Chapter 13
On Constraint Qualifications for Multiobjective Optimization Problems with Switching Constraints

Yogendra Pandey and Vinay Singh

Abstract In this chapter, we consider multiobjective optimization problems with switching constraint (MOPSC). We introduce linear independence constraint qualification (LICQ), Mangasarian–Fromovitz constraint qualification (MFCQ), Abadie constraint qualification (ACQ), and Guignard constraint qualification (GCQ) for multiobjective optimization problems with switching constraint (MOPSC). Further, we introduce the notion of Weak stationarity, Mordukhovich stationarity, and Strong stationarity, i.e., W-stationarity, M-stationarity, and S-stationarity, respectively, for the MOPSC. Also, we present a survey of the literature related to existing constraint qualifications and stationarity conditions for mathematical programs with equilibrium constraints (MPEC), mathematical programs with complementarity constraints (MPCC), mathematical programs with vanishing constraints (MPVC), and for mathematical programs with switching constraints (MPSC). We establish that the M-stationary conditions are sufficient optimality conditions for the MOPSC using generalized convexity. Further, we propose a Wolfe-type dual model for the MOPSC and establish weak duality and strong duality results under assumptions of generalized convexity.

Keywords Switching constraints · Constraint qualifications · Optimality conditions · Duality

13.1 Introduction

We consider the following multiobjective optimization problems with switching constraints (MOPSC):

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$$
\begin{aligned}
\text{(MOPSC)} \qquad & \min \ (f_1(x), \ldots, f_m(x)) \\
& \text{subject to}: \ g_i(x) \le 0, \ \ i = 1, \ldots, p, \\
& h_i(x) = 0, \ \ i = 1, \ldots, q, \\
& G_i(x) H_i(x) = 0, \quad i = 1, \ldots, l.
\end{aligned}
$$

All the functions $f_1, \ldots, f_m, g_1, \ldots, g_p, h_1, \ldots, h_q, G_1, \ldots, G_l, H_1, \ldots, H_l : \mathbb{R}^n \to \mathbb{R}$ are assumed to be continuously differentiable.

In optimal control, the concept of control switching became very important, for details see, [13, 14, 23, 27, 35, 59, 61, 63, 67] and references therein. Mathematical programs with switching constraints (MPSC) are related to mathematical programs with vanishing constraints (MPVC) and mathematical programs with complementarity constraints (MPCC) (see [38, 51]). Similarly, multiobjective optimization problems with switching constraints (MOPSC) are also closely related to multiobjective optimization problems with vanishing constraints (MOPVC), see [44]. Mehlitz [42] introduced the notions of weak, Mordukhovich, and strong stationarities for mathematical programs with switching constraints (MPSC). Recently, Kanzow et al. [34] proposed several relaxation schemes for the MPSC.

Constraint qualifications are regularity conditions for Kuhn–Tucker necessary optimality in nonlinear programming problems. The Slater constraint qualification, the weak Arrow–Hurwicz–Uzawa constraint qualification, the weak reverse convex constraint qualification, the Kuhn–Tucker constraint qualification, the linear independence constraint qualification (LICQ), the Mangasarian–Fromovitz constraint qualification (MFCQ), the Abadie constraint qualification (ACQ), and the Guignard constraint qualification (GCQ) are some of the important constraint qualifications among several constraint qualifications in nonlinear programming problems (see, [1, 24, 41]). Many authors studied these constraint qualifications and found relations for different types of optimization problems under smooth and nonsmooth environments. We refer to [9, 12, 22, 36, 37, 39, 40, 56, 57] for more details about several constraint qualifications and relationships among them for nonlinear programming problems and multiobjective programming problems.

Motivated by the above-mentioned works our aim is to study several constraint qualifications and stationarity conditions of the MOPSC. The chapter is structured as follows: We begin with some preliminary results in Sect. 13.2. Section 13.3 is dedicated to the study of constraint qualifications like LICQ, MFCQ, generalized ACQ, and generalized GCQ for the MOPSC. In Sect. 13.4, we introduce weak stationarity (W-stationarity), Mordukhovich stationarity (M-stationarity), and strong stationarity (S-stationarity) for the MOPSC. In Sect. 13.5, we establish that the M-stationary conditions are sufficient optimality conditions for the MOPSC using generalized convexity. In Sect. 13.6, we propose a Wolfe type dual for the MOPSC and establish weak duality and strong duality results under assumptions of generalized convexity. In Sect. 13.7, we discuss some future research work.

13.2 Preliminaries

This section contains some preliminaries which will be used throughout the chapter. Consider the following multiobjective optimization problem (MOP):

$$\begin{aligned} \text{(MOP)} \qquad & \hat{f}(x) := (\hat{f}_1(x), \ldots, \hat{f}_{\hat{m}}(x)) \\ s.t. \quad & \hat{g}_i(x) \leq 0, \forall i = 1, 2, \ldots, \hat{p}, \\ & \hat{h}_i(x) = 0, \forall i = 1, 2, \ldots, \hat{q}, \end{aligned} \tag{13.1}$$

where all the functions $\hat{f}_i, \hat{g}_i, \hat{h}_i : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable. Set F to be the feasible set of the MOP.

For each $k = \{1, \ldots, \hat{m}\} \subset \mathbb{N}$, the nonempty sets $\hat{S}^k$ and $\hat{S}$ are given as follows:

$$\begin{aligned} \hat{S}^k := \{x \in \mathbb{R}^n | \; & \hat{g}_i(x) \leq 0, \forall i = 1, 2, \ldots, \hat{p}, \\ & \hat{h}_i(x) = 0, \forall i = 1, 2, \ldots, \hat{q}, \\ & \hat{f}_i(x) \leq \hat{f}_i(\bar{x}), \forall 1, 2, \ldots, \hat{m}, \text{ and i} \neq \text{k}\}, \end{aligned} \tag{13.2}$$

and

$$\begin{aligned} \hat{S} := \{x \in \mathbb{R}^n | \; & \hat{g}_i(x) \leq 0, \forall i = 1, 2, \ldots, \hat{p}, \\ & \hat{h}_i(x) = 0, \forall i = 1, 2, \ldots, \hat{q}, \\ & \hat{f}_i(x) \leq \hat{f}_i(\bar{x}), \forall i = 1, 2, \ldots, \hat{m}\}. \end{aligned} \tag{13.3}$$

The following concept of the linearized cone to $\hat{S}$ at $\bar{x} \in \hat{S}$ was introduced in [39] for the MOP.

