Chapter 6
Set Order Relations, Set Optimization, and Ekeland's Variational Principle

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Abstract This chapter provides a brief survey on different kinds of set order relations which are used to compare the objective values of set-valued maps and play a key role to study set optimization problems. The solution concepts of set optimization problems and their relationships with respect to different kinds of set order relations are provided. The nonlinear scalarization functions for vector-valued maps as well as for set-valued maps are very useful to study the optimality solutions of vector optimization/set optimization problems. A survey of such nonlinear scalarization functions for vector-valued maps/set-valued maps is given. We give some new results on the existence of optimal solutions of set optimization problems. In the end, we gather some recent results, namely, Ekeland's variational principle and some equivalent variational principle for set-valued maps with respect to different kinds of set order relations.

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6.1 Introduction

An optimization problem whose objective function is a set-valued map is known as set optimization or set-valued optimization problem.

Let S be a nonempty subset of a vector space X, Y be a topological vector space, and $F : S \rightrightarrows Y$ be a set-valued map with nonempty values. The *set optimization problem* is defined as follows:

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$$\begin{aligned} &\min F(x) \\ &\text{subject to } x \in S. \end{aligned} \tag{SOP}$$

The study of such problems is known as set optimization or set-valued optimization. Since the set-valued maps include single-valued maps as well as vector-valued maps, the set optimization can be considered as an extension of scalar optimization and/or vector optimization. Since the middle of eighties, the theory of set optimization has received increasing interest in the optimization community and many authors have studied and investigated set optimization problems due to its extensive applications in different branches of applied mathematics, engineering, economics, finance, and medical sciences. Note that several problems from game theory [73], multivariate statistics [70, 106], radiotherapy treatment (medical image registration, intensity-modulated radiation therapy) [42, 104, 120], uncertain optimization [7, 105], welfare economics [16], socio-economics [132], mathematical finance [36, 69], optimal control [75], etc. can be written in the form of mathematical formulation of set optimization problems. Not only this, the robust optimization problems and stochastic/fuzzy programming problems can also be modeled as set optimization problems. For an overview and further detailed investigations, we refer to the books [68, 97].

Let us consider and distinguish simple examples of scalar optimization problem, vector optimization problem, and set optimization problem.

- To find the fastest bowler from a set of cricket players is the scalar optimization problem (maximization problem) where the objective function gives the speed of a player.
- To find the bowler(s) from a set of cricket players in such a way that he/she (they) is (are) having several qualities, namely, speed, in swing/out swing, etc., is a vector optimization problem. Consider an objective function from a set of players to the set of all such qualities, that is, the value of the objective function can be regarded as a vector whose coordinates consist of one's ability, speed, in swing/out swing, etc. In other words, the objective function is vector-valued.
- Consider the objective function whose values are teams and assume that a team is a set of players and each player is regarded as a vector whose coordinates consist of one's ability, speed, in swing/out swing, popularity, and so on. Then one can formulate the problem of choosing a good team for a cricket league in the form of set optimization problem with the objective function defined as above.

For a set optimization problem, it seems natural the first thing that has to be done is to decide how to define the solution of a set optimization problem. There are two popular approaches to define the solution concepts of a set optimization problem: one is the vector approach and another is the set approach.

In vector approach, one directly generalizes the concepts known from vector optimization to set optimization, that is, we try to find the best element, in some sense, of the union of all image sets of the set-valued objective map over the feasible set. In other words, in vector approach, a minimizer $(\bar{x}, \bar{y})$ depends on only certain special element $\bar{y}$ of $F(\bar{x})$ and other elements of $F(\bar{x})$ are ignored. That is, an element $\bar{x} \in S$

for which there exists at least one element $\bar{y} \in F(\bar{x})$ which is Pareto minimal point of the image set of F even if there exists many bad elements in $F(\bar{x})$ is a solution of the set optimization problem (SOP). The set optimization problems with vector approach have been studied and investigated by Corley [33, 34]; Luc [128]; Lin [123]; Jahn and Rauh [93]; Chen and Jahn [29]; Götz and Jahn [55]; Li [122]; Crespi, Ginchev, and Rocca [35], Alonso and Rodríguez-Marín [2], Hernández, Rodríguez-Marín and Sama [84], Hernández [79], etc. For more detail, we refer to the books [90, 97], the survey papers [41, 64, 79] and the references therein. Of course, vector optimization problems provide a very important special case of set optimization with numerous applications. Moreover, the answer to certain problems in vector optimization can be found, if the vector optimization problem is considered in a set-valued framework, see [67]. Note that the solution concept based on vector approach is of mathematical interest but it can not be often used in practice. This solution concept is not suitable to deal with the set optimization problem defined in the above example. For example, we can see that a team which has at least one good player is a solution, though most of the members of such teams are useless. Is it true that such team can achieve good results?

These solutions must be almost invalid and improper. This drawback gave birth to the set approach which is based on the comparison of values (sets) of objective set-valued map, that is, using the set approach, the sets $F(x)$ are compared by using some kinds of set order relations with the aim to choose the best one in some sense. The credit for the birth of set approach goes to Kuroiwa [109]. To resolve this problem, Kuroiwa [109] introduced six kinds of set order relations which are further studied and investigated in [1, 25, 72, 79–82, 84, 110, 111, 115, 132] and the references therein. Note that these set order relations were independently introduced in different fields, for example, in terms of algebraic structures by Young [150] in 1931, in the theory of fixed points of monotonic operators by Nishnianidze [134] in 1984, in interval arithmetic by Chiriaev and Walster [31] in 1998, and in theoretical computer science by Brink [22] in 1993. In 2011, Jahn and Ha [92] introduced the, so-called, minmax set order relations to deal with the solutions of the problem (SOP) where the above mentioned six kinds of set order relations fail. Since the notion of the set approach was introduced, there has been rapid growth in this field. On the contrary, the main disadvantage of the set approach over the vector approach is the loss of lineal structure. Hamel [67] studied the structure of the power set of Y by introducing a conlineal space. In order to avoid such a problem, several authors have considered specializations of F or tools to study the problem (SOP) via a structure well known or simpler than a conlineal space. For instance, Hernández [80] characterized the solutions of the problem (SOP) via nonlinear scalarization, see also [13, 129]. Kuroiwa and Nuriya [114] constructed an embedding vector space. Maeda's [130] work on n-dimensional Euclidean spaces shows that whenever the set-valued map is rectangle-valued (SOP), then it is equivalent to a pair of vector-valued optimization problems.

In general, there is no relation among solutions of the problem (SOP) obtained by vector approach and solutions obtained by set approach. Moreover, the existence of solutions by one approach does not imply the existence of solutions of the other approach, see [1, 63, 84, 99] and the references therein. Even though both criteria are different but under certain assumptions, the relation among solution concepts of the problem (SOP) with vector approach and set approach has been studied in [1, 79, 84, 129, 130] and the references therein.

In 2017, Chen et al. [30] introduced a set order relation called weighted set order relation. This weighted set order relation is the combination of Kuroiwa's [109] upper and lower set order relations. So, under some assumptions, this new set order relation is more general than Kuroiwa's upper and lower set order relations. It is useful for formulating solution concepts for researchers who do not specifically rely on either the upper or lower set order relation. Recently, Ansari et al. [6] studied Ekeland's variational principle and some equivalent results for set-valued maps by using weighted set order relations and gave some applications to order intervals.

In 2018, Karaman et al. [96] introduced set order relations on the family of sets based on the Minkowski difference. In comparison to Kuroiwa's set order relations, these set order relations are partial ordered on the family of bounded sets, and hence provide a new approach to study set optimization problems. Khushboo and Lalitha [99] studied the relationship among different kinds of solution sets of set optimization problems defined by means of Kuroiwa's set order relations and Karaman's set order relations. They also investigated that the solution sets of a set optimization problem defined by different kinds of set order relations are different. Therefore, it is interesting and important to investigate the set optimization problems by using Karaman's set order relations. Very recently, the set optimization problems have been investigated and studied in [12, 94–96, 99, 139] by using Karaman's set order relations.

Besides the set order relations with fixed ordering cone, the interest in set order relations with variable cone has increased during the last years due to some applications in different problems, see [7, 10, 20, 40, 42–44, 101, 102, 104, 105, 119, 120] and references therein. Therefore, in the order relations defined by convex cone to compare sets, the cone is replaced by a variable domination structure. This variable domination structure is defined by a set-valued map, called ordering map, which associates with each element of the space an individual set of preferred or dominated directions. In 2016, Eichfelder and Pilecka [42, 44] introduced the set order relations equipped with variable domination structures. They provided scalarization results for obtaining optimality conditions for the solutions of the problem (SOP). Further, Köbis [101, 102] introduced new set order relations equipped with variable domination structures and differentiated between a concept of domination and preference. In the recent years, set optimization problems with respect to variable domination structures have been studied and investigated in [7, 10, 20, 42, 44, 101, 102, 104, 119] and the references therein.

In the recent years, the set order relations has played an important role to deal with several problems from nonlinear analysis and optimization with set-valued maps, for instance, Ekeland's variational principle and related results [12, 13, 63, 72], continuity and convexity of set-valued maps [115, 117], minimax theorem for set-valued maps [116], well-posedness [62], stabilty [77], connectedness [78], concepts of efficiency for uncertain multi-objective optimiztion [88], optimality notions for (SOP) [1, 84, 94], and so on.

There are various techniques to deal with the set optimization problems, for instance, scalarization, vectorization, etc., see [13, 14, 72, 80, 91, 99, 151] and the references therein. One of the most and widely used techniques to deal with set optimization problems is the scalarization by which we can convert a set optimization problem into a scalar optimization problem, that is, by using scalarization, set optimization problem is replaced by a family of scalar optimization problems which allow to relate the solutions of both problems and solve the set optimization problem by a numerical method applicable for the scalar problems. To study set optimization problems, scalarization functions are one of the most essential tools from a theoretical as well as computational point of view. Several scalarization techniques for set optimization problems are available in the literature. Most of them are based on Gerstewitz function [54], oriented distance function [86], or their extensions [11, 13, 14, 72, 80, 96, 99, 151]. The original idea of the nonlinear scalarization functions was given by Krasnosel'skij [107] and Rubinov [140]. Krasnosel'skij [107] used them in order to establish necessary and sufficient conditions for a cone to be normal. Also, these types of functionals have been used in theoretical investigations within the framework of ordered linear spaces, see the book [38] by M. M. Day as an elegant tool for proof of the fact that the Hahn–Banach extension and a linear closure property imply the interpolation property. Furthermore, Feldman [48] and Rubinov [140] investigated the dual properties of such kinds of functionals, namely, their so-called support sets. The nonlinear scalarization functional for vector optimization with its concrete definition was given by Tammer (Gerstewitz) [52] in 1983 and applied to study separation theorems for not necessary convex sets by Tammer (Gerstewitz) and Iwanow [53] in 1985. Such nonlinear scalarization functions are now known as Gerstewitz nonlinear scalarization functional. Luc [125–127] also gave early contributions to this topic. On the other hand, Hiriart-Urruty [86] introduced the notion of oriented distance function to study optimality conditions of nonsmooth optimization problems from the geometric point of view. For more details on oriented distance function and their extensions, we refer [7, 8, 35, 60, 99, 151, 152] and the references therein.

The idea of nonlinear scalarization for sets was first investigated in 2000 by Tanaka–Georgiev [51]. In 2006, Hamel and Löhne [72] extended the above functions to two different functions on a power set of Y corresponding to the set order relations. Further, Hernández and Rodríguez-Marín, [80] investigated nonlinear scalarizing functions for sets by introducing cone-topological concepts, see [9]. Furthermore, in 2009, Knwano–Tanaka–Yamada [118] introduced a unified approach for such scalarizations for sets using Kuroiwa's set order relations. In the recent past, Araya [13, 14] investigated six types of nonlinear scalarizing functions for set-valued maps and their relationships. In the literature, expressions using inf–sup of the Gerstewitz function can be found in [60, 61] which were used to study necessary and sufficient optimality conditions in set optimization problems with set order relations. Khoshkhabar-amiranloo et al. [98] and Sach [141] also introduced slightly different nonlinear scalarization functions to study set optimization problmes. Recently, Karaman et al. [96] introduced nonlinear scalarization functions by using the set order relations defined by the Minkowski difference and studied optimality notions for (SOP). Very recently, Ansari et al. [6] introduced the notions of nonlinear scalarization functions by using weighted set order relations. Several applications of the nonlinear scalarization functions can be found in the literature, for instance, to study Ekeland's variational principle and related variational principle [6, 12, 13, 63, 65, 72]; nonconvex separation type theorems [13, 14, 54]; Gordan's type alternative theorems [13, 135]; equilibrium problems [9, 56]; minimax theorems [116]; vector variational inequalities [9, 56]; robustness and stochastic programming [100]; and stability and well-posedness [62, 77]. In the recent years, several authors have studied and investigated nonlinear scalarizing technique for set optimization problem, see [6, 12, 60, 61, 72, 97, 98, 141] and their references therein.

In recent years, scalarization functions with variable domination structures also gained increasing interest in the optimization community. Eichfelder and Pilecka [44] introduced a nonlinear scalarization function when the images of ordering maps are Bishop Phelps cones. Further, Köbis et al. [104] and Ansari et al. [7] introduced nonlinear scalarizing methods to characterize several set order relations and minimal solutions for set optimization problems equipped with variable domination structures with their applications in medical image registration and uncertain multi-objective optimization and to derive necessary optimality conditions for solutions of set optimization problems with respect to variable domination structures. Very recently, Kobis et al. [105] introduced a new nonlinear scalarization functional in set optimization equipped with variable domination structures, which are further studied by Ansari and Sharma [10] to obtain Ekeland's variational principle. For more details on scalarization functions with respect to variable domination structures, we refer [7, 10, 44, 104, 105] and the references therein.

The present chapter is organized as follows: In the next section, we recall some definitions and concepts which will be used in the sequel. In Sect. 6.3, we gather different kinds for set order relations with their properties. The relationships among these set order relations are provided along with theoretical and geometrical illustrations. In Sect. 6.4, a survey of nonlinear scalarization functions for vector-valued maps/set-valued maps is given. Such nonlinear scalarization functions for vector-valued maps as well as for set-valued maps are very useful to study the optimality solutions of vector optimization/set optimization problems and to study some set order relations. In Sect. 6.5, solution concepts for set optimization problems based on vector approach and set approach and relations among them are given. Several examples are given to illustrate each type of solution concept. Some new results on the existence of optimal solutions of set optimization problems are given in Sect. 6.6. In the last section, we investigate Ekeland's variational principle for set-valued maps in different settings and also by using different kinds of set order relations. Further, we investigate some other equivalent variational principles, namely minimal element theorem, Takahashi minimization theorem, and Caristi fixed point theorem for set-valued maps.

6.2 Preliminaries

Throughout the chapter, all vector spaces are assumed to be defined over the field of real numbers, and we adopt the following notations, unless otherwise specified.

We denote by $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ the set of all natural numbers, the set of all rational numbers, and the set of all real numbers, respectively, and $\mathbb{R}_+ = [0, \infty)$. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by $\mathbb{R}^n_+$ the nonnegative orthant in $\mathbb{R}^n$. The zero element in a vector space will be denoted by $\mathbf{0}$. Let Y be a topological vector space with its topological dual Y^*. We denote by 2^Y (respectively, $P(Y)$) and $\mathcal{B}(Y)$ the family of all (respectively, nonempty) subsets of Y and the family of all nonempty bounded subsets of Y, respectively. For a set $A \subseteq Y$, we denote by int$A$, $\overline{A}$ or clA, ∂A, and A^c, the interior, the closure, the boundary, and the complement of A, respectively.

For arbitrary nonempty sets X and Y, we denote by P_X and P_Y, the projection of $X \times Y$ onto X and Y, respectively, that is,

$$P_X(x, y) = x \quad \text{and} \quad P_Y(x, y) = y, \quad \text{for all } (x, y) \in X \times Y.$$

A function $F : X \to 2^Y$ is said to be a *set-valued map*, and it is denoted by $F : X \rightrightarrows Y$. For the set-valued map $F : X \rightrightarrows Y$, the *image* of F at $x \in X$ is a subset $F(x)$ of Y. The *domain* of F is

$$\text{dom } F = \{x \in X : F(x) \neq \emptyset\},$$

and the *image* of F is

$$\text{Im } F = \{y \in Y : \text{ there exists } x \in X \text{ such that } y \in F(x)\}.$$

The set-valued map $F : X \rightrightarrows Y$ can be identified by its *graph* which is defined as

$$\text{graph } F = \{(x, y) \in X \times Y : x \in X, \ y \in F(x)\}.$$

The *image* of the set $S \subseteq X$ under F is

$$F(S) := \bigcup_{x \in S} F(x),$$

so, Im $F = F(X)$. The set

$$\text{Graph } F = \{(x, V) \in X \times P(Y) : V = F(x)\}$$

is designated as *graph* of F by Hamel and Löhne [72].

A subset C of a vector space Y is said to be a *cone* if for all $x \in C$ and $\lambda \geq 0$, we have $\lambda y \in C$. The set C of Y is called a *convex cone* if it is convex and a cone, that is, for all $x, y \in C$ and $\lambda, \mu \geq 0$, we have $\lambda x + \mu y \in C$.

Definition 6.1 A cone C in Y is said to be

(a) *solid* if it has nonempty interior, that is, $\text{int}C \neq \emptyset$;
(b) *nontrivial* or *proper* if $C \neq \{\mathbf{0}\}$ and $C \neq Y$;
(c) *reproducing* if $C - C = Y$;
(d) *pointed* if for $\mathbf{0} \neq x \in C$, we have $-x \notin C$, that is, $C \cap (-C) = \{\mathbf{0}\}$;
(e) *closed cone* if it is also closed.

The *dual* of a cone $C \subseteq Y$ is defined by

$$C^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \text{ for all } y \in C\},$$

where $\langle y^*, y\rangle$ denotes the value of the functional y^* at y.

The convex cone $C \subseteq Y$ induces an ordering on Y as

$$x \preceq_C y \;\Leftrightarrow\; y - x \in C, \quad \text{for all } x, y \in Y.$$

If $\text{int}C \neq \emptyset$, then we have

$$x \prec_C y \;\Leftrightarrow\; y - x \in \text{int}C, \quad \text{for all } x, y \in Y.$$

Further, if C is pointed, then the ordering $\preceq_C$ is a partial ordering on Y.

Note that there is a one-to-one correspondence between an ordering and a convex cone (see [9]).

Definition 6.2 Let $A, B \in P(Y)$.

- The *algebraic sum* of A and B is defined as

$$A + B := \{a + b : a \in A,\ b \in B\}.$$

- The *algebraic difference* of A and B is defined as

$$A - B := \{a - b : a \in A,\ b \in B\}.$$

- The *Minkowski* (*Pontryagin*) *difference* of A and B is defined as

$$A \dot{-} B := \{y \in Y : y + B \subseteq A\} = \bigcap_{b \in B}(A - b).$$

- For $\lambda \in \mathbb{R}$, $\lambda A := \{\lambda x : x \in A\}$.

It is worth to mention that the set equation $A + A = 2A$ does not hold in general for a nonempty subset A of a vector space. The Minkowski difference of a set and a vector coincides with their algebraic difference, that is, $A \dot{-} a = A - a$ for all $A \in P(Y)$ and $a \in Y$.

Note that the *Minkowski (Pontryagin) difference* plays a very important role in many applications such as robot motion planning [124], morphological image analysis [143], and computer-aided design and manufacturing [121]. For further details on *Minkowski (Pontryagin) difference*, we refer to the book [136].

The following example illustrates different types of set addition and set difference.

Example 6.1 Let $A = [-1, 1] \times [-1, 1]$ and $B = [-1, 0] \times [-1, 0]$. Then,

$$A + B = [-2, 1] \times [-2, 1], \quad A - B = [-1, 2] \times [-1, 2], \quad \text{and} \quad A \dot{-} B = [0, 1] \times [0, 1].$$

See, Fig. 6.1 for an illustration of the sets A, B, $A + B$, $A - B$, and $A \dot{-} B$.

We present some basic properties of the *Minkowski (Pontryagin) difference*.

Proposition 6.1 [96] *Let Y be a normed space, $A, B \in P(Y)$, and $\alpha \in Y$. The following assertions hold.*

(a) $(\alpha + A) \dot{-} B = \alpha + (A \dot{-} B)$.
(b) $A \dot{-} (\alpha + B) = -\alpha + (A \dot{-} B)$.
(c) *If A is closed, then $A \dot{-} B$ is also closed.*
(d) *If A is bounded, then $A \dot{-} A = \{\mathbf{0}\}$.*

Definition 6.3 [9, 127] Let C be a closed convex cone in Y. A nonempty subset A of Y is said to be

(a) *C-proper* if $A + C \neq Y$;
(b) *C-closed* if $A + C$ is a closed set;
(c) *C-bounded* if, for each neighborhood U of $\mathbf{0} \in Y$, there is a positive number t such that $A \subset tU + C$;
(d) *C-compact* if each cover of A of the form $\{U_\lambda + C : U_\lambda \text{ is an open set}, \lambda \in \Lambda\}$ admits a finite subcover, where Λ denotes the index set.

Clearly, if C is a closed convex cone, so is $-C$. The replacement of C by $-C$ in the above definition produces $(-C)$-closed, $(-C)$-bounded, etc. For more detail and examples on cone-topological concepts, we refer to [9, 127].

We denote by Ω_C the family of all C-proper subsets of Y and by Ω_C^{cb} the family of all nonempty, C-proper, closed, and bounded subsets of Y.

6.3 Set Order Relations

This section deals with different kinds of set order relations to study set optimization problems.

