle 0.1154, 0.9479\rangle$
C_9	$\langle 0.2248, 0.8621\rangle$	$\langle 0.2349, 0.8497\rangle$
C_{10}	$\langle 0.3633, 0.7110\rangle$	$\langle 0.3687, 0.7032\rangle$

Table 6 Score function of criteria for cities

Criteria	Kuala Lumpur	Johor Bahru
C_1	0.2850	−0.6896
C_2	0.4546	0.4734
C_3	−0.6811	−0.4271
C_4	−0.1186	−0.0803
C_5	−0.3735	−0.4271
C_6	−0.8816	−0.5104
C_7	−0.8888	−0.5104
C_8	0.0769	−0.8852
C_9	−0.6927	−0.6667
C_{10}	−0.3735	−0.3586

Table 6 Aggregated weighted rating of Kuala Lumpur and Johor Bahru.

Step 4: Calculate the score function.

Equation (8) is used to calculate score function for the cities.

For example, the score function for Kuala Lumpur is calculated as,

$$s_{1,1} = 0.6576^2 - 0.3841^2 = 0.2850 \quad s_{2,1} = 0.7472^2 - 0.3221^2 = 0.4546$$

The remaining calculation for score functions are executed similarly. Summarily, the score functions of Kuala Lumpur and Johor Bahru are presented in Table 6.

Step 5: Obtain the relative weight of criteria using Eq. (9) and total weight of criteria using Eq. (10). The relative weights are presented in Table 7.

Step 6: Obtain the total weight of criteria $v(i)$ using Eq. (10)

Table 7 Relative weight and total weight of criteria

Criteria	Relative weight	Total Weight $v(i)$
C_1	0.0007	1.0000
C_2	0.1492	0.9993
C_3	0.0064	0.8501
C_4	0.0495	0.8436
C_5	0.0859	0.7941
C_6	0.1151	0.7083
C_7	0.1683	0.5932
C_8	0.1332	0.4248
C_9	0.1485	0.2917
C_{10}	0.1431	0.1431

For example,

$$
\begin{aligned}
v(1,2,3,4,5,6,7,8,9,10) &= 0.0007 + 0.1492 + 0.0064 + 0.0495 + 0.0859 \\
&\quad +0.1151 + 0.1683 + 0.1332 + 0.1485 + 0.1431 \\
&= 1.0000 \\
v(2,3,4,5,6,7,8,9,10) &= 0.1492 + 0.0064 + 0.0495 + 0.0859 \\
&\quad +0.1151 + 0.1683 + 0.1332 + 0.1485 + 0.1431 \\
&= 0.9993
\end{aligned}
$$

The total weight $v(i)$ of all criteria is presented in Table 7.

Step 7: Calculate the value of Choquet integral using Eq. (11).

For example,

$$
\begin{aligned}
x_1[v(1,\ldots,n) - v(2,\ldots,n)] &= 0.2850[1.0000 - 0.9993] = 0.0002 \\
x_2[v(2,\ldots,n) - v(3,\ldots,n)] &= 0.4546[0.9993 - 0.8501] = -0.0044
\end{aligned}
$$

The remaining calculation for $x_i[v(i,\ldots,n) - v(i+1,\ldots,n)]$ are implemented similarly.

From the calculation, it is found that the Choquet integral for Kuala Lumpur and Johor Bahru are $C_{v(1,\ldots,10),1}(x) = -0.3715$ and $C_{v(1,\ldots,10),2}(x) = -0.3867$, respectively.

The values of Choquet integral for Kuala Lumpur and Johor Bahru are very close to each other which indicates that the two major cities in Malaysia are not much different in managing solid waste. However, on close inspection, the values show that Kuala Lumpur provides is slightly better than Johor Bharu in managing solid waste.

5 Conclusions

In recent years, there has been an increasing interest in developing various types of aggregation operators based on many sets including PFS. These developments have led to a renewed interest in developing Choquet integral operations under PFS environment. In this chapter, we developed Choquet integral based on PFS by combining relative weight of criteria and a score function. In this aggregation, the Choquet integral has been considered to model the interaction between criteria. This new amalgamation has successfully overcome the issues of independence among criteria of decision problems under PFS environment. The proposed method has been applied to a case of sustainable SWM where establishing total weights of criteria and Choquet integral value are the ultimate decision. The proposed eight-step aggregation operator was implemented in the evaluation of two big cities in Malaysia pertaining to criteria of sustainable SWM. Kuala Lumpur is suggested as the better city in managing solid waste based on the values of Choquet integrals. This study has proved that the Choquet integral under PFS environment is one of the potent tools in aggregating incomplete and vague information. Nevertheless, this study has some recommendations for future research. The proposed aggregation operators can be extended by considering the other methods that directly deal with interrelationships among criteria. The Bonferroni mean [26] and the Shapley value [6, 32] are two aggregators that possibly could be utilized to deal with independence and interrelationship among criteria of decision problems. Another possible future research is developing linguistic interval-valued under Pythagorean fuzzy environment [20, 21].

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On Developing Pythagorean Fuzzy Dombi Geometric Bonferroni Mean Operators with Their Application to Multicriteria Decision Making

Nayana Deb and Animesh Biswas

1 Introduction

In recent days, multicriteria decision making (MCDM) has appeared as an active area of research for its capability to find best alternative from a given set of alternatives by evaluating them on the basis of satisfying criteria. Due to the presence of inherent vagueness in human perceptions as well as information collected from the sources, several kinds of imprecisions are involved with the decision values in the process of evaluation. Thus, it becomes difficult to put the decision values in the form of crisp numbers. In such situations, fuzzy sets [56] appeared as a successful tool for dealing with complex and imprecise circumstances. As an extension of fuzzy set [26, 29], intuitionistic fuzzy (IF) set (IFS) was introduced by Atanassov [3] with simultaneous consideration of membership degree as well as non-membership degree satisfying the condition that the sum of membership and non-membership degrees is less than or equals to 1.

More recently, Yager [53] extended the concept of IFS [6, 27, 28] to Pythagorean fuzzy (PF) set (PFS) by extending the domain of membership and non-membership degrees with the consideration that the square sum of membership and non-membership degrees might be less than or equals to 1. Considering this advantage, plenty of applications are made on PFSs. Some of them related to MCDM are described in the next paragraph.

In the context of MCDM the decision makers (DMs) put their decision values by evaluating the alternatives with respect to some criteria. In the process of evaluation, the DMs prefer to put their judgment values in the form of linguistic terms. Those linguistic terms are quantified using PF numbers (PFNs) for its capability to capture uncertainties in MCDM than any other variants of fuzzy sets. Considering this aspect, Cui et al. [11] introduced PF-based VIKOR method for the selection of construction

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_9

site for establishing electric vehicle charging station. He also proposed a generalized PF-ordered weighted standardized distance operator for ranking the alternatives. Introducing Hamy mean, Li et al. [33] generated some new PF aggregation operators to apply them in supplier selection problems with PFNs. Büyüközkan and Göçer [10] developed a novel approach for the purpose of selecting digital supply chain partners, combining analytic hierarchy process with complex proportional assessment under PF environment. Kumar et al. [30] introduced a new technique for solving transportation problems using PFNs. Based on PF VIKOR method, Gul et al. [21] proposed an approach for safety risk assessment in the mine industry. Using PF MULTI-MOORA, a process to evaluate passenger satisfaction of public transportation has recently been developed by Li et al. [32]. For information security risk analysis, Ak and Gul [1] presented an approach based on AHP-TOPSIS integration under PF environment. Mete et al. [38] applied a decision-support system for occupational risk assessment of a natural gas pipeline construction project based on PF-VIKOR method. A study relating to risk assessment was conducted by Muhammet [39] in the field of occupational health and safety using AHP and VIKOR methods under PF environment. Aycan et al. [4] presented an approach to investigate the problems of selecting the location of WEEE recycling plant based on PF AHP. Apart from those, there are several MCDM methods, viz., TOPSIS [8, 58], VIKOR [44], TODIM [45], ELECTRE [2], etc., for solving problems under PF environments.

Aggregation operators [17, 19, 31] play an important role in solving MCDM problems by depicting the ranking results in a better way by producing aggregated decision values of the alternatives; whereas, the traditional methods [7] can only reveal the ranking results. Under this perspective, Yager [54] proposed a range of aggregation operators for PFSs to solve MCDM problems. Peng and Yang [40] and Gou et al. [20] put forward the PF information aggregation process by introducing subtraction and division operations on PFNs. Based on Einstein t-conorm and t-norm, Garg [13, 14] proposed some generalized PF aggregation operators. Choquet integral was introduced by Peng and Yang [42] into PF data. Based on confidence levels, Garg [15] proposed some new PF averaging and geometric operators. Xu et al. [52] developed an induced generalized ordered weighted averaging (WA) operator to aggregate PF information. Rahman et al. [43] proposed PF Einstein weighted geometric aggregation operator and applied it to solve multi-attribute decision making (MADM) problems. Utilizing Hamacher operations, Wu and Wei [50] proposed an MADM method under PF environment. Zeng et al. [57] proposed an approach to solve MADM method by developing PF induced ordered weighted averaging-weighted average operator. A new logarithmic operational law for PFNs was introduced by Garg [16]. Further, Wei and Lu [48] developed PF power aggregation operators for solving MADM problems. Recently, some PF weighted, ordered weighted, and hybrid neutral averaging aggregation operators are introduced by Garg [18].

It is worthy to mention here that most of the aggregation operators cited in the above discussions consider independent arguments. But, in solving MCDM problems, interrelationship among input arguments is frequently observed. To tackle such situations, Bonferroni mean (BM) [9] is successfully applied to capture the interrelationship among aggregated arguments.

Using BM, Wei et al. [49] introduced an MCDM method. He et al. [23] proposed IF interaction BM operators and applied them in solving MADM problems. Combining geometric mean with BM, Xia et al. [51] proposed geometric BM (GBM) operator and applied it for solving MCDM problems under IF environments. Extending the concept of BM operator, Beliakova et al. [5] presented generalized BM operators in multicriteria aggregation processes. Later on, Liang et al. [34] developed a projection model based on GBM in the context of multicriteria group decision making under PF environment. Furthermore, Liang et al. [35] introduced BM operator to combine PFSs and proposed PF BM (PFBM) operator and its weighted variant.

Moreover, Dombi t-conorm and t-norm, defined by Dombi [12], possess the properties of Archimedean t-conorms and t-norms. Dombi operations are found to be superior than the other existing operations, viz., algebraic, Einstein, Hamacher, Frank, etc., for its ability to change preferences of the DMs by varying parameters associated with t-norm, t-conorm. So, it helps to make the information aggregation process more agile. From this viewpoint, Jana et al. [24] proposed bipolar fuzzy Dombi aggregation operators in MCDM. He [22] introduced Dombi operations to hesitant fuzzy sets and applied it to the process of aggregation in typhoon disaster assessment. Liu et al. [37] developed a multi-attribute group decision making method using Dombi BM operator under IF environment. Further, Peng and Smarandache [41] integrated Dombi operations with BM operators in single-valued neutrosophic sets. Recently, based on Dombi operations, [25] investigated some PF Dombi aggregation operators.

It is worthy to note here that, the characteristic of BM operator to capture the interrelationship among input arguments can overcome the limitation of many existing aggregation operators, which fail to consider the correlation of the arguments. Additionally, Dombi operations can successfully make aggregation processes more flexible. Thus, combination of BM aggregation operator with Dombi operations is a demanding area of research. As per authors' knowledge, up to now, this combination of BM operator with Dombi operations has not been developed for aggregating PF arguments. From this motivation, in this chapter two new PF aggregation operators are proposed utilizing the concept of GBM operator with Dombi operations. The objectives of this chapter are presented below:

- Two new PF aggregation operators, viz., PF Dombi GBM (PFDGBM) and PF weighted Dombi GBM (PFWDGBM) operators have been proposed.
- Some special cases and the properties of the proposed operators are discussed.
- An MCDM approach using the proposed operators has been developed.
- A numerical example is solved to show the validity and utility of the proposed approach.
- A comparative analysis is performed to describe the advantages of the newly described approach.

To fulfill the objectives, the rest of the chapter is organized in such a manner that in Sect. 2, some basic concepts and definitions are briefly reviewed. PF GBM operators based on Dombi t-conorm and t-norm, including PFDGBM operator and PFWDGBM operator are proposed in Sect. 3. In Sect. 4, a methodology for solving

MCDM problems using the proposed PFDGBM and PFWDGBM operators is developed under PF environment. Section 5 presents an illustrative example to show the effectiveness of the proposed methodology. In Sect. 6, some concluding remarks are included.

2 Preliminaries

Some concepts related to PFS [53, 55], BM operator [36], Dombi t-conorm, and t-norm [12] which are required to develop the proposed PFDGBM and PFWDGBM operators are briefly reviewed in this section.

2.1 Pythagorean Fuzzy Set

Definition 1. [53, 55] Let X be a fixed set. A PFS $\widetilde{P}$ on X is defined as: $\widetilde{P} = \left\{x, \mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x) | x \in X\right\}$, where $\mu_{\widetilde{P}}(x)$ denotes the degree of membership and $\nu_{\widetilde{P}}(x)$ denotes the degree of non-membership of the element $x \in X$ to the set $\widetilde{P}$, satisfying the condition that $0 \leq \left(\mu_{\widetilde{P}}(x)\right)^2 + \left(\nu_{\widetilde{P}}(x)\right)^2 \leq 1$.

The degree of indeterminacy is given by $\pi_{\widetilde{P}}(x) = \sqrt{1 - \left(\mu_{\widetilde{P}}(x)\right)^2 - \left(\nu_{\widetilde{P}}(x)\right)^2}$.

For given $x \in X$, $\left(\mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x)\right)$ is said to be a PFN [58], and for convenience, a PFN is denoted by $\tilde{p} = \left(\mu_{\tilde{p}}, \nu_{\tilde{p}}\right)$.

Definition 2. [47] For any PFN, $\tilde{p} = \left(\mu_{\tilde{p}}, \nu_{\tilde{p}}\right)$, the score function of $\tilde{p}$ is defined by.

$$S(\tilde{p}) = \frac{1}{2}\left(1 + \left(\mu_{\tilde{p}}\right)^2 - \left(\nu_{\tilde{p}}\right)^2\right), \tag{1}$$

where, $S(\tilde{p}) \in [0, 1]$.

For a PFN, $\tilde{p} = \left(\mu_{\tilde{p}}, \nu_{\tilde{p}}\right)$, the accuracy function of $\widetilde{p}$ is defined as

$$A(\tilde{p}) = \left(\mu_{\tilde{p}}\right)^2 + \left(\nu_{\tilde{p}}\right)^2, \tag{2}$$

where, $A(\tilde{p}) \in [0, 1]$.

Definition 3. [58] Let $\tilde{p}_1$ and $\tilde{p}_2$ be any two PFNs, then the ordering of those PFNs is done by the following principles:

- If $S(\tilde{p}_1) > S(\tilde{p}_2)$, then $\tilde{p}_1 \succ \tilde{p}_2$;
- If $S(\tilde{p}_1) = S(\tilde{p}_2)$, then

- if $A(\tilde{p}_1) > A(\tilde{p}_2)$, then $\tilde{p}_1 \succ \tilde{p}_2$;
- if $A(\tilde{p}_1) = A(\tilde{p}_2)$, then $\tilde{p}_1 \approx \tilde{p}_2$.

Definition 4. [53, 55] Let $\tilde{p} = (\mu, \nu)$, $\tilde{p}_1 = (\mu_1, \nu_1)$ and $\tilde{p}_2 = (\mu_2, \nu_2)$ be three PFNs, and $\lambda > 0$, then some basic operations are defined as follows:

(1) $\tilde{p}_1 \oplus \tilde{p}_2 = \left(\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \nu_1\nu_2\right)$;

(2) $\tilde{p}_1 \otimes \tilde{p}_2 = \left(\mu_1\mu_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2}\right)$;

(3) $\lambda\tilde{p} = \left(\sqrt{1 - \left(1 - \mu^2\right)^{\lambda}}, \nu^{\lambda}\right), \lambda > 0$;

(4) $\tilde{p}^{\lambda} = \left(\mu^{\lambda}, \sqrt{1 - \left(1 - \nu^2\right)^{\lambda}}\right), \lambda > 0$.

2.2 Geometric Bonferroni Mean Operator

The BM was originally introduced by Bonferroni [9], which is defined as follows:

Definition 5. [9] Let $p, q \geq 0$ and a_i, $(i = 1, 2, \ldots, n)$ be a collection of non-negative numbers. Then the operation

$$\mathrm{BM}^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i, j = 1 \\ i \neq j}}^{n} a_i^p a_j^q\right)^{\frac{1}{p+q}} \tag{3}$$

is called BM.

BM possesses the following properties:

(i) $\mathrm{BM}^{p,q}(0, 0, \ldots, 0) = 0$;
(ii) $\mathrm{BM}^{p,q}(a, a, \ldots, a) = a$;
(iii) $\mathrm{BM}^{p,q}(a_1, a_2, \ldots, a_n) \geq \mathrm{BM}^{p,q}(h_1, h_2, \ldots, h_n)$ if $a_i \geq h_i$ for all i;
(iv) $\min\{a_i\} \leq \mathrm{BM}^{p,q}(a_1, a_2, \ldots, a_n) \leq \max\{a_i\}$ for all i.

Based on geometric mean and BM, Xia et al. [51] introduced GBM as follows:

Definition 6. [51] Let $p, q > 0$ and $\{a_i\}$, $(i = 1, 2, \ldots, n)$ be a collection of non-negative numbers, then GBM of the collection $\{a_i\}$, $(i = 1, 2, \ldots, n)$ is defined as:

$$\mathrm{GBM}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q}\left(\prod_{\substack{i, j = 1 \\ i \neq j}}^{n} \left((pa_i) + \left(qa_j\right)\right)\right)^{\frac{1}{n(n-1)}}. \tag{4}$$

2.3 Dombi t-Conorm and t-Norm

Dombi operations, viz., Dombi sum and Dombi product are generated through Dombi t-conorm and t-norm [12], which are, respectively, shown as follows:

$$S_D(\mu, \nu) = 1 - \frac{1}{1 + \left(\left(\frac{\mu}{1-\mu}\right)^r + \left(\frac{\nu}{1-\nu}\right)^r\right)^{1/r}},$$

$$T_D(\mu, \nu) = \frac{1}{1 + \left(\left(\frac{1-\mu}{\mu}\right)^r + \left(\frac{1-\nu}{\nu}\right)^r\right)^{1/r}}, \tag{5}$$

where $r > 0$ and $\mu, \nu \in [0, 1]$

2.4 Operations of PFNs Based on Dombi t-Conorm and t-Norm

Based on Dombi t-conorm and t-norm, the operational rules of PFNs [25] are defined as follows:

Suppose $\kappa_i = (\gamma_i, \delta_i)$, $(i = 1, 2)$ and $\kappa = (\gamma, \delta)$, are any three PFNs and consider $r > 0$. Then the operational laws of PFNs based on Dombi t-conorm and t-norm can be defined as.

$$(1)\ \kappa_1 \oplus_D \kappa_2 = \left(\sqrt{1 - \frac{1}{1 + \left(\left(\frac{(\gamma_1)^2}{1-(\gamma_1)^2}\right)^r + \left(\frac{(\gamma_2)^2}{1-(\gamma_2)^2}\right)^r\right)^{\frac{1}{r}}}}, \frac{1}{\sqrt{1 + \left(\left(\frac{1-(\delta_1)^2}{(\delta_1)^2}\right)^r + \left(\frac{1-(\delta_2)^2}{(\delta_2)^2}\right)^r\right)^{\frac{1}{r}}}}\right); \tag{6}$$

$$(2)\ \kappa_1 \otimes_D \kappa_2 = \left(\frac{1}{\sqrt{1 + \left(\left(\frac{1-(\gamma_1)^2}{(\gamma_1)^2}\right)^r + \left(\frac{1-(\gamma_2)^2}{(\gamma_2)^2}\right)^r\right)^{\frac{1}{r}}}}, \sqrt{1 - \frac{1}{1 + \left(\left(\frac{(\delta_1)^2}{1-(\delta_1)^2}\right)^r + \left(\frac{(\delta_2)^2}{1-(\delta_2)^2}\right)^r\right)^{\frac{1}{r}}}}\right); \tag{7}$$

$$(3)\ \lambda\kappa = \left(\sqrt{1 - \frac{1}{1 + \left(\lambda\left(\frac{(\gamma)^2}{1-(\gamma)^2}\right)^r\right)^{\frac{1}{r}}}}, \frac{1}{\sqrt{1 + \left(\lambda\left(\frac{1-(\delta)^2}{(\delta)^2}\right)^r\right)^{\frac{1}{r}}}}\right), \lambda > 0; \tag{8}$$

$$(4)\ \kappa^\lambda = \left(\frac{1}{\sqrt{1 + \left(\lambda\left(\frac{1-(\gamma)^2}{(\gamma)^2}\right)^r\right)^{\frac{1}{r}}}}, \sqrt{1 - \frac{1}{1 + \left(\lambda\left(\frac{(\delta)^2}{1-(\delta)^2}\right)^r\right)^{\frac{1}{r}}}}\right), \lambda > 0; \tag{9}$$

3 Pythagorean Fuzzy Geometric Bonferroni Mean Operators Based on Dombi Operations

In this section, GBM is combined with Dombi operations under PF environment to generate PFDGBM and PFWDGBM operators. Several properties and special cases of those newly defined operators are also discussed.

3.1 Pythagorean Fuzzy Dombi Geometric Bonferroni Mean Operator

Definition 7. Let $\kappa_i = (\gamma_i, \delta_i)$, $(i = 1, 2, \ldots, n)$ be a collection of PFNs. Also let $p, q > 0$ be any two numbers. Then PFDGBM operator is given by.

$$\text{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n) = \frac{1}{p+q}\left(\otimes_D{}_{\substack{i,j=1 \\ i \neq j}}^{n}\left((p\kappa_i) \otimes_D (q\kappa_j)\right)^{\frac{1}{n(n-1)}}\right). \tag{10}$$

Theorem 1. *Let $\kappa_i = (\gamma_i, \delta_i)$, $(i = 1, 2, \ldots, n)$ be a collection of PFNs and $p, q > 0$, then the aggregated value using PFDGBM operator is also a* PFN *and can be given as.*

$$\text{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n) = \frac{1}{p+q}\left(\otimes_D{}_{\substack{i,j=1 \\ i \neq j}}^{n}\left((p\kappa_i) \otimes_D (q\kappa_j)\right)^{\frac{1}{n(n-1)}}\right) = (\gamma, \delta), \text{ where}$$

$$\gamma = \sqrt{1 - \left(1\Big/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1\Big/\sum_{\substack{i,j=1 \\ i \neq j}}^{n}\left(1\Big/\left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r + q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{1/r}\right)\right)},$$

$$\delta = \sqrt{1\Big/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1\Big/\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}\left(1\Big/\left(p\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r + q\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}. \tag{11}$$

Proof: From Eq. (8),

$$p\kappa_i = \left(\sqrt{1 - \left(1\Big/\left(1 + \left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r\right)^{1/r}\right)\right)}, 1\Big/\sqrt{1 + \left(p\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r\right)^{1/r}}\right) \text{ and}$$

$$q\kappa_j = \left(\sqrt{1 - \left(1\Big/\left(1 + \left(q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)^{1/r}\right)\right)}, 1\Big/\sqrt{1 + \left(q\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)^{1/r}}\right).$$

Let $\frac{(\gamma_i)^2}{1-(\gamma_i)^2} = u_i$, $\frac{(\gamma_j)^2}{1-(\gamma_j)^2} = u_j$,, $\frac{1-(\delta_i)^2}{(\delta_i)^2} = v_i$, and $\frac{1-(\delta_j)^2}{(\delta_j)^2} = v_j$, then

$$p\kappa_i = \left(\sqrt{1-\left(1/\left(1+p^{1/r}u_i\right)\right)}, 1/\sqrt{1+p^{1/r}v_i}\right),$$

$$q\kappa_j = \left(\sqrt{1-\left(1/\left(1+q^{1/r}u_j\right)\right)}, 1/\sqrt{1+q^{1/r}v_j}\right).$$

And then

$$(p\kappa_i) \oplus_D \left(q\kappa_j\right) = \left(\sqrt{1-1/\left(1+\left(pu_i^r+qu_j^r\right)^{\frac{1}{r}}\right)}, 1/\sqrt{1+\left(pv_i^r+qv_j^r\right)^{\frac{1}{r}}}\right).$$

So,

$$\left((p\kappa_i) \oplus_D \left(q\kappa_j\right)\right)^{\frac{1}{n(n-1)}} = \left(\sqrt{1/\left(1+\left(\frac{1}{n(n-1)}\left(1/\left(pu_i^r+qu_j^r\right)\right)\right)^{\frac{1}{r}}\right)},\right.$$
$$\left.\sqrt{1-\left(1/\left(1+\left(\frac{1}{n(n-1)}\left(1/\left(pv_i^r+qv_j^r\right)\right)\right)^{\frac{1}{r}}\right)\right)}\right). \tag{12}$$

Further,

$$\otimes_D{}_{\substack{i,j=1\\ i\neq j}}^{n}\left((p\kappa_i) \oplus_D \left(q\kappa_j\right)\right)^{\frac{1}{n(n-1)}} = \left(\sqrt{1/\left(1+\left(\frac{1}{n(n-1)}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(pu_i^r+qu_j^r\right)\right)\right)\right)^{1/r}\right)},\right.$$
$$\left.\sqrt{1-\left(1/\left(1+\left(\frac{1}{n(n-1)}\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(pv_i^r+qv_j^r\right)\right)\right)\right)^{1/r}\right)\right)}\right).$$

Then

$$\frac{1}{p+q}\left(\otimes_D{}_{\substack{i,j=1\\ i\neq j}}^{n}\left((p\kappa_i) \oplus_D \left(q\kappa_j\right)\right)^{\frac{1}{n(n-1)}}\right)$$
$$= \left(\sqrt{1-\left(1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(pu_i^r+qu_j^r\right)\right)\right)\right)^{1/r}\right)\right)}\right),$$

$$\sqrt{1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\left(\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(1/\left(pv_i^r+qv_j^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}. \tag{13}$$

Now, putting the values of u_i, u_j, v_i, and v_j in Eq. (13), the required expression follows.

Note 1. It is interesting to note here that if one expert provides evaluation value as $(\gamma_t, \delta_t) = (1, 0)$, for some t, then in the process of aggregating the input data by PFDGBM operator, the corresponding effect of that evaluation value on the summation related to the calculations of γ and δ would be zero. Thus there would be no effect for that attribute value. But, if all the input decision values are considered as (1,0), the aggregated value, (γ, δ) would also be (1,0). Further, for considering (0, 1), as an input evaluation value given by one expert, the aggregated preference value would be calculated as usual maintaining Eq. (11).

Example: Suppose $k_1 = (0.7, 0.5)$, $k_2 = (0.6, 0.5)$, $k_3 = (0.5, 0.8)$ are three PFNs. Then if PFDGBM operator is used to aggregate those three PFNs, considering $p = q = 1$ and $r = 2$, the aggregated PFN, (γ, δ), is obtained as follows:

Here, $\sum_{\substack{i,j=1\\i\neq j}}^{n}\frac{1}{\left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r+q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)}=\sum_{\substack{i,j=1\\i\neq j}}^{3}\frac{1}{\left(\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^2+\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^2\right)}$ $= 8.2254$.

Thus

$$\gamma=\sqrt{1-\left(1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\sum_{\substack{i,j=1\\i\neq j}}^{n}\left(1/\left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r+q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{1/r}\right)\right)}$$

$$=\sqrt{1-\left(1/\left(1+\left(\frac{3(3-1)}{1+1}\left(1/\sum_{\substack{i,j=1\\i\neq j}}^{3}\left(1/\left(\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^2+\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^2\right)\right)\right)\right)^{\frac{1}{2}}\right)\right)}=0.3765.$$

.

Similarly,

$$\delta=\sqrt{1/\left(1+\left(\frac{3(3-1)}{1+1}\left(1/\left(\sum_{\substack{i,j=1\\i\neq j}}^{3}\left(1/\left(\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r+\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}=0.5459.$$

Thus, $\text{PFDGBM}^{1,1}(k_1, k_2, k_3) = (0.3765, 0.5459)$.

3.2 Properties of Pythagorean Fuzzy Dombi Geometric Bonferroni Mean Operator

In this section, some desirable properties of the proposed PFDGBM operator are presented.

Property 1. (Idempotency). *Let* $\kappa_i = (\gamma_i, \delta_i)$, $(i = 1, 2, \ldots, n)$ *be a collection of PFNs. If all* κ_i,$(i = 1, 2, \ldots, n)$ *be equal to* $\kappa = (\gamma, \delta)$, *i.e.*, $\kappa_i = \kappa$, for all $i = 1, 2, \ldots, n$, *then*

$$\mathrm{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n) = \kappa. \tag{14}$$

Proof. From the given definition,

$$\mathrm{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n)$$

$$= \left(\sqrt{1 - \left(1/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1/\sum_{\substack{i,j=1 \\ i \neq j}}^{n}\left(1/\left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r + q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)}\right),$$

$$\sqrt{1/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1/\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}\left(1/\left(p\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r + q\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}\Bigg).$$

Since $\kappa_i = \kappa$, $(i = 1, 2, \ldots, n)$,

$$\mathrm{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n)$$

$$= \left(\sqrt{1 - \left(1/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1/\sum_{\substack{i,j=1 \\ i \neq j}}^{n}\left(1/\left(p\left(\frac{\gamma^2}{1-\gamma^2}\right)^r + q\left(\frac{\gamma^2}{1-\gamma^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)}\right),$$

$$\sqrt{1/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1/\left(\sum_{\substack{i,j=1 \\ i \neq j}}^{n}\left(1/\left(p\left(\frac{1-\delta^2}{\delta^2}\right)^r + q\left(\frac{1-\delta^2}{\delta^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}\Bigg)$$

$$= \left(\sqrt{1 - \left(1/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1/\left(\frac{n(n-1)}{(p+q)\left(\frac{\gamma^2}{1-\gamma^2}\right)^r}\right)\right)\right)^{\frac{1}{r}}\right)\right)}\right),$$

$$\sqrt{1/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1/\left(\frac{n(n-1)}{(p+q)\left(\frac{1-\delta^2}{\delta^2}\right)^r}\right)\right)\right)^{\frac{1}{r}}\right)}\Bigg)$$

$$= \left(\sqrt{1 - \left(1/\left(1 + \frac{\gamma^2}{1-\gamma^2}\right)\right)}, \sqrt{1/\left(1 + \frac{1-\delta^2}{\delta^2}\right)}\right)$$

$$= (\gamma, \delta) = \kappa.$$

Property 2. (Monotonicity). *Let $\kappa_i = (\gamma_i, \delta_i)$ and $\kappa_i' = \left(\gamma_i', \delta_i'\right)$, $(i = 1, 2, \ldots, n)$ be two collections of PFNs. If $k_i \le k_i'$ for all $i = 1, 2, \ldots, n$ then*

$$\mathrm{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n) \le \mathrm{PFDGBM}^{p,q}\left(\kappa_1', \kappa_2', \ldots, \kappa_n'\right). \tag{15}$$

Proof: From the given definition,

$$\mathrm{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n)$$

$$= \left(\sqrt{1 - \left(1/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1/\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(1/\left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r + q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)},\right.$$

$$\left.\sqrt{1/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1/\left(\sum_{\substack{i,j=1\\ i\ne j}}^{n}\left(1/\left(p\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r + q\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}\right).$$

Since $k_i \le k_i'$, then $\gamma_i \le \gamma_i'$, for all $i = 1, 2, \ldots, n$, $\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r \le \left(\frac{(\gamma_i')^2}{1-(\gamma_i')^2}\right)^r$, and so

$$p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r \le p\left(\frac{\left(\gamma_i'\right)^2}{1-\left(\gamma_i'\right)^2}\right)^r. \tag{16}$$

Similarly, $q\left(\frac{\left(\gamma_j\right)^2}{1-\left(\gamma_j\right)^2}\right)^r \le q\left(\frac{\left(\gamma_j'\right)^2}{1-\left(\gamma_j'\right)^2}\right)^r$, for all $j = 1, 2, \ldots, n$. (17)

From Eqs. (16) and (17) it is clear that

$$p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r + q\left(\frac{\left(\gamma_j\right)^2}{1-\left(\gamma_j\right)^2}\right)^r \le p\left(\frac{\left(\gamma_i'\right)^2}{1-\left(\gamma_i'\right)^2}\right)^r + q\left(\frac{\left(\gamma_j'\right)^2}{1-\left(\gamma_j'\right)^2}\right)^r$$

i.e.,

$$\sum_{\substack{i,\,j=1\\ i\neq j}}^{n}\left(1/\left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r+q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)$$

$$\geq \sum_{\substack{i,\,j=1\\ i\neq j}}^{n}\left(1/\left(p\left(\frac{\left(\gamma_i'\right)^2}{1-\left(\gamma_i'\right)^2}\right)^r+q\left(\frac{\left(\gamma_j'\right)^2}{1-\left(\gamma_j'\right)^2}\right)^r\right)\right)$$

i.e.,

$$1+\left(\frac{n(n-1)}{p+q}\left(1/\sum\nolimits_{\substack{i,\,j=1\\ i\neq j}}^{n}\left(1/\left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r+q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}$$

$$\leq 1+\left(\frac{n(n-1)}{p+q}\left(1/\sum\nolimits_{\substack{i,\,j=1\\ i\neq j}}^{n}\left(1/\left(p\left(\frac{(\gamma_i')^2}{1-(\gamma_i')^2}\right)^r+q\left(\frac{\left(\gamma_j'\right)^2}{1-\left(\gamma_j'\right)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}.$$

And so,

$$1-\left(1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\sum\nolimits_{\substack{i,\,j=1\\ i\neq j}}^{n}\left(1/\left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r+q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)\leq 1-$$
$$\left(1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\sum\nolimits_{\substack{i,\,j=1\\ i\neq j}}^{n}\left(1/\left(p\left(\frac{(\gamma_i')^2}{1-(\gamma_i')^2}\right)^r+q\left(\frac{\left(\gamma_j'\right)^2}{1-\left(\gamma_j'\right)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right). \tag{18}$$

In a similar manner, the following relationship is obtained, since $k_i \leq k_i'$; and so, $\delta_i \geq \delta_i'$ for all $i = 1, 2, \ldots, n$,

$$1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\left(\sum\nolimits_{\substack{i,\,j=1\\ i\neq j}}^{n}\left(1/\left(p\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r+q\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)\geq$$
$$1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\left(\sum\nolimits_{\substack{i,\,j=1\\ i\neq j}}^{n}\left(1/\left(p\left(\frac{1-\left(\delta_i'\right)^2}{\left(\delta_i'\right)^2}\right)^r+q\left(\frac{1-\left(\delta_j'\right)^2}{\left(\delta_j'\right)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right). \tag{19}$$

By Definition 3 and Eq. (18) and Eq. (19), the theorem is proved.

Property 3. (Boundary). *Let $\kappa_i = (\gamma_i, \delta_i)$, $(i = 1, 2, \ldots, n)$ be a collection of PFNs. Also, let*

$\gamma_+ = max_{i=1}^{n}\{\gamma_i\}$, $\gamma_- = min_{i=1}^{n}\{\gamma_i\}$, $\delta_+ = max_{i=1}^{n}\{\delta_i\}$, $\delta_- = min_{i=1}^{n}\{\delta_i\}$. Now, if $\kappa_- = (\gamma_-, \delta_+)$ and

$$\kappa_+ = (\gamma_+, \delta_-), \text{ then } \kappa_- \leq \mathrm{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n) \leq \kappa_+. \tag{20}$$

Proof. Since $\gamma_- \leq \gamma_i \leq \gamma_+$ and $\delta_- \leq \delta_i \leq \delta_+$, for all $i = 1, 2, \ldots, n$, then. $\kappa_- \leq \kappa_i$ for $i = 1, 2, \ldots, n$; and, therefore, from monotonicity,

$$\mathrm{PFDGBM}^{p,q}(\kappa_-, \kappa_-, \ldots, \kappa_-) \leq \mathrm{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n).$$

$$\text{Also, by idempotency, } \kappa_- \leq \mathrm{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n). \tag{21}$$

$$\text{Similarly, } \mathrm{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n) \leq \kappa_+. \tag{22}$$

Combining Eqs. (21) and (22), $\kappa_- \leq \mathrm{PFDGBM}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n) \leq \kappa_+$.

3.3 Some Special Cases of Pythagorean Fuzzy Dombi Geometric Bonferroni Mean Operator

In this section, some special cases of the proposed PFDGBM operator are studied with respect to the parameters p and q as follows:

(i) For $q \to 0, r > 0$,

$$\begin{aligned}
&\mathrm{PFDGBM}^{p,0}(\kappa_1, \kappa_2, \ldots, \kappa_n) = \\
&\lim_{q\to 0}\left(\sqrt{1 - \left(1/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1/\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r + q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)}\right),\\
&\sqrt{1/\left(1 + \left(\frac{n(n-1)}{p+q}\left(1/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(p\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r + q\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}\Bigg)\\
&= \left(\sqrt{1 - \left(1/\left(1 + \left(\frac{n(n-1)}{p}\left(1/\sum_{i=1}^{n}\left(1/\left(p\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)}\right),\\
&\sqrt{1/\left(1 + \left(\frac{n(n-1)}{p}\left(1/\left(\sum_{i=1}^{n}\left(1/\left(p\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}\Bigg).
\end{aligned} \tag{23}$$

(ii) For $p = 1, q \to 0, r > 0$,

$$\begin{aligned}
&\mathrm{PFDGBM}^{1,0}(\kappa_1, \kappa_2, \ldots, \kappa_n) = \left(\sqrt{1 - \left(1/\left(1 + \left(n(n-1)\left(1/\sum_{i=1}^{n}\left(1/\left(\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)}\right),\\
&\sqrt{1/\left(1 + \left(n(n-1)\left(1/\left(\sum_{i=1}^{n}\left(1/\left(\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}\Bigg).
\end{aligned} \tag{24}$$

(iii) For $p \to 0, r > 0$,

$$\mathrm{PFDGBM}^{0,q}(\kappa_1,\kappa_2,\ldots,\kappa_n)=\left(\sqrt{1-\left(1/\left(1+\left(\frac{n(n-1)}{q}\left(1/\sum_{j=1}^{n}\left(1/\left(q\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)},\right.$$
$$\left.\sqrt{1/\left(1+\left(\frac{n(n-1)}{q}\left(1/\left(\sum_{j=1}^{n}\left(1/\left(q\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}\right). \tag{25}$$

(iv) For $p = q = 1, r > 0$,

$$\mathrm{PFDGBM}^{1,1}(\kappa_1,\kappa_2,\ldots,\kappa_n)=$$
$$\left(\sqrt{1-\left(1/\left(1+\left(\frac{n(n-1)}{2}\left(1/\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r+\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)},\right.$$
$$\left.\sqrt{1/\left(1+\left(\frac{n(n-1)}{2}\left(1/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r+\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}\right). \tag{26}$$

3.4 Pythagorean Fuzzy Weighted Dombi Geometric Bonferroni Mean Operator

Definition 5. Let κ_i, $(i = 1, 2, \ldots, n)$ be a collection of PFNs and let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^t$ be the weight vector with $\omega_i \in [0, 1]$, $\sum_{i=1}^{n} \omega_i = 1$ and $p, q > 0$ be any numbers. If.

$$\mathrm{PFWDGBM}_{\omega}^{p,q}(\kappa_1,\kappa_2,\ldots,\kappa_n)=\frac{1}{p+q}\left(\otimes_D{}_{\substack{i,j=1\\ i\neq j}}^{n}\left(p(\kappa_i)^{\omega_i}\oplus_D q\left(\kappa_j\right)^{\omega_j}\right)^{\frac{1}{n(n-1)}}\right), \tag{27}$$

then $\mathrm{PFWDGBM}_{\omega}^{p,q}(\kappa_1, \kappa_2, \ldots, \kappa_n)$ is called PF weighted Dombi GBM (PFWDGBM) operator.

Theorem 2. Let $\kappa_i = (\gamma_i, \delta_i)$, $(i = 1, 2, \ldots, n)$ be a collection of PFNs, whose corresponding weights are given through the vector, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^t$, where, $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Let $p, q > 0$ be any two numbers. Then the aggregated value using PFWDGBM operator is also a PFN and is given by

$$\mathrm{PFWDGBM}_{\omega}^{p,q}(\kappa_1,\kappa_2,\ldots,\kappa_n)=$$

$$\left(\sqrt{1-\left(1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(\frac{p}{\omega_i}\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r+\frac{q}{\omega_j}\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)}\right),$$

$$\left.\sqrt{1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(\frac{p}{\omega_i}\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r+\frac{q}{\omega_j}\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}\right). \tag{28}$$

Proof: The proof is similar as Theorem 1.

Note 2. The proposed PFWDGBM operator also satisfies the following properties:

- Monotonicity,
- Boundedness.

4 An Approach to MCDM with Pythagorean Fuzzy Information

In the process of decision making, MCDM plays an important role for finding best alternative. For evaluating the alternatives DMs prefer to express decision values using PFNs. It is frequently observed that the input arguments provided by the DMs are dependent on each other. Sometimes, it is also found that the preferences of the DMs may be involved with the decision making processes. Considering all these aspects, in this section, a methodology for solving MCDM problems under PF environment is presented based on the developed PFWDGBM operator which possesses the capability to capture all of the above-mentioned characteristics. An MCDM problem under PF environment is described below:

Let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria with the weight vector representing weights corresponding to each criterion, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^t$, where $\omega_i \in [0, 1]$, $(i = 1, 2, \ldots, m)$ and $\sum_{i=1}^{n} \omega_i = 1$. After evaluating the alternatives with respect to criteria, DM provides the PF decision matrix (PFDM) as $\widetilde{D} = \left[\tilde{d}_{ij}\right]_{m\times n}$, whose elements are represented by PFNs with the form $\tilde{d}_{ij} = \left(\gamma_{ij}, \delta_{ij}\right)$. Each $\tilde{d}_{ij}$ indicates the decision value provided by DM corresponding to the alternative $x_i \in X$ evaluated on the basis of criterion $C_j \in C$. γ_{ij} represents the degree for which the alternative x_i satisfies the criterion C_j; and δ_{ij} represents the degree for which the alternative x_i does not satisfy the criterion C_j, according to the DM. Here, $\gamma_{ij}, \delta_{ij} \in [0, 1]$, with $\gamma_{ij}^2 + \delta_{ij}^2 \leq 1$, for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

The proposed methodology using the developed PFWDGBM operator is described through the following steps:

Step 1: Criteria are classified into two types, viz., benefit criteria and cost criteria. If the PFDM contains cost criteria along with benefit criteria, the cost criteria are converted into benefit criteria through normalization. Thus, the decision matrix $\widetilde{D} = \left[\tilde{d}_{ij}\right]_{m\times n}$, is converted into the normalized form, $R = \left(\tilde{r}_{ij}\right)_{m\times n}$ in the following way,

$$\tilde{r}_{ij} = \begin{cases} \tilde{d}_{ij} \text{ for benefit criteria } C_j \\ \tilde{d}_{ij}^{c} \text{ for cost criteria } C_j \end{cases}, \tag{29}$$

where $\tilde{d}_{ij}^{c}$ is the complement of $\tilde{d}_{ij}$; $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Step 2: Using PFWDGBM operator, the collective evaluation value corresponding to the alternative x_i with respect to the set of criteria C given in the normalized matrix R in the form of PFNs are aggregated considering criteria weight vector ω. The aggregation process is performed using some specific values of BM parameter p, q and Dombi parameter r as follows:

$$t_i = \mathrm{PFWDGBM}_{\omega}^{p,q}(r_{i1}, r_{i2}, \ldots, r_{in}) =$$

$$\left(\sqrt{1-\left(1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(\frac{p}{\omega_i}\left(\frac{(\gamma_i)^2}{1-(\gamma_i)^2}\right)^r+\frac{q}{\omega_j}\left(\frac{(\gamma_j)^2}{1-(\gamma_j)^2}\right)^r\right)\right)\right)\right)^{\frac{1}{r}}\right)\right)}\right),$$

$$\sqrt{1/\left(1+\left(\frac{n(n-1)}{p+q}\left(1/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(1/\left(\frac{p}{\omega_i}\left(\frac{1-(\delta_i)^2}{(\delta_i)^2}\right)^r+\frac{q}{\omega_j}\left(\frac{1-(\delta_j)^2}{(\delta_j)^2}\right)^r\right)\right)\right)\right)\right)^{\frac{1}{r}}\right)}.$$

Step 3: Calculate the score values S_{t_i} corresponding to the aggregated PFNs, t_i $(i = 1, 2, \ldots, m)$ based on Eq. (1). Compute the accuracy values using Eq. (2), if required.
Step 4: Rank all the alternatives x_i, $(i = 1, 2, \ldots, m)$ on the basis Definition 3 and find the best alternative.

5 An Illustrative Example

In this section, the methodology developed for solving MCDM problems is illustrated by solving the following case example.

5.1 *Description of the MCDM Problem*

The evaluation and selection processes of suppliers have become more and more complicated in recent days due to the imprecision associated with the evaluation of attributes as well as inherent fuzziness associated with the available data. Thus, decision making processes need to be advanced to tackle these situations. In supplier selection processes, DMs often deal with attributes having interrelationships among themselves. It has already been mentioned that the proposed PFWDGBM operator not only possesses the characteristics of capturing interrelationship among input arguments, but it can also make decision making process more general and feasible due to presence of the Dombi parameter r. Considering these aspects, a decision making problem adapted from an article presented by Wang et al. [46] has been considered and solved using the proposed methodology.

A manufacturer plans to choose appropriate supplier from four available suppliers, X_i, $i = 1, 2, 3, 4$. The selection process is performed on the basis of four criteria of the suppliers, viz., price of raw materials (U_1), quality of raw materials (U_2), reliability of each supplier (U_3) and quality of service (U_4). The weight vector corresponding to the weights of criteria is given by $\omega = (0.2, 0.3, 0.3, 0.2)^t$.

It is to be noted here that the criterion U_1 represents cost criterion; whereas, all other criteria represent benefit criteria.

After evaluating the attributes of each supplier, the following Pythagorean fuzzy decision matrix is presented in Table 1.

Regarding the formulation of the above table, it is important to mention here that DMs generally express their judgment values using linguistic terms. Those linguistic terms are converted into PFNs using some linguistic scale. Thus each PFN in the above table represents some linguistic hedges corresponding to the evaluation criteria. Although, same PFN designates same linguistic hedge, but the characteristic of evaluation may be different, i.e., one of them may represent benefit criterion, whereas, the other may signify cost criterion.

The above problem is executed step by step using the proposed method as follows:

Step 1: Since the criterion U_1, represents cost criterion, it is required to be converted to benefit criterion Thus, using Eq. (29) the following decision matrix is obtained after normalization as (Table 2).

Step 2: On utilizing PFWDGBM operator as presented in Eq. (28), all the attribute values for each supplier are aggregated. Varying the parameters p and q in between (0, 10], different aggregated attribute values for each supplier are obtained. In particular, if $p = q = 1$ and $r = 2$ are considered, the following aggregated attribute value of each supplier is achieved as:

$$t_1 = (0.4764, 0.2310), t_2 = (0.5870, 0.2321), t_3 = (0.6596, 0.1832),$$
$$t_4 = (0.7340, 0.2141).$$

Step 3: The score values of each t_i, $(i = 1, 2, 3, 4)$ using Eq. (1) are obtained as follows:

$$S_{t_1} = 0.1735, S_{t_2} = 0.2907, S_{t_3} = 0.4014, S_{t_4} = 0.4928.$$

Step 4: The alternatives are ranked using the achieved score values as.

$$X_4 \rangle X_3 \rangle X_2 \rangle X_1.$$

Therefore, the best supplier is identified as X_4.

5.2 Results and Discussions

Consequences of the parameters p, q, **and** r **on the decision making outcomes**.

Assigning different values of BM parameters, p and q in the proposed PFWDGBM operator, the effects on decision making process are observed. At first, varying the parameters p, q in (0, 10] and considering Dombi parameter, $r = 2$, the achieved results are presented via the following Figs. 1, 2, 3, and 4.

It is seen that using different values of p and q, the ranking results remain consistent and the best alternative is found as X_4. Thus the best alternative does not change for the presence of different aggregation parameters. It is worthy to mention here that for equal values of p and q, the highest score value corresponding to each alternative is achieved. Whereas, decreasing trends in score values are found with the increase of p or q having unequal values using the proposed PFWDGBM operator.

In the following, it has been shown that the preferences of the DMs highly depend on the Dombi parameter, r. So, keeping the BM parameter p and q fixed (taking $p = q = 1$, for convenience) and varying the Dombi parameter r for $0 < r \leq 15$, the changes in score values of the alternatives using PFWDGBM operator are visualized through Fig. 5.

Appropriate value of the parameter r can be chosen by the DMs according to their choice of preferences. From Fig. 5, it is evidenced that the ranking results remain the same for different values of the parameter r The figure depicts that the score values of the alternatives decrease when r increases. Thus for taking pessimistic decision, higher value of the parameter r is chosen; while, smaller value of the parameter r is chosen for taking optimistic decision by the DM.

5.3 Comparative Analysis

In this section, the proposed method is compared with existing methods [13, 46, 59]. Utilizing the decision information as presented in Table 1 [46], the aggregated results using PFWA operator [59], PFEWA operator [13], PFWBM operator [46], and the proposed PFWDGBM operator are presented in the following Table 3:

Table 1 Pythagorean fuzzy decision matrix given by the DM [46]

	U_1	U_2	U_3	U_4
X_1	(0.3, 0.4)	(0.5, 0.3)	(0.2, 0.3)	(0.3, 0.4)
X_2	(0.2, 0.5)	(0.6, 0.4)	(0.2, 0.4)	(0.4, 0.3)
X_3	(0.2, 0.6)	(0.5, 0.5)	(0.3, 0.2)	(0.6, 0.3)
X_4	(0.3, 0.6)	(0.7, 0.2)	(0.6, 0.3)	(0.5, 0.5)

Table 2 Transformed Pythagorean fuzzy decision matrix

	U_1	U_2	U_3	U_4
X_1	(0.4, 0.3)	(0.5, 0.3)	(0.2, 0.3)	(0.3, 0.4)
X_2	(0.5, 0.2)	(0.6, 0.4)	(0.2, 0.4)	(0.4, 0.3)
X_3	(0.6, 0.2)	(0.5, 0.5)	(0.3, 0.2)	(0.6, 0.3)
X_4	(0.6, 0.3)	(0.7, 0.2)	(0.6, 0.3)	(0.5, 0.5)

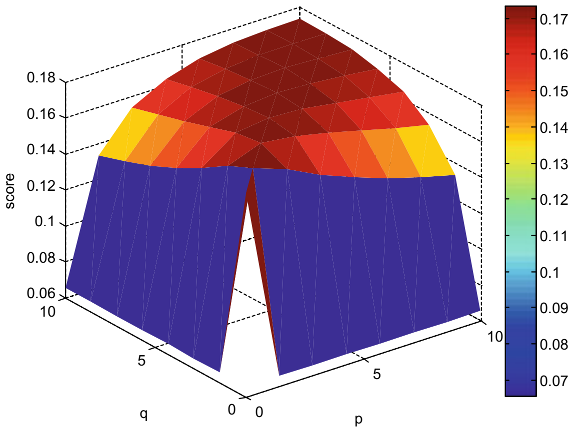


Fig. 1 Score of the alternative X_1 for p, q in (0, 10] using PFWDGBM operator

From Table 3, it signifies that the rankings of the alternatives are consistent and the best alternative X_4 remains the same for each case. This indicates the consistency of the proposed method.

It is to be noted here that the method developed by Zhang [13, 59] using their developed PFWA and PFEWA operators, respectively, could not capture interrelationship among the input arguments. Whereas, the proposed PFWDGBM operator has the capability to do so, due to the presence of BM operator in the developed aggregation function. Furthermore, having the flexible Dombi parameter, the proposed method provides more choices to the DM, for choosing different values of the parameter according to their needs. But, the other operators [13, 46, 59], under consideration fail to do so.

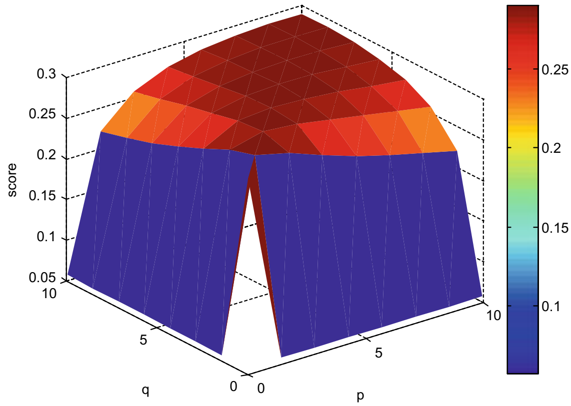


Fig. 2 Score of the alternative X_2 for p, q in (0, 10] using PFWDGBM operator

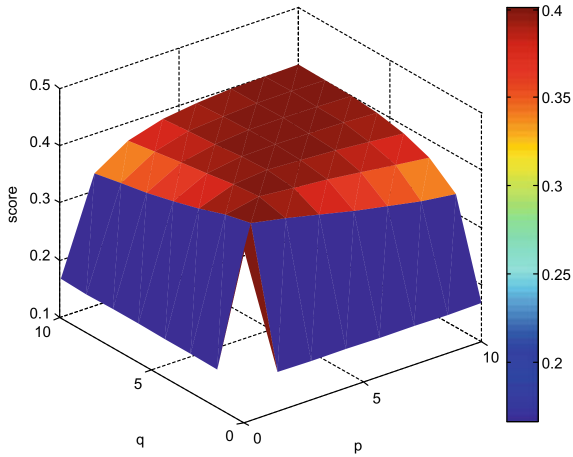


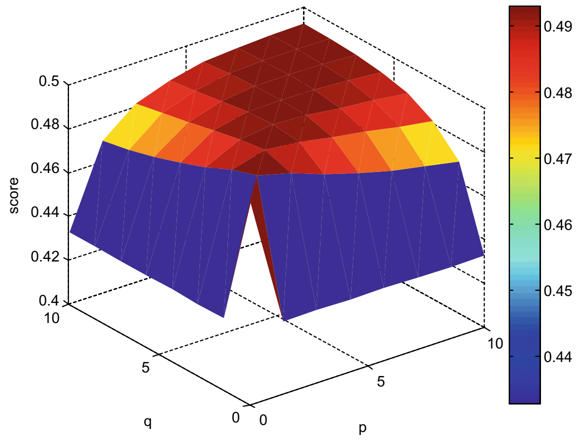
Fig. 3 Score of the alternative X_3 for p, q in (0, 10] using PFWDGBM operator

Further, it is interesting to note here that the method developed by Wang et al. [46] and the proposed method, both of them consider interrelationship of the attributes in the decision making process through BM. But the advantage of using the proposed operator is that it further includes the attitude of the DMs toward the decision options due to the appearance of Dombi parameter r.

It is the fact that larger differences in score values of consecutively ranked alternatives provide better ranking. From that viewpoint, in the following Fig. 6, the differences of score values between two consecutively ranked alternatives using proposed PFWDGBM operator and the other operators [13, 46, 59] under consideration is presented.

From Fig. 6 it is clear that using the proposed method, the differences of score values between two consecutively ranked alternatives are higher than other approaches [13, 46, 59]. This shows that the proposed method is more beneficial to rank the alternatives than the other techniques [13, 46, 59] in the process of MCDM.

Fig. 4 Score of the alternative X_4 for p, q in (0, 10] using PFWDGBM operator



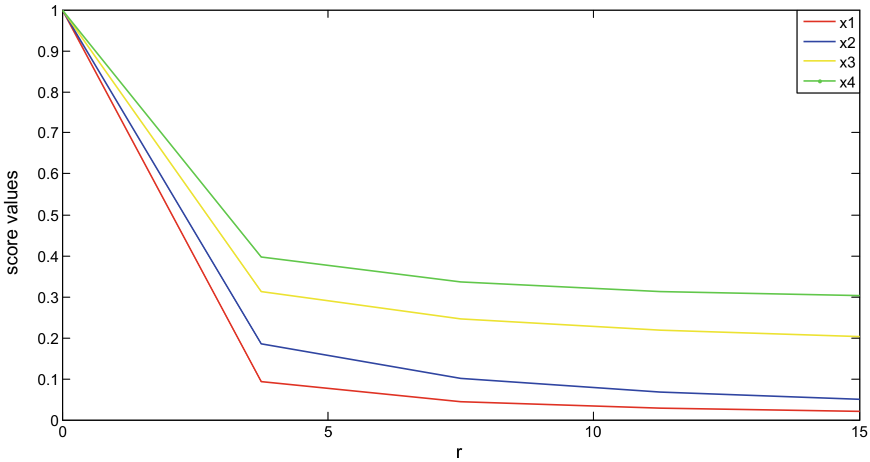


Fig. 5 Score values of the alternatives based on PFWDGBM operator for $r \in (0, 15]$.

6 Conclusions

In this chapter based on Dombi operations PFDGBM operator is defined for aggregating PF arguments and examined their special cases with properties. Further, introducing the weight vector of the attributes, PFWDGBM operator is developed. A method to deal with MCDM problems under PF environment is discussed. The novelty of the proposed PFWDGBM operator is that it can model the interrelationship among the input arguments in the decision making process for having BM parameters. Also, the newly defined operators possess the advantage to make the aggregation processes more flexible due to the presence of Dombi parameter r. Apart from those, the proposed method has the ability to depict DM's optismistic or pessimistic views

Table 3 Ranking results by different methods

Aggregation operators	Score values	Ranking
$PFWA$ operator [59]	$S_{t_1} = 0.5201$, $S_{t_2} = 0.5521$, $S_{t_3} = 0.5863$, $S_{t_4} = 0.6482$.	$X_4 \rangle X_3 \rangle X_2 \rangle X_1$
$PFEWA$ operator [13]	$S_{t_1} = 0.5185$, $S_{t_2} = 0.5483$, $S_{t_3} = 0.5831$, $S_{t_4} = 0.6464$.	$X_4 \rangle X_3 \rangle X_2 \rangle X_1$
$PFWBM$ operator [46]	$S_{t_1} = 0.4617$, $S_{t_2} = 0.4948$, $S_{t_3} = 0.5377$, $S_{t_4} = 0.5734$.	$X_4 \rangle X_3 \rangle X_2 \rangle X_1$
The proposed **PFWDGBM** operator	$S_{t_1} = 0.1735$, $S_{t_2} = 0.2907$, $S_{t_3} = 0.4014$, $S_{t_4} = 0.4928$.	$X_4 \rangle X_3 \rangle X_2 \rangle X_1$

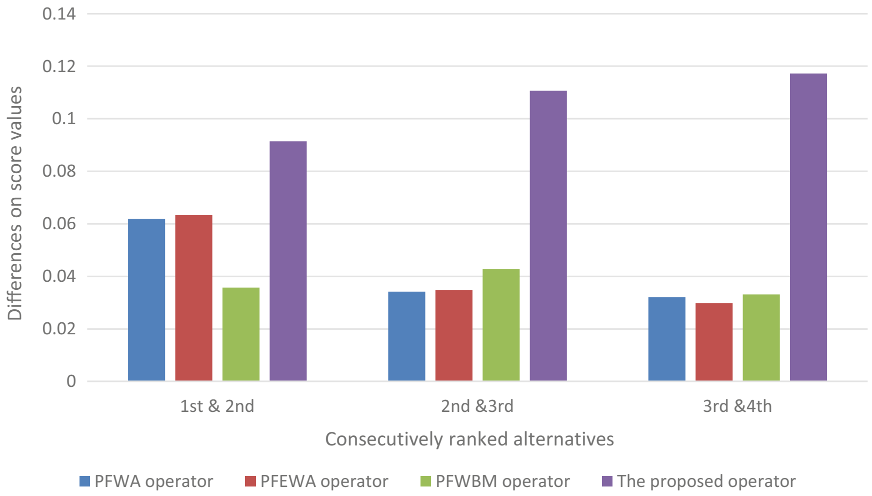


Fig. 6 Difference between two score values of the consecutively ranked alternatives

by choosing different values of the parameter. Furthermore, through the figures and achieved results, it has been established that the proposed method possesses the capability of ranking the alternatives in a better way than the existing ones. It is to be noted here that the proposed approach has some limitations in the sense that BM parameter involved with the proposed method considers interrelationship of each input attribute with all the remaining attributes. But, sometimes, it may happen, when some of the attributes may not be related to some other attributes. In such situations the developed operator does not work well.

In future research it would be significant to apply the Dombi Bonferroni mean operator into different imprecise domains, viz., hesitant fuzzy, interval-valued PF, and other variants. Moreover, it would be interesting to consider other types of interrelationship patterns among input attributes and thus new type of aggregation operators would be generated for solving real-life decision making problems, e.g., pattern recognition, cluster analysis, risk analysis, and others. However, it is hoped that the developed operators may be beneficial to future researchers to explore various emerging areas of research in MCDM arena.

Acknowledgements The authors remain grateful to the anonymous reviewers for their comments and suggestions to improve the quality of the manuscript.

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Schweizer–Sklar Muirhead Mean Aggregation Operators Based on Pythagorean Fuzzy Sets and Their Application in Multi-criteria Decision-Making

Tahir Mahmood and Zeeshan Ali

1 Introduction

MADM is a proficient technique that can provide the ranking results for the finite alternatives according to the attribute values of different alternatives and it is the important aspect of decision sciences. In recent years, the development of enterprises and social decision-making in all aspects are related to the issue of MADM. In real decision process, an important problem is how to express the attribute value more efficiently and accurately. In the real world, because of the complexity of decision-making problems and the fuzziness of decision-making environments, it is not enough to express attribute values of alternatives by exact values. For this the notion of fuzzy set (FS) was explored by Zadeh [1]. FS composes of the grade of supporting belonging to the unit interval, as a proficient technique to cope with unreliable and awkward information in realistic decision issues. At the point when a decision-maker gives the data as pair (0.5, 0.4) for truth and falsity grades, the notion of FS is not able to resolve it. For coping with such kind of issues, Atanassove [2] introduced the notion of intuitionistic FS (IFS) with the condition that the total of supporting and supporting against grades isn't surpassed from unit interval. IFS has gotten more consideration from researchers and a huge number of researchers have investigated their speculations [3–5].

There are some practical cases if the decision-maker gives 0.9 for positive grade and 0.3 for negative so their sum must be greater than 1, unlike the problem captured in IFS. Therefore, the work of Yager [6] called Pythagorean FS (PFS) can be successfully applied in different awkward fields because the sum of the square of positive

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_10

grade and square of negative grade is restricted to [0,1]. Due to its constraint, PFS becomes an essential tool to cope with awkward and difficult fuzzy information. Since it was established, it has received the attention of many researchers and it is utilized in the environment of aggregation operators, similarity measures, hybrid aggregation operators, and so on. The various existing works based on PFS are elaborated as follow as:

1. **Operators-based Approaches**: Based on the aggregation operators, many scholars have successfully utilized in the environment of PFS. For instance, Mohagheghi et al. [7] explored the new last aggregation evaluation based on PFSs, Ma and Xu [8] presented weighted averaging/geometric aggregation operators based on PFSs, Wei [9] discovered interaction aggregation operators based on PFSs, Garg [10] introduced information aggregation using Einstein operations based on PFSs, Garg [11] examined the confidential levels based on Pythagorean fuzzy aggregation operators, new logarithmic and aggregation operators based on PFSs were presented by Garg [12, 13] discovered the novel neutrality operation based on PFSs and their aggregation operators.
2. **Measures-based Approaches**: Similarity measure (SM) is a proficient technique to accurately examine the degree between any two objects. Many scholars have applied SM in different notions. For example, Wei and Wei [14] explored similarity measures based on PFSs, Garg [15] presented correlation coefficient based on PFSs, Biswas and Sarkar [16] discovered the similarity measures based on point operators by using PFSs, Li and Zeng [17] introduced distance measures based on PFSs and Zhang [18] examined similarity measures based on PFSs.
3. **Hybrid Operators-based Approaches**: To find the interrelationships between two objects, the hybrid aggregation operators play an essential role in the environment of realistic decision theory. Various scholars using the PFSs, explored different hybrid aggregation operators. For example, Liang et al. [19] discovered geometric Bonferroni mean operators based on PFSs, Prioritized aggregation operators [20], Bonferroni mean aggregation operators [21], Dombi aggregation operators [22], power aggregation operators [23], etc. [24–26].

After that, a more generalized operator was presented, that is, the Muirhead mean [26], which was added an alterable parametric vector P on the basis of considering interrelationships among multiple input parameters, and some existing operators are its special cases, for instance, arithmetic and geometric mean (GM) operators (not considering the correlations), Bonferroni mean (BM) operator, and Maclaurin symmetric mean (MSM). When dealing with MCDM problems, some aggregation operators cannot consider the relationship between any input parameters, while Muirhead mean (MM) operator can take into account the correlation between inputs by a variable parameter. Therefore, the MM operator is more superior when dealing with MCDM problems.

Multi-criteria decision-making refers to the use of existing decision information, in the case of multi-criteria that are in conflict with each other and cannot coexist, and in which the limited alternatives are ranked or selected in a certain way. Schweizer–Sklar operation uses a variable parameter to make their operations more effective and flexible. MCDM alludes to the utilization of existing choice data, for the situation of multi-models that are in strife with one another and can't exist together, and in which the constrained options are positioned or chosen with a specific goal in mind. SS activity utilizes a variable boundary to make their tasks increasingly successful and adaptable. What's more, PFS can deal with inadequate, uncertain, and conflicting data under fuzzy conditions. Subsequently, we directed further examination on SS tasks for PFS and applied SS activities to MCDM issues. Besides, in light of the fact that the MM operator thinks about interrelationships among different info boundaries by the alterable parametric vector, subsequently consolidating the MM operator with the SS activity gives some collection operators, and it was progressively significant to build up some new way to comprehend the MCDM issues in the Pythagorean fuzzy environment. As indicated by this, the reason furthermore, noteworthiness of this article are.

1. To build up various new MM operators by consolidating MM operators, SS activities, and PFS.
2. To talk about some significant properties and various instances of these operators set forward.
3. To manage a MCDM strategy for PFS data more adequately dependent on the operators set forward.
4. To show the feasibility and prevalence of the recently evolved strategy.

The aims of this manuscript are summarized as follow: In Sect. 2, we briefly state the fundamental conceptions of PFS, SS T-norm (SSTN), SS T-conorm (SSCTN), and MM operators. In Sect. 3, we explore the SS operators based on PFS and studied their score function, accuracy function, and their relationships. Further, based on these operators, the MM operators based on PFS are called Pythagorean fuzzy MM (PFMM) operator, Pythagorean fuzzy weighted MM (PFWMM) operator, and their special cases are presented. In Sect. 4, multi-attribute decision-making (MADM) problem is solved by using the explored operators based on PFS to observe the consistency and efficiency of the produced approach. Finally, the advantages, comparative analysis, and their geometrical representation are also discussed. The conclusion of this manuscript is discussed in Sect. 5.

2 Preliminaries

In this study we review some basic notions of PFSs and their fundamental laws. Throughout this manuscript, the universal set is expressed by .

2.1 Pythagorean Fuzzy Sets

Definition 1: [6] A PFS $\mathcal{R}_P$ is initiated by

$$\mathcal{R}_P = \left\{\left(\mu_{\mathcal{R}_P}(x), \eta_{\mathcal{R}_P}(x)\right) : x \in \mathcal{X}\right\} \tag{1}$$

where μ_P and η_P are the grades of supporting and supporting against, with a condition such that $0 \leq \mu^2_{Rp}(x) + \eta^2_{Rp}(x) \leq 1$. The refusal grade is of the form $\xi_{Rp}(x) = 1 - \left(\mu^2_{Rp}(x) + \eta^2_{Rp}(x)\right)^{\frac{1}{2}}$. The Pythagorean fuzzy number (PFN) is denoted by $\mathcal{R}_{P-i} = \left(\mu_{\mathcal{R}_{P-i}}, \eta_{\mathcal{R}_{P-i}}\right)$. Further, the score function and accuracy function for PFS $\mathcal{R}_{P-1} = \left(\mu_{\mathcal{R}_{P-1}}, \eta_{\mathcal{R}_{P-1}}\right)$ are initiated by

$$\mathcal{S}_{SF}(\mathcal{R}_{P-1}) = \mu^2_{\mathcal{R}_{P-1}} - \eta^2_{\mathcal{R}_{P-1}} \tag{2}$$

$$\mathcal{H}_{AF}(\mathcal{R}_{P-1}) = \mu^2_{\mathcal{R}_{P-1}} + \eta^2_{\mathcal{R}_{P-1}} \tag{3}$$

where $\mathcal{S}_{SF}(\mathcal{R}_{P-1}) \in [-1, 1]$ and $\mathcal{H}_{AF}(\mathcal{R}_{P-1}) \in [0, 1]$.

To examine the interrelationships between any two PFSs $\mathcal{R}_{P-1} = \left(\mu_{\mathcal{R}_{P-1}}, \eta_{\mathcal{R}_{P-1}}\right)$ and $\mathcal{R}_{P-2} = \left(\mu_{\mathcal{R}_{P-2}}, \eta_{\mathcal{R}_{P-2}}\right)$, then

$$\text{If } \mathcal{S}_{SF}(\mathcal{R}_{P-1}) > \mathcal{S}_{SF}(\mathcal{R}_{P-2}) \Rightarrow \mathcal{R}_{P-1} > \mathcal{R}_{P-2}; \tag{4}$$

$$\text{If } \mathcal{S}_{SF}(\mathcal{R}_{P-1}) = \mathcal{S}_{SF}(\mathcal{R}_{P-2}) \Rightarrow, \tag{5}$$

$$\text{If } \mathcal{H}_{AF}(\mathcal{R}_{P-1}) > \mathcal{H}_{AF}(\mathcal{R}_{P-2}) \Rightarrow \mathcal{R}_{P-1} > \mathcal{R}_{P-2}; \tag{6}$$

$$\text{If } \mathcal{H}_{AF}(\mathcal{R}_{P-1}) = \mathcal{H}_{AF}(\mathcal{R}_{P-2}) \Rightarrow \mathcal{R}_{P-1} = \mathcal{R}_{P-2}. \tag{7}$$

Further, we discussed the existing operator based on PFSs, which is discussed below. For any two PFSs $\mathcal{R}_{P-1} = \left(\mu_{\mathcal{R}_{P-1}}, \eta_{\mathcal{R}_{P-1}}\right)$ and $\mathcal{R}_{P-2} = \left(\mu_{\mathcal{R}_{P-2}}, \eta_{\mathcal{R}_{P-2}}\right)$, then.

1. $\mathcal{R}_{P-1} \oplus \mathcal{R}_{P-2} = \left(\left(\mu^2_{\mathcal{R}_{P-1}} + \mu^2_{\mathcal{R}_{P-2}} - \mu^2_{\mathcal{R}_{P-1}}\mu^2_{\mathcal{R}_{P-2}}\right)^{\frac{1}{2}}, \eta_{\mathcal{R}_{P-1}}\eta_{\mathcal{R}_{P-2}}\right)$;
2. $\mathcal{R}_{P-1} \otimes \mathcal{R}_{P-2} = \left(\mu_{\mathcal{R}_{P-1}}\mu_{\mathcal{R}_{P-2}}, \left(\eta^2_{\mathcal{R}_{P-1}} + \eta^2_{\mathcal{R}_{P-2}} - \eta^2_{\mathcal{R}_{P-1}}\eta^2_{\mathcal{R}_{P-2}}\right)^{\frac{1}{2}}\right)$;
3. $\delta\mathcal{R}_{P-1} = \left(\left(1 - \left(1 - \mu^2_{\mathcal{R}_{P-1}}\right)^{\delta}\right)^{\frac{1}{2}}, \eta^{\delta}_{\mathcal{R}_{P-1}}\right)$;
4. $\delta\mathcal{R}_{P-1} = \left(\mu^{\delta}_{\mathcal{R}_{P-1}}, \left(1 - \left(1 - \eta^2_{\mathcal{R}_{P-1}}\right)^{\delta}\right)^{\frac{1}{2}}\right)$.

2.2 Muirhead Mean Operator

Definition 2: [26] For any positive real numbers $\mathcal{R}_{P-i}(i = 1, 2, 3, \ldots, \xi)$, with their parameter vectors $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_l) \in R^l$, the MM operator is initiated by

$$MM^{\mathcal{P}}\left(\mathcal{R}_{P-1}, \mathcal{R}_{P-2}, \ldots, \mathcal{R}_{P-\xi}\right) = \left(\frac{1}{\xi!}\sum\nolimits_{\sigma\in S_\xi}\prod\nolimits_{i=1}^{\xi}\mathcal{R}_{P-\sigma(i)}^{\mathcal{P}_i}\right)^{\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}} \quad (8)$$

where $\sigma(i)$, $(i = 1, 2, .., \xi)$ expressed any permutations of $(i = 1, 2, .., \xi)$ and S_ξ expressed the family of permutations $(i = 1, 2, .., \xi)$. Additionally, Eq. (8), holds the following axioms.