Definition 13.1 The *linearized cone* to $\hat{S}$ at $\bar{x} \in \hat{S}$ is the set $L\left(\hat{S}; \bar{x}\right)$ given by

$$\begin{aligned} L\left(\hat{S}; \bar{x}\right) := \{d \in \mathbb{R}^n | \; & \nabla \hat{g}_i(\bar{x})^T d \leq 0, \forall i \in I_{\hat{g}}, \\ & \nabla \hat{h}_i(\bar{x})^T d = 0, \forall i \in I_{\hat{h}}, \\ & \nabla \hat{f}_i(\bar{x})^T d \leq 0, \forall i \in I_{\hat{f}}\}. \end{aligned}$$

where

$$\begin{aligned} I_{\hat{g}} &:= \{i \in \{1, \ldots, \hat{p}\} \mid \hat{g}_i(\bar{x}) = 0\}, \\ I_{\hat{h}} &:= \{1, \ldots, \hat{q}\}, \\ I_{\hat{f}} &:= \{1, \ldots, \hat{m}\}. \end{aligned}$$

Some of the important convex cones that play a vital role in optimization are the polar cone, tangent cone, and normal cone. The notion of tangent cones may be considered a generalization of the tangent concept in a smooth case to that in a nonsmooth case.

For the sake of convenience, let us recall the definition of a well-known concept having a crucial role to define constraint qualifications.

Definition 13.2 ([8, 58]) Let $\hat{S}$ be a nonempty subset of $\mathbb{R}^n$. The *tangent cone* to $\hat{S}$ at $\bar{x} \in cl\hat{S}$ is the set $T\left(\hat{S};\bar{x}\right)$ defined by

$$T(\hat{S};\bar{x}) := \left\{d \in \mathbb{R}^n | \exists \{x^n\} \subseteq \hat{S}, \{t_n\} \downarrow 0 : x^n \to \bar{x}, \frac{x^n - \bar{x}}{t_n} \to d\right\},$$

where $cl\hat{S}$ denotes the closure of $\hat{S}$.

The following definitions of constraint qualifications for the MOP are taken from [39].

Definition 13.3 Let $\bar{x} \in F$ be a feasible solution of the MOP. Then the *linear independence constraint qualification* (LICQ) holds at $\bar{x}$, if the gradients

$$\nabla \hat{f}_i(\bar{x}) \left(i \in I_{\hat{f}}\right),$$
$$\nabla \hat{g}_i(\bar{x}) \left(i \in I_{\hat{g}}\right),$$
$$\nabla \hat{h}_i(\bar{x}) \left(i \in I_{\hat{h}}\right),$$

are linearly independent.

Definition 13.4 Let $\bar{x} \in F$ be a feasible solution of the MOP. Then the *Mangasarian–Fromovitz constraint qualification* (MFCQ) holds at $\bar{x}$, if the gradients

$$\nabla \hat{f}_i(\bar{x}) \left(i \in I_{\hat{f}}\right),$$
$$\nabla \hat{h}_i(\bar{x}) \left(i \in I_{\hat{h}}\right),$$

are linearly independent, and the system

$$\nabla \hat{f}_i(\bar{x})^T d = 0 \forall i \in I_{\hat{f}},$$
$$\nabla \hat{g}_i(\bar{x})^T d < 0, \forall i \in I_{\hat{g}},$$
$$\nabla \hat{h}_i(\bar{x})^T d = 0, \forall i \in I_{\hat{h}},$$

has a solution $d \in \mathbb{R}^n$.

Definition 13.5 Let $\bar{x} \in F$ be a feasible solution of the MOP. Then the *Abadie constraint qualification* (ACQ) holds at $\bar{x}$ if

$$L\left(\hat{S};\bar{x}\right) \subseteq T\left(\hat{S};\bar{x}\right).$$

Definition 13.6 Let $\bar{x} \in F$ be a feasible solution of the MOP. Then the *generalized Abadie constraint qualification* (GACQ) holds at $\bar{x}$ if

$$L\left(\hat{S};\bar{x}\right) \subseteq \bigcap_{k=1}^{\hat{m}} T\left(\hat{S}^k;\bar{x}\right).$$

The following concept of efficiency was introduced by Pareto [52].

Definition 13.7 Let $\bar{x} \in F$ be a feasible solution of the MOP . Then $\bar{x}$ is said to be a *local efficient solution* of the MOP , if there exists a number $\delta > 0$ such that, there is no $x \in F \bigcap B(\bar{x};\delta)$ satisfying

$$\hat{f}_i(x) \leq \hat{f}_i(\bar{x}), \forall i = 1,\ldots,\hat{m},$$

$$\hat{f}_i(x) < \hat{f}_i(\bar{x}), \text{ at least one } i,$$

where $B(\bar{x};\delta)$ denotes the open ball of radius δ and centre $\bar{x}$.

Definition 13.8 Let $\bar{x} \in F$ be a feasible solution of the MOP. Then $\bar{x}$ is said to be an *efficient solution* of the MOP, if there is no $x \in F$ satisfying

$$\hat{f}_i(x) \leq \hat{f}_i(\bar{x}), \forall i = 1,\ldots,\hat{m},$$

$$\hat{f}_i(x) < \hat{f}_i(\bar{x}), \text{ at least one } i.$$

The following definitions and results are taken from [41].

Definition 13.9 Let f be a differentiable real-valued function defined on a nonempty open convex set $X \subseteq \mathbb{R}^n$. Then the function f is said to be *pseudoconvex* at $\bar{x} \in X$ if the following implication holds:

$$x,\bar{x} \in X, \ \langle \nabla f(\bar{x}), x - \bar{x}\rangle \geq 0 \Rightarrow f(x) \geq f(\bar{x}).$$

Equivalently,

$$x,\bar{x} \in X, \ f(x) < f(\bar{x}) \Rightarrow \langle \nabla f(\bar{x}), x - \bar{x}\rangle < 0.$$

Definition 13.10 Let f be a differentiable real-valued function defined on a nonempty open convex set $X \subseteq \mathbb{R}^n$. Then the function f is said to be quasiconvex at $\bar{x} \in X$ iff the following implication holds:

$$x,\bar{x} \in X, \quad f(x) \leq f(\bar{x}) \Rightarrow \langle \nabla f(\bar{x}), x - \bar{x}\rangle \leq 0.$$

13.3 Constraint Qualifications for Multiobjective Optimization Problems with Switching Constraint

The standard constraint qualifications for nonlinear optimization problems (LICQ or MFCQ) are always violated at every feasible point for mathematical programs with equilibrium constraints (MPEC)(see, [65]), for mathematical programs with complementarity constraints (MPCC) (see, [60]), for mathematical programs with vanishing constraints (MPVC)(see, [29]) and for mathematical programs with switching constraints (MPSC)(see,[42]).

Ye [66] introduced several constraint qualifications for the KKT-type necessary optimality conditions involving Mordukhovich co-derivatives for mathematical problems with variational inequality constraints (MPVIC). The standard Abadie constraint qualification is unlikely to be satisfied by the MPEC, the MPVC, and MPSC. Flegel and Kanzow [16] introduced the modified Abadie constraint qualification for the MPEC. Ye [64] proposed new constraint qualifications namely MPEC weak reverse convex constraint qualification, MPEC Arrow–Hurwicz–Uzawa constraint qualification, MPEC Zangwill constraint qualification, MPEC Kuhn–Tucker constraint qualification, MPEC Abadie constraint qualification. He also proved the relationship among them. For more details about several new constraint qualifications for the MPEC, the MPCC and the MPVIC, (see, [10, 11, 18–21, 25, 26]).