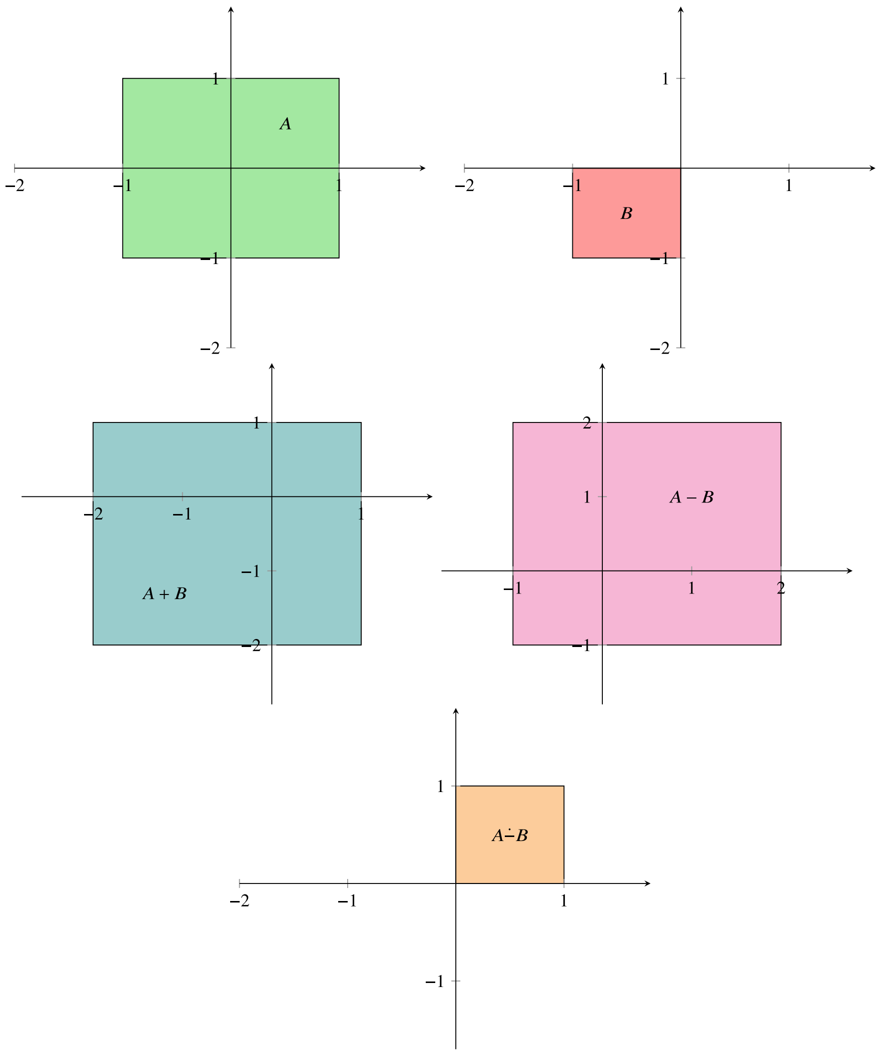


Fig. 6.1 Visualization of sets A, B, $A + B$, $A - B$, and $A \dot{-} B$

Definition 6.4 [92, 115] Let Y be a topological vector space, $A, B \in P(Y)$, and C be a proper convex cone in Y. The *set order relations* on $P(Y)$ with respect to C are defined as follows:

(a) The *lower set less order relation* $\preceq_C^l$ is defined by

$$A \preceq_C^l B \quad \Leftrightarrow \quad B \subseteq A + C,$$

or equivalently, for all $b \in B$, there exists $a \in A$ such that $a \preceq_C b$.

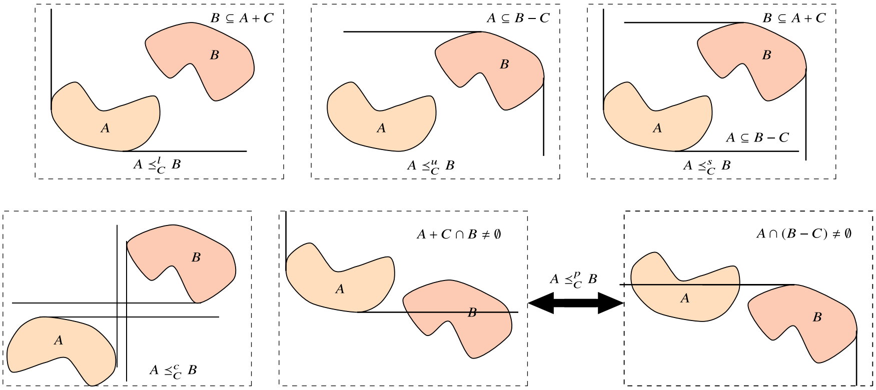


Fig. 6.2 Illustration of set order relations in $\mathbb{R}^2$ with $C = \mathbb{R}^2_+$

(b) The *upper set less order relation* $\preceq^u_C$ is defined by

$$A \preceq^u_C B \quad \Leftrightarrow \quad A \subseteq B - C,$$

or equivalently, for all $a \in A$, there exists $b \in B$ such that $a \preceq_C b$.

(c) The *set less order relation* $\preceq^s_C$ is defined by

$$A \preceq^s_C B \quad \Leftrightarrow \quad B \subseteq A + C \text{ and } A \subseteq B - C,$$

or equivalently, for all $b \in B$, there exists $a \in A$ such that $a \preceq_C b$, and for all $a \in A$, there exists $b \in B$ such that $a \preceq_C b$.

(d) The *certainly set less order relation* $\preceq^c_C$ is defined by

$$A \preceq^c_C B \Leftrightarrow (A = B) \text{ or } (A \neq B, \text{ for all } b \in B, \text{ for all } a \in A \text{ such that } a \preceq_C b),$$

or equivalently, $A = B$, or $B - A \subset C$ whenever $A \neq B$.

(e) The *possibly set less order relation* $\preceq^p_C$ is defined by

$$A \preceq^p_C B \quad \Leftrightarrow \quad \text{there exists } b \in B, \text{ there exists } a \in A \text{ such that } a \preceq_C b,$$

which is equivalent to $A \cap (B - C) \neq \emptyset$ or $B \cap (A + C) \neq \emptyset$.

When the cone C has nonempty interior, that is, int$C \neq \emptyset$, then we can define the corresponding weak set order relations $\prec^\alpha_C$, $\alpha \in \{l, u, s, c, p\}$ as the relations $\preceq^\alpha_C$, $\alpha \in \{l, u, s, c, p\}$ by replacing C with int$C$.

For instance, the weak lower set less order relation $\prec^l_C$ is defined as

$$A \prec^l_C B \quad \Leftrightarrow \quad B \subseteq A + \text{int}C,$$

and the weak upper set less order relation $\prec^u_C$ is defined as

$$A \prec_C^u B \quad \Leftrightarrow \quad A \subseteq B - \text{int}C.$$

Note that the set less order relation $\preceq_C^s$ has been independently introduced by Young [150] and Nishnianidze [134]. Chiriaev and Walster [31] used the set order relations $\preceq_C^s$, $\preceq_C^c$, and $\preceq_C^p$ in the interval arithmetic and implemented in the FORTRAN compiler f95 of SUN Microsystems [146]. These set order relations have been presented by Kuroiwa [115] in the modified form as defined in Definition 6.4. See Fig. 6.2 for an illustration of these set order relations.

The following proposition gives the properties of the set order relations defined as above.

Proposition 6.2 [92]

(a) *The set order relations $\preceq_C^l$, $\preceq_C^u$, and $\preceq_C^s$ are pre-order and compatible with respect to addition and scalar multiplication on $P(Y)$.*
(b) *The set order relation $\preceq_C^c$ is a pre-order and compatible with respect to addition and scalar multiplication on $P(Y)$. If the ordering cone C is pointed, then the set order relation $\preceq_C^c$ is antisymmetric and hence, a partial order relation.*
(c) *The set order relation $\preceq_C^p$ is reflexive and compatible with respect to addition and scalar multiplication on $P(Y)$. In general, it is not transitive and not antisymmetric.*
(d) *In general, the set order relations $\preceq_C^l$, $\preceq_C^u$, and $\preceq_C^s$ are not antisymmetric. More precisely, for arbitrary sets $A, B \in P(Y)$, we have*
    (1) $(A \preceq_C^l B$ *and* $B \preceq_C^l A) \Leftrightarrow A + C = B + C;$
    (2) $(A \preceq_C^u B$ *and* $B \preceq_C^u A) \Leftrightarrow A - C = B - C;$
    (3) $(A \preceq_C^s B$ *and* $B \preceq_C^s A) \Leftrightarrow (A + C = B + C$ *and* $A - C = B - C).$

Remark 6.1 The pointedness of the cone C in Proposition 6.2(b) cannot be relaxed. Indeed, let $Y = \mathbb{R}^2$ and $C = \mathbb{R} \times \{\mathbf{0}\}$. Then C is not pointed. For $A = [-1, 1] \times \{\mathbf{0}\}$ and $B = [3, 5] \times \{\mathbf{0}\}$, we have $A \preceq_C^c B$ and $B \preceq_C^c A$ but $A \neq B$.

The following example shows that the set order relation $\preceq_C^p$ is in fact not transitive and not antisymmetric.

Example 6.2 Let $Y = \mathbb{R}^2$ and $C = \mathbb{R}_+^2$. Consider the sets

$$\begin{aligned} A_1 &= \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 2^2, y_1 \geq 0, y_2 \geq 0\}, \\ A_2 &= \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - 3)^2 + (y_2 + 1)^2 \leq 1\}, \\ A_3 &= \text{conv}\{(4, -2), (6, -2), (6, -4)\}, \end{aligned}$$

where conv denotes the convex hull. One can easily see from Fig. 6.3 that

$$A_1 \preceq_C^p A_2 \quad \text{and} \quad A_2 \preceq_C^p A_1 \quad \text{but} \quad A_1 \neq A_2,$$

and

$$A_1 \preceq_C^p A_2 \quad \text{and} \quad A_2 \preceq_C^p A_3 \quad \text{but} \quad A_1 \npreceq_C^p A_3.$$

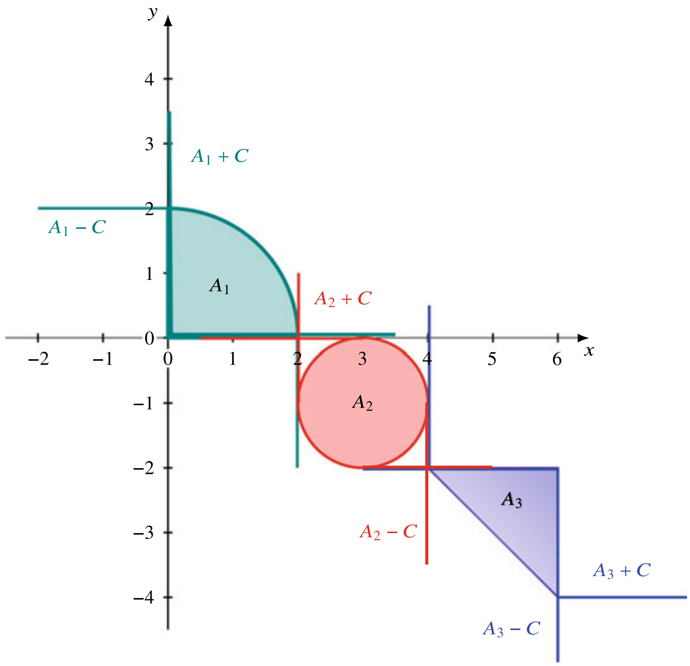


Fig. 6.3 Visualization of Example 6.2 with $C = \mathbb{R}^2_+$

We have the following relation between the lower set less order relation $\preceq^l_C$ and the upper set less order relation $\preceq^u_C$:

$$A \preceq^l_C B \;\Leftrightarrow\; B \subseteq A + C \;\Leftrightarrow\; B \subseteq A - (-C) \;\Leftrightarrow\; B \preceq^u_{-C} A \;\Leftrightarrow\; (-B) \preceq^u_C (-A).$$

Similarly,

$$A \prec^l_C B \;\Leftrightarrow\; B \subseteq A + \mathrm{int}C \;\Leftrightarrow\; B \subseteq A - (-\mathrm{int}C) \;\Leftrightarrow\; B \prec^u_{-C} A \;\Leftrightarrow\; (-B) \prec^u_C (-A).$$

Proposition 6.3 [92] *Let $A, B \in P(Y)$ with $A \neq B$. Then,*

(a) $A \preceq^s_C B \Rightarrow A \preceq^l_C B \Rightarrow A \preceq^p_C B$;
(b) $A \preceq^s_C B \Rightarrow A \preceq^u_C B \Rightarrow A \preceq^p_C B$;
(c) *$A \preceq^l_C B$ does not always imply $A \preceq^u_C B$, and $A \preceq^u_C B$ does not always imply $A \preceq^l_C B$.*

The following example shows that the implications in Proposition 6.3 are strict, that is, the converse implications do not hold.

Example 6.3 Let $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Consider the sets

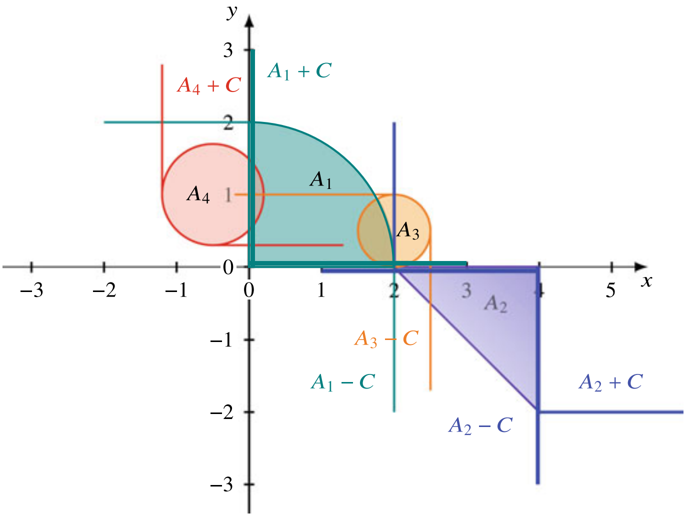


Fig. 6.4 Visualization of Example 6.3 with $C = \mathbb{R}^2_+$

$$
\begin{aligned}
A_1 &= \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 2^2, y_1 \geq 0, y_2 \geq 0\},\\
A_2 &= \text{conv}\{(2,0), (4,0), (4,-2)\},\\
A_3 &= \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - 2)^2 + (y_2 - 0.5)^2 \leq 0.5^2\},\\
A_4 &= \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 + 0.5)^2 + (y_2 - 1)^2 \leq 0.7^2\}.
\end{aligned}
$$

From Fig. 6.4, it can be easily visualized that

$$A_1 \preceq^p_C A_2 \quad \text{but} \quad A_1 \not\preceq^l_C A_2 \quad \text{and} \quad A_1 \not\preceq^u_C A_2,$$

$$A_1 \preceq^l_C A_3 \quad \text{but} \quad A_1 \not\preceq^u_C A_3 \quad \text{and hence} \quad A_1 \not\preceq^s_C A_3,$$

and

$$A_4 \preceq^u_C A_1 \quad \text{but} \quad A_4 \not\preceq^l_C A_1 \quad \text{and hence} \quad A_4 \not\preceq^s_C A_1.$$

Let us illustrate the set order relations by the following example with order intervals.

Example 6.4 [92] Let $a_1, a_2, b_1, b_2 \in Y$ be arbitrarily given with $a_1 \preceq_C a_2$ and $b_1 \preceq_C b_2$, and consider the intervals

$$A = [a_1, a_2] := \{y \in \mathbb{R} : a_1 \preceq_C y \preceq_C a_2\}$$

and

$$B = [b_1, b_2] := \{y \in \mathbb{R} : b_1 \preceq_C y \preceq_C b_2\}.$$

(a) $[a_1, a_2] \preceq_C^s [b_1, b_2] \Leftrightarrow a_1 \preceq_C b_1$ and $a_2 \preceq_C b_2$.
(b) $[a_1, a_2] \preceq_C^c [b_1, b_2] \Leftrightarrow a_2 \preceq_C b_1$.
(c) $[a_1, a_2] \preceq_C^p [b_1, b_2] \Leftrightarrow a_1 \preceq_C b_2$.
(d)

$$A \preceq_C^s B \Leftrightarrow \begin{cases} \min A \in \min B - C, & \min B \in \min A + C, \\ \max A \in \max B - C, & \max B \in \max A + C, \end{cases}$$
$$\Leftrightarrow \quad \min B - \min A \in C \text{ and } \max B - \max B \in C.$$

and

$$A \preceq_C^c B \Leftrightarrow \min B \in \max A + C \text{ and } \max A \in \min B - C,$$

where $\min A := \{a \in A : A \cap (a - C) = \{a\}\}$ and $\max A := \{a \in A : A \cap (a + C) = \{a\}\}$ are the sets of minimal elements and maximal elements, respectively, with respect to the convex pointed cone C.

This observation was one of the motivations to Jahn and Ha [92] to introduce new set order relations involving minimal and maximal elements.

From a practical point of view, the set order relations $\preceq_C^s$ and $\preceq_C^c$ are more appropriate in applications than the other set order relations. In the case of order intervals, the set order relations $\preceq_C^s$ and $\preceq_C^c$ are described by a pre-order of the minimal and maximal elements of these intervals. But for general nonempty sets A and B, which possess minimal elements and maximal elements, this property may not be fulfilled. The following figure illustrates two sets $A, B \in P(Y)$ with $A \preceq_C^s B$ and the properties $\max A \subseteq \max B - C$ but $\max B \nsubseteq \max A + C$. This means that there may be elements $b \in \max B$ and $a \in \max A$ which are not comparable with respect to the pre-order (see Fig. 6.5). In order to avoid this drawback, Jahn and Ha [92] defined new set order relations involving the minimal and maximal elements of a set. This leads to various definitions of "minmax less" set order relations. For further details, see [92].

We denote by $\Xi := \{A \in P(Y) : \min A \neq \emptyset \text{ and } \max A \neq \emptyset\}$, where $\min A := \{a \in A : A \cap (a - C) = \{a\}\}$ and $\max A := \{a \in A : A \cap (a + C) = \{a\}\}$ are the sets of minimal elements and maximal elements, respectively, with respect to the convex pointed cone C in a topological vector space Y.

Definition 6.5 [92] Let $A, B \in \Xi$ and C be a proper, convex, and pointed cone in a toplogical vector space Y. The minmax set order relations on Ξ with respect to C are defined as follows:

(a) The *minmax set less order relation* $\preceq_C^m$ is defined by

$$A \preceq_C^m B \quad \Leftrightarrow \quad \min A \preceq_C^s \min B \text{ and } \max A \preceq_C^s \max B.$$

(b) The *minmax certainly set less order relation* $\preceq_C^{mc}$ is defined by

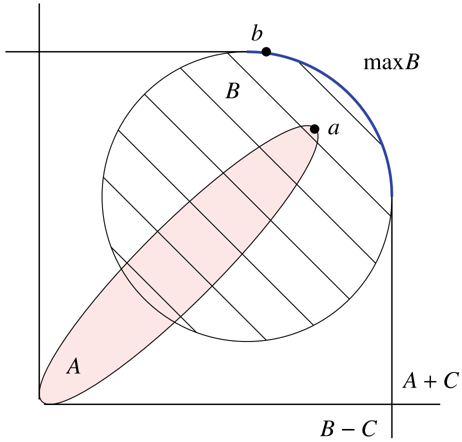


Fig. 6.5 Illustration of two sets $A,\ B \in P(Y)$ with $A \preceq_C^s B$, and $a \in \max A$ and $b \in \max B$ with $a \not\preceq_C b$ and $b \not\preceq_C a$, and $C = \mathbb{R}^2_+$

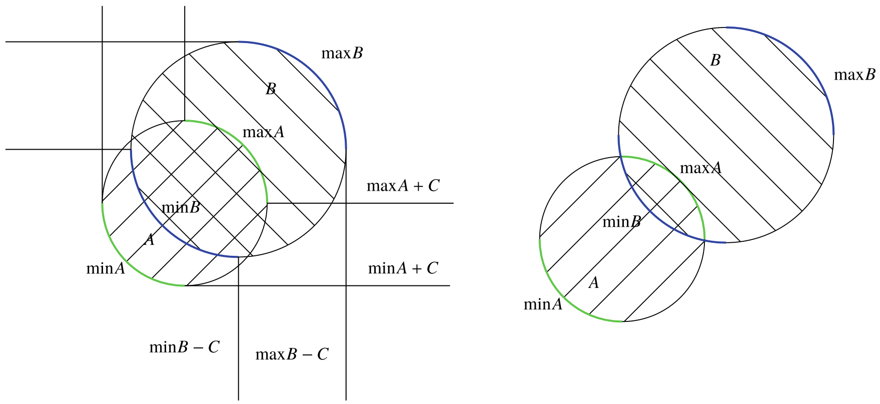


Fig. 6.6 Illustration of two sets $A,\ B \in \Xi$ with $A \preceq_C^m B$ and $A \preceq_C^{mc} B$, and $C = \mathbb{R}^2_+$

$$A \preceq_C^{mc} B \quad \Leftrightarrow \quad (A = B) \text{ or } (A \neq B,\ \min A \preceq_C^c \min B \text{ and } \max A \preceq_C^c \max B).$$

(c) The *minmax certainly nondominated set less order relation* $\preceq_C^{mn}$ is defined by

$$A \preceq_C^{mn} B \quad \Leftrightarrow \quad (A = B) \text{ or } (A \neq B,\ \max A \preceq_C^s \min B).$$

Neukel [132, 133] used set order relations defined in Definitions 6.4 and 6.5 to deal with the building conflict situation in the surroundings of the Frankfurt airport and cryptanalysis of substitution ciphers. See Fig. 6.2 and Fig. 6.6 for an illustration of these set order relations.