1. If $\mathcal{P} = (1, 0, 0, \ldots, 0)$, then the MM convert to $MM^{(1,0,0,\ldots,0)}\left(\mathcal{R}_{P-1}, \mathcal{R}_{P-2}, \ldots, \mathcal{R}_{P-\xi}\right) = \frac{1}{\xi}\sum_{i=1}^{\xi}\mathcal{R}_{P-i}$, which is expressed the arithmetic averaging operator.
2. If $\mathcal{P} = \left(\frac{1}{\xi}, \frac{1}{\xi}, \ldots., \frac{1}{\xi}\right)$, then the MM convert to $MM^{\left(\frac{1}{\xi}, \frac{1}{\xi}, \ldots., \frac{1}{\xi}\right)}\left(\mathcal{R}_{P-1}, \mathcal{R}_{P-2}, \ldots, \mathcal{R}_{P-\xi}\right) = \prod_{i=1}^{\xi}\mathcal{R}_{P-\sigma(i)}^{\frac{1}{\xi}}$, which is expressed the geometric mean operator.
3. If $\mathcal{P} = (1, 1, 0, \ldots, 0)$, then the MM convert to $MM^{(1,1,0,\ldots,0)}\left(\mathcal{R}_{P-1}, \mathcal{R}_{P-2}, \ldots, \mathcal{R}_{P-\xi}\right) = \left(\frac{1}{\xi(\xi-1)}\sum_{i\neq j=1}^{\xi}\mathcal{R}_{P-i}\mathcal{R}_{P-j}\right)^{\frac{1}{2}}$, which is expressed the Bonferroni mean operator.
4. If $\mathcal{P} = \left(\overbrace{1, 1, \ldots, 1, 1}^{k}\overbrace{0, 0, 0, 0, \ldots, 0}^{\xi-k}\right)$, then the MM convert to $MM^{\left(\overbrace{1, 1, \ldots, 1, 1}^{k}\overbrace{0, 0, 0, 0, \ldots, 0}^{\xi-k}\right)}\left(\mathcal{R}_{P-1}, \mathcal{R}_{P-2}, \ldots, \mathcal{R}_{P-\xi}\right) = \left(\frac{\oplus_{1\leq i_1\leq \ldots \leq i_k\leq \xi}\otimes_{j=1}^{k}\mathcal{R}_{P-ij}}{C_\xi^k}\right)^{\frac{1}{k}}$, which is expressed the Maclaurin symmetric mean operator.

From Def. (2) and the exceptional instances of the MM operator referenced above, we realize that the favorable position of the MM operator is that it can catch the general interrelationships among the numerous input boundaries and it is a speculation of some current total operators.

2.3 Schweizer–Sklar Operations

In this study, we review the basic notions of SS operations, which contain the SS sum and product based on T-norm and T-conorm.

Definition 3: [27] For any two PFSs $\mathcal{R}_{P-1} = \left(\mu_{\mathcal{R}_{P-1}}, \eta_{\mathcal{R}_{P-1}}\right)$ and $\mathcal{R}_{P-2} = \left(\mu_{\mathcal{R}_{P-2}}, \eta_{\mathcal{R}_{P-2}}\right)$, then.

$$\mathcal{R}_{P-1} \cap_{\mathcal{T},\mathcal{T}^*} \mathcal{R}_{P-2} = \left(\mathcal{T}\left(\mu_{\mathcal{R}_{P-1}}, \mu_{\mathcal{R}_{P-2}}\right), \mathcal{T}^*\left(\eta_{\mathcal{R}_{P-1}}, \eta_{\mathcal{R}_{P-2}}\right)\right) \tag{9}$$

$$\mathcal{R}_{P-1} \cup_{\mathcal{T},\mathcal{T}^*} \mathcal{R}_{P-2} = \left(\mathcal{T}^*\left(\mu_{\mathcal{R}_{P-1}}, \mu_{\mathcal{R}_{P-2}}\right), \mathcal{T}\left(\eta_{\mathcal{R}_{P-1}}, \eta_{\mathcal{R}_{P-2}}\right)\right) \tag{10}$$

where $\mathcal{T}$ and $\mathcal{T}^*$ are expressed the T-norm (TN) and T-conorm (TCN). Additionally, the SSTN and SSTCN are discussed below:

$$\mathcal{T}_{SS,\gamma}(x, y) = (x^\gamma + y^\gamma - 1)^{\frac{1}{\gamma}} \tag{11}$$

$$\mathcal{T}^*_{SS,\gamma}(x, y) = 1 - ((1-x)^\gamma + (1-y)^\gamma - 1)^{\frac{1}{\gamma}} \tag{12}$$

where $\gamma < 0, x, y, \in [0, 1]$. If $\gamma = 0$, then $\mathcal{T}_\gamma$ (x, y) = and $\mathcal{T}^*_\gamma$ (x, y) = x + y − xy, which are expressed the algebraic TN and TCN. Further, we have defined the SSTN and SSTCN for PFS, which are stated below:

Definition 4: For any two PFSs $\mathcal{R}_{P-1} = \left(\mu_{\mathcal{R}_{P-1}}, \eta_{\mathcal{R}_{P-1}}\right)$ and $\mathcal{R}_{P-2} = \left(\mu_{\mathcal{R}_{P-2}}, \eta_{\mathcal{R}_{P-2}}\right)$, then the generalized union and intersection using SS operational laws, such that

$$\mathcal{R}_{P-1} \otimes_{\mathcal{T},\mathcal{T}^*} \mathcal{R}_{P-2} = \left(\mathcal{T}\left(\mu^2_{\mathcal{R}_{P-1}}, \mu^2_{\mathcal{R}_{P-2}}\right), \mathcal{T}^*\left(\eta^2_{\mathcal{R}_{P-1}}, \eta^2_{\mathcal{R}_{P-2}}\right)\right) \tag{13}$$

$$\mathcal{R}_{P-1} \oplus_{\mathcal{T},\mathcal{T}^*} \mathcal{R}_{P-2} = \left(\mathcal{T}^*\left(\mu^2_{\mathcal{R}_{P-1}}, \mu^2_{\mathcal{R}_{P-2}}\right), \mathcal{T}\left(\eta^2_{\mathcal{R}_{P-1}}, \eta^2_{\mathcal{R}_{P-2}}\right)\right). \tag{14}$$

Based on Eqs. (11) and (12), we present SS operations based on PFSs, such that

$$\mathcal{R}_{P-1} \otimes_{SS} \mathcal{R}_{P-2} = \begin{pmatrix} \left(\mu^{2\gamma}_{\mathcal{R}_{P-1}} + \mu^{2\gamma}_{\mathcal{R}_{P-2}} - 1\right)^{\frac{1}{2\gamma}}, \\ \left(1 - \left(\left(1 - \eta^2_{\mathcal{R}_{P-1}}\right)^\gamma + \left(1 - \eta^2_{\mathcal{R}_{P-2}}\right)^\gamma - 1\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}} \end{pmatrix} \tag{15}$$

$$\mathcal{R}_{P-1} \oplus_{SS} \mathcal{R}_{P-2} = \begin{pmatrix} \left(1 - \left(\left(1 - \mu^2_{\mathcal{R}_{P-1}}\right)^\gamma + \left(1 - \mu^2_{\mathcal{R}_{P-2}}\right)^\gamma - 1\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}}, \\ \left(\eta^{2\gamma}_{\mathcal{R}_{P-1}} + \eta^{2\gamma}_{\mathcal{R}_{P-2}} - 1\right)^{\frac{1}{2\gamma}} \end{pmatrix} \tag{16}$$

$$\Delta\mathcal{R}_{P-1} = \begin{pmatrix} \left(1 - \left(\Delta\left(1 - \mu^2_{\mathcal{R}_{P-1}}\right)^\gamma - (\Delta - 1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}}, \\ \left(\Delta\eta^{2\gamma}_{\mathcal{R}_{P-1}} - (\Delta - 1)\right)^{\frac{1}{2\gamma}} \end{pmatrix} \tag{17}$$

$$\mathcal{R}_{P-1}^{\Delta} = \left(\begin{array}{c} \left(\Delta \mu_{\mathcal{R}_{P-1}}^{2\gamma} - (\Delta - 1)\right)^{\frac{1}{2\gamma}}, \\ \left(1 - \left(\Delta\left(1 - \eta_{\mathcal{R}_{P-1}}^{2}\right)^{\gamma} - (\Delta - 1)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}} \end{array} \right). \tag{18}$$

Theorem 1: *For any two PFSs* $\mathcal{R}_{P-1} = \left(\mu_{\mathcal{R}_{P-1}}, \eta_{\mathcal{R}_{P-1}}\right)$ *and* $\mathcal{R}_{P-2} = \left(\mu_{\mathcal{R}_{P-2}}, \eta_{\mathcal{R}_{P-2}}\right)$, *then*

1. $\mathcal{R}_{P-1} \oplus_{SS} \mathcal{R}_{P-2} = \mathcal{R}_{P-2} \oplus_{SS} \mathcal{R}_{P-1}$;
2. $\mathcal{R}_{P-1} \otimes_{SS} \mathcal{R}_{P-2} = \mathcal{R}_{P-2} \otimes_{SS} \mathcal{R}_{P-1}$;
3. $\Delta(\mathcal{R}_{P-1} \oplus_{SS} \mathcal{R}_{P-2}) = \Delta\mathcal{R}_{P-1} \oplus_{SS} \Delta\mathcal{R}_{P-2}$;
4. $\mathcal{R}_{P-1}^{\Delta_1+\Delta_2} = \Delta_1\mathcal{R}_{P-1} \otimes_{SS} \Delta_2\mathcal{R}_{P-1}$

Proof: Straightforward.

3 Pythagorean Fuzzy Schweizer–Sklar Muirhead Mean Aggregation Operations

The aim of this study is to present the MM operators based on PFS are called Pythagorean fuzzy MM (PFMM) operator, Pythagorean fuzzy weighted MM (PFWMM) operator, and their special cases are presented.

After that, a more generalized operator was presented, that is, the Muirhead mean [26], which was added an alterable parametric vector P on the basis of considering interrelationships among multiple input parameters, and some existing operators are its special cases, for instance, arithmetic and geometric mean (GM) operators (not considering the correlations), Bonferroni mean (BM) operator, and Maclaurin symmetric mean (MSM). When dealing with MCDM problems, some aggregation operators cannot consider the relationship between any input parameters, while Muirhead mean (MM) operator can take into account the correlation between inputs by a variable parameter. Therefore, the MM operator is more superior when dealing with MCDM problems.

Multi-criteria decision-making refers to the use of existing decision information, in the case of multi-criteria that are in conflict with each other and cannot coexist, and in which the limited alternatives are ranked or selected in a certain way. Schweizer–Sklar operation uses a variable parameter to make their operations more effective and flexible. MCDM alludes to the utilization of existing choice data, for the situation of multi-models that are in strife with one another and can't exist together, and in which the constrained options are positioned or chosen with a specific goal in mind. SS activity utilizes a variable boundary to make their tasks increasingly successful and adaptable. What's more, PFS can deal with inadequate, uncertain, and conflicting data under fuzzy conditions.

3.1 Pythagorean Fuzzy Schweizer–Sklar Muirhead Mean Operator

The aims of this study are to explore the idea of PFSSMM operator and their results to improve the quality of the explored work. The special cases of the explored work are also explored in this sub-section.

Definition 5: For any family of PFSs $\mathcal{R}_{P-i}(i = 1, 2, 3, \ldots, \xi)$, with their parameter vectors $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_l) \in R^l$, the PFSSMM operator is initiated by

$$PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1}, \mathcal{R}_{P-2}, \ldots, \mathcal{R}_{P-\xi}\right) = \left(\frac{1}{\xi!}\sum\nolimits_{\sigma\in S_\xi}\prod\nolimits_{i=1}^{\xi}\mathcal{R}_{P-\sigma(i)}^{\mathcal{P}_i}\right)^{\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}}, \tag{19}$$

where $\sigma(i), (i = 1, 2, .., \xi)$ expressed any permutations of $(i = 1, 2, .., \xi)$ and S_ξ expressed the family of permutations $(i = 1, 2, .., \xi)$.

By using Eq. (19) and the novel operational laws of SS, which are stated in the form of Def. (4), we get Theorem 2.

Theorem 2: *For any family of PFSs $\mathcal{R}_{P-i}(i = 1, 2, 3, \ldots, \xi)$, with their parameter vectors $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_l) \in R^l$, then by using Eq.* (19) *and Def.* (4), *we get Pythagorean fuzzy Schweizer–Sklar MM operator, such that*

$$PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1}, \mathcal{R}_{P-2}, \ldots, \mathcal{R}_{P-\xi}\right) = \\ \left(\begin{array}{c} \left(\begin{array}{c} 1+ \\ \frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\begin{array}{c} 1- \\ \left(\begin{array}{c} 1+ \\ \frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(\begin{array}{c} 1- \\ \left(\begin{array}{c} 1+ \\ \sum_{i=1}^{\xi}\mathcal{P}_i\left(\mu_{P-\sigma(i)}^{2\gamma}-1\right)\end{array}\right)^{\frac{1}{\gamma}}\end{array}\right)^{\gamma} - 2\right)\end{array}\right)^{\frac{1}{\gamma}}\end{array}\right)^{\gamma} - \frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\end{array}\right)^{\frac{1}{2\gamma}}, \\ \left(\begin{array}{c} 1- \\ \left(\begin{array}{c} 1+ \\ \frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\begin{array}{c} 1- \\ \left(\begin{array}{c} 1+ \\ \frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(\begin{array}{c} 1- \\ \left(\begin{array}{c} 1+ \\ \sum_{i=1}^{\xi}\mathcal{P}_i\left(\left(\begin{array}{c} 1- \\ \eta_{P-\sigma(i)}^{2}\end{array}\right)^{\gamma}-1\right)\end{array}\right)^{\frac{1}{\gamma}}\end{array}\right)^{\gamma} - 2\right)\end{array}\right)^{\frac{1}{\gamma}}\end{array}\right)^{\gamma} - \frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\end{array}\right)^{\frac{1}{\gamma}}\end{array}\right)^{\frac{1}{2}}\end{array}\right) \tag{20}$$

Proof: By using the Def. (4), we get

$$\mathcal{R}_{P-\sigma(i)}^{\mathcal{P}_i} = \left(\begin{array}{c}\left(\mathcal{P}_i\mu_{\mathcal{R}_{P-\sigma(i)}}^{2\gamma} - (\mathcal{P}_i - 1)\right)^{\frac{1}{\gamma}}, \\ \left(1 - \left(\mathcal{P}_i\left(1 - \eta_{\mathcal{R}_{P-\sigma(i)}}^{2}\right)^{\gamma} - (\mathcal{P}_i - 1)\right)^{\frac{1}{\gamma}}\right)\end{array}\right)$$

$$\prod_{i=1}^{\xi}\mathcal{R}_{P-\sigma(i)}^{\mathcal{P}_i}=\left(\begin{array}{c}\left(\sum_{i=1}^{\xi}\mathcal{P}_i\mu^{2\gamma}_{\mathcal{R}_{P-\sigma(i)}}-\sum_{i=1}^{\xi}\mathcal{P}_i+1\right)^{\frac{1}{\gamma}},\\ \left(1-\left(\sum_{i=1}^{\xi}\mathcal{P}_i\left(1-\eta^2_{\mathcal{R}_{P-\sigma(i)}}\right)^{\gamma}-\sum_{i=1}^{\xi}\mathcal{P}_i+1\right)^{\frac{1}{\gamma}}\right)\end{array}\right)$$

$$\sum_{\sigma\in S_\xi}\prod_{i=1}^{\xi}\mathcal{R}_{P-\sigma(i)}^{\mathcal{P}_i}=\left(\begin{array}{c}1-\left(\sum_{\sigma\in S_\xi}\left(1-\left(\sum_{i=1}^{\xi}\mathcal{P}_i\mu^{2\gamma}_{\mathcal{R}_{P-\sigma(i)}}-\sum_{i=1}^{\xi}\mathcal{P}_i+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}-1\right)^{\frac{1}{\gamma}},\\ \left(\sum_{\sigma\in S_\xi}\left(1-\left(\sum_{i=1}^{\xi}\mathcal{P}_i\left(1-\eta^2_{\mathcal{R}_{P-\sigma(i)}}\right)^{\gamma}-\sum_{i=1}^{\xi}\mathcal{P}_i+1\right)^{\frac{1}{\gamma}}\right)^{\gamma}-1\right)^{\frac{1}{\gamma}}\end{array}\right)$$

$$PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)=\left(\frac{1}{\xi!}\sum_{\sigma\in S_\xi}\prod_{i=1}^{\xi}\mathcal{R}_{P-\sigma(i)}^{\mathcal{P}_i}\right)^{\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}}=$$

$$\left(\begin{array}{c}\left(\genfrac{}{}{0pt}{}{1+}{\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\genfrac{}{}{0pt}{}{1-}{\left(\genfrac{}{}{0pt}{}{1+}{\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(\genfrac{}{}{0pt}{}{1-}{\left(\genfrac{}{}{0pt}{}{1+}{\sum_{i=1}^{\xi}\mathcal{P}_i\left(\mu^{2\gamma}_{P-\sigma(i)}-1\right)}\right)^{\frac{1}{\gamma}}}\right)^{\gamma}-2\right)}\right)^{\frac{1}{\gamma}}}\right)^{\gamma}-\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}}\right)^{\frac{1}{2\gamma}},\\ \left(\genfrac{}{}{0pt}{}{1-}{\left(\genfrac{}{}{0pt}{}{1+}{\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\genfrac{}{}{0pt}{}{1-}{\left(\genfrac{}{}{0pt}{}{1+}{\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(\genfrac{}{}{0pt}{}{1-}{\left(\genfrac{}{}{0pt}{}{1+}{\sum_{i=1}^{\xi}\mathcal{P}_i\left(\left(\genfrac{}{}{0pt}{}{1-}{\eta^2_{P-\sigma(i)}}\right)^{\gamma}-1\right)}\right)^{\frac{1}{\gamma}}}\right)^{\gamma}-2\right)}\right)^{\frac{1}{\gamma}}}\right)^{\gamma}-\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}}\right)^{\frac{1}{\gamma}}}\right)^{\frac{1}{2}}\end{array}\right)$$

Hence complete the proof.

Additionally, we have proved some properties like Monotonicity, Commutativity, and some special cases of the explored operators.

Theorem 3: *For any two families of PFSs $\mathcal{R}_{P-i}(i=1,2,3,\ldots,\xi)$ and $\mathcal{R}'_{P-i}$, with their parameter vectors $\mathcal{P}=(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_l)\in R^l$, if $\mu_{\mathcal{R}_{P-i}}\geq\mu'_{\mathcal{R}_{P-i}}$ and $\eta_{\mathcal{R}_{P-i}}\leq\eta'_{\mathcal{R}_{P-i}}$, then*

$$PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)\geq PFSSMM^{\mathcal{P}}\left(\mathcal{R}'_{P-1},\mathcal{R}'_{P-2},\ldots,\mathcal{R}'_{P-\xi}\right). \quad (21)$$

Proof: Let $PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)=(T,I)$ and $PFSSMM^{\mathcal{P}}\left(\mathcal{R}'_{P-1},\mathcal{R}'_{P-2},\ldots,\mathcal{R}'_{P-\xi}\right)=\left(T',I'\right)$, where

$$T=\left(1+\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\left(1-\left(1+\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(\left(1-\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left(\mu^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\right)^{\frac{1}{2\gamma}}$$

$$T'=\left(1+\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\left(1-\left(1+\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(\left(1-\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left({\mu'}^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\right)^{\frac{1}{2\gamma}}$$

and

$$I=\left(1-\left(1+\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\left(1-\left(1+\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(\left(1-\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left(\left(1-\eta^{2}_{P-\sigma(i)}\right)^{\gamma}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}}$$

$$I'=\left(1-\left(1+\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\left(1-\left(1+\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(\left(1-\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left(\left(1-{\eta'}^{2}_{P-\sigma(i)}\right)^{\gamma}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)-\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}}.$$

By hypothesis it's given that $\mu_{\mathcal{R}_{P-i}}\geq\mu'_{\mathcal{R}_{P-i}}$ and $\eta_{\mathcal{R}_{P-i}}\leq\eta'_{\mathcal{R}_{P-i}}$, then

$$\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left(\mu^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}\geq\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left({\mu'}^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}$$

$$\Rightarrow 1-\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left(\mu^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}\leq 1-\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left({\mu'}^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}$$

$$\Rightarrow\left(1-\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left(\mu^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\geq\left(1-\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left({\mu'}^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}$$

$$\Rightarrow\left(1+\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(1-\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left(\mu^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)^{\frac{1}{\gamma}}\leq\left(1+\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(1-\left(1+\sum_{i=1}^{\xi}\mathcal{P}_i\left({\mu'}^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)^{\frac{1}{\gamma}}$$

$$\Rightarrow\left(\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\left(\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(\begin{array}{c}1+\\1-\\\left(\begin{array}{c}1+\\1-\\\left(\begin{array}{c}1+\\\sum_{i=1}^{\xi}\mathcal{P}_i\left(\mu_{P-\sigma(i)}^{2\gamma}-1\right)\end{array}\right)^{\frac{1}{\gamma}}\end{array}\right)^{\gamma}\end{array}\right)-2\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\right)^{\frac{1}{2\gamma}}$$

$$\geq\left(\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\left(\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(\begin{array}{c}1+\\1-\\\left(\begin{array}{c}1+\\1-\\\left(\begin{array}{c}1+\\\sum_{i=1}^{\xi}\mathcal{P}_i\left({\mu'}_{P-\sigma(i)}^{2\gamma}-1\right)\end{array}\right)^{\frac{1}{\gamma}}\end{array}\right)^{\gamma}\end{array}\right)-2\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\right)^{\frac{1}{2\gamma}}.$$

Similarly, we examine the $\eta_{\mathcal{R}_{P-i}} \leq \eta'_{\mathcal{R}_{P-i}}$, then by combining the above both, we get

$$PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)\geq PFSSMM^{\mathcal{P}}\left(\mathcal{R}'_{P-1},\mathcal{R}'_{P-2},\ldots,\mathcal{R}'_{P-\xi}\right).$$

Theorem 4: *For any two families of PFSs $\mathcal{R}_{P-i}(i = 1, 2, 3, \ldots, \xi)$ and $\mathcal{R}'_{P-i}$, with their parameter vectors $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_l) \in R^l$, if $\mathcal{R}'_{P-i}$ is a permutations of $\mathcal{R}_{P-i}$, then*

$$PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)= PFSSMM^{\mathcal{P}}\left(\mathcal{R}'_{P-1},\mathcal{R}'_{P-2},\ldots,\mathcal{R}'_{P-\xi}\right). \quad (22)$$

Proof: Omitted.

Additionally, we will examine the special cases of the explored operators based on PFSs.

1. If $\mathcal{P} = (1, 0, 0, \ldots, 0)$, then the MM convert to Pythagorean fuzzy Schweizer–Sklar arithmetic averaging operator

$$\begin{aligned}PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)&=\frac{1}{\xi}\sum_{i=1}^{\xi}\mathcal{R}_{P-i}\\&=\left(\left(1-\left(1+\frac{1}{\xi}\left(\sum_{i=1}^{\xi}\left(\mu_{P-\sigma(i)}^{2}-1\right)^{\gamma}-2\right)\right)\right)^{\frac{1}{2\gamma}},\left(1+\frac{1}{\xi}\left(\sum_{\sigma\in S_\xi}\eta_{P-\sigma(i)}^{2\gamma}-2\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}}\right).\end{aligned} \quad (23)$$

2. If $\mathcal{P} = (\delta, 0, 0, \ldots, 0)$, then the MM convert to Pythagorean fuzzy Schweizer–Sklar generalized arithmetic averaging operator

$$PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)=\left(\frac{1}{\xi}\sum_{i=1}^{\xi}\mathcal{R}_{P-i}^{\delta}\right)^{\frac{1}{\delta}}$$

$$=\left(\begin{array}{c}\left(1+\frac{1}{\delta}\left(1-\left(1+\frac{1}{\xi}\left(\sum_{i=1}^{\xi}\left(1-\left(1+\delta\left(\mu^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-1\right)^{\frac{1}{2\gamma}}, \\ \left(1-\left(1+\frac{1}{\delta}\left(1-\left(1+\frac{1}{\xi}\left(\sum_{i=1}^{\xi}\left(1-\left(1+\delta\left(\left(1-\eta^{2}_{P-\sigma(i)}\right)^{\gamma}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-1\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}}\end{array}\right). \tag{24}$$

3. If $\mathcal{P}=(1,1,0,\ldots,0)$, then the MM convert to Pythagorean fuzzy Schweizer–Sklar Bonferroni mean operator

$$PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)=\left(\frac{1}{\xi(\xi-1)}\sum_{i\neq j=1}^{\xi}\mathcal{R}_{P-i}\mathcal{R}_{P-j}\right)^{\frac{1}{2}}$$

$$=\left(\begin{array}{c}\left(\frac{1}{2}\left(1+\left(1-\left(1+\frac{1}{\xi(\xi-1)}\left(\sum_{i\neq j=1}^{\xi}\left(1-\left(\mu^{2\gamma}_{P-\sigma(i)}+\mu^{2\gamma}_{P-\sigma(j)}-1\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)\right)^{\frac{1}{2\gamma}}, \\ \left(1-\left(\frac{1}{2}\left(1+\left(1-\left(1+\frac{1}{\xi(\xi-1)}\left(\sum_{i\neq j=1}^{\xi}\left(1-\left(\mu^{2\gamma}_{P-\sigma(i)}+\mu^{2\gamma}_{P-\sigma(j)}-1\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}}\end{array}\right). \tag{25}$$

4. If $\mathcal{P}=\left(\overbrace{1,1,\ldots,1,1}^{k}\overbrace{0,0,0,0,\ldots,0}^{\xi-k}\right)$, then the MM convert to Pythagorean fuzzy Schweizer–Sklar Maclaurin symmetric mean operator

$$PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)=\left(\frac{\oplus_{1\le i_1\le ....\le i_k\le\xi}\otimes_{j=1}^{k}\mathcal{R}_{P-ij}}{C_{\xi}^{k}}\right)^{\frac{1}{k}}$$

$$=\left(\begin{array}{c}\left(\genfrac{}{}{0pt}{}{1+}{\frac{1}{k}\left(\genfrac{}{}{0pt}{}{1-}{\left(\genfrac{}{}{0pt}{}{1+}{\frac{1}{C_{\xi}^{k}}\left(\sum_{1\le i_1\le ....\le i_k\le\xi}\left(\genfrac{}{}{0pt}{}{1-}{\left(\sum_{j=1}^{k}\left(\mu_{P-\sigma(i_j)}^{2\gamma}-1\right)\right)^{\frac{1}{\gamma}}}\right)^{\gamma}-2\right)}\right)^{\frac{1}{\gamma}}}\right)^{\gamma}-1}\right)^{\frac{1}{2\gamma}}, \\ \left(\genfrac{}{}{0pt}{}{1-}{\left(\genfrac{}{}{0pt}{}{1+}{\frac{1}{k}\left(\genfrac{}{}{0pt}{}{1-}{\left(\genfrac{}{}{0pt}{}{1+}{\frac{1}{C_{\xi}^{k}}\left(\sum_{1\le i_1\le ....\le i_k\le\xi}\left(\genfrac{}{}{0pt}{}{1-}{\left(\sum_{j=1}^{k}\left(\left(\genfrac{}{}{0pt}{}{1-}{\eta_{P-\sigma(i_j)}^{2}}\right)^{\gamma}-1\right)\right)^{\frac{1}{\gamma}}}\right)^{\gamma}-2\right)}\right)^{\frac{1}{\gamma}}}\right)^{\gamma}-1}\right)^{\frac{1}{\gamma}}}\right)^{\frac{1}{2}}\end{array}\right). \tag{26}$$

5. If $\mathcal{P}=\left(\frac{1}{\xi},\frac{1}{\xi},\ldots.,\frac{1}{\xi}\right)$, then the MM convert to Pythagorean fuzzy Schweizer–Sklar geometric mean operator

$$PFSSMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)=\prod_{i=1}^{\xi}\mathcal{R}_{P-\sigma(i)}^{\frac{1}{\xi}}$$
$$=\left(\left(1+\frac{1}{\xi}\left(\sum\nolimits_{\sigma\in S_{\xi}}\mu_{P-\sigma(i)}^{2\gamma}-2\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}},\left(1-\left(1+\frac{1}{\xi}\left(\sum\nolimits_{i=1}^{\xi}\left(\eta_{P-\sigma(i)}^{2}-1\right)^{\gamma}-2\right)\right)\right)^{\frac{1}{2\gamma}}\right). \tag{27}$$

3.2 *Pythagorean Fuzzy Schweizer–Sklar Weighted Muirhead Mean Operator*

The aim of this study is to explore the idea of PFSSWMM operator and their results to improve the quality of the explored work. The special cases of the explored work are also explored in this sub-section.

Definition 6: For any family of PFSs $\mathcal{R}_{P-i}(i=1,2,3,\ldots,\xi)$, with their parameter vectors $\mathcal{P}=(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_l)\in R^l$, the WMM operator is initiated by

$$PFSSWMM^{\mathcal{P}}\left(\mathcal{R}_{P-1},\mathcal{R}_{P-2},\ldots,\mathcal{R}_{P-\xi}\right)=\left(\frac{1}{\xi!}\sum\nolimits_{\sigma\in S_{\xi}}\prod\nolimits_{i=1}^{\xi}\mathcal{R}_{P-\sigma(i)}^{\mathcal{P}_i\omega_{W-i}}\right)^{\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}}, \tag{28}$$

where $\sigma(i),(i=1,2,..,\xi)$ expressed any permutations of $(i=1,2,..,\xi)$ and S_{ξ} expressed the family of permutations $(i=1,2,..,\xi)$. The weight vector is denoted and defined by: $\omega_W=\left(\omega_{W-1},\omega_{W-2},..,\omega_{W-\xi}\right)^T$ with a condition that is $\sum_{i=1}^{\xi}\omega_{W-i}=1,\omega_{W-i}\in[0,1],i=1,2,3,\ldots,\xi$.

By using Eq. (28) and the novel operational laws of SS, which are stated in the form of Def. (4), we get Theorem 5.

Theorem 5: *For any family of PFSs* $\mathcal{R}_{P-i}(i = 1, 2, 3, \ldots, \xi)$, *with their parameter vectors* $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_l) \in R^l$, *then by using Eq.* (28) *and Def.* (4), *we get Pythagorean fuzzy Schweizer–Sklar MM operator, such that*

$$PFSSWMM^{\mathcal{P}}(\mathcal{R}_{P-1}, \mathcal{R}_{P-2}, \ldots, \mathcal{R}_{P-\xi}) =$$

$$\left(\begin{array}{c}\left(\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\left(\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(1+1-1+1-1+\left(\sum_{i=1}^{\xi}\mathcal{P}_i\omega_{W-i}\left(\mu^{2\gamma}_{P-\sigma(i)}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\right)^{\frac{1}{2\gamma}}, \\ \left(\left(\left(\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\left(\left(\frac{1}{\xi!}\left(\sum_{\sigma\in S_\xi}\left(1-1+1-1+1-1+\left(\sum_{i=1}^{\xi}\mathcal{P}_i\omega_{W-i}\left(\left(1-\eta^{2}_{P-\sigma(i)}\right)^{\gamma}-1\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-2\right)\right)^{\frac{1}{\gamma}}\right)^{\gamma}-\frac{1}{\sum_{i=1}^{\xi}\mathcal{P}_i}\right)^{\frac{1}{\gamma}}\right)^{\frac{1}{2}}\right)\end{array}\right). \tag{29}$$

Proof: Straightforward. (The proof of this theorem id similar to the proof of Theorem 2).

Theorem 6: *For any two families of PFSs* $\mathcal{R}_{P-i}(i = 1, 2, 3, \ldots, \xi)$ *and* $\mathcal{R}'_{P-i}$, *with their parameter vectors* $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_l) \in R^l$, *if* $\mu_{\mathcal{R}_{P-i}} \geq \mu'_{\mathcal{R}_{P-i}}$ *and* $\eta_{\mathcal{R}_{P-i}} \leq \eta'_{\mathcal{R}_{P-i}}$, *then*

$$PFSSWMM^{\mathcal{P}}(\mathcal{R}_{P-1}, \mathcal{R}_{P-2}, \ldots, \mathcal{R}_{P-\xi}) \geq PFSSWMM^{\mathcal{P}}\left(\mathcal{R}'_{P-1}, \mathcal{R}'_{P-2}, \ldots, \mathcal{R}'_{P-\xi}\right). \tag{30}$$

Proof: Straightforward. (The proof of this theorem id similar to the proof of Theorem 3).

Theorem 7: *For any two families of PFSs* $\mathcal{R}_{P-i}(i = 1, 2, 3, \ldots, \xi)$ *and* $\mathcal{R}'_{P-i}$, *with their parameter vectors* $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_l) \in R^l$, *if* $\mathcal{R}'_{P-i}$ *is a permutations of* $\mathcal{R}_{P-i}$, *then*

$$PFSSWMM^{\mathcal{P}}(\mathcal{R}_{P-1}, \mathcal{R}_{P-2}, \ldots, \mathcal{R}_{P-\xi}) = PFSSWMM^{\mathcal{P}}\left(\mathcal{R}'_{P-1}, \mathcal{R}'_{P-2}, \ldots, \mathcal{R}'_{P-\xi}\right) \tag{31}$$

Proof: Straightforward. (The proof of this theorem id similar to the proof of Theorem 4).

4 Multi-criteria Decision-Making Problems Based on Pythagorean Fuzzy Schweizer–Sklar Muirhead Mean Aggregation Operations

To examine the proficiency of the explored operators in this manuscript, we presented the MCDM technique based on weighted MM operators by using Pythagorean fuzzy information's. To addressed effectively such kind of issues, we choose the family of alternatives and their criteria with weight vectors, whose expressions are summarized in the following ways: $\mathcal{A}_{AL} = \{\mathcal{A}_{AL-1}, \mathcal{A}_{AL-2}, \ldots, \mathcal{A}_{AL-m}\}$ and $\mathcal{C}_{CR} = \{\mathcal{C}_{CR-1}, \mathcal{C}_{CR-2}, \ldots, \mathcal{C}_{CR-\xi}\}$ with their weight vector $\omega_W = (\omega_{W-1}, \omega_{W-2}, \ldots, \omega_{W-\xi})^T$ with a condition that is $\sum_{i=1}^{\xi} \omega_{W-i} = 1, \omega_{W-i} \in [0, 1]$. For resolving such kind of issues, we construct the decision matrix, whose representation is of the form $R = (\mathcal{R}_{P-ij})_{m\times\xi}$, whose every entries in the form of Pythagorean fuzzy numbers that are $\mathcal{R}_{P-ij} = (\mu_{\mathcal{R}_{P-ij}}, \eta_{\mathcal{R}_{P-ij}})$. Then the steps of the decision-making technique is summarized in the following ways:

Step 1: By using Eq. (32), we normalized the decision matrix, if needed.

$$R = \begin{cases} (\mu_{\mathcal{R}_{P-ij}}, \eta_{\mathcal{R}_{P-ij}}) & for\ be\xi efit \\ (\eta_{\mathcal{R}_{P-ij}}, \mu_{\mathcal{R}_{P-ij}}) & for\ cost \end{cases} \tag{32}$$

Step 2: By using Eq. (29), we aggregated the normalized decision matrix.
Step 3: By using Eq. (2), examine the score values of the aggregated values.
Step 4: Rank to all alternatives, and examine the best one.
Step 5: The end.

Example 1: We allude to a case of MCDM to demonstrate the plausibility and legitimacy of the introduced technique. We allude to the decision-making issue in Ref. [28]. There is a speculation organization, which plans to pick the most ideal interest in the other options. There are four potential alternatives for the speculation organization to browse: (1) a vehicle organization $\mathcal{A}_{AL-1}$; (2) a food organization $\mathcal{A}_{AL-2}$; (3) a PC organization $\mathcal{A}_{AL-3}$; (4) an arms organization $\mathcal{A}_{AL-4}$. The venture organization will think about the accompanying three assessment records to settle on decisions: (1) the hazard investigation $\mathcal{C}_{CR-1}$; (2) the development examination $\mathcal{C}_{CR-2}$; and (3) the natural impact investigation. Among $\mathcal{C}_{CR-1}$ and $\mathcal{C}_{CR-2}$ are the advantage standards and $\mathcal{C}_{CR-3}$ is the cost basis. The weight vector of the standards is w = (0.5, 0.3, 0.2)T. The four potential options are assessed regarding the over three rules by the type of SVNSs, and single-esteemed neutrosophic choice network R is developed as recorded in Table 1.

Then the steps of the decision-making technique are summarized in the following ways:

Step 1: By using Eq. (32), we normalized the decision matrix, if needed, but it's not needed, see Table 2.

Table 1 Original decision matrix

Symbols	$\mathcal{C}_{CR-1}$	$\mathcal{C}_{CR-2}$	$\mathcal{C}_{CR-3}$
$\mathcal{A}_{AL-1}$	(0.4, 0.3)	(0.4, 0.3)	(0.2, 0.5)
$\mathcal{A}_{AL-2}$	(0.6, 0.2)	(0.6, 0.2)	(0.5, 0.2)
$\mathcal{A}_{AL-3}$	(0.3, 0.3)	(0.5, 0.3)	(0.5, 0.2)
$\mathcal{A}_{AL-4}$	(0.7, 0.1)	(0.6, 0.2)	(0.4, 0.2)

Table 2 Normalized decision matrix

Symbols	$\mathcal{C}_{CR-1}$	$\mathcal{C}_{CR-2}$	$\mathcal{C}_{CR-3}$
$\mathcal{A}_{AL-1}$	(0.4, 0.3)	(0.4, 0.3)	(0.2, 0.5)
$\mathcal{A}_{AL-2}$	(0.6, 0.2)	(0.6, 0.2)	(0.5, 0.2)
$\mathcal{A}_{AL-3}$	(0.3, 0.3)	(0.5, 0.3)	(0.5, 0.2)
$\mathcal{A}_{AL-4}$	(0.7, 0.1)	(0.6, 0.2)	(0.4, 0.2)

Step 2: By using Eq. (29), we aggregated the normalized decision matrix for $\mathcal{P} = (1, 1, 1), \gamma = -1$.

$$\mathcal{A}_{AL-1} = (0.5547, 0.4462), \mathcal{A}_{AL-2} = (0.4492, 0.4308), \mathcal{A}_{AL-3} = (0.5455, 0.4382), \mathcal{A}_{AL-4} = (0.4559, 0.4258)$$

Step 3: By using Eq. (2), examine the score values of the aggregated values.

$$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.1086, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.01613, \mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.1056, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.0265$$

Step 4: Rank to all alternatives, and examine the best one.

$$\mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$$

The best alternative is $\mathcal{A}_{AL-1}$.

Step 5: The end.

To check the effect of the boundaries vectors $\mathcal{P}$ and γ on the decision-making of the case, we select assorted boundaries vectors $\mathcal{P}$ and γ and give the arranging aftereffects of the other options. We can see the outcomes in Tables 3.

From the above analysis it is clear that for the different values of parameter the same ranking results are given, the best option is $\mathcal{A}_{AL-1}$.

4.1 Advantages of the Explored Operators

Additionally, to examine the reliability and proficiency of the explored operators, we choose the Pythagorean fuzzy kind of information's to find the accuracy and superiority of the explored operators.

Table 3 Examine the fluency of the parameter for $\gamma = -0.1$

Parameter $\mathcal{P}$	Score values	Ranking values
$(1, 1, 1)$	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.1086, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.0161,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.1056, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.0265$	$\mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
$(1, 2, 1)$	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.219, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.110,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.214, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.121$	$\mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
$(1, 1, 3)$	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.304, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.185,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.297, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.196$	$\mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
$(4, 1, 1)$	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.404, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.380,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.402, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.382$	$\mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
$(3, 4, 5)$	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.662, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.591,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.619, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.593$	$\mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
$(4, 5, 6)$	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.683, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.662,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.681, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.663$	$\mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$

Example 2: We allude to a case of MCDM to demonstrate the plausibility and legitimacy of the introduced technique. We allude to the decision-making issue in Ref. [28]. There is a speculation organization, which plans to pick the most ideal interest in the other options. There are four potential alternatives for the speculation organization to browse: (1) a vehicle organization $\mathcal{A}_{AL-1}$; (2) a food organization $\mathcal{A}_{AL-2}$; (3) a PC organization $\mathcal{A}_{AL-3}$; (4) an arms organization $\mathcal{A}_{AL-4}$. The venture organization will think about the accompanying three assessment records to settle on decisions: (1) the hazard investigation $\mathcal{C}_{CR-1}$; (2) the development examination $\mathcal{C}_{CR-2}$; and (3) the natural impact investigation. Among $\mathcal{C}_{CR-1}$ and $\mathcal{C}_{CR-2}$ are the advantage standards and $\mathcal{C}_{CR-3}$ is the cost basis. The weight vector of the standards is w = (0.5, 0.3, 0.2)T. The four potential options are assessed regarding the over three rules by the type of SVNSs, and single-esteemed neutrosophic choice network R is developed as recorded in Table 4.

Then the steps of the decision-making technique are summarized in the following ways:

Table 4 Original decision matrix

Symbols	$\mathcal{C}_{CR-1}$	$\mathcal{C}_{CR-2}$	$\mathcal{C}_{CR-3}$
$\mathcal{A}_{AL-1}$	$(0.9, 0.3)$	$(0.91, 0.3)$	$(0.92, 0.3)$
$\mathcal{A}_{AL-2}$	$(0.8, 0.4)$	$(0.81, 0.4)$	$(0.82, 0.4)$
$\mathcal{A}_{AL-3}$	$(0.7, 0.5)$	$(0.71, 0.5)$	$(0.72, 0.5)$
$\mathcal{A}_{AL-4}$	$(0.85, 0.35)$	$(0.86, 0.35)$	$(0.87, 0.35)$

Table 5 Normalized decision matrix

Symbols	$\mathcal{C}_{CR-1}$	$\mathcal{C}_{CR-2}$	$\mathcal{C}_{CR-3}$
$\mathcal{A}_{AL-1}$	$(0.9, 0.3)$	$(0.91, 0.3)$	$(0.92, 0.3)$
$\mathcal{A}_{AL-2}$	$(0.8, 0.4)$	$(0.81, 0.4)$	$(0.82, 0.4)$
$\mathcal{A}_{AL-3}$	$(0.7, 0.5)$	$(0.71, 0.5)$	$(0.72, 0.5)$
$\mathcal{A}_{AL-4}$	$(0.85, 0.35)$	$(0.86, 0.35)$	$(0.87, 0.35)$

Step 1: By using Eq. (32), we normalized the decision matrix, if needed, but it's not needed, see Table 5.

Step 2: By using Eq. (29), we aggregated the normalized decision matrix for $\mathcal{P} = (1, 1, 1), \gamma = -0.1$.

$$\mathcal{A}_{AL-1} = (0.9458, 0.0402), \mathcal{A}_{AL-2} = (0.9225, 0.0408), \mathcal{A}_{AL-3} = (0.9555, 0.0416), \mathcal{A}_{AL-4} = (0.9499, 0.0405)$$

Step 3: By using Eq. (2), examine the score values of the aggregated values.

$$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.8930, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.9056, \mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.9112, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.9008$$

Step 4: Rank to all alternatives, and examine the best one.

$$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$$

The best alternative is $\mathcal{A}_{AL-3}$.

Step 5: The end.

To check the effect of the boundaries vectors $\mathcal{P}$ and γ on the decision-making of the case, we select assorted boundaries vectors $\mathcal{P}$ and γ and give the arranging aftereffects of the other options. We can see the outcomes in Tables 6.

From the above analysis it is clear that for the different values of parameter the same ranking results are given, the best option is $\mathcal{A}_{AL-3}$.

Table 6 Examine the fluency of the parameter for $\gamma = -0.1$

Parameter $\mathcal{P}$	Score values	Ranking values
(1, 1, 1)	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.893, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.905,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.911, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.900$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$
(1, 2, 1)	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.918, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.928,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.932, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.924$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$
(1, 1, 3)	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.936, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.942,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.946, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.939$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$
(4, 1, 1)	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.955, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.958,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.959, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.957$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$
(3, 4, 5)	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.976, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.978,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.979, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.977$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$
(4, 5, 6)	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.982, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.983,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.983, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.982$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$

4.2 Comparative Analysis of the Explored Operators

The comparison of the explored approach with some existing approaches evaluates the reliability and effectiveness of the explored operators. The information of the existing operators are discussed below, for instance, Pythagorean fuzzy Schweizer–Sklar arithmetic averaging (PFSSAA) operator, Pythagorean fuzzy Schweizer–Sklar Bonferroni mean (PFSSBM) operator, Pythagorean fuzzy Schweizer–Sklar Maclaurin symmetric mean (PFSSMSM) operator, intuitionistic fuzzy Schweizer–Sklar arithmetic averaging (IFSSAA) operator, intuitionistic fuzzy Schweizer–Sklar Bonferroni mean (IFSSBM) operator, and intuitionistic fuzzy Schweizer–Sklar Maclaurin symmetric mean (IFSSMSM) operator. The comparative analysis of the explored work with some existing works is discussed in the form of Table 7, for Example 1.

From the above analysis it is clear that the explored operator and existing operators give the different ranking values, and the best one is $\mathcal{A}_{AL-2}$ and $\mathcal{A}_{AL-1}$.

Table 7 Comparative analysis for the information of Example 1

Method	Score values	Ranking values
IFSSAA operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = -0.844, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = -0.95,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = -0.843, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = -0.948$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
IFSSBM operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = -0.71, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = -0.93,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = -0.70, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = -0.93$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
IFSSMSM operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = -0.71, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = -0.93,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = -0.70, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = -0.93$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
Ma and Xu [8]	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = -0.271, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = -0.29,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = -0.23, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = -0.31$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
PFSSAA operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = -0.282, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = -0.32,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = -0.28, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = -0.31$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
PFSSBM operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = -0.044, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = -0.11,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = -0.045, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = -0.10$	$\mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
PFSSMSM operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = -0.044, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = -0.11,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = -0.045, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = -0.10$	$\mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$
Proposed operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.108, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.0161,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.1056, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.0256$	$\mathcal{A}_{AL-1} \geq \mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-2}$

The comparative analysis of the explored work with some existing works is discussed in the form of Table 8, for Example 2.

From the above analysis it is clear that the explored operator and existing operators give the same ranking values, and the best one is $\mathcal{A}_{AL-3}$.

Table 8 Comparative analysis for the information's of Example 2

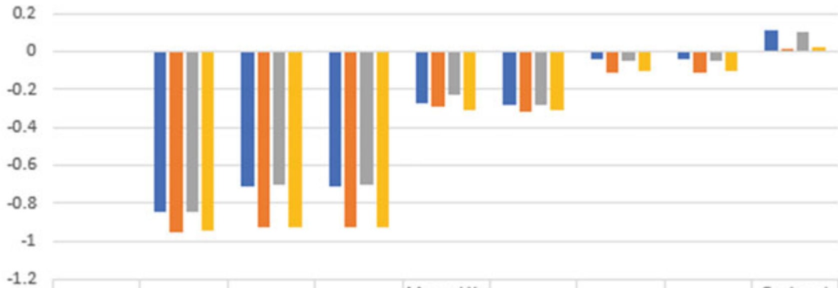
Method	Score values	Ranking values
IFSSAA operator	$Failed$	–
IFSSBM operator	$Failed$	–
IFSSMSM operator	$Failed$	–
Ma and Xu [8]	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.71, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.732,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.75, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.721$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$
PFSSAA operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.7053, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.7352,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.7486, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.7239$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$
PFSSBM operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.8430, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.8608,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.8687, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.8541$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$
PFSSMSM operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.8430, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.8608,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.8687, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.8541$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$
Proposed operator	$\mathcal{S}_{SF}(\mathcal{A}_{AL-1}) = 0.8930, \mathcal{S}_{SF}(\mathcal{A}_{AL-2}) = 0.9055,$ $\mathcal{S}_{SF}(\mathcal{A}_{AL-3}) = 0.9112, \mathcal{S}_{SF}(\mathcal{A}_{AL-4}) = 0.9008$	$\mathcal{A}_{AL-3} \geq \mathcal{A}_{AL-2} \geq \mathcal{A}_{AL-4} \geq \mathcal{A}_{AL-1}$

4.3 Graphical Representations of the Explored Operators

The graphical interpretation of the explored work with some existing approaches is discussed in the form of figures, to improve the quality of the research work, to examine the reliability and effectiveness of the explored work. The comparative analysis of the explored work with some existing works, which are discussed in the form of Table 7, is summarized with the help of Fig. 1.

From the above figure, it is clear that Fig. 1 contains five series which show different colors representing by the family of alternatives. There are many places which show the values are called the score function. By using these values we examine the best alternative from the family of alternatives.

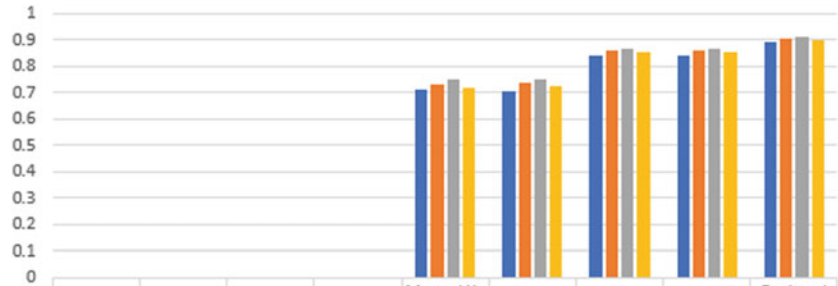
The comparative analysis of the explored work with some existing works, which are discussed in the form of Table 8, is summarized with the help of Fig. 2.



	Methods	IFSSAA	IFSSBM	IFSSMSM	Ma and Xu [8]	PFSSAA	PFSSBM	PFSSMSM	Explored Work
Series1	0	-0.844	-0.71	-0.71	-0.271	-0.282	-0.044	-0.044	0.108
Series2		-0.95	-0.93	-0.93	-0.29	-0.32	-0.11	-0.11	0.0161
Series3		-0.843	-0.7	-0.7	-0.23	-0.28	-0.045	-0.045	0.1056
Series4		-0.948	-0.93	-0.93	-0.31	-0.31	-0.1	-0.1	0.0256

Series1 Series2 Series3 Series4

Fig. 1 Geometrical interpretation of the explored work for Table 5



	Methods	IFSSAA	IFSSBM	IFSSMSM	Ma and Xu [8]	PFSSAA	PFSSBM	PFSSMSM	Explored Work
Series1	0	0	0	0	0.71	0.7053	0.843	0.843	0.893
Series2		0	0	0	0.732	0.7352	0.8608	0.8608	0.9055
Series3		0	0	0	0.75	0.7486	0.8687	0.8687	0.9112
Series4		0	0	0	0.721	0.7239	0.8541	0.8541	0.9008

Series1 Series2 Series3 Series4

Fig. 2 Geometrical interpretation of the explored work for Table 6

From the above figure, it is clear that Fig. 2 contains five series which show different colors representing by the family of alternatives. There are many places which show the values are called the score function. By using these values we examine the best alternative from the family of alternatives.

From the above analysis, it is clear that the explored operators based on PFSs are more proficient and more valuable than existing methods.

5 Conclusion

SS activity can make data conglomeration progressively adaptable, and the MM operator can consider the relationship among contributions by a variable parameter. Since customary MM is just accessible for genuine numbers and PFS can all the more likely express deficient and dubious data in choice frameworks. The objectives of this manuscript, first we explore the SS operators based on PFS and studied their score function, accuracy function, and their relationships. The limitation of the SS operators is discussed below:

1. Multi-criteria decision-making refers to the use of existing decision information, in the case of multi-criteria that are in conflict with each other and cannot coexist, and in which the limited alternatives are ranked or selected in a certain way. Schweizer–Sklar operation uses a variable parameter to make their operations more effective and flexible. In addition, PFS can handle incomplete, indeterminate, and inconsistent information under fuzzy environments. Therefore, we conducted further research on SS operations for PFS and applied SS operations to MCDM problems. Furthermore, because the MM operator considers interrelationships among multiple input parameters by the alterable parametric vector, hence combining the MM operator with the SS operation gives some aggregation operators, and it was more meaningful to develop some new means to solve the MCDM problems in the Pythagorean fuzzy environment. According to this, the purpose and significance of this article are
2. To develop a number of new MM operators by combining MM operators, SS operations, and PFS;
3. To discuss some meaningful properties and a number of cases of these operators put forward;
4. To deal with an MCDM method for PFS information more effectively based on the operators put forward;
5. To demonstrate the viability and superiority of the newly developed method.

Further, based on these operators, the MM operators based on PFS are called PFMM operator, PFWMM operator, and their special cases are presented. Additionally, MADM problem is solved by using the explored operators based on PFS to observe the consistency and efficiency of the produced approach. Finally, the advantages, comparative analysis, and their geometrical representation are also discussed.

In future, we will extend these ideas into complex fuzzy sets [29], picture hesitant fuzzy sets [30, 31], complex q-rung orthopair fuzzy sets [32–34], etc. [35–40]. By considering the superiority of new PFS, we can also extend them to some other aggregation operators, such as power mean aggregation operators, Bonferroni mean operators, Heronian mean operators, and so on.

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Pythagorean Fuzzy MCDM Method Based on CODAS

Xindong Peng

1 Introduction

Intuitionistic fuzzy sets (IFS) is an extension of fuzzy sets (FS) [1], explored by Atanassov [2]. Because IFS is described by membership and non-membership, it can describe the characteristics of uncertain data in a more comprehensive and detailed manner. The main feature of IFS is that it assigns members and non-members to each element, and its sum is ≤ 1. However, in some practical problems, the sum of the membership and non-membership of a given alternative that meets the criteria of an expert or decision-maker (DM) can be ≥ 1, and the sum of their squares ≤ 1.

Therefore, Yager [3] developed the Pythagorean fuzzy set (PFS) characterized by membership and non-membership, and it satisfies the corresponding non-membership and membership square sum ≤ 1. Yager and Abbasov [4] offered an example to state this situation: the alternative method for DMs or experts to provide approval for members is $\frac{\sqrt{3}}{2}$ and he opposed joining $\frac{1}{2}$, because its sum is >1, which is not a workable IFS although it can be adapted from PFS. Obviously, compared with IFS, PFS can simulate the uncertainty in real multi-criteria decision-making (MCDM) problems.

People have studied PFS from diverse angles [5], including decision-making techniques [6–23], aggregation operations [24–42], information measure [43–47]. Specifically, a brief literature review of the first two aspects is as follows:

(1) Decision-making technologies. Inspired by the modified TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) approach [48], Zhang and Xu [6] employed it in the MCDM problem in the Pythagorean fuzzy environment. A Pythagorean fuzzy-weighted average operator for dealing MCDM issue is proposed by Yager [7]. Peng and Yang [8] researched its relationship and

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_11

proposed a multi-criteria group decision-making (MCGDM) method based on SIR (superiority and inferiority ranking). Zhang [9] proposed the QUALIFLEX (qualitative flexible) approach, which uses a ranking method based on proximity index for PF decision analysis. Peng and Dai [10] presented stochastic MCDM methods under Pythagorean fuzzy environment with prospect theory and regret theory. Ren et al. [11] generalized the TODIM (an acronym in Portuguese for Interactive Multi-Criteria Decision Making) method to handle MCDM problems in the PF environment. Peng and Yang [12] proposed a new PF MCGDM method based on Choquet integral and MABAC (multi-attributive border approximation area comparison). Wan et al. [13] proposed a mathematical programming method for dealing MCGDM problems under the PF environment.

(2) Aggregation operators. Yager [4] introduced a large number of aggregation operators (AOs) and used them to solve MCDM issues. Garg [26, 27] developed a generalized Pythagorean fuzzy AOs through Einstein and geometric operations. Zeng et al. [28] studied mixed Pythagorean fuzzy aggregation information. Ma and Xu [29] discussed the Pythagorean fuzzy symmetric weighted average/geometric operator. Zeng [30] introduced Pythagorean fuzzy AOs based ordered weighted average and probability information. Peng and Yang [31] proposed some fundamental theories of interval-valued PF AOs. Pythagorean fuzzy MSM operators are given by Wei and Lu [32]. Zhang et al. [33] developed certain extended Pythagorean fuzzy BM AOs. Liu et al. [34] proposed PF interaction AOs and explored some properties.

Ghorabaee et al. [49] developed the CODAS (Combinative distance-based Assessment), which starts from the ideal-negative point and obtains the overall expression of an object through the Taxicab distance and the Euclidean distance. It uses discriminant Euclidean distance as an important scale for evaluation. When the Euclidean distances of two objects are not adjacent, the Taxicab distance is applied. The closeness of the Euclidean distance is adjusted by a threshold parameter. The Taxicab distance and Euclidean distance of l^1-norm and l^2-norm non-differential spaces are given [50]. Therefore, the algorithm is first evaluated in a l^2-norm non-differential space. When the given objects are not comparable, the l^1-norm non-differential space needs to be processed. In order to achieve the above process, each pair of objects should be compared. CODAS approach has been applied to market segmentation evaluation [51].

Due to the drawbacks of some existing PF MCDM methods [7, 26, 27, 29, 34, 45], it may be disadvantageous for DMs to obtain optimal alternative. Therefore, the objective of this article is to solve the above-mentioned shortcomings by proposing the MCDM method for processing preference information. This method can not only be sorted without complicated processes, but also without counterintuition to get the best choice.

For better discussion, the rest of the paper is as follows: In the second section, a brief review of the basic concepts of PFS is given. In the third section, we propose a new PF MCDM algorithm based on CODAS. A example is presented in the fourth section. The fifth part is the conclusion.

2 Preliminaries

Some basic definitions of PFS are shown below.

Definition 1 [3] Let X be a domain of discourse. The PFS P in X is shown as follows:

$$P = \{< x, \mu_P(x), \nu_P(x) >| \; x \in X\}, \tag{1}$$

where $\mu_P : X \rightarrow$ [0,1] signifies membership degree and $\nu_P : X \rightarrow$ [0,1] signifies non-membership degree of the element $x \in X$ to such set P, respectively. It should be fulfilled with the condition that $0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$. The hesitation degree $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}$. For short, Zhang and Xu [6] named $p = (\mu_P, \nu_P)$ as PFN.

Definition 2 [3, 6] For three PFNs $p = (\mu_p, \nu_p)$, $p_1 = (\mu_{p_1}, \nu_{p_1})$, $p_2 = (\mu_{p_2}, \nu_{p_2})$, the related operations are presented as follows:

(1) $p_1 \cup p_2 = (\max\{\mu_{p_1}, \mu_{p_2}\}, \min\{\nu_{p_1}, \nu_{p_2}\})$;
(2) $p_1 \cap p_2 = (\min\{\mu_{p_1}, \mu_{p_2}\}, \max\{\nu_{p_1}, \nu_{p_2}\})$;
(3) $p^c = (\nu_p, \mu_p)$;
(4) $p_1 \oplus p_2 = \left(\sqrt{\mu_{p_1}^2 + \mu_{p_2}^2 - \mu_{p_1}^2\mu_{p_2}^2}, \; \nu_{p_1}\nu_{p_2}\right)$;
(5) $p_1 \otimes p_2 = \left(\mu_{p_1}\mu_{p_2}, \; \sqrt{\nu_{p_1}^2 + \nu_{p_2}^2 - \nu_{p_1}^2\nu_{p_2}^2}\right)$;
(6) $\lambda p = \left(\sqrt{1 - (1 - \mu_p^2)^\lambda}, \; \nu_p^\lambda\right), \lambda > 0$;
(7) $p^\lambda = \left(\mu_p^\lambda, \; \sqrt{1 - (1 - \nu_p^2)^\lambda}\right), \lambda > 0$.

Definition 3 [6] For a PFN $p = (\mu_p, \nu_p)$, the score function of p is denoted as follows:

$$s(p) = \mu_p^2 - \nu_p^2, \tag{2}$$

where $s(p) \in [-1, 1]$.

For two PFNs p_1 and p_2,

(1) if $s(p_1) > s(p_2)$, then $p_1 \succ p_2$;
(2) if $s(p_1) = s(p_2)$, then $p_1 \sim p_2$.

3 Approach to PF MCDM Based on CODAS

For solving complicated real-life problems, we developed a PF MCDM method based on CODAS.

Table 1 The PF decision matrix $P = (p_{ij})_{m \times n}$

	C_1	C_2	$\cdots$	C_n
A_1	p_{11}	p_{12}	$\cdots$	p_{1n}
A_2	p_{21}	p_{22}	$\cdots$	p_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}

3.1 The Description Issue

In the next, we will discuss how to use the CODAS method to deal with the MCDM problem with the PF information. To illustrate this point, a brief description of existing issues will now be made.

The key to the MCDM problem of the PF information is to obtain the optimal alternative among a large number of alternatives, which are assessed by a set of criteria, where the assessed value is PFN. Then such problems can be described using mathematical symbols as follows:

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a finite set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a finite set of criteria. $w = \{w_1, w_2, \ldots, w_n\}$ is the weight vector of the criteria with $w_j \in [0, 1](j = 1, 2, \ldots, n)$ and $\sum_{j=1}^{n} w_j = 1$. The evaluation value over the alternative A_i along with the criterion C_j is denoted as PFN $p_{ij} = (\mu_{ij}, \nu_{ij})$, where μ_{ij} signifies the degree that alternative A_i agrees the criterion C_j, and ν_{ij} signifies the degree that the alternative A_i disagrees the criterion C_j. Hence, the MCDM issue can be denoted in PF decision matrix $P = (p_{ij})_{m \times n}$, which is shown in Table 1.

3.2 PF MCDM Method Based on CODAS

In order to solve the MCDM problem based on PF information, an improved PF-CODAS method is proposed. This algorithm is a new and efficient algorithm developed by Ghorabaee et al. [49]. The optimal choice is calculated through two indistinguishable spaces (l^1-norm and l^2). The combined model of Euclidean distance and taxi distance is used to calculate the evaluation scores of the alternatives based on the above non-difference space. However, some existing Euclidean distances and taxi distances depend on brittle or vague environments and fail to solve the problem of ambiguity in the Pythagorean theorem. To solve this problem, we use fuzzy-weighted Hamming distance and fuzzy-weighted Euclidean distance instead of brittle distance [49]. The focus of the research is to propose an improved PF-CODAS method.

Firstly, because of the existence of benefit standard and cost standard, the evaluation information is standardized. Such criteria react in reverse, that is, larger values indicate better performance for the benefit criteria but poorer performance for the cost criteria. Therefore, in order to ensure that all standards are compatible, we continue to convert the cost criteria into the revenue criteria through the following formula.

$$p'_{ij} = (\mu'_{ij}, \nu'_{ij}) = \begin{cases} (\mu_{ij}, \nu_{ij}), & C_j \text{ is benefit criterion,} \\ (\nu_{ij}, \mu_{ij}), & C_j \text{ is cost criterion.} \end{cases} \tag{3}$$

According to Eq. (3), we have the standard PF decision matrix $P' = (p'_{ij})_{m\times n}$.

Then, we calculate score function $t_{ij}(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ of p'_{ij} by Eq. (4),

$$t_{ij} = {\mu'_{ij}}^2 - {\nu'_{ij}}^2. \tag{4}$$

In order to calculate the weighted standard decision matrix $R = (r_{ij})_{m\times n}$, we use weighted normalized values to define it.

$$r_{ij} = w_j t_{ij}, \tag{5}$$

where $w_j(0 < w_j < 1)$ denotes the weight of jth criterion, and $\sum_{j=1}^{n} w_j = 1$.

The basic concept of the developed CODAS algorithm is the negative-ideal solution (NIS). Therefore, the negative-ideal solution is expressed as follows:

$$NIS = (nis_j)_{1\times n}, \tag{6}$$

$$nis_j = \min_i r_{ij}, i = 1, 2, \ldots, m. \tag{7}$$

Later, the Euclidean distance $E = (E_i)_{1\times m}$ and Taxicab distance $T = (T_i)_{1\times m}$ of alternative $A_i(i = 1, 2, \ldots, m)$ from negative-ideal solution are calculated as follows:

$$E_i = \sqrt{\sum_{j=1}^{n}(r_{ij} - nis_j)^2}, \tag{8}$$

$$T_i = \sum_{j=1}^{n} | r_{ij} - nis_j | . \tag{9}$$

According to the Euclidean distance and Taxicab distance, the relative assessment (RA) matrix is constructed in the following. Based on Euclid distance and taxi distance, the relative assessment (RA) matrix is established.

$$RA = (ra_{ik})_{m\times m}, \tag{10}$$

$$ra_{ik} = (E_i - E_k) + (\psi(E_i - E_k) \times (T_i - T_k)), i, k \in 1, 2, \ldots, m \tag{11}$$

where ψ represents a threshold function that identifies two different alternatives with equal Euclidean distance, as shown below.

$$\psi(x) = \begin{cases} 1, & \text{if} \quad | x | \geq \Theta, \\ 0, & \text{if} \quad | x | < \Theta. \end{cases} \tag{12}$$

In the above function, Θ is a threshold parameter, which can be determined and set by DMs. It is recommended to set this parameter between 0.01 and 0.05. When the difference between the two alternatives based on the Euclidean distance exceeds the given Θ, the taxi distance between the two alternatives will continue to be calculated. In this article, we set Θ = 0.02 for subsequent calculations.

Next, the evaluation score of i alternative A_i is as follows:

$$RA_i = \sum_{k=1}^{m} ra_{ik}. \tag{13}$$

Algorithm 1 : PF-CODAS

1: Set the Pythagorean fuzzy decision matrix $P = (P_{ij})_{m\times n} (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$.
2: Transform matrix $P = (P_{ij})_{m\times n}$ into the standard matrix $P' = (P'_{ij})_{m\times n}$ by Eq. (3).
3: Compute score matrix $T = (t_{ij})_{m\times n}$ of $P' = (P'_{ij})_{m\times n}$ by Eq. (4).
4:Determine the weighted normalized decision matrix r_{ij} by Eq. (5).
5: Calculate the NIS by Eq. (6).
6: Calculate Euclidean distance E and Taxicab distance T from the NIS by Eqs. (8) and (9).
7: Determine the matrix RA by Eq. (10).
8: Calculate the assessment score of ith alternative RA_i by Eq. (13).
9: Rank the alternatives by assessment score.

Remark 1 It should be pointed out that in the Pythagorean theorem, the evaluation information of the fuzzy CODAS method is expressed by PFNs. Expressed in terms of membership and non-membership, PFN is effective for the uncertainty of DMs in MCDM issues. Moreover, PF-CODAS method is an inestimable tool to manage MCDM problems with PFNs and has a strong ability to distinguish the best alternatives and obtain the best alternatives without counter-intuitive phenomenon. However, when using Pythagorean fuzzy information to solve MCDM problems, other methods do not have such ideal characteristics.

4 An Illustrative Example

Example 1 A university hope to use a teaching software for assisting the teaching. The software company provides six diverse potential softwares $A = \{A_i \mid (i = 1, 2, 3, 4, 5, 6)\}$, which maybe chose by the teachers. Assume that four criteria C_1 (operational), C_2 (functional), C_3 (security), and C_4 (economic), the weight vector of the criteria $C_j (j = 1, 2, 3, 4)$ is $w = (0.2, 0.4, 0.3, 0.1)$. Moreover, the C_1, C_2, and C_3 are benefit criteria, C_4 is cost criterion. Suppose that the TES A_i along with the criterion C_j is denoted by the PF matrix $P = (p_{ij})_{6\times4}$, which is presented in Table 2.

As we have discussed above, the PF-CODAS method is valid in dealing such MCDM issues. Next, we will take merit of the proposed method to model the MCDM process.

Step 1. Construct the PF decision matrix $P = (P_{ij})_{6\times4}$, shown in Table 2.

Step 2. Shift PF decision matrix $P = (P_{ij})_{6\times4}$ into a normalized matrix $P' = (P'_{ij})_{6\times4}$ by Eq. (3), presented in Table 3.

Step 3. Calculate the score matrix $T = (t_{ij})_{6\times4}$ of $P' = (P'_{ij})_{6\times4}$ by Eq. (4), shown as follows:

Table 2 The PF decision matrix $P = (p_{ij})_{6\times4}$

	C_1	C_2	C_3	C_4
A_1	(0.6, 0.2)	(0.6, 0.1)	(0.6, 0.4)	(0.7, 0.1)
A_2	(0.6, 0.2)	(0.7, 0.2)	(0.7, 0.2)	(0.4, 0.5)
A_3	(0.4, 0.4)	(0.6, 0.2)	(0.5, 0.7)	(0.3, 0.3)
A_4	(0.3, 0.4)	(0.7, 0.3)	(0.8, 0.2)	(0.5, 0.3)
A_5	(0.3, 0.2)	(0.6, 0.3)	(0.6, 0.2)	(0.4, 0.2)
A_6	(0.2, 0.3)	(0.6, 0.2)	(0.5, 0.2)	(0.5, 0.2)

Table 3 The normalized PF decision matrix P'

	C_1	C_2	C_3	C_4
A_1	(0.6, 0.2)	(0.6, 0.1)	(0.6, 0.4)	(0.1, 0.7)
A_2	(0.6, 0.2)	(0.7, 0.2)	(0.7, 0.2)	(0.5, 0.4)
A_3	(0.4, 0.4)	(0.6, 0.2)	(0.5, 0.7)	(0.3, 0.3)
A_4	(0.3, 0.4)	(0.7, 0.3)	(0.8, 0.2)	(0.3, 0.5)
A_5	(0.3, 0.2)	(0.6, 0.3)	(0.6, 0.2)	(0.2, 0.4)
A_6	(0.2, 0.3)	(0.6, 0.2)	(0.5, 0.2)	(0.2, 0.5)

$$T = \begin{pmatrix} 0.5120 & 0.5705 & 0.2960 & -0.7200 \\ 0.5120 & 0.6615 & 0.6615 & 0.1431 \\ 0.0000 & 0.5120 & -0.3024 & 0.0000 \\ -0.1225 & 0.5680 & 0.7920 & -0.2656 \\ 0.0935 & 0.4185 & 0.5120 & -0.2160 \\ -0.0935 & 0.5120 & 0.3591 & -0.3591 \end{pmatrix}.$$

Step 4. Compute the r_{ij} by Eq. (5), shown as follows:

$$R = \begin{pmatrix} 0.1024 & 0.2282 & 0.0888 & -0.0720 \\ 0.1024 & 0.2646 & 0.1984 & 0.0143 \\ 0.0000 & 0.2048 & -0.0907 & 0.0000 \\ -0.0245 & 0.2272 & 0.2376 & -0.0266 \\ 0.0187 & 0.1674 & 0.1536 & -0.0216 \\ -0.0187 & 0.2048 & 0.1077 & -0.0359 \end{pmatrix}.$$

Step 5. Compute the NIS by Eq. (6), shown as follows: $NIS = (-0.0245, 0.1674, -0.0907, -0.0720)$.

Step 6. Determine the Euclidean distance E and Taxicab distance T from the NIS by Eqs. (8) and (9), shown as follows: $E = (0.2281, 0.3415, 0.0848, 0.3368, 0.2532, 0.2052)$, $T = (0.3672, 0.5996, 0.1339, 0.4336, 0.3379, 0.2777)$.

Step 7. Construct the relative evaluation matrix RA by Eq. (10), shown as follows:

$$RA = \begin{pmatrix} 0.0000 & -0.2970 & 0.2782 & -0.1277 & -0.0121 & 0.0099 \\ 0.2970 & 0.0000 & 0.5751 & 0.0032 & 0.3142 & 0.3963 \\ -0.2782 & -0.5751 & 0.0000 & -0.4059 & -0.2609 & -0.1788 \\ 0.1277 & -0.0032 & 0.4059 & 0.0000 & 0.1450 & 0.2271 \\ 0.0121 & -0.3142 & 0.2609 & -0.1450 & 0.0000 & 0.0822 \\ -0.0099 & -0.3963 & 0.1788 & -0.2271 & -0.0822 & 0.0000 \end{pmatrix}.$$

Step 8. Compute the assessment score of ith teaching software RA_i by Eq. (13), presented as follows:

$$RA_1 = -0.1487, RA_2 = 1.5858, RA_3 = -1.6989,$$
$$RA_4 = 0.9026, RA_5 = -0.1040, RA_6 = -0.5368.$$

Step 9. The ordering of the TES is $A_2 \succ A_4 \succ A_5 \succ A_1 \succ A_6 \succ A_3$. Hence, A_2 is optimal teaching software.

Compared with the most advanced CODAS methods [49–51], the main advantage of this method is that it can not only process PF decision information, but also obtain the best alternative from counter-intuitive phenomena and has a strong ability.

Moreover, if we take a comparison with some PF decision-making methods (employing Examples 2 and 3 in Ref. [23]), we will find that the optimal alternative will result in major difference (counter-intuitive phenomena) due to the drawback of aggregation operators [24–27, 29, 32–35, 37, 38, 40]. Meanwhile, we will find that the proposed method has a strong ability [24–27, 29, 34, 35, 37, 38, 40] in obtaining a credible and discrepant ordering when decision-makers need.

5 Conclusion

CODAS is a very effective method for dealing with complex MCDM problems. It uses many criteria to evaluate many alternatives. In this paper, we developed PF MCDM method based on CODAS. Compared with the most advanced CODAS methods [49–51], the main advantage of this method is that it can not only process PF decision information, but also obtain the best alternative from counter-intuitive phenomena and has a strong ability. In the future, we will take other uncertain tool with CODAS for dealing more complex and special domain issues [52–60].

