Chapter 12
On Minty Variational Principle for Nonsmooth Interval-Valued Multiobjective Programming Problems

Balendu Bhooshan Upadhyay and Priyanka Mishra

Abstract In this chapter, we consider a class of nonsmooth interval-valued multiobjective programming problems and a class of approximate Minty and Stampacchia vector variational inequalities. Under generalized approximate LU-convexity hypotheses, we establish the relations between the solutions of approximate Minty and Stampacchia vector variational inequalities and the approximate LU-efficient solutions of the nonsmooth interval-valued multiobjective programming problem. The results of this chapter extend and unify the corresponding results of [14, 22, 23, 30, 33] for nonsmooth interval-valued multiobjective programming problems.

Keywords Approximate LU-convexity · Approximate LU-efficient solutions · Interval-valued programming problems

12.1 Introduction

In multiobjective programming problems, two or more objective functions are minimized on some set of constraints. Usually, optimization problems are considered to deal with deterministic values, and therefore, we get precise solutions. However, in many real-life applications, optimization problems occur with uncertainty. Interval-valued optimization is one of the deterministic optimization models to deal with inexact, imprecise, or uncertain data. In interval-valued optimization, the coefficients of objective and constraint functions are compact intervals. To deal with the functions with interval coefficients, Moore [25, 26] introduced the concept of

B. B. Upadhyay (✉) · P. Mishra
Department of Mathematics, Indian Institute of Technology Patna, Patna 801103, India
e-mail: bhoodhan@iitp.ac.in

P. Mishra
e-mail: priyanka.iitp14@gmail.com

V. Laha et al. (eds.), *Optimization, Variational Analysis and Applications*,
Springer Proceedings in Mathematics & Statistics 355,
https://doi.org/10.1007/978-981-16-1819-2_12

interval analysis. Wu [31] established the Karush–Kuhn–Tucker optimality conditions for interval-valued optimization problem. Antczak [1] established Fritz John and Karush–Kuhn–Tucker necessary and sufficient optimality conditions for nonsmooth interval-valued multiobjective programming problem. For more details about interval-valued optimization problems, we refer to [2, 8, 9, 16, 17] and the references cited therein.

The notion of efficiency or Pareto optimality is a widely used solution concept in multiobjective programming problems. Due to complexity of multiobjective programming problems, several variants of efficient solutions have been studied by many researchers, see [4, 5, 13, 15, 18] and the references cited therein. Loridan introduced the notion of ϵ-efficient solution for multiobjective programming problems. Recently, many authors have shown interest in the study of characterization and applications of approximate efficient solutions of multiobjective programming problems, see [12, 13, 21, 22] and the references cited therein.

In 1980, Giannessi [10] introduced the notion of vector variational inequality problems. Vector variational inequality problems have wider applications in optimization, optimal control, and economics equilibrium problems, see for example [7, 11, 19] and the references cited therein. The equivalence between the solutions of vector variational inequalities and solutions of multiobjective programming problems have been studied extensively by many authors, see [20, 24, 27–30, 32] and the references cited therein. Mishra and Laha [22] established the relations between the solutions of approximate vector variational inequalities and approximate efficient solution of the nonsmooth multiobjective programming problems. Further, Gupta and Mishra [14] extend the results of [22] for generalized approximate convex functions. Zhang et al. [33] established the relations between the solutions of interval-valued multiobjective programming problems and vector variational inequalities.

12.1.1 The Proposed Work

The novelty and contributions of our work are of three folds:

In the first fold, motivated by the work of Gupta and Mishra [14], we have introduced a new class of generalized approximate LU-convex functions, namely; approximate LU-pseudoconvex of type I, approximate LU-pseudoconvex of type II, approximate LU-quasiconvex of type I, and approximate LU-quasiconvex of type II functions. These classes of generalized approximate LU-convex functions are more general than the classes of generalized approximate convex functions used in Gupta and Mishra [14], Mishra and Laha [22] and Mishra and Upadhyay [23].

In the second fold, we extend the works of Lee and Lee [20], Mishra and Upadhyay [23] and Upadhyay et al. [30] for the class of interval-valued multiobjective programming problems.

In the third fold, we generalize the works of [14, 20, 23, 30] from Euclidean space to a more general space such as Banach space.

The rest of the chapter is organized as follows: In Sect. 12.2, some basic definitions and preliminaries are given which will be used throughout the sequel. In Sect. 12.3, we establish the relations between the solutions of approximate vector variational inequalities and approximate LU-efficient solutions of the nonsmooth interval-valued multiobjective programming problem by using generalized approximate LU-convex functions. The numerical example has also been given to justify the significance of these results.

12.2 Definition and Preliminaries

Let Ω be a Banach space and Ω^* be its dual space equipped with norms $\|.\|$ and $\|.\|_*$, respectively. Let $\langle ,.,\rangle$ denotes the dual pair between Ω and Ω^* and Γ be a nonempty subset of Ω. Let $B(z;\delta)$ be an open ball centered at z and radius $\delta > 0$. Let $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^n$.