Definition 6.6 [92] A set $A \in \Xi$ is said to have the *quasi domination property* if and only if the following equivalent conditions hold:

(a) $\min A + C = A + C$ and $\max A - C = A - C$.

(b) $A \subseteq \min A + C$ and $A \subseteq \max A - C$.

Proposition 6.4 [92]

(a) *The set order relations* $\preceq_C^m$ *and* $\preceq_C^{mc}$ *are pre-order on* Ξ *and compatible with respect to the scalar multiplication with nonnegative real numbers. In general, they are not antisymmetric.*

(b) *Let* $A, B \in \Xi$ *have the quasi domination property. The set order relation* $\preceq_C^{mn}$ *is pre-order on* Ξ *and compatible with respect to the scalar multiplication with nonnegative real numbers. If the ordering cone* C *is pointed, then the set order relation* $\preceq_C^{mn}$ *is antisymmetric.*

Remark 6.2 The pointedness of the cone C in Proposition 6.4(b) cannot be dropped. From Remark 6.1, it is easy to see that for the sets A and B, we have $A \preceq_C^{mn} B$ and $B \preceq_C^{mn} A$ but $A \neq B$.

More precisely, for any $A, B \in \Xi$, we have

(a) $(A \preceq_C^m B$ and $B \preceq_C^m A) \Leftrightarrow (\min A + C = \min B + C,\ \min A - C = \min B - C,\ \max A + C = \max B + C,$ and $\max A - C = \max B - C)$.

(b) If C is pointed, then

$$(A \preceq_C^{mc} B \text{ and } B \preceq_C^{mc} A) \quad \Leftrightarrow \quad (\min A = \min B \text{ and } \max A = \max B),$$

(c) If C is pointed and A, B have the quasi domination property, then

$$(A \preceq_C^{m} B \text{ and } B \preceq_C^{m} A) \quad \Leftrightarrow \quad (\min A = \min B \text{ and } \max A = \max B).$$

The following result provides the relation among different kinds of set order relations.

Proposition 6.5 [92] *Let* $A, B \in \Xi$ *with* $A \neq B$. *Suppose that* A *and* B *have the quasi domination property. Then,*

(a) $A \preceq_C^c B \Rightarrow A \preceq_C^{mc} B \Rightarrow A \preceq_C^m B \Rightarrow A \preceq_C^s B$;

(b) $A \preceq_C^c B \Rightarrow A \preceq_C^{mn} B \Rightarrow A \preceq_C^m B$;

(c) $A \preceq_C^{mn} B$ *does not always imply* $A \preceq_C^{mc} B$ *and* $A \preceq_C^{mc} B$ *does not always imply* $A \preceq_C^{mn} B$.

The following example illustrates that the implications in the above proposition are strict, that is, the converse implications do not hold.

Example 6.5 [92] Let $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Consider the sets

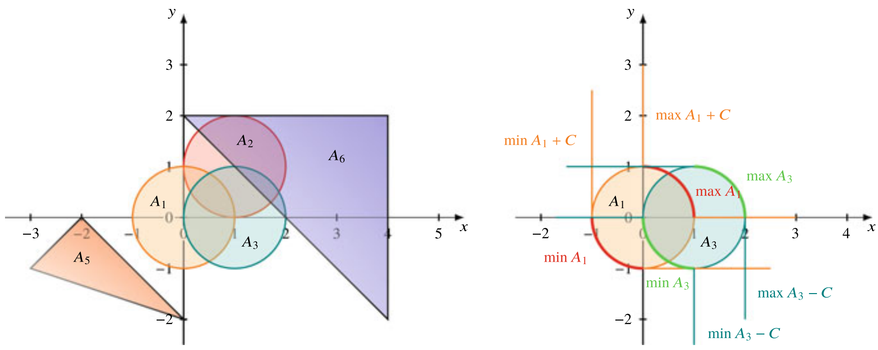


Fig. 6.7 Visualization of Example 6.5 with $C = \mathbb{R}^2_+$

$$A_1 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 1\},$$
$$A_2 = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - 1)^2 + (y_2 - 1)^2 \leq 1\},$$
$$A_3 = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - 1)^2 + y_2^2 \leq 1\},$$
$$A_4 = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - 1)^2 + (y_2 - 1)^2 \leq 1,\ y_1^2 + y_2^2 \geq 1\},$$
$$A_5 = \text{conv}\{(-2, 0), (-3, -1), (0, -2)\},$$
$$A_6 = \text{conv}\{(4, 2), (0, 2), (4, -2)\}.$$

From Fig. 6.7, one can easily visualize that

$$A_1 \preceq^{mc}_C A_2 \quad \text{but} \quad A_1 \npreceq^{mn}_C A_2 \quad \text{and} \quad A_1 \npreceq^{c}_C A_2,$$

$$A_1 \preceq^{m}_C A_3 \quad \text{but} \quad A_1 \npreceq^{mn}_C A_3 \quad \text{and hence} \quad A_1 \npreceq^{mc}_C A_3,$$

$$A_1 \preceq^{mn}_C A_4 \quad \text{but} \quad A_1 \npreceq^{c}_C A_4,$$

$$A_5 \preceq^{mn}_C A_6 \quad \text{but} \quad A_5 \npreceq^{mc}_C A_6.$$

Remark 6.3 From Propositions 6.2 and 6.5, it is clear that the set order relation $\preceq^p_C$ is the weakest one and the set order relation $\preceq^c_C$ is the strongest one. Furthermore, in contrast to the set order relations $\preceq^c_C$ and $\preceq^{mn}_C$, the set order relations $\preceq^\alpha_C$, $\alpha \in \{l, u, s, m, mc\}$ are generally not antisymmetric. To see this, it suffices to consider the case with the set order relation $\preceq^{mc}_C$ because this set order relation is the strongest one among all other set order relations.

Let $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Consider the sets

$$A_1 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 1\},$$
$$A_2 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 \leq 1, -1 \leq y_1 - y_2 \leq 1\}.$$

Then we can see that $A \neq B$, $A \preceq_C^{mc} B$, and $B \preceq_C^{mc} A$.

6.3.1 Set Order Relations in Terms of the Minkowski Difference

Recently, Karaman et al. [96] introduced the following set order relations on the family of sets by using the Minkowski difference.

Definition 6.7 [96] Let Y be a normed space and $A, B, K \in P(Y)$.

(a) The *m-upper set less order relation*, denoted by $\preceq_K^{mu}$, is defined as

$$A \preceq_K^{mu} B \quad \Leftrightarrow \quad (B \dot{-} A) \cap K \neq \emptyset.$$

(b) The *m-lower set less order relation*, denoted by $\preceq_K^{ml}$, is defined as

$$A \preceq_K^{ml} B \quad \Leftrightarrow \quad (A \dot{-} B) \cap (-K) \neq \emptyset.$$

If A and B are bounded and $A \dot{-} B \neq \emptyset$, $B \dot{-} A \neq \emptyset$, then $A \preceq_K^{mu} B$ if and only if $A \preceq_K^{ml} B$. If A and B are singleton sets and K is a convex and pointed cone with $\mathbf{0} \in K$, then $\preceq_K^{mu}$ and $\preceq_K^{ml}$ coincide with the vector order relation $\preceq_C$ on Y, that is, for any $a, b \in Y$, we have

$$\{a\} \preceq_K^{mu} \{b\} \quad \Leftrightarrow \quad \{a\} \preceq_K^{ml} \{b\} \quad \Leftrightarrow \quad a \preceq_K b.$$

It is pointed out in [96] that

(a) if K is a convex cone in Y and $\mathbf{0} \in K$, then $\preceq_K^{mu}$ and $\preceq_K^{ml}$ are pre-order on $P(Y)$;
(b) if K is a pointed convex cone in Y with $\mathbf{0} \in K$, then $\preceq_K^{mu}$ and $\preceq_K^{ml}$ are partial order on $\mathcal{B}(Y)$;
(c) $\preceq_K^{mu}$ and $\preceq_K^{ml}$ are compatible with addition;
(d) $\preceq_K^{mu}$ and $\preceq_K^{ml}$ are compatible with scalar multiplication if and only if K is a cone.

From now onward, we consider the ordering cone C on Y instead of K, then $\preceq_K^{mu}$ and $\preceq_K^{ml}$ turn to $\preceq_C^{mu}$ and $\preceq_C^{ml}$.

The set order relations $\preceq_C^{mu}$ and $\preceq_C^{ml}$ and the set order relations $\preceq_C^{u}$ and $\preceq_C^{l}$ have the following relations: For any $A, B \in P(Y)$,

$$A \preceq_C^{mu} B \Rightarrow A \preceq_C^{u} B \quad \text{and} \quad A \preceq_C^{ml} B \Rightarrow A \preceq_C^{l} B,$$

but the converse of the above implications may not be true.

The following example illustrates that the set order relation $\preceq_C^{u}$ does not imply $\preceq_C^{mu}$.

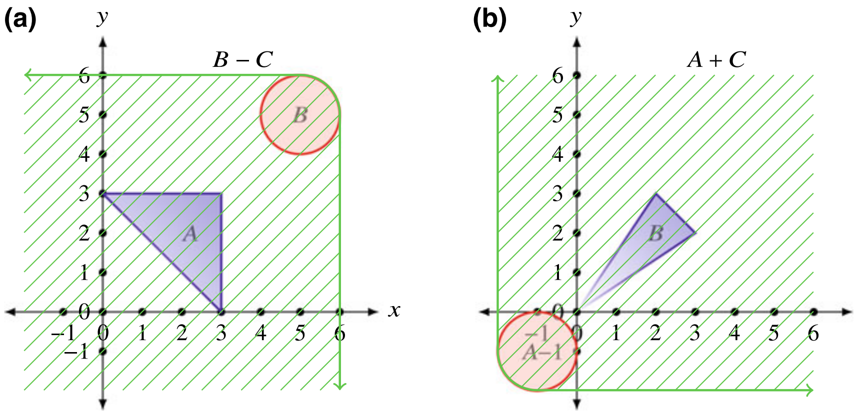


Fig. 6.8 **(a)** Illustration of sets in Example 6.6. **(b)** Illustration of sets in Example 6.7

Example 6.6 Let $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Consider the sets

$$A = \text{conv}\{(2, 0), (3, 3), (0, 2)\}$$

and

$$B = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - 5)^2 + (y_2 - 5)^2 \leq 1\}.$$

As in Fig. 6.8 (a), $A \subseteq B - C$ which gives us $A \preceq^u_C B$. On the other hand, there does not exist any $x \in \mathbb{R}^2$ such that $x + A \subseteq B$. Hence, we have $(B \dot{-} A) \cap C = \emptyset$, that is, $A \not\preceq^{mu}_C B$.

The following example shows that the set order relation $\preceq^l_C$ does not imply the set order relation $\preceq^{ml}_C$.

Example 6.7 Let $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Consider the sets

$$A = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 + 1)^2 + (y_2 + 1)^2 \leq 1\}$$

and

$$B = \text{conv}\{(0, 0), (3, 2), (2, 3)\}.$$

As in Fig. 6.8 (b), $B \subseteq A + C$ which gives us $A \preceq^l_C B$. On the other hand, there does not exist any $x \in \mathbb{R}^2$ such that $x + B \subseteq A$. Hence, we have $(A \dot{-} B) \cap (-C) = \emptyset$, that is, $A \not\preceq^{ml}_C B$.

The strict version of $\preceq^{mu}_C$ and $\preceq^{ml}_C$ is defined as follows:

Definition 6.8 [96] Let Y be a normed space, $A, B \in P(Y)$, and C be a convex cone in Y with $\text{int}C \neq \emptyset$.

(a) The *strictly m-upper set less order relation*, denoted by $\prec_C^{mu}$, is defined as

$$A \prec_C^{mu} B \quad \Leftrightarrow \quad (B \dot{-} A) \cap \operatorname{int} C \neq \emptyset.$$

(b) The *strictly m-lower set less order relation*, denoted by $\prec_C^{ml}$, is defined as

$$A \prec_C^{ml} B \quad \Leftrightarrow \quad (A \dot{-} B) \cap \operatorname{int}(-C) \neq \emptyset.$$

Remark 6.4 Let $\alpha \in \{mu, ml\}$ and $A, B \in P(Y)$. If $A \prec_C^{\alpha} B$, then $A \preceq_C^{\alpha} B$.

It is pointed out in [96] that

(a) $\prec_C^{mu}$ and $\prec_C^{ml}$ are compatible with addition;
(b) $\prec_C^{mu}$ and $\prec_C^{ml}$ are compatible with scalar multiplication.

If C is a pointed convex cone, even then the relations $\prec_C^{mu}$ and $\prec_C^{ml}$ are not reflexive unless $C = Y$, and hence, $\prec_C^{mu}$ and $\prec_C^{ml}$ are not partial order. The following example clarifies this fact.

Example 6.8 Let $Y=\mathbb{R}^2$, $C=\mathbb{R}^2_+$, and $A = \{(x, y) : x^2 + y^2 \leq 1\}$. Since $A \dot{-} A = \{\mathbf{0}\}$, we have $\{\mathbf{0}\} \cap \operatorname{int} C = \emptyset$ and $A \not\prec_C^{mu} A$. Similarly, since $(A \dot{-} A) \cap \operatorname{int}(-C) \neq \emptyset$, we obtain $A \not\prec_C^{ml} A$.

6.3.2 Set Order Relations with Respect to Variable Domination Structures

In the recent past, Eichfelder and Pilacka [42, 44] and Köbis et al. [7, 101, 102] are among major contributors to study set optimization problems with respect to variable ordering structures with applications to different real-world problems. The importance of incorporating variable ordering structures for intensity-modulated radiation therapy (IMRT) in order to allow an improved modeling of the decision-making problem is already discussed in [40, Chap. 10]. Another significant application of set optimization problems with respect to variable domination structures can be found in the theory of consumer demand [119], medical image registration [104], and uncertain optimization [7].

To study set optimization problems with respect to variable domination structures by using a set approach, we recall the following six kinds of generalized variable set order relations to compare sets in a topological vector space Y.

Definition 6.9 [104] Let $A, B \in P(Y)$ and $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map. The following binary relations on $P(Y)$ with respect to $\mathcal{K}$ are defined as follows:

(a) The *variable generalized lower set less order relation* $\preceq_l^{\mathcal{K}}$ is defined by

$$A \preceq_l^{\mathcal{K}} B \quad \Leftrightarrow \quad B \subseteq \bigcup_{a \in A} (a + \mathcal{K}(a)).$$

(b) The *variable generalized upper set less order relation* $\preceq_u^{\mathcal{K}}$ is defined by

$$A \preceq_u^{\mathcal{K}} B \quad \Leftrightarrow \quad A \subseteq \bigcup_{b \in B}(b - \mathcal{K}(b)).$$

(c) The *variable generalized certainly lower set less order relation* $\preceq_{cl}^{\mathcal{K}}$ is defined by

$$A \preceq_{cl}^{\mathcal{K}} B \quad \Leftrightarrow \quad B \subseteq \bigcap_{a \in A}(a + \mathcal{K}(a)).$$

(d) The *variable generalized certainly upper set less order relation* $\preceq_{cu}^{\mathcal{K}}$ is defined by

$$A \preceq_{cu}^{\mathcal{K}} B \quad \Leftrightarrow \quad A \subseteq \bigcap_{b \in B}(b - \mathcal{K}(b)).$$

(e) The *variable generalized possible lower set less order relation* $\preceq_{pl}^{\mathcal{K}}$ is defined by

$$A \preceq_{pl}^{\mathcal{K}} B \quad \Leftrightarrow \quad B \cap \bigcup_{a \in A}(a + \mathcal{K}(a)) \neq \emptyset.$$

(f) The *variable generalized possible upper set less order relation* $\preceq_{pu}^{\mathcal{K}}$ is defined by

$$A \preceq_{pu}^{\mathcal{K}} B \quad \Leftrightarrow \quad A \cap \bigcup_{b \in B}(b - \mathcal{K}(b)) \neq \emptyset.$$

Remark 6.5 For all $y \in Y$, if $\mathcal{K}(y) = C$ is a convex cone with $\text{int}C \neq \emptyset$ in Y, then the set order relations $\preceq_l^{\mathcal{K}}$ and $\preceq_u^{\mathcal{K}}$ reduce to the set order relations $\preceq_C^l$ and $\preceq_C^u$, respectively. See Fig. 6.9 for illustration of variable generalized set order relations.

Proposition 6.6 [119] *Let $A, B \in P(Y)$. Then, the following assertions hold:*

(a) $A \preceq_u^{\mathcal{K}} B \Leftrightarrow B \preceq_l^{-\mathcal{K}} A$;
(b) $A \preceq_{cu}^{\mathcal{K}} B \Leftrightarrow B \preceq_{cl}^{-\mathcal{K}} A$;
(c) $A \preceq_{pu}^{\mathcal{K}} B \Leftrightarrow B \preceq_{pl}^{-\mathcal{K}} A$;
(d) $A \preceq_{cl}^{\mathcal{K}} B \Rightarrow A \preceq_l^{\mathcal{K}} B \Rightarrow A \preceq_{pl}^{\mathcal{K}} B$;
(e) $A \preceq_{cu}^{\mathcal{K}} B \Rightarrow A \preceq_u^{\mathcal{K}} B \Rightarrow A \preceq_{pu}^{\mathcal{K}} B$.

Köbis et al. [104] established the following useful properties of the set order relations $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$

Proposition 6.7 [104] *Let $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map. The following statements hold:*

(a) *If $\mathbf{0} \in \mathcal{K}(y)$ for all $y \in Y$, then the set order relations $\preceq_l^{\mathcal{K}}$, $\preceq_u^{\mathcal{K}}$, $\preceq_{pl}^{\mathcal{K}}$, and $\preceq_{pu}^{\mathcal{K}}$ are reflexive.*

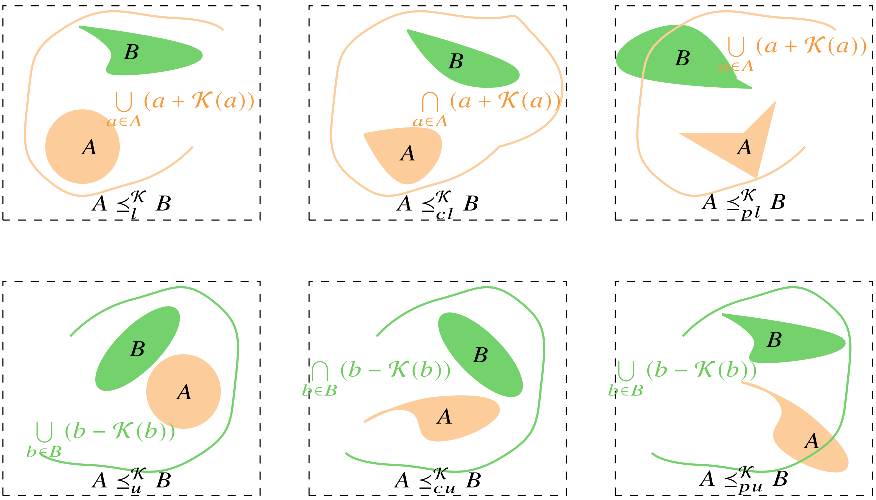


Fig. 6.9 Visualization of variable generalized set order relations defined in Definition 6.9

(b) *If* $\mathcal{K}(y) + \mathcal{K}(y) \subseteq \mathcal{K}(y)$ *for all* $y \in Y$ *and* $\mathcal{K}(y + d) \subseteq \mathcal{K}(y)$ *for all* $y \in Y$ *and all* $d \in \mathcal{K}(y)$*, then the set order relations* $\preceq_l^{\mathcal{K}}$ *and* $\preceq_{cl}^{\mathcal{K}}$ *are transitive.*
(c) *If* $\mathcal{K}(y) + \mathcal{K}(y) \subseteq \mathcal{K}(y)$ *for all* $y \in Y$ *and* $\mathcal{K}(y - d) \subseteq \mathcal{K}(y)$ *for all* $y \in Y$ *and all* $d \in \mathcal{K}(y)$*, then the set order relations* $\preceq_u^{\mathcal{K}}$ *and* $\preceq_{cu}^{\mathcal{K}}$ *are transitive.*
(d) *If* $\mathcal{K}(y) \cap (-\mathcal{K}(z)) = \{\mathbf{0}\}$ *for all* $y, z \in Y$*, then the set order relations* $\preceq_{cl}^{\mathcal{K}}$ *and* $\preceq_{cu}^{\mathcal{K}}$ *are antisymmetric.*

6.4 Nonlinear Scalarization Functions

We first recall the linear scalarization method for vectors. The most representative example of linear scalarizing functions is an inner product. For any $y, k \in Y$, in case of vector, the linear scalarizing function is defined by

$$h_k(y) := \langle y, k\rangle. \tag{6.1}$$

Based on this scalarization, we can consider the following scalarizing functions for a set $A \subseteq Y$ defined by

$$\varphi_k(A) := \inf_{y\in A}\langle y, k\rangle \quad \text{and} \quad \phi_k(A) := \sup_{y\in A}\langle y, k\rangle.$$

Rest of this section, we assume that C is a proper, solid, closed convex cone in a topological vector space Y and $k \in \operatorname{int} C$. The nonlinear scalarization functional $\varphi_{C,k} : Y \to (-\infty, \infty]$ is defined by

$$\varphi_{C,k}(y) = \inf\{t \in \mathbb{R} : y \preceq_C tk\} = \inf\{t \in \mathbb{R} : y \in tk - C\}, \quad \text{for all } y \in Y. \tag{6.2}$$

As mentioned in [103], Fig. 6.10 visualizes the functional $\varphi_{C,k}$ with $C = \mathbb{R}^2_+$ and $k \in \text{int}C$. We can see that the set $-C$ is moved along the line $\mathbb{R} \cdot k$ up until y belongs to $tk - C$. The functional $\varphi_{C,k}$ assigns the smallest value t such that the property $y \in tk - C$ is fulfilled.