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A Novel Pythagorean Fuzzy MULTIMOORA Applied to the Evaluation of Energy Storage Technologies

Iman Mohamad Sharaf

1 Introduction

Nowadays, the decision-making process in organizations confronts many challenges due to the extensive changes and increasing complexity of issues faced in a rapidly developing business environment. In recent decades, researchers introduced several methods for multi-criteria decision-making (MCDM). These methods deal with the complexities faced in decision-making and facilitate this process. They also increase efficiency and improve the quality of the process [11].

Real-life decision-making problems are mainly composed of imprecise, and uncertain data together with the subjective evaluations of the decision-makers (DMs) which is based on their perceptions which is sure to differ from one person to another. To face the unrealistic supposition that exact numerical values are proper to model and handle real-life decision-making problems, Zadeh [65] introduced the notion of fuzzy sets, later named type-1 fuzzy sets (T1FSs).

Due to the deficiency of T1FSs to express the uncertainties and cognitive limitations in decision-making, different types of fuzzy sets were proposed by various researchers to model diverse situations of vagueness and ambiguity. For example, type-2 fuzzy sets (T2FSs) [66], neutrosophic sets (NFSs) [53], hesitant fuzzy sets (HFSs) [56, 57], and spherical fuzzy sets (SFSs) [26].

Atanassov [2] defined intuitionistic fuzzy sets (IFSs) as a more general form of fuzzy sets by adding the non-membership degree under the constraint that the sum of the membership degree and the non-membership degree is less than or equal to one. The indeterminacy degree is treated as a residual term such that the sum of the three degrees is equal to one. Although IFSs are proficient at imprecise treatment and inexact data, still, there are certain difficulties IFSs cannot handle. Atanassov [3]

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_12

pointed out the possibility of changing the condition on the sum of the membership degree and the non-membership degree by increasing the power.

Yager and Abbasov [63] and Yager [62] discussed the model of Pythagorean fuzzy sets (PFSs) for a more human consistent reasoning under imperfect and imprecise data. In PFSs, the sum of the squares of the membership degree and the non-membership degree is less than or equal to one, and still, the indeterminacy degree is the square root of the residual term. The space of all PFSs includes IFSs. Thus, PFSs can be used more widely than IFSs in handling practical problems with imprecision and uncertainty.

Brauers and Zavadskas [5] proposed the MULTIMOORA (Multi-Objective Optimization by Ratio Analysis plus full multiplicative form) for MCDM. The MULTIMOORA method proved to be one of the most practical MCDM methods that have been applied in solving complex decision-making problems. It is a completely effective method for the evaluation and ranking of alternatives without subjective orientation in various phenomena [11]. It satisfies all the necessary robustness' conditions proposed by Brauers and Zavadskas [5] to become the most robust system of multi-objective optimization under the condition of support from the Ameliorated Nominal Group Technique and Delphi [5]. Due to its robustness and flexibility, MULTIMOORA was extended using different types of fuzzy sets and was applied to various practical fields. Throughout its application, it provided high efficiency and effectiveness in problem-solving [11].

The output of the MULTIMOORA results from ternary ranking techniques: ratio analysis, reference point theory, and full multiplicative form. Based on the scores of the three techniques, the alternatives are individually ranked in each technique. Then, using these individual rankings the rules of the dominance theory are applied to find the final ranking. However, the dominance theory has some limitations, e.g., multiple comparisons and circular reasoning [11].

As the ranking process using the dominance theory can be in some cases complex and challenging, it is not preferred in large-scale applications and the scores of the three techniques are aggregated instead [11, 68].

Up till now, when applying the reference point approach in the MULTIMOORA method, only the best solution is taken into consideration. Utilizing the two reference points, i.e., the best and worst solutions, was not previously studied.

When extended in the intuitionistic and Pythagorean fuzzy environment, the IF-MULTIMOORA and the PF-MULTIMOORA might have two main drawbacks. In the ratio analysis technique which exploits the additive utility, the weighted averaging operators play the main role. Most of these aggregation operators have a flaw that might result in a biased treatment and false ranking. For an alternative, a single criterion with the perfect rating (1, 0) will dominate regardless of its weight and abolish the effect of all the other criteria, which is not fair in the assessment process. Similarly, for the full multiplicative form in which the weighted geometric operators play the main role a single criterion with the worst rating (0, 1) will dominate regardless of its weight and abolish the effect of all the other criteria for an alternative, which is also not fair in an evaluation process.

In the last few decades, energy consumption increased and additional energy supplies are needed to balance the increasing demand. It was found that renewable energies are the best approach for the provision of energy due to their sustainable nature and broad utilization due to their diverse presence such as wind, solar, geothermal, bioenergy and hydropower. Yet, renewable sources usually cannot stand alone in power plants because of their intermittent nature and significant fluctuations, e.g., wind and solar energies. Energy storage technologies (ESTs) can solve this problem when coupled with renewable energy resources. ESTs improve the system's performance and increase the penetration of renewable energy sources. ESTs are continuously developed and different storage systems are established due to the multiple utilization of energy and the different types of applications [42]. The choice of a suitable EST is an MCDM problem since multiple technologies are defined for multiple conflicting criteria.

This chapter develops a new version MULTIMOORA in the Pythagorean fuzzy environment that eliminates the shortcomings of the previous versions. So far, when using the aggregation approach the result of each utility function is defuzzified using the score function before the aggregation for ranking. This can be mainly attributed to using the reference point approach which relies on the distance from the ideal situation that is always employed as a crisp value. In fact, being a distance between two fuzzy values it cannot be definitely determined. Hence, it is more appropriate to define distance on a fuzzy basis rather than a crisp basis. Consequently, this study will adopt the aggregation approach in which distances are utilized on a fuzzy basis. As a result, defuzzification is employed only in the final step for ranking. At this point, the accuracy function is also employed with the score function to make the comparison more discriminatory and to overcome any drawbacks that may be associated with the defuzzification using the score function solely. In addition, newly proposed aggregation operators are exploited. These operators guarantee fair treatment among the evaluation criteria since most of the aggregation operators have a flaw that might result in a biased treatment and false ranking in certain situations. A practical example that considers the evaluation of energy storage technologies is provided to illustrate the developed method and to make a comparative analysis.

From the previous discussion, the contribution of the study encompasses two main features. In the reference point approach, fuzzy distance is employed. This allows examining two reference points instead of one. The study also exploits aggregation operators that make the decision results more precise and exact.

The chapter is organized as follows. In Sect. 2, the literature is reviewed. Section 3 includes the basic concepts, definitions, and operators of PFSs together with the conventional MULTIMOORA. Section 4 explains the proposed PF-MULTIMOORA in detail. In Sect. 5, a practical example in the evaluation of energy storage technologies is provided to illustrate the newly developed method. Finally, the conclusion is given in Sect. 6.

2 Review of the Literature

2.1 The MULTIMOORA Method

Brauers and Zavadskas [4] proposed the MOORA method that combines two techniques, the ratio analysis, and the reference point methods. In the ratio analysis, the rating of each alternative to a criterion is compared to a denominator which is representative of all the alternatives concerning that criterion. While the reference point method measures the distance between the rating of an alternative for a criterion and a reference point. This reference point has the highest rating for maximization and the lowest for minimization. They applied the method to optimize privatization processes, especially for transition economies.

Brauers and Zavadskas [5] clarified that using two different methods of multi-objective optimization is more robust than using a single method. Also, using three methods is more robust than using two methods. Accordingly, they proposed the MULTIMOORA which is composed of the MOORA method and the full multiplicative form. In the full multiplicative form, the utility function is the multiplication the ratings of an alternative for the evaluation criteria. They applied the method for project management in a transition economy.

Since the introduction of MULTIMOORA, it has been applied in various practical sectors including industries, economics, civil services, and environmental policy-making, healthcare management, and information and communications technologies [28]. Several versions were also developed to handle uncertainty using different types of data. For a comprehensive review of the MULTIMOORA different versions applied to diverse practical problems until 2018, the reader is referred to Hafezalkotob et al. [28]. The most recent articles about MULTIMOORA are summarized by the type of data used as follows.

Using crisp data, Asante et al. [1] integrated MULTIMOORA with the Evaluation based on Distance from Average Solution (EDAS) method for the evaluation of renewable energy barriers in developing countries. Dizdar and Ünver [14] applied MULTIMOORA method for the assessment procedure of occupational safety and health based on the counts of occupational accidents and diseases. Fedajev et al. [17] used MULTIMOORA and the Shannon Entropy Index to rank and classify the European Union (EU) countries according to the progress achieved in the implementation of the "Europe 2020" strategy. Omrani et al. [45] proposed a new approach based on the Best–Worst Method (BWM) and MULTIMOORA methods to calculate semi-human development index (HDI). HDI is a useful tool for policymakers to understand the degree of development in their societies and establish new policies to improve it. Souzangarzadeh et al. [54] used the response surface methodology (RSM) D-optimal Design along with MULTIMOORA to find the optimum design of segmented tubes as energy absorbers in terms of various vehicles collision scenarios. Yörükoğlu and Aydın [64] applied MULTIMOORA for wind turbine selection problem according to qualitative and quantitative criteria.

As for type-1 fuzzy sets, Chen et al. [8] proposed an extended MULTIMOORA method using the ordered weighted geometric averaging (OWGA) operator and Choquet integral for failure mode and effects analysis (FMEA). Dai et al. [12] proposed a novel MULTIMOORA into the triangular fuzzy environment in which the period weights and attribute weights are completely unknown. Rahimi et al. [49] introduced a framework comprising Geographic Information System (GIS) techniques and fuzzy MCDM methods to select sustainable landfill site. The criteria weights were obtained using the fuzzy BWM. The suitability maps were generated based on the GIS analysis. The selected sites were then analyzed and ranked using the MULTIMOORA method. Tavana et al. [55] proposed an integrated approach for supply chain risk–benefit assessment and supplier selection that combines the fuzzy analytic hierarchy process (AHP) and the fuzzy MULTIMOORA. Dahooie et al. [11] applied an objective weight determination method called CCSD (Correlation Coefficient and Standard Deviation) to eliminate the limitations of the dominance theory and increase the robustness of the MULTIMOORA and enhance its performance by considering the importance level of the three different techniques (i.e., ratio system, reference point, and full multiplicative form).

In the context of probabilistic linguistic information (PLI), Chen et al. [7] proposed a MULTIMOORA and introduced an innovative two-step comparative method to evaluate cloud-based enterprise resource planning (ERP) systems. Liu and Li [39] established the prospect theory-based MULTIMOORA method. They used the probabilistic linguistic terms set (PLTS) to describe qualitative information not only to provide every possible evaluation value but also to give the weight of these values.

MULTIMOORA was also extended using the hesitant fuzzy linguistic term set (HFLTS). Liang et al. [34] designed a MULTIMOORA method using a dual hesitant fuzzy extended Bonferroni mean (DHFEBM) to select a renewable energy technology. Liao et al. [37] improved the MULTIMOORA method by integrating with the ORESTE (organísation, rangement et Synthèse de données relationnelles) method, and extended the method to the unbalanced hesitant fuzzy linguistic context based on an introduced score function to eliminate the defects of the subscript-based operations on HFLTSs.

Regarding picture fuzzy numbers (PFNs), Lin et al. [38] proposed a novel picture fuzzy MULTIMOORA to solve the site selection problem for car-sharing stations based on a novel score function and Borda rule.

In the intuitionistic fuzzy environment, Luo et al. [40] developed a distance-based IF-MULTIMOORA method integrating a novel weight-determining method to select medical equipment. Zhang et al. [68] proposed an IF-MULTIMOORA method for MCDM that involves information fusion to allow processing both crisp and fuzzy information.

On the subject of PFSs, Li et al. [33] proposed a new MULTIMOORA method to evaluate the passenger satisfaction level of the public transportation system under a large group environment. Xian et al. [60] developed a MULTIMOORA method using interval 2-tuple Pythagorean fuzzy linguistic sets to evaluate financial management performance in universities. Liang et al. [35] presented a MULTIMOORA method with interval-valued Pythagorean fuzzy sets (IVPFSs) to solve the selection problem of hospital open-source electronic health records (EHRs) systems for MedLab in Ghana.

Concerning the neutrosophic fuzzy environment, Liang et al. [36] proposed a MULTIMOORA approach based on linguistic neutrosophic numbers (LNNs) and applied this new approach to select the optimal mining method. Based on a neutrosophic MULTIMOORA technique, Siksnelyte et al. [52] presented an original framework for sustainable energy development indicators. They analyzed the trends of energy development across the Baltic Sea Region (BSR) countries from 2008 to 2015.

Lately, Gündoğdu [25] developed a MULTIMOORA method using spherical fuzzy sets (SFSs) to increase its efficiency at solving complex problems that require evaluation and estimation under unreliable data environment.

2.2 Pythagorean Fuzzy Operators

Various scholars have paid attention to MCDM problems under PF environment. To develop effective and efficient methods to solve these problems, PF operational laws and aggregation operators are crucial. Yager [61] developed the Pythagorean fuzzy weighted averaging (PFWA) operator and Pythagorean fuzzy weighted geometric (PFWG) operator to handle multiple attribute decision making (MADM) problems. Ma and Xu [41] defined some novel PFWG and PFWA operators for PF information which can treat the membership degree and the non-membership degree neutrally. Garg [19, 20] developed some generalized PF Einstein weighted and ordered weighted averaging operators. Zhang [69] presented a new PFWA operator and PF-ordered weighted averaging (PFOWA) operator to aggregate PFSs. Garg [18] presented the averaging and the geometric aggregation operators under the interval-valued PF environment. Peng and Yang [47] developed a PF Choquet integral operator for multiple attribute group decision-making (MAGDM) problems. Wei and Lu [59] proposed some PF Maclaurin symmetric mean operators for MADM. Garg [19] defined two new exponential operational laws for interval-valued Pythagorean fuzzy sets (IVPFSs) and their corresponding aggregation operators. Garg [22] developed some new logarithm operational laws (LOL) with a real number base for PFSs. Based on the properties of these LOL, various weighted averaging and geometric operators were developed and a decision-making method was introduced under PF information using the proposed operators. Garg [23] developed some new operational laws and their corresponding PFWGA operators by including the feature of the probability sum and the interaction coefficient into the analysis to get a neutral or a fair treatment to the membership and non-membership functions of PFSs. Later, Garg [24] defined some new PF weighted, ordered weighted, and hybrid neutral averaging aggregation operators for PF information. He utilized these operators that can neutrally treat the membership and non-membership degrees to present an algorithm to solve the MAGDM problems under the PF environment. Wang et al. [58] presented some PF interactive Hamacher power aggregation operators such as PF interactive

Hamacher power average, weighted average, ordered weighted average, PF interactive Hamacher power geometric, weighted geometric, and ordered geometric operators, respectively. In addition, they defined a PF entropy measure and established a method to determine the attribute weights. They explored a novel approach for MADM problems and assessed express service quality.

Shahzadi et al. [51] introduced six families of aggregation operators namely, PF Yager weighted averaging aggregation, PF Yager ordered weighted averaging aggregation, PF Yager hybrid weighted averaging aggregation, PF Yager weighted geometric aggregation, PF Yager ordered weighted geometric aggregation and PF Yager hybrid weighted geometric aggregation. These operators inherit the operational advantages of Yager parametric families.

2.3 Energy Storage Technologies

With the development of renewable energy, energy storage is becoming increasingly important; hence, finding and implementing cost-effective and sustainable energy storage and conversion systems is vital [13].

ESTs not only store the excess of energy but also increase renewable energy penetration and decrease its limitations as a power plant cannot depend solely on a renewable energy source without EST. This results in decreasing fuel consumption and CO_2 emissions. ESTs balance between the energy supply and demand while reducing renewable energy fluctuations due to its intermittent nature. They improve the overall efficiency of a power plant, thus reducing the operating cost in the long run. They also reduce the peak energy loads which will, in turn, decrease the risk of load shedding especially when the large capacity of storage is considered. The flexibility of ESTs makes it convenient and suitable to cover distant areas that suffer from the lack of electricity [42]. The major problem with ESTs is their investment cost and operational cost that should be within acceptable limits. Finding the possible low cost, efficient, and long term ESTs that don't harm the environment is a subject of extensive research [30].

The only way of storing electrical energy is by converting it to other forms of energy such as thermal energy, chemical energy, electrochemical energy, mechanical energy, and electromagnetic energy [31].

Thermal energy storage (TES): it is a technology that stores thermal energy by heating or cooling a storage medium and then utilizes this stored energy when needed. The stored energy can be used at a later time for heating and cooling applications and power generation [50]. Power is generated from this stored energy by applying a Rankine cycle turbine with the system. TES systems are applicable in diverse industrial and residential purposes, e.g., space heating or cooling, process heating and cooling, hot water production, and electricity generation. TES can be classified into three types: latent heat, sensible heat, and thermochemical heat storage. Sensible heat storage stores heat energy in any material depending on its heat capacity and the change of the material's temperature during the process of charging and discharging.

The main advantage of this type is that charging and discharging is completely reversible and have unlimited life cycles [30]. Latent heat storage is based on the amount of heat released or absorbed during the phase change of any material. Heat is stored in phase change materials which could be both organics and inorganics and can change their phase with varying heat [31]. The materials for storing the heat can be both liquid and solid. They are successfully integrated with solar energy systems [68].

Chemical energy storage (CES): Chemical energy is stored in the chemical bonds of atoms and molecules that can be only observed when released in a chemical reaction in the form of heat. After the release of chemical energy, the substance is often changed into a completely different substance. Chemical fuels are the dominant form of energy storage regarding electrical generation and energy transportation. Chemical energy storage is suitable for storing large amounts of energy and for longer durations [27]. CES includes hydrogen storage and biofuels [30]. Biofuels are produced by a biological process instead of geological processes. Biomass is an organic matter derived from the biodegradable fraction of energy crops, the waste matter of plants and animals. This biomass is used to produce biogas which can be converted through a generator to electricity. Biofuels include ethanol, biodiesel, bio-alcohols, bio-ether, green diesel, biofuel gasoline, vegetable oils, syngas, and solid biomass [30]. Hydrogen energy storage technology is one of the most prominent types. The process involves two steps: producing and storing hydrogen when there is excess power available, and then producing electricity from the stored hydrogen using fuel cells in case of power shortage [31].

Electrochemical energy storage (EcES): this storage technology converts electric energy into chemical energy and vice versa during energy storage and recovery. There are two main branches of EcES: electrochemical batteries and electrochemical capacitors. The type of the EcES differs according to the nature of the chemical reaction, structural features, and design. Electrochemical cells and batteries are classified according to three features. The first classification depends on the operation principle and contains 4 categories; primary cell or battery, secondary cell or battery, reserve cell, and fuel cell. In the primary batteries, the chemical once consumed cannot be recharged, while the secondary batteries can be charged and discharged many times. In power system applications, only secondary batteries are utilized [31]. The second classification is based on discharge depth, either shallow or deep cycle batteries. Deep cycle batteries are suitable for renewable applications. The third classification depends on the characteristic of the electrolyte in the battery, either flooded or wet and sealed. Flooded or wet batteries are vastly utilized in renewable applications [27]. EcES plays a vital role in our daily life since they are applied in small devices, e.g., laptops, tablets, and cell phones, and in larger devices, e.g., electric cars, to provide efficient and reliable use of energy. Battery energy storage is the most widespread storage method. It is available in different sizes ranging from tens of watts to megawatts [31]. Batteries have two main disadvantages. First, the long charging time since they have an intrinsically low power handling capability (<1 kW/kg, normalized by the device mass). Second, the short device cycle life. The specific power of modern batteries has been increased; yet, the cost of these advanced batteries is high, and they still do not fulfill the power demands of many applications, e.g., electric vehicles [32].

Electrical energy storage (EES): in this technology electrical energy is converted from a power network or source via an energy conversion module into another energy storage medium. This intermediate energy is stored for a limited time, then converted back into electrical energy when needed [43]. ESS is most suitable for any specific application in power systems. EES includes capacitors, supercapacitors, and superconducting magnetic energy storage (SMES). In supercapacitors the electric energy is stored in the form of the electrostatic field created in-between the two porous electrodes, separated by a separator. On the other hand, SMES stores electric energy in the electromagnetic field generated by a current following through a superconducting conductor [31]. The capacitors can be used for high currents, but for extremely short periods due to their relatively low capacitance generation. A Supercapacitor can replace a regular capacitor, but it offers very high capacitance in a small package. Superconducting magnetic energy storage systems are preferred on the outlet of power plants to stabilize the output or on industrial sites to accommodate peaks in energy consumption [27].

Mechanical energy storage (MES): this technology takes advantage of kinetic or gravitational forces to store energy. MES is easily adaptable to convert and store energy from water current, wave, and tidal sources [27]. Mechanical energy storage offers several advantages compared to other ESTs especially in terms of environmental impact, cost, and sustainability. In MES the energy is stored by doing some mechanical work, and then energy from mechanical work is exploited upon its requirement [30]. MES can be found in two forms according to the utilization of stored energy. The first form is pure mechanical if the system is directly used. The second form is mechanical–electrical when energy is transmitted via an electric motor-generator. The pure mechanical systems can provide mechanical work such as smoothing the rotation of a rotating mass; mechanical–electric systems are used to supply the grid with electricity. MES is classified by the working principle as follows: pressurized gas, forced springs, kinetic energy, and potential energy. The main types of MES are pumped hydroelectric storage (PHS), compressed air energy storage (CAES), and flywheel energy storage (FES) [42].

The wide range of ESTs, with each EST being different in terms of the scale of power, response time, energy/power density, discharge duration, and cost coupled with the complex characteristics matrices, makes it difficult to choose a particular EST for a specific application.

3 Preliminaries

3.1 Pythagorean Fuzzy Sets

In this section, the basic definitions, operations, and aggregation operators of PFSs are reviewed.

Definition 1 ([63]). A Pythagorean fuzzy set $\tilde{A}$ in a finite universe of discourse X is defined by

$$\tilde{A} = \left\{x, \mu_{\tilde{A}}(x), \upsilon_{\tilde{A}}(x) : x \in X\right\}, \tag{1}$$

where $\mu_{\tilde{A}}(x) : X \to [0, 1]$ denotes the membership degree,
$\upsilon_{\tilde{A}}(x) : X \to [0, 1]$ denotes the non-membership degree,
satisfying the constraint

$$0 \le \mu_{\tilde{A}}^2(x) + \upsilon_{\tilde{A}}^2(x) \le 1. \tag{2}$$

The hesitation margin, i.e., the degree of uncertainty, is represented by

$$\pi_{\tilde{A}}(x) = \sqrt{1 - \left(\mu_{\tilde{A}}^2(x) + \upsilon_{\tilde{A}}^2(x)\right)}. \tag{3}$$

Definition 2 ([62]). For the PFSs $\left\{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right\}$ having weights $(w_1, w_2, \ldots, w_n)$, with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, the Pythagorean fuzzy weighted averaging operator (PFWA_Y) and the Pythagorean fuzzy weighted geometric operator (PFWG_Y) are defined as follows:

(i)

$$\mathrm{PFWA}_Y = \left(\sum_{i=1}^{n} w_i \mu_{\tilde{A}_i}, \sum_{i=1}^{n} w_i \upsilon_{\tilde{A}_i}\right), \tag{4}$$

(ii)

$$\mathrm{PFWG}_Y = \left(\prod_{i=1}^{n} \mu_{\tilde{A}_i}^{w_i}, \prod_{i=1}^{n} \upsilon_{\tilde{A}_i}^{w_i}\right). \tag{5}$$

Definition 3 ([41]). For any two PFSs $\tilde{A} = \left(\mu_{\tilde{A}}, \upsilon_{\tilde{A}}\right)$ and $\tilde{B} = \left(\mu_{\tilde{B}}, \upsilon_{\tilde{B}}\right)$ the operational laws are given by

(i)

$$\tilde{A} \oplus \tilde{B} = \left(\sqrt{\mu_{\tilde{A}}^2 + \mu_{\tilde{B}}^2 - \mu_{\tilde{A}}^2\mu_{\tilde{B}}^2}, \upsilon_{\tilde{A}}\upsilon_{\tilde{B}}\right), \tag{6}$$

(ii)

$$\tilde{A} \otimes \tilde{B} = \left(\mu_{\tilde{A}}\mu_{\tilde{B}}, \sqrt{\upsilon_{\tilde{A}}^2 + \upsilon_{\tilde{B}}^2 - \upsilon_{\tilde{A}}^2\upsilon_{\tilde{B}}^2}\right), \tag{7}$$

(iii)

$$\lambda \odot \tilde{A} = \left(\sqrt{1 - \left(1 - \mu_{\tilde{A}}^2\right)^{\lambda}}, \upsilon_{\tilde{A}}^{\lambda}\right), \tag{8}$$

(iv)

$$\tilde{A}^{\lambda} = \left(\mu_{\tilde{A}}^{\lambda}, \sqrt{1 - \left(1 - \upsilon_{\tilde{A}}^2\right)^{\lambda}}\right), \text{ where } \lambda > 0 \text{ is a scalar.} \tag{9}$$

Based on the operational laws given in Definition 3, the PFWA_{MX} and the PFWG_{MX} are defined as follows:

Definition 4 ([41]). Consider the PFSs $\left\{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right\}$ with weights $(w_1, w_2, \ldots, w_n)$, where $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, the PFWA_{MX} and the PFWG_{MX} are defined as follows.

(i)

$$\begin{aligned}\mathrm{PFWA}_{MX}\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) &= \left(w_1 \odot \tilde{A}_1\right) \oplus \left(w_2 \odot \tilde{A}_2\right) \oplus \ldots \oplus \left(w_n \odot \tilde{A}_n\right) \\ &= \left(\left[1 - \prod_{i=1}^{n}\left(1 - \mu_{\tilde{A}_i}^2\right)^{w_i}\right]^{\frac{1}{2}}, \prod_{i=1}^{n} \upsilon_{\tilde{A}_i}^{w_i}\right),\end{aligned} \tag{10}$$

(ii)

$$\begin{aligned}\mathrm{PFWG}_{MX}\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) &= \tilde{A}_1^{w_1} \otimes \tilde{A}_2^{w_2} \otimes \ldots \otimes \tilde{A}_n^{w_n}. \\ &= \left(\prod_{i=1}^{n} \mu_{\tilde{A}_i}^{w_i}, \left[1 - \prod_{i=1}^{n}\left(1 - \upsilon_{\tilde{A}_i}^2\right)^{w_i}\right]^{\frac{1}{2}}\right)\end{aligned} \tag{11}$$

Definition 5 ([51]). For any two PFSs $\tilde{A} = \left(\mu_{\tilde{A}}, \upsilon_{\tilde{A}}\right)$ and $\tilde{B} = \left(\mu_{\tilde{B}}, \upsilon_{\tilde{B}}\right)$, $\theta > 0$ and $\lambda > 0$, Yager's t-norm and t-conorm operations are defined as follows:

(i)

$$\tilde{A} \boxplus \tilde{B} = \left(\sqrt{\min\left(1, \left(\mu_{\tilde{A}}^{2\theta} + \mu_{\tilde{B}}^{2\theta}\right)^{\frac{1}{\theta}}\right)}, \sqrt{1 - \min\left(1, \left(\left(1 - \upsilon_{\tilde{A}}^{2}\right)^{\theta} + \left(1 - \upsilon_{\tilde{B}}^{2}\right)^{\theta}\right)^{\frac{1}{\theta}}\right)} \right), \tag{12}$$

(ii)

$$\tilde{A} \boxtimes \tilde{B} = \left(\sqrt{1 - \min\left(1, \left(\left(1 - \mu_{\tilde{A}}^{2}\right)^{\theta} + \left(1 - \mu_{\tilde{B}}^{2}\right)^{\theta}\right)^{\frac{1}{\theta}}\right)}, \sqrt{\min\left(1, \left(\upsilon_{\tilde{A}}^{2\theta} + \upsilon_{\tilde{B}}^{2\theta}\right)^{\frac{1}{\theta}}\right)} \right), \tag{13}$$

(iii)

$$\lambda \boxdot \tilde{A} = \left(\sqrt{\min\left(1, \left(\lambda \mu_{\tilde{A}}^{2\theta}\right)^{\frac{1}{\theta}}\right)}, \sqrt{1 - \min\left(1, \left(\lambda\left(1 - \upsilon_{\tilde{A}}^{2}\right)^{\theta}\right)^{\frac{1}{\theta}}\right)} \right), \tag{14}$$

(iv)

$$\tilde{A}^{\lambda} = \left(\sqrt{1 - \min\left(1, \left(\lambda\left(1 - \mu_{\tilde{A}}^{2}\right)^{\theta}\right)^{\frac{1}{\theta}}\right)}, \sqrt{\min\left(1, \left(\lambda \upsilon_{\tilde{A}}^{2\theta}\right)^{\frac{1}{\theta}}\right)} \right). \tag{15}$$

Based on the operational laws given in Definition 5, the PFYWA and the PFYWG are defined as follows.

Definition 6 ([51]). For the PFSs $\left\{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right\}$ having weights $(w_1, w_2, \ldots, w_n)$, with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, the PFYWA and the PFYWG are given as follows:

$$\begin{aligned} \mathrm{PFYWA}\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) &= \left(w_1 \boxdot \tilde{A}_1\right) \boxplus \left(w_2 \boxdot \tilde{A}_2\right) \boxplus \ldots \boxplus \left(w_n \boxdot \tilde{A}_n\right) \\ &= \left(\sqrt{\min\left(1, \left(\sum_{i=1}^{n}\left(w_i \mu_{\tilde{A}_i}^{2\theta}\right)\right)^{\frac{1}{\theta}}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{i=1}^{n}\left(w_i\left(1 - \upsilon_{\tilde{A}_i}^{2}\right)^{\theta}\right)\right)^{\frac{1}{\theta}}\right)} \right), \end{aligned} \tag{16}$$

$$\begin{aligned} &\mathrm{PFYWG}\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \tilde{A}_1^{w_1} \boxtimes \tilde{A}_2^{w_2} \boxtimes \ldots \boxtimes \tilde{A}_n^{w_n} \\ &= \left(\sqrt{1 - \min\left(1, \left(\sum_{i=1}^{n}\left(w_i\left(1 - \mu_{\tilde{A}_i}^{2}\right)^{\theta}\right)\right)^{\frac{1}{\theta}}\right)}, \sqrt{\min\left(1, \left(\sum_{i=1}^{n}\left(w_i \upsilon_{\tilde{A}_i}^{2\theta}\right)\right)^{\frac{1}{\theta}}\right)} \right). \end{aligned} \tag{17}$$

For a PFS $\tilde{A} = \left(\mu_{\tilde{A}}, \upsilon_{\tilde{A}}\right)$, Zhang and Xu [70] proposed a score function to evaluate and compare PFSs and it is defined as follows:

$$\mathcal{S}\left(\tilde{A}\right) = \mu_{\tilde{A}}^2 - \upsilon_{\tilde{A}}^2, \text{ where } \mathcal{S}\left(\tilde{A}\right) \in [-1, 1]. \tag{18}$$

In addition, Peng and Yang [46] proposed an accuracy function to help in discrimination whenever a tie occurs. The accuracy function is defined as follows:

$$\mathcal{A}\left(\tilde{A}\right) = \mu_{\tilde{A}}^2 + \upsilon_{\tilde{A}}^2, \text{ where } \mathcal{A}\left(\tilde{A}\right) \in [0, 1]. \tag{19}$$

Definition 7 ([48]). The complement or the negation pf a PFS $\tilde{A} = \left(\mu_{\tilde{A}}, \upsilon_{\tilde{A}}\right)$ is denoted by

$$\tilde{A}^c = \left(\upsilon_{\tilde{A}}, \mu_{\tilde{A}}\right). \tag{20}$$

Definition 8 ([46]). Two PFSs $\tilde{A} = \left(\mu_{\tilde{A}}, \upsilon_{\tilde{A}}\right)$ and $\tilde{B} = \left(\mu_{\tilde{B}}, \upsilon_{\tilde{B}}\right)$ are compared as follows:

(a) If $\mathcal{S}\left(\tilde{A}\right) < \mathcal{S}\left(\tilde{B}\right)$, then $\tilde{A} < \tilde{B}$;

(b) If $\mathcal{S}\left(\tilde{A}\right) > \mathcal{S}\left(\tilde{B}\right)$, then $\tilde{A} > \tilde{B}$;

(c) If $\mathcal{S}\left(\tilde{A}\right) = \mathcal{S}\left(\tilde{B}\right)$, the accuracy function is employed

  (i) If $\mathcal{A}\left(\tilde{A}\right) < \mathcal{A}\left(\tilde{B}\right)$, then $\tilde{A} < \tilde{B}$;

  (ii) If $\mathcal{A}\left(\tilde{A}\right) > \mathcal{A}\left(\tilde{B}\right)$, then $\tilde{A} > \tilde{B}$;

  (iii) If $\mathcal{A}\left(\tilde{A}\right) = \mathcal{A}\left(\tilde{B}\right)$, then $\tilde{A} \approx \tilde{B}$.

Distance and similarity measures are important topics and have been extensively used in diverse fields such as pattern recognition, machine learning, and market prediction [68]. Some common metrics, e.g., Hamming distance and Euclidean distance, are widely used to find the distance between two PFSs. Initially, the membership degree and the non-membership degree were only considered in the distance formulas. Later, these formulas were modified to include the degree of hesitation as well. The distance formulas in a Pythagorean fuzzy environment are given as follows [29].

i. Hamming distance

$$d_{Hm}\left(\tilde{A}, \tilde{B}\right) = \frac{1}{2}\sum_{i=1}^{n}\left\{\left|\mu_{\tilde{A}}^2(x_i) - \mu_{\tilde{B}}^2(x_i)\right| + \left|\upsilon_{\tilde{A}}^2(x_i) - \upsilon_{\tilde{B}}^2(x_i)\right| + \left|\pi_{\tilde{A}}^2(x_i) - \pi_{\tilde{B}}^2(x_i)\right|\right\}. \tag{21}$$

ii. Normalized Hamming distance

$$d_{NHm}\left(\tilde{A},\tilde{B}\right)=\frac{1}{2n}\sum_{i=1}^{n}\left\{\left|\mu_{\tilde{A}}^{2}(x_i)-\mu_{\tilde{B}}^{2}(x_i)\right|+\left|\upsilon_{\tilde{A}}^{2}(x_i)-\upsilon_{\tilde{B}}^{2}(x_i)\right|+\left|\pi_{\tilde{A}}^{2}(x_i)-\pi_{\tilde{B}}^{2}(x_i)\right|\right\}. \tag{22}$$

iii. Euclidean distance

$$d_{E}\left(\tilde{A},\tilde{B}\right)=\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left\{\left(\mu_{\tilde{A}}^{2}(x_i)-\mu_{\tilde{B}}^{2}(x_i)\right)^2+\left(\upsilon_{\tilde{A}}^{2}(x_i)-\upsilon_{\tilde{B}}^{2}(x_i)\right)^2+\left(\pi_{\tilde{A}}^{2}(x_i)-\pi_{\tilde{B}}^{2}(x_i)\right)^2\right\}}. \tag{23}$$

iv. Normalized Euclidean distance

$$d_{NE}\left(\tilde{A},\tilde{B}\right)=\sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left\{\left(\mu_{\tilde{A}}^{2}(x_i)-\mu_{\tilde{B}}^{2}(x_i)\right)^2+\left(\upsilon_{\tilde{A}}^{2}(x_i)-\upsilon_{\tilde{B}}^{2}(x_i)\right)^2+\left(\pi_{\tilde{A}}^{2}(x_i)-\pi_{\tilde{B}}^{2}(x_i)\right)^2\right\}}. \tag{24}$$

v. Hausdorff distance

$$d_{Hs}\left(\tilde{A},\tilde{B}\right)=\sum_{i=1}^{n}\max\left\{\left|\mu_{\tilde{A}}^{2}(x_i)-\mu_{\tilde{B}}^{2}(x_i)\right|,\left|\upsilon_{\tilde{A}}^{2}(x_i)-\upsilon_{\tilde{B}}^{2}(x_i)\right|\right\}. \tag{25}$$

vi. Normalized Hausdorff distance

$$d_{NHs}\left(\tilde{A},\tilde{B}\right)=\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|\mu_{\tilde{A}}^{2}(x_i)-\mu_{\tilde{B}}^{2}(x_i)\right|+\left|\upsilon_{\tilde{A}}^{2}(x_i)-\upsilon_{\tilde{B}}^{2}(x_i)\right|\right\}. \tag{26}$$

The concepts of distance measure and similarity measure are dual concepts. Hence, the distance between two PFSs is used to define the similarity between two PFSs.

Proposition 1 ([16]). Let $d\left(\tilde{A},\tilde{B}\right)$ be the distance between two PFSs $\tilde{A}$ and $\tilde{B}$, then the similarity measure between the two PFSs is given as

$$S\left(\tilde{A},\tilde{B}\right)=1-d\left(\tilde{A},\tilde{B}\right). \tag{27}$$

Hussian and Yang [29] defined other similarity measures based on the Hausdorff metric beside the simple linear function (27). These similarity measures use a rational function and an exponential function as follows:

$$S\left(\tilde{A},\tilde{B}\right)=\frac{1-d_{Hs}\left(\tilde{A},\tilde{B}\right)}{1+d_{Hs}\left(\tilde{A},\tilde{B}\right)},\tag{28}$$

and

$$S\left(\tilde{A},\tilde{B}\right)=\frac{e^{d_{Hs}\left(\tilde{A},\tilde{B}\right)}-e^{-1}}{1-e^{-1}}.\tag{29}$$

3.2 The Classical MULTIMOORA Method

The MULTIMOORA method is an extension of the multi-objective optimization by ratio analysis (MOORA) method by incorporating the full multiplicative form of multiple objectives [5]. The MULTIMOORA method can be summarized as follows [5, 35]:

Suppose the general decision matrix of an MCDM problem is given by

$$\mathbf{D}=\left[X_{ij}\right]=\begin{array}{c} \\ X_1\\ X_2\\ \vdots\\ X_n\end{array}\begin{array}{c}\begin{array}{cccc}\mathrm{C}_1 & \mathrm{C}_2 & & \mathrm{C}_\mathrm{m}\end{array}\\ \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1m}\\ X_{21} & X_{22} & & X_{2m}\\ & \vdots & \ddots & \vdots\\ X_{n1} & X_{n2} & \cdots & X_{nm}\end{bmatrix}\end{array},$$

where its element X_{ij} is the rating of the alternative X_i; $i=1,2,\ldots,n$ for the criterion C_j; $j=1,2,\ldots,m$. First, the data of the general decision matrix is normalized by dividing the rating of an alternative for a criterion by the square root of the sum of squares of the ratings of the entire alternatives for that criterion,

$$X_{ij}^{N}=\frac{X_{ij}}{\sqrt{\sum_{i=1}^{n}X_{ij}^{2}}}.\tag{30}$$

Hence, the normalized general decision matrix $\mathbf{D}_\mathrm{N}=\left[X_{ij}^{N}\right]$ is formed.

In the ratio system technique, the elements X_{ij}^{N} are added in case of maximization and subtracted in case of minimization. Let g be the number of benefit criteria to be maximized and $m-g$ be the number of cost criteria to be minimized, the overall index of each alternative is:

$$R_i = \sum_{j=1}^{g} X_{ij}^N - \sum_{j=g+1}^{m} X_{ij}^N. \tag{31}$$

Using (31), the alternatives are ranked. The higher the value of R_i, the higher the rank.

In the reference point technique, the Reference Point Theory is applied with the Min–Max Metric of Chebyshev. The jth criterion reference point is defined by

$$X_j^* = \begin{cases} \max_i X_{ij}^N, & \text{for benefit criteria,} \\ \min_i X_{ij}^N, & \text{for cost criteria.} \end{cases} \tag{32}$$

The deviation of the normalized rating of each alternative from the reference point is calculated by

$$d_i = \min_i \left\{ \max_j \left| X_j^* - X_{ij}^N \right| \right\}. \tag{33}$$

Using (33), the alternatives are ranked. The lower the value of d_i, the higher the rank.

The full multiplicative form combines both maximization and minimization of the multiplicative utility function. The overall utility of each alternative is given by the dimensionless number

$$U_i = \frac{U_i^b}{U_i^c}, \tag{34}$$

where $U_i^b = \prod_{j=1}^{g} X_{ij}$ denotes the product of an alternative's ratings of benefit criteria, and $U_i^b = \prod_{j=g+1}^{m} X_{ij}$ denotes the product of an alternative's ratings of cost criteria. Using (34), the alternatives are ranked. The higher the value of U_i, the higher the rank.

Utilizing the dominance theory, the alternatives are ranked based on the previous three ranking lists and the final decision is made, i.e., the alternative with the highest appearance in the first place on all the ranking lists is the best.

4 The Proposed PF-MULTIMOORA

In this section, the MULTIMOORA method is utilized in PF environment due to its appealing features. The MULTIMOORA is one of the most practical MCDM methods. It is an effective, efficient, flexible, and robust method. It was successfully applied to various practical fields.

Consider an MCDM problem with n alternatives $\{X_1, X_2, \ldots, X_n\}$ and m criteria $\{C_1, C_2, \ldots, C_m\}$, with weights $(w_1, w_2, \ldots, w_m)$ satisfying $\sum_{i=1}^{m} w_i = 1$. The Pythagorean fuzzy general decision matrix is represented as

$$\tilde{\mathbf{D}} = \left[\tilde{\mathrm{X}}_{ij}\right] = \begin{array}{c} \\ \mathrm{X}_1 \\ \mathrm{X}_2 \\ \vdots \\ \mathrm{X}_n \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{C}_1 & \mathrm{C}_2 & & \mathrm{C}_m \end{array} \\ \begin{bmatrix} \tilde{\mathrm{X}}_{11} & \tilde{\mathrm{X}}_{12} & \cdots & \tilde{\mathrm{X}}_{1m} \\ \tilde{\mathrm{X}}_{21} & \tilde{\mathrm{X}}_{22} & & \tilde{\mathrm{X}}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathrm{X}}_{n1} & \tilde{\mathrm{X}}_{n2} & \cdots & \tilde{\mathrm{X}}_{nm} \end{bmatrix} \end{array},$$

where $\tilde{\mathrm{X}}_{ij} = \left(\mu_{ij}, \upsilon_{ij}\right)$ indicates the ratings of the alternatives for the assessment criteria expressed by PFSs. The value "μ_{ij}" indicates the degree to which an alternative X_i satisfies a criterion C_j, and the value "υ_{ij}" indicates the degree to which X_i fails to satisfy this criterion. In constructing the decision matrix, the complement of the PFS is used for the ratings of the cost criteria. Hence, the decision matrix needs no further processing and the three techniques are directly applied.

The three techniques are expressed in detail in the following subsections.

4.1 The Ratio System Technique

The ratio system is based on the additive utility function. As the ratings are already expressed by PFSs they do not need normalization. In addition, since the complement is used in case of cost criteria, all the criteria are treated as benefit ones and subtraction operation (31) is not required. A Pythagorean fuzzy weighted averaging aggregation operator is applied.

Most of the proposed weighted average aggregation operators cannot be applied in a certain situation that is illustrated by the following example. Consider a simple MCDM problem with two alternatives and three criteria. The criteria weights are 0.2, 0.3, and 0.5, respectively. The ratings of the alternatives for the criteria are given by the following decision matrix.

$$\tilde{\mathbf{D}} = \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} (1, 0) & (0.1, 0.9) & (0.1, 0.9) \\ (0.9, 0.1) & (0.8, 0.2) & (0.9, 0.1) \end{bmatrix} \end{array}$$

From the decision matrix, the performance of A_2 far exceeds that of A_1 for the second and third criteria that have larger weights, while the performance of A_1 is slightly better than that of A_2 for the first criterion that has the smallest weight. Therefore, it is obvious that A_2 is better than A_1 by intuition.

Using the PFWA$_{MX}$ (10) and the score function (18), we get PFWA$_{MX}\left(A_1|C_j\right) = (1,0)$ with $S(A_1) = 1$, while PFWA$_{MX}\left(A_2|C_j\right) = (0.8774, 0.1231)$ with $S(A_2) = 0.7547$. This result leads to the selection of A_1 despite being not the better choice by logic. Accordingly, it can be concluded that a single criterion with the perfect rating $(1, 0)$ will dominate regardless of its weight and abolish the effect of the rest of the evaluation criteria, which is not fair in the assessment process. In this case, the selection is biased to the alternative having a single perfect rating regardless of its ratings for the other criteria. Therefore, false ranking is obtained.

On the other hand, using the PFYWA (16) and the score function (18), we get PFYWA$\left(A_1|C_j\right) = (0.6688, 0.7222)$ with $S(A_1) = -0.0743$, while PFYWA$\left(A_2|C_j\right) = (0.8735, 0.1375)$ with $S(A_2) = 0.7441$. This result leads to the selection of A_2, which is the better alternative by intuition. Here, the obtained ranking is rational.

From the previous illustration, the PFYWA (16) is chosen for aggregation. Hence, the additive utility U_i^A of each alternative X_i is given by

$$\tilde{U}_i^A = \text{PFYWA}\left(\tilde{X}_{ij}|j = 1, 2, \ldots, m; w\right) = \left(w_1 \boxdot \tilde{X}_{i1}\right) \boxplus \left(w_2 \boxdot \tilde{X}_{i2}\right) \boxplus \ldots \boxplus \left(w_n \boxdot \tilde{X}_{im}\right)$$
$$= \left(\sqrt{\min\left(1, \left(\sum_{j=1}^{m}\left(w_i \mu_{\tilde{X}_{ij}}^{2\theta}\right)\right)^{\frac{1}{\theta}}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{j=1}^{m}\left(w_j\left(1 - \upsilon_{\tilde{X}_{ij}}^2\right)^{\theta}\right)\right)^{\frac{1}{\theta}}\right)}\right).$$

4.2 The Reference Point Technique

So far, the reference point technique proceeds by identifying a reference point for each criterion; this reference point indicates the best rating obtained by an alternative for a criterion. Then, the distance between the rating of each alternative for a criterion and the reference point is calculated using a distance formula. Therefore, the reference point approach yields a crisp value.

The reference point can be the theoretical reference point defined by $(1, 0, 0)$. Otherwise, it can be an empirical reference point, i.e., defined from the data of the problem. In this case, it is given by

$$\tilde{R}_j = \left(\mu_j^{'}, \upsilon_j^{'}\right), \text{ where } \mu_j^{'} = \max_i \mu_{ij}, \upsilon_j^{'} = \min_i \upsilon_{ij}, j = 1, 2, \ldots, m. \quad (35)$$

Actually, a distance between two fuzzy values cannot be definitely and uniquely defined. It is closer to be a fuzzy value rather than a crisp value. Therefore, it is more proper to define the distance between two PFSs with a PFS.

In this proposed PF-MULTIMOORA two reference points are considered, the best rating and the worst rating. We are in favor of an alternative according to its degree of similarity to the best rating and oppose this alternative according to its degree of similarity to the worst rating, and the degree of indeterminacy can be calculated residually. The reference point utility value is estimated as follows:

The weighted general decision matrix is calculated first. It is given by

$$\tilde{\mathbf{D}}_w = \left[\tilde{\mathcal{X}}_{ij}\right] = \begin{array}{c} \\ \mathrm{X}_1 \\ \mathrm{X}_2 \\ \vdots \\ \mathrm{X}_n \end{array} \begin{array}{c} \begin{array}{cccc} \mathrm{C}_1 & \mathrm{C}_2 & & \mathrm{C}_\mathrm{m} \end{array} \\ \begin{bmatrix} \tilde{\mathcal{X}}_{11} & \tilde{\mathcal{X}}_{12} & \cdots & \tilde{\mathcal{X}}_{1m} \\ \tilde{\mathcal{X}}_{21} & \tilde{\mathcal{X}}_{22} & & \tilde{\mathcal{X}}_{2m} \\ \vdots & & \ddots & \vdots \\ \tilde{\mathcal{X}}_{n1} & \tilde{\mathcal{X}}_{n2} & \cdots & \tilde{\mathcal{X}}_{nm} \end{bmatrix} \end{array}, \quad \text{where } \tilde{\mathcal{X}}_{ij} = w_j \odot \tilde{\mathcal{X}}_{ij}.$$

The theoretical reference point is employed to guarantee that the resulting value is a PFS. Let $\tilde{R}_j^+$ be the best rating (1, 0, 0) and $\tilde{R}_j^-$ be the worst rating (0, 1, 0). Applying the normalized Euclidean distance (24) to find the distance between the ratings of an alternative for the criteria and the best rating

$$d_i^+\left(\tilde{\mathcal{X}}_{ij}, \tilde{R}_j^+\right) = \sqrt{\frac{1}{2m}\sum_{j=1}^{m}\left\{\left(\mu^2_{\tilde{\mathcal{X}}_{ij}} - 1\right)^2 + \upsilon^4_{\tilde{\mathcal{X}}_{ij}} + \pi^4_{\tilde{\mathcal{X}}_{ij}}\right\}}. \tag{36}$$

Similarly, the normalized Euclidean distance (24) is applied to find the distance between the ratings of an alternative for the criteria and the worst rating

$$d_i^-\left(\tilde{\mathcal{X}}_{ij}, \tilde{R}_j^-\right) = \sqrt{\frac{1}{2m}\sum_{j=1}^{m}\left\{\mu^4_{\tilde{\mathcal{X}}_{ij}} + \left(\upsilon^2_{\tilde{\mathcal{X}}_{ij}} - 1\right)^2 + \pi^4_{\tilde{\mathcal{X}}_{ij}}\right\}}. \tag{37}$$

Then, the utility value based on the reference point approach is expressed by

$$\tilde{U}_i^R = (\mu_i, \upsilon_i) \tag{38}$$

where

$$\mu_i = S_i^+\left(\tilde{\mathcal{X}}_{ij}, \tilde{R}_j^+\right) = 1 - d_i^+\left(\tilde{\mathcal{X}}_{ij}, \tilde{R}_j^+\right) \tag{39}$$

represents the degree of agreement on an alternative for the assessment criteria regarding its closeness to the best rating,

$$\upsilon_i = S_i^-\left(\tilde{\mathcal{X}}_{ij}, \tilde{R}_j^-\right) = 1 - d_i^-\left(\tilde{\mathcal{X}}_{ij}, \tilde{R}_j^-\right) \tag{40}$$

represents the degree of disagreement on an alternative for the assessment criteria regarding its closeness to the worst rating.

4.3 The Full Multiplicative Form Technique

The full multiplicative technique is based on the multiplicative utility function. Since all the criteria are treated as benefit criteria after using the complement of the ratings of the cost criteria, the division operation (34) is no longer required. A Pythagorean fuzzy weighted geometric aggregation operator is applied.

For an MCDM problem with two alternatives and three criteria with weights 0.2, 0.3, and 0.5, respectively, the ratings of the alternatives for the criteria are given by the following decision matrix.

$$\tilde{\mathbf{D}}= \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{bmatrix} (0,1) & (0.9,0.1) & (0.9,0.1) \\ (0.1,0.9) & (0.2,0.8) & (0.1,0.9) \end{bmatrix} \end{array}.$$

From the decision matrix, the ratings of A_1 exceeds that of A_2 for the second and third criteria that have larger weights, and the rating of A_2 is slightly better than that of A_1 for the first criteria that has the smallest weight. Therefore, it is obvious that A_1 is better than A_2.

Using the PFWG$_{MX}$ (11) and the score function (18) we get $\text{PFWG}_{MX}\big(A_1|C_j\big) = (0,1)$ with $S(A_1) = -1$, while $\text{PFWG}_{MX}\big(A_2|C_j\big) = (0.1231, 0.4637)$ with $S(A_2) = -0.1999$. This result leads to the selection of A_2 although it is not the better choice by intuition. Therefore, it can be concluded that a single criterion with the worst performance (0, 1) will dominate regardless of its weight and abolish the effect of the rest of the evaluation criteria, which is also not fair in evaluation. In this case, the selection is biased against the alternative having this worst performance regardless of its performance for the other criteria. This leads to false ranking.

On the other hand, using the PFYWG (17) and the score function (18) we get $\text{PFYWG}\big(A_1|C_j\big) = (0.7222, 0.6688)$ with $S(A_1) = 0.0743$, while $\text{PFYWG}\big(A_2|C_j\big) = (0.1375, 0.8735)$ with $S(A_2) = -0.7441$. This result leads to the selection of A_1, which is actually the better alternative by intuition.

From the previous illustration, the PFYWG (17) is chosen for aggregation. Hence, the multiplicative utility U_i^M of each alternative X_i is given by

$$\tilde{U}_i^M = \text{PFYWG}\Big(\tilde{X}_{ij}|j=1,2,\ldots,m;w\Big) = \tilde{X}_{i1}^{w_1} \boxtimes \tilde{X}_{i2}^{w_2} \boxtimes \ldots \boxtimes \tilde{X}_{in}^{w_n}$$
$$= \left(\sqrt{1-\min\left(1, \left(\sum_{j=1}^{m}\left(w_i\left(1-\mu_{\tilde{X}_{ij}}^2\right)^{\theta}\right)\right)^{\frac{1}{\theta}}\right)}, \sqrt{\min\left(1, \left(\sum_{j=1}^{m}\left(w_i \upsilon_{\tilde{X}_{ij}}^{2\theta}\right)\right)^{\frac{1}{\theta}}\right)}\right).$$

4.4 The Overall Utility Score

Finally, the results of the three techniques are combined to get the overall utility value. In the early versions of the MULTIMOORA, the dominance theory was applied to rank the alternatives. When using the dominance theory several rules are utilized for discrimination: absolute dominance, general dominance, transitiveness, overall dominance, absolute equability, and partial equability [6]. In spite of all these rules, circular reasoning is possible. For example, consider the following alternatives in an MCDM problem with their ranking by the three techniques $X_1(11-20-14)$, $X_2(14-6-15)$, and $X_3(15-19-12)$. When applying the dominance rules we have: X_1 generally dominates X_2, X_2 generally dominates X_3, and X_3 generally dominates X_1. Accordingly, the same ranking is given to the three objects [6]. Therefore, the dominance theory in large scale applications has two main drawbacks: multiple comparisons and circular reasoning.

To overcome these drawbacks, recent researches proposed the aggregation of the three techniques to enhance the accuracy and efficiency of the MULTIMOORA method [11, 68].

Therefore, the three utility values are aggregated into the overall utility value. Here, the common trend is to defuzzify the utility values and then aggregate. This is accomplished by

$$U_i = \omega_A U_i^A + \omega_R U_i^R + \omega_M U_i^M, \tag{41}$$

where ω_A, ω_R, and ω_M are the coefficient of importance of the utility scores, and their sum is equal to one.

The main disadvantage of this trend is having equal scores for different PFSs which will surely affect the overall utility. For example, the PFSs $(0.6, 0.6)$ and $(0.3, 0.3)$ have the same score. Therefore, it is preferable to use a weighted average aggregation operator first, and then use the score function (18) for defuzzification. In this case, whenever we have equal scores the accuracy function (19) can be applied for discrimination.

The overall utility is computed by using the weighted average aggregation operator (4) for similar treatment of the membership and non-membership information for the three utility values:

$$\begin{aligned}\tilde{U}_i^T &= \mathrm{PFWA}_Y\left(\tilde{U}_i^A, \tilde{U}_i^R, \tilde{U}_i^M | \omega_A, \omega_R, \omega_M\right)\\ &= \left(\omega_A \mu_{\tilde{U}_i^A} + \omega_R \mu_{\tilde{U}_i^R} + \omega_M \mu_{\tilde{U}_i^M}, \omega_A \upsilon_{\tilde{U}_i^A} + \omega_R \upsilon_{\tilde{U}_i^R} + \omega_M \upsilon_{\tilde{U}_i^M}\right),\\ \omega_A &= \omega_R = \omega_M, \quad \text{and} \quad \omega_A + \omega_R + \omega_M = 1. \end{aligned} \tag{42}$$

Then, the score function (18) and the accuracy function (19) are used for ranking. The alternative with the highest overall utility score is the best.

The steps of the PF-MUTIMOORA are shown in Fig. 1.

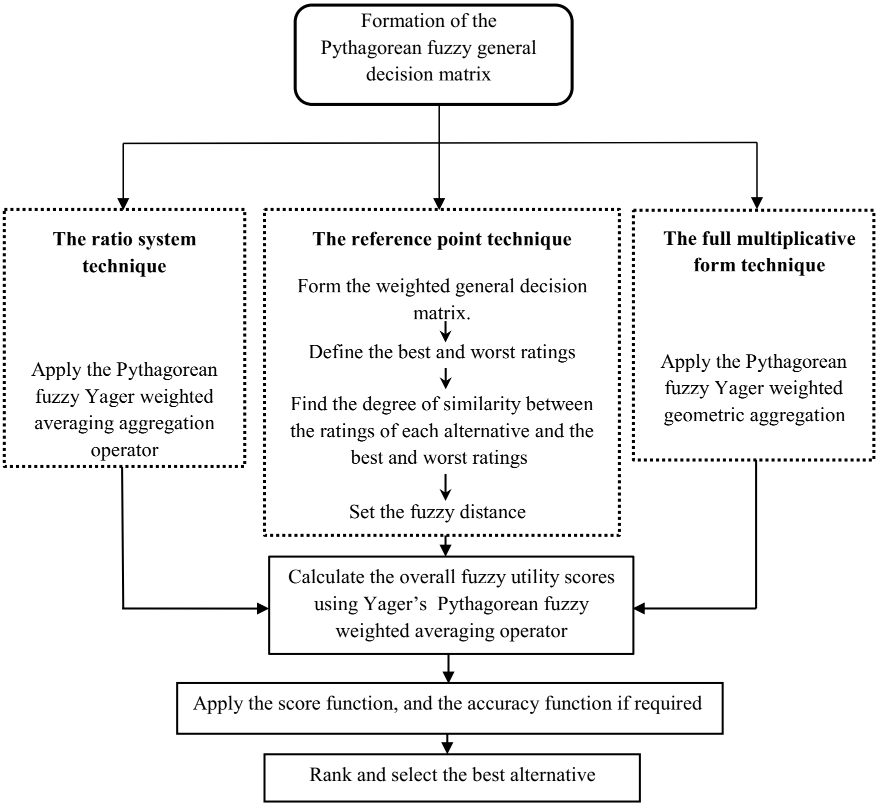

Fig. 1 The proposed PF-MULTIMOORA

5 Evaluation of Energy Storage Technologies

5.1 An Overview

As a result of industrialization and the growing population, energy demand has been increasing in the world. Renewable energy sources (RESs) are seen as effective alternatives to fulfill these increasing requirements. Since RESs are fluctuating and intermittent, energy storage technologies (ESTs) enable the storage of excess energy and utilize it when needed to secure energy supply [9]. ESTs provide a wide range of approaches to create a more resilient energy infrastructure and bring cost savings to utilities and consumers. Energy storage devices are charged when they absorb energy and discharged when they deliver the stored energy back into the grid. Charging and discharging processes normally require power conversion devices to transform electrical energy into a different energy form, e.g., chemical, electrochemical, electrical, mechanical, and thermal. In other words, energy storage enables supply and demand to be balanced even when the generation and consumption of energy do not happen at the same time [44].

Finding better ways to store energy is critical to becoming more energy efficient. Advances in energy storage can be achieved by finding new materials and understanding how current and new materials function. ESTs can be used in diverse applications. While some of them can be properly selected for specific applications, some others are applicable in wider frames. The key factor to success lies in matching the application to technology [27].

ESTs can be classified into five main categories thermal energy, chemical energy, electrochemical energy, mechanical energy, and electromagnetic energy storage as previously mentioned in Sect. 2. The form of converted energy determines the class of the EST. The power storing capacity, energy and power densities, response time, cost and economy scale, operating life, monitoring and control mechanisms, efficiency, and operating constraints are the critical parameters that govern the choice as to which type of technology.

The selection of an EST is an MCDM problem since the evaluation of ESTs is based on multiple conflicting criteria. The main criteria in the assessment of the performance of EST to achieve sustainability and energy security are technological, economic, and environmental criteria. Technological criteria allow assessing the reliability of the used technology and its ability to ensure safe energy supply. Economic criteria take into account competitiveness and affordability issues through the associated costs of installation and their impact on energy prices. Environmental criteria allow addressing environmental sustainability [68].

5.2 A Practical Example

The proposed PF-MULTIMOORA is utilized to rank a set of different ESTs. Fourteen alternatives are evaluated by using eleven criteria. The alternatives are given in Table 1. The criteria from one to eight are technological; the ninth and tenth criteria are economical, the eleventh criteria are environmental. The assessment criteria are defined as follows.

(C_1) **The power rating**: indicates the size of the power conversion subsystems resulting from the maximum power requirements of the electrical load on the discharging part (generation side) and the appearing excess power on the charging part (input side) [67]. The power rating is measured in megawatt (MW). High power rating indicates better EST [68].

(C_2) **The energy rating**: measured in hours, is the duration of discharge, i.e., the duration needed to empty the reservoir initially full at maximum outflow capacity [10]. It indicates how long a storage device can maintain output. Long discharge period is preferred since operating flexibility is required to manage variations in renewable energy generation and load to match demand. A Long duration EST refers to an EST with durations of 10 or more hours [15].

Table 1 The evaluated ESTs

Alternative	Name	Technology
X_1	Hydrogen	Chemical storage
X_2	Pumped hydroelectric storage (PHS)	Mechanical storage
X_3	Compressed air energy storage (CAES)	Mechanical storage
X_4	Flywheel	Mechanical storage
X_5	Superconducting magnetic energy storage (SMES)	Electrical storage
X_6	Supercapacitors (Supercap)	Electrical storage
X_7	Lead-acid (Pb-acid)	Electrochemical storage
X_8	Nickel–cadmium (NiCd)	Electrochemical storage
X_9	Lithium-ion (Li-ion)	Electrochemical storage
X_{10}	Sodium–Sulphur (NaS)	Electrochemical storage
X_{11}	Sodium–nickel chloride (NaNiCl)	Electrochemical storage
X_{12}	Vanadium redox (VRB)	Electrochemical storage
X_{13}	Zinc–bromine (ZnBr)	Electrochemical storage
X_{14}	Molten Salt	Thermal storage

(C_3) **The response time**: indicates the required time to activate the system, i.e., how quickly a storage technology can be brought into operation and discharge energy. ESTs with short response time provide electricity instantly, while ESTs with long response time provide electricity after a time interval [67]. It is measured on a linguistic scale. The lower this value the better the EST, since the rapid response is preferred [68].

(C_4) **The energy density**: the ratio of energy storage capacity to the system volume or mass [67]. It is measured in Wh/kg. High energy density indicates better EST [68].

(C_5) **The self-discharge time**: also known as idling losses, it is the losses occurring during the time in which energy remains stored [67]. It is measured in percentage per day, the lower the losses the better the EST [68].

(C_6) **The round-trip efficiency**: it is the ratio of input energy (in MWh) to the energy retrieved from storage (in MWh). It is measured in percentage. High round-trip efficiency is required [68].

(C_7) **The lifetime**: also known as the service period, it is expressed in years for a certain cycling rate, or in the total number of cycles, where a cycle is the time during which the system is fully charged and discharged. Long lifetime is required [68].

(C_8) **The number of cycles of operation**: the charge/discharge performance that represents the demands associated with a specific application placed on an EST.

(C_9) **The power cost**: it is the total costs of installation. It is measured in Eur/kW. Lower costs are desired [68].

(C_{10}) **The energy cost**: it is the costs of energy supply. It is measured in Eur/kWh. Lower costs are desired [68].

(C_{11}) **The environmental impact**: this encompasses the impacts of the construction, disposal/end of life, and usage of ESTs on the environment. For example, wastes from batteries manufacturing and recycling are a crucial and growing challenge for public health due to their toxicity, abundance and durability in the environment [13]. It is measured on a qualitative scale, and of course, the minimum impact must be attained.

From the previous illustration of the criteria, it is clear that they not only have quantitative and qualitative data but also the quantitative data have different units of measurement. Moreover, they differ in the objective. It is required to maximize $\{C_1, C_2, C_4, C_6, C_7, \text{and} C_8\}$, and minimize $\{C_3, C_5, C_9, C_{10}, \text{and} C_{11}\}$. Zhang et al. [68] transformed all the data into IFSs. In addition, the IFSs representing the criteria to be minimized was negated using (20). Therefore, the data given and used in this chapter is the data in its final form ready to be processed. The main difference is in the residual term, i.e., the hesitation margin, which is calculated under the PF condition. The problem data is given in Table 2. For detailed information about data transformation and fusion, the reader is referred to Zhang et al. [68].

Several weighting strategies were proposed by Zhang et al. [68] to test the effect of the different priorities on the selection of energy storage technologies. Each of the three strategies: technological, economic, and environmental is assigned a weight. Then, the weight of each strategy is equally distributed among its criteria.

First, they treated the main three strategies as equally important (balanced strategy). Hence, the weight of each dimension is (1/3). Therefore the weights of the criteria are

$$(0.042, 0.042, 0.042, 0.042, 0.042, 0.042, 0.042, 0.042, 0.167, 0.167, 0.333).$$

Second, they gave a high priority to the technological strategy (0.5) and (0.25) for the other two strategies. Then the weights of the criteria are given as follows:

$$(0.063, 0.063, 0.063, 0.063, 0.063, 0.063, 0.063, 0.063, 0.125, 0.125, 0.25).$$

Third, they gave a high priority to the economic strategy (0.5) and (0.25) for the other two strategies. Then the weights of the criteria are given as follows

$$(0.031, 0.031, 0.031, 0.031, 0.031, 0.031, 0.031, 0.031, 0.25, 0.25, 0.25).$$

Fourth, they gave a high priority to the environmental strategy (0.5) and (0.25) for the other two strategies. Then the weights of the criteria are given as follows:

$$(0.031, 0.031, 0.031, 0.031, 0.031, 0.031, 0.031, 0.031, 0.125, 0.125, 0.5).$$

The proposed PF-MULTIMOORA is applied to solve this problem for the differently proposed weights. The value of $\theta = 2$. The solution steps are demonstrated as follows:

Table 2 The PF ratings of the alternatives for the evaluation criteria

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
X_1	$\begin{pmatrix}0\\0.99\\0.1411\end{pmatrix}$	$\begin{pmatrix}0\\0.5223\\0.8528\end{pmatrix}$	$\begin{pmatrix}0.45\\0.5\\0.7399\end{pmatrix}$	$\begin{pmatrix}0.0796\\0.0046\\0.9968\end{pmatrix}$	$\begin{pmatrix}0.09829\\0.0043\\0.1841\end{pmatrix}$	$\begin{pmatrix}0.0477\\0.8808\\0.4711\end{pmatrix}$	$\begin{pmatrix}0.0354\\0.8937\\0.4473\end{pmatrix}$	$\begin{pmatrix}0.0001\\0.9999\\0.0141\end{pmatrix}$	$\begin{pmatrix}0.7731\\0.078\\0.6295\end{pmatrix}$	$\begin{pmatrix}0.9984\\0.0001\\0.0565\end{pmatrix}$	$\begin{pmatrix}0.2\\0.75\\0.6305\end{pmatrix}$
X_2	$\begin{pmatrix}0.0199\\0.0028\\0.9998\end{pmatrix}$	$\begin{pmatrix}0.0199\\0.5223\\0.8525\end{pmatrix}$	$\begin{pmatrix}0.45\\0.5\\0.7399\end{pmatrix}$	$\begin{pmatrix}0\\0.9999\\0.0141\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0.1788\\0.7973\\0.5765\end{pmatrix}$	$\begin{pmatrix}0.3544\\0.2912\\0.8886\end{pmatrix}$	$\begin{pmatrix}0.0002\\0.9995\\0.0316\end{pmatrix}$	$\begin{pmatrix}0.4895\\0.0709\\0.8691\end{pmatrix}$	$\begin{pmatrix}0.9838\\0.0054\\0.1792\end{pmatrix}$	$\begin{pmatrix}0.1\\0.9\\0.4243\end{pmatrix}$
X_3	$\begin{pmatrix}0.0199\\0.9402\\0.3400\end{pmatrix}$	$\begin{pmatrix}0.0199\\0.5223\\0.8525\end{pmatrix}$	$\begin{pmatrix}0.2\\0.75\\0.6305\end{pmatrix}$	$\begin{pmatrix}0.003\\0.994\\0.1093\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}0.1001\\0.8712\\0.4806\end{pmatrix}$	$\begin{pmatrix}0.1772\\0.7165\\0.6747\end{pmatrix}$	$\begin{pmatrix}0\\0.9995\\0.0316\end{pmatrix}$	$\begin{pmatrix}0.8369\\0.0567\\0.5444\end{pmatrix}$	$\begin{pmatrix}0.987\\0.0011\\0.1607\end{pmatrix}$	$\begin{pmatrix}0.1\\0.9\\0.4243\end{pmatrix}$
X_4	$\begin{pmatrix}0\\0.996\\0.0894\end{pmatrix}$	$\begin{pmatrix}0.00001\\0.995\\0.0999\end{pmatrix}$	$\begin{pmatrix}0.45\\0.5\\0.7399\end{pmatrix}$	$\begin{pmatrix}0.0005\\0.9871\\0.1601\end{pmatrix}$	$\begin{pmatrix}0.1452\\0.171\\0.9745\end{pmatrix}$	$\begin{pmatrix}0.2027\\0.7735\\0.6005\end{pmatrix}$	$\begin{pmatrix}0.1418\\0.8582\\0.4933\end{pmatrix}$	$\begin{pmatrix}0.001\\0.9005\\0.4349\end{pmatrix}$	$\begin{pmatrix}0.9575\\0.0142\\0.2881\end{pmatrix}$	$\begin{pmatrix}0.6219\\0.108\\0.7756\end{pmatrix}$	$\begin{pmatrix}0.9\\0.1\\0.4243\end{pmatrix}$
X_5	$\begin{pmatrix}0\\0.998\\0.0632\end{pmatrix}$	$\begin{pmatrix}0\\0.9983\\0.0583\end{pmatrix}$	$\begin{pmatrix}0.6\\0.35\\0.7194\end{pmatrix}$	$\begin{pmatrix}0\\0.9995\\0.0316\end{pmatrix}$	$\begin{pmatrix}0.8717\\0.0855\\0.4823\end{pmatrix}$	$\begin{pmatrix}0.2265\\0.7735\\0.5919\end{pmatrix}$	$\begin{pmatrix}0.1418\\0.8582\\0.4933\end{pmatrix}$	$\begin{pmatrix}0.0001\\0.9999\\0.0141\end{pmatrix}$	$\begin{pmatrix}0.9433\\0.0142\\0.3316\end{pmatrix}$	$\begin{pmatrix}0.2437\\0.0756\\0.9669\end{pmatrix}$	$\begin{pmatrix}0.1\\0.9\\0.4243\end{pmatrix}$

(continued)

Table 2 (continued)

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
X_6	$\begin{pmatrix}0\\0.9998\\0.0200\end{pmatrix}$	$\begin{pmatrix}0\\0.9801\\0.1985\end{pmatrix}$	$\begin{pmatrix}0.6\\0.35\\0.7194\end{pmatrix}$	$\begin{pmatrix}0\\0.9985\\0.0548\end{pmatrix}$	$\begin{pmatrix}0.6581\\0.0171\\0.7527\end{pmatrix}$	$\begin{pmatrix}0.2027\\0.7663\\0.6097\end{pmatrix}$	$\begin{pmatrix}0.1418\\0.8582\\0.4933\end{pmatrix}$	$\begin{pmatrix}0.0001\\0.005\\1\end{pmatrix}$	$\begin{pmatrix}0.9433\\0.0142\\0.3316\end{pmatrix}$	$\begin{pmatrix}0.5678\\0.0324\\0.8225\end{pmatrix}$	$\begin{pmatrix}0.6\\0.35\\0.7194\end{pmatrix}$
X_7	$\begin{pmatrix}0\\0.99\\0.1411\end{pmatrix}$	$\begin{pmatrix}0\\0.9403\\0.3403\end{pmatrix}$	$\begin{pmatrix}0.9\\0.1\\0.4243\end{pmatrix}$	$\begin{pmatrix}0.003\\0.995\\0.0998\end{pmatrix}$	$\begin{pmatrix}0.9974\\0.0009\\0.0721\end{pmatrix}$	$\begin{pmatrix}0.1431\\0.7735\\0.6174\end{pmatrix}$	$\begin{pmatrix}0.0213\\0.8937\\0.4482\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0.9078\\0.0284\\0.4184\end{pmatrix}$	$\begin{pmatrix}0.9676\\0.00054\\0.2524\end{pmatrix}$	$\begin{pmatrix}0.1\\0.9\\0.4243\end{pmatrix}$
X_8	$\begin{pmatrix}0\\0.992\\0.1262\end{pmatrix}$	$\begin{pmatrix}0\\0.9801\\0.1985\end{pmatrix}$	$\begin{pmatrix}0.9\\0.1\\0.4243\end{pmatrix}$	$\begin{pmatrix}0.004\\0.994\\0.1093\end{pmatrix}$	$\begin{pmatrix}0.9949\\0.0017\\0.1009\end{pmatrix}$	$\begin{pmatrix}0.1431\\0.783\\0.6053\end{pmatrix}$	$\begin{pmatrix}0.1063\\0.8582\\0.5022\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0.8582\\0.0496\\0.5109\end{pmatrix}$	$\begin{pmatrix}0.892\\0.0216\\0.4515\end{pmatrix}$	$\begin{pmatrix}0.9\\0.1\\0.4243\end{pmatrix}$
X_9	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0.003\\0.9801\\0.1985\end{pmatrix}$	$\begin{pmatrix}0.9\\0.1\\0.4243\end{pmatrix}$	$\begin{pmatrix}0.0075\\0.9751\\0.2216\end{pmatrix}$	$\begin{pmatrix}0.9974\\0.0009\\0.0721\end{pmatrix}$	$\begin{pmatrix}0.2027\\0.7616\\0.6155\end{pmatrix}$	$\begin{pmatrix}0.0354\\0.8937\\0.4473\end{pmatrix}$	$\begin{pmatrix}0\\0.9999\\0.0141\end{pmatrix}$	$\begin{pmatrix}0.5746\\0.0993\\0.8124\end{pmatrix}$	$\begin{pmatrix}0.8055\\0.0216\\0.5922\end{pmatrix}$	$\begin{pmatrix}0.6\\0.35\\0.7194\end{pmatrix}$
X_{10}	$\begin{pmatrix}0.0001\\0.99\\0.1411\end{pmatrix}$	$\begin{pmatrix}0\\0.9602\\0.2739\end{pmatrix}$	$\begin{pmatrix}0.9\\0.1\\0.4243\end{pmatrix}$	$\begin{pmatrix}0.0149\\0.9761\\0.2168\end{pmatrix}$	$\begin{pmatrix}0.829\\0.171\\0.5325\end{pmatrix}$	$\begin{pmatrix}0.2027\\0.7854\\0.5849\end{pmatrix}$	$\begin{pmatrix}0.0709\\0.8937\\0.4430\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0.7164\\0.0993\\0.6909\end{pmatrix}$	$\begin{pmatrix}0.9028\\0.0216\\0.4295\end{pmatrix}$	$\begin{pmatrix}0.6\\0.35\\0.7194\end{pmatrix}$
X_{11}	$\begin{pmatrix}0\\0.9998\\0.0200\end{pmatrix}$	$\begin{pmatrix}0.0003\\0.9801\\0.1985\end{pmatrix}$	$\begin{pmatrix}0.9\\0.1\\0.4243\end{pmatrix}$	$\begin{pmatrix}0.0124\\0.9876\\0.1565\end{pmatrix}$	$\begin{pmatrix}0.8718\\0.1282\\0.4728\end{pmatrix}$	$\begin{pmatrix}0.2146\\0.7854\\0.5806\end{pmatrix}$	$\begin{pmatrix}0.0709\\0.9008\\0.4284\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0.9716\\0.0142\\0.2362\end{pmatrix}$	$\begin{pmatrix}0.9838\\0.0076\\0.1791\end{pmatrix}$	$\begin{pmatrix}0.9\\0.1\\0.4243\end{pmatrix}$

(continued)

Table 2 (continued)

	C_1	C_2	C_3	C_4	C_5	C_6
X_{12}	$\begin{pmatrix}0\\0.9986\\0.0529\end{pmatrix}$	$\begin{pmatrix}0\\0.801\\0.5987\end{pmatrix}$	$\begin{pmatrix}0.6\\0.35\\0.7194\end{pmatrix}$	$\begin{pmatrix}0.0075\\0.9925\\0.1220\end{pmatrix}$	$\begin{pmatrix}0.9145\\0.1282\\0.4046\end{pmatrix}$	$\begin{pmatrix}0.2027\\0.7973\\0.5685\end{pmatrix}$
X_{13}	$\begin{pmatrix}0\\0.9996\\0.0283\end{pmatrix}$	$\begin{pmatrix}0\\0.801\\0.5987\end{pmatrix}$	$\begin{pmatrix}0.6\\0.35\\0.7194\end{pmatrix}$	$\begin{pmatrix}0.006\\0.992\\0.1261\end{pmatrix}$	$\begin{pmatrix}0.9915\\0.0085\\0.1298\end{pmatrix}$	$\begin{pmatrix}0.1669\\0.8212\\0.5457\end{pmatrix}$
X_{14}	$\begin{pmatrix}0.0002\\0.9701\\0.2427\end{pmatrix}$	$\begin{pmatrix}0.0199\\0.5223\\0.8525\end{pmatrix}$	$\begin{pmatrix}0.2\\0.75\\0.6305\end{pmatrix}$	$\begin{pmatrix}0.008\\0.9801\\0.1983\end{pmatrix}$	$\begin{pmatrix}0.9915\\0.0004\\0.1301\end{pmatrix}$	$\begin{pmatrix}0.1192\\0.8569\\0.5015\end{pmatrix}$

	C_7	C_8	C_9	C_{10}	C_{11}
X_{12}	$\begin{pmatrix}0.0354\\0.8582\\0.5121\end{pmatrix}$	$\begin{pmatrix}0.0001\\0.9999\\0.0141\end{pmatrix}$	$\begin{pmatrix}0.6455\\0.3545\\0.6765\end{pmatrix}$	$\begin{pmatrix}0.892\\0.0108\\0.4519\end{pmatrix}$	$\begin{pmatrix}0.1\\0.9\\0.4243\end{pmatrix}$
X_{13}	$\begin{pmatrix}0.0354\\0.9291\\0.3681\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0.7448\\0.0709\\0.6635\end{pmatrix}$	$\begin{pmatrix}0.9244\\0.0108\\0.3813\end{pmatrix}$	$\begin{pmatrix}0.9\\0.1\\0.4243\end{pmatrix}$
X_{14}	$\begin{pmatrix}0.0354\\0.8937\\0.4473\end{pmatrix}$	$\begin{pmatrix}0.0001\\0.9999\\0.0141\end{pmatrix}$	$\begin{pmatrix}0.9575\\0.0284\\0.2870\end{pmatrix}$	$\begin{pmatrix}0.9935\\0.0032\\0.1138\end{pmatrix}$	$\begin{pmatrix}0.6\\0.35\\0.7194\end{pmatrix}$

Step 1. Form the Pythagorean fuzzy general decision matrix and determine the weights of the criteria.