Hoheisel and Kanzow [30] introduced the Abadie and Guignard constraint qualifications for mathematical programs with vanishing constraints. Mishra et al. [44] introduced suitable modifications in constraint qualifications like Cottle constraint qualification, Slater constraint qualification, Mangasarian–Fromovitz constraint qualification, linear independence constraint qualification, linear objective constraint qualification, generalized Guignard constraint qualification for multiobjective optimization problems with vanishing constraints and established relationships among them. We refer to [2, 28, 29, 31, 32] and references their in for more details about constraint qualifications for the MPVC.

Recently, Ardakani et al. [3] introduced two new Abadie-type constraint qualifications and presented some necessary conditions for properly efficient solutions of the problem, using convex subdifferential for multiobjective optimization problems with nondifferentiable convex vanishing constraints. Mehlitz [42] introduced MPSC-tailored versions of MFCQ and LICQ and studied MPSC-tailored versions of the Abadie and Guignard constraint qualification for the MPSC.

Given a feasible point $\bar{x} \in S$, we consider the following index sets:

$$\begin{aligned}
&I_g(\bar{x}) := \{i = 1, 2, \ldots, p \, : g_i(\bar{x}) = 0\},\\
&\alpha := \alpha(\bar{x}) = \{i = 1, 2, \ldots, l \, : \, G_i(\bar{x}) = 0, \, H_i(\bar{x}) \neq 0\},\\
&\beta := \beta(\bar{x}) = \{i = 1, 2, \ldots, l \, : \, G_i(\bar{x}) = 0, \, H_i(\bar{x}) = 0\},\\
&\gamma := \gamma(\bar{x}) = \{i = 1, 2, \ldots, l \, : \, G_i(\bar{x}) \neq 0, \, H_i(\bar{x}) = 0\}.
\end{aligned}$$

Let us define a feasible set S of MOPSC by

$$S := \{x \in \mathbb{R}^n : \; g_i(x) \le 0, \forall i = 1, 2, \ldots, p, \\ h_i(x) = 0, \forall i = 1, 2, \ldots, q, \\ G_i(x) H_i(x) = 0, \forall i = 1, 2, \ldots, l\}.$$

Consider the following function:

$$\eta_i(x) := G_i(x) H_i(x), \forall i = 1, 2, \ldots, l \tag{13.4}$$

its gradient is given by

$$\nabla \eta_i(x) = G_i(x) \nabla H_i(x) + H_i(x) \nabla G_i(x), \forall i = 1, 2, \ldots, l. \tag{13.5}$$

By the definition of the index sets, we get

$$\nabla \eta_i(\bar{x}) = \begin{cases} H_i(\bar{x}) \nabla G_i(\bar{x}), & \text{if} \quad i \in \alpha, \\ 0, & \text{if} \quad i \in \beta, \\ G_i(\bar{x}) \nabla H_i(\bar{x}), & \text{if} \quad i \in \gamma, \end{cases} \tag{13.6}$$

For each $k = 1, 2, \ldots, m$, the nonempty sets S^k and S are defined as follows:

$$S^k := \{x \in \mathbb{R}^n | \; g_i(x) \le 0, \forall i = 1, 2, \ldots, p, \\ h_i(x) = 0, \forall i = 1, 2, \ldots, q, \\ G_i(x) H_i(x) = 0, \forall i = 1, 2, \ldots, r, \\ f_i(x) \le f_i(\bar{x}), \forall i = 1, 2, \ldots, m, i \neq k\},$$

and

$$S := \{x \in \mathbb{R}^n | \; g_i(x) \le 0, \forall i = 1, 2, \ldots, p, \\ h_i(x) = 0, \forall i = 1, 2, \ldots, q, \\ G_i(x) H_i(x) = 0, \forall i = 1, 2, \ldots, r, \\ f_i(x) \le f_i(\bar{x}), \forall i = 1, 2, \ldots, m, \}.$$

The following result gives the standard linearized cone to $S^k, k = 1, 2, \ldots, m$, at an efficient solution $\bar{x} \in S$ of the MOPSC .

Lemma 13.3.1 *Let $\bar{x} \in S$ be an efficient solution of the MOPSC. Then, the linearized cone to $S^k, k = 1, 2, \ldots, m$, at $\bar{x}$ is given by*

$$L(S^k;\bar{x}) = \{d \in \mathbb{R}^n |\ \begin{array}{l} \nabla f_i(\bar{x})^T d \le 0, \forall i \in I_f, i \neq k, \\ \nabla g_i(\bar{x})^T d \le 0, \forall i \in I_g, \\ \nabla h_i(\bar{x})^T d = 0, \forall i \in I_h, \\ \nabla H_i(\bar{x})^T d = 0, \forall i \in \gamma, \\ \nabla G_i(\bar{x})^T d = 0, \forall i \in \alpha\}. \end{array} \tag{13.7}$$

Proof Let $\eta_i(x) = G_i(x)H_i(x), \forall i = 1, 2, \ldots, r$. By the definitions of the index sets and in view of Definition of the linearized cone to $S^k, k = 1, \ldots, m$ at $\bar{x} \in S^k$ is given by

$$L(S^k;\bar{x}) = \{d \in \mathbb{R}^n |\ \begin{array}{l} \nabla f_i(\bar{x})^T d \le 0, \forall i \in I_f, i \neq k, \\ \nabla g_i(\bar{x})^T d \le 0, \forall i \in I_g, \\ \nabla h_i(\bar{x})^T d = 0, \forall i \in I_h, \\ \nabla \eta_i(\bar{x})^T d = 0, \forall i \in \alpha \cup \gamma\}. \end{array}$$

We know that $\nabla\eta_i(\bar{x}) = G_i(\bar{x})\nabla H_i(\bar{x}) + H_i(\bar{x})\nabla G_i(\bar{x})$,

$$\nabla\eta_i(\bar{x})^T d = 0$$

implies

$$G_i(\bar{x})\nabla H_i(\bar{x})^T d + H_i(\bar{x})\nabla G_i(\bar{x})^T d = 0.$$

Since, $G_i(\bar{x}) = 0,\ \forall i \in \alpha$, and $H_i(\bar{x}) = 0, ,\ \forall i \in \gamma$, we get

$$L(S^k;\bar{x}) = \{d \in \mathbb{R}^n |\ \begin{array}{l} \nabla f_i(\bar{x})^T d \le 0, \forall i \in I_f, i \neq k, \\ \nabla g_i(\bar{x})^T d \le 0, \forall i \in I_g, \\ \nabla h_i(\bar{x})^T d = 0, \forall i \in I_h, \\ \nabla H_i(\bar{x})^T d = 0, \forall i \in \gamma, \\ \nabla G_i(\bar{x})^T d = 0, \forall i \in \alpha\}. \end{array} \tag{13.8}$$

We introduce a tightened nonlinear multiobjective optimization problem (TNLMOP) derived from the MOPSC depending on an efficient solution $\bar{x} \in S$ as follows

$$\begin{array}{rl} \text{(TNLMOP)} & f(x) := (f_1(x), \ldots, f_m(x)) \\ s.t. & g_i(x) \le 0, \forall i = 1, 2, \ldots, p, \\ & h_i(x) = 0, \forall i = 1, 2, \ldots, q, \\ & G_i(x) = 0, \forall i \in \alpha \cup \beta, \\ & H_i(x) = 0, \forall i \in \gamma \cup \beta. \end{array} \tag{13.9}$$

The feasible set of the TNLMOP is a subset of the feasible set of MOPSC.