It can be shown that all minimal elements of a vector optimization problem can be found by means of $\varphi_{C,k}$ if $k \in C \setminus \{\mathbf{0}\}$, and all weakly minimal elements of a vector optimization problem can be determined if $k \in \text{int}C$ (see [54]). In Fig. 6.10, we can easily see that for the given cone $C = \mathbb{R}^2_+$, by a variation of the vector $k \in C \setminus \{\mathbf{0}\}$, all minimal elements of the vector optimization problem without any convexity assumptions can be found. The scalarizing functional $\varphi_{C,k}$ was used in [54] to prove nonconvex separation theorems and has applications in coherent risk measures in financial mathematics (see, for instance, [66, 85]).

We note that the set $\{t \in \mathbb{R} : y \in tk - C\}$ may be empty, and in this case $\varphi_{C,k}$ will take $+\infty$ as by convention $\inf \emptyset = +\infty$. For further details, see [56]. On the other hand, if $k \in C$, then the lower level set of $\varphi_{C,k}$ at each height t coincides with a parallel translation of $-C$ at offset tk, that is,

$$\{y \in Y : \varphi_{C,k}(y) \leq t\} = tk - C,$$

and hence $\varphi_{C,k}$ is the smallest strictly monotonic function with respect to the ordering cone C in case $k \in \text{int}C$. Also, this scalarization function has a dual form as follows:

$$-\varphi_{C,k}(-y) = \sup\{t \in \mathbb{R} : tk \preceq_C y\} = \sup\{t \in \mathbb{R} : y \in tk + C\}, \quad \text{for all } y \in Y.$$

The importance of this function is due to the fact that it characterizes, under some appropriate assumptions, the relation $\preceq_C$ as

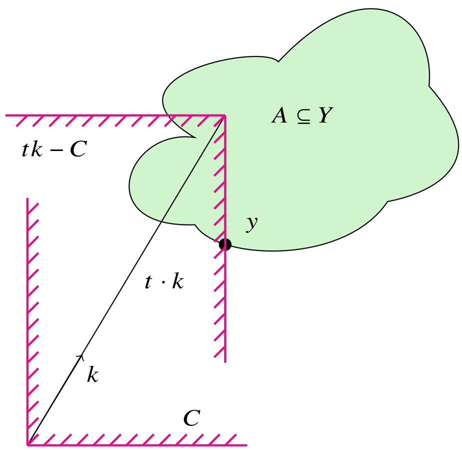


Fig. 6.10 Illustration of the functional (6.2) with $C = \mathbb{R}^2_+$

$$y_1 \preceq_C y_2 \quad \Leftrightarrow \quad \varphi_{C,k}(y_1 - y_2) \leq 0.$$

Another essential feature of this function is the so-called translativity property (see [56] for details), that is,

$$\text{for all } y \in Y \text{ and all } \alpha \in \mathbb{R}\colon \varphi_{C,k}(y + \alpha k) = \varphi_{C,k}(y) + \alpha.$$

In [148], functionals of type (6.2) have been applied in order to obtain vector-valued variants of Ekeland's variational principle. For this topic, see also [59] and [65]. Note that the originality of the approach in [54, 148] relies on the fact that the set C defining a functional via (6.2) was assumed neither to be a cone nor convex. In some papers, this functional has been treated and regarded as a generalization of the Chebyshev scalarization, see the books [9, 56, 127]. Essentially, it is equivalent to the smallest strictly monotonic function with respect to int$C$ defined by Luc in [127].

Recently, Köbis et al. [103] characterized the upper and lower set less order relations defined in Definition 6.4(a) and (b) by using the scalarization functional $\varphi_{C,k}$ as follows.

Theorem 6.4.1 [103, Theorems 3.3 and 3.8] *Let C be a proper closed convex cone in a topological vector space Y and $A, B \in P(Y)$.*

(a) *If $k_0 \in C\backslash\{\mathbf{0}\}$ is such that $\inf_{b\in B} \varphi_{C,k_0}(a - b)$ is attained for all $a \in A$, then*

$$\sup_{a\in A} \inf_{b\in B} \varphi_{C,k_0}(a - b) \leq 0 \quad \Leftrightarrow \quad A \subseteq B - C.$$

(b) *If $k_1 \in C\backslash\{\mathbf{0}\}$ is such that $\inf_{a\in A} \varphi_{C,k_1}(a - b)$ is attained for all $b \in B$, then*

$$\sup_{b\in B} \inf_{a\in A} \varphi_{C,k_1}(a - b) \leq 0 \quad \Leftrightarrow \quad B \subseteq A + C.$$

Recently, Köbis et al. [105] and Ansari et al. [10] studied and investigated new nonlinear scalarization functions for the relations $\preceq_l^{\mathcal{K}}$ and $\preceq_u^{\mathcal{K}}$ and discussed some of its properties.

Let $A, B \in P(Y)$ and $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map. For each $k \in Y \setminus \{\mathbf{0}\}$, let

$$[0, +\infty)k + \mathcal{K}(y) \subseteq \mathcal{K}(y), \quad \text{for all } y \in Y. \tag{6.3}$$

Let $B \in P(Y)$ be arbitrary but fixed. Consider the scalarization functionals $\varphi_{k,B} : P(Y) \to \mathbb{R} \cup \{+\infty\}$ and $\phi_{k,B} : P(Y) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi_{k,B}(A) := \inf\{t \geq 0 \; : \; A \preceq_u^{\mathcal{K}} tk + B\}, \quad \text{for all } A \in P(Y), \tag{6.4}$$

and

$$\phi_{k,B}(A) := \inf\{t \geq 0 \; : \; A \preceq_l^{\mathcal{K}} tk + B\}, \quad \text{for all } A \in P(Y), \tag{6.5}$$

respectively.

If we consider $B = \{\mathbf{0}\}$, then the scalarization functionals $\varphi_{k,B}$ and $\phi_{k,B}$ defined by (6.4) and (6.5) can be written as $h_k^u : P(Y) \to \mathbb{R} \cup \{+\infty\}$ and $h_k^l : P(Y) \to \mathbb{R} \cup \{+\infty\}$, respectively, and we have $h_k^u : P(Y) \to \mathbb{R} \cup \{+\infty\}$ and $h_k^l : P(Y) \to \mathbb{R} \cup \{+\infty\}$, respectively, and we have

$$h_k^u(A) = \inf\{t \geq 0 \,:\, A \preceq_u^{\mathcal{K}} tk\}, \tag{6.6}$$

and

$$h_k^l(A) = \inf\{t \geq 0 \,:\, A \preceq_l^{\mathcal{K}} tk\}, \tag{6.7}$$

for all $A \in P(Y)$.

For more details, we refer to [10, 105].

6.4.1 Weighted Set Order Relations

Rest of this subsection, we assume the following assumption.

Assumption 1 The ordering cone $C \neq Y$ is solid, closed, and convex in a Hausdorff topological vector space Y and $k \in \text{int}C$ is such that $\inf_{b\in B} \varphi_{C,k}(a-b)$ is attained for all $a \in A$ and $\inf_{a\in A} \varphi_{C,k}(a-b)$ is attained for all $b \in B$ whenever A and B are closed and bounded sets in Y.

By using the characterization of the set order relations $\preceq_C^l$ and $\preceq_C^u$ given in Theorem 6.4.1, Chen et al. [30] introduced the so-called weighted set order relations as follows.

Definition 6.10 Let $A, B \in \Omega^{cb}$ and $\lambda \in [0, 1]$. The *weighted set order relation* $\preceq_C^\lambda$ for sets $A, B \in P(Y)$ is defined by

$$A \preceq_C^\lambda B \quad \Leftrightarrow \quad \lambda g^u(A, B) + (1-\lambda) g^l(A, B) \leq 0,$$

where

$$g^u(A, B) := \sup_{a\in A} \inf_{b\in B} \varphi_{C,k}(a-b) \quad \text{and} \quad g^l(A, B) := \sup_{b\in B} \inf_{a\in A} \varphi_{C,k}(a-b).$$

Remark 6.6 [30] For any $\lambda \in [0, 1]$, the relation $\preceq_C^\lambda$ is reflexive and transitive, that is, $\preceq_C^\lambda$ is a pre-order. Moreover, the relation $\preceq_C^\lambda$ is compatible with nonnegative scalar multiplication, that is, for any $A, B \in P(Y)$ and $\alpha \geq 0$, one has

$$A \preceq_C^\lambda B \quad \Rightarrow \quad \alpha A \preceq_C^\lambda \alpha B.$$

Remark 6.7 For $\lambda = 1$, $\preceq_C^{\lambda}$ reduces to $\preceq_C^{u}$, and for $\lambda = 0$, $\preceq_C^{\lambda}$ reduces to $\preceq_C^{l}$. If $\preceq_C^{u}$ and $\preceq_C^{l}$ hold, then $\preceq_C^{\lambda}$ is true for all $\lambda \in [0, 1]$, but the converse is not true and this was exactly the intention of introducing $\preceq_C^{\lambda}$.

Note that the parameter λ serves as a weight vector which indicates the importance of either of the two relations $\preceq_C^{u}$ and $\preceq_C^{l}$. The relation which is more important should be associated with a higher weight factor. For instance, if $g^u(A, B) \leq 0$ and $g^l(A, B) > 0$, then, for large enough λ, $A \preceq_C^{\lambda} B$ can hold and the $A \preceq_C^{u} B$ "outweighs" the effects of $A \not\preceq_C^{l} B$.

Remark 6.8 Chen et al. [30] gave the definition of the weighted set order relation under the assumption of Y being a quasicompact topological space. Recalling that usually a Hausdorff (separated) topological space is called compact if it is quasi-compact, one may realize that [30, Definition 2.5] is basically empty (as well as the following results, for example, [30, Proposition 2.9]): Up to trivial examples, there are no (quasi) compact topological linear spaces; not even the real line with the usual topology satisfies this assumption. Therefore, we modified Assumption 2.4 in [30] to the version above: It is certainly satisfied in any finite-dimensional space when the usual topology since every closed bounded set in such a space is compact and the function $\varphi_{C,k}$ is continuous for $k \in \text{int}C$.

We provide an example below to illustrate the weighted set order relations $\preceq_C^{\lambda}$ and discuss the role of the parameter λ.

Example 6.9 [30] Let $A = [a, c]$ and $B = [b, d]$ be compact sets in $\mathbb{R}$. We choose $C = \mathbb{R}_+$ and $k = 1$. Then,

$$
\begin{aligned}
g^u(A, B) &= \sup_{a\in A} \inf_{b\in B} \varphi_{C,k}(a - b) = \sup_{a\in A} \inf_{b\in B} \inf\{t \in \mathbb{R} : a - b \leq t\} \\
&= \sup_{a\in A} \inf_{b\in B} (a - b) = \sup_{a\in A} a - \sup_{b\in B} b = c - d, \\
g^u(A, B) &= a - b, g^u(B, A) = d - c, g^l(B, A) = b - a.
\end{aligned}
$$

Consider $a = 5, c = 10, b = 0, and\, d = 11$. Then $B \not\preceq_C^{u} A$, but $B \preceq_C^{l} A$. Also, $A \preceq_C^{u} B$, but $A \not\preceq_C^{l} B$. However, we can see that the "amount" of B that is bigger than the supremum of A is very small compared to how the lower bound of B is smaller than the lower bound of A. In that sense, when a decision-maker has no clear understanding of how to choose a set, the weighted set order relation $\preceq_C^{\lambda}$ can be helpful. We have $g^u(A, B) = -1, g^l(A, B) = 5$. So, in order for $A \preceq_C^{\lambda} B$ to hold, $\lambda \in [\frac{5}{6}, 1]$. Similarly, as $g^u(B, A) = 1, g^l(B, A) = -5$, $\lambda \in [0, \frac{5}{6}]$ for $B \preceq_C^{\lambda} A$ to hold true.

6.5 Solution Concepts in Set Optimization

This section deals with the solution concepts of the set optimization problem (SOP) with vector approach and set approach. The solution concept based on the vector approach is of mathematical interest but it cannot be often used in practice.

In the vector approach, an element $\bar{x} \in S$ for which there exists at least one element $\bar{y} \in F(\bar{x})$ which is Pareto minimal point of the image set of F is a solution of the set optimization problem (SOP). In the past, solution concepts based on vector approach has been studied and investigated in [1, 2, 29, 33–35, 41, 55, 59, 61, 64, 80, 93, 122, 123] and the references therein.

Definition 6.11 [1, 80] An element $\bar{x} \in S$ is said to be

(a) a *minimal solution* of the problem (SOP) if there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y}$ is a minimal element of image set $F(S)$, that is,

$$(\{\bar{y}\} - C) \cap F(S) = \{\bar{y}\}.$$

(b) a *weak minimal solution* of the problem (SOP) if there exists $\bar{y} \in F(\bar{x})$ such that $\bar{y}$ is a weak minimal element of image set $F(S)$, that is,

$$(\{\bar{y}\} - \text{int}C) \cap F(S) = \emptyset.$$

We denote the set of minimal and weak minimal elements of (SOP) by $\text{Min}(F, S)$ and $\text{WMin}(F, S)$, respectively.

Recall that $\min A := \{a \in A : A \cap (a - C) = \{a\}\}$ and $\text{wmin}A := \{a \in A : A \cap (a - \text{int}C) = \emptyset\}$ are the sets of minimal elements and weak minimal elements, respectively, with respect to the convex pointed cone C in a topological vector space Y.

Note that Definition 6.11 can also be written as follows:
An element $\bar{x} \in S$ is said to be

(a) a *minimal solution* [99] of the problem (SOP) if there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(S) - \{\bar{y}\}) \cap (-C) = \{\mathbf{0}\};$$

(b) a *weak minimal solution* [99] of the problem (SOP) if there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(S) - \{\bar{y}\}) \cap (-\text{int}C) = \emptyset.$$

Another form of Definition 6.11 can also be written as follows:
An element $\bar{x} \in S$ is said to be

(a) a *minimal solution* [96] of the problem (SOP) if $F(\bar{x}) \cap \min F(S) \neq \emptyset$;
(b) a *weak minimal solution* [96] of the problem (SOP) if $F(\bar{x}) \cap \text{wmin}F(S) \neq \emptyset$.

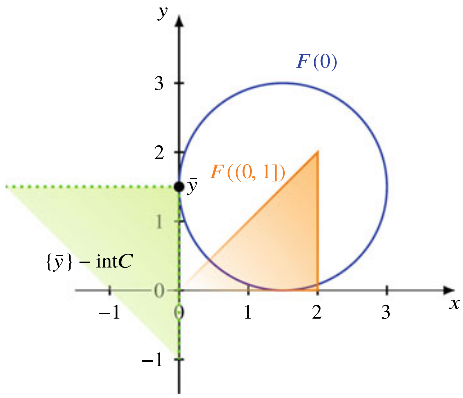


Fig. 6.11 Illustration of Example 6.10 with $C = \mathbb{R}^2_+$

It is clear that $\text{Min}(F, S) \subseteq \text{WMin}(F, S)$. However, the reverse inclusion may not hold.

Example 6.10 Let $X=\mathbb{R}$, $S=[0, 1]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - 3/2)^2 + (y_2 - 3/2)^2 = (3/2)^2\}, & \text{if } x = 0, \\ \text{conv}\{(0, 0), (2, 0), (2, 2)\}, & \text{otherwise.} \end{cases}$$

From Fig. 6.11, we can see that there exists $\bar{y} \in F(0)$ such that $(\{\bar{y}\} - \text{int}C) \cap F(S) = \emptyset$ and hence $0 \in \text{WMin}(F, S)$ but there does not exist any $\bar{y} \in F(0)$ such that $(\{\bar{y}\} - C) \cap F(S) = \{\bar{y}\}$, so $0 \notin \text{Min}(F, S)$.

As we have seen above, in the vector approach, we consider only a minimal element $\bar{y}$ of the image set $F(S)$. However, only one minimal element does not imply that the whole set $F(\bar{x})$ be in a certain sense minimal with respect to all sets $F(x)$ with $x \in S$. To overcome this drawback, the solution concepts based on the set approach are very helpful and important. In the set approach, solution concepts are defined by using different kinds of set order relations, and these solutions are based on the comparison of values of set-valued objective map using set order relations. In the recent past, solution concepts based on set approach have been studied and investigated in [1, 8, 25, 30, 67, 79–84, 92, 96, 109, 110, 112, 115] and the references therein.

As the set order relations $\preceq^l_C$, $\preceq^u_C$, $\preceq^s_C$, $\preceq^c_C$ on $P(Y)$; $\preceq^m_C$, $\preceq^{mc}_C$, $\preceq^{mn}_C$ on Ξ; and $\preceq^\lambda_C$ on Ω^{cb} are pre-order, we can define optimal solutions with respect to the pre-order $\preceq^t_C$, where $t \in \{l, u, s, c, m, mc, mn, \lambda\}$. For the set order relation $\preceq^t_C$, we assume the following condition:

$$F \text{ takes values on } \begin{cases} P(Y), & \text{if } t \in \{l, u, s, c\}, \\ \Xi, & \text{if } t \in \{m, mc, mn\}, \\ \Omega^{cb}, & \text{if } t = \lambda. \end{cases}$$

Definition 6.12 [30, 92] Let $t \in \{l, u, s, c, m, mc, mn, \lambda\}$. An element $\bar{x} \in S$ is said to be

(a) a *t-minimal solution* of the problem (SOP) with respect to the set order relation $\preceq^t_C$ if and only if

$$F(x) \preceq^t_C F(\bar{x}) \text{ for some } x \in S \quad \Rightarrow \quad F(\bar{x}) \preceq^t_C F(x);$$

(b) a *t-strongly minimal solution* of the problem (SOP) with respect to the set order relation $\preceq^t_C$ if and only if

$$F(\bar{x}) \preceq^t_C F(x), \quad \text{for all } x \in S;$$

(c) a *t-weak minimal solution* of the problem (SOP) with respect to the set order relation $\prec^t_C$, $t \neq \lambda$ if and only if

$$F(x) \prec^t_C F(\bar{x}) \text{ for some } x \in S \quad \Rightarrow \quad F(\bar{x}) \prec^t_C F(x).$$

We denote the family of t-minimal, t-strongly minimal, and t-weak minimal elements of S by $t - \text{Min}(F, S)$, $t - \text{SMin}(F, S)$, and $t - \text{WMin}(F, S)$, respectively, where $t \in \{l, u, s, c, m, mc, mn, \lambda\}$.

It is clear that $t - \text{Min}(F, S) \subseteq t - \text{WMin}(F, S)$ for $t \in \{l, u\}$. However, the reverse inclusion may not hold.

Example 6.11 Let $X = \mathbb{R}$, $S = [0, 1]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [(-1, -1), (1, 1)], & \text{if } x = 0, \\ \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq x^2\}, & \text{otherwise.} \end{cases}$$

Then it can be easily seen that $l - \text{Min}(F, S) = \{0\}$ and $l - \text{WMin}(F, S) = \{0, 1\}$.

Example 6.12 Let $X = \mathbb{R}$, $S = [-1, 0]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [0, -2] \times [0, -2], & \text{if } x = 0, \\ [0, -3) \times (0, -3), & \text{otherwise.} \end{cases}$$

Then we see that $0 \in u - \text{WMin}(F, S)$ but $0 \notin u - \text{Min}(F, S)$.

Definition 6.13 [77, 78] Let S be a nonempty convex subset of X. A set-valued map $F : S \rightrightarrows Y$ is said to be

(a) *strictly natural l-type C-quasi-convex* on S if for all $x_1, x_2 \in S$ with $x_1 \neq x_2$ and all $t \in (0, 1)$, there exists $\lambda \in [0, 1]$ such that

$$F(tx_1 + (1 - t)x_2) \prec^l_C \lambda F(x_1) + (1 - \lambda)F(x_2);$$

(b) *strictly natural u-type C-quasi-convex* on S if for all $x_1, x_2 \in S$ with $x_1 \neq x_2$ and all $t \in (0, 1)$, there exists $\lambda \in [0, 1]$ such that

$$F(tx_1 + (1-t)x_2) \prec_C^u \lambda F(x_1) + (1-\lambda)F(x_2).$$

Proposition 6.8 [77, 78] *Assume that S is a convex subset of X, $F : S \rightrightarrows Y$ is a strictly natural l-type C-quasi-convex map on S with nonempty compact values. Then,* $l - \mathrm{Min}(F, S) = l - \mathrm{WMin}(F, S)$.

Proposition 6.9 [77, 78] *Assume that S is a convex subset of X, $F : S \rightrightarrows Y$ is a strictly natural u-type C-quasi-convex map on S with nonempty compact values. Then,* $u - \mathrm{Min}(F, S) = u - \mathrm{WMin}(F, S)$.

The following examples show that there is no relation between minimal and l-minimal solutions.

Example 6.13 [79] Let $X = \mathbb{R}$, $S = \mathbb{R}_+$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(0,0)\}, & \text{if } x = 0, \\ [(0,0), (-x, \frac{1}{x})], & \text{otherwise.} \end{cases}$$

Then we can easily obtain $\mathrm{Min}(F, S) = S$ and $l - \mathrm{Min}(F, S) = \emptyset$.

Example 6.14 [79] Let $X = \mathbb{R}$, $S = [-1, 0]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(u, -u^2) \in \mathbb{R}^2 : -1 < u \leq 0\}, & \text{if } x = -1, \\ [(x, 0), (x, -x^2)], & \text{otherwise.} \end{cases}$$

After a short calculation, we get $\mathrm{Min}(F, S) = \emptyset$ and $l - \mathrm{Min}(F, S) = \{-1\}$.