The general decision matrix is given in Table 2.

Step 2. Apply the ratio system technique using (16).

$$\tilde{U}_i^A = \mathrm{PFYWA}\left(\tilde{X}_{ij} | j = 1, 2, \ldots, m; w\right).$$

Step 3. Apply the reference point technique.

(i) Form the weighted general decision matrix.
(ii) Define the best rating (1, 0, 0) and the worst rating (0, 1, 0).
(iii) Find the degree of similarity between the ratings of each alternative for the evaluation criteria and the best rating.

$$\mu_i = S_i^+\left(\widetilde{\mathcal{X}}_{ij}, \tilde{R}_j^+\right) = 1 - d_i^+\left(\widetilde{\mathcal{X}}_{ij}, \tilde{R}_j^+\right),$$

where $d_i^+\left(\widetilde{\mathcal{X}}_{ij}, \tilde{R}_j^+\right)$ is the normalized Euclidean distance (36).

(iv) Find the degree of similarity between the ratings of each alternative for the evaluation criteria and the worst rating.

$$\upsilon_i = S_i^-\left(\widetilde{\mathcal{X}}_{ij}, \tilde{R}_j^-\right) = 1 - d_i^-\left(\widetilde{\mathcal{X}}_{ij}, \tilde{R}_j^-\right),$$

where $d_i^-\left(\widetilde{\mathcal{X}}_{ij}, \tilde{R}_j^-\right)$ is the normalized Euclidean distance (37).

(v) Set the fuzzy distance from the reference point.

$$\tilde{U}_i^R = (\mu_i, \upsilon_i).$$

Step 4. Apply the full multiplicative form approach using (17).

$$\tilde{U}_i^M = PFYWG\left(\tilde{X}_{ij} | j = 1, 2, \ldots, m; w\right).$$

Step 5. Calculate the overall fuzzy utility scores using (4).

$$\tilde{U}_i^T = PFWA_Y\left(\tilde{U}_i^A, \tilde{U}_i^R, \tilde{U}_i^M | \omega_A, \omega_R, \omega_M\right),$$

where $\omega_A = \omega_R = \omega_M$, and$\omega_A + \omega_R + \omega_M = 1$.

Step 6. Rank the alternatives by the overall fuzzy utility scores using the score function (18) and the accuracy function (19).

The results of the three techniques, the overall fuzzy scores, and the ranking are summarized in Table 3.

The best ESTs obtained by the proposed PF-MULTIMOORA are NaNiCl, Ni–Cd, and ZnBr. Meanwhile, the worst ESTs are Pb-acid, VRB, and SMES. The best ESTs obtained by the IF-MULTIMOORA [68] is Molten salt, NaNiCl, and ZnBr. The worst three technologies coincide in the two methods.

The problem is resolved using the normalized Hausdorff and the normalized Hamming distances in the reference point approach to compare the ranking results. The rankings obtained by the PF-MULTIMOORA using these distance measures are given in Table 4. The results reveal that the ranking remains unchanged using different normalized distance measures.

Table 3 Results using the normalized Euclidean distance in the reference point approach

	AU	RPU (Euclidean)	MU	Total	Score	Rank
Hydrogen	(0.7189, 0.5219)	(0.2089, 0.7024)	(0.4667, 0.7039)	(0.4879, 0.6802)	−0.2247	9
PHS	(0.6773, 0.5314)	(0.2252, 0.6218)	(0.4121, 0.7553)	(0.4698, 0.6878)	−0.2523	10
CAES	(0.7291, 0.5866)	(0.2179, 0.6303)	(0.4551, 0.7882)	(0.5048, 0.7265)	−0.2729	11
Flywheel	(0.7880, 0.3809)	(0.1984, 0.7124)	(0.6129, 0.6607)	(0.5582, 0.6250)	−0.0790	4
SMES	(0.6351, 0.5884)	(0.1680, 0.7605)	(0.3724, 0.8070)	(0.4164, 0.7570)	−0.3997	14
Super cap	(0.6738, 0.3919)	(0.1977, 0.6978)	(0.5459, 0.6377)	(0.4907, 0.6113)	−0.1329	6
Pb-acid	(0.7573, 0.5816)	(0.2015, 0.7245)	(0.4856, 0.8034)	(0.5064, 0.7408)	−0.2922	12
Ni–Cd	(0.8339, 0.3646)	(0.2221, 0.6908)	(0.6852, 0.6696)	(0.6034, 0.6116)	−0.0099	2
Li-ion	(0.6695, 0.4192)	(0.1975, 0.7241)	(0.5530, 0.6742)	(0.4934, 0.6396)	−0.1656	8
NaS	(0.7039, 0.4217)	(0.1911, 0.7483)	(0.5832, 0.6721)	(0.5146, 0.6470)	−0.1538	7
NaNiCl	(0.8710, 0.3662)	(0.2282, 0.6812)	(0.6970, 0.6740)	(0.6261, 0.6148)	0.01399	1
VRB	(0.6418, 0.6096)	(0.1633, 0.7636)	(0.4352, 0.7970)	(0.4399, 0.7472)	−0.3649	13
ZnBr	(0.8135, 0.3711)	(0.2119, 0.6946)	(0.6621, 0.6628)	(0.5866, 0.6152)	−0.0345	3
Molten salt	(0.7887, 0.4279)	(0.2202, 0.6948)	(0.5887, 0.6592)	(0.5575, 0.6323)	−0.0890	5

Table 4 Results using normalized Hamming and normalized Hausdorff distances

	RPU (Hamming)	Total	Score	Rank	RPU (Hausdorff)	Total	Score	Rank
Hydrogen	(0.2796, 0.8169)	(0.4643, 0.6421)	−0.1967	9	(0.2237, 0.6674)	(0.4963, 0.6305)	−0.1772	9
PHS	(0.3215, 0.7788)	(0.4378, 0.6355)	−0.2123	10	(0.2468, 0.6031)	(0.4456, 0.6293)	−0.1975	10
CAES	(0.3319, 0.8069)	(0.4669, 0.6677)	−0.2278	11	(0.2302, 0.6039)	(0.4710, 0.6589)	−0.2124	11
Flywheel	(0.2754, 0.8353)	(0.5326, 0.5841)	−0.0575	4	(0.2120, 0.6730)	(0.5371, 0.5710)	−0.0375	4
SMES	(0.2430, 0.8779)	(0.3914, 0.7179)	−0.3622	14	(0.1854, 0.7336)	(0.3972, 0.7090)	−0.3448	14
Super cap	(0.2538, 0.8062)	(0.4720, 0.5752)	−0.1081	6	(0.2306, 0.6697)	(0.4830, 0.5659)	−0.0869	6
Pb-acid	(0.2779, 0.8395)	(0.4810,0.7025)	−0.2621	12	(0.2093, 0.6797)	(0.4836, 0.6875)	−0.2389	12
Ni–Cd	(0.2930, 0.8024)	(0.5798, 0.5744)	−0.0062	2	(0.2331, 0.6433)	(0.5835, 0.5586)	0.0284	2
Li-ion	(0.2593, 0.8272)	(0.4728, 0.6052)	−0.1427	8	(0.2171, 0.6916)	(0.4794, 0.5944)	−0.1236	8
NaS	(0.2581, 0.8491)	(0.4922, 0.6134)	−0.1340	7	(0.2032, 0.7096)	(0.4963, 0.6005)	−0.1143	7
NaNiCl	(0.3122, 0.8062)	(0.5981, 0.5732)	0.0292	1	(0.2306, 0.6254)	(0.5989, 0.6005)	0.0511	1
VRB	(0.2440, 0.8375)	(0.4130, 0.6927)	−0.3029	13	(0.2106, 0.6496)	(0.4288, 0.6847)	−0.2849	13
ZnBr	(0.2860, 0.8138)	(0.5619, 0.5756)	−0.0155	3	(0.2257, 0.6470)	(0.5665, 0.5597)	0.0077	3
Molten salt	(0.2967, 0.8116)	(0.5320, 0.5934)	−0.0691	5	(0.2271, 0.6513)	(0.5343, 0.5789)	−0.0496	5

The effect of the different strategies: technical, economic, and environmental strategies, on the ranking of the ESTs is also studied. The three distance measures are also used to rank the alternatives. The ranking using the environmental strategy is the same as the balanced strategy utilizing the three distance measures. Regarding the technical and economic strategies, the top-ranked three technologies and the worst-ranked three technologies are the same for the three distance measures. Slight differences are observed in the results from the balanced strategy in the moderately performing technologies. From four to six alternatives exchange rankings, within one or two places forward and backwards according to the used distance measure.

The ranking of the ESTs for the different strategies using the three distance measures is quite consistent. In the economic and environmental strategies, the ranking of the three distance measures is the same. In the technical strategy, only the 8th and 9th alternatives exchanged ranks. PHS is ranked the 8th and NaS is ranked the 9th using the normalized Euclidean and normalized Hamming distance. Meanwhile, PHS is ranked the 9th and NaS is ranked the 8th using the normalized Hausdorff distance. The results are summarized in Tables 5, 6, and 7.

The ranking of the proposed PF-MULTIMOORA and the IF-MULTIMOORA [68] for the different strategies are given in Table 8.

The results of the three approaches in Table 3 are defuzzified as given in Table 9, and then the alternatives are ranked using the dominance theory. From Table 10, the generally dominating rule is applied to rank the alternatives. The ranking is the

Table 5 Results using technical strategy

	Technical					
	Normalized Hamming distance		Normalized Euclidean distance		Normalized Hamming distance	
	Score	Rank	Score	Rank	Score	Rank
Hydrogen	−0.2662	10	−0.2371	10	−0.2164	10
PHS	−0.2649	9	−0.2258	8	−0.2126	8
CAES	−0.3246	11	−0.2777	11	−0.2694	11
Flywheel	−0.2107	5	−0.1889	5	−0.1716	5
SMES	−0.4378	14	−0.4080	14	−0.3932	14
Super cap	−0.2179	6	−0.1919	6	−0.1731	6
Pb-acid	−0.3315	12	−0.3033	12	−0.2815	12
Ni–Cd	−0.1250	2	−0.1074	2	−0.0869	2
Li-ion	−0.2424	7	−0.2207	7	−0.2028	7
NaS	−0.2469	8	−0.2308	9	−0.2133	9
NaNiCl	−0.1148	1	−0.0982	1	−0.0774	1
VRB	−0.4014	13	−0.3470	13	−0.3239	13
ZnBr	−0.1535	3	−0.1337	3	−0.1137	3
Molten salt	−0.1968	4	−0.1743	4	−0.1548	4

Table 6 Results using economic strategy

	Economic					
	Normalized Hamming distance		Normalized Euclidean distance		Normalized Hausdorff distance	
	Score	Rank	Score	Rank	Score	Rank
Hydrogen	−0.1146	9	−0.0887	9	−0.0704	9
PHS	−0.1470	11	−0.1118	11	−0.0943	11
CAES	−0.1357	10	−0.0998	10	−0.0824	10
Flywheel	−0.0154	5	0.0052	5	0.0268	5
SMES	−0.2878	14	−0.2455	14	−0.2251	14
Super cap	−0.0609	6	−0.0360	6	−0.0124	6
Pb-acid	−0.1610	12	−0.1321	12	−0.1066	12
Ni–Cd	0.0526	2	0.0677	2	0.0924	2
Li-ion	−0.1048	8	−0.0799	8	−0.0574	8
NaS	−0.0733	7	−0.0520	7	−0.0300	7
NaNiCl	0.0937	1	0.1060	1	0.1269	1
VRB	−0.2519	13	−0.2013	13	−0.1733	13
ZnBr	0.0260	3	0.0439	3	0.0691	3
Molten salt	0.0120	4	0.0288	4	0.0468	4

Table 7 Results using environmental strategy

	Environmental					
	Normalized Hamming distance		Normalized Euclidean distance		Normalized Hausdorff distance	
	Score	Rank	Score	Rank	Score	Rank
Hydrogen	−0.3009	9	−0.2729	9	−0.2538	9
PHS	−0.3537	10	−0.3081	10	−0.2970	10
CAES	−0.3669	11	−0.3160	11	−0.3031	11
Flywheel	−0.0066	4	0.0107	4	0.0283	4
SMES	−0.4814	14	−0.4483	14	−0.4335	14
Super cap	−0.1203	6	−0.0988	6	−0.0801	6
Pb-acid	−0.3942	12	−0.3654	12	−0.3438	12
Ni–Cd	0.0465	2	0.0594	2	0.0788	2
Li-ion	−0.1481	8	−0.1280	8	−0.1096	8
NaS	−0.1388	7	−0.1204	7	−0.1024	7
NaNiCl	0.0667	1	0.0788	1	0.0980	1
VRB	−0.4470	13	−0.3862	13	−0.3650	13
ZnBr	0.0281	3	0.0434	3	0.0636	3
Molten salt	−0.0836	5	−0.0656	5	−0.0477	5

Table 8 Ranking results for different strategies using PF-MULTIMOORA and IF-MULTIMOORA

Technology	Balanced		Technical		Economic		Environmental	
	PF	IF	PF	IF	PF	IF	PF	IF
Hydrogen	9	6	10	6	9	5	9	9
PHS	10	9	8	5	11	9	10	11
CAES	11	8	11	4	10	7	11	10
Flywheel	4	5	5	10	5	6	4	5
SMES	14	14	14	14	14	14	14	14
Super cap	6	11	6	11	6	11	6	8
Pb-acid	12	12	12	9	12	10	12	12
Ni–Cd	2	4	2	2	2	4	2	4
Li-ion	8	10	7	8	8	12	8	7
NaS	7	7	9	12	7	8	7	6
NaNiCl	1	2	1	7	1	2	1	1
VRB	13	13	13	13	13	13	13	13
ZnBr	3	3	3	3	3	3	3	2
Molten salt	5	1	4	1	4	1	5	3

Table 9 The scores of the three approaches using the Euclidean distance

Technology	AU score	RPU score	MU score
Hydrogen	0.2444	−0.4497	−0.2777
PHS	0.1763	−0.3360	−0.4007
CAES	0.1875	−0.3498	−0.4141
Flywheel	0.4759	−0.4682	−0.0609
SMES	0.0571	−0.5501	−0.5126
Super cap	0.3004	−0.4478	−0.1087
Pb-acid	0.2352	−0.4843	−0.4096
Ni–Cd	0.5625	−0.4284	−0.0211
Li-ion	0.2725	−0.4853	−0.1487
NaS	0.3176	−0.5234	−0.1116
NaNiCl	0.6245	−0.4120	0.0315
VRB	0.0403	−0.5564	−0.4458
ZnBr	0.5241	−0.4376	0
Molten salt	0.4389	−0.4343	−0.0880

same till the ninth place. A slight change is observed in the least ranked ESTs. PHS and Pb-acid exchange the 10th and 12th place, and SMES and VRB exchange the 13th and 14th place.

Table 10 The solution using the dominance theory

Rank	AU ranking	RPU ranking	MU ranking	Total rank
1	NaNiCl	PHS	NaNiCl	NaNiCl
2	Ni–Cd	CAES	Ni–Cd	Ni–Cd
3	ZnBr	NaNiCl	ZnBr	ZnBr
4	Flywheel	Ni–Cd	Flywheel	Flywheel
5	Molten salt	Molten salt	Molten salt	Molten salt
6	NaS	ZnBr	Super cap	Super cap
7	Super cap	Super cap	NaS	NaS
8	Li-ion	Hydrogen	Li-ion	Li-ion
9	Hydrogen	Flywheel	Hydrogen	Hydrogen
10	Pb-acid	Pb-acid	PHS	Pb-acid
11	CAES	Li-ion	Pb-acid	PHS
12	PHS	NaS	CAES	CAES
13	SMES	SMES	VRB	SMES
14	VRB	VRB	SMES	VRB

6 Conclusion

The MULTIMOORA method is one of the most practical MCDM methods that has been used to solve complicated decision-making problems. In this chapter, a new version of MULTIMOORA is developed to increase its efficiency and accuracy for solving large scale MCDM applications in the Pythagorean fuzzy environment. The proposed PF-MULTIMOORA exploits newly proposed aggregation operators that guarantee fair treatment among the evaluation criteria. Therefore, the proposed method avoids any biased treatment and false ranking that might occur in certain situations. When applying the reference point technique the distance is defined on a fuzzy basis rather than a crisp basis. Hence, instead of utilizing one reference point, i.e., the best rating, two reference points are utilized: the best and worst ratings. To avoid the complications of the dominance theory, the aggregation approach is applied. So, the aggregation approach is carried out using the fuzzy results of the three techniques. Thus, defuzzification is employed only in the final step for ranking in which the accuracy function can be also utilized with the score function to make the comparison more discriminatory.

Energy storage technologies were evaluated using the developed PF-MULTIMOORA. Sodium-nickel chloride, nickel–cadmium, and zinc–bromine were the top-ranked energy storage technologies. Meanwhile, lead-acid, vanadium redox, and superconducting magnetic energy storage were the worst-ranked technologies.

The dominance theory was applied to rank the alternatives instead of aggregating the three approaches. The result revealed that the alternatives till the ninth place are unchanged. A slight change is observed in the least ranked ESTs. It was clear that ranking by aggregating the three approaches is more direct and simpler than the dominance theory.

Besides the balanced weighting strategy, the effect of technical, economic, and environmental strategies on the ranking of the ESTs was studied. Only slight differences were observed in the results from the balanced strategy in the moderately performing technologies, while the three top technologies and three worst technologies remain unchanged.

In addition to the normalized Euclidean distance, another two distance measures were examined in the reference point technique, the normalized Hausdorff and the normalized Hamming distances. The ranking of the ESTs for the different weighting strategies using the two distance measures was quite similar to the ranking using the normalized Euclidean distance.

The contribution of the study can be summarized as follows. First, in the reference point technique, two reference points are used instead of one. Hence, the distance can be expressed using PFSs. Second, the study exploits aggregation operators in the ratio system approach and the full multiplicative form approach that prevent erroneous decisions.

The proposed PF-MULTIMOORA is restricted to using the theoretical reference point to guarantee that the resulting fuzzy distance is a PFS. The Empirical reference point cannot be utilized, which is a limitation in the proposed method. Future research will focus on expressing fuzzy distances using both theoretical and empirical reference points. Also, reference point techniques namely, PF-TOPSIS and PF-VIKOR will be implemented using fuzzy distances to study its performance compared with using crisp distances.

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Extensions of the Pythagorean Fuzzy Sets

Application of Linear Programming in Diet Problem Under Pythagorean Fuzzy Environment

Sapan Kumar Das and Seyyed Ahmad Edalatpanah

1 Introduction

The main significance of LP problems applied in real-world application such as management fields, engineering fields, economics sector, health sector, and transportation sector, etc. For decision-makers, it always happens that an individual, a group or a community faces the problem of maintaining good ratios among some crucial parameters. Due to rapid application in real-world application, many researchers showing interest and proposed many methods for solving the classical LP problem. In classical LP problem, all the parameters and variables are considered on exact values. However, when we applied in practical application, in that situation the information is undetermined. Due to uncertainty in real-life problems, the manager cannot always formulate the problem in a well-defined manner and exact. While addressing some real-world problems, the parameters and variables are imprecise and change rapidly due to several factors like labor, timing factor, innovation of new technology, etc. Therefore, uncertainty is an inborn part in decision making problems. Due to some drawbacks in classical LP problems, Zadeh [1] pioneered the idea of fuzzy set (F.S) in 1965, since then, researchers established a model of uncertainty arising in practical decision making problems. Bellman and Zadeh [2] proposed the definition of fuzzy decision. Since then many researchers have worked on fuzzy mathematical programming [3–16]. Many authors have introduced fuzzy programming approach to solve crisp linear programming problem [4, 6–8]. Some of them reduced fuzzy multi-objective linear programming to crisp programming problem using ranking functions [3, 10, 12]. Here, if the LP problem considers all the parameters were defined as fuzzy numbers. This type of problem is called fuzzy LP (FLP)

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_13

problems. So many researchers have considered [9, 17–20] the FLP problem and introduced several methods. Nassseri and Attari [10] introduced a method to solve FLP problem by using crisp LP problem and solved by classical technique. In that article, the authors introduced a new type of arithmetic for the symmetric trapezoidal fuzzy number. Lai and Hwang [18] considered FLP problem having triangular fuzzy numbers and they converted such problems to a multi-objective formulation to solve it. Lotfi et al. [21] applied lexicography and fuzzy approximate approaches to tackle linear programming with TFNs. Das et al. [3] proposed a new lexicographic method for solving FLP problem having TFNs. Das et al. [4] considered the same problem and solved it by using ranking function. Second, we deal with parameters taking TrFNs. Liu [9] introduced a method to measure the satisfaction of the constraints in linear programming with TrFNs. Maleki and Mashinchi [20] solved linear programming with TrFNs by a probabilistic approach. Allahviranloo et al. [22] proposed a ranking function-based approach for linear programming with TrFNs. Ebrahimnejad [52] proposed a revised fuzzy simplex method for linear programming with TrFNs and obtained some new results. Still, there exist a number of limitations for decision-maker to take decision while addressing in real-world problems and it requires all the parameters and variables are fuzzy numbers. Such types of problems are called as fully fuzzy LP (FFLP) problem. Das et al. [3] introduced a lexicographic method for solving fully fuzzy LP problem. In this article, the authors convert to multi-objective LP problem by using lexicographic technique and solved it. Recently, Das et al. [4] proposed a new method for solving fully fuzzy LP problem by using ranking function. Kumar et al. [19] also proposed another method for solving FFLP problem utilizing the ranking function.

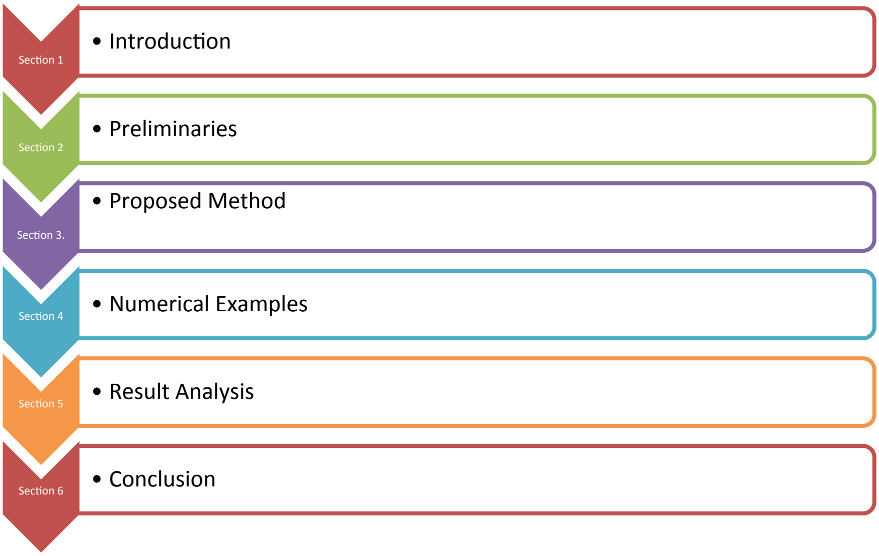
After successful implication of LP problem under fuzzy environment, the mathematicians had found some drawbacks when applied in real-life problem. The main drawbacks in fuzzy sets are the decision-makers consider only the membership function and ignore non-membership function. Here, the degree of non-membership function is just a compliment of the degree of membership function. Therefore, a new non-standard fuzzy subset is introduced by Yager [23] and called as Pythagorean fuzzy sets (PFS). So many researchers, [23–37] consider the PFNs for interval numbers and solved it. The main dominance of PFS comparing to fuzzy set is that the square sum of membership degree and non-membership degree is less than or equal to 1. There are various methods of PFS to solve decision making problem. Garg [38] proposed an interval-valued PF sets and using score function to solve it. Garg [39] considers PF systems with the help of aggregation Einstein operations and its application in decision making.

Motivation: Based on the above study on LP problem, there are no methods for solving LP problem under PF environment. Therefore, a deep study is required to establish a new method for Pythagorean fuzzy LP problem. To the best of our knowledge, there are no optimization models in literature for LP under Pythagorean fuzzy environment. This complete scenario has motivated us to come up with a new method for solving LP with the Pythagorean fuzzy range which are formulated and solved with the use of the proposed algorithm for the first time. Pythagorean set theory is documented technique to manage uncertainty in the optimization problem.

The main contributions of this paper are as follows:

1. New method helps to resolve a new set of problem with the use of PF number.
2. We characterize the LP issue under typical PF environmental factors and suggest an effective solution to find the relating crisp esteemed.
3. Inside the literature of PF set, we tend to present a scoring approach related to the proposed technique.

1.1 The remainder of the paper is orchestrated in an accompanying way.



2 Preliminaries

In this section, we present some definitions and key concepts that are very useful to the awareness of this paper. Most of all are well defined by [40, 41] and also used by many researchers.

Definition 1 [41] Let Z be a Pythagorean fuzzy (PF) sets defined on a universal set S is given by:

$$Z = \{\langle s, \sigma_Z(s), \eta_Z(s)\rangle | s \in S\}, \quad (1)$$

where the function $\sigma_Z(s) : Z \to [0, 1]$ and $\eta_Z(s) : Z \to [0, 1]$ are the degree of both membership function and non-membership function. Also for every $s \in S$, it holds that

$$(\sigma_Z(s))^2 + (\eta_Z(s))^2 \leq 1 \tag{2}$$

Definition 2 [42] Let $\tilde{u}_1^Z = (\sigma_r^Z, \eta_t^Z)$ and $\tilde{v}_1^Z = (\sigma_b^Z, \eta_c^Z)$ be two Pythagorean fuzzy numbers. Then the arithmetic operations are as follows:

(i) $\tilde{u}_1^Z \oplus \tilde{v}_1^Z = (\sqrt{(\sigma_r^Z)^2 + (\sigma_b^Z)^2 - (\sigma_r^Z)^2 \cdot (\sigma_b^Z)^2}, (\eta_t^Z) \cdot (\eta_c^Z))$

(ii) $\tilde{u}_1^Z \otimes \tilde{v}_1^Z = (\sigma_r^Z \cdot \sigma_b^Z, \sqrt{(\eta_t^Z)^2 + (\eta_c^Z)^2 - (\eta_t^Z)^2 \cdot (\eta_c^Z)^2})$

(iii) $h \cdot \tilde{u}_1^Z = \left(\sqrt{1 - (1 - \sigma_r^Z)^h}, (\eta_t^Z)^h\right)$, where h is a scalar product and non-negative, i.e., $h > 0$.

(iv) $(\tilde{u}_1^Z)^h = \left((\sigma_r^Z)^h, \sqrt{1 - (1 - (\eta_t^Z)^h}\right)$, where h is a scalar product and non-negative, i.e., $h > 0$.

Definition 3 [41] Let $\tilde{u}_1^Z = (\sigma_r^Z, \eta_t^Z)$ and $\tilde{v}_1^Z = (\sigma_b^Z, \eta_c^Z)$ be two Pythagorean fuzzy numbers. Then we define the score and accuracy functions are as follows:

(i) Score function: $Sr(\tilde{u}_1^Z) = \frac{1}{2}(1 - (\sigma_r^Z)^2 - (\eta_t^Z)^2)$
(ii) Accuracy function: $Acr(\tilde{u}_1^Z) = (\sigma_r^Z)^2 + (\eta_t^Z)^2)$

Based on the score and accuracy function of PFNs, a comparison method for two PFNs is defined as follows:

Situation 1: $\tilde{u}_1^Z > \tilde{v}_1^Z$ iff $Sr(\tilde{u}_1^Z) > Sr(\tilde{v}_1^Z)$
Situation 2: $\tilde{u}_1^Z < \tilde{v}_1^Z$ iff $Sr(\tilde{u}_1^Z) < Sr(\tilde{v}_1^Z)$
Situation 3: if $Sr(\tilde{u}_1^Z) = Sr(\tilde{v}_1^Z)$ and $Acr(\tilde{u}_1^Z) < Acr(\tilde{v}_1^Z)$ then $\tilde{u}_1^Z < \tilde{v}_1^Z$
Situation 4: if $Sr(\tilde{u}_1^Z) = Sr(\tilde{v}_1^Z)$ and $Acr(\tilde{u}_1^Z) > Acr(\tilde{v}_1^Z)$ then $\tilde{u}_1^Z > \tilde{v}_1^Z$
Situation 5: if $Sr(\tilde{u}_1^Z) = Sr(\tilde{v}_1^Z)$ and $Acr(\tilde{u}_1^Z) = Acr(\tilde{v}_1^Z)$ then $\tilde{u}_1^Z = \tilde{v}_1^Z$

Definition 4 [23] Let $\tilde{u}_1^Z = (\sigma_r^Z, \eta_t^Z)$ and $\tilde{v}_1^Z = (\sigma_b^Z, \eta_c^Z)$ be two Pythagorean fuzzy numbers, a nature quasi-ordering on the PF numbers is defined as follows: $\tilde{u}_1^Z \geq \tilde{v}_1^Z$ if and only if $\sigma_r^Z \geq \sigma_b^Z$ and $\eta_t^Z \leq \eta_c^Z$.

3 Proposed Method

Let us consider the standard form of linear programming (LP) problem with m constraints and n variables having all coefficients and resources are represented crisp numbers. In the LP problem, the objective functions are to either maximize the profit or minimize the cost of product from the source of destinations.

Here, we present the crisp LP model as follows:

$$\begin{aligned} &\text{maximize (minimize) } (c^t y) \\ &\text{s.t} \\ &\quad Dy \leq h, \end{aligned}$$

$$y \geq 0. \tag{3}$$

After all $D = \left[d_{ij}\right]_{m \times n}$ is the coefficient matrix, $h = [h_1, h_2, h_3, \ldots, h_m]^t$ is the available resource vector, $c = [c_1, c_2, c_3, \ldots, c_n]^t$ is the target coefficient and y is the selection variable vector.

Furthermore, if we replaced the parameter c^t from Eq. (3) into PF parameters, i.e., $(c^t)^Z$, then the new LP model obtained is called as Type-1 Pythagorean fuzzy LP (T1PFLP) problem, and we represent the model as follows:

$$\begin{aligned} &\text{maximize (minimize) } ((c^t)^Z y) \\ &\text{s. t} \\ &\quad Dy \leq h, \\ &\quad y \geq 0. \end{aligned} \tag{4}$$

After all $D = \left[d_{ij}\right]_{m \times n}$ is the coefficient matrix, $h = [h_1, h_2, h_3, \ldots, h_m]^t$ is the available resource vector, $c^Z = \left[c_1^Z, c_2^Z, c_3^Z, \ldots, c_n^Z\right]^t$ is the target coefficient and y is the selection variable vector.

Again, if the decision-maker will not be sure about the unit of coefficient matrix and resource vector, then we replace the parameters of D and h into PF numbers, i.e., $(D)^Z$ and $(h)^Z$, then the new LP model obtained is called as Type-2 Pythagorean fuzzy LP (T2PFLP) problem, and we represent the model as follows:

$$\begin{aligned} &\text{maximize (minimize) } ((c^t) y) \\ &\text{s. t} \\ &\quad (D)^Z y \leq (h)^Z, \\ &\quad y \geq 0. \end{aligned} \tag{5}$$

After all $(D)^Z = \left[(d_{ij})^Z\right]_{m \times n}$ is the coefficient matrix, $(h)^Z = \left[h_1^Z, h_2^Z, h_3^Z, \ldots, h_m^Z\right]^t$ is the available resource vector, $c^Z = [c_1, c_2, c_3, \ldots, c_n]^t$ is the target coefficient and y is the selection variable vector.

Finally, if the decision maker will not be sure about the coefficient matrix, parameters and resource vectors, we replace the parameter (c^t), D and h into PF numbers, i.e., $(c^t)^Z, (D)^Z$ and $(h)^Z$, then the new LP model obtained is called as Type-3 Pythagorean fuzzy LP (T3PFLP) problem, and we represent the model as follows:

$$\begin{aligned} &\text{maximize (minimize) } ((c^t)^Z y) \\ &\text{s. t} \\ &\quad (D)^Z y \leq (h)^Z, \\ &\quad y \geq 0. \end{aligned} \tag{6}$$

After all $(D)^Z = \left[(d_{ij})^Z\right]_{m\times n}$ is the coefficient matrix, $(h)^Z = \left[h_1^Z, h_2^Z, h_3^Z, \ldots, h_m^Z\right]^t$ is the available resource vector, $c^Z = \left[c_1^Z, c_2^Z, c_3^Z, \ldots, c_n^Z\right]^t$ is the target coefficient and y is the selection variable vector.

From the above three type of problem, i.e., Type-1, Type-2, and Type-3 PFLP problem, we propose a single main algorithm to solve all three types of PFLP problem. Now the steps are as follows:

Step 1: Firstly, we choose any of the models to solve the PFLP problem.

- Step 2.1: If we consider the Type-1 PF numbers, then we can transform the model Eq. (3) into the model of PFLP of Eq. (4).
- Step 2.1: If we consider the Type-2 PF numbers, then we can transform the model Eq. (3) into model Eq. (5).
- Step 2.2: If we consider the Type-3 PF numbers, then we can transform the model Eq. (3) into model Eq. (6).

Step 2: After choosing the model in Step 1, we utilizing the score function of each PF numbers.

Step 3: Replace all PFNs cost by its score function to obtain crisp LP problem.

Step 4: From Step 3, we use LINGO to take care of the fresh LP issue and get the ideal arrangement.

4 Numerical Example

In this section, we solved three types of examples to illustrate the potential application of the proposed method.

To best of our mind, still there is no direct method to solve PFNLP problem, therefore, in this section we consider a new method to solve PFNLP problem and compare it with fuzzy LP problem. The main drawback in fuzzy numbers is the manager only considers the membership degree; however, the PF numbers are considering both membership and non-membership degree.

Example 1 (T1PFN model) Consider the following PFLP problem:

$$\max = (0.4, 0.7)x_1 + (0.5, 0.4)x_2$$

Subject to

$$\begin{gathered} 5x_1 + 3x_2 \leq 12 \\ 2x_1 + 4x_2 \leq 7 \\ x_1, x_2 \geq 0 \end{gathered}$$

Now the problem is Type-1 PF number. As per our algorithm, we utilizing our step-2, new score function, the issue of T1PFN LP problem is converting into crisp LP problem and the problem will be as follows:

$$\max = 0.335x_1 + 0.545x_2$$

Subject to

$$\begin{gathered} 5x_1 + 3x_2 \leq 12 \\ 2x_1 + 4x_2 \leq 7 \\ x_1, x_2 \geq 0 \end{gathered}$$

Here, we can solve the crisp LP problem by using LINGO 18.0 we get the optimal solution.

The solution is: $x_1 = 1.98$, $x_2 = 0.78$ and $Z = 1.07$. Let us see in the following comparison table with existing method [4] and its very crystal clear that our method is always maximizing the result as the decision maker wanted (Table 1).

Example 2: (T2PFN) Consider the following type 2 PFNs of LP problems, where the objective functions are crisp, but in constraints part (except variables) are PFNs. We also remarked that the PFNs represent both membership and non-membership function, respectively. Additionally, they also satisfy all the conditions of PF, i.e., $0 \leq \sigma^Z \leq 1, 0 \leq \eta^Z \leq 1, 0 \leq (\sigma^Z)^2 + (\eta^Z)^2 \leq 1$.

$$\max = 0.0335x_1 + 0.0545x_2$$

Subject to

$$\begin{gathered} (0.4, 0.7)x_1 + (0.7, 0.3)x_2 \leq (0.7, 0.1) \\ (0.3, 0.8)x_1 + (0.1, 0.7)x_2 \leq (0.8, 0.1) \\ x_1, x_2 \geq 0 \end{gathered}$$

Now the problem is T2PFN. As per our algorithm, we utilizing our step-2, new core function, the issue of T2PFN LP problem is converting into crisp LP problem and the problem will be as follows:

$$\max = 0.0335x_1 + 0.0545x_2$$

Subject to

$$\begin{gathered} 0.335x_1 + 0.7x_2 \leq 0.74 \\ 0.775x_1 + 0.74x_2 \leq 0.815 \\ x_1, x_2 \geq 0 \end{gathered}$$

Table 1 Comparison with classical fuzzy LP

Solution	Proposed method	Fuzzy LP [4]
Optimal value	$x_1 = 1.98, x_2 = 0.78$	$x_1 = 1.85, x_2 = 0.52$
Optimal solution	$Z = 1.07$	$Z = 0.98$

Table 2 Comparison with classical fuzzy LP

Solution	Proposed method	Fuzzy LP [4]
Optimal value	$x_1 = 2.209, x_2 = 0$	$x_1 = 2.209, x_2 = 0$
Optimal solution	$Z = 0.74$	$Z = 0.74$

Here, we can solve the crisp LP problem by using LINGO 18.0 we get the optimal solution.

The solution is: $x_1 = 2.209, x_2 = 2.1229$ and $Z = 0.74$. Let us see in the following comparison table with existing method [4] and its very crystal clear that our method is always maximizing the result as the decision maker wanted (Table 2).

4.1 Diet Problem (T3PFN)

Currently, the whole world is suffering from pandemic Covid-19. It is very challenging for our scientists and doctors, to prevent this virus. They have suggested people to improve human immunity. To improve immunity, diet chat is very important to us. Diet chart not only improves our immunity but also helps many factors in our day-to-day basis. Therefore, we solved a real-life diet chart problem.

Below there is a diet chart that gives protein and carbohydrate content for 02 food items with two products like milk, roasted chicken. The Manager wants a diet with minimum cost. Likewise during the entire process, the manager falters in expectations of parameter values due to some uncontrollable factors. The diet chart is as follows:

Nutrition	Milk	Roasted chicken	Minimum nutrition required
Protein	(0.4, 0.7)	(0.7, 0.3)	(0.7, 0.1)
Carbohydrate	(0.3, 0.8)	(0.1, 0.7)	(0.8, 0.1)
Minimum product required	(0.4, 0.7)	(0.5, 0.4)	

Here, we consider the objective functions, constraints, and cost values are PFNs.

Now the problem is T3PFN. As per our algorithm, we utilizing our step-2, new core function, the issue of T3PFN LP problem is converting into crisp LP problem and the problem will be as follows:

$$\min = (0.4, 0.7)x_1 + (0.5, 0.4)x_2$$

Subject to,

$$\begin{gathered}(0.4, 0.7)x_1 + (0.7, 0.3)x_2 \leq (0.7, 0.1)\\ (0.3, 0.8)x_1 + (0.1, 0.7)x_2 \leq (0.8, 0.1)\\ x_1, x_2 \geq 0\end{gathered}$$

By utilizing our step-2, new ranking function, the issue of the PYFLP problem is converting into a crisp LP problem and the problem will be as follows:

$$\min = 0.335x_1 + 0.545x_2$$

Subject to,

$$\begin{gathered} 0.335x_1 + 0.7x_2 \leq 0.74 \\ 0.775x_1 + 0.74x_2 \leq 0.815 \\ x_1, x_2 \geq 0 \end{gathered}$$

Here, we can solve the crisp LP problem by using LINGO 18.0 we get the optimal solution.

The solution is: $x_1 = 0$, $x_2 = 0.2$ and $Z = 0.109$.

5 Result Analysis

It is seen that fuzzy linear programming does not ensure DM to give exact solutions. Because the fuzzy LP problems always concentrate the membership function and ignorance the non-membership function. Therefore, we proposed a solution technique to solve PFLP problem, which is considered both membership and non-membership functions. From the above three types of examples, we observed that PFLP is better than the fuzzy LP problem.

1. Our proposed model results are better than existing result [4]. As pers Table 1, the objective function is 1.07 and the existing result [4] is 0.98. As the problem is maximization taken by decision makers and our proposed objective solution results are also maximized.
2. In Example 1, our Type-2 of PFLP problem, we can see in Table 2 that our proposed results are equal to existing results, i.e., 0.74.
3. In our Example 2, we applied in Diet problem and solved it. Since in writing there is no immediate strategy for taking care of PFLP issue. In this way, we contrast our proposed outcome and fuzzy LP issue [4]. The consistent examination for all over three sorts of models, talked about models appears in the table.
4. In this table, we can see that the ideal estimation of PFLP issue is either equivalent or higher than the fluffy LP issue.
5. After above conversation, we may induce that our proposed technique is another approach to deal with the vulnerability in the fresh condition.

6 Conclusion

In this article, we consider a Pythagorean fuzzy LP and solved it. We proposed new arithmetic operations on Pythagorean fuzzy numbers and a score function is employed to get optimum solutions. We discussed three types of models of Pythagorean fuzzy LP problems. We use score function for converting into Pythagorean fuzzy numbers to its equivalent crisp numbers. After using this score function we solve the problem by using any standard method. We also illustrated solving a simple diet optimization problem with Pythagorean fuzzy linear programming problem. By comparing our proposed model with other existing fuzzy models, we concluded that our proposed model is simpler, efficient, and achieves better results than other researchers. In future, the proposed strategy can be applied in real-life applications like transportation problem, shortest path problem, assignment problem, job scheduling, etc. Additionally, our method can also extend to solve PF linear fractional programming problem.

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Maclaurin Symmetric Mean-Based Archimedean Aggregation Operators for Aggregating Hesitant Pythagorean Fuzzy Elements and Their Applications to Multicriteria Decision Making

Arun Sarkar and Animesh Biswas

1 Introduction

In real life decision making environments, multicriteria decision making (MCDM) processes play important roles to determine most suitable one from a limited number of alternatives. In this process, the alternatives are ranked by integrating evaluation information provided by the decision makers (DMs). In the process of evaluation of the alternatives with respect to criteria, DMs often utilize real numbers to express their judgment values. However, to capture uncertainties in MCDM problems, DMs gradually started to use fuzzy numbers for expressing their opinion after the initiation of fuzzy set [1]. Several methods for solving MCDM problems have already been developed [2–5].

Apart from wide advantages of using fuzzy sets, DMs sometimes face difficulties in assigning decision values in the form of fuzzy numbers due to increasing complexities and uncertainties in modern decision making processes. To overcome this situation, Atanassov [6] introduced the concept of intuitionistic fuzzy (IF) set (IFS) by incorporating non-membership grade with membership value. Many researchers studied decision making problems, extensively, in IF environments [7–12], after its inception.

Sometimes, the situation might go beyond the scope of IFSs, when the most pertinent feature of IFSs is violated, i.e., the sum of membership and non-membership grades exceeds 1. To tackle this issue, Yager [13, 14] introduced Pythagorean fuzzy (PF) set (PFS) by extending the concept of IFS. In PFS, the square sum of membership and non-membership grades does not exceed 1. After the introduction of PFS, theoretical as well as practical advancement on this subject area has been rapidly

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_14

progressed. For instance, Zhang and Xu [15] introduced elaborated mathematical notations for PFS, and proposed PF number (PFN). In aggregating PFNs, aggregation operators (AOs) are frequently used. Yager [14] developed some novel fuzzy weighted averaging (WA) and weighted geometric (WG) operators using PFSs. Peng and Yuan [16] used generalized mean [17] operator in PF environment to define generalized weighted AOs. For developing AOs, the operations are defined based on several t-conorms and t-norms. As an alternative presentation of algebraic t-conorms and t-norms, Garg [18] utilized the concept of Einstein operations and generalized mean to present Einstein operation-based geometric AOs under PF environment, viz., WG, Einstein ordered WG, generalized Einstein WG, and generalized Einstein ordered WG operators. Various properties and applications in MCDM problems using those operators are also discussed. For considering the relationship among the fused arguments, Wei and Lu [19] utilized power AOs to develop some PF power AOs, such as PF power averaging, WA, ordered WA, hybrid averaging operators, and correspondingly their geometric variants. Apart from those developed operators, many researchers presented different AOs [20–28] under PF contexts. Based on Einstein operations, Rahman et al. [29] introduced PF Einstein WG operator. Garg [30] further extended Einstein operations to develop weighted, ordered weighted, and hybrid geometric interactive AOs. Zeng et al. [31] presented PF ordered WA distance operator. Based on several concepts of aggregation operations, Garg, together with his co-researchers, explored different AOs [32–36] to enrich the area of PF information processing.

Sometimes, DMs prefer to express their decision values toward the alternatives with respect to some criteria in terms of a set of possible membership values, rather than to express decision values in terms of a single value. From that viewpoint, Torra and Narukawa [37] and Torra [38] introduced an effective concept, viz., hesitant fuzzy (HF) sets (HFSs), by considering a set of possible membership values within [0, 1]. Extending the concept of HFS in PF context, Liang and Xu [39] proposed hesitant Pythagorean fuzzy (HPF) sets (HPFSs), and defined operations on HPF elements (HPFEs). In HPFS, the degrees of membership and non-membership corresponding to an element to a given set are represented by two sets of numbers satisfying the condition that the square sum of the greatest membership and non-membership values is less than or equal to 1. In HPF environments, Lu et al. [40] proposed HPF Hamacher WA (HPFHWA), HPF Hamacher ordered WA (HPFHOWA) operators, and their corresponding geometric operators. Garg [41] developed HPF WA and WG, ordered WA and WG, and hybrid WA and WG operators. Under HPF context, Oztaysi et al. [42] presented a multi-expert and multicriteria HPF decision analysis to select best fit water treatment technology.

Further, it is to be noted here that the above AOs are almost based on the hypothesis that input arguments are independent. In some real-life situations, there may be interactions among different attributes. Considering this aspect, Maclaurin symmetric mean (MSM) [43] was proposed by Maclaurin. The main advantage of this operation is that it can capture interrelationship among several input arguments than the Bonferroni mean (BM), which can only consider interrelationship between only two arguments. Wei and Lu [44] established Maclaurin symmetric mean (MSM) operator to PF environment and proposed PF MSM (PFMSM) and PF weighted MSM (PFWMSM) operators. Garg [45, 46] extended MSM operator to HPF environment and developed HPF MSM (HPFMSM) operator for aggregating HPF information.

It is well known that Archimedean t-conorms and t-norms (At-CN&t-Ns) [47, 48] possess greater capability of generating various aggregation operators. Recently, Sarkar and Biswas [49] introduced At-CN&t-N operations on Pythagorean HF (PHF) sets and defined At-CN&t-N-based PHF WA and WG AOs. Inspired by the work of Sarkar and Biswas [49], this chapter proposes At-CN&t-N-based operational rules on HPFSs and investigates their properties. Utilizing these operational laws and taking the advantage of MSM operator, At-CN&t-N-based HPF MSM (AHPFMSM) and weighted MSM (AHPFWMSM) AOs are proposed for dealing with the aggregation of HPFEs. A methodology has been developed for solving MCDM problems with HPF information based on the proposed operators to capture correlation between input arguments provided by the DMs.

The rest of this chapter is organized as follows: in Sect. 2, a brief review on basic concepts of PFSs, HPFSs, MSM operator, and At-CN&t-Ns are presented. Section 3 provides some operations of HPFEs based on At-CN&t-Ns. Utilizing those operations, MSM-based AOs, viz., AHPFMSM and AHPFWMSM operators are introduced and their desirable properties are investigated. Several forms of operators, viz., HPF weighted MSM (HPFWMSM), HPF Einstein weighted MSM (HPFEWMSM), HPF Hamacher weighted MSM (HPFHWMSM), and HPF Frank weighted MSM (HPFFWMSM) operators, which can be derived from the developed operators, are also presented in this section. In Sect. 4, an MCDM method based on the proposed HPFHWMSM and HPFFWMSM operators is developed. An example to illustrate the effectiveness of the proposed MCDM method is presented in Sect. 5. Comparison with the existing MCDM methods [44, 45] has been described in Sect. 6. Finally, Sect. 7 discusses the conclusions and scope for future studies.

2 Preliminaries

2.1 PFS

The basic concepts of PFSs [13, 14] and their properties, which are required to develop the proposed method, are briefly reviewed in this section.

Definition 1 ([13, 14]) Let X be a universe of discourse. A PFS P in X is given by

$$P = \left\{ < x, \mu_p(x), \nu_p(x) | x \in X > \right\}, \tag{1}$$

where $\mu_p : X \to [0, 1]$ denotes the degree of membership and $\nu_p : X \to [0, 1]$ denotes the degree of non-membership of the element $x \in X$ to the set P, with the condition that $0 \leq \left(\mu_p(x)\right)^2 + \left(\nu_p(x)\right)^2 \leq 1$.

The degree of indeterminacy of the PFS P is given by $\pi_p(x) = \sqrt{1 - \left(\mu_p(x)\right)^2 - \left(\nu_p(x)\right)^2}$. For convenience, Zhang and Xu [15] called $\left(\mu_p(x), \nu_p(x)\right)$ as a PFN and denoted it by $p = (\mu, \nu)$.

Definition 2 ([15]) For any PFN $p = (\mu, \nu)$, the score function $S(p)$ of p is defined as

$$S(p) = (\mu)^2 - (\nu)^2,$$

where $S(p) \in [-1, 1]$.

The accuracy function $A(p)$ of p is defined as

$$A(p) = (\mu)^2 + (\nu)^2.$$

Definition 3 ([15]) Let $p = (\mu, \nu)$, $p_1 = (\mu_1, \nu_1)$ and $p_2 = (\mu_2, \nu_2)$ be three PFNs, and $\lambda > 0$. Then the operations on PFNs are defined as follows:

(1) $p_1 \oplus p_2 = \left(\sqrt{{\mu_1}^2 + {\mu_2}^2 - {\mu_1}^2{\mu_2}^2}, \nu_1\nu_2\right)$,

(2) $p_1 \otimes p_2 = \left(\mu_1\mu_2, \sqrt{{\nu_1}^2 + {\nu_2}^2 - {\nu_1}^2{\nu_2}^2}\right)$,

(3) $\lambda p = \left(\sqrt{1 - \left(1 - \mu^2\right)^{\lambda}}, \nu^{\lambda}\right)$,

(4) $p^{\lambda} = \left(\mu^{\lambda}, \sqrt{1 - \left(1 - \nu^2\right)^{\lambda}}\right)$.

2.2 HPFSs

Extending the notion of PFSs by simultaneously considering several membership and non-membership degrees for a PFN, the concept of HPFS [39, 45] is developed. It is expressed by the following definition.

Definition 4 ([39, 45]) Let X be a fixed set. An HPFS, $\tilde{K}$ on X is described as

$$\tilde{K} = \left\{\langle x, \tilde{h}_K(x), \tilde{g}(x)\rangle | x \in X\right\}, \tag{2}$$

where $\tilde{h}_K(x) : X \to [0, 1]$ and $\tilde{g}_K(x) : X \to [0, 1]$, respectively, denote two sets of possible membership and non-membership values of the element $x \in X$ to the set $\tilde{K}$ satisfying the condition that $0 \leq \left(\xi^+\right)^2 + \left(\eta^+\right)^2 \leq 1$, where $\xi^+ = \max_{\xi \in \tilde{h}_K(x)} \{\xi\}$, and

$\eta^+ = \max_{\eta \in \widetilde{\mathrm{g}}_K(x)} \{\eta\}$ for all $x \in X$. For convenience, the pair $\left(\tilde{h}_K(x), \widetilde{\mathrm{g}}_K(x)\right)$ is called an HPFE and is denoted by $\widetilde{\kappa} = \left(\tilde{h}, \widetilde{\mathrm{g}}\right)$.

Definition 5 ([39, 45]) Let $\widetilde{\kappa} = \left(\tilde{h}, \widetilde{\mathrm{g}}\right)$ be an HPFE. Then the score function of $\widetilde{\kappa}$ is defined as

$$S\left(\widetilde{\kappa}\right) = \left(\frac{1}{\left|\tilde{h}\right|}\sum\nolimits_{\xi \in \tilde{h}}\left(\xi^2\right) - \frac{1}{\left|\widetilde{\mathrm{g}}\right|}\sum\nolimits_{\eta \in \widetilde{\mathrm{g}}}\left(\eta^2\right)\right), \tag{3}$$

and the accuracy function of $\widetilde{\kappa}$ is defined as

$$A\left(\widetilde{\kappa}\right) = \left(\frac{1}{\left|\tilde{h}\right|}\sum\nolimits_{\xi \in \tilde{h}}\left(\xi^2\right) + \frac{1}{\left|\widetilde{\mathrm{g}}\right|}\sum\nolimits_{\eta \in \widetilde{\mathrm{g}}}\left(\eta^2\right)\right), \tag{4}$$

where $\left|\tilde{h}\right|$ and $\left|\widetilde{\mathrm{g}}\right|$ denote the number of elements in $\tilde{h}$ and $\widetilde{\mathrm{g}}$, respectively.

Definition 6 ([39, 45]) Let $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ be any two HPFEs, then

(i) if $S\left(\widetilde{\kappa}_1\right) > S\left(\widetilde{\kappa}_2\right)$ then $\widetilde{\kappa}_1 > \widetilde{\kappa}_2$,
(ii) if $S\left(\widetilde{\kappa}_1\right) = S\left(\widetilde{\kappa}_2\right)$ then

if $A\left(\widetilde{\kappa}_1\right) > A\left(\widetilde{\kappa}_2\right)$ then $\widetilde{\kappa}_1 > \widetilde{\kappa}_2$; if $A\left(\widetilde{\kappa}_1\right) = A\left(\widetilde{\kappa}_2\right)$ then $\widetilde{\kappa}_1 = \widetilde{\kappa}_2$.

To establish the interrelationship among the arguments to be aggregated, Maclaurin [43] proposed MSM operator which is presented as follows.

2.3 MSM Operator

Definition 7 ([43]) Let $a_j \geq 0$ $(j = 1, 2, \ldots, n)$ be a set of n real numbers . If

$$MSM^{(r)}(a_1, a_2, \ldots, a_n) = \left(\frac{\sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} \prod_{j=1}^{r} a_{i_j}}{C_r^n}\right)^{\frac{1}{r}}, \tag{5}$$

then $MSM^{(r)}$ is called the MSM operator, where $r = 1, 2, \ldots, n$; and $(i_1, i_2, \ldots, i_r)$ is a permutation of $(1, 2, \ldots, n)$, taking r number of ordered elements at a time maintaining the relationship $1 \leq i_1 < i_2 < \ldots < i_r \leq n$, and $C_r^n = \frac{n!}{r!(n-r)!}$ indicates the binomial coefficient.

The MSM operator has the following characteristics [43]:

(i) If $a \geq 0$ then $MSM^{(r)}(a, a, \ldots, a) = a$. (Idempotency)

(ii) $MSM^{(r)}(a_1, a_2, \ldots, a_n) \leq MSM^{(r)}(b_1, b_2, \ldots, b_n)$, if $a_i \leq b_i$ for all $i = 1, 2, \ldots, n$. (Monotonicity)

(iii) $\min_i\{a_i\} \leq MSM^{(r)}(a_1, a_2, \ldots, a_n) \leq \max_i\{a_i\}$. (Boundedness)

2.4 At-CN&t-Ns

Definition 8 ([47, 48]) An Archimedean t-conorm is generated by an increasing generator, g such that

$$U(a, b) = \mathrm{g}^{(-1)}(\mathrm{g}(a) + \mathrm{g}(b)) \text{for all } a, b \in [0, 1], \tag{6}$$

and an Archimedean t-norm is constructed by a decreasing generator, f such that

$$I(a, b) = f^{(-1)}(f(a) + f(b)) \text{with } \mathrm{g}(t) = f(1-t) \text{for all } a, b, t \in [0, 1]. \tag{7}$$

Considering some specific forms of the decreasing generator, f, Klement [50] proposed some t-conorms and t-norms as presented in Table 1.

Now, if $\tau = 1$ is considered, then Hamacher t-conorms and t-norms are reduced to algebraic t-conorm and t-norm, respectively. Also if $\tau = 2$ is considered, then

Table 1 t-conorms and t-norms generating from decreasing generator f

Generating function	Classes of t-conorms and t-norms	t-conorms	t-norms
$f(t) = -\log t$	Algebraic	$U^A(a, b) = a + b - ab$	$I^A(a, b) = ab$
$f(t) = -\log\left(\frac{2-t}{t}\right)$	Einstein	$U^E(a, b) = \frac{a+b}{1+ab}$	$I^E(a, b) = \frac{ab}{1+(1-a)(1-b)}$
$f(t) = \log\left(\frac{\tau+(1-\tau)t}{t}\right)$, $\tau > 0$	Hamacher	$U_\tau^H(a, b) = \frac{a+b-ab-(1-\tau)ab}{1-(1-\tau)ab}$	$I_\tau^H(a, b) = \frac{ab}{\tau+(1-\tau)(a+b-ab)}$
$f(t) = \log\left(\frac{\rho-1}{\rho^t-1}\right)$, $\rho > 1$	Frank	$U_\rho^F(a, b) = 1 - \log_\rho\left(1 + \frac{(\rho^{1-a}-1)(\rho^{1-b}-1)}{\rho-1}\right)$	$I_\rho^F(a, b) = \log_\rho\left(1 + \frac{(\rho^a-1)(\rho^b-1)}{\rho-1}\right)$

Hamacher t-conorms and t-norms take the form of Einstein t-conorm and t-norm, respectively.

3 Development of At-CN&t-N-Based MSM Operators for HPFEs

To develop AOs more flexible, At-CN&t-Ns are used in HPF environment. The advantage of introducing At-CN&t-Ns into HPFS is that it can produce various operations between HPFEs to make operational rules for AOs suppler.

It is also important to consider interactive argument among the criteria in MCDM contexts. To resolve this issue, MSM operation is merged with At-CN&t-Ns to develop several numbers of AOs by considering interactive relationships among the input arguments under HPF situation. The proposed AOs would be more flexible in the sense that it is applicable in such situations when there is no interaction among arguments, interaction between two arguments and also interaction among multiple arguments.

Now, based on At-CN&t-Ns and MSM, AOs are developed, and their properties are studied. Various operators which can be derived from those operators are also discussed, subsequently.

Definition 9 Let $\widetilde{\kappa}_i = \left\{\tilde{h}_i, \widetilde{\mathrm{g}}_i\right\}$ $(i = 1, 2)$ and $\widetilde{\kappa} = \left\{\tilde{h}, \widetilde{\mathrm{g}}\right\}$ be any three HPFEs. Now some new operational laws for HPFEs based on At-CN&t-Ns by using Eqs. (6) and (7) are defined as follows:

$$(1)\ \tilde{k}_1 \oplus_A \tilde{k}_2 = \left(\mathrm{U}_{\xi_1\in\tilde{h}_1,\xi_2\in\tilde{h}_2}\left\{\sqrt{U\left(\xi_1^2,\xi_2^2\right)}\right\}, \mathrm{U}_{\eta_1\in\tilde{g}_1,\eta_2\in\tilde{g}_2}\left\{\sqrt{I\left(\eta_1^2,\eta_2^2\right)}\right\}\right)$$

$$= \left(\mathrm{U}_{\substack{\xi_1\in\tilde{h}_1,\\ \xi_2\in\tilde{h}_2}}\left\{\sqrt{g^{-1}\left(g(\gamma_1^2)+g(\gamma_2^2)\right)}\right\}, \mathrm{U}_{\substack{\eta_1\in\tilde{g}_1,\\ \eta_2\in\tilde{g}_2}}\left\{\sqrt{f^{-1}\left(f(\eta_1^2)+f((\eta_2^2))\right)}\right\}\right),$$

$$(2)\ \tilde{k}_1 \oplus_A \tilde{k}_2 = \left(\mathrm{U}_{\xi_1\in\tilde{h}_1,\xi_2\in\tilde{h}_2}\left\{\sqrt{I\left(\xi_1^2,\xi_2^2\right)}\right\}, \mathrm{U}_{\eta_1\in\tilde{g}_1,\eta_2\in\tilde{g}_2}\left\{\sqrt{U\left(\eta_1^2,\eta_2^2\right)}\right\}\right)$$

$$= \left(\mathrm{U}_{\substack{\xi_1\in\tilde{h}_1,\\ \xi_2\in\tilde{h}_2}}\left\{\sqrt{f^{-1}\left(f(\xi_1^2)+f(\xi_2^2)\right)}\right\}, \mathrm{U}_{\substack{\eta_1\in\tilde{g}_1,\\ \eta_2\in\tilde{g}_2}}\left\{\sqrt{g^{-1}\left(g(\eta_1^2)+g((\eta_2^2))\right)}\right\}\right),$$

$$(3)\quad \lambda\,\widetilde{\kappa} = \left(\bigcup_{\xi\in\tilde{h}}\left\{\sqrt{\mathrm{g}^{-1}\left(\lambda \mathrm{g}(\xi^2)\right)}\right\}, \bigcup_{\eta\in\widetilde{\mathrm{g}}}\left\{\sqrt{f^{-1}\left(\lambda f(\eta^2)\right)}\right\}\right), \lambda > 0,$$

$$(4)\quad \widetilde{\kappa}^{\lambda} = \left(\bigcup_{\xi\in\tilde{h}}\left\{\sqrt{f^{-1}\left(\lambda f(\xi^2)\right)}\right\}, \bigcup_{\eta\in\widetilde{\mathrm{g}}}\left\{\sqrt{\mathrm{g}^{-1}\left(\lambda \mathrm{g}(\eta^2)\right)}\right\}\right), \lambda > 0.$$

Using the above definition and by extending the concept of MSM operator in HPF environment to aggregate all the possible arguments and to capture the interrelationship between them, the AHPFMSM operator is developed as follows.

Definition 10 Let $\widetilde{\kappa}_i = \left(\tilde{h}_i, \tilde{\mathrm{g}}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs and $r = 1, 2, \ldots, n$ be any number. If

$$AHPFMSM^{(r)}\left(\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_n\right) = \left(\frac{\oplus_{A\, 1 \le i_1 < i_2 < \ldots < i_r \le n} \otimes_{A\, j=1}^{r} \widetilde{\kappa}_{i_j}}{C_r^n}\right)^{\frac{1}{r}}, \quad (8)$$

then $AHPFMSM^{(r)}\left(\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_n\right)$ is called AHPFMSM operator.

Theorem 1 Let $\widetilde{\kappa}_i = \left(\tilde{h}_i, \tilde{\mathrm{g}}_i\right)$ $(i = 1, 2, \ldots, n)$ be a set of HPFEs, and $r = 1, 2, \ldots, n$ be any number. Then the aggregated value using AHPFMSM is also an HPFE and

$$AHPFMSM^{(r)}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \left(\frac{\oplus_{A\, 1 \le i_1 < i_2 < \ldots < i_r \le n} \otimes_{A\, j=1}^{r} \tilde{\kappa}_{i_j}}{C_r^n}\right)^{\frac{1}{r}}$$

$$\left(\bigcup_{\xi_{i_j} \in \tilde{h}_{i_j}} \left\{\left(f^{-1}\left(\frac{1}{r} f\left(\mathrm{g}^{-1}\left(\frac{1}{C_r^n} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} \mathrm{g}\left(f^{-1}\left(\sum_{j=1}^{r} f\left(\xi_{i_j}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}, \right.$$

$$\left. \bigcup_{\eta_{i_j} \in \tilde{g}_{i_j}} \left\{\left(\mathrm{g}^{-1}\left(\frac{1}{r} \mathrm{g}\left(f^{-1}\left(\frac{1}{C_r^n} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} f\left(\mathrm{g}^{-1}\left(\sum_{j=1}^{r} \mathrm{g}\left(\eta_{i_j}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}\right). \quad (9)$$

Proof First, applying mathematical induction, it is necessary to prove that

$$\otimes_{A\, j=1}^{r} \widetilde{\kappa}_{i_j} = \left(\bigcup_{\xi_{i_j} \in \tilde{h}_{i_j}} \left\{\left(f^{-1}\left(\sum_{j=1}^{r} f\left({\xi_{i_j}}^2\right)\right)\right)^{\frac{1}{2}}\right\}, \bigcup_{\eta_{i_j} \in \widetilde{g}_{i_j}} \left\{\left(\mathrm{g}^{-1}\left(\sum_{j=1}^{r} \mathrm{g}\left({\eta_{i_j}}^2\right)\right)\right)^{\frac{1}{2}}\right\}\right). \quad (10)$$

Now, for

$$r = 2, \otimes_{A\, j=1}^{2} \tilde{\kappa}_{i_j} = \left(\tilde{h}_{i_1}, \tilde{g}_{i_1}\right) \otimes_A \left(\tilde{h}_{i_2}, \tilde{g}_{i_2}\right)$$

$$= \left(\bigcup_{\substack{\xi_1 \in \tilde{h}_1, \\ \xi_2 \in \tilde{h}_2}} \left\{\left(f^{-1}\left(f\left(\xi_1^2\right) + f\left(\xi_2^2\right)\right)\right)^{\frac{1}{2}}\right\}, \bigcup_{\substack{\eta_1 \in \tilde{g}_1, \\ \eta_2 \in \tilde{g}_2}} \left\{\left(g^{-1}\left(g\left(\eta_1^2\right) + g\left(\eta_2^2\right)\right)\right)^{\frac{1}{2}}\right\}\right)$$

$$= \left(\bigcup_{\xi_{i_j} \in \tilde{h}_{i_j}} \left\{\left(f^{-1}\left(\sum_{j=1}^{2} f\left(\xi_{i_j}^2\right)\right)\right)^{\frac{1}{2}}\right\}, \bigcup_{\eta_{i_j} \in \tilde{g}_{i_j}} \left\{\left(g^{-1}\left(\sum_{j=1}^{2} g\left(\eta_{i_j}^2\right)\right)\right)^{\frac{1}{2}}\right\}\right).$$

Let Eq. (10) be true for $r = p$, i.e.,

$$\otimes_A{}_{j=1}^{p}\widetilde{\kappa}_{i_j}=\left(\bigcup_{\xi_{i_j}\in\tilde{h}_{i_j}}\left\{\left(f^{-1}\left(\sum_{j=1}^{p}f\left(\xi_{i_j}{}^2\right)\right)\right)^{\frac{1}{2}}\right\},\bigcup_{\eta_{i_j}\in\widetilde{g}_{i_j}}\left\{\left(\mathrm{g}^{-1}\left(\sum_{j=1}^{p}\mathrm{g}\left(\eta_{i_j}{}^2\right)\right)\right)^{\frac{1}{2}}\right\}\right).$$

Now, for $r=p+1$,

$$\otimes_{Aj=1}^{p+1}\tilde{\kappa}_{i_j}=\left(\otimes_{Aj=1}^{p}\tilde{\kappa}_{i_j}\right)\otimes_A\tilde{\kappa}_{i_{p+1}}$$

$$=\left(\cup_{\xi_{i_j}\in\tilde{h}_{i_j}}\left\{\left(f^{-1}\left(\sum_{j=1}^{p}f\left(\xi_{i_j}^2\right)\right)\right)^{\frac{1}{2}}\right\},\cup_{\eta_{i_j}\in\tilde{g}_{i_j}}\left\{\left(g^{-1}\left(\sum_{j=1}^{p}g\left(\eta_{i_j}^2\right)\right)\right)^{\frac{1}{2}}\right\}\right)$$

$$\otimes_A\left(\cup_{\xi_{i_{p+1}}\in\tilde{h}_{i_{p+1}}}\left\{\xi_{i_{p+1}}\right\},\cup_{\eta_{i_{p+1}}\in\tilde{g}_{i_{p+1}}}\left\{\eta_{i_{p+1}}\right\}\right)$$

$$=\left(\cup_{\xi_{i_j}\in\tilde{h}_{i_j}}\left\{\left(f^{-1}\left(\sum_{j=1}^{p}f\left(\xi_{i_j}^2\right)+f\left(\xi_{i_{p+1}}^2\right)\right)\right)^{\frac{1}{2}}\right\},\cup_{\eta_{i_j}\in\tilde{g}_{i_j}}\left\{\left(g^{-1}\left(\sum_{j=1}^{p}g\left(\eta_{i_j}^2\right)+g\left(\eta_{i_{p+1}}^2\right)\right)\right)^{2}\right\}\right)$$

$$=\left(\cup_{\xi_{i_j}\in\tilde{h}_{i_j}}\left\{\left(f^{-1}\left(\sum_{j=1}^{p+1}f\left(\xi_{i_j}^2\right)\right)\right)^{\frac{1}{2}}\right\},\cup_{\eta_{i_j}\in\tilde{g}_{i_j}}\left\{\left(g^{-1}\left(\sum_{j=1}^{p+1}g\left(\eta_{i_j}^2\right)\right)\right)^{\frac{1}{2}}\right\}\right).$$

Therefore, Eq. (10) is true for all r.