Definition 13.11 Let $\bar{x} \in S$ be a feasible point of the MOPSC. If LICQ holds for TNLMOP at $\bar{x}$. Then $\bar{x}$ is said to satisfy LICQ–MOPSC.

Definition 13.12 Let $\bar{x} \in S$ be a feasible point of the MOPSC. If MFCQ holds for TNLMOP at $\bar{x}$. Then $\bar{x}$ is said to satisfy MFCQ–MOPSC.

From the Definitions 13.3 and 13.4 [39] for TNLMOP, one has

$$\text{LICQ} \implies \text{MFCQ}.$$

Therefore,

$$\text{LICQ–MOPSC} \implies \text{MFCQ-MOPSC}.$$

13.3.1 A Generalized Guignard and Abadie CQ for MOPSC

For each $k = 1, 2, \ldots, m$, the nonempty sets $\overline{S}^k$ and $\overline{S}$ are defined as follows:

$$\begin{aligned} \overline{S}^k := \{x \in \mathbb{R}^n |\ & g_i(x) \leq 0, \forall i = 1, 2, \ldots, p, \\ & h_i(x) = 0, \forall i = 1, 2, \ldots, q, \\ & G_i(x) = 0, \forall i \in \alpha \cup \beta, \\ & H_i(x) = 0, \forall i \in \gamma \cup \beta, \\ & f_i(x) \leq f_i(\bar{x}), \forall i = 1, 2, \ldots, m, i \neq k\}, \end{aligned}$$

and

$$\begin{aligned} \overline{S} := \{x \in \mathbb{R}^n |\ & g_i(x) \leq 0, \forall i = 1, 2, \ldots, p, \\ & h_i(x) = 0, \forall i = 1, 2, \ldots, q, \\ & G_i(x) = 0, \forall i \in \alpha \cup \beta, \\ & H_i(x) = 0, \forall i \in \gamma \cup \beta, \\ & f_i(x) \leq f_i(\bar{x}), \forall i = 1, 2, \ldots, m\}. \end{aligned}$$

The linearized cone to $\overline{S}^k$ at $\bar{x} \in \overline{S}^k$ is given by

$$\begin{aligned} L(\overline{S}^k; \bar{x}) = \{d \in \mathbb{R}^n |\ & \nabla f_i(\bar{x})^T d \leq 0, \forall i = 1, \ldots, m, i \neq k, \\ & \nabla g_i(\bar{x})^T d \leq 0, \forall i \in I_g, \\ & \nabla h_i(\bar{x})^T d = 0, \forall i \in I_h, \\ & \nabla G_i(\bar{x})^T d = 0, \forall i \in \alpha \cup \beta, \\ & \nabla H_i(\bar{x})^T d = 0, \forall i \in \gamma \cup \beta\}. \end{aligned} \tag{13.10}$$

$$L(\overline{S};\bar{x}) = \{d \in \mathbb{R}^n |\ \nabla f_i(\bar{x})^T d \leq 0, \forall i = 1,\ldots,m, \\ \nabla g_i(\bar{x})^T d \leq 0, \forall i \in I_g, \\ \nabla h_i(\bar{x})^T d = 0, \forall i \in I_h, \\ \nabla G_i(\bar{x})^T d = 0, \forall i \in \alpha \cup \beta, \\ \nabla H_i(\bar{x})^T d = 0, \forall i \in \gamma \cup \beta\}. \tag{13.11}$$

We have the following relation:

$$L\left(\overline{S};\bar{x}\right) = \bigcap_{k=1}^{m} L\left(\overline{S}^k;\bar{x}\right). \tag{13.12}$$

Definition 13.13 Let $\bar{x} \in X$ be any feasible solution to the MOPSC. Then, a *Generalized Abadie Constraint Qualification* (GACQ) for the MOPSC, denoted by GACQ–MOPSC, holds at $\bar{x}$, if

$$L\left(\overline{S};\bar{x}\right) \subseteq \bigcap_{k=1}^{m} T\left(S^k;\bar{x}\right).$$

The following constraint qualification gives a sufficient condition to the GACQ–MOPVC.

Definition 13.14 Let $\bar{x} \in X$ be any feasible solution to the TNLMOP. Then a *Generalized Abadie Constraint Qualification* (GACQ) for the TNLMOP, denoted by GACQ–TNLMOP, holds at $\bar{x}$, if

$$L\left(S;\bar{x}\right) \subseteq \bigcap_{k=1}^{m} T\left(\overline{S}^k;\bar{x}\right).$$

Note 13.1 The standard GACQ gives a sufficient condition for the GACQ–MOPVC to hold. Since $L\left(\overline{S};\bar{x}\right) \subseteq L\left(S;\bar{x}\right)$.

The following lemma is about relationships between GACQ–TNLMOP and GACQ–MOPSC.

Lemma 13.3.2 *If the GACQ–TNLMOP holds at $\bar{x}$ then the standard GACQ and the GACQ–MOPVC both are satisfied at $\bar{x}$.*

Proof We know that

$$\overline{S}^k \subset S^k \ \ \forall k = 1, 2 \ldots m$$

and

$$T(\overline{S}^k; \bar{x}) \subset T(S^k; \bar{x}) \ \forall k = 1, 2 \dots m.$$

Hence,

$$\bigcap_{k=1}^{m} T\left(\overline{S}^k; \bar{x}\right) \subset \bigcap_{k=1}^{m} T\left(S^k; \bar{x}\right).$$

From Definition 13.14, we have

$$L\left(\overline{S}; \bar{x}\right) \subseteq L\left(S; \bar{x}\right) \subseteq \bigcap_{k=1}^{m} T\left(\overline{S}^k; \bar{x}\right) \subset \bigcap_{k=1}^{m} T\left(S^k; \bar{x}\right).$$

Therefore, GACQ–MOPSC holds at $\bar{x}$.
By Definitions 13.13 and 13.14, we obtain

$$\text{GACQ–TNLMOP} \implies \text{GACQ–MOPSC}$$

Now, we discuss the relationship between tangent cone $T\left(\overline{S}^k; \bar{x}\right)$, k=1, 2, …, m, and the linearized cone $L(S; \bar{x})$.