The following examples show that there is no relation between minimal and u-minimal solutions.

Example 6.15 [1] Let $X = \mathbb{R}$, $S = [0, 1]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = x^2, v > 0\}, & \text{if } x \neq -1, 0, \\ (-1/2, 1), & \text{if } x = -1, \\ (1/2, 1), & \text{if } x = 0. \end{cases}$$

We can easily check that $\mathrm{Min}(F, S) = \emptyset$ and $u - \mathrm{Min}(F, S) = \{-1\}$.

Example 6.16 Let $X = \mathbb{R}$, $S = [0, 1]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [[(2,2),(3,3)]], & \text{if } x = 0, \\ [[(0,0),(4,4)]], & \text{otherwise.} \end{cases}$$

where $[[(a,b),(c,d)]] = \{(y_1, y_2) : a \leq y_1 \leq c, b \leq y_2 \leq d\}$. After a short calculation, we get $\text{Min}(F, S) = (0, 1]$ and $u - \text{Min}(F, S) = \{0\}$.

We now recall the notions of optimal solutions of the problem (SOP) with respect to the relations $\preceq_C^*$ and $\prec_C^*$, where $* \in \{ml, mu\}$. For the set order relations $\preceq_C^*$ and $\prec_C^*$, we assume that Y is a normed space, $F(x) \in \mathcal{B}(Y)$ for all $x \in S$, $K := C$ is a closed convex and pointed cone with $\text{int}C \neq \emptyset$ and $F(x) \neq \emptyset$ for all $x \in X$.

Definition 6.14 Let $* \in \{ml, mu\}$. An element $\bar{x} \in S$ is called

(a) a $*$-*minimal solution* of the problem (SOP) with respect to $\preceq_C^*$ if there does not exist any $x \in S$ such that $F(x) \preceq_C^* F(\bar{x})$ and $F(x) \neq F(\bar{x})$, that is, either $F(x) \npreceq_C^* F(\bar{x})$ or $F(x) = F(\bar{x})$ for any $x \in S$;
(b) a $*$-*weak minimal solution* of the problem (SOP) with respect to $\prec_C^*$ if

$$F(x) \prec_C^* F(\bar{x}) \text{ for some } x \in S \quad \Rightarrow \quad F(\bar{x}) \prec_C^* F(x).$$

We denote the set of $*$-minimal and $*$-weak minimal solutions of the problem (SOP) by $* - \text{Min}(F, S)$ and $* - \text{WMin}(F, S)$, respectively.

Remark 6.9 (a) Since the set order relations $\preceq_C^*$, $* \in \{ml, mu\}$, are partial order, Definition 6.14(a) can also be written as follows:
An element $\bar{x} \in S$ is said to be a $*$-*minimal solution* of the problem (SOP) if

$$F(x) \preceq_C^* F(\bar{x}) \text{ for some } x \in S \quad \Rightarrow \quad F(\bar{x}) = F(x).$$

Furthermore, if $\preceq_C^t$ is partial order for any $t \in \{l, u, s, c, m, mc, mn, \lambda\}$, then the above also holds true for Definition 6.12(a).
(b) Definition 6.12(c) and Definition 6.14(b) can also be written as follows:
An element $\bar{x} \in S$ is said to be a t-weak minimal solution ($*$-weak minimal solution) of the problem (SOP) if there does not exist any $x \in S$ such that $F(x) \prec_C^t F(\bar{x})$ ($F(x) \prec_C^* F(\bar{x})$).

Clearly, $* - \text{Min}(F, S) \subseteq * - \text{WMin}(F, S)$. However, the reverse inclusion may not hold.

Example 6.17 Let $X = \mathbb{R}$, $S = [0, 1]$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 4, u > 0, v > 0\}, & \text{if } x = 0, \\ (0, 3) \times (0, 3), & \text{otherwise.} \end{cases}$$

Then, $0 \in ml - \text{WMin}(F, S)$ but $0 \notin ml - \text{Min}(F, S)$.

Example 6.18 Let $X = \mathbb{R}, S = [-1, 0], Y = \mathbb{R}^2, C = \mathbb{R}^2_+,$ and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 9, u < 0, v < 0\}, & \text{if } x = 0, \\ (0, -1) \times (0, -1), & \text{otherwise.} \end{cases}$$

Then, $0 \in mu - \text{WMin}(F, S)$ but $0 \notin mu - \text{Min}(F, S)$.

Definition 6.15 Let S be a nonempty convex subset of X. A set-valued map $F : S \rightrightarrows Y$ is said to be

(a) *strictly natural ml-type C-quasi-convex* on S if for all $x_1, x_2 \in S$ with $x_1 \neq x_2$ and all $t \in (0, 1)$, there exists $\lambda \in [0, 1]$ such that

$$F(tx_1 + (1-t)x_2) \prec_C^{ml} \lambda F(x_1) + (1-\lambda)F(x_2).$$

(b) *strictly natural u-type C-quasi-convex* on S if for all $x_1, x_2 \in S$ with $x_1 \neq x_2$ and all $t \in (0, 1)$, there exists $\lambda \in [0, 1]$ such that

$$F(tx_1 + (1-t)x_2) \prec_C^{mu} \lambda F(x_1) + (1-\lambda)F(x_2).$$

Proposition 6.10 *Assume that S is a convex subset of X, $F : S \rightrightarrows Y$ is a strictly natural l-type C-quasi-convex map on S with nonempty compact values. Then, $ml -$* $\text{Min}(F, S) = ml - \text{WMin}(F, S)$.

Proposition 6.11 *Assume that S is a convex subset of X, $F : S \rightrightarrows Y$ is a strictly natural u-type C-quasi-convex map on S with nonempty compact values. Then, $mu -$* $\text{Min}(F, S) = mu - \text{WMin}(F, S)$.

We now give the following example to show that an ml-minimal solution may not be a minimal solution and vice-versa.

Example 6.19 Let $X = \mathbb{R},\ S = [-1, 1],\ Y = \mathbb{R}^2,\ C = \mathbb{R}^2_+,$ and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} (0, 4) \times (0, 4), & \text{if } x = -1, \\ [0, 2] \times [0, 2], & \text{otherwise.} \end{cases}$$

Then, $\text{Min}(F, S) = (-1, 1]$ and $ml - \text{Min}(F, S) = [-1, 1]$.

In the above example, if $F(-1) = [0, 4] \times [0, 4]$ is replaced by $F(-1) = (0, 4) \times (0, 4)$, then $\text{Min}(F, S) = [-1, 1]$ and $ml - \text{Min}(F, S) = \{-1\}$.

We now give the following examples to show that an mu-minimal solution may not be a minimal solution and vice-versa.

Example 6.20 Let $X = \mathbb{R}, S = \{0, 1\}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+,$ and $F : S \rightrightarrows Y$ be defined by

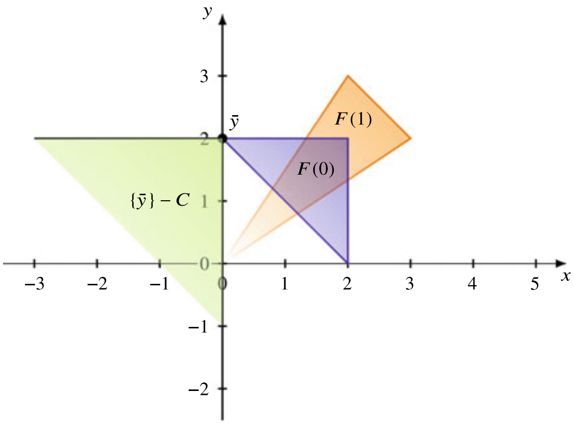


Fig. 6.12 Illustration of Example 6.20 with $C = \mathbb{R}^2_+$

$$F(x) = \begin{cases} \text{conv}\{(0,0), (2,3), (3,2)\}, & \text{if } x = 0, \\ \text{conv}\{(2,0), (0,2), (2,2)\}, & \text{if } x = 1. \end{cases}$$

From Fig. 6.12, we can see that there does not exist any $\bar{y} \in F(1)$ such that $(\{\bar{y}\} - C) \cap F(S) = \{\bar{y}\}$. Therefore, $1 \notin \text{Min}(F, S)$ but $1 \in mu - \text{Min}(F, S)$ because $F(0) \not\preceq^{mu}_C F(1)$.

Example 6.21 Let $X = \mathbb{R}$, $S = \{0, 1\}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, and $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}, & \text{if } x = 0, \\ \{(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})\}, & \text{if } x = 1. \end{cases}$$

Then we can easily obtain $\text{Min}(F, S) = \{0, 1\}$ but $ml - \text{Min}(F, S) = \{1\}$. Indeed, $F(1) \dot{-} F(0) = \emptyset$. Therefore, $F(1) \dot{-} F(0) \cap C = \emptyset$. Thus, $F(0) \not\preceq^{mu}_C F(1)$ and therefore $1 \in mu - \text{Min}(F, S)$. Moreover, $F(0) \dot{-} F(1) \neq \emptyset$ (see Fig. 6.13). Therefore, $F(1) \dot{-} F(0) \cap C \neq \emptyset$. Thus, $F(1) \preceq^{mu}_C F(0)$ and therefore $0 \notin mu - \text{Min}(F, S)$.

The following example shows that an mu-minimal solution may not be a u-minimal solution and vice-versa.

Example 6.22 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [0, 1]$, and $C = \mathbb{R}^2_+$. Let $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [0, -1] \times [0, -1], & \text{if } x = 0, \\ (0, -2) \times (0, -2), & \text{otherwise.} \end{cases}$$

Then,

$$mu - \text{Min} F = [0, 1] \quad \text{and} \quad u - \text{Min} F = (0, 1].$$

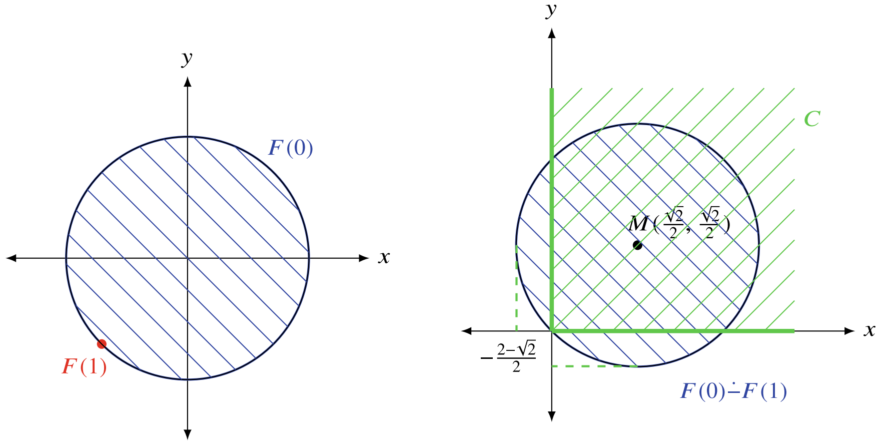


Fig. 6.13 Illustration of Example 6.21 with $C = \mathbb{R}^2_+$

Furthermore, if we replace the value of $F(x)$ for all $x \in (0, 1]$ by $[0, -2] \times [0, -2]$, then

$$mu - \text{Min} F = \{0\} \quad \text{and} \quad u - \text{Min} F = [0, 1].$$

Example 6.23 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [-1, 1]$, and $C = \mathbb{R}^2_+$. Let $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} (0, 1) \times (0, 1), & \text{if } x = -1, \\ [0, 1/2] \times [0, 1/2], & \text{otherwise.} \end{cases}$$

We can easily see that

$$ml - \text{Min} F = [-1, 1] \quad \text{and} \quad l - \text{Min} F = (-1, 1].$$

Furthermore, if we replace the value of $F(-1)$ by $[0, 1] \times [0, 1]$, then

$$ml - \text{Min} F = \{-1\} \quad \text{and} \quad l - \text{Min} F = [-1, 1].$$

The following theorem shows that every weak minimal solution of (SOP) is a ml-weak minimal solution of the problem (SOP).

Theorem 6.5.1 [99] *If C is a closed convex pointed cone in Y with* int$C \neq \emptyset$*, then*

$$\text{WMin}(F, S) \subseteq ml - \text{WMin}(F, S).$$

From Example 6.17, it is clear that the reverse inclusion of the above theorem fails because $0 \in ml - \text{WMin}(F, S)$ but $0 \notin \text{WMin}(F, S)$.

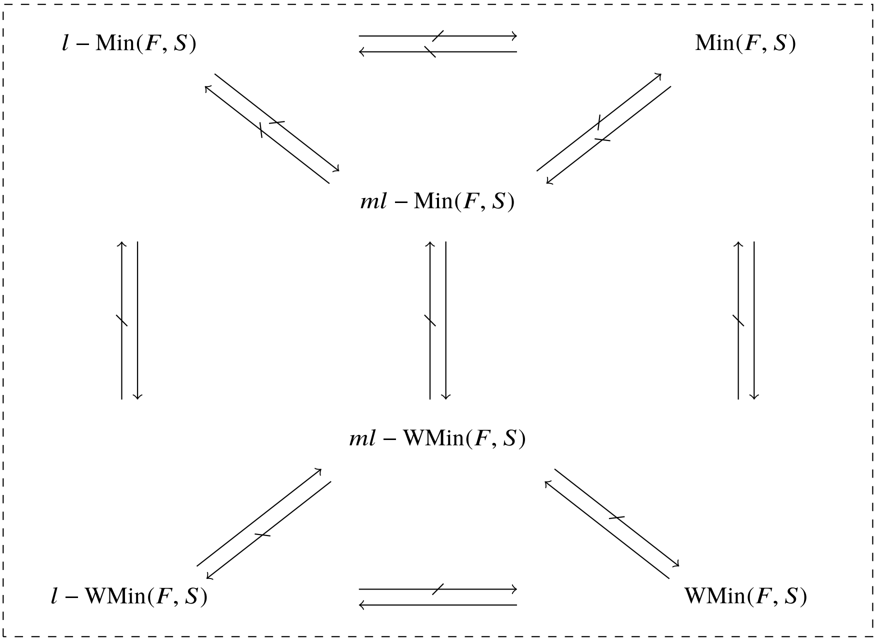


Fig. 6.14 Relationship between different kinds of solution concepts of the problem (SOP)

The following theorem shows that every l-weak (u-weak) minimal solution of (SOP) is a ml-weak (mu-weak) minimal solution of the problem (SOP).

Theorem 6.5.2 [99] *If C is a closed convex pointed cone in Y with $\text{int}C \neq \emptyset$, then*

$$u - \text{WMin}(F, S) \subseteq mu - \text{WMin}(F, S) \quad \text{and} \quad l - \text{WMin}(F, S) \subseteq ml - \text{WMin}(F, S).$$

However, the converse of the above theorem may not hold. For instance, in Example 6.22, $0 \in mu - \text{WMin}(F, S)$ but $0 \notin u - \text{WMin}(F, S)$ and in Example 6.23, $-1 \in ml - \text{WMin}(F, S)$ but $-1 \notin l - \text{WMin}(F, S)$.

In the following diagram, we summarize the relations among various notions of minimal and weak minimal solutions involving the set order relations $\preceq_C^l$ and $\preceq_C^{ml}$. In a similar way, we can establish relations among various notions of minimal and weak minimal solutions using different set order relations (Fig. 6.14).

6.5.1 Solution Concepts in Set Optimization with Respect to Variable Domination Structures

This section introduces different concepts for minimal elements of a family of sets and solution concepts for the problem (SOP) with respect to variable ordering structures.

These concepts are defined based on set relations introduced in Definition 6.9. In addition, we present the relationship between the sets of different minimal elements.

Definition 6.16 [104] Let $\mathcal{A}$ be a family of nonempty sets in Y and $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map.

(a) A set $\bar{A} \in \mathcal{A}$ is called a *minimal element* of $\mathcal{A}$ with respect to $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$, if

$$\forall\, A \in \mathcal{A}, \quad A \preceq_t^{\mathcal{K}} \bar{A} \quad \Rightarrow \quad \bar{A} \preceq_t^{\mathcal{K}} A.$$

(b) A set $\bar{A} \in \mathcal{A}$ is called a *strong minimal element* of $\mathcal{A}$ with respect to $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$, if

$$\forall\, A \in \mathcal{A}, \quad \bar{A} \preceq_t^{\mathcal{K}} A.$$

(c) A set $\bar{A} \in \mathcal{A}$ is called a *strict minimal element* of $\mathcal{A}$ with respect to $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$, if

$$\forall\, A \in \mathcal{A}, \quad A \preceq_t^{\mathcal{K}} \bar{A} \quad \Rightarrow \quad \bar{A} = A.$$

The sets of all minimal, strong minimal, and strict minimal elements of $\mathcal{A}$ with respect to $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$, are denoted by $\mathrm{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $\mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$, and $\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$, respectively.

Remark 6.10 (a) When $\mathcal{A}$ is a family of singleton sets and $\mathcal{K}(y)$ is a closed, convex and pointed cone for each $y \in Y$, then the definition of strictly minimal element of $\mathcal{A}$ with respect to $\preceq_t^{\mathcal{K}}$ reduces to the definition of nondominated element of $\mathcal{A}$ with respect to $\mathcal{K}$ (see [40, Definition 2.7]).

(b) If $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}})$, then for all $B \sim \bar{A}$, we have $B \in \mathrm{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}})$. From Definition 6.16, we obtain

$$\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \quad \text{and} \quad \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}}).$$

However, neither $\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ nor $\mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ always holds (see [104]).

(b)

$$\mathrm{SoMin}(\mathcal{A}, \preceq_{cl}^{\mathcal{K}}) \subseteq \mathrm{SoMin}(\mathcal{A}, \preceq_{l}^{\mathcal{K}}) \subseteq \mathrm{SoMin}(\mathcal{A}, \preceq_{pl}^{\mathcal{K}})$$

$$\mathrm{SiMin}(\mathcal{A}, \preceq_{pl}^{\mathcal{K}}) \subseteq \mathrm{SiMin}(\mathcal{A}, \preceq_{l}^{\mathcal{K}}) \subseteq \mathrm{SiMin}(\mathcal{A}, \preceq_{cl}^{\mathcal{K}})$$

$$\mathrm{SoMin}(\mathcal{A}, \preceq_{cu}^{\mathcal{K}}) \subseteq \mathrm{SoMin}(\mathcal{A}, \preceq_{u}^{\mathcal{K}}) \subseteq \mathrm{SoMin}(\mathcal{A}, \preceq_{pu}^{\mathcal{K}})$$

$$\mathrm{SiMin}(\mathcal{A}, \preceq_{pu}^{\mathcal{K}}) \subseteq \mathrm{SiMin}(\mathcal{A}, \preceq_{u}^{\mathcal{K}}) \subseteq \mathrm{SiMin}(\mathcal{A}, \preceq_{cu}^{\mathcal{K}}).$$

The following example illustrates that neither $\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ nor $\mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ always holds.

Example 6.24 [104] Let

$$\begin{aligned} A_1 &= \{(y_1, y_2) \in \mathbb{R}^2 : 2 \leq y_1, y_2 \leq 3,\ y_1 + y_2 \leq 5\},\\ A_2 &= \{(2, y_2) \in \mathbb{R}^2 : 2 \leq y_2 \leq 3\} \cup \{(y_1, 2) \in \mathbb{R}^2 : 2 \leq y_1 \leq 3\},\\ A_3 &= \{(5, 5)\},\\ A_4 &= \{(y_1, y_2) \in \mathbb{R}^2 : 3 \leq y_1 \leq 5,\ 0 \leq y_2 \leq 1\}, \end{aligned}$$

and the set-valued map $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be defined as

$$\mathcal{K}(t) = \begin{cases} \{(d_1, d_2) : 0 \leq d_1 \leq 2d_2\}, & \text{if} t \in \mathbb{R}^2 \setminus \{(1, 3)\},\\ \mathbb{R}^2_+, & \text{if} t = \{(1, 3)\}. \end{cases}$$

From Fig. 6.15, we can easily see that

$$A_1 \preceq_l^{\mathcal{K}} A_2, \quad A_1 \preceq_l^{\mathcal{K}} A_3, \quad A_1 \not\preceq_l^{\mathcal{K}} A_4,$$

$$A_2 \preceq_l^{\mathcal{K}} A_1, \quad A_2 \preceq_l^{\mathcal{K}} A_3, \quad A_1 \not\preceq_2^{\mathcal{K}} A_4,$$

$$A_3 \preceq_l^{\mathcal{K}} A_1, \quad A_3 \preceq_l^{\mathcal{K}} A_2, \quad A_3 \not\preceq_l^{\mathcal{K}} A_4,$$

$$A_4 \not\preceq_l^{\mathcal{K}} A_1, \quad A_4 \not\preceq_l^{\mathcal{K}} A_2, \quad A_4 \preceq_l^{\mathcal{K}} A_1.$$

Let $\mathcal{A} := \{A_1, A_2, A_3\}$. Then we have

$$\text{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}}) = \{A_1, A_2\}, \quad \text{SoMin}(\mathcal{A}, \preceq_l^{\mathcal{K}}) = \{A_1, A_2\}, \quad \text{SiMin}(\mathcal{A}, \preceq_l^{\mathcal{K}}) = \emptyset.$$

Let $\mathcal{A}' := \{A_1, A_2, A_3, A_4\}$. Then we have

$$\text{Min}(\mathcal{A}', \preceq_l^{\mathcal{K}}) = \{A_1, A_2, A_4\}, \quad \text{SoMin}(\mathcal{A}', \preceq_l^{\mathcal{K}}) = \{A_1, A_2\}, \quad \text{SiMin}(\mathcal{A}', \preceq_l^{\mathcal{K}}) = \{A_4\}.$$

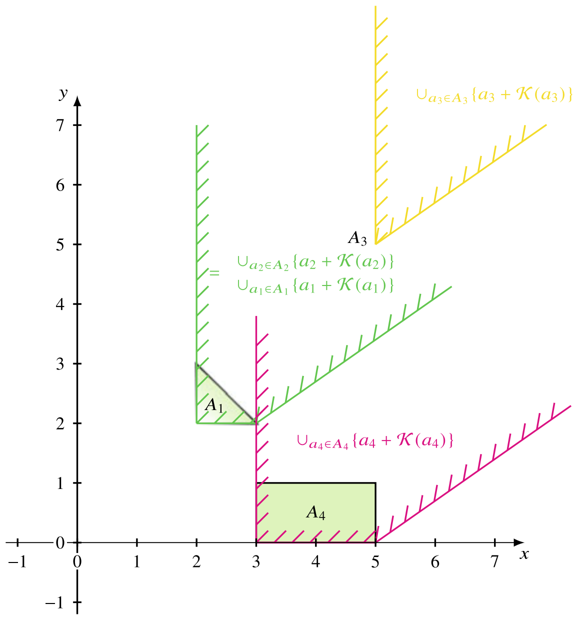
Let $\mathcal{A}'' := \{A_3, A_4\}$. Then we have

$$\text{Min}(\mathcal{A}'', \preceq_l^{\mathcal{K}}) = \text{SoMin}(\mathcal{A}'', \preceq_l^{\mathcal{K}}) = \text{SiMin}(\mathcal{A}'', \preceq_l^{\mathcal{K}}) = \{A_4\}.$$

Proposition 6.12 *Let $\mathcal{A}$ be a family of sets in $P(Y)$, $S \in P(Y)$, and $|S|$ denote the number of elements in S. Then, for $t \in \{l, u, cl, cu, pl, pu\}$, the following statements hold.*

(a) *If* $|\text{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| > 1$, *then* $\text{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) = \emptyset$.
(b) *If* $|\text{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| > 1$, *then* $\text{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) = \emptyset$.
(c) *If* $\text{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \cap \text{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \neq \emptyset$, *then*

Fig. 6.15 Illustration of Example 6.24



$$\begin{cases} |\text{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| = |\text{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| = 1, \\ \text{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) = \text{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}). \end{cases}$$

Now, we define the solution concepts of set optimization problem (SOP) with respect to the set order relations $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$. Note that the solution concepts in the following definition are given in the preimage space X, whereas the solution concepts in Definition 6.16 are formulated in the image space Y.