Now, using operational laws of HPFEs as defined in Definition 9, it can be easily shown that

$$\oplus_{A1\le i_1<i_2<\ldots<i_r\le n}\otimes_{Aj=1}^{r}\tilde{\kappa}_{i_j}$$

$$\left(\bigcup_{\xi_{i_j}\in\tilde{h}_{i_j}}\left\{\left(g^{-1}\left(\sum_{j=1}^{\binom{n}{r}}g\left(f^{-1}\left(\sum_{j=1}^{r}f\left(\xi_{i_j}^2\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\},\right.$$

$$\left.\bigcup_{\eta_{i_j}\in\tilde{g}_{i_j}}\left\{\left(f^{-1}\left(\sum_{j=1}^{\binom{n}{r}}f\left(g^{-1}\left(\sum_{j=1}^{r}g\left(\eta_{i_j}^2\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}\right).\tag{11}$$

Also, using Eqs. (10) and (11),

$$\frac{\oplus_{A1\le i_1<i_2<\ldots<i_r\le n}\otimes_{Aj=1}^{r}\tilde{\kappa}_{i_j}}{C_r^n}$$

$$=\left(\bigcup_{\xi_{i_j}\in\tilde{h}_{i_j}}\left\{\left(g^{-1}\left(\frac{1}{C_r^n}g\left(g^{-1}\left(\sum_{1\le i_1<i_2<\ldots<i_r\le n}g\left(f^{-1}\left(\sum_{j=1}^{r}f\left(\xi_{i_j}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\},\right.$$

$$\underset{\eta_{i_j}\in\tilde{g}_{i_j}}{\mathrm{U}}\left\{\left(f^{-1}\left(\frac{1}{C_r^n}f\left(f^{-1}\left(\sum_{1\le i_1<i_2<\ldots<i_r\le n}f\left(g^{-1}\left(\sum_{j=1}^r g\left(\eta_{i_j}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}\Bigg)$$

$$=\left(\underset{\xi_{i_j}\in\tilde{h}_{i_j}}{\mathrm{U}}\left\{\left(g^{-1}\left(\frac{1}{C_r^n}\sum_{1\le i_1<i_2<\ldots<i_r\le n}g\left(f^{-1}\left(\sum_{j=1}^r f\left(\xi_{i_j}^2\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\},\right.$$

$$\left.\underset{\eta_{i_j}\in\tilde{g}_{i_j}}{\mathrm{U}}\left\{\left(f^{-1}\left(\frac{1}{C_r^n}\sum_{1\le i_1<i_2<\ldots<i_r\le n}f\left(g^{-1}\left(\sum_{j=1}^r g\left(\eta_{i_j}^2\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}\right).$$

Finally,

$$AHPFMSM^{(r)}(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\left(\frac{\oplus A1\le i_1<i_2<\ldots<i_r\le n\otimes^r_{A j=1}\tilde{\kappa}_{i_j}}{C_r^n}\right)^{\frac{1}{r}}$$

$$=\left(\underset{\xi_{i_j}\in\tilde{h}_{i_j}}{\mathrm{U}}\left\{\left(f^{-1}\left(\frac{1}{r}f\left(g^{-1}\left(\frac{1}{C_r^n}\sum_{1\le i_1<i_2<\ldots<i_r\le n}g\left(f^{-1}\left(\sum_{j=1}^r f\left(\xi_{i_j}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\},\right.$$

$$\left.\underset{\eta_{i_j}\in\tilde{g}_{i_j}}{\mathrm{U}}\left\{\left(g^{-1}\left(\frac{1}{r}g\left(f^{-1}\left(\frac{1}{C_r^n}\sum_{1\le i_1<i_2<\ldots<i_r\le n}f\left(g^{-1}\left(\sum_{j=1}^r g\left(\eta_{i_j}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}\right).$$

Hence, the theorem is proved.

Theorem 2 (*Idempotency*) If $\widetilde{\kappa}_i=\left(\tilde{h}_i,\widetilde{\mathrm{g}}_i\right)$ are all equal to $\widetilde{\kappa}=\left(\tilde{h},\widetilde{\mathrm{g}}\right)$, for $i=1,2,\ldots,n$, then
$AHPFMSM^{(r)}\left(\widetilde{\kappa}_1,\widetilde{\kappa}_2,\ldots,\widetilde{\kappa}_n\right)=\widetilde{\kappa}$.

Proof From Theorem 1,

$$AHPFMSM^{(r)}(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)$$

$$=\left(\underset{\xi_{i_j}\in\tilde{h}_{i_j}}{\mathrm{U}}\left\{\left(f^{-1}\left(\frac{1}{r}f\left(g^{-1}\left(\frac{1}{C_r^n}\sum_{1\le i_1<\ldots<i_r\le n}g\left(f^{-1}\left(\sum_{j=1}^r f\left(\xi_{i_j}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\},\right.$$

$$\left.\underset{\eta_{i_j}\in\tilde{g}_{i_j}}{\mathrm{U}}\left\{\left(g^{-1}\left(\frac{1}{r}g\left(f^{-1}\left(\frac{1}{C_r^n}\sum_{1\le i_1<\ldots<i_r\le n}f\left(g^{-1}\left(\sum_{j=1}^r g\left(\eta_{i_j}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}\right).$$

Now, here $\widetilde{\kappa}_i=\widetilde{\kappa}=\left(\tilde{h},\widetilde{\mathrm{g}}\right)$ for all $i=1,2,\ldots,n$; so, $\xi_{i_j}=\xi$ and $\eta_{i_j}=\eta$ for all $i=1,2,\ldots,n$. Therefore,

$$AHPFMSM^{(r)}(\tilde{\kappa}, \tilde{\kappa}, \ldots, \tilde{\kappa}) = \left(\underset{\xi \in \tilde{h}}{\mathrm{U}} \left\{ \left(f^{-1}\left(\frac{1}{r} f\left(g^{-1}\left(\frac{1}{C_r^n} C_r^n g\left(f^{-1}\left(rf\left(\xi^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}} \right\}, \right.$$

$$\left. \underset{\eta \in \tilde{g}}{\mathrm{U}} \left\{ \left(g^{-1}\left(\frac{1}{r} g\left(f^{-1}\left(\frac{1}{C_r^n} C_r^n f\left(g^{-1}\left(rg\left(\eta^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}} \right\} \right)$$

$$= \left(\underset{\xi \in \tilde{h}}{\mathrm{U}} \left\{ \left(f^{-1}\left(\frac{1}{r} f\left(g^{-1}\left(g\left(f^{-1}\left(rf\left(\xi^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}} \right\}, \underset{\eta \in \tilde{g}}{\mathrm{U}} \left\{ \left(g^{-1}\left(\frac{1}{r} g\left(f^{-1}\left(f\left(g^{-1}\left(rg\left(\eta^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}} \right\} \right)$$

$$= \left(\underset{\xi \in \tilde{h}}{\mathrm{U}} \{\xi\}, \underset{\eta \in \tilde{g}}{\mathrm{U}} \{\eta\} \right) = \tilde{\kappa}.$$

Hence the proof.

Theorem 3 (Monotonicity) Suppose $\{\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n\}$ and $\left\{\tilde{\kappa}'_1, \tilde{\kappa}'_2, \ldots, \tilde{\kappa}'_n\right\}$ are two collections of HPFEs where $\tilde{\kappa}_i = \left(\tilde{h}_i, \tilde{g}_i\right)$, $\tilde{\kappa}'_i = \left(\tilde{h}'_i, \tilde{g}'_i\right)$. If $\xi_i \leq \xi'_i$ and $\eta_i \geq \eta'_i$ for all $i = 1, 2, \ldots, n$, where $\xi_i \in \tilde{h}_i$, $\xi'_i \in \tilde{h}'_i$, $\eta_i \in \tilde{g}_i$, and $\eta'_i \in \tilde{g}'_i$ represent the possible degrees of membership and non-membership, respectively, then

$$AHPFMSM^{(r)}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) \leq AHPFMSM^{(r)}\left(\tilde{\kappa}'_1, \tilde{\kappa}'_2, \ldots, \tilde{\kappa}'_n\right). \tag{12}$$

Proof Since f is a decreasing generator,

$$f^{-1}\left(\sum_{j=1}^{r} f\left(\left(\xi_{i_j}\right)^2 \right) \right) \leq f^{-1}\left(\sum_{j=1}^{r} f\left(\left(\xi'_{i_j}\right)^2 \right) \right).$$

Also, since g is an increasing generator,

$$\mathrm{g}\left(f^{-1}\left(\sum_{j=1}^{r} f\left(\left(\xi_{i_j}\right)^2 \right) \right) \right) \leq \mathrm{g}\left(f^{-1}\left(\sum_{j=1}^{r} f\left(\left(\xi'_{i_j}\right)^2 \right) \right) \right),$$

i.e.,

$$\mathrm{g}^{-1}\left(\frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} \mathrm{g}\left(f^{-1}\left(\sum_{j=1}^{r} f\left(\left(\xi_{i_j}\right)^2 \right) \right) \right) \right) \leq \mathrm{g}^{-1}\left(\frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} \mathrm{g}\left(f^{-1}\left(\sum_{j=1}^{r} f\left(\left(\xi'_{i_j}\right)^2 \right) \right) \right) \right),$$

i.e.,

$$f^{-1}\left(\frac{1}{r} f\left(\mathrm{g}^{-1}\left(\frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} \mathrm{g}\left(f^{-1}\left(\sum_{j=1}^{r} f\left(\left(\xi_{i_j}\right)^2 \right) \right) \right) \right) \right) \right)$$
$$\leq f^{-1}\left(\frac{1}{r} f\left(\mathrm{g}^{-1}\left(\frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} \mathrm{g}\left(f^{-1}\left(\sum_{j=1}^{r} f\left(\left(\xi'_{i_j}\right)^2 \right) \right) \right) \right) \right) \right). \tag{13}$$

Again, since g is an increasing generator, from the given condition,

$$\sum_{j=1}^{r} g\left(\left(\eta_{i_j}\right)^2\right) \geq \sum_{j=1}^{r} g\left(\left(\eta'_{i_j}\right)^2\right),$$

i.e.,

$$\frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} f\left(g^{-1}\left(\sum_{j=1}^{r} g\left(\left(\eta_{i_j}\right)^2\right)\right)\right) \leq \frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} f\left(g^{-1}\left(\sum_{j=1}^{r} g\left(\left(\eta'_{i_j}\right)^2\right)\right)\right),$$

i.e.,

$$\frac{1}{r} g\left(f^{-1}\left(\frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} f\left(g^{-1}\left(\sum_{j=1}^{r} g\left(\left(\eta_{i_j}\right)^2\right)\right)\right)\right)\right)$$
$$\geq \frac{1}{r} g\left(f^{-1}\left(\frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} f\left(g^{-1}\left(\sum_{j=1}^{r} g\left(\left(\eta'_{i_j}\right)^2\right)\right)\right)\right)\right),$$

i.e.,

$$\left(g^{-1}\left(\frac{1}{r} g\left(f^{-1}\left(\frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} f\left(g^{-1}\left(\sum_{j=1}^{r} g\left(\left(\eta_{i_j}\right)^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}$$
$$\geq \left(g^{-1}\left(\frac{1}{r} g\left(f^{-1}\left(\frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} f\left(g^{-1}\left(\sum_{j=1}^{r} g\left(\left(\eta'_{i_j}\right)^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}. \quad (14)$$

From (13) and (14), the theorem follows.

Theorem 4 (Boundedness) Suppose $\{\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n\}$ be any collection of HPFEs, where $\tilde{\kappa}_i = \left(\tilde{h}_i, \tilde{g}_i\right)$. Let $\xi_- = min\left\{\xi_{i_j}\right\}$, $\xi_+ = max\left\{\xi_{i_j}\right\}$, $\eta_- = min\left\{\eta_{i_j}\right\}$, and $\eta_+ = max\left\{\eta_{i_j}\right\}$, where $\xi_i \in \tilde{h}_i$, $\xi'_i \in \tilde{h}'_i$, $\eta_i \in \tilde{g}_i$, and $\eta'_i \in \tilde{g}'_i$ represent the possible degrees of membership and non-membership, respectively.

Then $\tilde{\kappa}_- \leq AHPFMSM^{(r)}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) \leq \tilde{\kappa}_+$, where $\tilde{\kappa}_- = (\xi_-, \eta_+)$ and $\tilde{\kappa}_+ = (\xi_+, \eta_-)$.

Proof

Since $\xi_- \leq \xi_{i_j} \leq \xi_+$ for all $i = 1, 2, \ldots, n$, so

$$\sum_{j=1}^{r} f\left((\xi_-)^2\right) \geq \sum_{j=1}^{r} f\left(\xi_{i_j}^2\right) \geq \sum_{j=1}^{r} f\left((\xi_+)^2\right)$$

$$\text{i.e., } g\left(f^{-1}\left(rf\left((\xi_-)^2\right)\right)\right) \leq \frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} g\left(f^{-1}\left(\sum_{j=1}^{r} f\left(\xi_{i_j}^2\right)\right)\right) \leq g\left(f^{-1}\left(rf\left((\xi_+)^2\right)\right)\right)$$

$$\text{i.e., } f\left((\xi_-)^2\right) \geq \frac{1}{r} f\left(g^{-1}\left(\frac{1}{C_r^n} \sum_{1 \leq i_1 < \ldots < i_r \leq n} g\left(f^{-1}\left(\sum_{j=1}^{r} f\left(\xi_{i_j}^2\right)\right)\right)\right)\right) \geq f\left((\xi_+)^2\right)$$

$$\text{i.e., } (\xi_-)^2 \leq f^{-1}\left(\frac{1}{r} f\left(g^{-1}\left(\frac{1}{C_r^n} \sum_{1\leq i_1<\ldots<i_r\leq n} g\left(f^{-1}\left(\sum_{j=1}^{r} f\left(\xi_{i_j}^2\right)\right)\right)\right)\right)\right) \leq (\xi_+)^2. \tag{15}$$

In a similar way, it can be proved that

$$(\eta_-)^2 \leq g^{-1}\left(\frac{1}{r} g\left(f^{-1}\left(\frac{1}{C_r^n} \sum_{1\leq i_1<\ldots<i_r\leq n} f\left(g^{-1}\left(\sum\nolimits_{j=1}^{r} g\left({\eta_{i_j}}^2\right)\right)\right)\right)\right)\right) \leq (\eta_+)^2. \tag{16}$$

Using Definitions 5, 6 and following Eqs. (15) and (16), the theorem holds

Theorem 5 (Commutativity) Let $\left\{\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_n\right\}$ be a set of HPFEs and $\left(\widetilde{\kappa}^*_1, \widetilde{\kappa}^*_2, \ldots, \widetilde{\kappa}^*_n\right)$ indicates any permutation of $\left(\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_n\right)$, then

$$AHPFMSM^{(r)}\left(\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_n\right) = AHPFMSM^{(r)}\left(\widetilde{\kappa}^*_1, \widetilde{\kappa}^*_2, \ldots, \widetilde{\kappa}^*_n\right).$$

Proof As $\left(\widetilde{\kappa}^*_1, \widetilde{\kappa}^*_2, \ldots, \widetilde{\kappa}^*_n\right)$ indicates any permutation of $\left\{\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_n\right\}$, then utilizing the definition of AHPFMSM, it is obtained that

$$AHPFMSM^{(r)}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \left(\frac{\oplus_{A 1\leq i_1<i_2<\ldots<i_r\leq n} \otimes_{A j=1}^{r} \tilde{\kappa}_{i_j}}{C_r^n}\right)^{\frac{1}{r}}$$
$$= \left(\frac{\oplus_{A 1\leq i_1<i_2<\ldots<i_r\leq n} \otimes_{A j=1}^{r} \tilde{\kappa}^*_{i_j}}{C_r^n}\right)^{\frac{1}{r}} = AHPFMSM^{(r)}\left(\tilde{\kappa}^*_1, \tilde{\kappa}^*_2, \ldots, \tilde{\kappa}^*_n\right).$$

Hence the theorem.

Now, some special cases of the AHPFMSM operator are discussed by varying the parameter, r.

- When $r = 1$, the proposed AHPFMSM operator becomes Archimedean HPF averaging (AHPFA) operator and is shown as follows:

$$AHPFMSM^{(1)}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \left(\frac{\oplus_{A 1\leq i_1\leq n} \otimes_{A j=1}^{1} \tilde{\kappa}_{i_j}}{C_1^n}\right)^{\frac{1}{1}}$$
$$= \left(\bigcup_{\xi_{i_j}\in\tilde{h}_{i_j}} \left\{\left(f^{-1}\left(f\left(g^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} g\left(\xi_i^2\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\},\right.$$

$$\underset{\eta_{i_j}\in\tilde{g}_{i_j}}{\mathrm{U}}\left\{\left(g^{-1}\left(g\left(f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}f\left(\left(\eta_i^2\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}\right). \tag{17}$$

- When $r = 2$, AHPFMSM operator reduces to Archimedean HPF BM (AHPFBM) operator as follows:

$$\begin{aligned}
&AHPFMSM^{(2)}(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\left(\frac{\oplus_{A1\le i_1<i_2\le n}\otimes_{Aj=1}^{2}\tilde{\kappa}_{i_j}}{C_2^n}\right)^{\frac{1}{2}}\\
&=\left(\underset{\xi_{i_j}\in\tilde{h}_{i_j}}{\mathrm{U}}\left\{\left(f^{-1}\left(\frac{1}{2}f\left(g^{-1}\left(\frac{1}{C_2^n}\sum_{1\le i_1<i_2\le n}g\left(f^{-1}\left(\sum_{j=1}^{2}f\left(\xi_{i_j}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\},\right.\\
&\left.\underset{\eta_{i_j}\in\tilde{g}_{i_j}}{\mathrm{U}}\left\{\left(g^{-1}\left(\frac{1}{2}g\left(f^{-1}\left(\frac{1}{C_2^n}\sum_{1\le i_1<i_2\le n}f\left(g^{-1}\left(\sum_{j=1}^{2}g\left(\eta_{i_j}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}\right)\\
&=\left(\underset{\xi_{i_j}\in\tilde{h}_{i_j}}{\mathrm{U}}\left\{\left(f^{-1}\left(\frac{1}{2}f\left(g^{-1}\left(\frac{2}{n(n-1)}\sum_{1\le i_1<i_2\le n}g\left(f^{-1}\left(f\left(\xi_{i_1}^2\right)+f\left(\xi_{i_2}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\},\right.\\
&\left.\underset{\eta_{i_j}\in\tilde{g}_{i_j}}{\mathrm{U}}\left\{\left(g^{-1}\left(\frac{1}{2}g\left(f^{-1}\left(\frac{2}{n(n-1)}\sum_{1\le i_1<i_2\le n}f\left(g^{-1}\left(g\left(\eta_{i_1}^2\right)+g\left(\eta_{i_2}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}\right)\\
&=\left(\underset{\xi_{i_j}\in\tilde{h}_{i_j}}{\mathrm{U}}\left\{\left(f^{-1}\left(\frac{1}{2}f\left(g^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i_1,i_2=1\\ i_1\ne i_2}}^{n}g\left(f^{-1}\left(f\left(\xi_{i_1}^2\right)+f\left(\xi_{i_2}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\},\right.\\
&\left.\underset{\eta_{i_j}\in\tilde{g}_{i_j}}{\mathrm{U}}\left\{\left(g^{-1}\left(\frac{1}{2}g\left(f^{-1}\left(\frac{1}{n(n-1)}\sum_{\substack{i_1,i_2=1\\ i_1\ne i_2}}^{n}f\left(g^{-1}\left(g\left(\eta_{i_1}^2\right)+g\left(\eta_{i_2}^2\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{2}}\right\}\right)
\end{aligned} \tag{18}$$

$$= AHPFBM^{1,1}\left(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n\right).$$

- When $r = n$, AHPFMSM operator reduces to At-CN&t-N-based HPF geometric mean (AHPFGM) operator and is presented as follows:

$$AHPFMSM^{(n)}(\tilde{\kappa}_1,\tilde{\kappa}_2,\ldots,\tilde{\kappa}_n)=\left(\frac{\oplus_{A1\le i_1<i_2<\ldots<i_n\le n}\otimes_{Aj=1}^{n}\tilde{\kappa}_{i_j}}{C_n^n}\right)^{\frac{1}{n}}=\left(\otimes_{Aj=1}^{n}\tilde{\kappa}_{i_j}\right)^{\frac{1}{n}}$$

$$= \left(\underset{\xi_{i_j} \in \tilde{h}_{i_j}}{\mathrm{U}} \left\{ \left(f^{-1}\left(\frac{1}{n} f\left(g^{-1}\left(g\left(f^{-1}\left(\sum_{j=1}^{n} f\left(\xi_{i_j}^2\right) \right) \right) \right) \right) \right) \right)^{\frac{1}{2}} \right\},$$

$$\underset{\eta_{i_j} \in \tilde{g}_{i_j}}{\mathrm{U}} \left\{ \left(g^{-1}\left(\frac{1}{n} g\left(f^{-1}\left(f\left(g^{-1}\left(\sum_{j=1}^{n} g\left(\eta_{i_j}^2\right) \right) \right) \right) \right) \right) \right)^{\frac{1}{2}} \right\} \Bigg)$$

$$= \left(\underset{\xi_i \in \tilde{h}_i}{\mathrm{U}} \left\{ \left(f^{-1}\left(\frac{1}{n} f\left(f^{-1}\left(\sum_{i=1}^{n} f\left(\xi_i^2\right) \right) \right) \right) \right)^{\frac{1}{2}} \right\},$$

$$\underset{\eta_i \in \tilde{g}_i}{\mathrm{U}} \left\{ \left(g^{-1}\left(\frac{1}{n} g\left(g^{-1}\left(\sum_{i=1}^{n} g\left(\eta_i^2\right) \right) \right) \right) \right)^{\frac{1}{2}} \right\} \Bigg) \text{(considering } i_j = i)$$

$$= \left(\underset{\xi_i \in \tilde{h}_i}{\mathrm{U}} \left\{ \left(f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f\left(\xi_i^2\right) \right) \right)^{\frac{1}{2}} \right\}, \underset{\eta_i \in \tilde{g}_i}{\mathrm{U}} \left\{ \left(g^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} g\left(\eta_i^2\right) \right) \right)^{\frac{1}{2}} \right\} \right). \quad (19)$$

Now, AHPFWMSM operator is developed by considering the preferences of the attributes toward the decision-making process to provide the importance of the aggregated arguments.

Definition 11 Let $\widetilde{\kappa}_i = \left(\tilde{h}_i, \widetilde{g}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector such that $\omega_i > 0$, $\sum_i^n \omega_i = 1$, and $r = 1, 2, \ldots, n$ be any number. Now, if

$$AHPFWMSM_{\omega}^{(r)}\left(\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_n\right) = \left(\frac{\oplus_{A\, 1 \le i_1 < i_2 < \ldots < i_r \le n} \otimes_{A\, j=1}^{r} \left(\omega_{i_j} \widetilde{\kappa}_{i_j}\right)}{C_r^n} \right)^{\frac{1}{r}}, \quad (20)$$

then $AHPFWMSM_{\omega}^{(r)}\left(\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_n\right)$ is called AHPFWMSM operator.

As like AHPFMSM operator, the following theorem holds for AHPFWMSM operator also.

Theorem 6 Let $\widetilde{\kappa}_i = \left(\tilde{h}_i, \widetilde{g}_i\right)$ $(i = 1, 2, \ldots, n)$ be a collection of HPFEs, and $r = 1, 2, \ldots, n$ be any number. Then the aggregated value using AHPFWMSM operator is also an HPFE and

$$AHPFWMSM_{\omega}^{(r)}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n) = \left(\frac{\oplus_{A\, 1 \le i_1 < i_2 < \ldots < i_r \le n} \otimes_{A\, j=1}^{r} \left(\omega_{i_j} \tilde{\kappa}_{i_j}\right)}{C_r^n} \right)^{\frac{1}{r}}$$

$$= \left(\underset{\xi_{i_j} \in \tilde{h}_{i_j}}{\text{U}} \left\{ f^{-1}\left(\frac{1}{r} f\left(g^{-1}\left(\frac{1}{C_r^n} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} g\left(f^{-1}\left(\sum_{j=1}^{r} \omega_{i_j} f\left(\xi_{i_j}\right)\right)\right)\right)\right)\right) \right\}, \right.$$
$$\left. \underset{\eta_{i_j} \in \tilde{g}_{i_j}}{\text{U}} \left\{ g^{-1}\left(\frac{1}{r} g\left(f^{-1}\left(\frac{1}{C_r^n} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} f\left(g^{-1}\left(\sum_{j=1}^{r} \omega_{i_j} g\left(\eta_{i_j}\right)\right)\right)\right)\right)\right) \right\} \right). \tag{21}$$

Proof The proof of Theorem 6 is analogous to the proof of Theorem 1.

By considering different forms of the decreasing generator, f, several forms of AHPFWMSM operator are derived as follows.

Case 1 (Algebraic) If $f(t) = -\log t$, then AHPFWMSM operator reduces to HPF weighted MSM operator (HPFWMSM) [45] as follows:

$$HPFWMSM_{\omega}^{(r)}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$$
$$= \left(\underset{\xi_{i_j} \in \tilde{h}_{i_j}}{\text{U}} \left\{ \left(\left(1 - \prod_{1 \le i_1 < i_2 < \ldots < i_r \le n} \left(1 - \prod_{j=1}^{r} \left(\xi_{i_j}\right)^{\omega_{i_j}} \right)^{\frac{1}{C_r^n}} \right)^{\frac{1}{r}} \right)^{\frac{1}{2}} \right\}, \right.$$
$$\left. \underset{\eta_{i_j} \in \tilde{g}_{i_j}}{\text{U}} \left\{ \left(1 - \left(1 - \prod_{1 \le i_1 < i_2 < \ldots < i_r \le n} \left(1 - \prod_{j=1}^{r} \left(1 - \eta_{i_j}\right)^{\omega_{i_j}} \right)^{\frac{1}{C_r^n}} \right)^{\frac{1}{r}} \right)^{\frac{1}{2}} \right\} \right). \tag{22}$$

Case 2 (Einstein) If $f(t) = \log\left(\frac{2-t}{t}\right)$, then AHPFWMSM operator takes the form of Einstein t-conorm and t-norm-based HPFEWMSM operator as follows:

$$HPFEWMSM_{\omega}^{(r)}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$$
$$= \left(\underset{\xi_{i_j} \in \tilde{h}_{i_j}}{\text{U}} \left\{ \sqrt{\frac{2\left(A_{i_j} + 3\right)^{\frac{1}{r}}}{\left(A_{i_j} - 1\right)^{\frac{1}{r}} + \left(A_{i_j} + 3\right)^{\frac{1}{r}}}} \right\}, \underset{\eta_{i_j} \in \tilde{g}_{i_j}}{\text{U}} \left\{ \sqrt{\frac{\left(B_{i_j} + 3\right)^{\frac{1}{r}} - \left(B_{i_j} - 1\right)^{\frac{1}{r}}}{\left(B_{i_j} + 3\right)^{\frac{1}{r}} + \left(B_{i_j} - 1\right)^{\frac{1}{r}}}} \right\} \right), \tag{23}$$

where A_{i_j} and B_{i_j} are defined as

$$A_{i_j} = \prod_{1 \le i_1 < i_2 < \ldots < i_r \le n} \left(\frac{\prod_{j=1}^{r}\left(2 - \xi_{i_j}^2\right)^{-\omega_{i_j}} + 3\prod_{j=1}^{r}\left(\xi_{i_j}^2\right)^{-\omega_{i_j}}}{\prod_{j=1}^{r}\left(2 - \xi_{i_j}^2\right)^{-\omega_{i_j}} - \prod_{j=1}^{r}\left(\xi_{i_j}^2\right)^{-\omega_{i_j}}} \right)^{\frac{1}{C_r^n}}$$

$$\text{and } B_{i_j} = \prod_{1 \le i_1 < i_2 < \ldots < i_r \le n} \left(\frac{\prod_{j=1}^{r} \left(1 + \eta_{i_j}^2\right)^{\omega_{i_j}} - \prod_{j=1}^{r} \left(1 - \eta_{i_j}^2\right)^{\omega_{i_j}}}{\prod_{j=1}^{r} \left(1 + \eta_{i_j}^2\right)^{\omega_{i_j}} + 3 \prod_{j=1}^{r} \left(1 - \eta_{i_j}^2\right)^{\omega_{i_j}}} \right)^{\frac{1}{C_r^n}}.$$

Case 3 (Hamacher)

Now, if $f(t) = \log\left(\frac{\tau + (1-\tau)t}{t}\right)$, $\tau > 0$, then AHPFWMSM operator is converted into Hamacher t-conorm and t-norm-based HPFHWMSM operator and is presented as follows:

$$HPFHWMSM_{\omega}^{(r)}(\tilde{\kappa}_1, \tilde{\kappa}_2, \ldots, \tilde{\kappa}_n)$$
$$= \left(\underset{\xi_{i_j} \in \tilde{h}_{i_j}}{\mathrm{U}} \left\{ \frac{\tau (M)^{\frac{1}{r}}}{(\tau + (1-\tau)M)^{\frac{1}{r}} + (\tau - 1)(M)^{\frac{1}{r}}} \right\}, \underset{\eta_{i_j} \in \tilde{g}_{i_j}}{\mathrm{U}} \left\{ \frac{(\tau + (1-\tau)(1-N))^{\frac{1}{r}} - (1-N)^{\frac{1}{r}}}{(\tau + (1-\tau)(1-N))^{\frac{1}{r}} + (\tau - 1)(1-N)^{\frac{1}{r}}} \right\} \right), \tag{24}$$

where $M = \frac{\prod_{1 \le i_1 < i_2 < \ldots < i_r \le n} \left(A + (\tau^2 - 1)B\right)^{\frac{1}{C_r^n}} - \prod_{1 \le i_1 < i_2 < \ldots < i_r \le n} (A - B)^{\frac{1}{C_r^n}}}{\prod_{1 \le i_1 < i_2 < \ldots < i_r \le n} \left(A + (\tau^2 - 1)B\right)^{\frac{1}{C_r^n}} + (\tau - 1) \prod_{1 \le i_1 < i_2 < \ldots < i_r \le n} (A - B)^{\frac{1}{C_r^n}}}$,

in which $A = \prod_{j=1}^{r} \left(\tau + (1-\tau)\xi_{i_j}\right)^{\omega_{i_j}}$ and $B = \prod_{j=1}^{r} \left(\xi_{i_j}\right)^{\omega_{i_j}}$;
and $= \frac{tF}{G + (t-1)F}$,

where

$$F = \prod_{1 \le i_1 < i_2 < \ldots < i_r \le n} \left(\prod_{j=1}^{r} \left(\tau + (1-\tau)\left(1 - \eta_{i_j}\right)\right)^{\omega_{i_j}} - \prod_{j=1}^{r} \left(1 - \eta_{i_j}\right)^{\omega_{i_j}} \right)^{\frac{1}{C_r^n}}$$

and

$$G = \prod_{1 \le i_1 < i_2 < \ldots < i_r \le n} \left(\prod_{j=1}^{r} \left(\tau + (1-\tau)\left(1 - \eta_{i_j}\right)\right)^{\omega_{i_j}} + \left(\tau^2 - 1\right) \prod_{j=1}^{r} \left(1 - \eta_{i_j}\right)^{\omega_{i_j}} \right)^{\frac{1}{C_r^n}}$$

Case 4 (Frank) If $f(t) = \log\left(\frac{\rho - 1}{\rho^t - 1}\right)$, $\rho > 1$, then AHPFWMSM operator reduces to HPFFWMSM operator and is defined as

$$HPFFWMSM_{\omega}^{(r)}\left(\widetilde{\kappa}_1, \widetilde{\kappa}_2, \ldots, \widetilde{\kappa}_n\right) = (M, N), \tag{25}$$

where

$$M=\frac{\log\left(\frac{(\rho-1)+\left(\frac{\rho-1}{\rho^{1-\frac{\log\left(\frac{(\rho-1)+A_{i_j}}{A_{i_j}}\right)}{\log\rho}}-1}\right)^{\frac{1}{r}}}{\left(\frac{\rho-1}{\rho^{1-\frac{\log\left(\frac{(\rho-1)+A_{i_j}}{A_{i_j}}\right)}{\log\rho}}-1}\right)^{\frac{1}{r}}}\right)}{\log\rho}\text{ and }N=1-\frac{\log\left(\frac{(\rho-1)+\left(\frac{\rho-1}{\rho^{1-\frac{\log\left(\frac{(\rho-1)+B_{i_j}}{B_{i_j}}\right)}{\log\rho}}-1}\right)^{\frac{1}{r}}}{\left(\frac{\rho-1}{\rho^{1-\frac{\log\left(\frac{(\rho-1)+B_{i_j}}{B_{i_j}}\right)}{\log\rho}}-1}\right)^{\frac{1}{r}}}\right)}{\log\rho},$$

in which

$$A_{i_j}=\prod_{1\le i_1<i_2<\ldots<i_r\le n}\left(\frac{\rho-1}{\rho^{1-\frac{\log\left(\frac{(\rho-1)+\prod_{j=1}^{r}\left(\frac{\rho-1}{\rho^{\xi_{i_j}}-1}\right)^{\omega_{i_j}}}{\prod_{j=1}^{r}\left(\frac{\rho-1}{\rho^{\xi_{i_j}}-1}\right)^{\omega_{i_j}}}\right)}{\log\rho}}-1}\right)^{\frac{1}{C_r^n}}\text{ and}$$

$$B_{i_j}=\prod_{1\le i_1<i_2<\ldots<i_r\le n}\left(\frac{\rho-1}{\rho^{1-\frac{\log\left(\frac{(\rho-1)+\prod_{j=1}^{r}\left(\frac{\rho-1}{\rho^{1-\eta_{i_j}}-1}\right)^{\omega_{i_j}}}{\prod_{j=1}^{r}\left(\frac{\rho-1}{\rho^{1-\eta_{i_j}}-1}\right)^{\omega_{i_j}}}\right)}{\log\rho}}-1}\right)^{\frac{1}{C_r^n}}.$$

4 An Approach to MCDM with HPF Information

In this section, the developed HPFHWMSM and HPFFWMSM operators are used to solve MCDM problems. An MCDM problem under HPF environment is described below.

Let $X=\{x_1,x_2,\ldots,x_m\}$ be a set of alternatives and $C=\{C_1,C_2,\ldots,C_n\}$ be a collection of criteria. Let $\tilde{D}=\left[\tilde{d}_{ij}\right]_{m\times n}$ be an HPF decision matrix (HPFDM), whose elements are represented by HPFEs with the form $\tilde{d}_{ij}=\left(\tilde{h}_{ij},\widetilde{g}_{ij}\right)$, where $\tilde{h}_{ij}$ represents the membership degree of the alternative x_i that satisfies the attribute C_j;

and $\widetilde{\mathrm{g}}_{ij}$ indicates the degree of the alternative x_i that does not satisfy the attribute C_j, $i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$.

Then, the established HPFHWMSM (or HPFFWMSM) operators are used to develop an approach for solving MCDM problems in an HPF environment. The proposed methodology is described through the following steps:

Step 1: If the HPFDM contains some cost criteria, then the HPFDM is converted into the normalized HPFDM, $\tilde{R} = \left[\tilde{r}_{ij}\right]_{m\times n}$ by applying

$$\tilde{r}_{ij} = \begin{cases} \tilde{d}_{ij}\, for\ benefit\ attribute\ C_j \\ \tilde{d}_{ij}^{c}\, for\ cost\ attribute\ C_j \end{cases}, \tag{26}$$

where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; and $\tilde{d}_{ij}^{c}$ represent the complement of $\tilde{d}_{ij}$.

Step 2: Utilize the decision information given in matrix $\tilde{R}$, to aggregate the HPFEs $\tilde{r}_{ij}$ for each alternative x_i using HPFHWMSM (or HPFFWMSM) operator which has associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$, as follows:

$$\tilde{r}_i = HPFHWMSM_{\omega}^{(r)}(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in})$$

or

$$\tilde{r}_i' = HPFFWMSM_{\omega}^{(r)}(\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{in})$$
$$i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n.$$

Step 3: Calculate the score values $S(\tilde{r}_i)$ (or $S(\tilde{r}_i'))$ $(i = 1, 2, \ldots, m)$ of the collective overall HPF preference values of each alternative x_i using Definition 11.

Step 4: Rank the alternatives based on the achieved score values.

5 Illustrative Examples

In this section, two examples are considered and solved to show the validity and advantages of the developed MCDM method.

Example 1 This problem is aimed to find best emerging technology enterprise from a set of five possible emerging technology enterprises, $\{A_1, A_2, A_3, A_4, A_5\}$, which is adapted from [44]. There are four attributes for evaluating the alternatives with the associated weight vector, $\omega = (0.2, 0.1, 0.3, 0.4)^T$. The attributes are listed as technical advancement (C_1), potential market and market risk (C_2), industrialization infrastructure, human resources, financial conditions (C_3), employment creation, and development of science and technology (C_4). After evaluating the alternatives, DM constructed the decision matrix $\tilde{R} = \left[\tilde{r}_{ij}\right]_{m\times n}$ using HPFEs $\tilde{r}_{ij}$ $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3, 4)$ and is presented by the following HPFDM as described in Table 2.

The step-by-step method for solving the example using the proposed methodology is presented as follows:

Step 1. Since all the criteria values represent benefit criteria, normalization is not required here. Thus $\tilde{R} = \tilde{D}$.

Step 2. Using HPFHWMSM operator displayed in Eq. (24), the collective preference values, $\tilde{r}_{ij}$, corresponding to the alternative, A_i (without loss of generality, taking Hamacher parameter $\tau = 2$, and MSM parameter $r = 2$), and the overall evaluation values $\tilde{r}_i$ $(i = 1, 2, 3, 4, 5)$ are shown as follows:

$$\tilde{r}_1 = (\{0.8155, 0.8277, 0.8404, 0.8516, 0.8256, 0.8375, 0.8499, 0.8608\}, \{0.1466, 0.1794, 0.1611, 0.1939, 0.1759, 0.2091, 0.1556, 0.1882, 0.1702, 0.2025, 0.1851, 0.2176, 0.1703, 0.2023, 0.1848, 0.2161, 0.1999, 0.2309, 0.1791, 0.2106, 0.1935, 0.2238, 0.2086, 0.2384\}),$$

$$\tilde{r}_2 = (\{0.8647, 0.8736, 0.8823, 0.8780, 0.8867, 0.8953, 0.8713, 0.8801, 0.8887, 0.8849, 0.8934, 0.9018\}, \{0.2208, 0.2389, 0.2287, 0.2472, 0.2323, 0.2509, 0.2394, 0.2582\}),$$

$$\tilde{r}_3 = (\{0.8570, 0.8679, 0.8929, 0.9023, 0.8846, 0.8948, 0.9172, 0.9255\}, \{0.1566, 0.1715, 0.1706, 0.1860\}),$$

$$\tilde{r}_4 = (\{0.8319, 0.8660, 0.8371, 0.8711, 0.8516, 0.8838, 0.8566, 0.8886\}, \{0.1494, 0.1662, 0.1585, 0.1754, 0.1684, 0.1856\}),$$

and

$$\tilde{r}_5 = (\{0.8742, 0.9030, 0.9006, 0.9257, 0.8780, 0.9065, 0.9044, 0.9290, 0.8953, 0.9216, 0.9207, 0.9426, 0.8999, 0.9254, 0.9248, 0.9458\}, \{0.1383, 0.1546, 0.1529, 0.1692\}).$$

Step 3. Calculate the score value of $\tilde{r}_i$ $(i = 1, 2, 3, 4, 5)$ as $S(\tilde{r}_1) = 0.6655$, $S(\tilde{r}_2) = 0.7230$, $S(\tilde{r}_3) = 0.7681$, $S(\tilde{r}_4) = 0.7133$, and $S(\tilde{r}_5) = 0.8090$.

Step 4. Since, $S(\tilde{r}_5) \succ S(\tilde{r}_3) \succ S(\tilde{r}_2) \succ S(\tilde{r}_4) \succ S(\tilde{r}_1)$, based on the score values, the ordering of the alternatives appears as $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$. So, the best alternative is found as A_5.

Again, this problem is solved by utilizing HPFFWMSM operator, and the process is presented step by step as follows:

Step. $1'$ Similar to Step 1 as described, previously.

Step. $2'$ Obtain the collective preference values by applying HPFFWMSM operator taking $\rho = 2$, $r = 2$, in Eq. (25) as

$$\tilde{r}_1' = (\{0.8829, 0.8912, 0.8997, 0.9073, 0.8896, 0.8978, 0.9061, 0.9135\}, \{0.1490, 0.1827, 0.1642, 0.1978, 0.1798, 0.2139, 0.1590, 0.1926, 0.1743, 0.2074, 0.1901, 0.2235, 0.1741, 0.2069, 0.1893, 0.2211, 0.2053, 0.2368, 0.1841, 0.2162, 0.1990, 0.2298, 0.2151, 0.2452\}),$$

$$\tilde{r}_2' = (\{0.9160, 0.9220, 0.9277, 0.9252, 0.9310, 0.9368, 0.9203, 0.9262, 0.9320, 0.9298, 0.9355, 0.9411\}, \{0.2260, 0.2453, 0.2347, 0.2544, 0.2381, 0.2579, 0.2458, 0.2659\}),$$

Table 2 HPFDM

	C_1	C_2	C_3	C_4
A_1	$(\{0.4\}, \{0.3, 0.5\})$	$(\{0.3, 0.5\}, \{0.5, 0.6\})$	$(\{0.2, 0.4\}, \{0.3, 0.4, 0.5\})$	$(\{0.4, 0.5\}, \{0.2, 0.4\})$
A_2	$(\{0.6, 0.7\}, \{0.3, 0.4\})$	$(\{0.3, 0.6\}, \{0.4, 0.5\})$	$(\{0.5, 0.6, 0.7\}, \{0.4\})$	$(\{0.4\}, \{0.6, 0.7\})$
A_3	$(\{0.4, 0.8\}, \{0.2, 0.3\})$	$(\{0.4\}, \{0.1\})$	$(\{0.3, 0.7\}, \{0.4, 0.5\})$	$(\{0.6, 0.7\}, \{0.4\})$
A_4	$(\{0.2, 0.4\}, \{0.3\})$	$(\{0.3, 0.4\}, \{0.5, 0.6, 0.7\})$	$(\{0.4\}, \{0.2\})$	$(\{0.5, 0.8\}, \{0.3, 0.4\})$
A_5	$(\{0.6, 0.9\}, \{0.3\})$	$(\{0.5, 0.6\}, \{0.4\})$	$(\{0.5, 0.8\}, \{0.3, 0.4\})$	$(\{0.4, 0.7\}, \{0.2, 0.3\})$

$$\tilde{r}_3' = (\{0.8829, 0.8997, 0.8997, 0.9073, 0.8896, 0.8978, 0.9061, 0.9135\}, \{0.1589, 0.1746, 0.1734, 0.1897\}),$$

$$\tilde{r}_4' = (\{0.8942, 0.9161, 0.8976, 0.9196, 0.9073, 0.9280, 0.9107, 0.9314\}, \{0.1519, 0.1692, 0.1620, 0.1796, 0.1733, 0.1912\}),$$

and

$$\tilde{r}_5' = (\{0.9225, 0.9417, 0.9399, 0.9563, 0.9252, 0.9440, 0.9426, 0.9585, 0.9358, 0.9532, 0.9527, 0.9667, 0.9390, 0.9558, 0.9556, 0.9690\}, \{0.1400, 0.1566, 0.1551, 0.1718\}).$$

Step. 3$'$ Calculate the score value of $\tilde{r}_i'$ $(i = 1, 2, 3, 4, 5)$ as $S(\tilde{r}_1') = 0.7675$, $S(\tilde{r}_2') = 0.8017$, $S(\tilde{r}_3') = 0.7770$, $S(\tilde{r}_4') = 0.8045$, $S(\tilde{r}_5') = 0.8733$.

Step. 4$'$. Based on the score values, the ranking is obtained as $S(\tilde{r}_5') \succ S(\tilde{r}_4') \succ S(\tilde{r}_2') \succ S(\tilde{r}_3') \succ S(\tilde{r}_1')$. Thus, the ordering of the alternatives appears as $A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$. Though this ranking result is slightly different compared to HPFHWMSM operator, still the best alternative is the same as A_5.

- Influence of different parameters on decision-making results.

Now, the influence of Hamacher parameter, τ, Frank parameter, ρ, and MSM parameter, r on decision-making results obtained through HPFHWMSM and HPFFWMSM AOs is investigated. Those parameters play vital roles in the ranking of alternatives. Different score values are obtained by assigning different values of those parameters. The ranking results are shown in Tables 3 and 4 by varying the values of τ, ρ, and r.

From Figs. 1, 2, 3, and 4, it is observed that the score values increase with the increase of the parameter τ in 0 to 20, when Hamacher t-conorm and t-norm-based HPFHWMSM AO is used. Whereas its reverse situation is found for changing the MSM parameter, r. In such situation, if the value of the MSM parameter is considered as $r = 1, 2, 3, 4$, the score values decrease. Though different ranking results are derived using HPFHWMSM operator with changing parameters τ and r, the best alternative A_5 and the worst alternative A_1 remain unchanged. This conveys that the proposed HPFHWMSM operator has an excellent capacity to adapt with the situation with variation of the parameters τ and r.

A graphical interpretation of score values of the alternatives is presented in Fig. 1 by varying the Hamacher parameter τ in HPFHWMSM operator, keeping the value $r = 1$ as fixed. The ranking result, in this case, is found as $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$ for $0 < \tau \leq 20$.

Again, from Fig. 2 it is observed that, when the same AO is utilized by keeping the fixed value $r = 2$, some different orderings of the alternatives are found for changing the value of τ in 0 to 20. The ranking result of the alternatives is found as $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$, when $0 < \tau < 4.1911$. Whereas when the value of τ lies in $(4.1911, \ 20]$, the ranking result slightly differs as $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$. But the best alternative remains the same as A_5 for both the sub-intervals.

As like Figs. 1 and 3 also provide the same ranking result, i.e., $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$, for varying the Hamacher parameter τ with $r = 3$.

Table 3 Scores and the rankings results using developed method for different values of τ and different values of MSM parameter r using HPFHWMSM operator

	Parameter τ	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	Ranking result
$r = 1$	$\tau = 1$	0.6286	0.7370	0.7591	0.6789	0.8121	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$\tau = 2$	0.6917	0.7760	0.7983	0.7365	0.8367	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$\tau = 3$	0.7310	0.8018	0.8231	0.7714	0.8538	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$\tau = 4$	0.7588	0.8206	0.8407	0.7955	0.8665	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
$r = 2$	$\tau = 1$	0.5934	0.6729	0.7221	0.6476	0.7764	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$\tau = 2$	0.6655	0.7230	0.7681	0.7133	0.8090	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$\tau = 3$	0.7085	0.7550	0.7962	0.7515	0.8302	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$\tau = 4$	0.7384	0.7781	0.8159	0.7777	0.8455	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
$r = 3$	$\tau = 1$	0.5801	0.6478	0.7080	0.7274	0.7607	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 2$	0.6572	0.7064	0.7583	0.7749	0.7991	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 3$	0.7025	0.7424	0.7883	0.8033	0.8228	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 4$	0.7337	0.7677	0.8092	0.8229	0.8395	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
$r = 4$	$\tau = 1$	0.5716	0.6285	0.7003	0.6335	0.7497	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
	$\tau = 2$	0.6508	0.6944	0.7535	0.7044	0.7927	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
	$\tau = 3$	0.6974	0.7338	0.7848	0.7450	0.8183	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
	$\tau = 4$	0.7295	0.7611	0.8063	0.7725	0.8361	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$

Further, from Fig. 4, it is clear that the ranking result is obtained as $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$ for $0 < \tau < 0.6416$ and the value $r = 4$ is considered when HPFHWMSM operator is used. Whereas if the value of the parameter lies in $0.6416 < \tau \leq 20$, the ordering becomes $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$.

On the other side, when the problem is solved by Frank t-conorm and t-norm-based HPFFWMSM AO, different ranking results are found which are represented in Figs. 5, 6, 7, and 8. Considering the value of the MSM parameter $r = 1, 2, 3, 4$ with varying the Frank parameter ρ in 1.001 to 20, the score values of the alternatives are calculated and graphically displayed in Figs. 5, 6, 7, and 8.

From Fig. 5, it is observed that the ranking appears as $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$ when $\rho \in [1.001, 20]$, whereas, when $r = 2$ is considered, different ranking results of the alternatives are obtained and is shown in Fig. 6. The ranking result $A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$ is found for varying ρ in $[1.001, 1.2471]$, and when ρ is considered in $1.2471 < \rho \leq 20$, the ordering is found as $A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$.

Again, from Figs. 7 and 8, it is noticed that the achieved ranking results are the same, i.e., $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$, for taking $r = 3$ and 4. However, for other values, the ordering may slightly differ, but the best alternative is fixed as A_5.

To compare the proposed method with the other method developed in HPF environment, the following example, considered previously by Garg [45], is solved, using the developed method in this chapter.

Table 4 Scores and the ranking results using developed method for different values of ρ and different values of MSM parameter r using HPFHWMSM operator

	Parameter ρ	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	$S(A_5)$	Ordering
$r = 1$	$\rho = 1.001$	0.7748	0.8227	0.8619	0.8067	0.8863	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$\rho = 2$	0.7871	0.8294	0.8682	0.8178	0.8898	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$\rho = 3$	0.7931	0.8326	0.8713	0.8232	0.8914	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$\rho = 4$	0.7968	0.8345	0.8731	0.8266	0.8925	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
$r = 2$	$\rho = 1.001$	0.7529	0.7930	0.7629	0.7916	0.8684	$A_5 \succ A_2 \succ A_4 \succ A_3 \succ A_1$
	$\rho = 2$	0.7675	0.8017	0.7770	0.8045	0.8733	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$
	$\rho = 3$	0.7746	0.8059	0.7838	0.8107	0.8756	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$
	$\rho = 4$	0.7791	0.8084	0.7880	0.8145	0.8771	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$
$r = 3$	$\rho = 1.001$	0.7452	0.7753	0.8253	0.7861	0.8598	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
	$\rho = 2$	0.7611	0.7863	0.8342	0.7999	0.8659	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
	$\rho = 3$	0.7689	0.7915	0.8385	0.8065	0.8687	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
	$\rho = 4$	0.7737	0.7948	0.8411	0.8106	0.8705	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
$r = 4$	$\rho = 1.001$	0.7409	0.7623	0.8208	0.8083	0.8544	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
	$\rho = 2$	0.7576	0.7757	0.8305	0.8157	0.8614	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
	$\rho = 3$	0.7657	0.7821	0.8351	0.8192	0.8648	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
	$\rho = 4$	0.7708	0.7860	0.8379	0.8214	0.8668	$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$

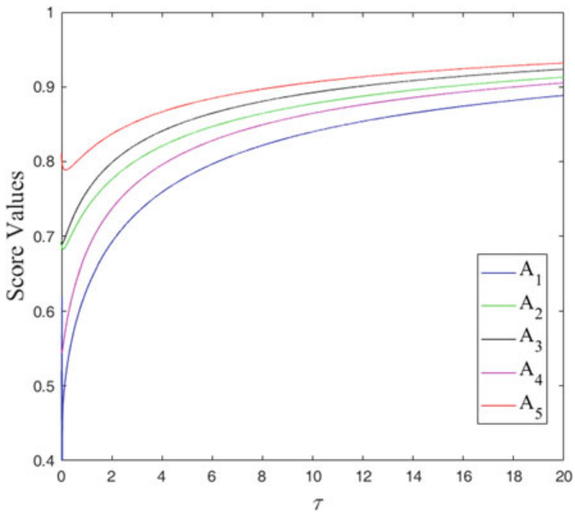


Fig. 1 Score values of alternative A_i for variation in parameter τ by considering $r = 1$, using HPFHWMSM operator

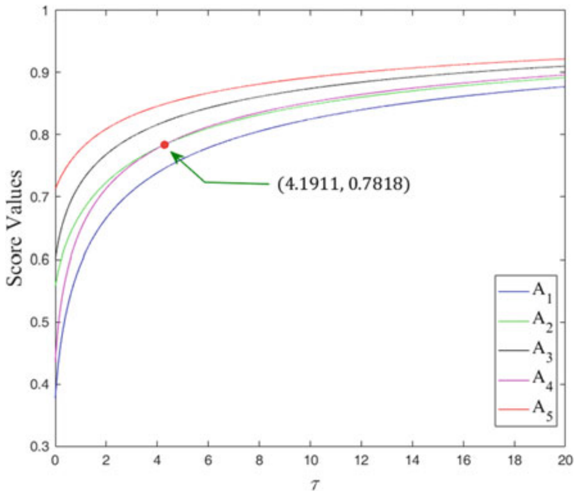


Fig. 2 Score values of alternative A_i for variation in parameter τ by considering $r = 2$, using HPFHWMSM operator

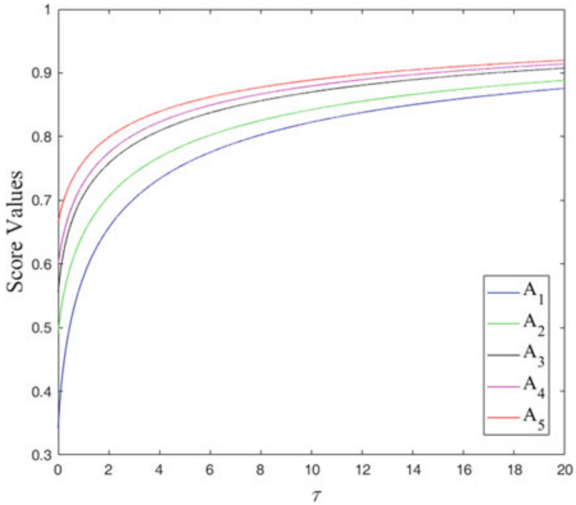


Fig. 3 Score values of alternative A_i for variation in parameter τ by considering $r = 3$, using HPFHWMSM operator

Example 2 This problem is related to find most profitable market for investment by an investor in India under HPF environment. In this respect, five markets, viz., A_1, A_2, A_3, A_4, and A_5 are evaluated by an expert with respect to four criteria, viz., C_1, C_2, C_3, C_4, with the corresponding weight vector, $\omega = (0.20, 0.15, 0.35, 0.30)^T$. It is to be mentioned here that C_1 and C_3 are benefit criteria, whereas C_2 and C_4 are cost criteria.

After evaluation, decision values in the form of HPFEs are provided by the experts, which are displayed in Table 5.

This problem is solved using the aforementioned steps as shown in Example 1.

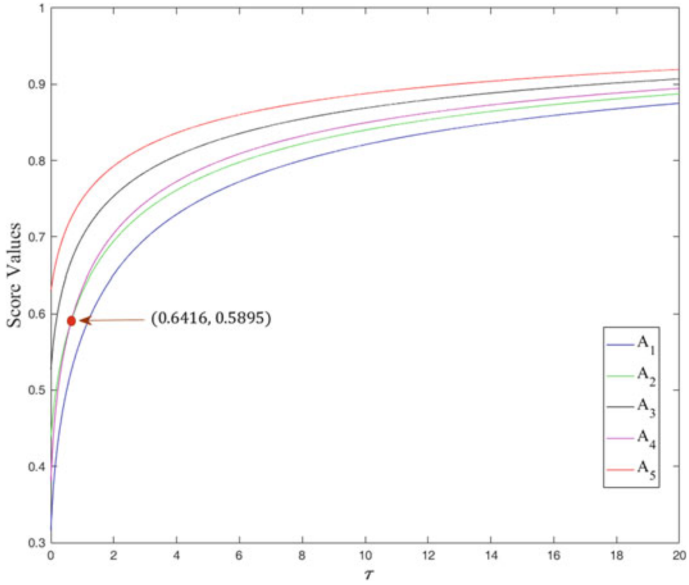
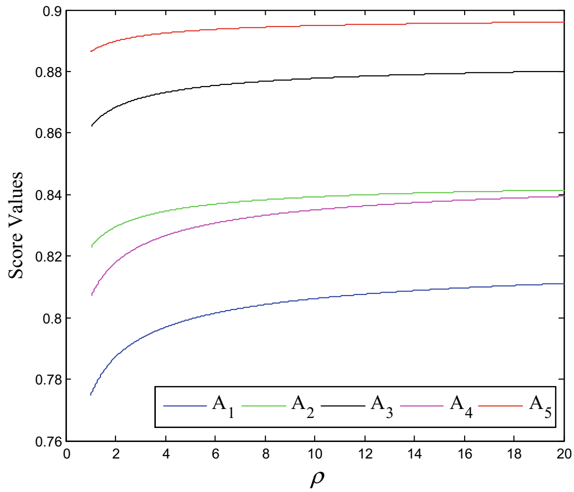


Fig. 4 Score values of alternative A_i for variation in parameter τ by considering $r = 4$, using HPFHWMSM operator

Fig. 5 Score values of alternative A_i for variation in parameter ρ by considering $r = 1$, using HPFFWMSM operator



Using HPFHWMSM and HPFFWMSM operators, the achieved results for different values corresponding to MSM parameter, r, Hamacher parameter, τ, and Frank parameter, ρ, are presented in Table 6.

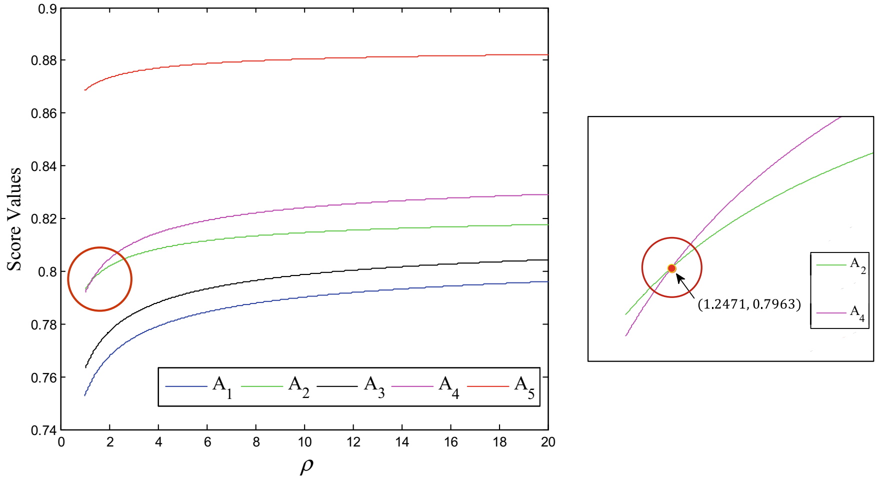


Fig. 6 Score values of alternative A_i for variation in parameter ρ by considering $r = 2$, using HPFFWMSM operator

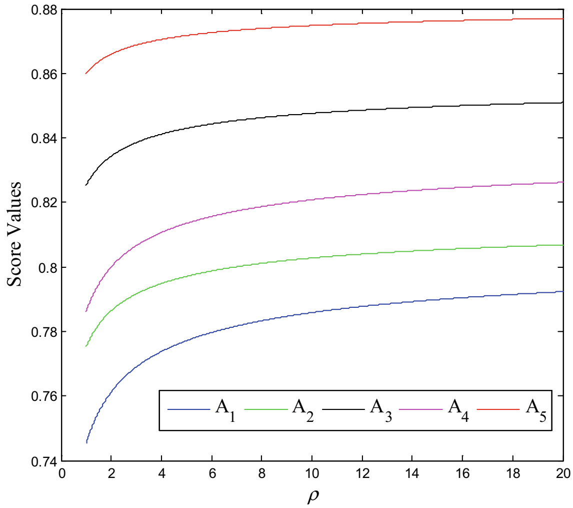


Fig. 7 Score values of alternative A_i for variation in parameter ρ by considering $r = 3$, using HPFFWMSM operator

From Table 6, it is observed that the influence of different parameters involved with the developed method based on HPFHWMSM and HPFFWMSM operators cannot be ignored as like the previous example, Example 1.

The influence of those parameters on the ranking order of the alternatives is discussed by varying those parameters, continuously, in some specified intervals, and is shown in Figs. 9, 10, 11, 12, 13, 14, 15, and 16.

If the problem is solved using HPFHWMSM operator, by varying the Hamacher parameter, τ, between 0 and 20, several ranking results are obtained. Figures 9, 10,

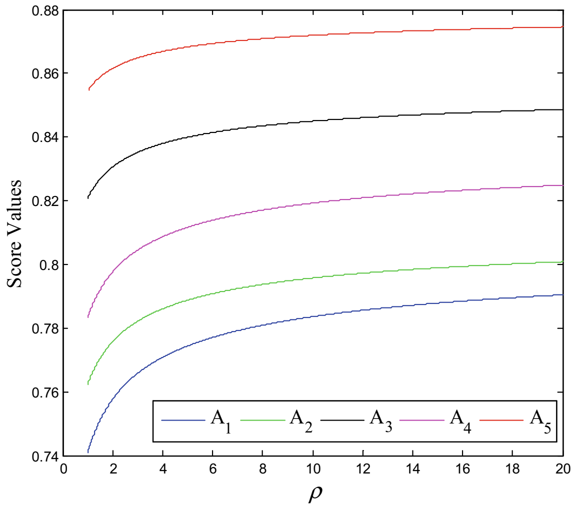


Fig. 8 Score values of alternative A_i for variation in parameter ρ by considering $r = 4$, using HPFFWMSM operator

Table 5 HPFDM [45]

	C_1	C_2	C_3	C_4
A_1	$(\{0.3, 0.4\}, \{0.6\})$	$(\{0.3, 0.4\}, \{0.4, 0.5\})$	$(\{0.2, 0.3\}, \{0.7\})$	$(\{0.5\}, \{0.4, 0.5\})$
A_2	$(\{0.6\}, \{0.4\})$	$(\{0.4\}, \{0.2, 0.4, 0.5\})$	$(\{0.2\}, \{0.6, 0.7, 0.8\})$	$(\{0.4, 0.5\}, \{0.5\})$
A_3	$(\{0.5, 0.7\}, \{0.2\})$	$(\{0.7, 0.8\}, \{0.2\})$	$(\{0.2, 0.3, 0.4\}, \{0.6\})$	$(\{0.3\}, \{0.5, 0.6, 0.7\})$
A_4	$(\{0.7\}, \{0.3\})$	$(\{0.2\}, \{0.6, 0.7, 0.8\})$	$(\{0.1, 0.2\}, \{0.3\})$	$(\{0.6, 0.7, 0.8\}, \{0.1\})$
A_5	$(\{0.6, 0.7\}, \{0.2\})$	$(\{0.5\}, \{0.2, 0.3, 0.4\})$	$(\{0.0.4, 0.5\}, \{0.2\})$	$(\{0.5\}, \{0.2, 0.3, 0.4\})$

11, and 12 represent the graphical interpretation of score values of the alternatives for choosing the values of MSM parameter, $r = 1, 2, 3, 4$, respectively.

From Fig. 9, it is noticed that, for $r = 1$, several ranking results are found, as τ changes from 0 to 20. When $\tau \in (0, 1.3361)$, the ordering of alternatives is obtained as $A_4 \succ A_3 \succ A_5 \succ A_2 \succ A_1$, and when $\tau \in (1.3361, 1.3741)$ best alternative is changed to A_3 from A_4, with the ordering $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$. For $\tau = 1.3361$, score value of alternatives A_3 and A_4 remains the same as $S(A_3) = S(A_4) = 0.6947$, and the ranking is found as $A_3 \approx A_4 \succ A_5 \succ A_2 \succ A_1$. If the problem is solved using Einstein operation-based HPFEWMSM operator, i.e., by considering the value of $\tau = 2$ in the developed HPFHWMSM operator, ordering of alternatives is found as $A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$, i.e., the best alternative is A_3. Again, for $\tau \in (1.3741, 3.3316)$, the ordering is obtained as $A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$ and the ordering, $A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$, is found for $\tau \in (3.3316, 20]$.

Now, considering $r = 2$, Fig. 10 indicates that several ranking results are found for different span of τ as $A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$ for $\tau \in (0, 0.1865)$; $A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$ for $\tau \in (0.1865, 0.3877)$; $A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$

Table 6 Ranking results by the proposed method with the variation of different parameters

	Parameter τ	By HPFHWMSM operator	Parameter ρ	By HPFFWMSM operator
$r = 1$	$\tau = 0.001$	$A_4 \succ A_3 \succ A_5 \succ A_2 \succ A_1$	$\rho = 1.001$	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 5$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 5$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 10$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 10$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 15$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 15$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 20$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 20$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
$r = 2$	$\tau = 0.001$	$A_4 \succ A_5 \succ A_2 \succ A_3 \succ A_1$	$\rho = 1.001$	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 5$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 5$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 10$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 10$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 15$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 15$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 20$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 20$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
$r = 3$	$\tau = 0.001$	$A_5 \succ A_2 \succ A_3 \succ A_1 \succ A_4$	$\rho = 1.001$	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 5$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 5$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 10$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 10$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 15$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 15$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$\tau = 20$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 20$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
$r = 4$	$\tau = 0.001$	$A_4 \succ A_5 \succ A_2 \succ A_3 \succ A_1$	$\rho = 1.001$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
	$\tau = 5$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 5$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
	$\tau = 10$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 10$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
	$\tau = 15$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 15$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
	$\tau = 20$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$	$\rho = 20$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$

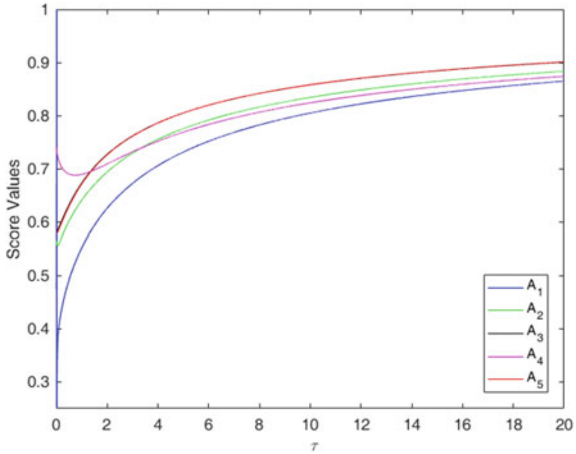


Fig. 9 Score values of alternatives for variation in ρ when $r = 1$ using HPFHWMSM operator

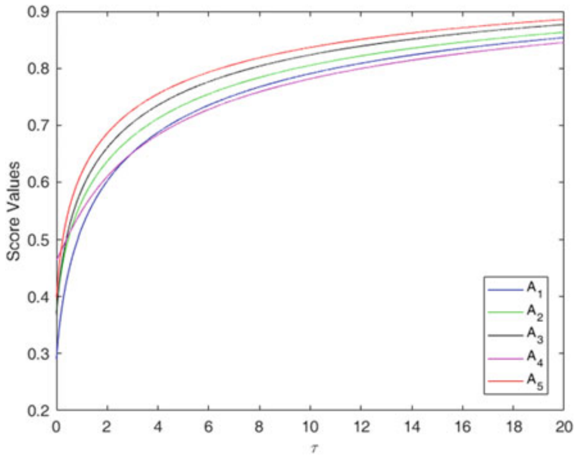


Fig. 10 Score values of alternatives for variation in ρ when $r = 2$ using HPFHWMSM operator

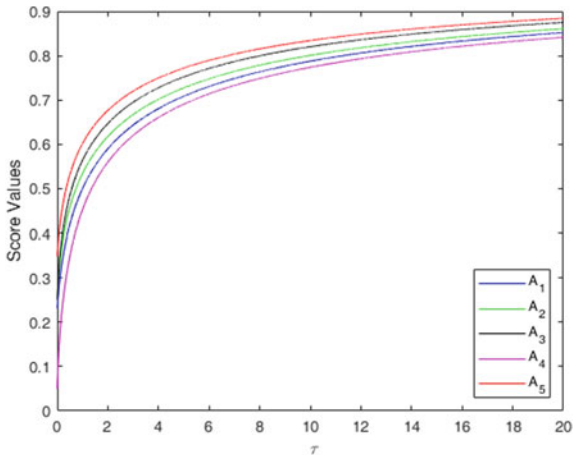


Fig. 11 Score values of alternatives for variation in ρ when $r = 3$ using HPFHWMSM operator

for $\tau \in (0.3877, 0.5541)$; $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$ for $\tau \in (0.5541, 2.9418)$; $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ for $\tau \in (2.9418,\ 20]$.

Corresponding to $r = 3$ and 4, ranking results remain stable over $\tau \in (0.001, 20]$, with the ranking $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$, as displayed in Figs. 11 and 12.

Again, if the problem under consideration is solved by HPFFWMSM operator, keeping MSM parameter $r = 1, 2, 3, 4$, several ranking results are obtained which are presented in Figs. 13, 14, 15, 16, respectively.

When HPFFWMSM operator is used taking $r = 1$, ranking result is found as $A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$ for Frank parameter $\rho \in (1.001, 1.4875)$, but ordering place of A_4 and A_5 interchanges when ρ varying from 1.4875 to 20, which is realized from Fig. 13. But, ranking result still remains the same as $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$ (as displayed in Fig. 14) for using HPFFWMSM operator along with $r = 2$.

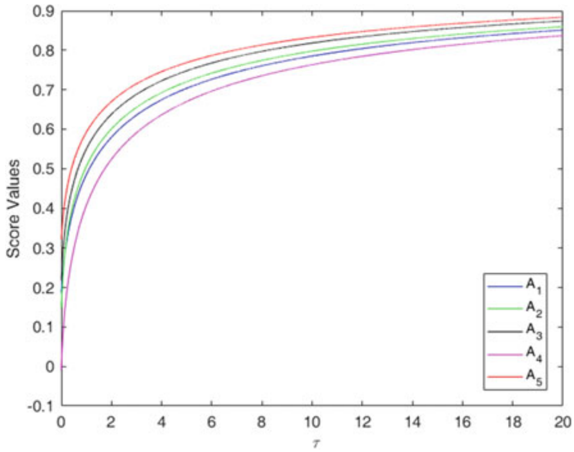


Fig. 12 Score values of alternatives for variation in ρ when $r = 4$ using HPFHWMSM operator

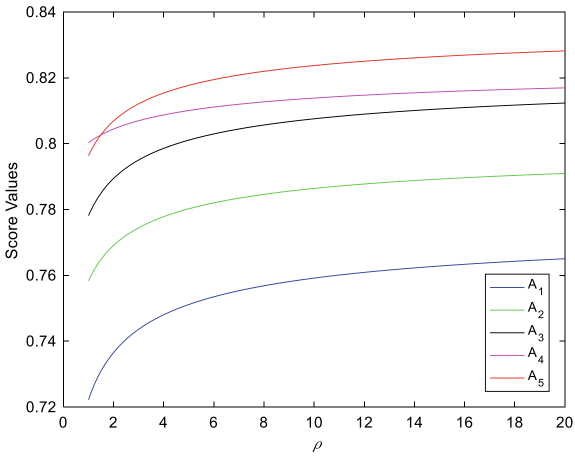


Fig. 13 Score values of alternatives for variation in ρ when $r = 1$ using HPFFWMSM operator

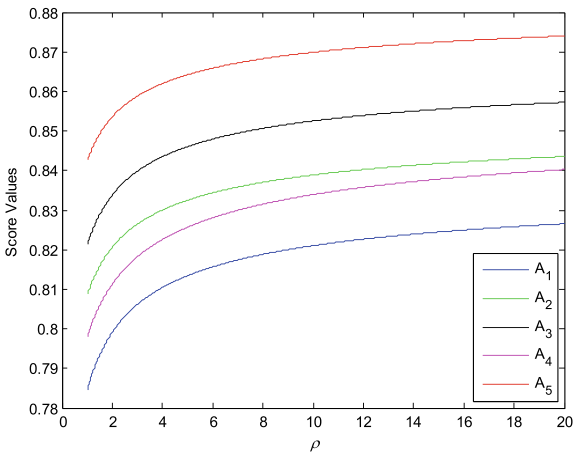


Fig. 14 Score values of alternatives for variation in ρ when $r = 2$ using HPFFWMSM operator

Fig. 15 Score values of alternatives for variation in ρ when $r = 3$ using HPFFWMSM operator

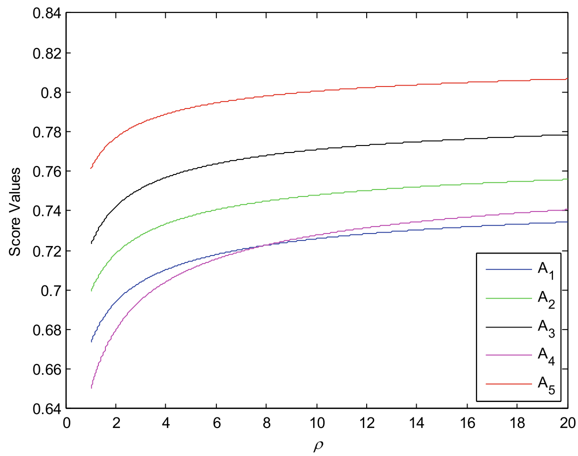
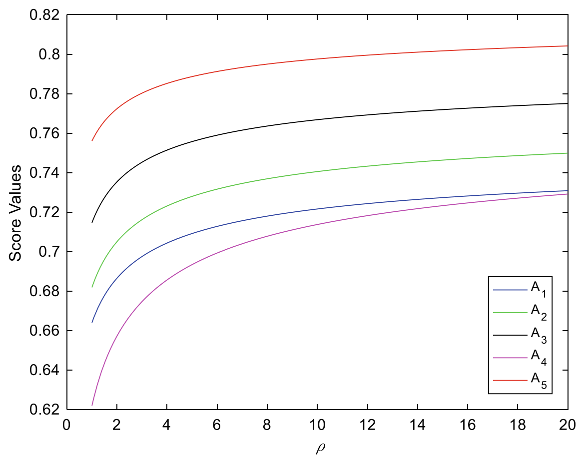


Fig. 16 Score values of alternatives for variation in ρ when $r = 4$ using HPFFWMSM operator



Further according to Fig. 15, if $r = 3$ is taken as the value of MSM parameter, ordering of the alternatives is found as $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ and $A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$ for $\rho \in (1.001, 7.9174)$ and $\rho \in (7.9174, 20]$, respectively.