Lemma 13.3.3 *Let $\bar{x} \in$ be a feasible solution of the MOPSC. Then we have*

$$\bigcap_{k=1}^{m} clcoT\left(\overline{S}^k; \bar{x}\right) \subseteq L\left(S; \bar{x}\right).$$

Proof The proof follows on the lines of the proof of Lemma 3.1 [42]. □

Definition 13.15 Let $\bar{x} \in X$ be any feasible solution to the TNLMOP. Then a *Generalized Guignard Constraint Qualification* (GGCQ) for the TNLMOP, denoted by GGCQ–TNLMOP, holds at $\bar{x}$, if

$$L\left(S; \bar{x}\right) \subseteq \bigcap_{k=1}^{m} clcoT\left(\overline{S}^k; \bar{x}\right).$$

Definition 13.16 Let $\bar{x} \in X$ be any feasible solution to the MOPSC. Then, a *Generalized Guignard Constraint Qualification* (GGCQ) for the MOPSC, denoted by GGCQ–MOPSC, holds at $\bar{x}$, if

$$L\left(\overline{S}; \bar{x}\right) \subseteq \bigcap_{k=1}^{m} clcoT\left(S^k; \bar{x}\right).$$

The following result gives the relationship between the GGCQ–TNLMOP and the GGCQ–MOPVC.

Lemma 13.3.4 *Let $\bar{x} \in X$ be any feasible solution of the MOPVC. If the GGCQ-TNLMOP holds at $\bar{x}$, then the GGCQ–MOPVC also holds at $\bar{x} \in X$.*

Proof Assume that $\bar{x} \in X$ is a feasible solution of the MOPSC and GGCQ–TNLMOP holds at $\bar{x}$, then

$$L\left(S;\bar{x}\right) \subseteq \bigcap_{k=1}^{m} clcoT\left(\overline{S}^{k};\bar{x}\right). \tag{13.13}$$

Also,

$$\overline{S}^{k} \subset S^{k} \ \ \forall k = 1, 2 \ldots m$$

and

$$T(\overline{S}^{k};\bar{x}) \subset T(S^{k};\bar{x}) \ \ \forall k = 1, 2 \ldots m.$$

Hence

$$\bigcap_{k=1}^{m} clcoT\left(\overline{S}^{k};\bar{x}\right) \subset \bigcap_{k=1}^{m} clcoT\left(S^{k};\bar{x}\right). \tag{13.14}$$

We always have

$$L\left(\overline{S};\bar{x}\right) \subseteq L\left(S;\bar{x}\right). \tag{13.15}$$

From Eqs. (13.13), (13.14) and (13.15), we get

$$L\left(\overline{S};\bar{x}\right) \subseteq \bigcap_{k=1}^{m} clcoT\left(S^{k};\bar{x}\right).$$

Therefore, GGCQ–MOPVC holds at $\bar{x} \in X$. This completes the proof.

In the following lemma, we derive a relationship between the GACQ–MOPSC and the GGCQ–MOPSC.

Lemma 13.3.5 *Let $\bar{x} \in X$ be a feasible solution of the MOPSC. If the GACQ–MOPSC holds at $\bar{x}$ then the GGCQ-MOPSC is satisfied.*

Proof Assume $\bar{x} \in X$ be a feasible solution of the MOPSC and that GACQ–MOPSC holds at $\bar{x}$. From Definition 13.13, we have

$$L\left(\overline{S};\bar{x}\right) \subseteq \bigcap_{k=1}^{m} T\left(S^{k};\bar{x}\right).$$

Since

$$T\left(S^{k};\bar{x}\right) \subseteq clcoT\left(S^{k};\bar{x}\right),$$

we have

$$\bigcap_{k=1}^{m} T\left(S^k;\bar{x}\right) \subseteq \bigcap_{k=1}^{m} clcoT\left(S^k;\bar{x}\right).$$

Which implies

$$L\left(\overline{S};\bar{x}\right) \subseteq \bigcap_{k=1}^{m} clcoT\left(S^k;\bar{x}\right).$$

Therefore, the GGCQ–MOPSC is satisfied at $\bar{x}$. This completes the proof. □

By Lemma 13.3.5, we have

$$\text{GACQ–MOPSC} \implies \text{GGCQ–MOPSC.}$$

13.4 Stationary Conditions for MOPSC

The standard nonlinear programming has only one dual stationary condition, i.e., the Karush–Kuhn–Tucker condition, but we have various stationarity concepts for mathematical programs with equilibrium constraints (MPEC), mathematical program with complementarity constraints (MPCC), mathematical program with vanishing constraints (MPVC), and mathematical program with switching constraints (MPSC).

Outrata [50] introduced the notion of Mordukhovich stationary point (M-stationary) for mathematical programs with equilibrium constraints (MPEC). Scheel and Scholtes [60] introduced the concept of strong-stationary point (S-stationary) and Clarke-stationary (C-stationary) for the mathematical program with complementarity constraints (MPCC). Flegel and Kanzow [15] introduced the concept of Alternatively, stationary point (A-stationary) for the MPEC. Further, Flegel and Kanzow [17] proved that M-stationarity is the first-order optimality condition under a weak Abadie-type constraint qualification for the MPEC.

Ye [64] introduced various stationarity conditions and obtained new constraint qualifications for the considered MPEC. Hoheisel and Kanzow [29] introduced several stationarity conditions for mathematical programs with vanishing constraints (MPVC) using weak constraint qualifications. Ardali et al. [4] studied several new constraint qualifications, GS-stationarity concepts, and optimality conditions for a nonsmooth mathematical program with equilibrium constraints based on the convexificators. Mehlitz [42] introduced notions of weak stationary point (W-stationary), Mordukhovich stationary point (M-stationary), strong stationary point (S-stationary) for mathematical program with vanishing constraints (MPVC) and obtain that the S-stationarity conditions of the MPSC equal its KKT conditions in a certain sense.

In this section, we introduce the notion of weak stationarity, Mordukhovich stationarity, and strong stationarity, i.e., W-stationarity, M-stationarity, and S-stationarity, respectively for the MOPSC.

The following stationarity conditions can be treated as a multiobjective analog of the stationarity conditions for scalar optimization problem with switching constraint introduced in [42].