Definition 6.17 [104] Let $F : X \rightrightarrows Y$ and $\mathcal{K} : Y \rightrightarrows Y$ be two set-valued maps such that $F(x)$ and $\mathcal{K}(y)$ are nonempty sets for all $x \in X$, $y \in Y$.

(a) An element $\bar{x} \in X$ is called a *minimal element* of (SOP) with respect to $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$, if

$$x \in X, \quad F(x) \preceq_t^{\mathcal{K}} F(\bar{x}) \quad \Rightarrow \quad F(\bar{x}) \preceq_t^{\mathcal{K}} F(x).$$

(b) An element $\bar{x} \in X$ is called a *strong minimal element* of (SOP) with respect to $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$, if

$$\forall x \in X \setminus \{\bar{x}\}, \quad F(\bar{x}) \preceq_t^{\mathcal{K}} F(x).$$

(c) An element $\bar{x} \in X$ is called a *strict minimal element* of (SOP) with respect to $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$, if

$$x \in X, \quad F(x) \preceq_t^{\mathcal{K}} F(\bar{x}) \text{ or } F(x) = F(\bar{x}) \quad \Rightarrow \quad x = \bar{x}. \tag{6.8}$$

The sets of all minimal, strong minimal, and strict minimal elements of (SOP) with respect to $\preceq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$, are denoted by $\text{Min}(F(X), \preceq_t^{\mathcal{K}})$, $\text{SoMin}(F(X), \preceq_t^{\mathcal{K}})$, and $\text{SiMin}(F(X), \preceq_t^{\mathcal{K}})$, respectively.

Remark 6.11 [104]

(a) If the relation $\preceq_t^{\mathcal{K}}$ is reflexive, then Definition 6.17(c) is equivalent to

$$x \in X, \quad F(x) \preceq_t^{\mathcal{K}} F(\bar{x}) \quad \Rightarrow \quad x = \bar{x}.$$

(b) Definition 6.17 implies that $\text{SoMin}(F(X), \preceq_t^{\mathcal{K}})$ and $\text{SiMin}(F(X), \preceq_t^{\mathcal{K}})$ are subsets of $\text{Min}(F(X), \preceq_t^{\mathcal{K}})$. Furthermore, the following relations for the sets of minimal solutions of (SOP) with respect to the lower set order relations $\preceq_l^{\mathcal{K}}$, $\preceq_{cl}^{\mathcal{K}}$, and $\preceq_{pl}^{\mathcal{K}}$ hold:

$$\text{SoMin}(F(X), \preceq_{cl}^{\mathcal{K}}) \subseteq \text{SoMin}(F(X), \preceq_l^{\mathcal{K}}) \subseteq \text{SoMin}(F(X), \preceq_{pl}^{\mathcal{K}})$$

and

$$\text{SiMin}(F(X), \preceq_{pl}^{\mathcal{K}}) \subseteq \text{SiMin}(F(X), \preceq_l^{\mathcal{K}}) \subseteq \text{SiMin}(F(X), \preceq_{cl}^{\mathcal{K}}).$$

Similarly, the following relations for the sets of minimal solutions of (SOP) with respect to the upper set order relations $\preceq_u^{\mathcal{K}}$, $\preceq_{cu}^{\mathcal{K}}$, and $\preceq_{pu}^{\mathcal{K}}$ hold:

$$\text{SoMin}(F(X), \preceq_{cu}^{\mathcal{K}}) \subseteq \text{SoMin}(F(X), \preceq_u^{\mathcal{K}}) \subseteq \text{SoMin}(F(X), \preceq_{pu}^{\mathcal{K}})$$

$$\text{SiMin}(F(X), \preceq_{pu}^{\mathcal{K}}) \subseteq \text{SiMin}(F(X), \preceq_u^{\mathcal{K}}) \subseteq \text{SiMin}(F(X), \preceq_{cu}^{\mathcal{K}}).$$

6.6 Existence of Solutions

It is well known that the semicontinuity for set-valued maps plays a significant role to study the set optimization problems. Kuroiwa [113] and Jahn and Ha [92] extended the concept of semicontinuities for set-valued maps by using the set order relations $\preceq_C^l$ and $\preceq_C^u$ and applied them to obtain the existence of solutions for set optimization problems. Hernández et al. [84] further used and investigated the semicontinuity for set-valued maps to study the existence of solutions of the problem (SOP) and the relation among solutions using vector approach and set approach. Very recently, Zhang and Huang [153] introduced the notion of lower semicontinuity from above and used it to obtain the existence of results and discussed the link between solutions of the problem (SOP) obtained by vector approach and set approach.

6.6.1 Generalized Semicontinuity for Set-Valued Maps

In this subsection, we introduce the notions of generalized semicontinuity for set-valued maps involving the partial set order relation $\preceq_C^{ml}$. Further, we study some properties of the generalized semicontinuity for set-valued maps, which are then applied to study the existence of solutions for set optimization problems.

Throughout this subsection, we assume that S is a nonempty subset of a Hausdorff topological vector space X and Y is a real normed space. Further, we assume that $F(x) \in \mathcal{B}(Y)$ for all $x \in S$, C is a closed convex and pointed cone with $\text{int}C \neq \emptyset$ and $F(x) \neq \emptyset$ for all $x \in X$.

Definition 6.18 The set-valued map $F : X \rightrightarrows Y$ is said to have

(a) $\preceq_C^{ml}$*-lower property* at $\bar{x} \in S$ if there exists a point $x \in S$ such that $F(x) \preceq_C^{ml} F(\bar{x})$;
(b) $\prec_C^{ml}$*-lower property* at $\bar{x} \in S$ if there exists a point $x \in S$ such that $F(x) \prec_C^{ml} F(\bar{x})$;
(c) *strictly* $\prec_C^{ml}$*-lower property* at $\bar{x} \in S$ if there exists a point $x \in S$ such that $F(x) \cap F(\bar{x}) = \emptyset$ and $F(x) \prec_C^{ml} F(\bar{x})$.

Definition 6.19 Let $\{A_\alpha\}_{\alpha \in I}$ be a net and $(I, <)$ be a directed set. The net $\{A_\alpha\}_{\alpha \in I}$ is said to be

(a) $\preceq_C^{ml}$*-increasing* if for $\alpha, \beta \in I$ with $\alpha < \beta$, we have $A_\alpha \preceq_C^{ml} A_\beta$;
(b) $\preceq_C^{ml}$*-decreasing* if for $\alpha, \beta \in I$ with $\alpha < \beta$, we have $A_\beta \preceq_C^{ml} A_\alpha$.

Definition 6.20 A set-valued map $F : S \rightrightarrows Y$ is said to be *ml-type Demi-lower semicontinuous* at $\bar{x} \in S$ if for any net $\{x_\alpha\}_{\alpha \in I}$ in S such that $x_\alpha \to \bar{x}$ and $\{F(x_\alpha)\}_{\alpha \in I}$ is a $\preceq_C^{ml}$-decreasing net, the following condition holds:

$$F(\bar{x}) \preceq_C^{ml} \text{Limsup}_\alpha (F(x_\alpha) + C),$$

where $\text{Limsup}_\alpha (F(x_\alpha) + C)$ denotes the set of all cluster points of $\{y_\alpha : y_\alpha \in (F(x_\alpha) + C)\}_{\alpha \in I}$.

We say that F is ml-type Demi-lower semicontinuous on S if it is ml-type Demi-lower semicontinuous at each point $\bar{x} \in S$.

Definition 6.21 Let X be a topological space. A set-valued map $F : X \rightrightarrows Y$ is said to be

(a) $\preceq_C^{ml}$*-lower semicontinuous from above* at $\bar{x} \in X$ if for any net $\{x_\alpha\}_{\alpha \in I}$ in X with $x_\alpha \to \bar{x}$ such that $\{F(x_\alpha)\}_{\alpha \in I}$ is a $\preceq_C^{ml}$-decreasing net, one has $F(\bar{x}) \preceq_C^{ml} F(x_\alpha)$ for all $\alpha \in I$;
(b) $\preceq_C^{ml}$*-upper semicontinuous from below* at $\bar{x} \in X$ if for any net $\{x_\alpha\}_{\alpha \in I}$ in X with $x_\alpha \to \bar{x}$ such that $\{F(x_\alpha)\}_{\alpha \in I}$ is a $\preceq_C^{ml}$-increasing net, one has $F(x_\alpha) \preceq_C^{ml} F(\bar{x})$ for all $\alpha \in I$.

We say that F is $\preceq_C^{ml}$-lower semicontinuous from above (respectively, $\preceq_C^{ml}$-upper semicontinuous from below) on X if it is $\preceq_C^{ml}$-lower semicontinuous from above (respectively, $\preceq_C^{ml}$-upper semicontinuous from below) at each point $\bar{x} \in X$.

Remark 6.12 The ml-type Demi-lower semicontinuity implies the $\preceq_C^{ml}$-lower semicontinuity from above, but the following example shows that the converse is not true.

Example 6.25 Let $S = \mathbb{R}^2$, $Y = \mathbb{R}^2$, and $C = \mathbb{R}^2_+$. Let $F : S \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{(0, 1)\}, & \text{if } x > 0, \\ \{(0, \varepsilon) : 0 < \varepsilon < 2\}, & \text{if } x = 0, \\ \{(0, -1)\}, & \text{if } x < 0. \end{cases}$$

At $\bar{x} = 0$, one can easily see that for any net $\{x_\alpha\}_{\alpha \in I}$ in S with $x_\alpha \to 0$, $\{F(x_\alpha)\}_{\alpha \in I}$ is a $\preceq_C^{ml}$-decreasing net and $F(0) \preceq_C^{ml} F(x_\alpha)$ for all $\alpha \in I$. Hence, F is $\preceq_C^{ml}$-lower semicontinuous from above at $\bar{x} = 0$. However, F is not ml-type Demi-lower semicontinuous at $\bar{x} = 0$. Indeed, taking a sequence $\{x_n\} = \{\frac{1}{n}\}_{n \in \mathbb{N}}$, we get $\text{Limsup}_{n \to +\infty}(F(x_n) + C) = C$. After a short calculation, we obtain $F(0) \npreceq_C^{ml} \text{Limsup}_{n \to +\infty}(F(x_n) + C)$. Hence, F is not ml-type Demi-lower semicontinuous at $\bar{x} = 0$.

Definition 6.22 A set-valued map $F : X \rightrightarrows Y$ is said to be *$\preceq_C^{ml}$-lower semicontinuous* at $\bar{x} \in X$ if the set $\{x \in X : F(x) \preceq_C^{ml} F(\bar{x})\}$ is closed. We say that F is $\preceq_C^{ml}$-lower semicontinuous on X if it is $\preceq_C^{ml}$-lower semicontinuous at each point $\bar{x} \in X$.

Proposition 6.13 *If the set-valued map F is $\preceq_C^{ml}$-lower semicontinuous on X, then it is $\preceq_C^{ml}$-lower semicontinuous from above on X.*

Proof Let $\{x_\alpha\}_{\alpha \in I}$ be a net in X such that $x_\alpha \to \bar{x}$ and $F(x_\beta) \preceq_C^{ml} F(x_\alpha)$ for $\alpha < \beta$ with $\alpha, \beta \in I$. Then for each $\alpha \in I$, the net $\{x_\beta\}_{\alpha < \beta}$ satisfies $x_\beta \to \bar{x}$. By $\preceq_C^{ml}$-lower semicontinuity of F, one has $\bar{x} \in \{x \in X : F(x) \preceq_C^{ml} F(x_\alpha)\}$ for all $\alpha \in I$. This shows that F is $\preceq_C^{ml}$-lower semicontinuous from above on X.

□

The following example shows that the reverse of the above proposition is not true.

Example 6.26 Let $X = \mathbb{R}$, $Y = \mathbb{R}$, and $C = \mathbb{R}_+$. Let $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} \{0\}, & \text{if } x > 0, \\ [0, 2), & \text{if } -1 < x \leq 0, \\ \{1\}, & \text{if } x \leq -1. \end{cases}$$

At $\bar{x} = 0$, one can easily see that for any net $\{x_\alpha\}_{\alpha \in I}$ in X with $x_\alpha \to 0$, $\{F(x_\alpha)\}_{\alpha \in I}$ is $\preceq_C^{ml}$-decreasing net and $F(0) \preceq_C^{ml} F(x_\alpha)$ for all $\alpha \in I$. Hence, F is $\preceq_C^{ml}$-lower semicontinuous from above at $\bar{x} = 0$. However, the set $\{x \in X : F(x) \preceq_C^{ml} F(0)\} = \{x \in X : -1 < x \leq 0\}$ is not closed. Hence, F is not $\preceq_C^{ml}$-lower semicontinuous at $\bar{x} = 0$.

Remark 6.13 In a similar way, we can introduce the notions of generalized semi-continuity with respect to other different kinds of set order relations.

6.6.2 Existence of Solutions in Set Optimization Problems

In this subsection, we study the existence of results for solutions of set optimization problems with respect to the partial set order relation $\preceq_C^{ml}$ by using generalized semicontinuity. Since existence results for other set order relations can be obtained in a similar way, we skip such a study.

Throughout this subsection, unless otherwise specified, we assume that S is a nonempty subset of a Hausdorff topological vector space X and Y is a real normed space. Further, we assume that $F(x) \in \mathcal{B}(Y)$ for all $x \in S$, C is a closed convex and pointed cone with $\text{int}C \neq \emptyset$ and $F(x) \neq \emptyset$ for all $x \in X$.

Let $A, B \in P(Y)$ and $\bar{x} \in S$, we write

$$A \sim B \quad \Leftrightarrow \quad A \preceq_C^{ml} B \text{ and } B \preceq_C^{ml} A,$$

$$E(\bar{x}, \preceq_C^{ml}) = \{x \in S : F(\bar{x}) \sim F(x)\},$$

and the level set of F at $\bar{x} \in S$ is given by

$$L(\bar{x}, \preceq_C^{ml}) = \{x \in S : F(x) \preceq_C^{ml} F(\bar{x})\}.$$

It is simple to verify that $E(\bar{x}, \preceq_C^{ml}) \subseteq L(\bar{x}, \preceq_C^{ml})$. The converse holds for a *ml*-minimal solution of the problem (SOP).

Proposition 6.14 *$\bar{x} \in ml - Min(F, S)$ if and only if $E(\bar{x}, \preceq_C^{ml}) = L(\bar{x}, \preceq_C^{ml})$.*

The following result is obvious and so we skip its proof.

Proposition 6.15 *If $\bar{x} \in ml - Min(F, S)$, then $E(\bar{x}, \preceq_C^{ml}) \subseteq ml - Min(F, S)$.*

Theorem 6.6.1 *Let S be a nonempty compact subset of a Hausdorff topological vector space X. If the set-valued map $F : S \rightrightarrows Y$ is $\preceq_C^{ml}$-lower semicontinuous from above on S, then the problem (SOP) has ml-minimal solution.*

Proof We define a relation $\preceq$ on the quotient set $P(Y)/\sim$ as follows: For any $[A]$ and $[B]$ in $P(Y)/\sim$, $[A] \preceq [B] \Leftrightarrow A \preceq_C^{ml} B$. Let $\{[F(x_\alpha)]\}_{\alpha \in I}$ be a totally ordered set in the quotient set $P(Y)/\sim$. Without loss of generality, let $\alpha, \beta \in I$ with $\alpha < \beta$ such that $[F(x_\beta)] \preceq [F(x_\alpha)]$. Then the compactness of S implies that there exist $\tilde{I} \subseteq I$ and a subnet $\{x_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \tilde{I}}$ of $\{x_\alpha\}_{\alpha \in I}$ such that $x_{\tilde{\alpha}} \to \bar{x}$. Thus, by the $\preceq_C^{ml}$-lower semicontinuous from above of F, we know that $F(\bar{x}) \preceq_C^{ml} F(x_{\tilde{\alpha}})$ for all $\tilde{\alpha} \in \tilde{I}$. Hence, $[F(\bar{x})] \preceq [F(x_{\tilde{\alpha}})]$ for all $\tilde{\alpha} \in \tilde{I}$.

Next we prove that $[F(\bar{x})] \preceq [F(x_\alpha)]$ for all $\alpha \in I$. If it is not true, then there exists $\bar{\alpha} \in I$ such that $[F(\bar{x})] \npreceq [F(x_{\bar{\alpha}})]$. For each $\alpha' \in \tilde{I}$ with $\bar{\alpha} < \alpha'$, we have $[F(\bar{x}_{\alpha'})] \preceq$

$[F(x_{\tilde{\alpha}})]$. Since $[F(\bar{x})] \preceq [F(x_{\tilde{\alpha}})]$ for all $\tilde{\alpha} \in \tilde{I}$, one has $[F(\bar{x})] \preceq [F(x_{\alpha'})]$ and so $[F(\bar{x})] \preceq [F(x_{\bar{\alpha}})]$, which is a contradiction. Therefore, $[F(\bar{x})] \preceq [F(x_{\alpha})]$ for all $\alpha \in I$. Now by Zorn's lemma, we know that $\{[F(\bar{x})]\}_{x \in S}$ has a minimal element. That is, the problem (SOP) has a ml-minimal set.

□

Definition 6.23 Let S be a nonempty subset of a Hausdorff topological vector space and $F : S \rightrightarrows Y$ be a set-valued map. We say that S satisfies the condition (A) if for each net $\{x_\alpha\}_{\alpha \in I}$ in S such that $\{F(x_\alpha)\}_{\alpha \in I}$ is a $\preceq_C^{ml}$-decreasing net, there exist $\bar{I} \subseteq I$ and a subnet $\{x_{\bar{\alpha}}\}_{\bar{\alpha} \in \bar{I}}$ of $\{x_\alpha\}_{\alpha \in I}$ such that $\{x_{\bar{\alpha}}\} \to \bar{x} \in S$.

Similar to the proof of Theorem 6.6.1, we can obtain the following theorem.

Theorem 6.6.2 *Let S be a nonempty subset of a Hausdorff topological vector space and $F : S \rightrightarrows Y$ be a set-valued map. If S satisfies the condition (A) and F is $\preceq_C^{ml}$-lower semicontinuous from above on S, then the problem (SOP) has a ml-minimal solution.*

6.6.3 Relation Between Minimal Solutions with Respect to Vector and Set Approach

In this subsection, we study the relations between minimal solutions for set optimization problems with respect to vector approach and set approach involving the partial set order relation $\preceq_C^{ml}$.

Throughout this subsection, unless otherwise specified, we assume that S is a nonempty subset of a Hausdorff topological vector space X and Y is a real normed space. Further, we assume that $F(x) \in \mathcal{B}(Y)$ for all $x \in S$, C is a closed convex and pointed cone with $\text{int}C \neq \emptyset$ and $F(x) \neq \emptyset$ for all $x \in X$.

Now, we show how the set relation $\preceq_C^{ml}$ can help to find the minimal solutions by vector approach.