Figure 16 indicates that only one ordering, $A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$ is obtained by varying ρ from 1.001 to 20.

6 Comparison and Discussions

From the above results, it is evidenced that the developed AOs have the higher capability, not only to cover the concepts of different existing operators, but also a large number of AOs can be developed based on those operators.

- Compared with PFWMSM operator [44]

It is worthy to mention here that different ranking results of alternatives are found using the proposed method and which also covers the result of Wei and Lu [44]. The technique developed by Wei and Lu [44] is based on algebraic operation under PF environment, whereas the proposed approach is based on At-CN&t-Ns using HPF information. So, it is claimed that the approach of Wei and Lu [44] is a particular case of the proposed method. Thus, the proposed methodology is more consistent than the technique developed by Wei and Lu [44].

- Compared with WHPFMSM operator [45]

In comparison to WHPFMSM AO [45], it is found that the solution achieved through the developed HPFHWMSM operator, by considering the value of Hamacher parameter, $\tau = 1$, the ranking results of the alternatives are found as the same as achieved by Garg [45], as shown in Table 7. Therefore, the method introduced by Garg [45] now appeared as a special case of the proposed method. Again, if the Hamacher

Table 7 Comparison of the ordering of alternatives with the existing operator [45]

Methods	MSM parameter	Ordering
WHPFMSM [45] operator	$r = 1$	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$
	$r = 2$	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$r = 3, 4$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
Proposed HPFHWMSM operator	$r = 1$	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$
		$A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$
		$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$
		$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$
	$r = 2$	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$
		$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
		$A_5 \succ A_3 \succ A_4 \succ A_2 \succ A_1$
		$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
		$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
	$r = 3, 4$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
Proposed HPFFWMSM operator	$r = 1$	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$
		$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$
	$r = 2$	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$r = 3$	$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$
		$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
	$r = 4$	$A_5 \succ A_3 \succ A_2 \succ A_4 \succ A_1$
		$A_5 \succ A_3 \succ A_2 \succ A_1 \succ A_4$

parameter, $\tau = 1$, is considered together with MSM parameter, $r = 2$, the developed HPFHWMSM operator is converted into weighted PF BM operator [51].

Apart from that, several number of AOs, viz., dual HPF BM [52], dual HF weighted MSM [53], and others can be constructed from the developed HPFHWMSM operator. The score values of alternatives are also evaluated for different values of the parameter τ in the range (0,20] and different ranking orders of alternatives are found.

Moreover, when this problem is solved using HPFFWMSM AO, several ranking orders of the alternatives are found as like as the ordering achieved using Hamacher class of AOs.

Thus, the proposed method is flexible enough to capture the concept of a large number of AOs by varying different parameters associated with it.

7 Conclusions

The concept of MSM is extended in this article under HPF environment to construct AHPFMSM and AHPFWMSM AOs based on At-CN&t-Ns. Several properties of the developed operators have also been studied. The developed AOs possess the capacity of acquiring concepts of interrelationships between input arguments. It has already been shown in Sect. 6 that the developed operators are efficient enough to capture the concepts of other types of AOs in different variants of fuzzy environments. Many new types of AOs can be generated from the developed AOs, viz., AHPFA, AHPFBM, and AHPFGM operators by changing the value of MSM parameter. Again considering different decreasing generators in AHPFMSM, several AOs, viz., HPFWMSM, HPFEWMSM, HPFHWMSM, and HPFFWMSM operators can be derived. Using the developed operators, several methods for solving MCDM problems with HPF information are presented. The developed approaches can capture the preferences of the DMs using different parameters involved with the methods.

Two illustrative examples have been solved to establish potentiality of the proposed approaches in the contexts of the selection of emerging technology enterprise and profitable market for investment by an investor. Solving those it has been proved that several existing operators now appeared as special cases of the developed operators by keeping the same ranking results or with slight variations. The developed approaches for solving MCDM problems can be applied to solve various real-life decision-making problems, like uncertain programming, pattern recognition, cluster analysis, etc. under HPF environments. Further, the concept of the developed operators may also be extended to other fuzzy contexts, like q-rung, hesitant q-rung orthopair fuzzy, and other variants. However, it is hoped that the developed methods may open up new direction of AOs under different variants of fuzzy environments by integrating a large number of AOs in the context of solving MCDM problems.

Acknowledgements The authors remain grateful to the reviewers for their constructive comments and suggestions to improve the quality of the manuscript. Authors are also thankful to Dr. H. Garg, the editor, for his continuous effort for publication of this book chapter.

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Extensions of Linguistic Pythagorean Fuzzy Sets and Their Applications in Multi-attribute Group Decision-Making

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1 Introduction

In modern decision-making sciences, multi-attribute group decision-making (MAGDM) refers to a series of decision issues where domain decision experts (DEs) are required to provide their evaluations over feasible alternatives under multiple attributes [1–5]. Some techniques or methods are then applied to help DEs to obtain the ranking order of candidate alternatives so that the optimal one(s) is obtained accordingly. However, it is quite difficult to deal with the inherent fuzziness and uncertainties in practical MAGDM problems. Numerous scholars and scientists devoted themselves to discover tools that can effectively handle fuzzy information in data. Intuitionistic fuzzy sets (IFSs) [6], originally generated by Atanassov have the capability of representing uncertain information and they depict fuzzy phenomenon from both positive and negative perspectives. In other words, by simultaneously incorporating membership degree (MD) and non-membership degree (NMD), IFSs can more comprehensively and accurately describe fuzzy and vague decision-making information. Due to this characteristic, IFSs-based MAGDM theory and methods

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_15

have been active fields, and quite a few new achievements have been made in the past couple of years. For example, in [7] Xu originated the theory of aggregation operators (AOs) for intuitionistic fuzzy numbers (IFNs). After it, some other AOs for IFNs, such as intuitionistic fuzzy Bonferroni mean [8], intuitionistic fuzzy point operators [9], intuitionistic fuzzy Maclaurin symmetric mean [10], intuitionistic fuzzy Muirhead mean [11], intuitionistic fuzzy power average operators [12], intuitionistic fuzzy power Bonferroni mean [13], etc., have been proposed one after another.

MDs and NMDs of IFSs are denoted by crisp numbers, however, which is sometimes difficult to be determined by DEs. As MAGDM problems in practice are becoming more and more complicated, rather than crisp numbers, DEs would like to employ linguistic terms to denote MDs and NMDs to express their assessments. Motivated by this fact, Chen et al. [14] generalized IFSs and proposed linguistic intuitionistic fuzzy sets (LIFSs), which use linguistic terms to denote MD and NMD. Linguistic terms are quite similar to natural language and so that it is more convenient for DEs to use LIFSs to provide their evaluation values. After the appearance of LIFS, Ou et al. [15] introduced the LIFS-TOPSIS method to deal with MAGDM problems. Yuan et al. [16] proposed a series of linguistic intuitionistic fuzzy (LIF) Shapely AOs to handle decision-making issues with LIF numbers (LIFNs). Meng et al. [17] studied preference relations under LIFSs and applied them in decision-making problems. Liu and You [18] proposed a collection of novel LIF Heronian mean AOs under Einstein t-norm and t-conorm. Liu and Qin [19] put forward LIF power average operator in order to reduce the negative influence of unduly high or low aggregated LIFNs on the final results. Liu and Qin [20], and Liu and Liu [21] further presented LIF Maclaurin symmetric mean operators and LIF Hamy mean operators to capture the interrelationship among multiple aggregated LIFNs. To effectively handle heterogeneous interrelationship among LIFNs, Liu and his colleagues [22] defined LIF partitioned Heronian mean operators. Garg and Kumar [23] further introduced novel LIF power average operators and the corresponding decision-making method based on set pair analysis. Zhang et al. [24] proposed the LIF-ELECTRE decision-making method and applied in coal mine safety evaluation problems. Peng and Wang [25] introduced a LIF MAGDM method based on cloud model and studied its application in selecting sustainable energy crop. For more studies on LIFSs-based MAGDM methods, readers are suggested to refer [26–30].

Aforementioned literatures reveal that LIFSs are capable to depict fuzzy evaluation values provided by DEs effectively, however, they still have drawbacks when dealing with some practical decision-making situations. The definition of LIFS is as follows: let $\tilde{S} = \{s_\alpha | s_0 \leq s_\alpha \leq s_t, \alpha \in [0, t]\}$ be a predefined continuous linguistic term set, then a LIFS defined on $\tilde{S}$ can be expressed as $A = \{(x, s_\theta(x), s_\sigma(x)) | x \in X\}$. As is known, A should satisfy the constraint that $\theta + \sigma \leq t$, which cannot be always strictly satisfied in some practical MAGDM problems. For example, an ordered pair (s_3, s_4) is used to depict a DE's evaluation value, where s_l is a linguistic term and $l \in [0, 6]$. As $3 + 4 \leq 6$, the evaluation value (s_3, s_4) cannot be handled by LIFSs, which illustrates the weakness of LIFSs. In order to more accurately capture DEs' complicated and uncertain decision information and handle more difficult decision situations, Garg [31] proposed a new tool, called linguistic Pythagorean fuzzy sets (LPFSs), which are motivated by Pythagorean fuzzy sets (PFSs) that introduced by Yager [32]. As is widely known, the constraint of PFSs is that the square sum of MD and NMD is not greater than one and this characteristic makes PFSs more powerful and flexible than IFSs, gaining great interests from scholars [33–42]. In LPFSs, MDs and NMDs are denoted by linguistic terms, satisfying the constraint that $\theta^2 + \sigma^2 \leq t^2$ (s_θ and s_σ denote the MD and NMD, respectively, and s_t is the largest linguistic term of the predefined linguistic term set), and due to this reason, LPFSs can describe lager information span than LIFSs. In [31], Garg proposed basic operational linguistic Pythagorean fuzzy (LPF) values, presented their fundamental AOs, and applied in MAGDM problems. After it, Liu et al. [43] studied LPF operational rules, AOs, and MAGDM method based on Archimedean t-norm and t-conorm. Han et al. [44] introduced distance and entropy measures of LPFSs and based on which, an LPF-TOPSIS decision-making method was originated. To felicitously handle MAGDM issues where interrelationship among attributes is heterogeneous, Lin et al. [45] put forward the LPF partitioned Bonferroni mean operators.

There exists high indeterminacy and hesitancy when providing MDs and NMDs of evaluation values in the most practical MAGDM processes. Hence, the key problem is to effectively deal with the inherent fuzziness and uncertainties of data and DEs' hesitancy. For instance, Torra [46] proposed the concept of hesitant fuzzy sets (HFSs) by considering multiple possible MDs in an evaluation element. Compared with the classical fuzzy set theory [47], HFSs can better depict DEs' hesitancy. Similarly, Zhu et al. [48] introduced the dual HFSs (DHFSs) by taking not only multiple MDs but also NMDs into account. DHFSs are regarded as an extension of IFS, as they emphasize multiple values of degrees instead single ones. Another example is dual hesitant Pythagorean fuzzy sets introduced by Wei et al. [49], which consider more than one Pythagorean fuzzy MDs and NMDs in the traditional PFSs. These publications remind scholars and DEs an extensively existing phenomenon that most decisions are made in a hesitant fuzzy environment and DEs' hesitancy should be taken into consideration before determining the rank of feasible alternatives. In LPF, decision-making environment, we always encounter MAGDM situations wherein DEs are hesitant among a set of linguistic terms when giving the MDs and NMDs an evaluation value. Motivated by DHFSs which allow the existence of multiple MDs and NMDs in a decision evaluation value, this paper extends the traditional LPFSs to a hesitant fuzzy environment and propose dual hesitant linguistic Pythagorean fuzzy sets (DHLPFSs), which permit MDs and NMDs to be denoted by a set of linguistic terms. Compared with LPFSs, DHLPFSs are more flexible and can depict attribute values more accurately. In this chapter. We first give the definition, operational rules, comparison method, and AOs of DHLPFSs and propose a novel MAGDM method with DHLPFSs. We also show the performance of the proposed new method through illustrative examples.

Additionally, to more accurately capture DEs' evaluation value in hesitant fuzzy decision-making environment, not only each member in an evaluation element but also its probabilistic information should be considered. For example, Zhang et al. [50] capture the probability of each member in HFSs and proposed the probabilistic

HFSs (PHFSs). Compared with HFSs, the PHFSs not only depict the hesitant fuzzy MDs but also the corresponding probabilistic information. Afterward, PHFSs have been successfully applied in various fields, such as public company efficiency evaluation [51], hospital evaluation [52], virtual reality project declaration evaluation [53], selection of the most influential teacher [54], etc. Similarly, Hao et al. [55] extended DHFSs to probabilistic DHFSs by taking the probabilities of possible MDs and NMDs into account. The ability of efficiency of probabilistic DHFSs to depict DEs' evaluation values is further studied in [56–58]. Additionally, some other new information representation tools were also proposed, such as probabilistic linguistic terms set [59], probabilistic linguistic dual hesitant fuzzy sets [60], and probabilistic single-valued neutrosophic hesitant fuzzy sets [61]. These publications motivate us to further extend DHLPFSs to a more generalized form, i.e., probabilistic DHLPFSs (PDHLPFSs). The advantages of PDHLPFSs are outstanding. First, they allow the MD and NMD to be denoted by two collections of possible linguistic terms, which can comprehensively describe DEs' high hesitancy. Second, they also consider the corresponding probabilistic information of each linguistic term, which can more effectively depict group's evaluation values. For the sake of applications of PDHLPFSs in MAGDM, the basic operational laws, comparison method, and AOs of PDHLPFSs are studied. Finally, the main steps of solving a MAGDM problem under PDHLPFSs are presented.

The main motivations of our works are to propose novel MAGDM methods, that only not more accurately depict DEs' evaluation information but also help them to appropriately determine the optimal alternatives. The main contributions of this chapter are four-fold. First, two information expression tools were proposed, namely DHLPFSs and PDHLPFSs. These two fuzzy set theories have obvious advantages and superiorities in depicting DEs' evaluations. Second, we proposed a series of AOs to fuse DHLPFSs and PDHLPFSs, which are potential for introducing novel decision-making methods. Third, we proposed two new MAGDM methods. Finally, real MAGDM problems were employed to prove the validity of our methods. The rest of this chapter is organized as follows. Section 2 recalls basic notions which will be used in the following sections. Section 3 introduces DHLPFSs and the corresponding MAGDM method. Section 4 further proposes DHLPFSs and studies their applications in MAGDM. Conclusion remarks can be found in Sect. 5.

2 Basic Concepts

This section briefly reviews basic notions that will be used in the following sections.

Definition 1 ([31]) Let X be a fixed set and $\tilde{S} = \{s_0, s_1, s_2 \ldots, s_l\}$ be a continuous linguistic term set with odd cardinality. A linguistic Pythagorean fuzzy set (LPFS) defined in X is given as

$$\gamma = \big\{\big(x, s_\alpha(x), s_\beta(x)\big)|x \in X\big\}, \tag{1}$$

where $s_\alpha(x), s_\beta(x) \in S_{[0,l]}$, s_α, and s_β represent the linguistic MD and linguistic NMD, respectively, such that $0 \le \alpha^2 + \beta^2 \le l^2$. For convenience, the ordered pair $\gamma = \left(s_\alpha, s_\beta\right)$ is called a linguistic Pythagorean fuzzy value (LPFV). The linguistic indeterminacy degree of γ is expressed as $\pi(x) = s_{\left(l^2-\alpha^2-\beta^2\right)^{1/2}}$.

The basic operational rules of LPFVs are presented as follows.

Definition 2 ([31]) Let $\gamma_1 = \left(s_{\alpha_1}, s_{\beta_1}\right)$, $\gamma_2 = \left(s_{\alpha_2}, s_{\beta_2}\right)$, and $\gamma = \left(s_\alpha, s_\beta\right)$ be LPFVs and λ be a positive real number, then

(1) $\gamma_1 \oplus \gamma_2 = \left(s_{\left(\alpha_1^2+\alpha_2^2-\alpha_1^2\alpha_2^2/l^2\right)^{1/2}}, s_{(\beta_1\beta_2/l)}\right)$;

(2) $\gamma_1 \otimes \gamma_2 = \left(s_{(\alpha_1\alpha_2/l)}, s_{\left(\beta_1^2+\beta_2^2-\beta_1^2\beta_2^2/l^2\right)^{1/2}}\right)$;

(3) $\lambda\gamma = \left(s_{l\left(1-\left(1-\alpha^2/l^2\right)^\lambda\right)^{1/2}}, s_{l(\beta/l)^\lambda}\right)$;

(4) $\gamma^\lambda = \left(s_{l(\alpha/l)^\lambda}, s_{l\left(1-\left(1-\beta^2/l^2\right)^\lambda\right)^{1/2}}\right)$.

Garg [31] proposed a method to compare any two LPFVs.

Definition 3 ([31]) Let $\gamma = \left(s_\alpha, s_\beta\right) \in \Gamma_{[0,l]}$ be a LPFV, then the score function $S(\gamma)$ of γ is expressed as

$$S(\gamma) = s_{\sqrt{\left(l^2+\alpha^2-\beta^2\right)/2}}, \tag{2}$$

and the accuracy function $H(\gamma)$ is defined as

$$H(\gamma) = s_{\sqrt{\alpha^2+\beta^2}}, \tag{3}$$

where $S(\gamma), H(\gamma) \in S$. Let $\gamma_1 = \left(s_{\alpha_1}, s_{\beta_1}\right)$ and $\gamma_2 = \left(s_{\alpha_2}, s_{\beta_2}\right)$ be any two LPFVs, then

(1) If $S(\gamma_1) > S(\gamma_2)$, then $\gamma_1 > \gamma_2$;

(2) If $S(\gamma_1) = S(\gamma_2)$, then

If $H(\gamma_1) > H(\gamma_2)$, then $\gamma_1 > \gamma_2$;
If $H(\gamma_1) = H(\gamma_2)$, then $\gamma_1 = \gamma_2$.

3 Dual Hesitant Linguistic Pythagorean Fuzzy Sets and Their Applications in MAGDM

In this section, we introduce the notion of DHLPFSs and study their application in MAGDM problems. For this purpose, we first introduce the motivations to explain why we propose DHLPFSs and why we need them. Afterward, some related notions, such as operational rules, comparison method, and AOs are proposed. Finally, we employ DHLPFSs as well as their AOs to solve MAGDM problems.

3.1 Motivations and Necessity of Proposing DHLPFSs

In LPFSs, MD and NMD are denoted by two linguistic terms. As is known, linguistic terms set and linguistic terms are similar to natural language so that LPFSs provide DEs a convenient and natural manner to express their evaluation values. Due to this reason, LPFSs are more suitable than PFSs to depict DEs' fuzzy and complex evaluation information. However, the traditional LPFSs still have limitations in some practical MAGDM problems. As real decision-making problems are very complicated, sometimes it is difficult for DEs to provide single linguistic terms for MD and NMD. Actually, DEs are always hesitant among a collection of possible linguistic terms when determining MDs and NMDs of their evaluation values. To better demonstrate this phenomenon, we provide the following example.

Example 1 Suppose there are three professors and they are invited to evaluate the innovativeness of a doctoral student's research proposal. To more accurately and effectively evaluate the quality and innovation, DEs are permitted to use multiple values to denote MDs and NMDs of their evaluation values. Let S be a given linguistic term set, where $S = \{s_0 =$ very poor, $s_1 =$ poor, $s_2 =$ slightly poor, $s_3 =$ fair, s_4 $=$ slightly good, $s_5 =$ good, $s_6 =$ slightly good$\}$. the three professors use multiple linguistic terms to express their evaluation information and DEs' evaluation opinions are listed in Table 1.

Take the first professor as an example, as seen in Table 1, he/she is hesitant between s_4 and s_5 when giving MD and s_1, s_2 and s_3 when providing MD. It is obvious that in the framework of LPFSs, the overall evaluation values of the decision group cannot be denoted. This is because LPFS theory only allows single MD and NMD, and so that it is insufficient to handle Example 1. In practical MAGDM problems, due to many reasons, such as lacking prior knowledge or time, DEs often hesitate among several values when providing MDs and NMDs and obviously LPFSs are incapable to handle these situations. Therefore, it is necessary to study LPFSs under hesitant fuzzy decision environment.

Table 1 The evaluation information provided by DEs in Example 1

	Possible MDs	Possible NMDs
The first professor	$\{s_4, s_5\}$	$\{s_1, s_2, s_3\}$
The second professor	$\{s_2, s_3, s_4\}$	$\{s_0, s_3\}$
The third professor	$\{s_1, s_2, s_4\}$	$\{s_0, s_2, s_3\}$

3.2 Definition of DHLPFSs

Definition 4 Let X be a fixed set and $\tilde{S} = \{s_\alpha | 0 \le \alpha \le l\}$ be continuous linguistic term set with odd cardinality. A dual hesitant linguistic Pythagorean fuzzy set (DHLPFSs) defined on X is expressed as

$$D = \{\langle x, h_D(x), g_A(x)\rangle | x \in X\}, \tag{4}$$

where $h_A(x), g_A(x) \subseteq \tilde{S}$ are two sets of some linguistic terms, denoting the possible linguistic MDs and linguistic NMDs of element $x \in X$ to the set D, respectively, such that $\sigma^2 + \eta^2 \le l^2$, where $s_\sigma \in h_D(x)$, and $s_\eta \in g_D(x)$ for $x \in X$. For convenience, we call the ordered pair $d(x) = (h_D(x), g_D(x))$ a dual hesitant linguistic Pythagorean fuzzy element (DHLPFE), which can be denoted as $d = (h, g)$, where $s_\sigma \in h, s_\eta \in g$ and $\sigma^2 + \eta^2 \le l^2$.

Definition 4 reveals that DHLPFSs can be regarded as a generalized form of LPFSs, by considering situations of the existence of multiple MDs and NMDs. In other words, LPFS is a special case of DHLPFSs. Hence, the proposed DHLPFSs are more suitable to deal with decision-making cases wherein DEs have different opinions and they cannot reach an agreement. In Example 1, if we use DHLPFSs to denote the overall evaluation value, then it should be $d = \{\{s_1, s_2, s_3, s_4, s_5\}, \{s_0, s_1, s_2, s_3\}\}$, which is obviously a DHLPFE.

3.3 Operations of DHLPFEs

Based on the definition of DHLPFEs and the operations principle of DHFEs, we propose some basic operational rules of DHLPFEs.

Definition 5 Let $d_1 = (h_1, g_1), d_2 = (h_1, g_2)$, and $d = (h, g)$ be any three DHLPFEs and λ be a positive real number, then

(1) $d_1 \oplus d_2 = \bigcup_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{\left\{s_{\left(\sigma_1^2 + \sigma_2^2 - \sigma_1^2\sigma_2^2/l^2\right)^{1/2}}\right\}, \left\{s_{(\eta_1\eta_2/l)}\right\}\right\}$;

(2) $d_1 \otimes d_2 = \bigcup_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{\left\{s_{(\sigma_1\sigma_2/l)}\right\}, \left\{s_{\left(\eta_1^2 + \eta_2^2 - \eta_1^2\eta_2^2/l^2\right)^{1/2}}\right\}\right\}$;

(3) $\lambda d = \bigcup_{\sigma \in h, \eta \in g} \left\{\left\{s_{l\left(1-\left(1-\sigma^2/l^2\right)^\lambda\right)^{1/2}}\right\}, \left\{s_{l(\eta/l)^\lambda}\right\}\right\}$;

(4) $d^\lambda = \bigcup_{\sigma \in h, \eta \in g} \left\{\left\{s_{l(\sigma/l)^\lambda}\right\}, \left\{s_{l\left(1-\left(1-\eta^2/l^2\right)^\lambda\right)^{1/2}}\right\}\right\}$.

Example 2 Let $d_1 = \{\{s_3, s_4, s_5\}, \{s_2, s_3\}\}$ and $d_2 = \{\{s_2, s_4\}, \{s_4\}\}$ be two DHLPFEs derived from a pre-defined linguistic term set $\tilde{S} = \{s_\alpha | 0 \le \alpha \le 6\}$, then

$d_1 \oplus d_2 = \{\{s_{3.4641}, s_{4.5826}, s_{4.2687}, s_{4.9889}, s_{5.1208}, s_{5.4671}\}, \{s_{1.3333}, s_{2.0000}\}\}$;

$d_1 \otimes d_2 = \{\{s_{1.0000}, s_{1.333}, s_{2.6667}, s_{3.3333}\}, \{s_{4.2687}, , s_{4.5826}\}\};$
$3d_1 = \{\{s_{4.5621}, , s_{5.4614}, s_{5.9138}\}, \{s_{0.2222}, s_{0.7500}\}\};$
$d_1^3 = \{\{s_{0.7500}, s_{1.7778}, s_{3.4722}\}, \{s_{3.2735}, s_{4.5621}\}\}.$

Theorem 1 *Let $d_1 = (h_1, g_1)$, $d_2 = (h_1, g_2)$, and $d = (h, g)$ be any three DHLPFEs, then*

(1) $d_1 \oplus d_2 = d_2 \oplus d_2$;
(2) $d_1 \otimes d_2 = d_2 \otimes d_1$;
(3) $\lambda(d_1 \oplus d_2) = \lambda d_1 \oplus \lambda d_2$;
(4) $\lambda_1 d \oplus \lambda_2 d = (\lambda_1 + \lambda_2)d, (\lambda_1, \lambda_2 \geq 0)$;
(5) $d^{\lambda_1} \otimes d^{\lambda_2} = d^{\lambda_1+\lambda_2}, (\lambda_1, \lambda_2 \geq 0)$;
(6) $d_1^{\lambda} \otimes d_2^{\lambda} = (d_1 \otimes d_2)^{\lambda}, (\lambda \geq 0)$.

Proof It is easy to prove that (1) and (2) hold, in the following we attempt to prove other formulas. According to Definition 5, we have

$$\lambda(d_1 \oplus d_2) = \bigcup_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ s_{l\left(1-\left(\left(1-\sigma_1^2/l^2\right)\left(1-\sigma_2^2/l^2\right)\right)^{\lambda}\right)^{1/2}} \right\}, \left\{ s_{l\left(\eta_1\eta_2/l^2\right)^{\lambda}} \right\} \right\},$$

and

$$\begin{aligned}\lambda d_1 \oplus \lambda d_2 &= \bigcup_{\sigma_1 \in h_1, \eta_1 \in g_1} \left\{ \left\{ s_{l\left(1-\left(1-\sigma_1^2/l^2\right)^{\lambda}\right)^{1/2}} \right\}, \left\{ s_{l(\eta_1/l)^{\lambda}} \right\} \right\} \\ &\oplus \bigcup_{\sigma_2 \in h_2, \eta_2 \in g_2} \left\{ \left\{ s_{l\left(1-\left(1-\sigma_2^2/l^2\right)^{\lambda}\right)^{1/2}} \right\}, \left\{ s_{l(\eta_2/l)^{\lambda}} \right\} \right\} \\ &= \bigcup_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ s_{l\left(1-\left((1-\sigma_1^2/l^2)(1-\sigma_2^2/l^2)\right)^{\lambda}\right)^{1/2}} \right\}, \left\{ s_{l(\eta_1\eta_2/l^2)^{\lambda}} \right\} \right\} = \lambda(d_1 \oplus d_2),\end{aligned}$$

which proves the correctness of (3).

Meanwhile, we can obtain that

$$\begin{aligned}&\lambda_1 d \oplus \lambda_2 d \\ &= \bigcup_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l\left(1-(1-\sigma^2/l^2)^{\lambda_1}\right)^{1/2}} \right\}, \left\{ s_{l(\eta/l)^{\lambda_1}} \right\} \right\} \oplus \bigcup_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l\left(1-(1-\sigma^2/l^2)^{\lambda_2}\right)^{1/2}} \right\}, \left\{ s_{l(\eta/l)^{\lambda_2}} \right\} \right\} \\ &= \bigcup_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l\left(1-(1-\sigma^2/l^2)^{\lambda_2+\lambda_1}\right)^{1/2}} \right\}, \left\{ s_{(l(\eta/l)^{\lambda_1+\lambda_2})} \right\} \right\},\end{aligned}$$

and

$$(\lambda_1 + \lambda_2)d = \bigcup_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l\left(1-\left(1-\sigma^2/l^2\right)^{\lambda_1+\lambda_2}\right)^{1/2}} \right\}, \left\{ s_{l(\eta/l)^{\lambda_1+\lambda_2}} \right\} \right\} = \lambda_1 d \oplus \lambda_2 d,$$

which proves the validity of (4).

Moreover,

$$\begin{aligned} d^{\lambda_1} \otimes d^{\lambda_2} &= \bigcup\nolimits_{\sigma\in h,\eta\in g} \left\{ \left\{ s_{l(\sigma/l)^{\lambda_1}} \right\}, \left\{ s_{l\left(1-\left(1-\eta^2/l^2\right)^{\lambda_1}\right)^{1/2}} \right\} \right\} \\ &\quad \otimes \bigcup\nolimits_{\sigma\in h,\eta\in g} \left\{ \left\{ s_{l(\sigma/l)^{\lambda_2}} \right\}, \left\{ s_{l\left(1-\left(1-\eta^2/l^2\right)^{\lambda_2}\right)^{1/2}} \right\} \right\}, \\ &= \bigcup\nolimits_{\sigma\in h,\eta\in g} \left\{ \left\{ s_{l(\sigma/l)^{\lambda_1+\lambda_2}} \right\}, \left\{ s_{l\left(1-\left(1-\eta^2/l^2\right)^{\lambda_1+\lambda_2}\right)^{1/2}} \right\} \right\} \\ d^{\lambda_1+\lambda_2} &= \bigcup\nolimits_{\sigma\in h,\eta\in g} \left\{ \left\{ s_{l(\sigma/l)^{\lambda_1+\lambda_2}} \right\}, \left\{ s_{l\left(1-\left(1-\eta^2/l^2\right)^{\lambda_1+\lambda_2}\right)^{1/2}} \right\} \right\} = d^{\lambda_1} \otimes d^{\lambda_2}, \end{aligned}$$

which proves the rightness of (5).

Besides,

$$\begin{aligned} d_1^{\lambda} \otimes d_2^{\lambda} &= \bigcup\nolimits_{\sigma_1\in h_1,\eta_1\in g_1} \left\{ \left\{ s_{l(\sigma_1/l)^{\lambda}} \right\}, \left\{ s_{l\left(1-\left(1-\eta_1^2/l^2\right)^{\lambda}\right)^{1/2}} \right\} \right\} \\ &\quad \otimes \bigcup\nolimits_{\sigma_2\in h_2,\eta_2\in g_2} \left\{ \left\{ s_{l(\sigma_2/l)^{\lambda}} \right\}, \left\{ s_{l\left(1-\left(1-\eta_2^2/l^2\right)^{\lambda}\right)^{1/2}} \right\} \right\}, \\ &= \bigcup\nolimits_{\sigma_1\in h_1,\sigma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2} \left\{ \left\{ s_{l\left(\sigma_1\sigma_2/l^2\right)^{\lambda}} \right\}, \left\{ s_{l\left(1-\left(\left(1-\eta_1^2/l^2\right)\left(1-\eta_2^2/l^2\right)\right)^{\lambda}\right)^{1/2}} \right\} \right\}. \end{aligned}$$

In addition,

$$\begin{aligned} (d_1 \otimes d_2)^{\lambda} &= \left(\bigcup\nolimits_{\sigma_1\in h_1,\sigma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2} \left\{ \left\{ s_{\sigma_1\sigma_2/l} \right\}, \left\{ s_{\left(\eta_1^2+\eta_2^2-\eta_1^2\eta_2^2/l^2\right)^{1/2}} \right\} \right\} \right)^{\lambda} \\ &= \bigcup\nolimits_{\sigma_1\in h_1,\sigma_2\in h_2,\eta_1\in g_1,\eta_2\in g_2} \left\{ \left\{ s_{l\left(\sigma_1\sigma_2/l^2\right)^{\lambda}} \right\}, \left\{ s_{l\left(1-\left(\left(1-\eta_1^2/l^2\right)\left(1-\eta_2^2/l^2\right)\right)^{\lambda}\right)^{1/2}} \right\} \right\} = d_1^{\lambda} \otimes d_2^{\lambda}, \end{aligned}$$

which demonstrates (6) holds.

3.4 Comparison Method of DHLPFEs

To rank DHLPFEs, we provide the following comparison method.

Definition 6 Let $d = (h, g)$ be a DHLPFEs, the score function $\tau(d)$ of d is expressed as

$$\tau(d) = s_{\sqrt{\left(l^2+\left(\sum_{i=1,\sigma\in h}^{\#h} \sigma_i^2\right)/\#h-\left(\sum_{j=1,\eta\in g}^{\#g} \eta_j^2\right)/\#g\right)/2}}, \tag{5}$$

and the accuracy function $\varphi(d)$ is defined as

$$\varphi(d) = s_{\sqrt{\left(\left(\sum\limits_{i=1,\sigma\in h}^{\#h}\sigma_i^2\right)/\#h+\left(\sum\limits_{j=1,\eta\in g}^{\#g}\eta_j^2\right)/\#g\right)/2}}. \tag{6}$$

where $\#h$ and $\#g$ denote the numbers of elements in h and g. For any two DHLPFEs d_1 and d_2,

(3) If $\tau(d_1) > \tau(d_2)$, then $d_1 > d_2$;
(4) If $\tau(d_1) = \tau(d_2)$, then

If $\varphi(d_1) > \varphi(d_2)$, then $d_1 > d_2$;
If $\varphi(d_1) = \varphi(d_2)$, then $d_1 = d_2$.

Example 3 Let $\tilde{S} = \{s_\alpha | 0 \leq \alpha \leq 6\}$ be a pre-defined continuous linguistic term set, and $d_1 = \{\{s_0, s_2, s_3\}, \{s_4, s_5\}\}$ and $d_2 = \{\{s_1, s_3\}, \{s_2, s_3, s_5\}\}$ be two DHLPFEs defined on $\tilde{S}$, then we have

$$\tau(d_1) = s_{\sqrt{\left(6^2+\left(0^2+2^2+3^2\right)/3-\left(4^2+5^2\right)/2\right)/2}} = s_{3.1491}, \quad \varphi(d_1) = s_{\sqrt{\left(\left(0^2+2^2+3^2\right)/3+\left(4^2+5^2\right)/2\right)/2}} = s_{3.5237}$$
$$\tau(d_2) = s_{\sqrt{\left(6^2+\left(1^2+3^2\right)/2-\left(2^2+3^2+5^2\right)/3\right)/2}} = s_{3.7639}, \quad \varphi(d_2) = s_{\sqrt{\left(\left(1^2+3^2\right)/2+\left(2^2+3^2+5^2\right)/3\right)/2}} = s_{2.9721}$$

According to Definition 6, we can get $d_2 > d_1$.

3.5 Some Basic Aggregation Operators of DHLPFEs

To aggregate attribute values under DHLPFSs, we propose a series of weighted AOs for DHLPFEs and discuss their properties.

Definition 7 Let $d_i = (h_i, g_i)(i = 1, 2, \ldots, n)$ be a collection of DHLPFEs, and let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector, such that $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$. The dual hesitant linguistic Pythagorean fuzzy weighted average (DHLPFWA) operator is defined as

$$DHLPFWA(d_1, d_2, \ldots, d_n) = \oplus_{i=1}^{n} w_i d_i, \tag{7}$$

Based on the operations of DHLPFEs, the following aggregated value can be obtained.

Theorem 2 *Let $d_i = (h_i, g_i)(i = 1, 2, \ldots, n)$ be collection of DHLPFEs, then the aggregation result by using the DHLPFWA operator is also a DHLPFE and*

$$DHLPFWA(d_1, d_2, \ldots, d_n) = \bigcup_{\sigma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ s_{l\left(1-\prod_{i=1}^{n}\left(1-\sigma_i^2/l^2\right)^{w_i}\right)^{1/2}} \right\}, \left\{ s_{l\prod_{i=1}^{n}(\eta_i/l)^{w_i}} \right\} \right\}, \quad (8)$$

Proof When $n = 2$, then

$$w_1 d_1 = \bigcup_{\sigma_1 \in h_1, \eta_1 \in g_1} \left\{ \left\{ s_{l\left(1-\left(1-\sigma_1^2/l^2\right)^{w_1}\right)^{1/2}} \right\}, \left\{ s_{l(\eta_1/l)^{w_1}} \right\} \right\},$$

and

$$w_2 d_2 = \bigcup_{\sigma_2 \in h_2, \eta_2 \in g_2} \left\{ \left\{ s_{l\left(1-\left(1-\sigma_2^2/l^2\right)^{w_2}\right)^{1/2}} \right\}, \left\{ s_{l(\eta_2/l)^{w_2}} \right\} \right\}.$$

Then

$$\begin{aligned} DHLPFWA(d_1, d_2) &= w_1 d_1 \oplus w_2 d_2 \\ &= \bigcup_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ s_{l\left(1-\left(1-\sigma_1^2/l^2\right)^{w_1}\left(1-\sigma_2^2/l^2\right)^{w_2}\right)^{1/2}} \right\}, \right. \\ &\qquad \left. \left\{ s_{\left(l(\eta_1/l)^{w_1}(\eta_2/l)^{w_2}\right)} \right\} \right\}, \end{aligned}$$

which implies that Eq. (8) holds for $n = 2$.

In addition, we assume Eq. (8) holds for $n = k$, i.e.,

$$DHLPFWA(d_1, d_2, \ldots, d_k) = \bigcup_{\sigma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ s_{l\left(1-\prod_{i=1}^{k}\left(1-\sigma_i^2/l^2\right)^{w_i}\right)^{1/2}} \right\}, \left\{ s_{l\prod_{i=1}^{k}(\eta_i/l)^{w_i}} \right\} \right\},$$

then when $n = k + 1$, we can obtain

$$\begin{aligned} DHLPFWA(d_1, d_2, \ldots, d_k, d_{k+1}) &= \oplus_{i=1}^{k} w_i d_i \\ &\oplus w_{k+1} d_{k+1} \bigcup_{\sigma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ s_{l\left(1-\prod_{i=1}^{k}\left(1-\sigma_i^2/l^2\right)^{w_i}\right)^{1/2}} \right\}, \left\{ s_{l\prod_{i=1}^{k}(\eta_i/l)^{w_i}} \right\} \right\} \\ &\oplus \bigcup_{\sigma_{k+1} \in h_{k+1}, \eta_{k+1} \in g_{k+1}} \left\{ \left\{ s_{l\left(1-\left(1-\sigma_{k+1}^2/l^2\right)^{\lambda}\right)^{1/2}} \right\}, \left\{ s_{l\left(\eta_{k+1}^2/l\right)^{\lambda}} \right\} \right\} \\ &= \bigcup_{\sigma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ s_{l\left(1-\prod_{i=1}^{k+1}\left(1-\sigma_i^2/l^2\right)^{w_i}\right)^{1/2}} \right\}, \left\{ s_{l\prod_{i=1}^{k+1}(\eta_i/l)^{w_i}} \right\} \right\}, \end{aligned}$$

i.e., Eq. (8) holds for $n = k + 1$. Therefore, (12) holds for all n. The proof of Theorem 2 is completed.

In the following, we investigate some properties of DHLPFWA operator.

Theorem 3 (Monotonicity) *Let $d_i = (h_i, g_i)$ and $d_i^* = \left(h_i^*, g_i^*\right)(i = 1, 2, \ldots, n)$ be two collections of DHLPFEs, where $s_{\sigma_i} \in h_i$, $s_{\eta_i} \in g_i$ and $s_{\sigma_i^*} \in h_i^*$, $s_{\eta_i^*} \in g_i^*$. For $\forall i = 1, 2, \ldots, n$, if $s_{\sigma_i} \le s_{\sigma_i^*}$ and $s_{\eta_i} \ge s_{\eta_i^*}$, then*

$$DHLPFWA(d_1, d_2, \ldots, d_n) \le DHLPFWA\left(d_1^*, d_2^*, \ldots, d_n^*\right). \tag{9}$$

Proof For any i, there are $s_{\sigma_i} \le s_{\sigma_i^*}$ and $s_{\eta_i} \ge s_{\eta_i^*}$. For the terms in the aggregated results, we have

$$s_{l\left(1-\prod_{i=1}^{n}\left(1-\sigma_i^2/l^2\right)^{w_i}\right)^{1/2}} \le s_{l\left(1-\prod_{i=1}^{n}\left(1-\sigma_i^{*2}/l^2\right)^{w_i}\right)^{1/2}} \text{ and } s_{l\prod_{i=1}^{n}(\eta_i/l)^{w_i}} \ge s_{l\prod_{i=1}^{n}\left(\eta_i^*/l\right)^{w_i}}.$$

According to Definition 6, we can get $DHLPFWA(d_1, d_2, \ldots, d_n) \le DHLPFWA\left(d_1^*, d_2^*, \ldots, d_n^*\right)$ with equality if and only if $s_{\sigma_i} = s_{\sigma_i^*}$ and $s_{\eta_i} = s_{\eta_i^*}$ for all i.

Theorem 4 (Boundedness) *Let $d_i = (h_i, g_i)(i = 1, 2, \ldots, n)$ be a collection of DHLPFEs. For each $s_{\sigma_i} \in h_i$, $s_{\eta_i} \in g_i$ $(i = 1, 2, \ldots, n)$, let $d^- = \left(s_{\min\{\sigma_i\}}, s_{\max\{\eta_i\}}\right)$, $d^+ = \left(s_{\max\{\sigma_i\}}, s_{\min\{\eta_i\}}\right)$. Then*

$$\begin{aligned} DHLPFWA\left(d^-, d^-, \ldots, d^-\right) &\le DHLPFWA(d_1, d_2, \ldots, d_n) \\ &\le DHLPFWA\left(d^+, d^+, \ldots, d^+\right). \end{aligned} \tag{10}$$

Proof For $\forall i = 1, 2, \ldots, n$, we have $s_{\min\{\sigma_i\}} \le s_{\sigma_i} \le s_{\max\{\sigma_i\}}$, $s_{\min\{\eta_i\}} \le s_{\eta_i} \le s_{\max\{\eta_i\}}$. Then

$$s_{l\left(1-\prod_{i=1}^{n}\left(1-\sigma_i^2/l^2\right)^{w_i}\right)^{\frac{1}{2}}} \ge s_{l\left(1-\prod_{i=1}^{n}\left(1-(\min\{\sigma_i\}/l)^2\right)^{w_i}\right)^{\frac{1}{2}}} = s_{l\left(1-\left(1-(\min\{\sigma_i\}/l)^2\right)^{\sum_{i=1}^{n} w_i}\right)^{\frac{1}{2}}} = s_{\min\{\sigma_i\}},$$

$$\Rightarrow s_{l\prod_{i=1}^{n}(\eta_i/l)^{w_i}} \le s_{l\prod_{i=1}^{n}(\max\{\eta_i\}/l)^{w_i}} = s_{\max\{\eta_i\}},$$

$$\Rightarrow s_{l\left(1-\prod_{i=1}^{n}\left(1-\sigma_i^2/l^2\right)^{w_i}\right)^{\frac{1}{2}}} \le s_{l\left(1-\prod_{i=1}^{n}\left(1-(\max\{\sigma_i\}/l)^2\right)^{w_i}\right)^{\frac{1}{2}}} = s_{\max\{\sigma_i\}},$$

$$\Rightarrow s_{l\prod_{i=1}^{n}(\eta_i/l)^{w_i}} \ge s_{l\prod_{i=1}^{n}(\min\{\eta_i\}/l)^{w_i}} = s_{\min\{\eta_i\}}.$$

Further,

$$DHLPFWA\left(d^-, d^-, \ldots, d^-\right) = \cup\left\{\left\{s_{l\left(1-\prod_{i=1}^{n}\left(1-(\min\{\sigma_i\}/l)^2\right)^{w_i}\right)^{1/2}}\right\}, \left\{s_{l\prod_{i=1}^{n}(\max\{\eta_i\}/l)^{w_i}}\right\}\right\}$$

$$= \left\{ s_{\min\{\sigma_i\}}, s_{\max\{\eta_i\}} \right\};$$

$$DHLPFWA\left(d^+, d^+, \ldots, d^+\right) = \cup \left\{ \left\{ s_{l\left(1 - \prod_{i=1}^{n}\left(1 - (\max\{\sigma_i\}/l)^2\right)^{w_i}\right)^{1/2}} \right\}, \left\{ s_{l \prod_{i=1}^{n} (\min\{\eta_i\}/l)^{w_i}} \right\} \right\}$$

$$= \left\{ s_{\max\{\sigma_i\}}, s_{\min\{\eta_i\}} \right\};$$

According to Definition 6, we have $DHLPFWA(d_1, d_2, \ldots, d_n) \geq DHLPFWA\left(d^-, d^-, \ldots, d^-\right)$ with equality if and only if d_i is same as d^-. Similarly, $DHLPFWA(d_1, d_2, \ldots, d_n) \leq DHLPFWA\left(d^+, d^+, \ldots, d^+\right)$ with equality if and only if d_i is same as d^+ can be obtained. So, the proof of the theorem is completed.

Definition 8 Let $d_i = (h_i, g_i)(i = 1, 2, \ldots, n)$ be a collection of DHLPFEs, and let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector, such that $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$. The dual hesitant linguistic Pythagorean fuzzy weighted geometric (DHLPFWG) operator is defined as

$$DHLPFWG(d_1, d_2, \ldots, d_n) = \otimes_{i=1}^{n} d_i^{w_i}, \tag{11}$$

Theorem 5 *Let $d_i = (h_i, g_i)(i = 1, 2, \ldots, n)$ be collection of DHLPFEs, then the aggregation result by using the DHLPFWG operator is also a DHLPFE and*

$$DHLPFWG(d_1, d_2, \ldots, d_n) = \bigcup_{\sigma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ s_{l \prod_{i=1}^{n} (\eta_i/l)^{w_i}} \right\}, \left\{ s_{l\left(1 - \prod_{i=1}^{n}\left(1 - \sigma_i^2/l^2\right)^{w_i}\right)^{1/2}} \right\} \right\}, \tag{12}$$

The proof of Theorem 5 is similar to that of Theorem 2, which is omitted here. In addition, DHLPFWG operator has the following properties and the proofs are similar to those of Theorems 3 and 4.

Theorem 6 (Monotonicity) *Let $d_i = (h_i, g_i)$ and $d_i^* = \left(h_i^*, g_i^*\right)(i = 1, 2, \ldots, n)$ be two collections of DHLPFEs, where $s_{\sigma_i} \in h_i$, $s_{\eta_i} \in g_i$ and $s_{\sigma_i^*} \in h_i^*$, $s_{\eta_i^*} \in g_i^*$. For $\forall i = 1, 2, \ldots, n$, if $s_{\sigma_i} \leq s_{\sigma_i^*}$ and $s_{\eta_i} \geq s_{\eta_i^*}$, then*

$$DHLPFWG(d_1, d_2, \ldots, d_n) \leq DHLPFWG\left(d_1^*, d_2^*, \ldots, d_n^*\right). \tag{13}$$

Theorem 7 (Boundedness) *Let $d_i = (h_i, g_i)(i = 1, 2, \ldots, n)$ be a collection of DHLPFEs. For each $s_{\sigma_i} \in h_i$, $s_{\eta_i} \in g_i$ $(i = 1, 2, \ldots, n)$, let $d^- = \left(s_{\min\{\sigma_i\}}, s_{\max\{\eta_i\}}\right)$, $d^+ = \left(s_{\max\{\sigma_i\}|}, s_{\min\{\eta_i\}}\right)$. Then*

$$d^- \leq DHLPFWG(d_1, d_2, \ldots, d_n) \leq d^+. \tag{14}$$

3.6 A MAGDM Method Based on DHLPFSs

In this section, we study DHLPFSs and their AOs in MAGDM problems and propose a new MAGDM method. We further provide a real decision-making example to illustrate the effectiveness of the new method.

3.6.1 Description of a Typical MAGDM Problem Under DHLPFSs

A typical MAGDM problem under DHLPFSs can be described as follows: Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of candidates and $G = \{G_1, G_2, \ldots, G_n\}$ be set of attributes. The weight vector of attributes is $w = (w_1, w_2, \ldots, w_n)^T$, such that $\sum_{j=1}^{n} w_j = 1$ and $0 \leq w_j \leq 1$. A group of DEs $D = \{D_1, D_2, \ldots, D_t\}$ is invited to assess the performance of all the alternatives. The weight vector of DEs is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)^T$, such that $0 \leq \lambda_e \leq 1$ and $\sum_{e=1}^{t} \lambda_e = 1$. Let $\tilde{S} = \{s_h | h \in [0, l]\}$ be a pre-defined continuous linguistic term set. To properly evaluate the feasible alternatives, for attribute $G_j(j = 1, 2, \ldots, n)$ of $A_i(i = 1, 2, \ldots, m)$, DE $D_e(e = 1, 2, \ldots, t)$ uses a DHLPFEs $d_{ij}^e = \left(h_{ij}^e, g_{ij}^e\right)$ defined on $\tilde{S}$ to express his/her evaluation information. Finally, a series of dual hesitant linguistic Pythagorean fuzzy decision matrices are obtained. In the following, based on the proposed AOs we further present a method to solve this problem.

3.6.2 The Steps of a Novel MAGDM Method Based on DHLPFEs

Step 1. Normalize the original decision matrix. In most practical MAGDM problems, there are two types of attributes, i.e., benefit type and cost type. Hence, the original decision matrices should be normalized according to the following formula:

$$d_{ij}^e = \begin{cases} \left(h_{ij}^e, g_{ij}^e\right) & for\, benefit\, attribute \\ \left(g_{ij}^e, h_{ij}^e\right) & for\, cost\, attribute \end{cases}. \tag{15}$$

Step 2. Compute the overall decision matrix. For alternative $X_i(i = 1, 2, \ldots, m)$, use DHLPFWA operator

$$d_{ij} = DHLPFWA\left(d_{ij}^1, d_{ij}^2, \ldots, d_{ij}^t\right), \tag{16}$$

or the DHLPWG operator

$$d_{ij} = DHLPFWG\left(d_{ij}^1, d_{ij}^2, \ldots, d_{ij}^t\right), \tag{17}$$

to determine the comprehensive evaluation matrix.

Step 3. Compute the final overall evaluation values of alternatives. For alternative $X_i(i = 1, 2, \ldots, m)$, use DHLPFWA operator

$$d_i = DHLPFWA(d_{i1}, d_{i2}, \ldots, d_{in}), \tag{18}$$

or the DHLPWG operator

$$d_i = DHLPFWG(d_{i1}, d_{i2}, \ldots, d_{in}), \tag{19}$$

to compute its comprehensive evaluation value.

Step 4. Calculate the score value $S(d_i)$ and accuracy value $H(d_i)$ of d_i.

Step 5. Rank all the alternatives according to the score and accuracy values.

3.6.3 An Illustrative Example

Example 5 In order to improve the accommodation conditions of students, a university plans to install air conditioners in student dormitories. After primary evaluation, there are four suppliers to be select (A_1, A_2, A_3, and A_4). In order to choose the optimal air conditioners supplier, the university arranges an expert group composed of students and teachers to evaluate all candidate alternatives. All the five possible candidates are evaluated under four attributes, namely reputation (G_1), competitive power (G_2), quality of products (G_3), and price advantage (G_4). The weight vector of attributes is $w = (0.3, 0.1, 0.2, 0.4)^T$. We assume there are three DEs (D_1, D_2, and D_3) whose weight is $\delta = (0.243, 0.514, 0.243)^T$. Let $S = \{s_0 =$ extremely poor, $s_1 =$ very poor, $s_2 =$ poor, $s_3 =$ slightly poor, $s_4 =$ fair, $s_5 =$ slightly good, $s_6 =$ good, $s_7 =$ very good, $s_8 =$ extremely good$\}$ be a linguistic term set, and DEs use DHLPFEs to express their evaluation values. The original decision matrices are listed in Tables 2, 3, 4. In the following, we use the proposed MAGDM method to determine the most suitable air conditioners supplier.

Table 2 The original decision matrix provided by D_1 in Example 5

	G_1	G_2	G_3	G_4
A_1	$\{\{s_6, s_7\}, \{s_2, s_8\}\}$	$\{\{s_7\}, \{s_2, s_4\}\}$	$\{\{s_5\}, \{s_2\}\}$	$\{\{s_6, s_7\}, \{s_3\}\}$
A_2	$\{\{s_5\}, \{s_4\}\}$	$\{\{s_1\}, \{s_7, s_6\}\}$	$\{\{s_6, s_3\}, \{s_3\}\}$	$\{\{s_3\}, \{s_3\}\}$
A_3	$\{\{s_5, s_8\}, \{s_2\}\}$	$\{\{s_5\}, \{s_2\}\}$	$\{\{s_1\}, \{s_7\}\}$	$\{\{s_2, s_4\}, \{s_6\}\}$
A_4	$\{\{s_3\}, \{s_8\}\}$	$\{\{s_2, s_3\}, \{s_6\}\}$	$\{\{s_1\}, \{s_6\}\}$	$\{\{s_2\}, \{s_6\}\}$

Table 3 The original decision matrix provided by D_2 in Example 5

	G_1	G_2	G_3	G_4
A_1	$\{\{s_2, s_3\}, \{s_2\}\}$	$\{\{s_5, s_7\}, \{s_4\}\}$	$\{\{s_3\}, \{s_2, s_4\}\}$	$\{\{s_6\}, \{s_2, s_4\}\}$
A_2	$\{\{s_1, s_2\}, \{s_4\}\}$	$\{\{s_1, s_2\}, \{s_6\}\}$	$\{\{s_3\}, \{s_1, s_6\}\}$	$\{\{s_5\}, \{s_3\}\}$
A_3	$\{\{s_8\}, \{s_2, s_4\}\}$	$\{\{s_5\}, \{s_2\}\}$	$\{\{s_1\}, \{s_5\}\}$	$\{\{s_4\}, \{s_3, s_6\}\}$
A_4	$\{\{s_2, s_3\}, \{s_5, s_8\}\}$	$\{\{s_2\}, \{s_5\}\}$	$\{\{s_4, s_5\}, \{s_1\}\}$	$\{\{s_2, s_5\}, \{s_8\}\}$

Table 4 The original decision matrix provided by D_3 in Example 5

	G_1	G_2	G_3	G_4
A_1	$\{\{s_3\}, \{s_6\}\}$	$\{\{s_2, s_7\}, \{s_4\}\}$	$\{\{s_3\}, \{s_2, s_5\}\}$	$\{\{s_6\}, \{s_3, s_4\}\}$
A_2	$\{\{s_1, s_5\}, \{s_3, s_4\}\}$	$\{\{s_1, s_3\}, \{s_6\}\}$	$\{\{s_3\}, \{s_3\}\}$	$\{\{s_5, s_7\}, \{s_1\}\}$
A_3	$\{\{s_8\}, \{s_2\}\}$	$\{\{s_5, s_6\}, \{s_1\}\}$	$\{\{s_4\}, \{s_6\}\}$	$\{\{s_4\}, \{s_4, s_6\}\}$
A_4	$\{\{s_3, s_5\}, \{s_8\}\}$	$\{\{s_2\}, \{s_4\}\}$	$\{\{s_1\}, \{s_6\}\}$	$\{\{s_1\}, \{s_8\}\}$

Step 1. It is easy to find out that all attributes are benefit types and hence the original decision matrices do not need to be normalized.

Step 2. Use the DHLPFWA operator to compute the comprehensive decision matrix, which is shown in Table 5.

Step 3. Use the DHLPFWA to compute the comprehensive evaluation values of alternatives. As the final overall evaluation values of all the feasible alternatives are too complicated, we omit them here.

Table 5 The comprehensive decision matrix of Example 5 using the DHLPFWA operator

	G_1	G_2
A_1	$\{\{s_{4.8878}, s_{4.6964}, s_{4.1619}, s_{3.8970}\}, \{s_{2.6120}, s_{3.6582}\}\}$	$\{\{s_{7.0}, s_{6.5568}, s_{6.2839}, s_{5.4469}\}, \{s_{3.3799}, s_{4.0}\}\}$
A_2	$\{\{s_{3.7547}, s_{2.8160}, s_{3.9158}, s_{3.0527}\}, \{s_{4.0}, s_{3.7299}\}\}$	$\{\{s_{1.0}, s_{1.7450}, s_{1.60}, s_{2.1342}\}, \{s_{5.9591}, s_{5.74}\}\}$
A_3	$\{\{s_{8.0}\}, \{s_{2.0}\}\}$	$\{\{s_{5.0}, s_{5.5291}\}, \{s_{1.69}\}\}$
A_4	$\{\{s_{2.5464}, s_{3.3307}, s_{3.0}, s_{0.36617}\}, \{s_{8.0}, s_{6.2831}\}\}$	$\{\{s_{2.2924}, s_{2.0}\}, \{s_{4.9506}\}\}$
	G_3	G_4
A_1	$\{\{s_{3.6617}\}, \{s_{2.0}, s_{2.4988}, s_{2.8560}, s_{3.5683}\}\}$	$\{\{s_{6.3198}, s_{6.0}\}, \{s_{2.4356}, s_{2.6120}, s_{3.4781}, s_{3.7299}\}\}$
A_2	$\{\{s_{4.1619}, s_{3.0}\}, \{s_{1.7056}, s_{4.284}\}\}$	$\{\{s_{3.6478}, s_{5.5125}\}, \{s_{2.2971}\}\}$
A_3	$\{\{s_{2.2424}\}, \{s_{5.6718}\}\}$	$\{\{s_{3.6504}, s_{4.0}\}, \{s_{5.437}, s_{6.0}, s_{3.8074}, s_{4.2017}\}\}$
A_4	$\{\{s_{3.0361}, s_{3.8423}\}, \{s_{2.3888}\}\}$	$\{\{s_{1.8123}, s_{3.9171}\}, \{s_{7.4598}\}\}$

Table 6 The comprehensive decision matrix of Example 5 using the DHLPFWG operator

	G_1	G_2
A_1	$\{\{s_{3.6859}, s_{2.9925}, s_{3.5504}, s_{2.8824}\}, \{s_{3.7608}, s_{8.0}\}\}$	$\{\{s_{7.0}, s_{5.1628}, s_{5.8883}, s_{4.3429}\}, \{s_{3.6504}, s_{4.0}\}\}$
A_2	$\{\{s_{2.1862}, s_{1.4786}, s_{3.1220}, s_{2.1114}\}, \{s_{4.0}, s_{3.7928}\}\}$	$\{\{s_{1.0}, s_{1.3060}, s_{1.4280}, s_{1.8650}\}, \{s_{1.6582}, s_{5.8011}\}\}$
A_3	$\{\{s_{7.1368}\}, \{s_{2.0}\}\}$	$\{\{s_{5.0}, s_{5.2265}\}, \{s_{1.8123}\}\}$
A_4	$\{\{s_{2.4356}, s_{2.7575}, s_{3.0}, s_{3.3965}\}, \{s_{8.0}, s_{6.2831}\}\}$	$\{\{s_{2.2071}, s_{2.0}\}, \{s_{5.1142}\}\}$
	G_3	G_4
A_1	$\{\{s_{3.3965}\}, \{s_{2.0}, s_{3.1565}, s_{3.2406}, s_{3.9773}\}\}$	$\{\{s_{6.2290}, s_{6.0}\}, \{s_{2.5464}, s_{2.8906}, s_{3.5561}, s_{3.7928}\}\}$
A_2	$\{\{s_{3.5504}, s_{3.0}\}, \{s_{2.2414}, s_{5.0126}\}\}$	$\{\{s_{4.4163}, s_{4.7926}\}, \{s_{2.6748}\}\}$
A_3	$\{\{s_{1.4006}\}, \{s_{5.9559}\}\}$	$\{\{s_{3.3799}, s_{4.0}\}, \{s_{5.6641}, s_{6.0}, s_{4.3406}, s_{4.9381}\}\}$
A_4	$\{\{s_{2.0392}, s_{2.2870}\}, \{s_{4.6390}\}\}$	$\{\{s_{1.6900}, s_{2.7066}\}, \{s_{8.0}\}\}$

Step 4. Compute score values of alternatives according to Definition 6 and we can get the following results:

$$\tau(d_1) = 6.4920,\ \tau(d_2) = 6.0067,\ \tau(d_3) = 7.6187,\ \tau(d_4) = 4.5884$$

Step 5. Based on the score values presented in the afore step, we get the ranking order of alternatives, i.e., $A_3 \succ A_1 \succ A_2 \succ A_4$, and A_3 is the the best alternative.

In step 2, if the DHLPFWG operator is used to compute the comprehensive decision matrix, then we can get the following results (see Table 6).

Then, we continue to use the DHLPFWG operator to compute the overall values and calculate the score values according to Definition 6, we have

$$\tau(d_1) = 4.7904,\ \tau(d_2) = 5.3967,\ \tau(d_3) = 5.3515,\ \tau(d_4) = 1.6702$$

Therefore, the ranking order of alternatives is $A_2 \succ A_3 \succ A_1 \succ A_4$, and A_2 is best alternative.

3.6.4 Further Discussion

This section proposes a new MAGDM method wherein DHLPFSs are used to property DEs' evaluation values. The main advantages of our developed decision-making method are two-fold. First, it allows attribute values or DEs' evaluation values to be denoted by linguistic terms, which provides DEs a flexible and reliable manner to express their assessments. In actual decision-making situations, DEs usually would

Table 7 Characteristics of different MAGDM methods

	Whether it depicts DEs' qualitative evaluation	Whether it depicts DEs' qualitative evaluation	Whether it permits multiple MDs and NMDs	The degree of flexibility it provides for DEs
Gag's [31] method based on LPFSs	Yes	Yes	No	Medium
Yu et al.'s [62] method based on DHFSs	Yes	No	Yes	Medium
Our proposed method	Yes	Yes	Yes	High

like to use linguistic terms numbers to evaluate the performance of possible alternatives. Hence, our MAGDM method provides DEs, scientists, and practitioners a practical approach to make reasonable decisions. Second, our method permits the attribute values by several possible linguistic terms, which effectively handle DEs' hesitancy. Therefore, our method is more practical and powerful than some existing decision-making methods. First, it is more useful than that proposed by Garg [31] based on LPFSs. Garg's [31] MAGDM method only allows the MD and NMD of attribute values to be denoted by single linguistic terms, which overlooks DEs' high hesitancy in complicated MAGDM situations. In addition, our method can solve decision-making problems in which attribute values are in the form of LPFSs. However, the decision-making method introduced by Garg [31] is unable to handle MAGDM problems in DHLPFSs. Additionally, our proposed method is also more powerful than that put forward by Yu et al.'s [62] based on DHFSs. Similar to DHLPFSs, DHFSs can also effectively deal with DEs' high hesitancy in alternatives' performance evaluation process. However, in DHFSs the possible MDs and NMDs are represented by crisp numbers while in DHLPFSs all MDs and NMDs are denoted by linguistic terms. In other words, DHFSs can only describe DEs' quantitative evaluation values, while DHLPFSs depict both DEs' quantitative and qualitative evaluation information. Hence, our method is also better than Yu et al.'s [62] MAGDM method. We provide Table 7 to better illustrate the advantages of our proposed method.

4 Probabilistic Dual Hesitant Linguistic Pythagorean Fuzzy Sets and Their Applications

In this section, we introduce another new concept, called PDHLPFSs, for depicting DEs' evaluation information. We first introduce the motivations of proposing PDHLPFSs. Then, the definition of PDHLPFSs, as well as some other notions, such as operational rules, comparison method, and AOs are studied. Based on these notions, a new MAGDM method is proposed and its actual performance in realistic decision-making problems is illustrated through numerical examples.

4.1 Motivations of Proposing PDHLPFSs

As discussed above, DHLPFSs permit multiple linguistic MDs and NMDs, which can more effectively describe DEs' evaluation values. However, as realistic MAGDM problems and very complicated, there are quite a few situations that cannot be handled by DHLPFSs. In DHLPFSs, all possible values provided by DEs have importance, which is not consistent with real decision-making situations. Actually, each member in DHLPFEs has a different degree of importance. We provide the following examples to better explain the drawbacks of DHLPFSs.

Example 6 A professor is invited to evaluate the innovation of a student's thesis (which can be denoted as A for convenience). Let $S = \{s_0 = \text{very poor}, s_1 = \text{poor}, s_2 = \text{slightly poor}, s_3 = \text{fair}, s_4 = \text{slightly good}, s_5 = \text{good}, s_6 = \text{slightly good}\}$ be a pre-defined linguistic term set. The professor may express that he/she is 30% sure that the MD should be s_4, and 70% sure that the MD should be s_5. In addition, he/she is 40% sure that the NMD should be s_0, 40% and 20% sure that the NMD should be s_2 and s_3, respectively. Then the evaluation value of the expert can be expressed as

$$\text{Innovation } (A) = \{\{s_4(0.3), s_5(0.7)\}, \{s_0(0.4), s_2(0.4), s_3(0.4)\}\}$$

Example 7 There are a hundred teachers and students and they are required to express how they feel about the satisfied degree of the design scheme of new campus (which can be denoted as B convenience). Let $S = \{s_0 = \text{very poor}, s_1 = \text{poor}, s_2 = \text{slightly poor}, s_3 = \text{fair}, s_4 = \text{slightly good}, s_5 = \text{good}, s_6 = \text{slightly good}\}$ be a pre-defined linguistic term set. For the MDs, 28 of them state it should be s_3, 34 of them argue it should be s_4, and the other 38insist it should be s_5. For the NMDs, 57 of them state it should be s_1 and the other 42 of them insist it should be s_3. Then the overall evaluation value can be expressed as

$$\text{Satisfied degree}(B) = \{\{s_3(0.28), s_4(0.34), s_5(0.38)\}, \{s_1(0.57), s_3(0.43)\}\}$$

The above examples reveal that in order to more accurately capture DEs' evaluation values, not only multiple MDs and NMDs but also their corresponding probabilistic information should be taken into account. As a matter of fact, some scholars have noticed this phenomenon and some effective information description tools have been proposed, such as probabilistic linguistic sets, probabilistic hesitant fuzzy sets, and probabilistic dual hesitant fuzzy sets. Motivated by these fuzzy set theories, we genialize DHLPFSs into PDHLPFSs.

4.2 Definition of PDHLPFSs

Definition 9 Let X be a fixed set and $\tilde{S} = \{s_\alpha | 0 \leq \alpha \leq l\}$ be continuous linguistic term set with odd cardinality. A probabilistic dual hesitant linguistic Pythagorean fuzzy set (PDHLPFSs) E is expressed as

$$E = \{\langle x, h_E(x)|p(x), g_E(x)|t(x)\rangle | x \in X\}. \tag{20}$$

The component $h_E(x)|p(x)$ and $g_E(x)|t(x)$ are two sets of some possible values, where $h_E(x), g_E(x) \subseteq \tilde{S}$, denoting the possible linguistic MDs and NMDs of the element $x \in X$ to the set E, respectively, such that $\sigma^q + \eta^q \leq l^q (q \geq 1)$, where $s_\sigma \in h_E(x)$, and $s_\eta \in g_E(x)$ for $x \in X$. $p(x)$ and $t(x)$ are corresponding probabilistic information of $h_E(x)$ and $g_E(x)$, respectively, such that $0 \leq p_i \leq 1$, $0 \leq t_j \leq 1$, $\sum_{i=1}^{\#h} p_i = 1$, and $\sum_{j=1}^{\#g} t_j = 1$. For convenience, we call the ordered paper $e(x) = (h_E(x)|p(x), g_E(x)|t(x))$ a probabilistic dual hesitant linguistic Pythagorean fuzzy element (PDHLPFE), which can be denoted as $e = (h|p, g|t)$ for simplicity.

From Definition 9, it is seen that PDHLPFS is a generalized form of DHLPFS and DHLPFS is a special case of PDHLPFS, where the importance degrees of all members are equal. In the framework of PDHLPFSs, the overall evaluation value of DEs' in Example 1 can be express as $d = \{\{s_1 | 0.125, s_2 | 0.250, s_3 | 0.125, s_4 | 0.375, s_5 | 0.125\}, \{s_0 | 0.250, s_1 | 0.125, s_2 | 0.250, s_3 | 0.375\}\}$, which is a PDHLPFE.

4.3 Operation of PDHLPFEs

Definition 10 Let $e_1 = \left(h_1 \middle| p_{h_1}, g_1 \middle| t_{g_1}\right)$, $e_2 = \left(h_2 \middle| p_{h_2}, g_2 \middle| t_{g_2}\right)$, and $e = (h|p, g|t)$ be any three PDHLPFEs and λ be a positive real number,

(5) $e_1 \oplus e_2 = \bigcup_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{\left\{s_{(\sigma_1^2+\sigma_2^2-\sigma_1^2\sigma_2^2/l^2)^{1/2}} | p_{\sigma_1} p_{\sigma_2}\right\}, \left\{s_{(\eta_1\eta_2/l)} | t_{\eta_1} t_{\eta_2}\right\}\right\}$;

(6) $e_1 \otimes e_2 = \bigcup_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{\left\{s_{(\sigma_1\sigma_2/l)} | p_{\sigma_1} p_{\sigma_2}\right\}, \left\{s_{(\eta_1^2+\eta_2^2-\eta_1^2\eta_2^2/l^2)^{1/2}} | t_{\eta_1} t_{\eta_2}\right\}\right\}$;

(7) $\lambda e = \bigcup_{\sigma \in h, \eta \in g} \left\{\left\{s_{l\left(1-\left(1-\sigma^2/l^2\right)^\lambda\right)^{1/2}} | p_\sigma\right\}, \left\{s_{l(\eta/l)^\lambda} | t_\eta\right\}\right\}$;

(8) $e^\lambda = \bigcup_{\sigma \in h, \eta \in g} \left\{\left\{s_{l(\sigma/l)^\lambda} | p_\sigma\right\}, \left\{s_{l\left(1-\left(1-\eta^2/l^2\right)^\lambda\right)^{1/2}} | t_\eta\right\}\right\}$.

Example 8 Let $e_1 = \{\{s_3|0.5, s_4|0.2, s_5|0.3\}, \{s_2|0.6, s_3|0.4\}\}$ and $e_2 = \{\{s_2|0.3, s_4|0.7\}, \{s_4|1\}\}$ be two PDHLPFEs defined on a pre-given continuous linguistic set $\tilde{S} = \{s_\alpha | 0 \leq \alpha \leq 6\}$, then

$$e_1 \oplus e_2 = \left\{\begin{array}{c} \{s_{3.4641}|0.1500, s_{4.5826}|0.3500, s_{4.2687}|0.0600, s_{4.9889}|0.1400, s_{5.1208}|0.0900, s_{5.4671}|0.2100\}, \\ \{s_{1.3333}|0.6000, s_{2.0000}|0.4000\} \end{array}\right\};$$

$$e_1 \otimes e_2 = \left\{ \begin{array}{c} \{s_{1.0000}|0.1500, s_{1.333}|0.0600, s_{1.6667}|0.0900, s_{2.0000}|0.3500, s_{2.6667}|0.1400, s_{3.3333}|0.2100\}, \\ \{s_{4.2687}|0.6000, s_{4.5826}|0.4000\} \end{array} \right\};$$

$$3e_1 = \{\{s_{4.5621}|0.5000, s_{5.4614}|0.2000, s_{5.9138}|0.3000\}, \{s_{0.2222}|0.6000, s_{0.7500}|0.4000\}\};$$

$$e_2^3 = \{\{s_{0.2222}|0.3000, s_{1.7778}|0.7000\}, \{s_{5.4614}|1\}\}$$

Theorem 8 *Let* $e_1 = \left(h_1\middle|p_{h_1}, g_1\middle|t_{g_1}\right)$, $e_2 = \left(h_2\middle|p_{h_2}, g_2\middle|t_{g_2}\right)$, *and* $e = (h|p, g|t)$ *be any three PDHLPFEs, then*

(1) $e_1 \oplus e_2 = e_2 \oplus e_2$;
(2) $e_1 \otimes e_2 = e_2 \otimes e_1$;
(3) $\lambda(e_1 \oplus e_2) = \lambda e_1 \oplus \lambda e_2$;
(4) $\lambda_1 e \oplus \lambda_2 e = (\lambda_1 + \lambda_2)e, (\lambda_1, \lambda_2 \geq 0)$;
(5) $e^{\lambda_1} \otimes e^{\lambda_2} = e^{\lambda_1 + \lambda_2}, (\lambda_1, \lambda_2 \geq 0)$;
(6) $e_1^{\lambda} \otimes e_2^{\lambda} = (e_1 \otimes e_2)^{\lambda}, (\lambda \geq 0)$.