Definition 13.17 (*W-stationary point*) A feasible point $\bar{x}$ of MOPSC is called a *weak stationary point* (W-stationary point) if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2l}$, and $\theta_i > 0, i \in \{1, \ldots, m\}$ such that following conditions hold:

$$0 = \sum_{i=1}^{m} \theta_i \nabla f_i(\bar{x}) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{x}) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(\bar{x}) + \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(\bar{x}) + \lambda_i^H \nabla H_i(\bar{x})],$$

$$\forall i \in I^g(\bar{x}) : \lambda_i^g \geq 0,$$

$$\forall i \in \alpha(\bar{x}) : \lambda_i^H = 0,$$

$$\forall i \in \gamma(\bar{x}) : \lambda_i^G = 0.$$

Definition 13.18 (*M-stationary point*) A feasible point $\bar{x}$ of MOPSC is called a *Mordukhovich stationary point* (M-stationary point) if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2l}$, and $\theta_i > 0, i \in \{1, \ldots, m\}$ such that following conditions hold:

$$0 = \sum_{i=1}^{m} \theta_i \nabla f_i(\bar{x}) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{x}) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(\bar{x}) + \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(\bar{x}) + \lambda_i^H \nabla H_i(\bar{x})],$$

$$\forall i \in I^g(\bar{x}) : \lambda_i^g \geq 0,$$

$$\forall i \in \alpha(\bar{x}) : \lambda_i^H = 0,$$

$$\forall i \in \gamma(\bar{x}) : \lambda_i^G = 0,$$

$$\forall i \in \beta(\bar{x}) : \lambda_i^G \lambda_i^H = 0.$$

Definition 13.19 (*S-stationary point*) A feasible point $\bar{x}$ of MOPSC is called a *strong stationary point* (S-stationary point) if there exists $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2l}$, and $\theta_i > 0, i \in \{1, \ldots, m\}$ such that following conditions hold:

$$0 = \sum_{i=1}^{m} \theta_i \nabla f_i(\bar{x}) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{x}) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(\bar{x}) + \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(\bar{x}) + \lambda_i^H \nabla H_i(\bar{x})],$$

$$\forall i \in I^g(\bar{x}) : \lambda_i^g \geq 0,$$

$$\forall i \in \alpha(\bar{x}) : \lambda_i^H = 0,$$

$$\forall i \in \gamma(\bar{x}) : \lambda_i^G = 0,$$

$$\forall i \in \beta(\bar{x}) : \lambda_i^G = 0 \text{ and } \lambda_i^H = 0.$$

By Definitions 13.17, 13.18 and 13.19, we have

$$S - stationarity \implies M - stationarity \implies W - stationarity.$$

13.5 Sufficient Optimality Conditions for the MOPSC

Mordukhovich [46] established necessary optimality conditions for multiobjective equilibrium programs with equilibrium constraints in finite-dimensional spaces based on advanced generalized differential tools of variational analysis. Bao et al. [5] studied multiobjective optimization problems with equilibrium constraints (MOPECs) described by generalized equations in the form

$$0 \in G(x, y) + Q(x, y),$$

where mappings G and Q are set-valued.

Bao et al. [5] established a necessary optimality conditions for the MOPEC using tools of variational analysis and generalized differentiation. Mordukhovich [48] derived new qualified necessary optimality conditions for the MOPEC in finite- and infinite-dimensional spaces. Movahedian and Nobakhtian [49] derived a necessary optimality result on any Asplund space and established sufficient optimality conditions for nonsmooth MPEC in Banach spaces. Recently, Pandey and Mishra [53] introduced the concept of Mordukhovich stationary point in terms of the Clarke subdifferentials and established that M-stationarity conditions are strong KKT-type sufficient optimality conditions for the multiobjective semi-infinite mathematical programming problem with equilibrium constraints.

We divide the index sets as follows. Let
$T^+ := \{i : \lambda_i^h > 0\}, \quad T^- := \{i : \lambda_i^h < 0\}$
$\beta^+ := \{i \in \beta : \lambda_i^G > 0, \lambda_i^H > 0\},$
$\beta_G^+ := \{i \in \beta : \lambda_i^G = 0, \lambda_i^H > 0\}, \quad \beta_G^- := \{i \in \beta : \lambda_i^G = 0, \lambda_i^H < 0\},$
$\beta_H^+ := \{i \in \beta : \lambda_i^H = 0, \lambda_i^G > 0\}, \quad \beta_H^- := \{i \in \beta : \lambda_i^H = 0, \lambda_i^G < 0\},$
$\alpha^+ := \{i \in \alpha : \lambda_i^G > 0\}, \quad \alpha^- := \{i \in \alpha : \lambda_i^G < 0\},$
$\gamma^+ := \{i \in \gamma : \lambda_i^H > 0\}, \quad \gamma^- := \{i \in \gamma : \lambda_i^H < 0\}.$

Definition 13.20 Let $\bar{x} \in X$ be a feasible point of the MOPSC. We say that the *No Nonzero Abnormal Multiplier Constraint Qualification* (NNAMCQ) is satisfied at

$\bar{x}$, if there is no nonzero vector $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{p+q+2l}$, such that

$$0 \in \sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{x}) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(\bar{x}) + \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(\bar{x}) + \lambda_i^H \nabla H_i(\bar{x})],$$

$$\forall i \in I^g(\bar{x}) : \lambda_i^g \geq 0,$$

$$\forall i \in \alpha(\bar{x}) : \lambda_i^H = 0,$$

$$\forall i \in \gamma(\bar{x}) : \lambda_i^G = 0,$$

and

$$\forall i \in \beta(\bar{x}) : \lambda_i^G \lambda_i^H = 0.$$

The following theorem shows that the MOPSC M-stationary conditions are a KKT type sufficient optimality conditions for weakly efficient solution of the MOPSC.

Theorem 13.5.1 *Let $\bar{x} \in X$ be a feasible point of the MOPSC and the M-stationarity conditions hold at $\bar{x}$. Suppose that each $f_i (i = 1, \ldots, m)$ is pseudoconvex at $\bar{x}$, $g_j (j \in J(\bar{x}))$, $h_i (i \in T^+)$, $-h_i (i \in T^-)$, $G_i (i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, $-G_i$ $(i \in \alpha^- \cup \beta_H^-)$, $H_i (i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$, $-H_i (i \in \gamma^- \cup \beta_G^-)$ are quasiconvex at $\bar{x}$. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then $\bar{x}$ is a weakly efficient solution for MOPSC.*

Proof Assume that $\bar{x}$ is not a weakly efficient solution for MOPSC. Then there exists a feasible point x for MOPSC such that such that

$$f_i(x) < f_i(\bar{x}) \quad \forall i = 1, \ldots, m.$$

Since each f_i is pseudoconvex, we have

$$\langle \nabla f_i(\bar{x}), x - \bar{x} \rangle < 0. \tag{13.16}$$

Also $\eta_i > 0$ for all $i \in \{1, \ldots, m\}$, we get

$$\left\langle \sum_{i=1}^{m} \eta_i \nabla f_i(\bar{x}), x - \bar{x} \right\rangle < 0. \tag{13.17}$$