Lemma 6.6.1 *If $\bar{x} \in Min(F, S)$ with $F(x) \preceq_C^{ml} F(\bar{x})$ for each $x \in S$, then $x \in Min(F, S)$.*

Proof Assume that $\bar{x} \in \text{Min}(F, S)$. Let $\bar{y} \in F(\bar{x})$ be such that $\bar{y} \in \min F(S)$. We only need to show that $\bar{y} \in F(x)$. Assume that $\bar{y} \notin F(x)$. Then by the hypothesis, we have $F(x) \preceq_C^{ml} F(\bar{x})$, that is, $F(x) \dot{-} F(\bar{x}) \cap (-C) \neq \emptyset$. Hence, there exists a $c \in C$ such that $-c \in F(x) \dot{-} F(\bar{x})$, equivalently $-c + F(\bar{x}) \subseteq F(x)$. Therefore, there exists $\bar{y} \in F(\bar{x})$ such that $-c + \bar{y} = y$ for some $y \in F(x)$. Then we have $\bar{y} = \tilde{y} + c$ for some $\tilde{y} \in F(x)$. This implies that $\bar{y} \in \tilde{y} + C$, that is, $\tilde{y} \preceq_C \bar{y}$ for some $\tilde{y} \in F(x)$ which contradicts to $\bar{y} \in \min F(S)$. Thus $\bar{y} \in F(x)$ and hence $x \in \text{Min}(F, S)$.

□

Proposition 6.16 *Let $\bar{x} \in Min(F, S)$. Then, only one of the following two assertions holds.*

(a) *$\bar{x}$ is a ml-minimal solution of the problem (SOP).*
(b) *There exists a minimal solution $\hat{x} \in Min(F, S)$ of the problem (SOP) such that $F(\hat{x}) \preceq_C^{ml} F(\bar{x})$ and $F(\hat{x}) \nsim F(\bar{x})$.*

Proof By the definition of ml-minimal solution, (b) is false if (a) holds. Assume that (a) does not hold. Then there exists $\hat{x} \in S$ such that $F(\hat{x}) \preceq_C^{ml} F(\bar{x})$ and $F(\hat{x}) \nsim F(\bar{x})$. Since $\bar{x} \in \mathrm{Min}(F, S)$, by Lemma 6.6.1, we have that $\hat{x} \in \mathrm{Min}(F, S)$ and (b) holds.

□

Definition 6.24 A set-valued map $F : S \rightrightarrows Y$ is said to be *strongly injective* on S if for any $x_1, x_2 \in S$, $F(x_2) \preceq_C^{ml} F(x_1)$ and $F(x_1) \npreceq_C^{ml} F(x_2)$ imply that $F(x_1) \cap F(x_2) \neq \emptyset$.

Lemma 6.6.2 *If $\bar{x} \in Min(F, S)$ and F is strongly injective on S, then $\bar{x} \in ml - Min(F, S)$.*

Proof Let $\bar{x} \in \mathrm{Min}(F, S)$. Assume that $\bar{x} \notin ml - \mathrm{Min}(F, S)$. Then there exists $\tilde{x} \in S$ such that $F(\tilde{x}) \preceq_C^{ml} F(\bar{x})$ and $F(\bar{x}) \neq F(\tilde{x})$. Since $\bar{x} \in \mathrm{Min}(F, S)$, we can choose $\bar{y} \in F(\bar{x})$ such that $\bar{y} \in \min F(S)$. By Lemma 6.6.1, we have $\bar{y} \in F(\tilde{x})$, which contradicts the fact that F is strongly injective. Therefore, $\bar{x} \in ml - \mathrm{Min}(F, S)$.

□

Theorem 6.6.3 *Let S be a nonempty subset of a Hausdorff topological vector space and $F : S \rightrightarrows Y$ satisfy $\preceq_C^{ml}$-lower property at $\bar{x} \in S$ with $F(\bar{x}) \cap F(\tilde{x}) = \emptyset$, $\tilde{x} \in S$. If $\min F(\bar{x}) \neq \emptyset$ and $\bar{x} \in ml - Min(F, S)$, then $\bar{x} \in Min(F, S)$.*

Proof Let $\bar{x} \in ml - \mathrm{Min}(F, S)$ with $\min F(\bar{x}) \neq \emptyset$. Assume to the contrary that $\bar{x} \notin \mathrm{Min}(F, S)$. Then $F(\bar{x}) \cap \min F(S) = \emptyset$. Since $\min F(\bar{x}) \subseteq F(\bar{x}) \subseteq F(S)$, we have $\min F(\bar{x}) \cap \min F(S) = \emptyset$. Since F has the $\preceq_C^{ml}$-lower property at $\bar{x} \in S$, there exists $\tilde{x} \in S$ such that $F(\tilde{x}) \prec_C^{ml} F(\bar{x})$. Then, there exists a $c \in C$ such that $-c + F(\bar{x}) \subseteq F(\tilde{x})$. Therefore, there exists $\bar{y} \in F(\bar{x})$ such that $-c + \bar{y} = \tilde{y}$ for some $\tilde{y} \in F(\tilde{x})$. This implies that $\bar{y} \in \tilde{y} + C$, that is, $\tilde{y} \preceq_C \bar{y}$ for some $\tilde{y} \in F(\tilde{x})$. Since $\bar{x} \in ml - \mathrm{Min}(F, S)$, we have $F(\bar{x}) \prec_C^{ml} F(\tilde{x})$ and $F(\bar{x}) \neq F(\tilde{x})$. By $F(\bar{x}) \cap F(\tilde{x}) = \emptyset$ and $\tilde{y} \in F(\tilde{x})$, there exists a $\hat{y} \in F(\bar{x})$ such that $\hat{y} \preceq_C \tilde{y}$. Using the transitivity of the order relation $\preceq_C$, we get $\hat{y} \preceq_C \bar{y}$. Thus $\bar{y} \notin \min F(\bar{x})$, which contradicts the fact that $\min F(\bar{x}) \neq \emptyset$. Thus, we have $F(\bar{x}) \cap \min F(S) \neq \emptyset$ and hence $\bar{x} \in \mathrm{Min}(F, S)$. □

6.7 Ekeland's Variational Principle for Set-Valued Maps

Ekeland's variational principle (in short, EVP) is one of the fundamental results from nonlinear analysis which was developed in the pioneer papers [45–47] by

I. Ekeland. One of the most important ideas of EVP is that in the absence of a known minimum, one can use EVP to reach close to a minimum. It is found that several other fundamental results from nonlinear analysis, namely, Caristi's fixed point theorem [23, 24], Takahashi's minimization theorem [147], Phelps's minimal element theorem [137, 138], etc., are equivalent to EVP in the sense that they can be achieved by using EVP and vice-versa. The EVP is one of the most powerful tools to deal with many applications in optimization, optimal control, global analysis, mathematical economy, partial differential equations, etc., see [3, 39, 45–47]. During the last three decades, EVP has been extended for vector-valued/set-valued maps and also under different space settings, see, for example, [3–6, 10, 12, 15, 17, 18, 20, 26–28, 50, 56, 58, 61, 63, 65, 71, 72, 76, 87, 89, 97, 131, 148, 149] and the references therein.

Ekeland's variational principle for vector-valued maps was explored by Németh [131], Tammer [148], and Isac [89]. However, each of these vector-valued versions have different conditions on the involved function. In 1998, Chen and Huang [27] unified these results. In [58, 59], a variational principle for a vector-valued map was presented as a consequence of the minimal point theorem on the product space. It is worth to mention that the minimal element theorems were established by Göpfert and Tammer [57] and further generalized by Göpfert, et al. [58, 59], Hamel and Löhne [71], Hamel [65], and Hamel and Tammer [74] on the product space $X \times Y$ in different settings. Such theorems played an important role to derive Ekeland's variational principle for vector-valued maps. These minimal element theorems are the extension of Phelps's minimal element theorem [137, 138].

Hamel and Löhne [72] considered a subset $\mathcal{A} \subseteq X \times P(Y)$, where X is a separated uniform space and Y is a topological vector space and introduced the following notation:

$$\mathcal{V}(\mathcal{A}) := \{V \in P(Y) : \exists x \in X : (x, V) \in \mathcal{A}\}.$$

Let Λ be the directed set and $\{q_\lambda\}_{\lambda \in \Lambda}$ be the family of quasi-metrics which generates the topology of the uniform space X. We write q_Λ if and only if an assertion holds for all $\lambda \in \Lambda$. Using the relation $\preceq^l_C$ and $\preceq^u_C$, Hamel and Löhne [72] introduced the following ordering relations on $X \times P(Y)$: For all $x_1, x_2 \in X$, $V_1, V_2 \in P(Y)$, and $k \in C \setminus -\mathrm{cl}C$,

$$(x_1, V_1) \preceq^k_l (x_2, V_2) \quad \Leftrightarrow \quad V_1 + q_\Lambda(x_1, x_2)k \preceq^l_C V_2,$$

and

$$(x_1, V_1) \preceq^k_u (x_2, V_2) \quad \Leftrightarrow \quad V_1 + q_\Lambda(x_1, x_2)k \preceq^u_C V_2.$$

Note that the previous relations can be read as

$$\text{for all } \lambda \in \Lambda, \quad V_1 + q_\lambda(x_1, x_2)k \preceq^l_C V_2,$$

and

$$\text{for all } \lambda \in \Lambda, \quad V_1 + q_\lambda(x_1, x_2)k \preceq^u_C V_2.$$

The relations $\preceq_l^k$ and $\preceq_u^k$ are reflexive and transitive on $X \times P(Y)$.

Hamel and Löhne [72] introduced the minimal element theorems for set-valued maps in the separated uniform spaces involving the set order relations $\preceq_C^l$ and $\preceq_C^u$. Such minimal element theorems are the extensions of minimal element theorems presented in [58, 59].

Moreover, they [72] introduced the concept of the domain of a set-valued map F for the set order relations $\preceq_C^l$ and $\preceq_C^u$ in the following way:

$$\preceq_C^l - \text{dom}\, F := \{x \in X : F(x) \preceq_C^l V \text{ for some nonempty } V \subseteq Y\},$$

and

$$\preceq_C^u - \text{dom}\, F := \{x \in X : F(x) \preceq_C^l V \text{ for some topologically bounded } V \subseteq Y\}.$$

They derived variational principle for set-valued maps involving the set order relation $\preceq_C^l / \preceq_C^u$.

In [12], we studied minimal element theorem, Ekeland's variational principle, Caristi's fixed point theorem, and Takahashi's minimization theorem involving set order relations $\preceq_k^{ml}$ and $\preceq_k^{mu}$ defined on $X \times P(Y)$ by using $\preceq_C^{ml}$ and $\preceq_C^{mu}$ as follows:

Let C be a solid convex cone in a normed space Y and (X, d) be a metric space. For all $x_1, x_2 \in X$, $V_1, V_2 \in P(Y)$, and $k \in \text{int}C$, define

$$(x_1, V_1) \preceq_k^{ml} (x_2, V_2) \quad \Leftrightarrow \quad V_1 + d(x_1, x_2)k \preceq_C^{ml} V_2,$$

and

$$(x_1, V_1) \preceq_k^{mu} (x_2, V_2) \quad \Leftrightarrow \quad V_1 + d(x_1, x_2)k \preceq_C^{mu} V_2.$$

It can be easily seen that the relations $\preceq_k^{ml}$ and $\preceq_k^{mu}$ are reflexive and transitive on $X \times P(Y)$.

We now consider the following assumptions.

Assumption 2 Let (X, d) be a complete metric space, Y be a real normed vector space, C be a solid closed convex pointed cone in Y, and $k \in \text{int}C$. Let $F : X \to P(Y)$ be a closed-valued map such that

(i) F is ml-bounded below (that is, there exists $V \in P(Y)$ such that $V \preceq_C^{ml} F(x)$ for all $x \in X$),
(ii) $\widetilde{S}(x) = \{\tilde{x} \in X : (\tilde{x}, F(\tilde{x})) \preceq_k^{ml} (x, F(x))\}$ is closed for all $x \in X$.

Assumption 3 Let (X, d) be a complete metric space, Y be a real normed vector space, C be a solid closed convex pointed cone in Y, and $k \in \text{int}C$. Let $F : X \to P(Y)$ be a closed-valued map such that

(i) F is mu-bounded below (that is, there exists $V \in P(Y)$ such that $V \preceq_C^{mu} F(x)$ for all $x \in X$),
(ii) $\widehat{S}(x) = \{\hat{x} \in X : (\hat{x}, F(\hat{x})) \preceq_k^{mu} (x, F(x))\}$ is closed for all $x \in X$.

The minimal element theorems involving the set order relations $\preceq_k^{ml}$ and $\preceq_k^{mu}$ on $X \times P(Y)$ are presented in [12]. Here, we mention such result only for the set order relation $\preceq_C^{ml}$.

Theorem 6.7.1 [12] *Let (X, d) be a complete metric space, Y be a real normed space, C be a solid closed convex pointed cone in Y, $k \in \text{int}C$, and $\mathcal{A} \subset X \times P(Y)$ be a nonempty set. Assume that the following conditions hold:*

(i) *$\mathcal{A}$ is ml-bounded below (that is, there exists $V \in P(Y)$ such that $V \preceq_C^{ml} P_{P(Y)}(\mathcal{A})$ for all $x \in X$);*
(ii) *For all $\preceq_k^{ml}$ −decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \mathcal{A}$ (that is, $(x_{n+1}, V_{n+1}) \preceq_k^{ml} (x_n, V_n)$ for all $n \in \mathbb{N}$), there exists $(x, V) \in \mathcal{A}$ such that $(x, V) \preceq_k^{ml} (x_n, V_n)$ for all $n \in \mathbb{N}$.*

Then for every $(x_0, V_0) \in \mathcal{A}$, there exists $(\bar{x}, \bar{V}) \in \mathcal{A}$ such that

(a) $(\bar{x}, \bar{V}) \preceq_k^{ml} (x_0, V_0)$,
(b) *for any $(\tilde{x}, \tilde{V}) \in \mathcal{A}$ such that $(\tilde{x}, \tilde{V}) \preceq_k^{ml} (\bar{x}, \bar{V})$, then $\tilde{x} = \bar{x}$.*

In [12], we established Ekeland's variational principle for set-valued maps involving the set order relations $\preceq_C^{ml}$ and $\preceq_C^{mu}$. Here we mention such result only for the set order relation $\preceq_C^{ml}$.

Theorem 6.7.2 [12] *Assume that the Assumption 2 holds. If for $k \in \text{int}C$ and $x_0 \in X$, $F(x_0) \not\subset F(X) + k + \text{int}C$, then there exists $\bar{x} \in X$ such that*

(a) $F(\bar{x}) + d(\bar{x}, x_0)k \preceq_C^{ml} F(x_0)$,
(b) $F(x) + d(\bar{x}, x)k \not\preceq_C^{ml} F(\bar{x})$ *for all* $x \neq \bar{x}$,
(c) $d(\bar{x}, x_0) \leq 1$.

In [12], we further obtained Caristi's fixed point theorems for set-valued maps under the set order relations $\preceq_C^{ml}$ and $\preceq_C^{mu}$. Here we mention such result only for the set order relation $\preceq_C^{ml}$.

Theorem 6.7.3 [12] *Suppose that the Assumption 2 and the following condition hold.*

(Caristi-$\preceq_C^{ml}$) Condition. *Let $T : X \to 2^X$ be a set-valued map such that for every $x \in X$, there exists $y \in T(x)$ such that*

$$F(y) + d(x, y)k \preceq_C^{ml} F(x).$$

Then T has a fixed point in X, that is, there exists $\bar{x} \in X$ with $\bar{x} \in T(\bar{x})$.

In [12], we also obtained Takahashi's minimization theorems for set-valued maps under the set order relations $\preceq_C^{ml}$ and $\preceq_C^{mu}$. Here we mention such result only for the set order relation $\preceq_C^{ml}$.

Theorem 6.7.4 [12] *Suppose that the Assumption 2 and the following condition hold.*

(Takahashi-$\preceq_C^{ml}$ Condition). *For every $y \in X$ with $F(y) \notin ml - \mathrm{WMin}(F, X)$, there exists $z \in X \setminus \{y\}$ such that*

$$F(z) + d(y,z)k \preceq_C^{ml} F(y).$$

Then there exists $\bar{x} \in X$ such that $F(\bar{x}) \in ml - \mathrm{WMin}(F, X)$.

We remark that the following implications hold

$$\text{Theorem 6.7.2} \quad \Leftrightarrow \quad \text{Theorem 6.7.3} \quad \Leftrightarrow \quad \text{Theorem 6.7.4}.$$

6.7.1 A Minimal Element Theorem and Ekeland's Principle with Mixed Set Order Relations

Throughout this subsection, unless otherwise specified, we assume that Y is a Hausdorff topological vector space and C is a nontrivial, solid convex cone. Let W be a nonempty set with a transitive relation $\preceq$ on W. We say that the sequence $\{w_n\}_{n\in\mathbb{N}} \subset W$ is $\preceq$-decreasing [76] if $w_{n+1} \preceq w_n$ for all $n \in \mathbb{N}$. We set $S_\preceq(w_0) := \{w \in W : w \preceq w_0\}$ for each $w_0 \in W$. Of course, $S_\preceq : W \rightrightarrows W$ is a set-valued map whose domain is $\mathrm{dom}\, S_\preceq := \{w_0 \in W : S_\preceq(w_0) \neq \emptyset\}$. Clearly, $w \in S_\preceq(w_0) \Rightarrow S_\preceq(w) \subset S_\preceq(w_0)$ and $\mathrm{dom}\, S_\preceq = W$ when $\preceq$ is a pre-order, that is, a reflexive and transitive order relation on W.

The following variational principle for minimal points on a pre-ordered set played a key role to establish the main results of this subsection.

Theorem 6.7.5 (Extended Brézis–Browder Principle) [21, 76] *Let $\preceq$ be a transitive relation and $\phi : W \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ be a function such that the following conditions hold.*

(i) *ϕ is $\preceq$-increasing (that is, $w_1 \preceq w_2$ implies $\phi(w_1) \leq \phi(w_2)$);*
(ii) *For every $\preceq$-decreasing sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq W$, there exists $w \in W$ such that $w \preceq w_n$ for all $n \in \mathbb{N}$.*

Then for every $w_0 \in \mathrm{dom}\, S_\preceq$, there exists $\bar{w} \in S_\preceq(w_0)$ such that $\phi(\hat{w}) = \phi(\bar{w})$ for all $\hat{w} \in S_\preceq(\bar{w})$.

Let (X, d) be a metric space and $\lambda \in [0, 1]$. For $x_1, x_2 \in X$, $V_1, V_2 \in \Omega_C^{cb}$, and $k \in \mathrm{int}C$, in [6], we introduced the following set order relation $\preceq_k^\lambda$ on $X \times \Omega^{cb}$ as follows:

$$(x_1, V_1) \preceq_k^\lambda (x_2, V_2) \quad \Leftrightarrow \quad V_1 + d(x_1, x_2)k \preceq_C^\lambda V_2.$$

It can be easily seen that the set order relation $\preceq_k^\lambda$ is reflexive and transitive on $X \times \Omega^{cb}$.

By using the technique of [12, 72], but for the weighted set order relation $\preceq_k^\lambda$ on $X \times \Omega_C^{cb}$, we [6] established the following minimal element theorem. It is worth to mention that Hamel and Löhne [72] used the set relations $\preceq_k^l$ and $\preceq_k^u$, which are special cases of the set relation $\preceq_k^\lambda$, to obtain a minimal element theorem.

Let (X, d) be a complete metric space. For a set $\mathcal{A} \subseteq X \times \Omega^{cb}$, we denote by $P_{\Omega^{cb}}(\mathcal{A})$ the projection of $\mathcal{A}$ onto its second component, that is,

$$P_{\Omega^{cb}}(\mathcal{A}) = \{A \in \Omega^{cb} : \exists x \in X \text{ with } (x, A) \in \mathcal{A}\}.$$

Theorem 6.7.6 [6] *Let $\mathcal{A} \subset X \times \Omega^{cb}$ be a nonempty set. Assume that the following condition holds:*

(M) *For all $\preceq_k^\lambda$-decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \mathcal{A}$ (that is, $(x_{n+1}, V_{n+1}) \preceq_k^\lambda (x_n, V_n)$ for all $n \in \mathbb{N}$), there exists $(x, V) \in \mathcal{A}$ such that $(x, V) \preceq_k^\lambda (x_n, V_n)$ for all $n \in \mathbb{N}$.*

Then for every $(x_0, V_0) \in \mathcal{A}$, there exists $(\bar{x}, \bar{V}) \in \mathcal{A}$ such that

(a) $(\bar{x}, \bar{V}) \preceq_k^\lambda (x_0, V_0)$,
(b) *if $(\hat{x}, \hat{V}) \in \mathcal{A}$ such that $(\hat{x}, \hat{V}) \preceq_k^\lambda (\bar{x}, \bar{V})$, then $\hat{x} = \bar{x}$.*

Assumption 4 Let $F : X \to \Omega^{cb}$ be a $C-$closed-valued map such that

(i) F is bounded below (that is, there exists $V \in \Omega^{cb}$ such that $V \preceq_C^\lambda F(x)$ for all $x \in X$),
(ii) $\widehat{S}(x) = \{\hat{x} \in X : (\hat{x}, F(\hat{x})) \preceq_k^\lambda (x, F(x))\}$ is closed for all $x \in X$.

In [6], we established Ekeland's variational principle for set-valued maps involving set order relation $\preceq_C^\lambda$.

Theorem 6.7.7 *Assume that Assumption 4 is satisfied. If for $k \in \text{int}C$ and $x_0 \in X$, $F(X) + k \not\preceq_C^\lambda F(x_0)$ holds, then there exists $\bar{x} \in X$ such that*

(a) $F(\bar{x}) + d(\bar{x}, x_0)k \preceq_C^\lambda F(x_0)$,
(b) $F(x) + d(\bar{x}, x)k \not\preceq_C^\lambda F(\bar{x})$ *for* $x \neq \bar{x}$,
(c) $d(\bar{x}, x_0) \leq 1$.

Remark 6.14 It is worth to mention that Theorems 6.7.6 and 6.7.7 are more general than [12, Theorems 4.4 and 4.6], [72, Theorems 5.1 and 6.1], and [76, Theorem 5.1] under certain assumptions. However, due to strong assumptions on the order relation $\preceq_C^\lambda$, Theorems 6.7.6 and 6.7.7 are not completely comparable with the results mentioned above.