Proof It is easy to prove that (1) and (2) hold. In the following, we try to prove the correctness of the following equations. According to the operational rules for PDHLPFEs presented in Definition 10, we have

$$\lambda(e_1 \oplus e_2) = \bigcup\nolimits_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ s_{l\left(1-\left((1-\sigma_1^2/l^2)(1-\sigma_2^2/l^2)\right)^{\lambda}\right)^{1/2}} | p_{\sigma_1} p_{\sigma_2} \right\}, \left\{ s_{l(\eta_1\eta_2/l^2)^{\lambda}} | t_{\eta_1} t_{\eta_2} \right\} \right\},$$

and

$$\begin{aligned} \lambda e_1 \oplus \lambda e_2 &= \bigcup\nolimits_{\sigma_1 \in h_1, \eta_1 \in g_1} \left\{ \left\{ s_{l\left(1-(1-\sigma_1^2/l^2)^{\lambda}\right)^{1/2}} | p_{\sigma_1} \right\}, \left\{ s_{l(\eta_1/l)^{\lambda}} | t_{\eta_1} \right\} \right\} \\ &\oplus \bigcup\nolimits_{\sigma_2 \in h_2, \eta_2 \in g_2} \left\{ \left\{ s_{l\left(1-(1-\sigma_2^2/l^2)^{\lambda}\right)^{1/2}} | p_{\sigma_2} \right\}, \left\{ s_{l(\eta_2/l)^{\lambda}} | t_{\eta_2} \right\} \right\} \\ &= \bigcup\nolimits_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ s_{l\left(1-\left((1-\sigma_1^2/l^2)(1-\sigma_2^2/l^2)\right)^{\lambda}\right)^{1/2}} | p_{\sigma_1} p_{\sigma_2} \right\}, \left\{ s_{l(\eta_1\eta_2/l^2)^{\lambda}} | t_{\eta_1} t_{\eta_2} \right\} \right\} \\ &= \lambda(e_1 \oplus e_2). \end{aligned}$$

which proves that (3) holds.

Meanwhile, we can obtain that

$$\begin{aligned} &\lambda_1 e \oplus \lambda_2 e \\ &= \bigcup\nolimits_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l\left(1-\left(1-\sigma^2/l^2\right)^{\lambda_1}\right)^{1/2}} | p_{\sigma} \right\}, \left\{ s_{l(\eta/l)^{\lambda_1}} | t_{\eta} \right\} \right\} \\ &\oplus \bigcup\nolimits_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l\left(1-\left(1-\sigma^2/l^2\right)^{\lambda_2}\right)^{1/2}} | p_{\sigma} \right\}, \left\{ s_{l(\eta/l)^{\lambda_2}} | t_{\eta} \right\} \right\} \end{aligned}$$

$$= \bigcup_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l\left(1-\left(1-\sigma^2/l^2\right)^{\lambda_2+\lambda_1}\right)^{1/2}} | p_\sigma \right\}, \left\{ s_{\left(l(\eta/l)^{\lambda_1+\lambda_2}\right)} | t_\eta \right\} \right\},$$

and

$$(\lambda_1 + \lambda_2)e = \bigcup_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l\left(1-(1-\sigma^2/l^2)^{\lambda_1+\lambda_2}\right)^{1/2}} | p_\sigma \right\}, \left\{ s_{l(\eta/l)^{\lambda_1+\lambda_2}} | t_\eta \right\} \right\} = \lambda_1 e \oplus \lambda_2 e,$$

which illustrates the validity of (4).

Moreover,

$$\begin{aligned} e^{\lambda_1} \otimes e^{\lambda_2} &= \bigcup_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l(\sigma/l)^{\lambda_1}} | p_\sigma \right\}, \left\{ s_{l\left(1-\left(1-\eta^2/l^2\right)^{\lambda_1}\right)^{1/2}} | t_\eta \right\} \right\} \\ &\quad \otimes \bigcup_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l(\sigma/l)^{\lambda_2}} | p_\sigma \right\}, \left\{ s_{l\left(1-\left(1-\eta^2/l^2\right)^{\lambda_2}\right)^{1/2}} | t_\eta \right\} \right\}, \\ &= \bigcup_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l(\sigma/l)^{\lambda_1+\lambda_2}} | p_\sigma \right\}, \left\{ s_{l\left(1-\left(1-\eta^2/l^2\right)^{\lambda_1+\lambda_2}\right)^{1/2}} | t_\eta \right\} \right\}, \end{aligned}$$

and

$$e^{\lambda_1+\lambda_2} = \bigcup_{\sigma \in h, \eta \in g} \left\{ \left\{ s_{l(\sigma/l)^{\lambda_1+\lambda_2}} | p_\sigma \right\}, \left\{ s_{l\left(1-\left(1-\eta^2/l^2\right)^{\lambda_1+\lambda_2}\right)^{1/2}} | t_\eta \right\} \right\} = e^{\lambda_1} \otimes e^{\lambda_2}.$$

Hence, (5) holds.

Finally,

$$\begin{aligned} e_1^\lambda \otimes e_2^\lambda &= \bigcup_{\sigma_1 \in h_1, \eta_1 \in g_1} \left\{ \left\{ s_{l(\sigma_1/l)^\lambda} | p_{\sigma_1} \right\}, \left\{ s_{l\left(1-\left(1-\eta_1^2/l^2\right)^\lambda\right)^{1/2}} | t_{\eta_1} \right\} \right\} \\ &\quad \otimes \bigcup_{\sigma_2 \in h_2, \eta_2 \in g_2} \left\{ \left\{ s_{l(\sigma_2/l)^\lambda} | p_{\sigma_2} \right\}, \left\{ s_{l\left(1-\left(1-\eta_2^2/l^2\right)^\lambda\right)^{1/2}} | t_{\eta_2} \right\} \right\}, \\ &= \bigcup_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ s_{l\left(\sigma_1\sigma_2/l^2\right)^\lambda} | p_{\sigma_1} p_{\sigma_2} \right\}, \left\{ s_{l\left(1-\left(\left(1-\eta_1^2/l^2\right)\left(1-\eta_2^2/l^2\right)\right)^\lambda\right)^{1/2}} | t_{\eta_1} t_{\eta_2} \right\} \right\}, \end{aligned}$$

and

$$\begin{aligned} (e_1 \otimes e_2)^\lambda &= \left(\bigcup_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ s_{\sigma_1\sigma_2/l} | p_{\sigma_1} p_{\sigma_2} \right\}, \left\{ s_{\left(\eta_1^2+\eta_2^2-\eta_1^2\eta_2^2/l^2\right)^{1/2}} | t_{\eta_1} t_{\eta_2} \right\} \right\} \right)^\lambda \\ &= \bigcup_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \left\{ \left\{ s_{l\left(\sigma_1\sigma_2/l^2\right)^\lambda} | p_{\sigma_1} p_{\sigma_2} \right\}, \left\{ s_{l\left(1-\left(\left(1-\eta_1^2/l^2\right)\left(1-\eta_2^2/l^2\right)\right)^\lambda\right)^{1/2}} | t_{\eta_1} t_{\eta_2} \right\} \right\} \\ &= e_1^\lambda \otimes e_2^\lambda, \end{aligned}$$

which demonstrates the correctness of (6).

4.4 Comparison Method of PDHLPFEs

Definition 11 Let $e = (h|p, g|t)$ be a PDHLPFEs, the score function $\Gamma(e)$ of e is expressed as

$$\Gamma(e) = s_{\sqrt{\left(l^2 + \sum\limits_{i=1,\sigma\in h}^{\#h} \sigma_i^2 p_{\sigma_i} - \sum\limits_{j=1,\eta\in g}^{\#g} \eta_j^2 t_{\eta_j}\right)/2}}, \tag{21}$$

and the accuracy function $\Omega(e)$ is defined as

$$\Omega(e) = s_{\sqrt{\left(\sum\limits_{i=1,\sigma\in h}^{\#h} \sigma_i^2 p_{\sigma_i} + \sum\limits_{j=1,\eta\in g}^{\#g} \eta_j^2 t_{\eta_j}\right)/2}}, \tag{22}$$

where $\#h$ and $\#g$ denote the numbers of elements in h and g. For any two PDHLPFEs e_1 and e_2, then

(1) If $\Gamma(e_1) > \Gamma(e_2)$, then $e_1 > e_2$;
(2) If $\Gamma(e_1) = \Gamma(e_2)$, then

If $\Omega(e_1) > \Omega(e_2)$, then $e_1 > e_2$;
If $\Omega(e_1) = \Omega(e_2)$, then $e_1 = e_2$.

Example 9 Let $\tilde{S} = \{s_\alpha | 0 \le \alpha \le 6\}$ be a pre-defined continuous linguistic term set, and $e_1 = \{\{s_0|0.2, s_2|0.3, s_3|0.5\}, \{s_4|0.4, s_5|0.6\}\}$ and $e_2 = \{\{s_1|0.5, s_3|0.5\}, \{s_2|0.2, s_3|0.7, s_5|0.1\}\}$ be two PDHLPFEs defined on $\tilde{S}$, then we have

$$\Gamma(e_1) = s_{\sqrt{(6^2+(0^2*0.2^2+2^2*0.3^2+3^2*0.5^2)/3-(4^2*0.4^2+5^2*0.6^2)/2)/2}} = s_{3.9427},$$

$$\Omega(e_1) = s_{\sqrt{((0^2*0.2^2+2^2*0.3^2+3^2*0.5^2)/3+(4^2*0.4^2+5^2*0.6^2)/2)/2}} = s_{1.8235},$$

$$\Gamma(e_2) = s_{\sqrt{(6^2+(1^2*0.5^2+3^2*0.5^2)/2-(2^2*0.2^2+3^2*0.7^2+5^2*0.1^2)/3)/2}} = s_{4.2216},$$

$$\Omega(e_2) = s_{\sqrt{((1^2*0.5^2+3^2*0.5^2)/2+(2^2*0.2^2+3^2*0.7^2+5^2*0.1^2)/3)/2}} = s_{1.1951}.$$

According to Definition 11, we can get $e_2 > e_1$.

4.5 Aggregation Operators of PDHLPFEs

Definition 12 Let $e_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)(i = 1, 2, \ldots, n)$ be a collection of PDHLPFEs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector, such that $0 \le w_i \le 1$ and $\sum_{i=1}^{n} w_i = 1$. The probabilistic dual hesitant linguistic Pythagorean fuzzy weighted average (PDHLPFWA) operator is expressed as

$$PDHLPFWA(e_1, e_2, \ldots, e_n) = \oplus_{i=1}^{n} w_i e_i, \tag{23}$$

Theorem 9 *Let $e_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)(i = 1, 2, \ldots, n)$ be collection of PDHLPFEs, then the aggregation result by using the PDHLPFWA operator is also a PDHLPFEs and*

$$\begin{aligned} &PDHLPFWA(e_1, e_2, \ldots, e_n) \\ &= \bigcup\nolimits_{\sigma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ s_{l\left(1-\prod_{i=1}^{n}(1-\sigma_i^2/l^2)^{w_i}\right)^{1/2}} \middle| \prod_{i=1}^{n} p_{\sigma_i} \right\}, \left\{ s_{l\prod_{i=1}^{n}(\eta_i/l)^{w_i}} \middle| \prod_{i=1}^{n} t_{\eta_i} \right\} \right\}, \end{aligned} \tag{24}$$

Proof We first prove that (24) holds for $n = 2$. Since

$$w_1 e_1 = \bigcup\nolimits_{\sigma_1 \in h_1, \eta_1 \in g_1} \left\{ \left\{ s_{l(1-(1-\sigma_1^2/l^2)^{w_1})^{1/2}} | p_{\sigma_1} \right\}, \left\{ s_{l(\eta_1/l)^{w_1}} | t_{\eta_1} \right\} \right\},$$

and

$$w_2 e_2 = \bigcup\nolimits_{\sigma_2 \in h_2, \eta_2 \in g_2} \left\{ \left\{ s_{l(1-(1-\sigma_2^2/l^2)^{w_2})^{1/2}} | p_{\sigma_2} \right\}, \left\{ s_{l(\eta_2/l)^{w_2}} | t_{\eta_2} \right\} \right\}.$$

Then

$$\begin{aligned} PDHLPFWA(e_1, e_2) &= w_1 e_1 \oplus w_2 e_2 \\ &= \bigcup\nolimits_{\sigma_1 \in h_1, \sigma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \\ &\left\{ \left\{ s_{l\left(1-\left(1-\sigma_1^2/l^2\right)^{w_1}\left(1-\sigma_2^2/l^2\right)^{w_2}\right)^{1/2}} | p_{\sigma_1} p_{\sigma_2} \right\}, \left\{ s_{\left(l(\eta_1/l)^{w_1}(\eta_2/l)^{w_2}\right)} | t_{\eta_1} t_{\eta_2} \right\} \right\}. \end{aligned}$$

which demonstrates that Eq. (24) holds for $n = 2$.

If Eq. (24) holds for $n = k$, i.e.,

$$\begin{aligned} &PDHLPFWA(e_1, e_2, \ldots, e_k) \\ &= \bigcup\nolimits_{\sigma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ s_{l\left(1-\prod_{i=1}^{k}(1-\sigma_i^2/l^2)^{w_i}\right)^{1/2}} \middle| \prod_{i=1}^{k} p_{\sigma_i} \right\}, \left\{ s_{l\prod_{i=1}^{k}(\eta_i/l)^{w_i}} \middle| \prod_{i=1}^{k} t_{\eta_i} \right\} \right\}, \end{aligned}$$

then when $n = k + 1$, we can obtain

$$\begin{aligned}
PDHLPFWA(e_1, e_2, \ldots, e_{k+1}) &= \oplus_{i=1}^{k} w_i e_i \oplus w_{k+1} e_{k+1} \\
&= \bigcup_{\sigma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ s_{l\left(1-\prod_{i=1}^{k}(1-\sigma_i^2/l^2)^{w_i}\right)^{1/2}} | \prod_{i=1}^{k} p_{\sigma_i} \right\}, \left\{ s_{l\prod_{i=1}^{k}(\eta_i/l)^{w_i}} | \prod_{i=1}^{k} t_{\eta_i} \right\} \right\} \\
&\oplus \bigcup_{\sigma_{k+1} \in h_{k+1}, \eta_{k+1} \in g_{k+1}} \left\{ \left\{ s_{l\left(1-(1-\sigma_{k+1}^2/l^2)^{\lambda}\right)^{1/2}} | p_{\sigma_{k+1}} \right\}, \left\{ s_{l(\eta_{k+1}^2/l)^{\lambda}} | t_{\eta_{k+1}} \right\} \right\} \\
&= \bigcup_{\sigma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ s_{l\left(1-\prod_{i=1}^{k+1}(1-\sigma_i^2/l^2)^{w_i}\right)^{1/2}} | \prod_{i=1}^{k+1} p_{\sigma_i} \right\}, \left\{ s_{l\prod_{i=1}^{k+1}(\eta_i/l)^{w_i}} | \prod_{i=1}^{k+1} t_{\eta_i} \right\} \right\},
\end{aligned}$$

i.e., Eq. (24) holds for $n = k + 1$. Therefore, Eq. (24) holds for all n. The proof of Theorem 9 is completed.

Theorem 10 (Monotonicity) *Let* $e_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)$ *and* $e_i^* = \left(h_i^* \middle| p_{h_i^*}, g_i^* \middle| t_{g_i^*}\right)(i = 1, 2, \ldots, n)$ *be two collections of PDHLPFEs, where* $s_{\sigma_i} \in h_i$, $s_{\eta_i} \in g_i$ *and* $s_{\sigma_i^*} \in h_i^*$, $s_{\eta_i^*} \in g_i^*$. *For* $\forall i = 1, 2, \ldots, n$, *if* $s_{\sigma_i} \leq s_{\sigma_i^*}$ *and* $s_{\eta_i} \geq s_{\eta_i^*}$, *while the probabilities are the same, i.e.,* $p_{\sigma_i} = p_{\sigma_i^*}, t_{\eta_i} = t_{\eta_i^*}$, *then*

$$PDHLPFWA(e_1, e_2, \ldots, e_n) \leq PDHLPFWA\left(e_1^*, e_2^*, \ldots, e_n^*\right). \tag{25}$$

Proof For any i, there are $s_{\sigma_i} \leq s_{\sigma_i^*}$ and $s_{\eta_i} \geq s_{\eta_i^*}$. For the terms in the aggregated results, we have

$$s_{l\left(1-\prod_{i=1}^{n}(1-\sigma_i^2/l^2)^{w_i}\right)^{1/2}} \leq s_{l\left(1-\prod_{i=1}^{n}(1-\sigma_i^{*2}/l^2)^{w_i}\right)^{1/2}} \text{ and } s_{l\prod_{i=1}^{n}(\eta_i/l)^{w_i}} \geq s_{l\prod_{i=1}^{n}(\eta_i^*/l)^{w_i}}.$$

According to the score function in Definition 11, we can get $PDHLPFWA(e_1, e_2, \ldots, e_n) \leq PDHLPFWA\left(e_1^*, e_2^*, \ldots, e_n^*\right)$ with equality if and only if $s_{\sigma_i} = s_{\sigma_i^*}$ and $s_{\eta_i} = s_{\eta_i^*}$ for all i.

Theorem 11 (Boundedness) *Let* $e_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)(i = 1, 2, \ldots, n)$ *be a collection of PDHLPFEs. For each* $s_{\sigma_i} \in h_i$, $s_{\eta_i} \in g_i (i = 1, 2, \ldots, n)$, *let* $e^- = \left(s_{\min\{\sigma_i\}} \middle| p_{\min\{\sigma_i\}}, s_{\max\{\eta_i\}} \middle| t_{\max\{\eta_i\}}\right)$, $e^+ = \left(s_{\max\{\sigma_i\}} \middle| p_{\max\{\sigma_i\}}, s_{\min\{\eta_i\}} \middle| t_{\min\{\eta_i\}}\right)$. *Then*

$$PDHLPFWA\left(e^-, e^-, \ldots, e^-\right) \leq PDHLPFWA(e_1, e_2, \ldots, e_n) \leq PDHLPFWA\left(e^+, e^+, \ldots, e^+\right). \tag{26}$$

Proof For $\forall i = 1, 2, \ldots, n$, we have $s_{\min\{\sigma_i\}} \leq s_{\sigma_i} \leq s_{\max\{\sigma_i\}}$, $s_{\min\{\eta_i\}} \leq s_{\eta_i} \leq s_{\max\{\eta_i\}}$, $p_{\min\{\sigma_i\}} \leq p_{\sigma_i} \leq p_{\max\{\sigma_i\}}$, $t_{\min\{\eta_i\}} \leq t_{\eta_i} \leq t_{\max\{\eta_i\}}$. Then

$$s_{l\left(1-\prod_{i=1}^{n}(1-\sigma_i^2/l^2)^{w_i}\right)^{\frac{1}{2}}} \geq s_{l\left(1-\prod_{i=1}^{n}(1-(\min\{\sigma_i\}/l)^2)^{w_i}\right)^{\frac{1}{2}}} = s_{l\left(1-(1-(\min\{\sigma_i\}/l)^2)^{\sum_{i=1}^{n} w_i}\right)^{\frac{1}{2}}} = s_{\min\{\sigma_i\}},$$

$$s_{l\prod_{i=1}^{n}(\eta_i/l)^{w_i}} \leq s_{l\prod_{i=1}^{n}(\max\{\eta_i\}/l)^{w_i}} = s_{\max\{\eta_i\}},$$

$$s_{l\left(1-\prod_{i=1}^{n}\left(1-\sigma_i^2/l^2\right)^{w_i}\right)^{\frac{1}{2}}} \leq s_{l\left(1-\prod_{i=1}^{n}\left(1-(\max\{\sigma_i\}/l)^2\right)^{w_i}\right)^{\frac{1}{2}}} = s_{\max\{\sigma_i\}},$$

$$s_{l\prod_{i=1}^{n}(\eta_i/l)^{w_i}} \geq s_{l\prod_{i=1}^{n}(\min\{\eta_i\}/l)^{w_i}} = s_{\min\{\eta_i\}}.$$

For the probabilities:

$$\prod_{i=1}^{n} p_{\sigma_i} \geq \prod_{i=1}^{n} p_{\min\{\sigma_i\}},\ \prod_{i=1}^{n} t_{\sigma_i} \geq \prod_{i=1}^{n} t_{\min\{\sigma_i\}}$$

and

$$\prod_{i=1}^{n} p_{\sigma_i} \leq \prod_{i=1}^{n} p_{\max\{\sigma_i\}},\ \prod_{i=1}^{n} t_{\sigma_i} \leq \prod_{i=1}^{n} t_{\max\{\sigma_i\}}.$$

And,

$$\begin{aligned}
&PDHLPFWA\left(e^-, e^-, \ldots, e^-\right) \\
&= \bigcup\left\{\left\{s_{l\left(1-\prod_{i=1}^{n}\left(1-(\min\{\sigma_i\}/l)^2\right)^{w_i}\right)^{1/2}} \Big| \prod_{i=1}^{n} p_{\min\{\sigma_i\}}\right\}, \left\{s_{l\prod_{i=1}^{n}(\max\{\eta_i\}/l)^{w_i}} \Big| \prod_{i=1}^{n} t_{\max\{\eta_i\}}\right\}\right\} \\
&= \left\{s_{\min\{\sigma_i\}}\left|\prod_{i=1}^{n} p_{\min\{\sigma_i\}}, s_{\max\{\eta_i\}}\right|\prod_{i=1}^{n} t_{\max\{\eta_i\}}\right\}
\end{aligned}$$

$$\begin{aligned}
&PDHLPFWA\left(e^+, e^+, \ldots, e^+\right) \\
&= \bigcup\left\{\left\{s_{l\left(1-\prod_{i=1}^{n}\left(1-(\max\{\sigma_i\}/l)^2\right)^{w_i}\right)^{1/2}} \Big| \prod_{i=1}^{n} p_{\min\{\sigma_i\}}\right\}, \left\{s_{l\prod_{i=1}^{n}(\min\{\eta_i\}/l)^{w_i}} \Big| \prod_{i=1}^{n} t_{\max\{\eta_i\}}\right\}\right\} \\
&= \left\{s_{\max\{\sigma_i\}}\left|\prod_{i=1}^{n} p_{\min\{\sigma_i\}}, s_{\min\{\eta_i\}}\right|\prod_{i=1}^{n} t_{\max\{\eta_i\}}\right\}
\end{aligned}$$

According to the score function in Definition 11, we have $PDHLPFWA(e_1, e_2, \ldots, e_n) \geq PDHLPFWA\left(e^-, e^-, \ldots, e^-\right)$ with equality if and only if e_i is same as e^-. Similarly, $PDHLPFWA(e_1, e_2, \ldots, e_n) \leq PDHLPFWA\left(e^+, e^+, \ldots, e^+\right)$ with equality if and only if e_i is same as e^+ can be obtained. So, the proof of the theorem is completed.

Definition 13 Let $e_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)(i = 1, 2, \ldots, n)$ be a collection of PDHLPFEs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector, such that $0 \le w_i \le 1$ and $\sum_{i=1}^{n} w_i = 1$. The probabilistic dual hesitant linguistic Pythagorean fuzzy weighted geometric (PDHLPFWG) operator is expressed as

$$PDHLPFWG(e_1, e_2, \ldots, e_n) = \otimes_{i=1}^{n} e_i^{w_i}, \tag{27}$$

Theorem 12 *Let $e_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)(i = 1, 2, \ldots, n)$ be collection of PDHLPFEs, then the aggregation result by using the PDHLPFWG operator is also a PDHLPFEs and*

$$\begin{aligned} & PDHLPFWG(e_1, e_2, \ldots, e_n) \\ & = \bigcup\nolimits_{\sigma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ s_{l \prod_{i=1}^{n} (\eta_i / l)^{w_i}} \middle| \prod_{i=1}^{n} p_{\sigma_i} \right\}, \left\{ s_{l \left(1 - \prod_{i=1}^{n} \left(1 - \sigma_i^2 / l^2\right)^{w_i}\right)^{1/2}} \middle| \prod_{i=1}^{n} t_{\eta_i} \right\} \right\}, \end{aligned} \tag{28}$$

The proof of Theorem 12 is similar to that of Theorem, which is omitted here. In addition, it is easy to prove that PDHLPFWG operator has the following properties.

Theorem 13 (Monotonicity) *Let $e_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)$ and $e_i^* = \left(h_i^* \middle| p_{h_i^*}, g_i^* \middle| t_{g_i^*}\right)(i = 1, 2, \ldots, n)$ be two collections of PDHLPFEs, where $s_{\sigma_i} \in h_i$, $s_{\eta_i} \in g_i$ and $s_{\sigma_i^*} \in h_i^*$, $s_{\eta_i^*} \in g_i^*$. For $\forall i = 1, 2, \ldots, n$, if $s_{\sigma_i} \le s_{\sigma_i^*}$ and $s_{\eta_i} \ge s_{\eta_i^*}$, while the probabilities are the same, i.e., $p_{\sigma_i} = p_{\sigma_i^*}, t_{\eta_i} = t_{\eta_i^*}$, then*

$$PDHLPFWG(e_1, e_2, \ldots, e_n) \le PDHLPFWG\left(e_1^*, e_2^*, \ldots, e_n^*\right). \tag{29}$$

Theorem 14 (Boundedness) *Let $e_i = \left(h_i \middle| p_{h_i}, g_i \middle| t_{g_i}\right)(i = 1, 2, \ldots, n)$ be a collection of PDHLPFEs. For each $s_{\sigma_i} \in h_i$, $s_{\eta_i} \in g_i (i = 1, 2, \ldots, n)$, let $e^- = \left(s_{\min\{\sigma_i\}} \middle| p_{\min\{\sigma_i\}}, s_{\max\{\eta_i\}} \middle| t_{\max\{\eta_i\}}\right)$, $e^+ = \left(s_{\max\{\sigma_i\}} \middle| p_{\max\{\sigma_i\}}, s_{\min\{\eta_i\}} \middle| t_{\min\{\eta_i\}}\right)$. Then,*

$$e^- \le PDHLPFWG(e_1, e_2, \ldots, e_n) \le e^+. \tag{30}$$

4.6 MAGDM Based on PDHLPFEs

In this section, we introduce a new MAGDM method under PDHLPFSs based on the proposed AOs. Further, a numerical example is presented to show the effectiveness of our proposed method.

4.6.1 The Main Steps of a MAGDM Approach Under PDHLPFSs

A representative probabilistic dual hesitant linguistic Pythagorean fuzzy MAGDM problem is described as follows: Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of candidates and $G = \{G_1, G_2, \ldots, G_n\}$ be set of attributes. The weight vector of attributes is $w = (w_1, w_2, \ldots, w_n)^T$, such that $\sum_{j=1}^{n} w_j = 1$ and $0 \le w_j \le 1$. Several DEs $D = \{D_1, D_2, \ldots, D_t\}$ are invited to form a group to evaluate the efficiency of all the feasible alternatives. The weight vector of DEs is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)^T$, such that $0 \le \lambda_e \le 1$ and $\sum_{e=1}^{t} \lambda_e = 1$. Let $\tilde{S} = \{s_h | h \in [0, l]\}$ be a predefined continuous linguistic term set. For attribute $G_j (j = 1, 2, \ldots, n)$ of alternative $A_i (i = 1, 2, \ldots, m)$, the DE $D_k (k = 1, 2, \ldots, t)$ uses a PDHLPFE $e_{ij}^k = \left(h_{ij}^k \middle| p_{h_{ij}^k}, g_{ij}^k \middle| t_{g_{ij}^k}\right)$ to express his/her evaluation. Finally, a set of probabilistic dual hesitant linguistic Pythagorean fuzzy decision matrices are gotten. In the following, we use the proposed AOs to introduce a novel MAGDM method.

4.6.2 The Steps of a Novel MAGDM Method Based on PDHLPFEs

Step 1. Normalize the original decision matrix. In most practical MAGDM problems, there are two types of attributes, i.e., benefit type and cost type. Hence, the original decision matrices should be normalized according to the following formula:

$$e_{ij}^k = \begin{cases} \left(h_{ij}^k \middle| p_{h_{ij}^k}, g_{ij}^k \middle| t_{g_{ij}^k}\right) & for\, benefit\, attribute \\ \left(g_{ij}^k \middle| t_{g_{ij}^k}, h_{ij}^k \middle| p_{h_{ij}^k}\right) & for\, cost\, attribute \end{cases}. \tag{31}$$

Step 2. Compute the overall decision matrix. For alternative $A_i (i = 1, 2, \ldots, m)$, use the PDHLPFWA operator

$$e_{ij} = PDHLPFWA\left(e_{ij}^1, e_{ij}^2, \ldots, e_{ij}^f\right), \tag{32}$$

or the PDHLPFWG operator

$$e_{ij} = PDHLPFWG\left(e_{ij}^1, e_{ij}^2, \ldots, e_{ij}^f\right), \tag{33}$$

to determine the comprehensive evaluation matrix.

Step 3. Compute the final overall evaluation values of alternatives. For alternative $A_i (i = 1, 2, \ldots, m)$, use the PDHLPFWA operator

$$e_i = PDHLPFWA(e_{i1}, e_{i2}, \ldots, e_{in}), \tag{34}$$

or the PDHLPWG operator

$$e_i = PDHLPFWG(e_{i1}, e_{i2}, \ldots, e_{in}), \tag{35}$$

to compute its comprehensive evaluation value.

Step 4. Calculate the score value $S(e_i)$ and accuracy value $H(e_i)$ of e_i.

Step 5. Rank all the alternatives according to the score and accuracy values.

4.6.3 A Real Application of the Proposed Method

Example 10 In order to stimulate the enthusiasm for research of doctoral students, a college plans to evaluate the quality of doctoral students' theses and selects the best one and gives the author scholarship. After primary selection, there are five theses authored by five students, and three professors are required to evaluate the five theses. The weight vector of the three professors is $\lambda = (0.243, 0.514, 0.243)^T$. The five theses are assessed under four attributes, i.e., the significant degree of the study (G1), the degree of innovation (G2), the degree of compliance with academic norms (G3), and the significant degree methodology (G4). The weight vector of attributes is $w = (0.3, 0.1, 0.2, 0.4)^T$. Let $S = \{s_0 =$ extremely poor, $s_1 =$ very poor, $s_2 =$ poor, $s_3 =$ slightly poor, $s_4 =$ fair, $s_5 =$ slightly good, $s_6 =$ good, $s_7 =$ very good, s_8 $=$ extremely good$\}$ be a linguistic term set, and DEs use PDHLPFEs to express their evaluation values. The original decision matrices are presented in Tables 8, 9, 10.

Step 1. As the original decision matrices are benefit types, the original decision matrices do not need to be normalized.

Step 2. Use the PDHLPFWA operator to determine the comprehensive decision matrix, and the result is listed in Table 11.

Step 3. Use the PDHLPFWA operator to calculate the overall evaluation values of alternatives. As the comprehensive evaluations are too complicated, we omit them here.

Step 4. Calculate the score values of alternatives, we can obtain

$$\Gamma(e_1) = s_{5.6466},\ \Gamma(e_2) = s_{5.6735},\ \Gamma(e_3) = s_{5.5948},\ \Gamma(e_4) = s_{5.6687}$$

Step 5. Rank the alternatives and we can get $A_2 \succ A_4 \succ A_1 \succ A_3$, which implies A_2 is the optimal alternative.

If we calculate the comprehensive decision matrix and overall evaluation values by the PDHLPFWG operator, then the score values of alternatives are $\Gamma(e_1) = s_{5.6266}$, $\Gamma(e_2) = s_{5.6583}$, $\Gamma(e_3) = s_{5.5646}$, and $\Gamma(e_4) = s_{5.6453}$, and the ranking order is $A_2 \succ A_4 \succ A_1 \succ A_3$, which indicates A_2 is the best alternative.

Table 8 The original decision matrix provided by D_1 in Example 10

	G_1	G_2	G_3	G_4
A_1	$\{\{s_7\|0.3, s_6\|0.3, s_5\|0.4\}, \{s_2\|1\}\}$	$\{\{s_7\|1\}, \{s_2\|1\}\}$	$\{\{s_2\|1\}, \{s_2\|1\}\}$	$\{\{s_7\|0.5, s_6\|0.5\}, \{s_3\|1\}\}$
A_2	$\{\{s_1\|1\}, \{s_4\|1\}\}$	$\{\{s_3\|1\}, \{s_7\|1\}\}$	$\{\{s_7\|1\}, \{s_3\|0.5, s_2\|0.5\}\}$	$\{\{s_3\|1\}, \{s_3\|1\}\}$
A_3	$\{\{s_6\|1\}, \{s_3\|1\}\}$	$\{\{s_5\|1\}, \{s_2\|1\}\}$	$\{\{s_1\|1\}, \{s_7\|1\}\}$	$\{\{s_4\|0.4, s_2\|0.6\}, \{s_4\|1\}\}$
A_4	$\{\{s_5\|0.7, s_2\|0.3\}, \{s_5\|1\}\}$	$\{\{s_3\|0.5, s_2\|0.5\}, \{s_6\|1\}\}$	$\{\{s_8\|1\}, \{s_1\|1\}\}$	$\{\{s_2\|1\}, \{s_6\|1\}\}$

Table 9 The original decision matrix provided by D_2 in Example 10

	G_1	G_2	G_3	G_4
A_1	$\{\{s_7 \mid 1\}, \{s_2 \mid 0.1, s_5 0.9\}\}$	$\{\{s_3 \mid 1\}, \{s_2 \mid 0.4, s_6 0.6\}\}$	$\{\{s_3 \mid 1\}, \{s_5 \mid 1\}\}$	$\{\{s_8 \mid 1\}, \{s_5 \mid 0.3, s_7 \mid 0.7\}\}$
A_2	$\{\{s_8 \mid 1\}, \{s_7 \mid 0.3, s_8 \mid 0.7\}\}$	$\{\{s_6 \mid 1\}, \{s_2 \mid 0.5, s_6 0.5\}\}$	$\{\{s_6 \mid 0.7, s_4 \mid 0.3\}, \{s_4 \mid 1\}\}$	$\{\{s_4 \mid 0.4, s_6 0.6\}, \{s_8 \mid 0.1, s_6 \mid 0.9\}\}$
A_3	$\{\{s_7 \mid 0.6, s_1 \mid 0.4\}, \{s_3 \mid 1\}\}$	$\{\{s_2 \mid 0.5, s_6 \mid 0.3, s_5 \mid 0.2\}, \{s_1 \mid 0.2, s_6 0.8\}\}$	$\{\{s_2 \mid 1\}, \{s_5 \mid 1\}\}$	$\{\{s_7 \mid 1\}, \{s_5 \mid 0.4, s_6 \mid 0.6\}\}$
A_4	$\{\{s_5 \mid 0.6, s_3 \mid 0.4\}, \{s_6 \mid 0.4, s_2 \mid 0.6\}\}$	$\{\{s_3 \mid 1\}, \{s_1 \mid 0.3, s_8 0.7\}\}$	$\{\{s_7 \mid 1\}, \{s_3 \mid 1\}\}$	$\{\{s_5 \mid 1\}, \{s_5 \mid 1\}\}$

Table 10 The original decision matrix provided by D_3 in Example 10

	G_1	G_2	G_3	G_4
A_1	$\{\{s_4\|1\}, \{s_3\|1\}\}$	$\{\{s_3\|1\}, \{s_4\|1\}\}$	$\{\{s_5\|0.3, s_6\|0.1, s_3\|0.6\}, \{s_4\|1\}\}$	$\{\{s_3\|1\}, \{s_4\|1\}\}$
A_2	$\{\{s_3\|0.4, s_6\|0.6\}, \{s_5\|1\}\}$	$\{\{s_2\|1\}, \{s_4\|0.4, s_7\|0.6\}\}$	$\{\{s_5\|1\}, \{s_6\|1\}\}$	$\{\{s_3\|0.2, s_6\|0.8\}, \{s_6\|1\}\}$
A_3	$\{\{s_7\|1\}, \{s_2\|1\}\}$	$\{\{s_1\|1\}, \{s_5\|1\}\}$	$\{\{s_1\|1\}, \{s_7\|1\}\}$	$\{\{s_2\|0.6, s_4\|0.4\}, \{s_4\|1\}\}$
A_4	$\{\{s_6\|1\}, \{s_5\|1\}\}$	$\{\{s_2\|0.3, s_1\|0.7\}, \{s_3\|0.3, s_7\|0.7\}\}$	$\{\{s_5\|1\}, \{s_3\|0.3, s_5\|0.7\}\}$	$\{\{s_4\|1\}, \{s_1\|1\}\}$

Table 11 The comprehensive evaluation matrix of Example 10 by using the PDHLPFWA operator

	G_1	G_2
A_1	$\{\{s_{6.6408}\|0.3, s_{6.3907}\|0.3, s_{6.2370}\|0.4\}, \{s_{2.2071}\|0.1, s_{3.5348}\|0.9\}\}$	$\{\{s_{4.8878}\|1\}, \{s_{2.3669}\|0.4, s_{4.1631}6\}\}$
A_2	$\{\{s_{8}\|1\}, \{s_{5.6303}\|0.3, s_{6.0303}\|0.7\}\}$	$\{\{s_{8}\|1\}, \{s_{5.6303}\|0.3, s_{6.0303}\|0.7\}\}$
A_3	$\{\{s_{6.8223}\|0.6, s_{5.2439}\|0.4\}, \{s_{2.7185}\|1\}\}$	$\{\{s_{4.9291}\|1\}, \{s_{3.2091}\|0.2, s_{3.6766}\|0.3, s_{5.6445}\|0.2, s_{6.4668}\|0.3\}\}$
A$_4$	$\{\{s_{5.2931}\|0.42, s_{4.5894}\|0.28, s_{4.9882}\|0.18, s_{4.1619}\|0.12\}, \{s_{5.4912}\|0.4, s_{3.1220}\|0.6\}\}$	$\{\{s_{2.7973}\|0.15, s_{2.6748}\|0.35, s_{2.5738}\|0.15, s_{2.4371}\|0.35\}, \{s_{2.0185}\|0.09, s_{2.48}\|0.21, s_{5.8779}\|0.21, s_{7.2217}\|0.49\}\}$
	G_3	G_4
A_1	$\{\{s_{3.5109}\|0.3, s_{4.0403}\|0.1, s_{2.7797}\|0.6\}, \{s_{3.7907}\|1\}\}$	$\{\{s_{8}\|1\}, \{s_{4.1832}\|0.3, s_{4.9730}\|0.7\}\}$
A_2	$\{\{s_{6.1582}\|0.7, s_{5.4405}\|0.3\}, \{s_{4.1661}\|0.5, s_{3.7299}0.5\}\}$	$\{\{s_{3.5661}\|0.08, s_{4.5251}\|0.32, s_{5.0126}\|0.12, s_{5.5685}\|0.48\}, \{s_{5.8779}\|0.1, s_{5.0669}\|0.9\}\}$
A_3	$\{\{s_{1.6}\|1\}, \{s_{5.8883}\|1\}\}$	$\{\{s_{5.8802}\|0.36, s_{6.0108}\|0.48, s_{6.1319}\|0.16\}, \{s_{4.4861}\|0.4, s_{4.9269}\|0.6\}\}$
A_4	$\{\{s_{8}\|1\}, \{s_{2.2971}\|0.3, s_{2.6007}\|0.7\}\}$	$\{\{s_{4.2959}\|1\}, \{s_{3.5348}\|1\}\}$

4.6.4 Further Discussion

This section proposes a novel MAGDM method wherein DEs' evaluation information or attribute values are expressed by PDHLPFSs. To sum up, the main advantages of our proposed decision-making method have three aspects. First, it employs linguistic terms to denote the possible MDs and NMDs, which makes it easy to denote DEs' assessment information both quantificationally and qualitatively. Second, it allows the existence of multiple MDs and NMDs, which is more capable of describing DEs' high hesitancy in realistic complex decision-making problems. Third, it can also describe probabilistic information of each linguistic term, making it smoother to describe a decision-making group's overall evaluation information. These three merits make our decision-making method more flexible and powerful. In addition, our MAGDM method is also more powerful than some existing decision-making approaches. First, compared with aforementioned decision-making method based on DHLPFSs, the newly developed MAGDM method under PDHLPFSs can more accurately depict DEs' evaluation information, as it not only considers multiple MDs and NMDs but also take the probabilistic values of MDs and NMDs into consideration. Second, it is also more powerful than the method introduced by Hao et al. [55]. It is noted that the method proposed by Hao et al. [55] is based on PDHFSs, which use crisp numbers to denote the possible MDs and NMDs. Hence, Hao et al.'s [55] method only consider DEs' quantitative evaluation values. In our proposed PDHLPFSs, the possible MDs and NMDs are denoted by linguistic terms, and hence our method can describe DEs' evaluation values both quantificationally and qualitatively. We provide Table 12 to better demonstrate the advantages of our developed MAGDM method.

Table 12 Characteristics of some MAGDM methods

	Whether it considers the probabilistic information of MDs and NMDs	Whether it considers DEs' quantitative and qualitative evaluation values simultaneously	Its degree of flexibility when dealing with practical MAGDM problems
The aforementioned decision-making method based on DHLPFSs	No	Yes	Medium
Hao et al.'s [55] decision-making method based on PDHFSs	Yes	No	Medium
Our developed method based on PDHLPFSs	Yes	Yes	High

5 Conclusion Remarks

The LPFSs can effectively describe DEs' evaluation values in complicated decision-making situations. However, the main drawbacks of LPFSs are they overlook multiple MDs and NMDs as well as their corresponding probabilistic information. The aim is to overcome the two shortcomings by probing extensions of LPFSs. We first proposed the DHLPFSs, which have the ability of effectively dealing with multiple MDs and NMDs. After it, we proposed AOs for DHLPFSs and applied in MAGDM method. we continued to consider the fact that members in DHLPFSs may have different frequencies, occurrences, and degrees of importance, we further generalize DHLPFSs to PDHLPFSs by considering not multiple MDs and NMDs but also their probabilities. We further showed how to use to PDHLPFSs to solve practical MAGDM problems. Numerical examples have demonstrated the effectiveness of our proposed novel MAGDM methods. In future research, we shall continue our study from two aspects. First, we are studying applications of our methods in more actual-life MAGDM methods. Second, we will study more extensions of LPFSs, such as cubic LPFSs, hesitant LPFSs, uncertain LPFSs, etc., to accommodate more complex decision-making environments and propose novel MAGDM methods to aid practitioners to make wise decisions.

Acknowledgments This work was supported by Funds for First-class Discipline Construction (XK1802-5).

Conflict of Interests We declare that there is no conflict of interest with regard to the publication of this chapter.

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Pythagorean Fuzzy Soft Sets-Based MADM

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1 Introduction

The fuzzy morphological methodology delivers promising yields in quite a lot of areas, whose narrative is pretty qualitative. The inspiration for the use of words or sentences in preference to numbers is that philological descriptions or cataloging are frequently fewer absolute than arithmetical or algebraic ones. Problems that are equipped with unreliable conditions commonly occur in taking decisions, nonetheless are challenging due to perplexing condition of modeling and handling that arises with such uncertainties. To tackle multifarious and complicated problems in day-to-day life situations, the modus operandi customarily utilized as discussed in literature of classical mathematics is not of assistance each time because of the presence of uncertainties and indistinctness. There are abundant procedures that can be believed as mathematical models for coping with imprecision, inexactness, and uncertainties. Inauspiciously, all these simulations are fitted out with technical hitches and complications. To get control on these sorts of insufficiencies, Zadeh [64] brought together the notion of fuzzy sets (FSs). An FS is a substantial mathematical model for stamping an assembling of articles having unintelligible boundary. Atanassov [3–5] moved one step ahead by proposing intuitionistic fuzzy sets (IFSs). Atanassov [6] presented geometrical version of the components of intuitionistic fuzzy objects. Yager [60], by altering the condition on parameters, unveiled Pythagorean fuzzy subsets. Yager and Abbasov [61] studied Pythagorean membership grades. Later, Yager [62] employed these grades in decision-making. Molodtsov [41] patented the perception of a novel sort of model for sorting out uncertainties, traditionally acknowledged as soft sets (SSs).

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H. Garg (ed.), *Pythagorean Fuzzy Sets*,
https://doi.org/10.1007/978-981-16-1989-2_16

FSs, SSs, and their further expansions are resilient mathematical models for solving many real-world problems. The researchers have coined various mathematical models to deal with real-world problems. Çağman et al. [7] explored fuzzy soft sets with applications. Feng et al. [15] presented an adjustable approach to fuzzy soft set-based decision-making. Majumdar and Samanta [39] presented generalized fuzzy soft sets. Feng et al. [16] promote the study of SSs pooled with FSs and rough sets. Davvaz and Sadrabadi [11] presented usage of IFSs in medicine. Maji et al. [38] acquainted with the notion of intuitionistic fuzzy soft sets (IFSSs). Feng et al. [17] presented an additional outlook on generalized IFSSs and associated multi-attribute decision-making methods. Li and Cui [36] studied topological structure of IFSSs. Osmanoglu and Tokat [48] also presented IFS topology independently. Garg and Arora [20] devised a nonlinear-programming methodology for multi-attribute decision-making problem with interval-valued IFSSs information. Guleria and Bajaj [25] used matrices to represent Pythagorean fuzzy soft sets. Naz et al. [45] extended the notion of PFSs to PF graphs. Akram and Naz [2] presented energy of PF graphs with applications. Peng and Yang [49] deliberated some results for PFSs. Peng et al. [50] familiarized some PF information measures with their useful implementations. Peng et al. [51] globalized PFSs to corresponding SSs and solidified their uses. Peng and Selvachandran summed up the notions of Pythagorean fuzzy sets in [52]. Riaz and Naeem [55, 56] obtained some indispensable philosophies of SSs organized with soft σ-algebra and put on show some employments of soft mappings. Fei et al. [14] discussed Pythagorean fuzzy decision-making using soft likelihood functions. Fei and Deng [13], recently, studied multi-criteria decision-making in Pythagorean fuzzy environment.

The decision-taking techniques of TOPSIS and VIKOR have been deliberated by voluminous researchers including Hwang and Yoon [28], Adeel et al. [1], Eraslan and Karaaslan [12], Naeem et al. [44], Liu et al. [37], Kumar and Garg [33], Riaz et al. [54, 57], Li and Nan [34], Opricovic and Tzeng [46, 47], Mohd and Abdullah [40], Naeem et al. [43], Kalkan et al. [30], and Zhang and Xu [65]. Garg and Arora [18] presented generalized IFS power aggregation operator along with its practical usage. Garg and Arora [19] explored dual hesitant fuzzy soft aggregation operators with applications. Garg and Arora [23] explored TOPSIS method based on correlation coefficient for solving decision-making problems with IFSS information. Garg [22] presented, for the purpose of multiple attribute group decision analysis, novel neutrality operations based-Pythagorean fuzzy geometric aggregation operators. Li et al. [35] studied some novel Pythagorean hybrid weighted aggregation operators using Pythagorean fuzzy numbers along with their applications to decision-making. Recently, Garg [21] unveiled Pythagorean fuzzy aggregation operators based upon neutrality operations and rendered its utilizations in the process of multiple attribute group decision-making. Garg and Arora [24] presented Maclaurin symmetric mean aggregation operators based on t-norm operations for the dual hesitant fuzzy soft set.

The notion of similarity measure is indispensably significant in nearly every arena of science and technology. It is ordinarily forged for testing the validity of an object, situation, or document. Similarity measure serves as a substantial tool to decide the level of alikeness between two or more data sets. The similarity measures established by means of the notions of FSs, SSs, IFSs, and PFSs are broadly and efficiently applied in medical diagnosis, pattern recognition, signal detection, image processing, security verification systems, artificial intelligence, machine learning, etc. Similarity measures on various models are explored by Hong and Kim [26], Kharal [32], Kamaci [31], Hung and Yang [27], Hyung et al. [29], Ye [63], Chen [8, 9], Chen et al. [10], Wang et al. [58], and Muthukumar and Krishnan [42]. In recent times, Peng and Garg [53] made public multiparametric similarity measures on PFSs with applications to pattern recognition.

The goal of this chapter is to study Pythagorean fuzzy soft sets (PFSSs) and their practical implementations. We make use of different techniques including choice value method PFS-TOPSIS, VIKOR, and similarity measures for modeling uncertainties in decision-making problems. PFSSs offer a plenty of uses in decision-taking problems of daily life situations ranging from micro to high-level decisions. The chapter is organized as follows: Sect. 2 gives access to essential operations and fundamental characteristics of PFSSs. We devote Sect. 3 for an application of multi-criteria group decision-making (MCGDM) utilizing PFS matrices. In the very next section, we propose PFS-TOPSIS algorithm accompanied by its application in choosing appropriate persons for key ministries in a government. In Sect. 5, we propose PFS-VIKOR and utilize it on the selection of brand ambassadors for a multi-national company. In Sect. 6, we devise a similarity measure (SM) and weighted similarity measure for PFSSs. Based on this SM, we present an application in life sciences. In conclusion, we summarize our work in Sect. 7.

For better understanding of this unit, the reader is suggested to see [5, 7, 38, 41, 60–62, 64] for preliminary notions.

2 Structure of Pythagorean Fuzzy Soft Sets

Peng et al. [51] floated the notion of *Pythagorean fuzzy soft set* (PFSSs) and presented some of their applications. Later, Guleria and Bajaj [25] made use of matrices to represent PFSSs. The matrices used are known as *Pythagorean fuzzy soft matrices* (PFS matrices).

In this segment, we study some fundamental concepts, basic properties, and algebraic operations on PFSSs. X will represent the universe of discourse and E the aggregate of attributes with $A, A_1, A_2, A_3 \subseteq E$, in this section.

Definition 2.1 A *Pythagorean fuzzy soft set* (PFSS) on X is a family of the form

$$\begin{aligned}(\curlywedge, A) &= \Big\{\big(e, \{\partial, \sigma_{\curlywedge_A}(\partial), \varrho_{\curlywedge_A}(\partial)\}\big) : e \in A, \partial \in X\Big\}\\ &= \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\curlywedge_A}(\partial), \varrho_{\curlywedge_A}(\partial))}\right\}\right) : e \in A, \partial \in X\right\}\\ &= \left\{\left(e, \left\{\frac{(\sigma_{\curlywedge_A}(\partial), \varrho_{\curlywedge_A}(\partial))}{\partial}\right\}\right) : e \in A, \partial \in X\right\},\end{aligned}$$

Table 1 Tabulatory representation of PFSS $\curlywedge_A$

$\curlywedge_A$	e_1	e_2	$\cdots$	e_n
∂_1	$(\sigma_{11}, \varrho_{11})$	$(\sigma_{12}, \varrho_{12})$	$\cdots$	$(\sigma_{1n}, \varrho_{1n})$
∂_2	$(\sigma_{21}, \varrho_{21})$	$(\sigma_{22}, \varrho_{22})$	$\cdots$	$(\sigma_{2n}, \varrho_{2n})$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
∂_m	$(\sigma_{m1}, \varrho_{m1})$	$(\sigma_{m2}, \varrho_{m2})$	$\cdots$	$(\sigma_{mn}, \varrho_{mn})$

where $\sigma_{\curlywedge_A}$ and $\varrho_{\curlywedge_A}$ are mappings dragging members of X to [0, 1], obeying the requirement

$$0 \leq \sigma^2_{\curlywedge_A}(\partial) + \varrho^2_{\curlywedge_A}(\partial) \leq 1$$

.

If we write $\sigma_{ij} = \sigma_{\curlywedge_A}(e_j)(\partial_i)$ and $\varrho_{ij} = \varrho_{\curlywedge_A}(e_j)(\partial_i), i = 1, \ldots, m; j = 1, \ldots, n,$ then the PFSS $\curlywedge_A$ may be expressed in tabular array as in Table 1.

The matrix representing PFSS $\curlywedge_A$ is termed as *Pythagorean fuzzy soft matrix* (PFS matrix), and has form

$$\begin{aligned}\curlywedge_A &= [(\sigma_{ij}, \varrho_{ij})]_{m\times n}\\ &= \begin{pmatrix} (\sigma_{11}, \varrho_{11}) & (\sigma_{12}, \varrho_{12}) & \cdots & (\sigma_{1n}, \varrho_{1n}) \\ (\sigma_{21}, \varrho_{21}) & (\sigma_{22}, \varrho_{22}) & \cdots & (\sigma_{2n}, \varrho_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\sigma_{m1}, \varrho_{m1}) & (\sigma_{m2}, \varrho_{m2}) & \cdots & (\sigma_{mn}, \varrho_{mn}) \end{pmatrix}\end{aligned}$$

Example 2.2 Let $X = \{\partial_i : i = 1, \cdots, 4\}$ and $E = \{e_i : i = 1, 2, \ldots, 5\}$. Take $A = \{e_2, e_5\}$. Then,

$$\curlywedge_A = \left\{\left(e_2, \left\{\left(\frac{\partial_1}{(0.13, 0.91)}\right), \left(\frac{\partial_3}{(0.25, 0.62)}\right), \left(\frac{\partial_4}{(0.24, 0.89)}\right)\right\}\right), \left(e_5, \left\{\left(\frac{\partial_1}{(0.71, 0.29)}\right), \left(\frac{\partial_2}{(0.41, 0.06)}\right)\right\}\right)\right\}$$

is a PFSS over X. The tabular representation of $\curlywedge_A$ is given in Table 2.

Table 2 Tabular representation of $\curlywedge_A$

$\curlywedge_A$	e_1	e_2	e_3	e_4	e_5
∂_1	(0,1)	(0.13,0.91)	(0,1)	(0,1)	(0.71,0.29)
∂_2	(0,1)	(0,1)	(0,1)	(0,1)	(0.41,0.06)
∂_3	(0,1)	(0.25,0.62)	(0,1)	(0,1)	(0,1)
∂_4	(0,1)	(0.24,0.89)	(0,1)	(0,1)	(0,1)

The corresponding PFS matrix is

$$
\begin{aligned}
\angle_A &= [\sigma_{ij}, \varrho_{ij}]_{4\times 5} \\
&= \begin{pmatrix}
(0,1) & (0.13, 0.91) & (0,1) & (0,1) & (0.71, 0.29) \\
(0,1) & (0,1) & (0,1) & (0,1) & (0.41, 0.06) \\
(0,1) & (0.25, 0.62) & (0,1) & (0,1) & (0,1) \\
(0,1) & (0.24, 0.89) & (0,1) & (0,1) & (0,1)
\end{pmatrix}
\end{aligned}
$$

Definition 2.3 A PFSS $\angle^{(1)}_{A_1}$ is called *PFS subset* of $\angle^{(2)}_{A_2}$, i.e., $\angle^{(1)}_{A_1} \tilde{\subseteq} \angle^{(2)}_{A_2}$, if

(i) $A_1 \subseteq A_2$, and
(ii) $\angle^{(1)}(e)$ is PFS subset of $\angle^{(2)}(e)$, for all $e \in A_1$.

It is remarkable to notice that $\angle_A \tilde{\subseteq} G_B$ by no means requires that each member of $\angle^{(1)}_A$ must also present in $\angle^{(1)}_B$, contrary to classical set theory.

Definition 2.4 The *union* of two PFSSs $(\angle_1, A_1)$ and $(\angle_2, A_2)$ defined over X is given as $(\angle, A_1 \cup A_2) = (\angle_1, A_1) \tilde{\cup} (\angle_2, A_2)$, and for all $e \in A$,

$$
\angle(e) = \begin{cases}
\angle_1(e), & \text{if } e \in A_1 \;\&\; e \notin A_2 \\
\angle_2(e), & \text{if } e \in A_2 \;\&\; e \notin A_1 \\
\angle_1(e) \cup \angle_2(e), & \text{if } e \in A_1 \cap A_2,
\end{cases}
$$

where $\angle_1(e) \cup \angle_2(e)$ is the union of two PFSSs.

Definition 2.5 The *intersection* of two PFSSs $(\angle_1, A_1)$ and $(\angle_2, A_2)$ is another PFSS $(\angle, A_1 \cap A_2) = (\angle_1, A_1) \tilde{\cap} (\angle_2, A_2)$, where $\angle(e) = \angle_1(e) \cap \angle_2(e)$ for all $e \in A_1 \cap A_2$.

Definition 2.6 The *difference* of two PFSSs $(\angle_1, A_1)$ and $(\angle_2, A_2)$ over X is defined as

$$
(\angle_1, A_1) \tilde{\setminus} (\angle_2, A_2) = \left\{ \left(e, \left\{ \partial, \min\{\sigma_{\angle_1(e)}(\partial), \varrho_{\angle_2(e)}(\partial)\}, \max\{\varrho_{\angle_1(e)}(\partial), \sigma_{\angle_2(e)}(\partial)\} \right\} \right) : \partial \in X, e \in E \right\}.
$$

Definition 2.7 The *complement* of a PFSS $(\angle, A)$ is a mapping $\angle^c : A \to PF^X$ given by $\angle^c(e) = [\angle(e)]^c$, for all $e \in A$. It is represented as $(\angle, A)^c$ or sometimes by $(\angle^c, A)$. Thus, if

$$
\angle(e) = \{(\partial, \sigma_{\angle(e)}(\partial), \varrho_{\angle(e)}(\partial)) : \partial \in X\}
$$

then

$$
\angle^c(e) = \{(\partial, \varrho_{\angle(e)}(\partial), \sigma_{\angle(e)}(\partial)) : \partial \in X\}
$$

for all $e \in A$.

Definition 2.8 A PFSS defined over X is termed as *null PFSS* if it is in the form

$$\Phi = \left\{\left(e, \left\{\frac{\partial}{(0,1)}\right\}\right) : e \in E, \partial \in X\right\}.$$

Definition 2.9 A PFSS defined over X is termed as *absolute PFSS* if it is in the form

$$\breve{X} = \left\{\left(e, \left\{\frac{\partial}{(1,0)}\right\}\right) : e \in E, \partial \in X\right\}.$$

Definition 2.10 If

$$\leftthreetimes^{(1)}_{A_1} = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes^{(1)}_{A_1}}(\partial), \varrho_{\leftthreetimes^{(1)}_{A_1}}(\partial))}\right\}\right) : e \in A_1, \partial \in X\right\}$$

and

$$\leftthreetimes^{(2)}_{A_2} = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes^{(2)}_{A_2}}(\partial), \varrho_{\leftthreetimes^{(2)}_{A_2}}(\partial))}\right\}\right) : e \in A_2, \partial \in X\right\}$$

are two PFSSs, then

$$\leftthreetimes^{(1)}_{A_1} \widetilde{\oplus} \leftthreetimes^{(2)}_{A_2} = \left\{\left(e, \left\{\frac{\partial}{(\sqrt{(\sigma_{\leftthreetimes^{(1)}_{A_1}}(\partial))^2 + (\sigma_{\leftthreetimes^{(2)}_{A_2}}(\partial))^2 - (\sigma_{\leftthreetimes^{(1)}_{A_1}}(\partial)\sigma_{\leftthreetimes^{(2)}_{A_2}}(\partial))^2}, \varrho_{\leftthreetimes^{(1)}_{A_1}}(\partial)\varrho_{\leftthreetimes^{(2)}_{A_2}}(\partial))}\right\}\right) : e \in E, \partial \in X\right\}$$

and

$$\leftthreetimes^{(1)}_{A_1} \widetilde{\otimes} \leftthreetimes^{(2)}_{A_2} = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes^{(1)}_{A_1}}(\partial)\sigma_{\leftthreetimes^{(2)}_{A_2}}(\partial), \sqrt{(\varrho_{\leftthreetimes^{(1)}_{A_1}}(\partial))^2 + (\varrho_{\leftthreetimes^{(2)}_{A_2}}(\partial))^2 - (\varrho_{\leftthreetimes^{(1)}_{A_1}}(\partial)\varrho_{\leftthreetimes^{(2)}_{A_2}}(\partial))^2})}\right\}\right) : e \in E, \partial \in X\right\}$$

Definition 2.11 The *necessity operator* on the PFSS

$$\leftthreetimes_A = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes_A}(\partial), \varrho_{\leftthreetimes_A}(\partial))}\right\}\right) : e \in A, \partial \in X\right\}$$

is defined as

$$\widetilde{\Box}\leftthreetimes_A = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes_A}(\partial), \sqrt{1-\sigma^2_{\leftthreetimes_A}(\partial)})}\right\}\right) : e \in A, \partial \in X\right\}.$$

Definition 2.12 The *possibility operator* on the PFSS

$$\leftthreetimes_A = \left\{\left(e, \left\{\frac{\partial}{(\sigma_{\leftthreetimes_A}(\partial), \varrho_{\leftthreetimes_A}(\partial))}\right\}\right) : e \in A, \partial \in X\right\}$$

is defined as

$$\widetilde{\diamond}\curlywedge_A = \left\{\left(e, \left\{\frac{\partial}{(\sqrt{1-\varrho^2_{\curlywedge_A}(\partial)}, \varrho_{\curlywedge_A}(\partial))}\right\}\right) : e \in A, \partial \in X\right\}$$

Remark The modal operators presented in Definition 2.11 and 2.12 transform any PFSS to the corresponding FSS.

We elaborate the notions presented above with the help of following example.

Example 2.13 Take $X = \{\partial_1, \cdots, \partial_4\}$ and $E = \{e_1, e_2, \cdots, e_6\}$. Assume that $A_1 = \{e_2, e_4\}$, $A_2 = \{e_1, e_4, e_5\}$ and $A_3 = \{e_2, e_4, e_6\}$. Consider the PFSSs

$$\curlywedge^{(1)}_{A_1} = \begin{pmatrix} (0,1) & (0.27,0.78) & (0,1) & (0.39,0.48) & (0,1) & (0,1) \\ (0,1) & (0.11,0.04) & (0,1) & (0.73,0.54) & (0,1) & (0,1) \\ (0,1) & (0.56,0.60) & (0,1) & (0.59,0.51) & (0,1) & (0,1) \\ (0,1) & (0.62,0.62) & (0,1) & (0.37,0.56) & (0,1) & (0,1) \end{pmatrix},$$

$$\curlywedge^{(2)}_{A_2} = \begin{pmatrix} (0.56,0.27) & (0,1) & (0,1) & (0.45,0.58) & (0.33,0.78) & (0,1) \\ (0.11,0.85) & (0,1) & (0,1) & (0.09,0.28) & (0.42,0.51) & (0,1) \\ (0.76,0.49) & (0,1) & (0,1) & (0.62,0.67) & (0.92,0.21) & (0,1) \\ (0.54,0.71) & (0,1) & (0,1) & (0.54,0.82) & (0.87,0.48) & (0,1) \end{pmatrix}$$

and

$$\curlywedge^{(3)}_{A_3} = \begin{pmatrix} (0,1) & (0.31,0.54) & (0,1) & (0.39,0.01) & (0,1) & (0.22,0.87) \\ (0,1) & (0.25,0.04) & (0,1) & (0.76,0.21) & (0,1) & (0.53,0.16) \\ (0,1) & (0.49,0.32) & (0,1) & (0.62,0.37) & (0,1) & (0.42,0.19) \\ (0,1) & (0.63,0.45) & (0,1) & (0.46,0.54) & (0,1) & (0.88,0.32) \end{pmatrix}.$$

It may be observed that $\curlywedge^{(1)}_{A_1} \widetilde{\subseteq} \curlywedge^{(3)}_{A_3}$, whereas neither $\curlywedge^{(1)}_{A_1} \widetilde{\subseteq} \curlywedge^{(2)}_{A_2}$ nor $\curlywedge^{(2)}_{A_2} \widetilde{\subseteq} \curlywedge^{(3)}_{A_3}$. Moreover,

$$\curlywedge^{(1)}_{A_1} \widetilde{\cup} \curlywedge^{(2)}_{A_2} = \begin{pmatrix} (0.56,0.27) & (0.27,0.78) & (0,1) & (0.45,0.48) & (0.33,0.78) & (0,1) \\ (0.11,0.85) & (0.11,0.04) & (0,1) & (0.73,0.28) & (0.42,0.51) & (0,1) \\ (0.76,0.49) & (0.56,0.60) & (0,1) & (0.62,0.51) & (0.92,0.21) & (0,1) \\ (0.54,0.71) & (0.62,0.62) & (0,1) & (0.54,0.56) & (0.87,0.48) & (0,1) \end{pmatrix},$$

$$\curlywedge^{(1)}_{A_1} \widetilde{\cap} \curlywedge^{(2)}_{A_2} = \begin{pmatrix} (0,1) & (0,1) & (0,1) & (0.39,0.58) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.09,0.54) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.59,0.67) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.37,0.82) & (0,1) & (0,1) \end{pmatrix},$$

$$\left(\leftthreetimes_{A_1}^{(1)}\right)^c = \begin{pmatrix} (1,0) & (0.78,0.27) & (1,0) & (0.48,0.39) & (1,0) & (1,0) \\ (1,0) & (0.04,0.11) & (1,0) & (0.54,0.73) & (1,0) & (1,0) \\ (1,0) & (0.60,0.56) & (1,0) & (0.51,0.59) & (1,0) & (1,0) \\ (1,0) & (0.62,0.62) & (1,0) & (0.56,0.37) & (1,0) & (1,0) \end{pmatrix},$$

$$\leftthreetimes_{A_1}^{(1)} \widetilde{\setminus} \leftthreetimes_{A_2}^{(2)} = \begin{pmatrix} (0,1) & (0,1) & (0,1) & (0.39,0.48) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.28,0.54) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.59,0.62) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.37,0.56) & (0,1) & (0,1) \end{pmatrix},$$

$$\widetilde{\Box} \leftthreetimes_{A_1}^{(1)} = \begin{pmatrix} (0,1) & (0.27,0.96) & (0,1) & (0.39,0.92) & (0,1) & (0,1) \\ (0,1) & (0.11,0.99) & (0,1) & (0.73,0.68) & (0,1) & (0,1) \\ (0,1) & (0.56,0.83) & (0,1) & (0.59,0.81) & (0,1) & (0,1) \\ (0,1) & (0.62,0.78) & (0,1) & (0.37,0.93) & (0,1) & (0,1) \end{pmatrix},$$

$$\widetilde{\Diamond} \leftthreetimes_{A_1}^{(1)} = \begin{pmatrix} (0,1) & (0.62,0.78) & (0,1) & (0.88,0.48) & (0,1) & (0,1) \\ (0,1) & (0.99,0.04) & (0,1) & (0.84,0.54) & (0,1) & (0,1) \\ (0,1) & (0.80,0.60) & (0,1) & (0.86,0.51) & (0,1) & (0,1) \\ (0,1) & (0.78,0.62) & (0,1) & (0.83,0.56) & (0,1) & (0,1) \end{pmatrix},$$

$$\leftthreetimes_{A_1}^{(1)} \widetilde{\oplus} \leftthreetimes_{A_2}^{(2)} = \begin{pmatrix} (0.56,0.27) & (0.27,0.78) & (0,1) & (0.57,0.28) & (0.33,0.78) & (0,1) \\ (0.11,0.85) & (0.11,0.04) & (0,1) & (0.73,0.15) & (0.42,0.51) & (0,1) \\ (0.76,0.49) & (0.56,0.60) & (0,1) & (0.77,0.34) & (0.92,0.21) & (0,1) \\ (0.54,0.71) & (0.62,0.62) & (0,1) & (0.62,0.46) & (0.87,0.48) & (0,1) \end{pmatrix}$$

and

$$\leftthreetimes_{A_1}^{(1)} \widetilde{\otimes} \leftthreetimes_{A_2}^{(2)} = \begin{pmatrix} (0,1) & (0,1) & (0,1) & (0.18,0.70) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.06,0.59) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.36,0.77) & (0,1) & (0,1) \\ (0,1) & (0,1) & (0,1) & (0.20,0.88) & (0,1) & (0,1) \end{pmatrix}.$$

Proposition 2.14 *Every PFSS $\leftthreetimes_A$ may be sandwiched between Φ and $\breve{X}$, i.e., $\Phi \widetilde{\subseteq} \leftthreetimes_A \widetilde{\subseteq} \breve{X}$.*

Proposition 2.15 *If $\leftthreetimes_{A_1}^{(1)}$, $\leftthreetimes_{A_2}^{(2)}$ and $\leftthreetimes_{A_3}^{(3)}$ are three PFSSs over X, then*

(i) $\leftthreetimes_{A_1}^{(1)} \widetilde{\cap} \leftthreetimes_{A_1}^{(1)} = \leftthreetimes_{A_1}^{(1)}$.
(ii) $\leftthreetimes_{A_1}^{(1)} \widetilde{\cup} \leftthreetimes_{A_1}^{(1)} = \leftthreetimes_{A_1}^{(1)}$.
(iii) $\leftthreetimes_{A_1}^{(1)} \widetilde{\cap} \leftthreetimes_{A_2}^{(2)} = \leftthreetimes_{A_2}^{(2)} \widetilde{\cap} \leftthreetimes_{A_1}^{(1)}$.

(iv) $\leftthreetimes_{A_1}^{(1)} \widetilde{\cup} \leftthreetimes_{A_2}^{(2)} = \leftthreetimes_{A_2}^{(2)} \widetilde{\cup} \leftthreetimes_{A_1}^{(1)}$.
(v) $\leftthreetimes_{A_1}^{(1)} \widetilde{\cap} (\leftthreetimes_{A_2}^{(2)} \widetilde{\cap} \leftthreetimes_{A_3}^{(3)}) = (\leftthreetimes_{A_1}^{(1)} \widetilde{\cap} \leftthreetimes_{A_2}^{(2)}) \widetilde{\cap} \leftthreetimes_{A_3}^{(3)}$.
(vi) $\leftthreetimes_{A_1}^{(1)} \widetilde{\cup} (\leftthreetimes_{A_2}^{(2)} \widetilde{\cup} \leftthreetimes_{A_3}^{(3)}) = (\leftthreetimes_{A_1}^{(1)} \widetilde{\cup} \leftthreetimes_{A_2}^{(2)}) \widetilde{\cup} \leftthreetimes_{A_3}^{(3)}$.
(vii) $\leftthreetimes_{A_1}^{(1)} \widetilde{\cup} (\leftthreetimes_{A_2}^{(2)} \widetilde{\cap} \leftthreetimes_{A_3}^{(3)}) = (\leftthreetimes_{A_1}^{(1)} \widetilde{\cup} \leftthreetimes_{A_2}^{(2)}) \widetilde{\cap} (\leftthreetimes_{A_1}^{(1)} \widetilde{\cup} \leftthreetimes_{A_3}^{(3)})$.
(viii) $\leftthreetimes_{A_1}^{(1)} \widetilde{\cap} (\leftthreetimes_{A_2}^{(2)} \widetilde{\cup} \leftthreetimes_{A_3}^{(3)}) = (\leftthreetimes_{A_1}^{(1)} \widetilde{\cap} \leftthreetimes_{A_2}^{(2)}) \widetilde{\cup} (\leftthreetimes_{A_1}^{(1)} \widetilde{\cap} \leftthreetimes_{A_3}^{(3)})$.

Proposition 2.16 *If $\leftthreetimes_{A_1}^{(1)}$ and $\leftthreetimes_{A_2}^{(2)}$ are two PFSSs over X, then*

(i) $\leftthreetimes_{A_1}^{(1)} \widetilde{\cap} \leftthreetimes_{A_2}^{(2)} \widetilde{\subseteq} \leftthreetimes_{A_1}^{(1)} \widetilde{\subseteq} \leftthreetimes_{A_1}^{(1)} \widetilde{\cup} \leftthreetimes_{A_2}^{(2)}$
(ii) $\leftthreetimes_{A_1}^{(1)} \widetilde{\cap} \leftthreetimes_{A_2}^{(2)} \widetilde{\subseteq} \leftthreetimes_{A_2}^{(2)} \widetilde{\subseteq} \leftthreetimes_{A_1}^{(1)} \widetilde{\cup} \leftthreetimes_{A_2}^{(2)}$.

The above propositions are easy consequences of definition.