Since $\bar{x}$ is MOPSC M-stationary point, we have

$$-\sum_{i \in I_g} \lambda_i^g \nabla g_i(\bar{x}) - \sum_{i=1}^{q} \lambda_i^h \nabla h_i(\bar{x}) - \sum_{\alpha \cup \beta} \lambda_i^G \nabla G_i(\bar{x}) - \sum_{\beta \cup \gamma} \lambda_i^H \nabla H_i(\bar{x}) = \sum_{i=1}^{m} \eta_i \nabla f_i(\bar{x}). \tag{13.18}$$

By Eq. (13.17), we get

$$\left\langle \left(\sum_{i\in I_g} \lambda_i^g \nabla g_i(\bar{x}) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(\bar{x}) + \sum_{\alpha\cup\beta} \lambda_i^G \nabla G_i(\bar{x}) + \sum_{\beta\cup\gamma} \lambda_i^H \nabla H_i(\bar{x}) \right), x - \bar{x} \right\rangle > 0. \tag{13.19}$$

For each $i \in I_g(\bar{x}),\ g_i(x) \le 0 = g_i(\bar{x})$. Hence, by quasiconvexity of g_i, we have

$$\langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \le 0\,. \tag{13.20}$$

For any feasible point x of MOPSC and for each $i \in T^-,\ 0 = -h_i(\bar{x}) = h_i(x)$, by quasiconvexity of h_i, we get

$$\langle \nabla h_i(\bar{x}), x - \bar{x} \rangle \ge 0,\ \forall i \in T^-. \tag{13.21}$$

Similarly, we have

$$\langle \nabla h_i(\bar{x}), x - \bar{x} \rangle \le 0,\ \forall i \in T^+. \tag{13.22}$$

Also $G_i(x) \le G_i(\bar{x}),\ \forall i \in \alpha^+ \cup \beta_H^+$, and $H_i(x) \le H_i(\bar{x}),\ \forall i \in \gamma^+ \cup \beta_G^+$. Since all of these functions are quasiconvex, we get

$$\langle \nabla G_i(\bar{x}), x - \bar{x} \rangle \le 0,\ \forall i \in \alpha^+ \cup \beta_H^+, \tag{13.23}$$

$$\langle \nabla H_i(\bar{x}), x - \bar{x} \rangle \le 0,\ \forall i \in \gamma^+ \cup \beta_G^+. \tag{13.24}$$

From Eqs. (13.20)–(13.24), we have

$$\begin{aligned}
&\langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \le 0,\ \forall i \in I_g(\bar{x}),\\
&\langle \nabla h_i(\bar{x}), x - \bar{x} \rangle \le 0,\ \forall i \in T^+,\\
&\langle \nabla h_i(\bar{x}), x - \bar{x} \rangle \ge 0,\ i \in T^-,\\
&\langle \nabla G_i(\bar{x}), x - \bar{x} \rangle \le 0,\ \forall i \in \alpha^+ \cup \beta_H^+,\\
&\langle \nabla H_i(\bar{x}), x - \bar{x} \rangle \le 0,\ \forall i \in \gamma^+ \cup \beta_G^+.
\end{aligned}$$

Since $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, we get

$$\left\langle \sum_{\alpha\cup\beta} \lambda_i^G \nabla G_i(\bar{x}), x - \bar{x} \right\rangle \le 0,\ \left\langle \sum_{\beta\cup\gamma} \lambda_i^H \nabla H_i(\bar{x}), x - \bar{x} \right\rangle \le 0,$$

$$\left\langle \sum_{i\in I_g(\bar{x})} \lambda_i^g \nabla g_i(\bar{x}), x - \bar{x} \right\rangle \le 0,\ \left\langle \sum_{i=1}^{q} \lambda_i^h \nabla h_i(\bar{x}), x - \bar{x} \right\rangle \le 0.$$

So,

$$\left\langle \left(\sum_{i\in I_g(\bar{x})} \lambda_i^g \nabla g_i(\bar{x}) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(\bar{x}) + \sum_{\alpha\cup\beta} \lambda_i^G \nabla G_i(\bar{x}) + \sum_{\beta\cup\gamma} \lambda_i^H \nabla H_i(\bar{x}) \right), x - \bar{x} \right\rangle \le 0,$$

which contradicts (13.19). Hence, $\bar{x}$ is a weakly efficient solution for MOPSC. This completes the proof. □

Theorem 13.5.2 *Let $\bar{x}$ be a feasible point of MOPSC and the M-stationarity conditions hold at $\bar{x}$. Suppose that each $f_i(i = 1, \ldots, m)$ is strictly pseudoconvex at $\bar{x}$, $g_i(i \in I_g(\bar{x}))$, $h_i(i \in T^+)$, $-h_i(i \in T^-)$, $G_i(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, $-G_i$ $(i \in \alpha^- \cup \beta_H^-)$, $H_i(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$, $-H_i(i \in \gamma^- \cup \beta_G^-)$ are quasiconvex at $\bar{x}$. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, then $\bar{x}$ is efficient solution for MOPSC.*

Proof The proof follows the lines of the proof of Theorem 13.5.1. □

13.6 Duality

In this section, we formulate and study a Wolfe-type dual problem for the MOPSC under the generalized convexity assumption. The Wolfe-type dual problem is formulated as follows:

$$WDMOPSC(\bar{x}) \quad \max_{u,\lambda} \quad f(u) + \Big[\sum_{i\in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) + \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]\Big] e$$

subject to:

$$0 \in \sum_{i=1}^{m} \rho_i \nabla f_i(u) + \sum_{i\in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(u) + \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)], \quad (13.25)$$

$$\forall i \in I^g(\bar{x}) : \lambda_i^g \geq 0,$$

$$\forall i \in \alpha(\bar{x}) : \lambda_i^H = 0,$$

$$\forall i \in \gamma(\bar{x}) : \lambda_i^G = 0,$$

$$\forall i \in \beta(\bar{x}) : \lambda_i^G \lambda_i^H = 0,$$

where, $e := (1, \ldots, 1) \in \mathbb{R}^m$, $\lambda = (\lambda^g, \lambda^h, \lambda^G, \lambda^H) \in \mathbb{R}^{k+p+2l}$, $\rho = (\rho_1, \ldots, \rho_m) \geq 0$ and $\sum_i^m \rho_i = 1$.