We derived in [6] the following Caristi fixed point theorem for set-valued maps with set order relation $\preceq_C^\lambda$.

Theorem 6.7.8 *Suppose that Assumption 4 and the following condition hold.*

(Caristi-$\preceq_C^\lambda$) Condition. *Let $T : X \rightrightarrows X$ be a set-valued map with nonempty values such that for every $x \in X$, there exists $y \in T(x)$ satisfying*

$$F(y) + d(x, y)k \preceq_C^\lambda F(x).$$

Then T has a fixed point in X, that is, there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$.

We further obtained in [6] the following Takahashi minimization theorem for set-valued maps with mixed set order relation $\preceq_C^{\lambda}$.

Theorem 6.7.9 *Suppose that Assumption 4 and the following condition hold.*

(Takahashi-$\preceq_C^{\lambda}$ Condition). *For every $y \in X$ with $F(y) \notin \lambda - \mathrm{Min}(F, X)$, there exists $z \in X \setminus \{y\}$ such that*

$$F(z) + d(y, z)k \preceq_C^{\lambda} F(y).$$

Then there exists $\bar{x} \in X$ such that $F(\bar{x}) \in \lambda\text{-}\mathrm{Min}(F, X)$.

In [6], we verified that Theorems 6.7.7, 6.7.8, and 6.7.9 are equivalent to each other in the sense that each one can be derived by using the other.

6.7.2 Ekeland's Variational Principle for Set-Valued Maps in Quasi-Metric Spaces

We recall the definition of a quasi-metric space. For further details and definitions, we refer to [32].

Definition 6.25 Let X be a nonempty set. A *quasi-metric* on X is a function $q : X \times X \to \mathbb{R}_+ := [0, +\infty)$ that satisfies the following conditions:

(Q1) $q(x, y) \geq 0$ and $q(x, x) = 0$ for all $x \in X$;
(Q2) $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$;
(Q3) $q(x, y) = q(y, x) = 0 \Rightarrow x = y$ for all $x, y \in X$.

The set X equipped with a quasi-metric q is called a quasi-metric space and it is denoted by (X, q). If, in addition, the quasi-metric q satisfies the symmetry property, that is, $q(x, y) = q(y, x)$ for all $x, y \in X$, then q is called a metric. The topological space equipped with a quasi-metric is known as the Sorgenfrey line. Every quasi-metric space (X, q) can be viewed as a topological space on which the topology is induced by taking the collection of balls $\{\mathbb{B}_r(x) : r > 0\}$ as a base of the neighborhood filter for every $x \in X$, where the (left) ball $\mathbb{B}_r(x)$ is defined by

$$\mathbb{B}_r(x) := \{y \in X : q(x, y) < r\}.$$

We present some basic notions from quasi-metric spaces, which are needed in this subsection.

Definition 6.26 Let (X, q) be a quasi-metric space and Ω be a nonempty subset of X.

(a) We say that the sequence $\{x_n\} \subset X$ *(left-sequentially) converges* to $\bar{x} \in X$ if $\lim_{n\to\infty} q(x_k, x^*) = 0$, and it is denoted by $x_n \to x^* \in X$.

(b) We say that the set Ω is *left-sequentially closed* if for any sequence $x_n \to x^*$ with $\{x_n\} \subset \Omega$, $x^* \in \Omega$.

(c) We say that the sequence $\{x_n\} \subset X$ is *left-sequentially Cauchy* if for each $\beta \in \mathbb{N}$, there is a natural number N_β such that

$$q(x_n, x_m) < 1/\beta, \quad \text{for all } m \geq n \geq N_\beta.$$

(d) We say that the quasi-metric space (X, q) is *left-sequentially complete* if each left-sequentially Cauchy sequence is convergent and its limit belongs to X.

(e) A quasi-metric space is the *Hausdorff topological space* if

$$\left[\lim_{n\to\infty} q(x_n, \bar{x}) = 0 \text{ and } \lim_{n\to\infty} q(x_n, \bar{u}) = 0\right] \quad \Rightarrow \quad \bar{x} = \bar{u}. \tag{6.9}$$

(f) A quasi-metric space (X, q) ordered by a pre-order $\preceq$ (that is, a reflexive and transitive relation) is said to satisfy the *Hausdorff decreasing condition* if for every decreasing sequence $\{x_n\} \subset X$ and $\bar{x}, \bar{u} \in X$ with $\bar{x} \preceq \bar{u}$ the implication in (6.9) holds.

In 1983, Dancs, Hegedüs, and Medvegyev [37] (in short, DHM) established a fixed point theorem for set-valued maps on a complete metric space by using the generalized Picard iteration under some appropriate assumptions. Since then, many authors have generalized this fixed point theorem under different assumptions and in different settings. Recently, Bao et al. [17] extended DHM's fixed point theorem for parametric dynamic systems in quasi-metric spaces. Motivated by the result in [17], we, in [10], introduced the extended Picard iterative process for set-valued maps on the product spaces and obtained the extended version of DHM's fixed point theorem. We defined the extended Picard sequence in the following way:

Let X be a nonempty set, Y be a topological vector space, and $\Phi : X \times P(Y) \rightrightarrows X \times P(Y)$ be a set-valued map. We say that the sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}}$ is an extended Picard sequence/iterative process if

$$(x_2, V_2) \in \Phi(x_1, V_1),\ (x_3, V_3) \in \Phi(x_2, V_2),\ \ldots,\ (x_n, V_n) \in \Phi(x_{n-1}, V_{n-1}),$$

for all $n \in \mathbb{N}$.

In [10], we established the following extended parametric fixed point theorem on the product space $X \times P(Y)$.

Theorem 6.7.10 *Let (X, q) be a complete Hausdorff quasi-metric space, Y be a topological vector space, and $\emptyset \neq \Gamma \subset X \times P(Y)$. Assume that the parametric dynamical system $\Phi : X \times P(Y) \rightrightarrows X \times P(Y)$ satisfies the following conditions:*

(F1) $(x, V) \in \Phi(x, V)$ *for all* $(x, V) \in \Gamma$.

(F2) *For all* $(x_1, V_1), (x_2, V_2) \in \Gamma$ *such that* $(x_2, V_2) \in \Phi(x_1, V_1)$, *we have* $\Phi(x_2, V_2) \subset \Phi(x_1, V_1)$.

(F3) *For each extended Picard sequence* $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \Gamma$ *with* $x_n \to x^*$ *as* $n \to \infty$, *there exists* $V^* \in P(Y)$ *such that*

$$(x^*, V^*) \in \Gamma \textit{ and } (x^*, V^*) \in \Phi(x_n, V_n), \quad \textit{for all } n \in \mathbb{N}, \tag{6.10}$$

and

$$(x^*, V) \in \Gamma \cap \Phi(x^*, V^*) \textit{ implies } V = V^*. \tag{6.11}$$

(F4) *For each extended Picard sequence* $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \Gamma$*,* $q(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$.

Then for every $(x_0, V_0) \in \Gamma$*, there is an extended Picard sequence* $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \Gamma$ *starting from* (x_0, V_0) *and ending at a fixed point* (x^*, V^*) *of* Φ *in the sense that* $\Phi(x^*, V^*) = \{(x^*, V^*)\}$.

From now onward, we assume that $\mathcal{K} : Y \rightrightarrows Y$ is a set-valued map and the following conditions hold.

- $\mathbf{0} \in \mathcal{K}(y)$.
- $\mathcal{K}(y) + \mathcal{K}(y) \subseteq \mathcal{K}(y)$ for all $y \in Y$.
- $[0, +\infty)k + \mathcal{K}(y) \subseteq \mathcal{K}(y)$ for all $y \in Y$ and all $k \in Y \setminus \{\mathbf{0}\}$.
- For all $y \in Y$ and all $v \in \mathcal{K}(y)$, we have

$$\mathcal{K}(y - v) \subseteq \mathcal{K}(y). \tag{6.12}$$

- For all $A, B, D, E \in P(Y)$, we have

$$\forall b \in B,\ \forall e \in E \text{ we have } \mathcal{K}(b) + \mathcal{K}(e) \subseteq \mathcal{K}(b + e). \tag{6.13}$$

- For all $y \in Y$ and all $v \in \mathcal{K}(y)$, we have

$$\mathcal{K}(y + v) \subseteq \mathcal{K}(y). \tag{6.14}$$

- For all $A, B, D, E \in P(Y)$, we have

$$\forall a \in A \text{ and } \forall d \in D \text{ we have } \mathcal{K}(a) + \mathcal{K}(d) \subseteq \mathcal{K}(a + d). \tag{6.15}$$

Let X be a quasi-metric space. For all $x_1, x_2 \in X$ and $V_1, V_2 \in P(Y)$, we [10] defined the set order relations $\preceq_k^u$ and $\preceq_k^l$ on $X \times P(Y)$ as follows:

$$(x_1, V_1) \preceq_k^u (x_2, V_2) \quad \Leftrightarrow \quad V_1 + q(x_2, x_1)k \preceq_u^{\mathcal{K}} V_2, \tag{6.16}$$

and

$$(x_1, V_1) \preceq_k^l (x_2, V_2) \quad \Leftrightarrow \quad V_1 + q(x_2, x_1)k \preceq_l^{\mathcal{K}} V_2. \tag{6.17}$$

We note that the above set order relations on $X \times P(Y)$ are pre-order. Note that the relation $\preceq_k^u$ is reflexive and transitive on $X \times P(Y)$ if (6.12) and (6.13) hold, and the relation $\preceq_k^l$ is reflexive and transitive on $X \times P(Y)$ if (6.14) and (6.15) hold.

Based on the idea of [49, 56, 59, 149], we, in [10], also defined the following order relations on $X \times P(Y)$, which are stronger than $\preceq_k^u$ and $\preceq_k^l$.

$$(x_1, V_1) \preceq_{k,h_k^u}^u (x_2, V_2) \quad \Leftrightarrow \quad \begin{cases} (x_1, V_1) = (x_2, V_2), \\ \text{or} \\ (x_1, V_1) \preceq_k^u (x_2, V_2) \text{ and } h_k^u(V_1) < h_k^u(V_2). \end{cases}$$

$$(x_1, V_1) \preceq_{k,h_k^l}^l (x_2, V_2) \quad \Leftrightarrow \quad \begin{cases} (x_1, V_1) = (x_2, V_2), \\ \text{or} \\ (x_1, V_1) \preceq_k^l (x_2, V_2) \text{ and } h_k^l(V_1) < h_k^l(V_2). \end{cases}$$

It can be easily seen that $\preceq_{k,h_k^u}^u$ and $\preceq_{k,h_k^l}^l$ are reflexive and transitive on $X \times P(Y)$.

Recently, Bao et al. [17, 19] have developed a constructive dynamical approach to prove the existence of a minimal element of a nonempty subset of the product space $X \times Y$ ordered by some preference. Such a result is called parametric minimal point theorem. They applied the parametric minimal point theorem to derive the Ekeland type variational principle for set-valued maps in the setting of quasi-metric spaces. They used the preferences given by a set-valued map $\mathcal{K} : Y \rightrightarrows Y$, but their approach depends on the vector approach. It is worth to mention that the set approach is used in [12, 72] to obtain minimal element theorems and the Ekeland type variational principle for set-valued maps in the setting of complete metric spaces with a constant ordering cone. Since the set-valued optimization problems with variable domination structures have their own importance not only in the theoretical areas but also in real-life applications (see, [40, 42, 44, 104]), we [10] extended the results of [17, 19] in the setting of product space $X \times P(Y)$ for the generalized variable upper/lower less set order relations $\preceq_u^{\mathcal{K}}/\preceq_l^{\mathcal{K}}$.

Definition 6.27 [10] Let X be a nonempty set, Y be a topological vector space, and $\Gamma \subset X \times P(Y)$ be a nonempty set pre-ordered by $\preceq_k^u$ $[\preceq_k^l]$.

(a) A sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \Gamma$ is said to be *decreasing with respect to the pre-order* $\preceq_k^u$ $[\preceq_k^l]$ if $(x_n, V_n) \preceq_k^u (x_{n-1}, V_{n-1})$ $[(x_n, V_n) \preceq_k^l (x_{n-1}, V_{n-1})]$ for all $n \in \mathbb{N}$.
(b) An element $(\bar{x}, \bar{V})$ of Γ is said to be *partial minimal element with respect to the pre-order* $\preceq_k^u$ $[\preceq_k^l]$ if $(x, V) \in \Gamma$ and $(x, V) \preceq_k^u (\bar{x}, \bar{V})$ $[(x, V) \preceq_k^l (\bar{x}, \bar{V})]$, then $x = \bar{x}$.
(c) An element $(\bar{x}, \bar{V})$ of Γ is said to be *minimal element with respect to the pre-order* $\preceq_k^u$ $[\preceq_k^l]$ if $(x, V) \in \Gamma$ and $(x, V) \preceq_k^u (\bar{x}, \bar{V})$ $[(x, V) \preceq_k^l (\bar{x}, \bar{V})]$, then $(x, V) = (\bar{x}, \bar{V})$.

In [10], we derived the following minimal element theorem for the set order relation $\preceq_u^{\mathcal{K}}$.

Theorem 6.7.11 [10] *Let* (X, q) *be a Hausdorff quasi-metric space,* Y *be a topological vector space,* $\mathcal{K} : Y \rightrightarrows Y$ *be a set-valued map that satisfies* (6.12) *and* (6.13)*, and* $\Gamma \subset X \times P(Y)$ *be a nonempty set. For a given* $(x_0, V_0) \in \Gamma$*, define the set*

$$\mathcal{A}_0 := \mathcal{A}(x_0, V_0) = \{(\tilde{x}, \tilde{V}) \in \Gamma : (\tilde{x}, \tilde{V}) \preceq_k^u (x_0, V_0)\}.$$

Let $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \mathcal{A}_0$ *be a* $\preceq_k^u$*-decreasing sequence such that the following conditions hold.*

(M1) $q(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$.
(M2) *If* $\{x_n\}$ *is a left-sequentially Cauchy sequence, then there exists* $(\bar{x}, \bar{V}) \in \mathcal{A}_0$ *such that* $(\bar{x}, \bar{V}) \preceq_k^u (x_n, V_n)$ *for all* $n \in \mathbb{N}$.

Then there is a decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \Gamma$ *starting from* (x_0, V_0) *and ending at a partially minimal element* $(\bar{x}, \bar{V})$ *of* Γ *with respect to* $\preceq_k^u$*. If, furthermore,* $(\bar{x}, \bar{V})$ *satisfies the domination condition*

$$(\bar{x}, V) \preceq_k^u (\bar{x}, \bar{V}) \quad \Rightarrow \quad V = \bar{V}, \quad \text{for all } (\bar{x}, V) \in \mathcal{A}_0, \tag{6.18}$$

then it can be chosen as a minimal element of the set Γ *with respect to* $\preceq_k^u$.

Moreover, if we replace $\preceq_k^u$ *by* $\preceq_{k,h_k^u}^u$*, then, under the assumption* (6.18)*, there is a decreasing sequence* $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \Gamma$ *starting from* (x_0, V_0) *and ending at a minimal point* $(\bar{x}, \bar{V})$ *of* Ω *with respect to* $\preceq_{k,h_k^u}^u$.

We also derived the following minimal element theorem for the set order relation $\preceq_l^{\mathcal{K}}$.

Theorem 6.7.12 [10] *Let* (X, q) *be a Hausdorff quasi-metric space,* Y *be a topological vector space,* $\mathcal{K} : Y \rightrightarrows Y$ *be a set-valued map that satisfy* (6.14) *and* (6.15)*, and* $\Gamma \subset X \times P(Y)$ *be a nonempty set. For a given* $(x_0, V_0) \in \Gamma$*, define the set*

$$\mathcal{A}_0 := \mathcal{A}(x_0, V_0) = \{(\tilde{x}, \tilde{V}) \in \Gamma : (\tilde{x}, \tilde{V}) \preceq_k^l (x_0, V_0)\}.$$

Let $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \mathcal{A}_0$ *be a* $\preceq_k^l$*-decreasing sequence such that the following conditions hold.*

(M1′) $q(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$.
(M2′) *If* $\{x_n\}$ *is a left-sequentially Cauchy sequence, then there exists* $(\bar{x}, \bar{V}) \in \mathcal{A}_0$ *such that* $(\bar{x}, \bar{V}) \preceq_k^l (x_n, V_n)$ *for all* $n \in \mathbb{N}$.

Then, there is a decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \Gamma$ *starting from* (x_0, V_0) *and ending at a partially minimal element* $(\bar{x}, \bar{V})$ *of* Γ *with respect to* $\preceq_k^l$*. If, furthermore,* $(\bar{x}, \bar{V})$ *satisfies the domination condition*

$$(\bar{x}, V) \preceq_k^l (\bar{x}, \bar{V}) \quad \Rightarrow \quad V = \bar{V} \quad \text{for all } (\bar{x}, V) \in \mathcal{A}_0, \tag{6.19}$$

then it can be chosen as a minimal element of the set Γ *with respect to* $\preceq_k^l$.

Moreover, if we replace $\preceq_k^u$ *by* $\preceq_{k,h_k^l}^l$*, then, under the assumption* (6.19)*, there is a decreasing sequence* $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \Gamma$ *starting from* (x_0, V_0) *and ending at a minimal element* $(\bar{x}, \bar{V})$ *of* Γ *with respect to* $\preceq_{k,h_k^l}^l$.

Recall that (X, q) be a quasi-metric space and Y be a topological vector space. A set-valued map $F : X \to P(Y)$ is said to be

(a) level-decreasingly-closed on dom$F$ with respect to $\preceq_u^{\mathcal{K}}$ $[\preceq_l^{\mathcal{K}}]$ if for any sequence $\{(x_n, V_n)\} \subset$ Graph$F$ such that $x_n \to \bar{x} \in X$ as $n \to \infty$ and $\{V_n\}$ is a sequence decreasing with respect to $\preceq_u^{\mathcal{K}}$ $[\preceq_l^{\mathcal{K}}]$, there exists $\bar{V} = F(\bar{x}) \in \text{Min}(F(X); \preceq_u^{\mathcal{K}}$ $[\preceq_l^{\mathcal{K}}])$ such that $\bar{V} \preceq_u^{\mathcal{K}}$ $[\preceq_l^{\mathcal{K}}] V_n$ for all $n \in \mathbb{N}$.
(b) quasi-bounded from below with respect to a closed convex cone C in Y if there is a bounded subset $M \subset Y$ such that $F(X) \subseteq M + C$.

In [10], we established the following Ekeland type variational principle for set-valued maps under variable ordering structures.

Theorem 6.7.13 [10] *Let* (X, q) *be a complete Hausdorff quasi-metric space,* Y *be a topological vector space, and* $\mathcal{K} : Y \rightrightarrows Y$ *be a set-valued map such that* (6.12) *and* (6.13) *hold. Let* C *be a convex cone in* Y *and* $F : X \to P(Y)$ *be a set-valued map which is quasi-bounded from below with respect to* C*. Assume that the following conditions are satisfied:*

(E1) *For every* $y \in Y$, $\mathcal{K}(y)$ *is a closed set in* Y.
(E2) *For any* $A, B \in P(Y)$*, if* $A \preceq_u^{\mathcal{K}} B$*, then* $\mathcal{K}(a) + \mathcal{K}(b) \subseteq \mathcal{K}(b)$ *for all* $a \in A$ *and* $b \in B$.
(E3) *For any sequence* $\{(x_n, V_n)\} \subset$ *Graph*F *such that* $x_n \to \bar{x} \in X$ *as* $n \to \infty$ *and* $\{V_n\}$ *is decreasing with respect to* $\preceq_u^{\mathcal{K}}$*, there exists a minimal element* $\bar{V} \in \text{Min}(F(X); \preceq_u^{\mathcal{K}})$ *for which* $\bar{V} \preceq_u^{\mathcal{K}} V_n$ *for all* $n \in \mathbb{N}$.
(E4) $k \notin \text{cl}(-C - \mathcal{K}(V_0)))$.

Then, for every $(x_0, F(x_0)) \in$ *Graph*F*, there exists* $(\bar{x}, F(\bar{x})) \in$ *Graph*F *with* $F(\bar{x}) \in \text{Min}(F(X); \preceq_u^{\mathcal{K}})$ *such that*

(a) $F(\bar{x}) + q(x_0, \bar{x})k \preceq_u^{\mathcal{K}} F(x_0)$,
(b) $F(x) + q(\bar{x}, x)k \npreceq_u^{\mathcal{K}} F(\bar{x})$ *for all* $x \neq \bar{x}$.
Furthermore, assume that $F(\bar{x}) + k \npreceq_u^{\mathcal{K}} F(x_0)$ *for all* $k \in Y \setminus \{\mathbf{0}\}$ *and* $x_0 \in X$*, then*
(c) $q(x_0, \bar{x}) \leq 1$.

In [10], we also obtained the Ekeland type variational principle for set-valued maps involving variable order structure $\preceq_l^{\mathcal{K}}$.

Remark 6.15 Bao et al. [17] considered the main issues of Sen's capability theory [108, 142] and the variational rationality model of human behavior. They developed dynamical aspects of capability theory and discussed the major findings in this directions by applying the parametric fixed point theorem, parametric minimal element theorem, and Ekeland's variational principle. By using variational rationality [17, 18, 144, 145] technique, we can consider modeling the functionings/preferences dynamics in term of acceptable stays and changes, which mainly relates to the extended parametric fixed point theorem. Then we can find the functionings/preferences dynamics in term of worthwhile stays and changes, which relates to the obtained variational principle for maps with variable domination structures. Very recently, Bao et al. [20] also introduced a new version of Ekeland's variational principle in set optimization with domination structure and gave some applications to career development theories; in particular, changing the job process.

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