Remark Consider the PFSSs $\leftthreetimes_{A_1}^{(1)}$ and $\leftthreetimes_{A_2}^{(2)}$ given in Example 2.13. We have

$$\left(\leftthreetimes_{A_1}^{(1)} \widetilde{\cup} \leftthreetimes_{A_2}^{(2)}\right)^c = \begin{pmatrix} (0.27, 0.56) & (0.78, 0.27) & (1, 0) & (0.48, 0.45) & (0.78, 0.33) & (1, 0) \\ (0.85, 0.11) & (0.04, 0.11) & (1, 0) & (0.28, 0.73) & (0.51, 0.42) & (1, 0) \\ (0.49, 0.76) & (0.60, 0.56) & (1, 0) & (0.51, 0.62) & (0.21, 0.92) & (1, 0) \\ (0.71, 0.54) & (0.62, 0.62) & (1, 0) & (0.56, 0.54) & (0.48, 0.87) & (1, 0) \end{pmatrix} \tag{1}$$

$$\left(\leftthreetimes_{A_1}^{(1)}\right)^c = \begin{pmatrix} (1, 0) & (0.78, 0.27) & (1, 0) & (0.48, 0.39) & (1, 0) & (1, 0) \\ (1, 0) & (0.04, 0.11) & (1, 0) & (0.54, 0.73) & (1, 0) & (1, 0) \\ (1, 0) & (0.60, 0.56) & (1, 0) & (0.51, 0.59) & (1, 0) & (1, 0) \\ (1, 0) & (0.62, 0.62) & (1, 0) & (0.56, 0.37) & (1, 0) & (1, 0) \end{pmatrix},$$

$$(\leftthreetimes_{A_2}^{(2)})^c = \begin{pmatrix} (0.27, 0.56) & (1, 0) & (1, 0) & (0.58, 0.45) & (0.78, 0.33) & (1, 0) \\ (0.85, 0.11) & (1, 0) & (1, 0) & (0.28, 0.09) & (0.51, 0.42) & (1, 0) \\ (0.49, 0.76) & (1, 0) & (1, 0) & (0.67, 0.62) & (0.21, 0.92) & (1, 0) \\ (0.71, 0.54) & (1, 0) & (1, 0) & (0.82, 0.54) & (0.48, 0.87) & (1, 0) \end{pmatrix}$$

Since $A_1 \cap A_2 = \{e_4\}$, so

$$\left(\leftthreetimes_{A_1}^{(1)}\right)^c \widetilde{\cap} \left(\leftthreetimes_{A_2}^{(2)}\right)^c = \begin{pmatrix} (0, 1) & (0, 1) & (0, 1) & (0.48, 0.45) & (0, 1) & (0, 1) \\ (0, 1) & (0, 1) & (0, 1) & (0.28, 0.73) & (0, 1) & (0, 1) \\ (0, 1) & (0, 1) & (0, 1) & (0.51, 0.62) & (0, 1) & (0, 1) \\ (0, 1) & (0, 1) & (0, 1) & (0.56, 0.54) & (0, 1) & (0, 1) \end{pmatrix} \tag{2}$$

From (1) & (2), we conclude that De Morgan's laws do not make sense in PFSS theory.

Theorem 2.17 *If $(\leftthreetimes_1, A_1)$ and $(\leftthreetimes_2, A_2)$ are two PFSSs over X, then*

(a) $\Big((\leftthreetimes_1, A_1) \widetilde{\cup} (\leftthreetimes_2, A_2)\Big)^c \neq (\leftthreetimes_1, A_1)^c \widetilde{\cap} (\leftthreetimes_2, A_2)^c$, *and*

(b) $\Big((\leftthreetimes_1, A_1)\widetilde{\cap}(\leftthreetimes_2, A_2)\Big)^c \neq (\leftthreetimes_1, A_1)^c\widetilde{\cup}(\leftthreetimes_2, A_2)^c$.

Remark Consider again the PFSS $\leftthreetimes^{(1)}_{A_1}$ given in Example 2.13. We have

$$\leftthreetimes^{(1)}_{A_1} = \begin{pmatrix} (0,1) & (0.27, 0.78) & (0,1) & (0.39, 0.48) & (0,1) & (0,1) \\ (0,1) & (0.11, 0.04) & (0,1) & (0.73, 0.54) & (0,1) & (0,1) \\ (0,1) & (0.56, 0.60) & (0,1) & (0.59, 0.51) & (0,1) & (0,1) \\ (0,1) & (0.62, 0.62) & (0,1) & (0.37, 0.56) & (0,1) & (0,1) \end{pmatrix}$$

$$\therefore (\leftthreetimes^{(1)}_{A_1})^c = \begin{pmatrix} (1,0) & (0.78, 0.27) & (1,0) & (0.48, 0.39) & (1,0) & (1,0) \\ (1,0) & (0.04, 0.11) & (1,0) & (0.54, 0.73) & (1,0) & (1,0) \\ (1,0) & (0.60, 0.56) & (1,0) & (0.51, 0.59) & (1,0) & (1,0) \\ (1,0) & (0.62, 0.62) & (1,0) & (0.56, 0.37) & (1,0) & (1,0) \end{pmatrix}$$

Now,

$$\leftthreetimes^{(1)}_{A_1}\widetilde{\cup}(\leftthreetimes^{(1)}_{A_1})^c = \begin{pmatrix} (1,0) & (0.78, 0.27) & (1,0) & (0.48, 0.39) & (1,0) & (1,0) \\ (1,0) & (0.11, 0.04) & (1,0) & (0.73, 0.54) & (1,0) & (1,0) \\ (1,0) & (0.60, 0.56) & (1,0) & (0.59, 0.51) & (1,0) & (1,0) \\ (1,0) & (0.62, 0.62) & (1,0) & (0.56, 0.37) & (1,0) & (1,0) \end{pmatrix} \neq \breve{X}$$

and

$$\leftthreetimes^{(1)}_{A_1}\widetilde{\cap}(\leftthreetimes^{(1)}_{A_1})^c = \begin{pmatrix} (0,1) & (0.27, 0.78) & (0,1) & (0.39, 0.48) & (0,1) & (0,1) \\ (0,1) & (0.04, 0.11) & (0,1) & (0.54, 0.73) & (0,1) & (0,1) \\ (0,1) & (0.56, 0.60) & (0,1) & (0.51, 0.59) & (0,1) & (0,1) \\ (0,1) & (0.62, 0.62) & (0,1) & (0.37, 0.56) & (0,1) & (0,1) \end{pmatrix} \neq \Phi$$

These observations lead to the following theorem.

Theorem 2.18 *If* $(\leftthreetimes, A)$ *is any PFSS over* X*, then*

(1) $\leftthreetimes_A\widetilde{\cup}\leftthreetimes^c_A \neq \breve{X}$*, and*
(2) $\leftthreetimes_A\widetilde{\cap}\leftthreetimes^c_A \neq \Phi$.

Definition 2.19 We know that

$$\leftthreetimes^{(1)}_{A_1}\widetilde{\otimes}\leftthreetimes^{(2)}_{A_2} = \left\{\left(e, \left\{\frac{\partial}{\left(\sigma_{\leftthreetimes^{(1)}_{A_1}}(\partial)\sigma_{\leftthreetimes^{(2)}_{A_2}}(\partial), \sqrt{(\varrho_{\leftthreetimes^{(1)}_{A_1}}(\partial))^2+(\varrho_{\leftthreetimes^{(2)}_{A_2}}(\partial))^2-(\varrho_{\leftthreetimes^{(1)}_{A_1}}(\partial)\varrho_{\leftthreetimes^{(2)}_{A_2}}(\partial))^2}\right)}\right\}\right) : e \in E, \partial \in X\right\}.$$

If we substitute $\leftthreetimes^{(1)}_{A_1} = \leftthreetimes^{(2)}_{A_2} = \leftthreetimes_A$, then

$$\curlywedge_A \widetilde{\otimes} \curlywedge_A = \left\{\left(e, \left\{\frac{\partial}{(\sigma^2_{\curlywedge_A}(\partial), \sqrt{2\varrho^2_{\curlywedge_A}(\partial) - \varrho^4_{\curlywedge_A}(\partial)})}\right\}\right) : e \in E, \partial \in X\right\}.$$

That is,

$$(\curlywedge_A)^2 = \left\{\left(e, \left\{\frac{\partial}{\left(\sigma^2_{\curlywedge_A}(\partial), \sqrt{1 - (1 - \varrho^2_{\curlywedge_A})^2}\right)}\right\}\right) : e \in E, \partial \in X\right\}.$$

In general, if k is any non-negative real number, then

$$(\curlywedge_A)^k = \left\{\left(e, \left\{\frac{\partial}{\left(\sigma^k_{\curlywedge_A}(\partial), \sqrt{1 - (1 - \varrho^2_{\curlywedge_A})^k}\right)}\right\}\right) : e \in E, \partial \in X\right\}.$$

In particular, for $k = \frac{1}{2}$, we have

$$(\curlywedge_A)^{\frac{1}{2}} = \left\{\left(e, \left\{\frac{\partial}{\left(\sqrt{\sigma_{\curlywedge_A}(\partial)}, \sqrt{1 - \sqrt{1 - \varrho^2_{\curlywedge_A}}}\right)}\right\}\right) : e \in E, \partial \in X\right\}$$

$(\curlywedge_A)^2$ is called *concentration* of $\curlywedge_A$, denoted as $con(\curlywedge_A)$ whereas $(\curlywedge_A)^{\frac{1}{2}}$ is entitled as *dilation* of $\curlywedge_A$, denoted as $dil(\curlywedge_A)$.

Example 2.20 For PFSS $\curlywedge^{(1)}_{A_1} = \curlywedge_A$ given in Example 2.13, concentration and dilation are

$$con(\curlywedge_A) = \begin{pmatrix} (0,1) & (0.07,0.92) & (0,1) & (0.15,0.64) & (0,1) & (0,1) \\ (0,1) & (0.01,0.06) & (0,1) & (0.53,0.71) & (0,1) & (0,1) \\ (0,1) & (0.31,0.77) & (0,1) & (0.35,0.67) & (0,1) & (0,1) \\ (0,1) & (0.38,0.79) & (0,1) & (0.14,0.73) & (0,1) & (0,1) \end{pmatrix}$$

and

$$dil(\curlywedge_A) = \begin{pmatrix} (0,1) & (0.52,0.61) & (0,1) & (0.62,0.35) & (0,1) & (0,1) \\ (0,1) & (0.33,0.03) & (0,1) & (0.85,0.40) & (0,1) & (0,1) \\ (0,1) & (0.75,0.45) & (0,1) & (0.77,0.37) & (0,1) & (0,1) \\ (0,1) & (0.79,0.46) & (0,1) & (0.61,0.41) & (0,1) & (0,1) \end{pmatrix}$$

respectively.

We observe that in concentration of the PFSS, the value of membership function is reduced and that of non-membership function exceeds the corresponding original values. On the other hand, in case of dilation of the PFSS, the value of membership function exceeds and that of non-membership function reduces as compared to the corresponding original values.

Keeping in mind this observation, we may link phonetic terms like "*very*", "*moderate*", "*highly*", and "*not*" with the PFSS $\curlywedge_A$ by giving different non-negative real values to k. For Example,

$$
\begin{aligned}
k &= \frac{1}{2} \Rightarrow \text{“}very\text{”}\\
k &= \frac{3}{4} \Rightarrow \text{“}moderate\text{”}\\
k &= \frac{1}{5} \Rightarrow \text{“}highly\text{”}\\
k &= 4 \Rightarrow \text{“}not\text{”}
\end{aligned}
$$

For conceiving these notions effectively, consider the following example.

Example 2.21 Choose $X = \{\text{Angelica, Smith, Adina, Paul}\}$ as the class of students and $E = \{e_1, \cdots, e_5\}$ as the collection of attributes, where

$$
\begin{aligned}
e_1 &= \text{Sharp in Mathematics}\\
e_2 &= \text{Sharp in Physics}\\
e_3 &= \text{Sharp in Chemistry}\\
e_4 &= \text{Obedient}\\
e_5 &= \text{Active in physical games}
\end{aligned}
$$

Assume that the PFSS representing members of X and the value of trait e_j in the form of PFNs is

$$
\curlywedge_A = \begin{pmatrix}
(0.83, 0.28) & (0.54, 0.21) & (0.37, 0.64) & (0.59, 0.16) & (0.86, 0.11)\\
(0.31, 0.26) & (0.56, 0.57) & (0.43, 0.32) & (0.74, 0.25) & (0.13, 0.05)\\
(0.52, 0.27) & (0.64, 0.12) & (0.45, 0.57) & (0.61, 0.35) & (0.29, 0.51)\\
(0.48, 0.59) & (0.35, 0.21) & (0.57, 0.13) & (0.21, 0.21) & (0.88, 0.41)
\end{pmatrix}
$$

The entry at (1, 1) position, i.e., $(0.83, 0.28)$ shows that Angelica's tendency towards sharpness in mathematics is 83% whereas against it is 28%.

Now,

$$
very(\curlywedge_A) = \begin{pmatrix}
(0.91, 0.20) & (0.73, 0.15) & (0.61, 0.48) & (0.77, 0.11) & (0.93, 0.08)\\
(0.56, 0.19) & (0.75, 0.42) & (0.66, 0.23) & (0.86, 0.18) & (0.36, 0.04)\\
(0.72, 0.19) & (0.80, 0.09) & (0.67, 0.42) & (0.78, 0.25) & (0.54, 0.37)\\
(0.69, 0.44) & (0.59, 0.15) & (0.75, 0.09) & (0.46, 0.15) & (0.94, 0.30)
\end{pmatrix},
$$

$$
moderate(\curlywedge_A) = \begin{pmatrix}
(0.87, 0.24) & (0.63, 0.18) & (0.47, 0.57) & (0.67, 0.14) & (0.89, 0.10)\\
(0.42, 0.23) & (0.65, 0.51) & (0.53, 0.28) & (0.80, 0.22) & (0.22, 0.04)\\
(0.61, 0.23) & (0.72, 0.10) & (0.55, 0.51) & (0.69, 0.31) & (0.40, 0.45)\\
(0.58, 0.52) & (0.46, 0.18) & (0.66, 0.11) & (0.31, 0.18) & (0.91, 0.36)
\end{pmatrix},
$$

$$highly(\measuredangle_A) = \begin{pmatrix} (0.96, 0.13) & (0.88, 0.09) & (0.82, 0.32) & (0.90, 0.07) & (0.97, 0.05) \\ (0.79, 0.12) & (0.89, 0.27) & (0.84, 0.15) & (0.94, 0.11) & (0.66, 0.02) \\ (0.88, 0.12) & (0.91, 0.05) & (0.85, 0.27) & (0.91, 0.16) & (0.78, 0.24) \\ (0.86, 0.29) & (0.81, 0.09) & (0.89, 0.06) & (0.73, 0.09) & (0.97, 0.19) \end{pmatrix}$$

and

$$not(\measuredangle_A) = \begin{pmatrix} (0.47, 0.53) & (0.09, 0.41) & (0.02, 0.94) & (0.12, 0.31) & (0.55, 0.22) \\ (0.01, 0.49) & (0.10, 0.89) & (0.03, 0.59) & (0.30, 0.48) & (0.00, 0.10) \\ (0.07, 0.51) & (0.17, 0.24) & (0.04, 0.89) & (0.14, 0.64) & (0.01, 0.84) \\ (0.05, 0.91) & (0.02, 0.41) & (0.11, 0.26) & (0.00, 0.41) & (0.60, 0.72) \end{pmatrix}.$$

Definition 2.22 A PFSS $(\measuredangle, A)$ is termed as a *Pythagorean fuzzy soft point* (PFS point), denoted as $\vartheta_{\measuredangle}$, if for the element $\vartheta \in A$ we have

(i) $\measuredangle(\vartheta) \neq \Phi$, and
(ii) $\measuredangle(\vartheta') = \check{X}$, for all $\vartheta' \in A - \{\vartheta\}$.

Definition 2.23 A PFS point $\vartheta_{\measuredangle}\widetilde{\in}(\measuredangle, A)$ is said to be in PFSS $(\measuredangle_1, A_1)$, i.e., $\vartheta_{\measuredangle}\widetilde{\in}(\measuredangle_1, A_1)$ if $\vartheta \in A_1 \Rightarrow \measuredangle(\vartheta) \subseteq \measuredangle_1(\vartheta)$.

Example 2.24 Let $X = \{i, n, k\}$ and $E = \{\vartheta_1, \vartheta_2\}$, then

$$\vartheta_{\measuredangle_1} = \{(\vartheta_1, \{(i, 0.42, 0.57), (k, 0.43, 0.42)\})\},$$

and

$$\vartheta_{\measuredangle_2} = \{(\vartheta_2, \{(n, 0.37, 0.56), (k, 0.68, 0.29)\})\}$$

are two distinct PFS points contained in the PFSS

$$\measuredangle_E = \left\{(\vartheta_1, \{(i, 0.42, 0.57), (k, 0.43, 0.42)\}), (\vartheta_2, \{(n, 0.37, 0.56), (k, 0.68, 0.29)\})\right\}.$$

Notice that $\measuredangle_E = \vartheta_{\measuredangle_1}\widetilde{\cup}\vartheta_{\measuredangle_2}$, i.e., a PFS is union of its PFS points.

3 Multi-criteria Group Decision-Making Using Pythagorean Fuzzy Soft Information

There are lots of expressions that we casually use in daily life that have fuzzy structure. Usually we use numerical or, sometimes, verbal expressions to explain an event, refer to something, evaluating expertise of someone, and in many other situations include fuzziness. It is customary to use lingual expressions. These expressions generally do not express cast-iron certainty when deciding on a situation or elucidating some event. For example, the words poor, middle class, lower middle class, upper middle class, upper class and rich, etc. are used according to the income of an individual. We use the word "fast" to express a speed of 80 km/h while traveling on a rough road but call it "slow" while moving on motorways. These examples illustrate how human brain works and decides in ambiguous and uncertain situations, and how it recognizes, assesses, and commands events.

Science and technology have made tremendous developments with the advent of FSs. The mathematics of FSs has gained a large number of practical implementations in both theoretical and applied studies ranging from life sciences to artificial intelligence, and from physical sciences to engineering and humanities.

Often, we face problems in daily life situations which are not precise and clear. This issue leads us to different sorts of decision-making mechanisms. We endeavor to reach at some flawless and intellectual decision employing these mechanisms. For that reason, it is the need of the hour to have improved mathematical models and techniques for handling uncertainty and imprecision.

Shrewdly choice making is an energetic portion of trade, financial matters, social sciences, and other real-world issues. It marks out from day-by-day moo level operational appraisals at low-ranking administration level to long-term key arranging confronted by senior members of any organization. Conclusions that are delivered at any level can cause genuine or awful results, but is there an unequivocal format that choice producers ought to embrace in arrange to guarantee victory, or ought to supersede the standard plans of tackling a problem?

The choice producers ought to contract numerous components into consideration before reaching a unanimous and consistent choice. So it is basic to discover all these components are taken before the assurance is finalized. In parliamentary law to guarantee that all the vital realities and figures are scrutinized, it is irreplaceable to arrange the choice making advancement with an ordered demeanor.

Above and beyond other colossal applications, the science of mathematics helps us too in coming to conclusions on logical evidence. PFSSs take a broad view of both of IFSs and SSs in the sense that all intuitionistic fuzzy numbers used to express membership and non-membership degrees are part of Pythagorean fuzzy numbers. In daily decision-taking problems, PFSSs cover a greater membership space than the IFSS. As a consequence, PFSSs are more capable than IFSSs to model imprecision and uncertainty in choice making problems.

In this segment, we present an algorithm for handling multiple criteria group decision-making problem using choice value method under the umbrella of PFSSs supported by an illustration.

Algorithm 1:

Step 1: Input $X = \{\partial_i : i = 1, 2, \cdots, m\}$ as an aggregate of choices and $E = \{e_i : i = 1, 2, \cdots, n\}$ as a collection of attributes.
Step 2: Construct the PFS matrix with the assistance of experts.
Step 3: Compute relative importance, i.e., weight w_i of each attribute such that $\Sigma_{i=1}^{n} w_i = 1$.
Step 4: Compute the matrix of choice values using $\mathrm{C} = \curlywedge_E \times W^t$.
Step 5: Compute the score value s for each alternative using $s_j = n_{\sigma_j} - n_{\varrho_j}$, where n_{σ_j} denotes number of times σ_j goes beyond or equals other values of σ_k, $k \neq j$.
Step 6: The alternative for which score value is highest is the requisite choice.

Table 3 Tabular representation of $\leftthreetimes_E$

$\leftthreetimes_E$	e_1	e_2	e_3	e_4	e_5
∂_1	(0.42,0.56)	(0.37,0.54)	(0.59,0.11)	(0.23,0.59)	(0.11,0.92)
∂_2	(0.34,0.13)	(0.52,0.41)	(0.54,0.11)	(0.33,0.02)	(0.22,0.14)
∂_3	(0.89,0.24)	(0.77,0.31)	(0.56,0.15)	(0.50,0.13)	(0.28,0.13)
∂_4	(0.43,0.44)	(0.56,0.67)	(0.83,0.29)	(0.47,0.58)	(0.37,0.09)
∂_5	(0.56,0.67)	(0.49,0.52)	(0.57,0.38)	(0.21,0.34)	(0.38,0.36)
∂_6	(0.79,0.34)	(0.44,0.43)	(0.56,0.58)	(0.91,0.39)	(0.33,0.39)
∂_7	(0.54,0.24)	(0.51,0.42)	(0.55,0.55)	(0.11,0.09)	(0.39,0.56)

As a case study, we employ Algorithm 1 in stock exchange investment problem using hypothetical information.

Example 3.1 Suppose that an investor wishes to invest some money in some business with least risk. Let $X = \{\partial_i : i = 1, \cdots, 7\}$ be the collection of choices under consideration. For the purpose of reducing the risk factor, he decides to invest his capital in the ratio 3:2 in accordance with the top ranked two businesses. After getting advice from his four financial advisors, he chooses the set of attributes as $E = \{e_i : i = 1, \cdots, 5\}$, where

$$\begin{aligned}
e_1 &= \text{Standing reputation of the business}\\
e_2 &= \text{Impact on market}\\
e_3 &= \text{Prospects}\\
e_4 &= \text{Product Viability}\\
e_5 &= \text{Investment Safety}
\end{aligned}$$

Studying the history and trends of these businesses, the members of the technical team of the investor arranges the gathered information in the form of Table 3 of the PFS-set $\leftthreetimes_E$.

This information may be put in the form of PFS matrix as

$$\leftthreetimes_E = \begin{pmatrix}
(0.42, 0.56) & (0.37, 0.54) & (0.59, 0.11) & (0.23, 0.59) & (0.11, 0.92)\\
(0.34, 0.13) & (0.52, 0.41) & (0.54, 0.11) & (0.33, 0.02) & (0.22, 0.14)\\
(0.89, 0.24) & (0.77, 0.31) & (0.56, 0.15) & (0.50, 0.13) & (0.28, 0.13)\\
(0.43, 0.44) & (0.56, 0.67) & (0.83, 0.29) & (0.47, 0.58) & (0.37, 0.09)\\
(0.56, 0.67) & (0.49, 0.52) & (0.57, 0.38) & (0.21, 0.34) & (0.38, 0.36)\\
(0.79, 0.34) & (0.44, 0.43) & (0.56, 0.58) & (0.91, 0.39) & (0.33, 0.39)\\
(0.54, 0.24) & (0.51, 0.42) & (0.55, 0.55) & (0.11, 0.09) & (0.39, 0.56)
\end{pmatrix}$$

Assume that the four financial advisors provide the relative importance, i.e., weights, which are fuzzified, to each attribute and are given in the form of following matrix:

$$M = \begin{pmatrix} 0.54 & 0.38 & 0.59 & 0.89 & 0.76 \\ 0.37 & 0.47 & 0.48 & 0.94 & 0.88 \\ 0.82 & 0.46 & 0.76 & 0.23 & 0.79 \\ 0.18 & 0.32 & 0.57 & 0.46 & 0.69 \end{pmatrix}$$

After normalizing the entries of M, the normalized matrix appears to be

$$\widehat{M} = \begin{pmatrix} 0.507 & 0.461 & 0.485 & 0.639 & 0.485 \\ 0.348 & 0.570 & 0.394 & 0.675 & 0.562 \\ 0.770 & 0.558 & 0.625 & 0.165 & 0.504 \\ 0.169 & 0.388 & 0.468 & 0.330 & 0.441 \end{pmatrix}$$

Thus, the weighted values for the attributes are

$$W(e_1) = 0.188,\ W(e_2) = 0.207,\ W(e_3) = 0.207,\ W(e_4) = 0.190,\ W(e_5) = 0.209.$$

Hence, the weight vector is

$$W = \begin{pmatrix} 0.188 & 0.207 & 0.207 & 0.190 & 0.209 \end{pmatrix}$$

Thus, the PF-matrix for choice values is

$$\begin{aligned} \mathrm{C} &= \curlywedge_E \times W^t \\ &= \begin{pmatrix} (0.42, 0.56) & (0.37, 0.54) & (0.59, 0.11) & (0.23, 0.59) & (0.11, 0.92) \\ (0.34, 0.13) & (0.52, 0.41) & (0.54, 0.11) & (0.33, 0.02) & (0.22, 0.14) \\ (0.89, 0.24) & (0.77, 0.31) & (0.56, 0.15) & (0.50, 0.13) & (0.28, 0.13) \\ (0.43, 0.44) & (0.56, 0.67) & (0.83, 0.29) & (0.47, 0.58) & (0.37, 0.09) \\ (0.56, 0.67) & (0.49, 0.52) & (0.57, 0.38) & (0.21, 0.34) & (0.38, 0.36) \\ (0.79, 0.34) & (0.44, 0.43) & (0.56, 0.58) & (0.91, 0.39) & (0.33, 0.39) \\ (0.54, 0.24) & (0.51, 0.42) & (0.55, 0.55) & (0.11, 0.09) & (0.39, 0.56) \end{pmatrix} \begin{pmatrix} 0.188 \\ 0.207 \\ 0.207 \\ 0.190 \\ 0.209 \end{pmatrix} \\ &= \begin{pmatrix} (0.3444, 0.5442) \\ (0.3920, 0.1651) \\ (0.5962, 0.1922) \\ (0.5352, 0.4105) \\ (0.4440, 0.4521) \\ (0.5974, 0.4286) \\ (0.4234, 0.3801) \end{pmatrix} \end{aligned}$$

The values of the score function along with ranking are given in Table 4. Table 4 demonstrates that

$$\partial_3 \succ \partial_6 \succ \partial_2 = \partial_4 \succ \partial_7 \succ \partial_5 \succ \partial_1$$

This ranking is depicted in Fig. 1.

Table 4 Values of score function and ranking

X	s	Ranking
∂_1	$0-6=-6$	6
∂_2	$1-0=1$	3
∂_3	$5-1=4$	1
∂_4	$4-3=1$	3
∂_5	$3-5=-2$	5
∂_6	$6-4=2$	2
∂_7	$2-2=0$	4

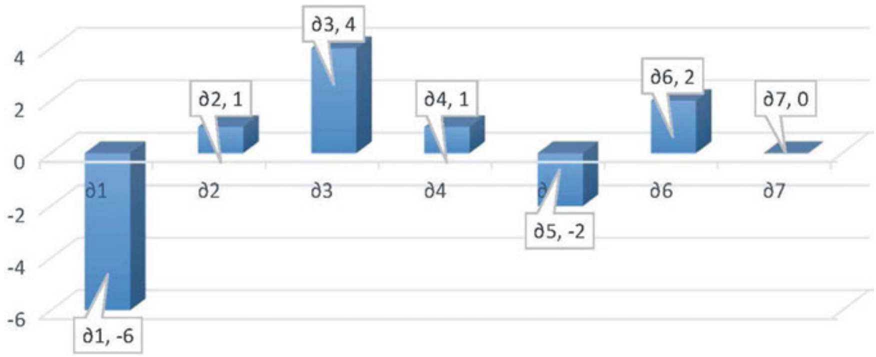

Fig. 1 Ranking of companies

In view of this ranking, it may be concluded that the investor should invest 60% of the capital on ∂_3, and 40% on ∂_6.

3.1 Comparison Analysis

We compare the results of our proposed Algorithm 1 with that of some existing methods. The results obtained are shown in Table 5.

The results portrayed in Table 5 approve the validity of the proposed technique.

Table 5 Comparison of results of suggested algorithm 1 with some existing techniques

Method	Ranking of choices
Algorithm 1 (Suggested)	$\partial_3 \succ \partial_6 \succ \partial_2 = \partial_4$
Guleria and Bajaj (Case-I) [25]	$\partial_3 \succ \partial_6 \succ \partial_2 \succ \partial_4$
Guleria and Bajaj (Case-II) [25]	$\partial_3 \succ \partial_6 \succ \partial_2 \succ \partial_4$
Peng et al. [51]	$\partial_3 \succ \partial_6 \succ \partial_4 \succ \partial_2$

Table 6 Phonetic labels for assessing alternatives

Linguistic Terms	Fuzzy Weights
Very Necessary (VN)	(0.80, 1]
Mandatory (M)	(0.50, 0.80]
More or Less Required (MLR)	(0.20, 0.50]
Average Requirement (AR)	(0.10, 0.20]
Of No Use (ONU)	[0, 0.10]

4 TOPSIS Approach for Choice Making with Pythagorean Fuzzy Soft Sets

In this section, we study the utilization of PFSSs in intelligent decision-taking. For this purpose, we first extend TOPSIS to PFSS. The proposed version will be called PFS-TOPSIS. Afterwards, we shall consider a problem of choosing suitable candidates for key ministries of a country, where PFSSs may be used.

We launch by illuminating the offered modus operandi a step at a time. The suggested PFS-TOPSIS is generality of fuzzy soft TOPSIS suggested in [12] by Eraslan and Karaaslan.

Algorithm 2:

Step 1: Recognizing the problem: Suppose that $DM = \{\mathcal{D}_i : i = 1, \cdots, n\}$ is team of decision experts, $C = \{\ddot{c}_i = 1, \cdots, l\}$ is the assemblage of choices and $Q = \{q_j : j = 1, \cdots, m\}$ is family of attributes.

Step 2: Picking the phonetic terms as given in Table 6, prepare weighted parameter matrix as $[w_{ij}]_{n\times m}$, where w_{ij} is the weight allocated by the decision expert $\mathcal{D}_i$ to the attribute q_j.

Step 3: Normalize the weighted matrix to get $\hat{N} = [\hat{n}_{ij}]_{n\times m}$, where $\hat{n}_{ij} = \frac{w_{ij}}{\sqrt{\Sigma_{i=1}^{n} w_{ij}^2}}$ and obtaining the weight vector $\mathcal{W} = (\mathfrak{w}_1, \mathfrak{w}_2, \cdots, \mathfrak{w}_m)$, where $\mathfrak{w}_j = \frac{\sum_{i=1}^{n} \hat{n}_{ij}}{m \sum_{k=1}^{m} \hat{n}_{ik}}$.

Step 4: Construct PFS matrix

$$D_i = [v^i_{jk}]_{l\times m} = \begin{pmatrix} v^i_{11} & v^i_{12} & \cdots & v^i_{1m} \\ v^i_{21} & v^i_{22} & \cdots & v^i_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ v^i_{j1} & v^i_{j2} & \cdots & v^i_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ v^i_{l1} & v^i_{l2} & \cdots & v^i_{lm}, \end{pmatrix}$$

where v^i_{jk} is a PFS-element, provided by ith decision expert. Then obtain the aggregated matrix

$$D = \frac{D_1 + D_2 + \cdots + D_n}{n} = [\dot{v}_{jk}]_{l\times m}.$$

Step 5: Achieve the weighted PFS matrix

$$D_w = [\ddot{r}_{jk}]_{l \times m} = \begin{pmatrix} \ddot{r}_{11} & \ddot{r}_{12} & \cdots & \ddot{r}_{1m} \\ \ddot{r}_{21} & \ddot{r}_{22} & \cdots & \ddot{r}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \ddot{r}_{j1} & \ddot{r}_{j2} & \cdots & \ddot{r}_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ \ddot{r}_{l1} & \ddot{r}_{l2} & \cdots & \ddot{r}_{lm}, \end{pmatrix}$$

where $\ddot{r}_{jk} = \mathfrak{w}_k \times \dot{v}_{jk}$.

Step 6: Track the PFS-valued positive ideal solution (PFSV-PIS) and PFS-valued negative ideal solution (PFSV-NIS). For this purpose, we employ in order

$$\begin{aligned} \text{PFSV-PIS} &= \{\ddot{r}_1^+, \ddot{r}_2^+, \cdots, \ddot{r}_m^+\} \\ &= \left\{(\vee_k \ddot{r}_{jk}, \wedge_k \ddot{r}_{jk}); k = 1, \cdots, m\right\} \\ &= \left\{(\sigma_k^+, \varrho_k^+) : k = 1, \cdots, m\right\} \end{aligned}$$

and

$$\begin{aligned} \text{PFSV-NIS} &= \{\ddot{r}_1^-, \ddot{r}_2^-, \cdots, \ddot{r}_m^-\} \\ &= \left\{(\wedge_k \ddot{r}_{jk}, \vee_k \ddot{r}_{jk}); k = 1, \cdots, m\right\} \\ &= \left\{(\sigma_k^-, \varrho_k^-) : k = 1, \cdots, m\right\}, \end{aligned}$$

where $\vee$ stands for PFS union and $\wedge$ represents PFS intersection.

Step 7: Compute distances of each alternative from PFSV-PIS and PFSV-NIS, respectively, utilizing

$$\mathfrak{z}_j^+ = \sqrt{\Sigma_{k=1}^m \left\{\left(\sigma_{jk} - \sigma_k^+\right)^2 + \left(\varrho_{jk} - \varrho_k^+\right)^2\right\}}$$

and

$$\mathfrak{z}_j^- = \sqrt{\Sigma_{k=1}^m \left\{\left(\sigma_{jk} - \sigma_k^-\right)^2 + \left(\varrho_{jk} - \varrho_k^-\right)^2\right\}}.$$

Step 8: Attain the closeness coefficient of each alternative with ideal solution by making use of

$$\mathcal{C}_j^* = \frac{\mathfrak{z}_j^-}{\mathfrak{z}_j^+ + \mathfrak{z}_j^-} \in [0, 1].$$

Step 9: Arrange the ranking of choices in decreasing (or increasing) for obtaining the priority order of the choices.

As an illustration of Algorithm 2, we discuss a state managerial problem following the procedural steps given in Algorithm 2.

Example 4.1 Suppose that a political party clean sweeps in general elections in a country. The party has got chance for the first time to make national government and wishes to prove that it is the best. The party chairman wants to deliver to the people of the country his best. The party wants to fill the positions of key ministries by choosing ministers, who should also be competent, well educated/trained and meritorious in their respective fields. The party's top leadership constitutes a committee of experts to help him solve this riddle on scientific grounds. They also decide that no member should be given more than one ministry. Assume that

$$C = \{c_1, c_2, \ldots, c_6\}$$

is the set of candidates who are to be deputed in different key ministries (ministries of foreign affairs, defence, finance, and information & broadcasting in order). Further suppose that

$$Q = \{q_1, q_2, \ldots, q_5\}$$

is the set of qualification/merit mandatory for filling a position. The committee interviews each candidate carefully to see who is appropriate for which ministry.

Picking the weights from Table 6, the experts provide the following weighted parameter matrix

$$\mathcal{P} = \begin{pmatrix} \text{VN} & \text{MLR} & \text{MLR} & \text{ONU} & \text{VN} \\ \text{M} & \text{AR} & \text{AR} & \text{AR} & \text{VN} \\ \text{M} & \text{M} & \text{VN} & \text{M} & \text{M} \\ \text{MLR} & \text{AR} & \text{MLR} & \text{AR} & \text{VN} \end{pmatrix}$$
$$= \begin{pmatrix} 0.90 & 0.40 & 0.30 & 0.10 & 0.90 \\ 0.70 & 0.15 & 0.20 & 0.15 & 0.85 \\ 0.60 & 0.70 & 0.90 & 0.80 & 0.75 \\ 0.40 & 0.15 & 0.40 & 0.15 & 0.90 \end{pmatrix}$$

The normalized weighted matrix is

$$\hat{N} = \begin{pmatrix} 0.667 & 0.480 & 0.286 & 0.120 & 0.528 \\ 0.519 & 0.180 & 0.191 & 0.180 & 0.499 \\ 0.445 & 0.840 & 0.858 & 0.960 & 0.440 \\ 0.296 & 0.180 & 0.381 & 0.180 & 0.528 \end{pmatrix}$$

and hence the weight vector is $\mathcal{W} = (0.220, 0.192, 0.196, 0.164, 0.228)$.

Assume that the four experts provide the following PFS matrices in which the PFN at (i, j)th position demarcated grades of candidates row-wise and the attribute column-wise.

$$D_1 = \begin{pmatrix} (0.57, 0.39) & (0.49, 0.74) & (0.77, 0.38) & (0.54, 0.21) & (0.12, 0.48) \\ (0.66, 0.51) & (0.54, 0.54) & (0.32, 0.13) & (0.99, 0.13) & (0.54, 0.07) \\ (0.15, 0.68) & (0.19, 0.32) & (0.76, 0.41) & (0.45, 0.15) & (0.11, 0.49) \\ (0.67, 0.74) & (0.09, 0.83) & (0.59, 0.31) & (0.84, 0.16) & (0.37, 0.21) \\ (0.59, 0.17) & (0.33, 0.67) & (0.34, 0.68) & (0.52, 0.19) & (0.58, 0.61) \\ (0.27, 0.54) & (0.49, 0.46) & (0.48, 0.59) & (0.55, 0.54) & (0.38, 0.01) \end{pmatrix}$$

$$D_2 = \begin{pmatrix} (0.34, 0.52) & (0.58, 0.21) & (0.47, 0.21) & (0.70, 0.31) & (0.11, 0.34) \\ (0.47, 0.33) & (0.39, 0.32) & (0.56, 0.20) & (0.38, 0.11) & (0.26, 0.18) \\ (0.59, 0.17) & (0.33, 0.17) & (0.19, 0.28) & (0.59, 0.06) & (0.78, 0.16) \\ (0.44, 0.17) & (0.38, 0.23) & (0.58, 0.27) & (0.71, 0.24) & (0.54, 0.02) \\ (0.32, 0.28) & (0.56, 0.11) & (0.44, 0.37) & (0.49, 0.29) & (0.55, 0.55) \\ (0.34, 0.47) & (0.52, 0.37) & (0.11, 0.18) & (0.47, 0.13) & (0.47, 0.27) \end{pmatrix}$$

$$D_3 = \begin{pmatrix} (0.11, 0.58) & (0.37, 0.22) & (0.56, 0.11) & (0.21, 0.69) & (0.79, 0.32) \\ (0.13, 0.67) & (0.46, 0.13) & (0.36, 0.54) & (0.56, 0.27) & (0.46, 0.61) \\ (0.59, 0.13) & (0.25, 0.11) & (0.62, 0.33) & (0.47, 0.28) & (0.28, 0.47) \\ (0.11, 0.49) & (0.23, 0.05) & (0.50, 0.28) & (0.34, 0.48) & (0.61, 0.54) \\ (0.17, 0.29) & (0.82, 0.34) & (0.56, 0.51) & (0.50, 0.28) & (0.49, 0.12) \\ (0.33, 0.69) & (0.57, 0.61) & (0.48, 0.57) & (0.33, 0.02) & (0.46, 0.31) \end{pmatrix}$$

$$D_4 = \begin{pmatrix} (0.40, 0.59) & (0.41, 0.32) & (0.49, 0.12) & (0.35, 0.65) & (0.39, 0.12) \\ (0.25, 0.17) & (0.38, 0.10) & (0.85, 0.26) & (0.44, 0.57) & (0.92, 0.14) \\ (0.38, 0.51) & (0.36, 0.11) & (0.52, 0.29) & (0.48, 0.38) & (0.52, 0.35) \\ (0.56, 0.11) & (0.73, 0.16) & (0.35, 0.27) & (0.58, 0.62) & (0.62, 0.63) \\ (0.11, 0.01) & (0.33, 0.37) & (0.28, 0.38) & (0.47, 0.32) & (0.71, 0.19) \\ (0.58, 0.17) & (0.44, 0.15) & (0.56, 0.16) & (0.33, 0.21) & (0.88, 0.26) \end{pmatrix}$$

Thus, the aggregated matrix is

$$D = \begin{pmatrix} (0.355, 0.520) & (0.463, 0.373) & (0.573, 0.205) & (0.450, 0.465) & (0.353, 0.315) \\ (0.378, 0.420) & (0.443, 0.273) & (0.523, 0.283) & (0.593, 0.270) & (0.545, 0.250) \\ (0.428, 0.373) & (0.373, 0.178) & (0.523, 0.328) & (0.498, 0.218) & (0.423, 0.368) \\ (0.445, 0.378) & (0.358, 0.318) & (0.505, 0.283) & (0.618, 0.375) & (0.535, 0.350) \\ (0.298, 0.188) & (0.510, 0.373) & (0.405, 0.485) & (0.495, 0.270) & (0.583, 0.368) \\ (0.380, 0.468) & (0.505, 0.398) & (0.408, 0.375) & (0.420, 0.225) & (0.548, 0.213) \end{pmatrix}$$

and hence the weighted PFS matrix is

Table 7 Distance & closeness coefficient of each candidate

Candidate	$\mathfrak{Z}_j^+$	$\mathfrak{Z}_j^-$	C_j^*
c_1	0.1137	0.0714	0.3857
c_2	0.0620	0.0843	0.5762
c_3	0.0774	0.0830	0.5175
c_4	0.0741	0.0782	0.5135
c_5	0.0909	0.0934	0.5068
c_6	0.0953	0.0725	0.4321

$$D_w = \begin{pmatrix} (0.078, 0.114) & (0.089, 0.072) & (0.112, 0.040) & (0.074, 0.076) & (0.080, 0.072) \\ (0.083, 0.092) & (0.085, 0.052) & (0.103, 0.055) & (0.097, 0.044) & (0.124, 0.057) \\ (0.094, 0.082) & (0.072, 0.034) & (0.103, 0.064) & (0.082, 0.036) & (0.096, 0.084) \\ (0.098, 0.083) & (0.069, 0.061) & (0.099, 0.055) & (0.101, 0.062) & (0.122, 0.080) \\ (0.066, 0.041) & (0.098, 0.072) & (0.079, 0.095) & (0.081, 0.044) & (0.133, 0.084) \\ (0.084, 0.103) & (0.097, 0.076) & (0.080, 0.074) & (0.069, 0.037) & (0.125, 0.049) \end{pmatrix}$$

The positive and negative ideal solutions are

$$\begin{aligned} \text{PFSV-PIS} &= \{\ddot{r}_1^+, \ddot{r}_2^+, \cdots, \ddot{r}_5^+\} \\ &= \{(0.098, 0.041), (0.098, 0.034), (0.112, 0.040), (0.101, 0.036), (0.133, 0.049)\} \end{aligned}$$

and

$$\begin{aligned} \text{PFSV-NIS} &= \{\ddot{r}_1^-, \ddot{r}_2^-, \cdots, \ddot{r}_5^-\} \\ &= \{(0.066, 0.114), (0.069, 0.076), (0.079, 0.095), (0.069, 0.076), (0.096, 0.084)\} \end{aligned}$$

respectively.

The distance of each candidate from PFSV-PIS and PFSV-NIS accompanied by their relative closeness coefficients are displayed in Table 7.

Hence, the ranking preference is

$$c_2 \succ c_3 \succ c_4 \succ c_5 \succ c_6 \succ c_1$$

. This preference order is depicted in Fig. 2.

The above priority order advocates that the ministry of foreign affairs should be given to c_2, defence to c_3, finance to c_4 and the ministry of information & broadcasting to c_5.

Fig. 2 Ranking of candidates

5 Multiple Criteria Group Decision-Making Using PFS-VIKOR Method

The word VIKOR is abbreviated version of "Vlse Kriterijumska Optimizacija Kompromisno Resenje" from Serbian language to mean manifold-criteria analysis (or optimization) and middle ground way out. This technique was devised by Serafim Opricovic to handle choice making problems having dissenting and non-commensurable principles, with the assumption that finding the middle grounds is apt for resolving any clash. The team of experts rummages around for a solution that neighbors the superlative idyllic solution, and the choices are evaluated following all recognized rules. VIKOR has transpired as a widely held multi-criteria decision-making technique mainly because of its computational straightforwardness and scrupulousness of solution.

We elucidate the suggested technique bit by bit as below. First six steps of PFS-VIKOR are the same as of PFS-TOPSIS given in Algorithm 2, so we skip them.

Algorithm 3 (PFS-VIKOR):

Step 7: Use the formulae

$$S_i = \Sigma_{j=1}^{m} \mathfrak{w}_j \left(\frac{d\left(\ddot{r}_j^+, \ddot{r}_{ij}\right)}{d\left(\ddot{r}_j^+, \ddot{r}_j^-\right)} \right)$$

$$R_i = \max_{j=1}^{m} \mathfrak{w}_j \left(\frac{d\left(\ddot{r}_j^+, \ddot{r}_{ij}\right)}{d\left(\ddot{r}_j^+, \ddot{r}_j^-\right)} \right)$$

$$Q_i = \kappa \left(\frac{S_i - S^-}{S^+ - S^-} \right) + (1-\kappa) \left(\frac{R_i - R^-}{R^+ - R^-} \right),$$

where $S^+ = \max_i S_i$, $S^- = \min_i S_i$, $R^+ = \max_i R_i$, and $R^- = \min_i R_i$, to get the values of group utility S_i, individual regret R_i, and compromise Q_i. The real number κ is termed as coefficient of decision mechanism. The role of the coefficient κ is that if compromise solution is to be selected by majority, we choose $\kappa > 0.5$; for consensus we use $\kappa = 0.5$, and $\kappa < 0.5$ represents veto. $\mathfrak{w}_j$ represents the weight of the jth criteria, which expresses its relative importance.

Step 8: Arrange S_i, R_i, and Q_i in ascending array. The choice $\ddot{r}_\alpha$ would be announced middle ground solution if it has the minimum value of Q_i and further gratifies the following two necessities in chorus:

(a) If $\ddot{r}_{\alpha_1}$ and $\ddot{r}_{\alpha_2}$ are two best choices regarding Q_i, then

$$Q(\ddot{r}_{\alpha_2}) - Q(\ddot{r}_{\alpha_1}) \geq \frac{1}{n-1}$$

n being the number of attributes.

(b) The choice $\ddot{r}_{\alpha_1}$ must be best ranked by at least one of R_i and S_i.

There will exist multiple compromise solutions otherwise, which may be located as under:

(i) $\ddot{r}_{\alpha_1}$ and α_2 will be the compromise solutions in case merely (a) is gratified.

(ii) $\ddot{r}_{\alpha_1}, \ddot{r}_{\alpha_2}, \cdots, \ddot{r}_{\alpha_u}$ would be the compromise solutions in case (a) is not fulfilled, where $\ddot{r}_{\alpha_u}$ may be found employing

$$Q(\ddot{r}_{\alpha_u}) - Q(\ddot{r}_{\alpha_1}) \geq \frac{1}{n-1}.$$

Example 5.1 Assume that a multi-national company wants to choose some brand ambassadors for advertisement of its products. The CEO of that company constitutes a committee of four experts to give recommendations about the selection of ambassadors. The number of ambassadors may vary from one to any reasonable number. The CEO needs a unanimous decision about their selection. The committee decides to work on scientific grounds. Assume that

$$C = \{a_1, a_2, \cdots, a_6\}$$

is the set of persons under consideration as embassador. Further suppose that

$$Q = \{q_1, q_2, \cdots, q_5\}$$

is the set of qualities under consideration for the selection of any individual. The committee ponders on the personalities and the effectiveness of those individuals on the mob.

Picking the weights from Table 6, the experts provide the following weighted parameter matrix

$$\mathcal{P} = \begin{pmatrix} \text{VN} & \text{MLR} & \text{MLR} & \text{ONU} & \text{VN} \\ \text{M} & \text{AR} & \text{AR} & \text{AR} & \text{VN} \\ \text{M} & \text{M} & \text{VN} & \text{M} & \text{M} \\ \text{MLR} & \text{AR} & \text{MLR} & \text{AR} & \text{VN} \end{pmatrix}$$

$$= \begin{pmatrix} 0.90 & 0.40 & 0.30 & 0.10 & 0.90 \\ 0.70 & 0.15 & 0.20 & 0.15 & 0.85 \\ 0.60 & 0.70 & 0.90 & 0.80 & 0.75 \\ 0.40 & 0.15 & 0.40 & 0.15 & 0.90 \end{pmatrix}$$

The normalized weighted matrix is

$$\hat{N} = \begin{pmatrix} 0.667 & 0.480 & 0.286 & 0.120 & 0.528 \\ 0.519 & 0.180 & 0.191 & 0.180 & 0.499 \\ 0.445 & 0.840 & 0.858 & 0.960 & 0.440 \\ 0.296 & 0.180 & 0.381 & 0.180 & 0.528 \end{pmatrix}$$

and hence the weight vector is $\mathcal{W} = (0.220, 0.192, 0.196, 0.164, 0.228)$.

Assume that the four experts provide the following PFS matrices in which the PFN at (i, j)th position demarcated grades of candidates row-wise and the attribute column-wise.

$$D_1 = \begin{pmatrix} (0.57, 0.39) & (0.49, 0.74) & (0.77, 0.38) & (0.54, 0.21) & (0.12, 0.48) \\ (0.66, 0.51) & (0.54, 0.54) & (0.32, 0.13) & (0.99, 0.13) & (0.54, 0.07) \\ (0.15, 0.68) & (0.19, 0.32) & (0.76, 0.41) & (0.45, 0.15) & (0.11, 0.49) \\ (0.67, 0.74) & (0.09, 0.83) & (0.59, 0.31) & (0.84, 0.16) & (0.37, 0.21) \\ (0.59, 0.17) & (0.33, 0.67) & (0.34, 0.68) & (0.52, 0.19) & (0.58, 0.61) \\ (0.27, 0.54) & (0.49, 0.46) & (0.48, 0.59) & (0.55, 0.54) & (0.38, 0.01) \end{pmatrix}$$

$$D_2 = \begin{pmatrix} (0.34, 0.52) & (0.58, 0.21) & (0.47, 0.21) & (0.70, 0.31) & (0.11, 0.34) \\ (0.47, 0.33) & (0.39, 0.32) & (0.56, 0.20) & (0.38, 0.11) & (0.26, 0.18) \\ (0.59, 0.17) & (0.33, 0.17) & (0.19, 0.28) & (0.59, 0.06) & (0.78, 0.16) \\ (0.44, 0.17) & (0.38, 0.23) & (0.58, 0.27) & (0.71, 0.24) & (0.54, 0.02) \\ (0.32, 0.28) & (0.56, 0.11) & (0.44, 0.37) & (0.49, 0.29) & (0.55, 0.55) \\ (0.34, 0.47) & (0.52, 0.37) & (0.11, 0.18) & (0.47, 0.13) & (0.47, 0.27) \end{pmatrix}$$

$$D_3 = \begin{pmatrix} (0.11, 0.58) & (0.37, 0.22) & (0.56, 0.11) & (0.21, 0.69) & (0.79, 0.32) \\ (0.13, 0.67) & (0.46, 0.13) & (0.36, 0.54) & (0.56, 0.27) & (0.46, 0.61) \\ (0.59, 0.13) & (0.25, 0.11) & (0.62, 0.33) & (0.47, 0.28) & (0.28, 0.47) \\ (0.11, 0.49) & (0.23, 0.05) & (0.50, 0.28) & (0.34, 0.48) & (0.61, 0.54) \\ (0.17, 0.29) & (0.82, 0.34) & (0.56, 0.51) & (0.50, 0.28) & (0.49, 0.12) \\ (0.33, 0.69) & (0.57, 0.61) & (0.48, 0.57) & (0.33, 0.02) & (0.46, 0.31) \end{pmatrix}$$

$$D_4 = \begin{pmatrix} (0.40, 0.59) & (0.41, 0.32) & (0.49, 0.12) & (0.35, 0.65) & (0.39, 0.12) \\ (0.25, 0.17) & (0.38, 0.10) & (0.85, 0.26) & (0.44, 0.57) & (0.92, 0.14) \\ (0.38, 0.51) & (0.36, 0.11) & (0.52, 0.29) & (0.48, 0.38) & (0.52, 0.35) \\ (0.56, 0.11) & (0.73, 0.16) & (0.35, 0.27) & (0.58, 0.62) & (0.62, 0.63) \\ (0.11, 0.01) & (0.33, 0.37) & (0.28, 0.38) & (0.47, 0.32) & (0.71, 0.19) \\ (0.58, 0.17) & (0.44, 0.15) & (0.56, 0.16) & (0.33, 0.21) & (0.88, 0.26) \end{pmatrix}$$

Thus, the aggregated matrix is

$$D = \begin{pmatrix} (0.355, 0.520) & (0.463, 0.373) & (0.573, 0.205) & (0.450, 0.465) & (0.353, 0.315) \\ (0.378, 0.420) & (0.443, 0.273) & (0.523, 0.283) & (0.593, 0.270) & (0.545, 0.250) \\ (0.428, 0.373) & (0.373, 0.178) & (0.523, 0.328) & (0.498, 0.218) & (0.423, 0.368) \\ (0.445, 0.378) & (0.358, 0.318) & (0.505, 0.283) & (0.618, 0.375) & (0.535, 0.350) \\ (0.298, 0.188) & (0.510, 0.373) & (0.405, 0.485) & (0.495, 0.270) & (0.583, 0.368) \\ (0.380, 0.468) & (0.505, 0.398) & (0.408, 0.375) & (0.420, 0.225) & (0.548, 0.213) \end{pmatrix}$$

and hence the weighted PFS matrix is

$$D_w = \begin{pmatrix} (0.078, 0.114) & (0.089, 0.072) & (0.112, 0.040) & (0.074, 0.076) & (0.080, 0.072) \\ (0.083, 0.092) & (0.085, 0.052) & (0.103, 0.055) & (0.097, 0.044) & (0.124, 0.057) \\ (0.094, 0.082) & (0.072, 0.034) & (0.103, 0.064) & (0.082, 0.036) & (0.096, 0.084) \\ (0.098, 0.083) & (0.069, 0.061) & (0.099, 0.055) & (0.101, 0.062) & (0.122, 0.080) \\ (0.066, 0.041) & (0.098, 0.072) & (0.079, 0.095) & (0.081, 0.044) & (0.133, 0.084) \\ (0.084, 0.103) & (0.097, 0.076) & (0.080, 0.074) & (0.069, 0.037) & (0.125, 0.049) \end{pmatrix}$$

The positive and negative ideal solutions are

$$\begin{aligned} \text{PFSV-PIS} &= \{\ddot{r}_1^+, \ddot{r}_2^+, \cdots, \ddot{r}_5^+\} \\ &= \{(0.098, 0.041), (0.098, 0.034), (0.112, 0.040), (0.101, 0.036), (0.133, 0.049)\} \end{aligned}$$

and

$$\begin{aligned} \text{PFSV-NIS} &= \{\ddot{r}_1^-, \ddot{r}_2^-, \cdots, \ddot{r}_5^-\} \\ &= \{(0.066, 0.114), (0.069, 0.076), (0.079, 0.095), (0.069, 0.076), (0.096, 0.084)\} \end{aligned}$$

respectively.

Choosing $\kappa = 0.5$, the values of S_i, R_i, and Q_i for each choice $\ddot{r}_i$ are calculated utilizing

$$\begin{aligned} S_i &= \Sigma_{j=1}^5 \mathfrak{w}_j \left(\frac{d\left(\ddot{r}_j^+, \ddot{r}_{ij}\right)}{d\left(\ddot{r}_j^+, \ddot{r}_j^-\right)} \right) \\ R_i &= \max_{j=1}^{5} \mathfrak{w}_j \left(\frac{d\left(\ddot{r}_j^+, \ddot{r}_{ij}\right)}{d\left(\ddot{r}_j^+, \ddot{r}_j^-\right)} \right) \\ Q_i &= \kappa \left(\frac{S_i - S^-}{S^+ - S^-} \right) + (1-\kappa) \left(\frac{R_i - R^-}{R^+ - R^-} \right) \end{aligned}$$

Table 8 Values of S_i, R_i, and Q_i for alternatives

Alternative	S_i	R_i	Q_i
a_1	0.7698	0.2589	1.0000
a_2	0.3663	0.1469	0.0000
a_3	0.5788	0.2280	0.6260
a_4	0.5562	0.1491	0.2457
a_5	0.6531	0.1025	0.5754
a_6	0.6148	0.0358	0.4368

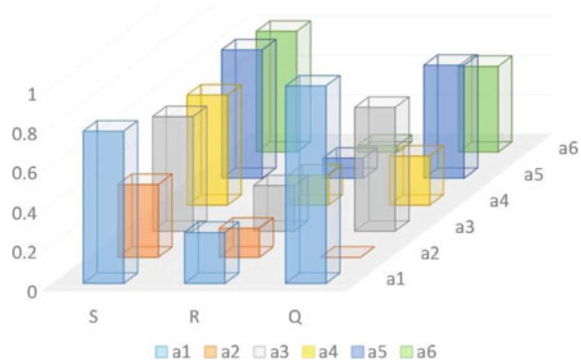

Fig. 3 3D column chart of rankings

and are given in Table 8 below:

The rank of choices is as under:

$$\text{By } Q_i : \quad a_2 \prec a_4 \prec a_6 \prec a_5 \prec a_3 \prec a_1$$

$$\text{By } S_i : \quad a_2 \prec a_4 \prec a_3 \prec a_6 \prec a_5 \prec a_1$$

$$\text{By } R_i : \quad a_6 \prec a_5 \prec a_2 \prec a_4 \prec a_3 \prec a_1$$

Since

$$Q(a_4) - Q(a_2) = 0.2457 \ngeq \frac{1}{4}$$

so (a) is not gratified. Further,

$$Q(a_6) - (a_2) = 0.4368 \geq \frac{1}{4}$$

Thus, the committee recommends that the persons a_2, a_4, and a_6 must be chosen as brand ambassadors. These rankings are depicted in Fig. 3.

6 A Similarity Measure for PFSSs

In this section, we propose a new similarity measure for PFSSs based on cosine similarity measure and Frobenius inner product of matrices and render some of its characteristics.

Definition 6.1 Let $X=\{\partial_i : i=1, \cdots, m\}$ be a crisp set and $E = \{e_j : j = 1, \cdots, n\}$ be the aggregate of attributes. If

$$\angle_1 = \begin{pmatrix} (\sigma_{11}, \varrho_{11})_{\angle_1} & (\sigma_{12}, \varrho_{12})_{\angle_1} & \cdots & (\sigma_{1n}, \varrho_{1n})_{\angle_1} \\ (\sigma_{21}, \varrho_{21})_{\angle_1} & (\sigma_{22}, \varrho_{22})_{\angle_1} & \cdots & (\sigma_{2n}, \varrho_{2n})_{\angle_1} \\ \vdots & \vdots & \ddots & \vdots \\ (\sigma_{m1}, \varrho_{m1})_{\angle_1} & (\sigma_{m2}, \varrho_{m2})_{\angle_1} & \cdots & (\sigma_{mn}, \varrho_{mn})_{\angle_1} \end{pmatrix}$$

and

$$\angle_2 = \begin{pmatrix} (\sigma_{11}, \varrho_{11})_{\angle_2} & (\sigma_{12}, \varrho_{12})_{\angle_2} & \cdots & (\sigma_{1n}, \varrho_{1n})_{\angle_2} \\ (\sigma_{21}, \varrho_{21})_{\angle_2} & (\sigma_{22}, \varrho_{22})_{\angle_2} & \cdots & (\sigma_{2n}, \varrho_{2n})_{\angle_2} \\ \vdots & \vdots & \ddots & \vdots \\ (\sigma_{m1}, \varrho_{m1})_{\angle_2} & (\sigma_{m2}, \varrho_{m2})_{\angle_2} & \cdots & (\sigma_{mn}, \varrho_{mn})_{\angle_2} \end{pmatrix}$$

are PFS matrices of PFSSs $(\angle_1, E)$ and $(\angle_2, E)$, then *similarity measure* between $(\angle_1, E)$ and $(\angle_2, E)$ is given as

$$Sim(\angle_1, \angle_2) = \frac{< \angle_1, \angle_2 >}{\| \angle_1 \| \| \angle_2 \|},$$

where

$$< \angle_1, \angle_2 > = tr(\angle_1^T \angle_2)$$
$$\| \angle_1 \| = \sqrt{< \angle_1, \angle_1 >}.$$

Here $tr(\angle_1^T \angle_2)$ (known as *trace* of the matrix $\angle_1^T \angle_2$) denotes the sum of elements at principal diagonal of the matrix $\angle_1^T \angle_2$. The above definition holds good if hesitation margin ε_{ij} is also taken into account. Moreover, this similarity measure satisfies the following:

(1) $0 \leq Sim(\angle_1, \angle_2) \leq 1$.
(2) $Sim(\angle_1, \angle_2) = 1 \Leftrightarrow \angle_1 = \angle_2$.
(3) $Sim(\angle_1, \angle_2) = Sim(\angle_2, \angle_1)$.
(4) $Sim(\angle, \angle^c) = 1$ iff $\angle$ is a crisp set.
(5) If $(\angle_1, E)\widetilde{\sqsubseteq}(\angle_2, E)\widetilde{\sqsubseteq}(\angle_3, E)$, then $Sim(\angle_1, \angle_3) \leq Sim(\angle_2, \angle_3)$.

Example 6.2 Let $X = \{\partial_1, \cdots, \partial_4\}$ be the universe and $E = \{e_i \mid i = 1, 2, 3\}$ be the aggregate of attributes. Consider the PFS matrices

$$\measuredangle_1 = \begin{pmatrix} (0.95, 0.21) & (0.73, 0.46) & (0.53, 0.71) \\ (0.38, 0.82) & (1, 0) & (0.67, 0.52) \\ (0.28, 0.57) & (0.58, 0.31) & (0.62, 0.79) \\ (0, 1) & (0.91, 0.19) & (0.63, 0.74) \end{pmatrix}$$

and

$$\measuredangle_2 = \begin{pmatrix} (0.54, 0.29) & (0.61, 0.67) & (0.76, 0.02) \\ (0.07, 0.53) & (0.56, 0.11) & (0.39, 0.79) \\ (0.58, 0.17) & (0.36, 0.34) & (0.17, 0.58) \\ (0.21, 0.83) & (0.49, 0.48) & (0.21, 0.87) \end{pmatrix}$$

representing PFSSs $(\measuredangle_1, E)$ and $(\measuredangle_2, E)$, respectively. Now,

$$\begin{aligned} < \measuredangle_1, \measuredangle_2 > &= (0.95, 0.21).(0.54, 0.29) + (0.73, 0.46).(0.61, 0.67) + \cdots + (0.63, 0.74).(0.21, 0.87) \\ &= 6.7180, \\ \| \measuredangle_1 \| &= \sqrt{0.95^2 + 0.21^2 + 0.73^2 + \cdots + 0.74^2} \\ &= 3.1089, \\ \| \measuredangle_2 \| &= \sqrt{0.54^2 + 0.29^2 + 0.61^2 + \cdots + 0.87^2} \\ &= 2.4784. \\ \therefore Sim(\measuredangle_1, \measuredangle_2) &= \frac{< \measuredangle_1, \measuredangle_2 >}{\| \measuredangle_1 \| \| \measuredangle_2 \|} \\ &= \frac{6.7180}{3.1089 \times 2.4784} \\ &= 0.8719. \end{aligned}$$

Example 6.3 Let $X = \{\partial_1, \partial_2, \partial_3\}$ and $E = \{e_1, e_2, e_3\}$. Let

$$\measuredangle_1 = \begin{pmatrix} (0.52, 0.73, 0.44) & (0.89, 0.15, 0.43) & (0.62, 0.59, 0.52) \\ (0.46, 0.73, 0.50) & (1, 0, 0) & (0.51, 0.51, 0.69) \\ (0.32, 0.19, 0.93) & (0.64, 0.27, 0.72) & (0.87, 0.03, 0.49) \end{pmatrix}$$
$$\measuredangle_2 = \begin{pmatrix} (0.68, 0.52, 0.52) & (0.31, 0.69, 0.65) & (0.44, 0.02, 0.90) \\ (0.61, 0.50, 0.61) & (0.33, 0.57, 0.75) & (0.81, 0.16, 0.56) \\ (0.52, 0.28, 0.81) & (0.29, 0.22, 0.93) & (0.21, 0.39, 0.90) \end{pmatrix}$$

be the PFS matrices representing the PFSSs $(\measuredangle_1, E)$ and $(\measuredangle_2, E)$, respectively. Now,

$$\begin{aligned} < \measuredangle_1, \measuredangle_2 > &= (0.52, 0.73, 0.44).(0.68, 0.52, 0.52) + \cdots + (0.74, 0.63, 0.49).(0.35, 0.54, 0.90) \\ &= 7.0581, \\ \| \measuredangle_1 \| &= \sqrt{0.52^2 + 0.73^2 + 0.44^2 + \cdots + 0.49^2} \\ &= 2.9987, \\ \| \measuredangle_2 \| &= \sqrt{0.68^2 + 0.52^2 + 0.31^2 + \cdots + 0.54^2} \\ &= 2.9994 \end{aligned}$$

$$\begin{aligned}\therefore Sim(\measuredangle_1, \measuredangle_2) &= \frac{<\measuredangle_1, \measuredangle_2>}{\parallel \measuredangle_1 \parallel \parallel \measuredangle_2 \parallel}\\ &= \frac{7.0581}{2.9987 \times 2.9994}\\ &= 0.7847.\end{aligned}$$

Example 6.4 Let $X = \{\partial_1, \partial_2, \partial_3\}$ and $E = \{e_1, e_2, e_3\}$. Consider the PFS matrices

$$(\measuredangle_1, E) = \begin{pmatrix} (0.27, 0.39) & (0.42, 0.51) & (0.61, 0.43) \\ (0.25, 0.56) & (0.58, 0.49) & (0.92, 0.36) \\ (0.76, 0.23) & (0.46, 0.48) & (0.54, 0.21) \end{pmatrix}$$
$$(\measuredangle_2, E) = \begin{pmatrix} (0.45, 0.21) & (0.26, 0.89) & (0.54, 0.39) \\ (0.29, 0.28) & (0.46, 0.44) & (0.64, 0.31) \\ (0.27, 0.54) & (0.28, 0.33) & (0.89, 0.16) \end{pmatrix}$$
$$(\measuredangle_3, E) = \begin{pmatrix} (0.93, 0.15) & (0.45, 0.59) & (0.33, 0.14) \\ (0.39, 0.28) & (0.51, 0.55) & (0.64, 0.27) \\ (0.71, 0.32) & (0.33, 0.18) & (0.09, 0.56) \end{pmatrix}$$

Then $Sim(\measuredangle_2, \measuredangle_1) = 0.8925 > 0.82$ and $Sim(\measuredangle_1, \measuredangle_3) = 0.8491 > 0.82$ but $Sim(\measuredangle_2, \measuredangle_3) = 0.8027 \ngtr 0.82$. This advocates that the relation of being similar is not transitive.

Definition 6.5 Two PFSSs $(\measuredangle_1, E_1)$ and $(\measuredangle_2, E_2)$ defined over (X, E) are called λ-*similar*, denoted as $(\measuredangle_1, E_1) \approx^{\lambda} (\measuredangle_2, E_2)$, if $Sim(\measuredangle_1, \measuredangle_2) \geq \lambda$ for some $0 < \lambda < 1$.

Proposition 6.6 *The relation of being λ-similar is reflexive and symmetric, but not transitive.*

Corollary 6.7 *The relation of being λ-similar is not an equivalence relation.*

6.1 Weighted Similarity Measure for PFSSs

In this subsection, we present weighted similarity measure between two PFSSs and give some of its peculiar characteristics.

Definition 6.8 Let $\measuredangle_1$ and $\measuredangle_2$ be as given in Definition 6.1. Assume that the weight of e_j is $w_j \in [0, 1]$ for $j = 1, 2, \cdots, n$. The *weighted similarity measure* between $\measuredangle_1$ and $\measuredangle_2$ is given as

$$Sim_W(\measuredangle_1, \measuredangle_2) = \frac{<\measuredangle_1, \measuredangle_2>}{\parallel \measuredangle_1 \parallel \parallel \measuredangle_2 \parallel},$$

where

$$< \measuredangle_1, \measuredangle_2 > = \frac{\Sigma_{i,j} w_j (\sigma_{ij}, \varrho_{ij})_{\measuredangle_1} . (\sigma_{ij}, \varrho_{ij})_{\measuredangle_2}}{\Sigma_j w_j}$$

$$\| \measuredangle_1 \| = \sqrt{< \measuredangle_1, \measuredangle_1 >}.$$

This weighted similarity measure satisfies the same properties as given in Definition 6.1.

Example 6.9 Consider the PFSSs given by the PFS matrices

$$\measuredangle_1 = \begin{pmatrix} (0.52, 0.73) & (0.89, 0.15) & (0.62, 0.59) \\ (0.46, 0.73) & (1, 0) & (0.51, 0.51) \\ (0.32, 0.19) & (0.64, 0.27) & (0.87, 0.03) \end{pmatrix}$$

$$\measuredangle_2 = \begin{pmatrix} (0.68, 0.52) & (0.31, 0.69) & (0.44, 0.02) \\ (0.61, 0.50) & (0.33, 0.57) & (0.81, 0.16) \\ (0.52, 0.28) & (0.29, 0.22) & (0.21, 0.39) \end{pmatrix}$$

Assume that the weights of the attributes e_1, e_2, and e_3 are $w_1 = 0.52$, $w_2 = 0.31$, and $w_3 = 0.47$, respectively. Then,

$$\begin{aligned} < \measuredangle_1, \measuredangle_2 > &= 1.7672 \\ \| \measuredangle_1 \| &= 1.7020 \\ \| \measuredangle_2 \| &= 1.3479 \\ \therefore Sim_W(\measuredangle_1, \measuredangle_2) &= 0.7703 \end{aligned}$$

6.2 Practical Implementation of Proposed Similarity Measure in Life Sciences

As a model, in this subsection, we employ proposed similarity measure to diagnose whether a person has hepatitis or not. As earlier, we first propose Algorithm 4 before heading towards numerical example where proposed similarity measure may be successfully employed as follows:

Algorithm 4

Step 1: Choose the set $X = \{\eta_1 = \text{hepatitis}, \eta_2 = \text{no hepatitis}\}$.
Step 2: Choose the set of symptoms $E = \{e_1, e_2, \cdots, e_n\}$.
Step 3: Choose a model PFS matrix $(\measuredangle, E)$ with which similarity is to be computed.
Step 4: Choose PFS matrix $(\measuredangle_1, E)$ for the patient.
Step 5: Compute similarity between $(\measuredangle_1, E)$ and $(\measuredangle, E)$.
Step 6: Decide the threshold value $\lambda \in]0, 1[$.
Step 7: The patient is diseased if $Sim(\measuredangle, \measuredangle_1) \geq \lambda$.

Example 6.10 Presume that $X = \{\eta_1 = \text{hepatitis}, \eta_2 = \text{no hepatitis}\}$. Let's choose the set of parameters containing the collection of some detectible symptoms, say, $E = \{e_i : i = 1, 2, \cdots, 5\}$, where

$$\begin{aligned} e_1 &= \text{vomiting,} \\ e_2 &= \text{jaundice,} \\ e_3 &= \text{light/clay-colored stool,} \\ e_4 &= \text{abdominal discomfort, and} \\ e_5 &= \text{dark urine.} \end{aligned}$$

The PFS matrix $(\measuredangle, E)$ over X for hepatitis is given as under, which may be constructed with the aid of clinical/medical experts:

$$(\measuredangle, E) = \begin{pmatrix} (0.62, 0.47) & (0.36, 0.57) \\ (0.89, 0.41) & (0.27, 0.93) \\ (0.58, 0.25) & (0.31, 0.54) \\ (0.51, 0.62) & (0.49, 0.38) \\ (0.63, 0.45) & (0.53, 0.41) \end{pmatrix}$$

The PFS matrix $(\measuredangle_1, E)$ over X for hepatitis based upon an ill person is given as follows:

$$(\measuredangle_1, E) = \begin{pmatrix} (0.11, 0.07) & (0.92, 0.15) \\ (0.14, 0.05) & (0.86, 0.26) \\ (0.08, 0.96) & (0.57, 0.02) \\ (0.36, 0.69) & (0.83, 0.19) \\ (0.46, 0.37) & (0.29, 0.84) \end{pmatrix}$$

Let's decide the threshold value $\lambda = 0.75$. The similarity measure between $(\measuredangle, E)$ and $(\measuredangle_1, E)$ is $Sim(\measuredangle, \measuredangle_1) = 0.6497 < \lambda$, so we conclude that the person does not seem to be victim of hepatitis.

7 Conclusion

We studied some elementary notions of Pythagorean fuzzy soft sets in this chapter. Some fundamental operations and their prime characteristics are also examined with the assistance of elaborative examples. We proposed four algorithms, i.e., choice value method, PFS-TOPSIS, VIKOR, and method of similarity measures, for modeling uncertainties in MADM problems based upon PFSSs. The proposed Algorithms have been efficaciously applied on ranking different alternatives. To comprehend the final rankings, we have made use of statistical charts. The proposed models have tremendous potential for further exploration in theoretical besides application perspective and may be efficiently applied in other hybrid structures of fuzzy sets including Pythagorean m-polar fuzzy sets, Pythagorean m-polar fuzzy soft sets, q-rung fuzzy soft sets, neutrosophic soft sets, and Pythagorean fuzzy parameterized soft sets, etc. with slight amendments. The ideas may be efficiently employed in handling uncertainties in different sectors of real-life situations including business, artificial intelligence, marketing, shortest route problem, image processing, electoral system, pattern recognition, machine learning, medical diagnosis, trade analysis, game theory, forecasting, agri-business analysis, robotics, coding theory, recruitment problems, and many other problems.

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