Theorem 13.6.3 (Weak Duality) *Let $\bar{x}$ be feasible for* MOPSC*, (u, ρ, λ) feasible for* WDMOPSC $(\bar{x})$ *and index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Suppose that each $f_i(i = 1, \ldots, m)$, $g_i(i \in I_g(\bar{x}))$, $h_i(i \in T^+)$, $-h_i(i \in T^-)$, $G_i(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, $-G_i$ $(i \in \alpha^- \cup \beta_H^-)$, $H_i(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ and $-H_i(i \in \gamma^- \cup \beta_G^-)$ are pseudoconvex at at u. If $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$, Then,*

$$f(x) \nleq f(u) + \Big[\sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) + \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]\Big] e.$$

Proof Let

$$f(x) \leq f(u) + \Big[\sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) + \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]\Big] e.$$

Then there exist n such that

$$f_n(x) < f_n(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) + \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]$$

and

$$f_i(x) \leqq f_i(u) + \sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) + \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)], \forall i \neq n.$$

From the Definition 13.9 and above inequality, we have

$$\left\langle \left(\sum_{i=1}^{m} \rho_i \nabla f_i(u) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(u) + \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)] \right), x - u \right\rangle < 0.$$

Then,

$$\sum_{i=1}^{m} \rho_i \nabla f_i(u) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(u) + \sum_{i=1}^{q} \lambda_i^h \nabla h_i(u) + \sum_{i=1}^{l} [\lambda_i^G \nabla G_i(u) + \lambda_i^H \nabla H_i(u)] < 0.$$

Which is a contradiction to the feasibility of the (u, ρ, λ) for the WDMOPSC, therefore

$$f(x) \nleq f(u) + \Big[\sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) + \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]\Big] e.$$

This complete the proof. □

Theorem 13.6.4 (Strong Duality) *If $\bar{x}$ is a efficient solution of* MOPSC*, such that* NNAMCQ *is satisfied at $\bar{x}$ and index sets $I_g, \alpha, \beta, \gamma$ defined accordingly. Let $f_i(i =$*

$1, \ldots m)$, $g_i(i \in I_g)$, $h_i(i \in J^+)$, $-h_i(i \in J^-)$, $G_i(i \in \alpha^- \cup \beta_H^-)$, $-G_i(i \in \alpha^+ \cup \beta_H^+ \cup \beta^+)$, $H_i(i \in \gamma^- \cup \beta_G^-)$, $-H_i(i \in \gamma^+ \cup \beta_G^+ \cup \beta^+)$ *satisfy the assumption of the Theorem 13.6.3 and If* $\alpha^- \cup \gamma^- \cup \beta_G^- \cup \beta_H^- = \phi$. *Then, there exists* $(\bar{\rho}, \bar{\lambda})$, *such that* $(\bar{x}, \bar{\rho}, \bar{\lambda})$ *is an efficient solution of* WDMOPSC $(\bar{x})$ *and respective objective values are equal.*

Proof Since, $\bar{x}$ is an efficient solution of MOPSC and the NNAMCQ is satisfied at $\bar{x}$, hence, $\exists\, \bar{\lambda} = (\bar{\lambda}^g, \bar{\lambda}^h, \bar{\lambda}^G, \bar{\lambda}^H) \in \mathbb{R}^{p+q+2l}$, such that the M-stationarity conditions for MOPSC are satisfied, that is,

$$0 = \sum_{i=1}^{m} \bar{\rho}_i \nabla f_i(\bar{x}) + \sum_{i \in I_g} \bar{\lambda}_i^g \nabla g_i(\bar{x}) + \sum_{i=1}^{q} \bar{\lambda}_i^h \nabla h_i(\bar{x}) + \sum_{i=1}^{l} [\bar{\lambda}_i^G \nabla G_i(\bar{x}) + \bar{\lambda}_i^H \nabla H_i(\bar{x})].$$

$$\forall i \in I^g(\bar{x}) : \lambda_i^g \geq 0, \forall i \in \alpha(\bar{x}) : \lambda_i^H = 0, \forall i \in \gamma(\bar{x}) : \lambda_i^G = 0, \forall i \in \beta(\bar{x}) : \lambda_i^G \lambda_i^H = 0.$$

Therefore, $(\bar{x}, \bar{\rho}, \bar{\lambda})$ is feasible for WDMOPSC $(\bar{x})$. By Theorem 13.6.3, from the feasibility condition of MOPSC and WDMOPSC $(\bar{x})$, we have

$$f(\bar{x}) = f(\bar{x}) + \Big[\sum_{i \in I_g} \bar{\lambda}_i^g g_i(\bar{x}) + \sum_{i=1}^{q} \bar{\lambda}_i^h h_i(\bar{x}) + \sum_{i=1}^{l} [\bar{\lambda}_i^G G_i(\bar{x}) + \bar{\lambda}_i^H H_i(\bar{x})]\Big] e. \tag{13.26}$$

Using Theorem 13.6.3 and from Eq. (13.26), we have

$$f(\bar{x}) = f(\bar{x}) + \Big[\sum_{i \in I_g} \bar{\lambda}_i^g g_i(\bar{x}) + \sum_{i=1}^{q} \bar{\lambda}_i^h h_i(\bar{x}) + \sum_{i=1}^{l} [\bar{\lambda}_i^G G_i(\bar{x}) + \bar{\lambda}_i^H H_i(\bar{x})]\Big] e$$

$$\nless f(u) + \Big[\sum_{i \in I_g} \lambda_i^g g_i(u) + \sum_{i=1}^{q} \lambda_i^h h_i(u) + \sum_{i=1}^{l} [\lambda_i^G G_i(u) + \lambda_i^H H_i(u)]\Big] e.$$

Hence, $(\bar{x}, \bar{\rho}, \bar{\lambda})$ is an efficient solution for WDMOPSC $(\bar{x})$ and the respective objective values are equal. □

13.7 Future Research Work

In the future, the concept of weak stationarity, Mordukhovich stationarity, and strong stationarity, i.e., W-stationarity, M-stationarity, and S-stationarity may be extended for nonsmooth multiobjective optimization problems with switching constraint using Mordukhovich limiting subdifferential and Michel–Penot subdifferential (see, [33, 45, 47]). Bao et al. [6] established new weak and strong suboptimality conditions for the general MPEC problems in finite-dimensional and infinite-dimensional spaces that do not assume the existence of optimal solutions. Bao and Mordukhovich [7] established necessary optimality conditions to super efficiency using variational principles for multiobjective optimization problems with equilibrium constraints. It will be interesting to obtain super efficiency, strong suboptimality conditions, and established necessary conditions for nonsmooth multiobjective optimization problems with switching constraints in the future.

Duality is an important subject in the study of mathematical programming problems as the weak duality provides a lower bound to the objective function of the primal problem. Pandey and Mishra [54, 55] formulated a Mond–Weir-type dual problem and established weak duality theorems, strong duality theorems under generalized standard Abadie constraint qualification for nonsmooth optimization problems with equilibrium constraints and semi-infinite mathematical programming problems with equilibrium constraints, respectively. Further, Mishra et al. [43] obtained a several duality theorems for mathematical programs with vanishing constraints. Recently, Su and Dinh [62] introduced the Mangasarian–Fromovitz-type regularity condition and the two Wolfe and Mond–Weir dual models for interval-valued pseudoconvex optimization problem with equilibrium constraints, as well as provided weak and strong duality theorems for the same using the notion of contingent epiderivatives with pseudoconvex functions in real Banach spaces. It will be interesting to study duality results in real Banach spaces for nonsmooth multiobjective optimization problems with switching constraint.

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