For $z, y \in \mathbb{R}^n$, following notion for equality and inequalities will be used throughout the sequel:

(i) $z = y, \iff z_i = y_i,\ \forall i = 1, 2, \ldots, n;$
(ii) $z < y, \iff z_i < y_i,\ \forall i = 1, 2, \ldots, n;$
(iii) $z \leqq y, \iff z_i \le y_i,\ \forall i = 1, 2, \ldots, n;$
(iv) $z \le y, \iff z_i \le y_i,\ \forall i = 1, 2, \ldots, n,\ i \neq j$ and $z_j < y_j$ for some j.

The following notions of interval analysis are from Moore [25].
Let $\mathcal{I}$ denotes the class of all closed intervals in $\mathbb{R}$. $A = [a^L, a^U] \in \mathcal{I}$ denotes a closed interval, where a^L and a^U denote the lower and upper bounds of A, respectively.

For $A = [a^L, a^U],\ B = [b^L, b^U] \in \mathcal{I}$, we have

(i) $A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U];$
(ii) $-A = \{-a : a \in A\} = [-a^U, -a^L];$
(iii) $A \times B = \{ab : a \in A \text{ and } b \in B\} = [\min_{ab}, \max_{ab}]$, where $\min_{ab} = \min\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$ and $\max_{ab} = \max\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$.

Then, we can show that

$$A - B = A + (-B) = [a^L - b^U, a^U - b^L],$$
$$kA = \{ka : a \in A\} = \begin{cases} [ka^L, ka^U], & k \geq 0, \\ |k|[-a^U, -a^L], & k < 0, \end{cases} \tag{12.1}$$

where $k \in \mathbb{R}$. The real number a can be considered as a closed interval $A_a = [a, a]$.

Let $A = [a^L, a^U],\ B = [b^L, b^U] \in \mathcal{I}$, then we define

1. $A \preceq_{LU} B \iff a^L \le b^L$ and $a^U \le b^U$,
2. $A \prec_{LU} B \iff A \preceq_{LU} B$ and $A \neq B$, that is, one of the following is satisfied:

   a. $a^L < b^L$ and $a^U < b^U$; or

b. $a^L \leq b^L$ and $a^U < b^U$; or
c. $a^L < b^L$ and $a^U \leq b^U$.

Remark 12.1 $A = [a^L, a^U],\ B = [b^L, b^U] \in \mathcal{I}$ are comparable if and only if $A \preceq_{LU} B$ or $A \succeq_{LU} B$. A and B are not comparable if one of the following holds:

$$a^L \leq b^L \text{ and } a^U > b^U;\ a^L < b^L \text{ and } a^U \geq b^U;\ a^L < b^L \text{ and } a^U > b^U;$$

$$a^L \geq b^L \text{ and } a^U < b^U;\ a^L > b^L \text{ and } a^U \leq b^U;\ a^L > b^L \text{ and } a^U < b^U.$$

Let $\mathbf{A} = (A_1, \ldots, A_n)$ be an interval-valued vector, where each component $A_k = [a_k^L, a_k^U],\ k = 1, 2, \ldots, n$ is a closed interval. Let $\mathbf{A}$ and $\mathbf{B}$ be two interval-valued vectors, if A_k and B_k are comparable for each $k = 1, 2, \ldots, n,$ then

1. $\mathbf{A} \preceq_{LU} \mathbf{B}$ if and only if $A_k \preceq_{LU} B_k$ for each $k = 1, 2, \ldots, n$;
2. $\mathbf{A} \prec_{LU} \mathbf{B}$ if and only if $A_k \preceq_{LU} B_k$ for each $k = 1, 2, \ldots, n$, and $A_r \prec_{LU} B_r$ for at least one index r.

The function $g : \mathbb{R}^n \to \mathcal{I}$ is called an interval-valued function, if $g(z) = [g^L(z), g^U(z)]$, where g^L and g^U are real-valued functions defined on $\mathbb{R}^n$ satisfying $g^L(z) \leq g^U(z)$, for all $z \in \mathbb{R}^n$.

Definition 12.1 ([24]) The set Γ is said to be a *convex set*, if for all $z, y \in \Gamma$, one has

$$z + \lambda(y - z) \in \Gamma,\ \forall \lambda \in [0, 1].$$

The following notions are from [6].

Definition 12.2 A function $g : \Gamma \to \mathbb{R}$ is said to be *Lipschitz* near $z_\circ \in \Gamma$, if there exist two positive constants $L, \delta > 0$, such that for all $y, z \in B(z_\circ; \delta) \cap \Gamma$, one has

$$|g(y) - g(z)| \leq L\|y - z\|.$$

The function g is *locally Lipschitz* on Γ, if it is Lipschitz near every $z \in \Gamma$.

Definition 12.3 Let $g : \Gamma \to \mathbb{R}$ be Lipschitz near $z \in \Gamma$. The *Clarke generalized directional derivative* of g at $z \in \Gamma$ in the direction $d \in \Omega$, is given as

$$g^\circ(z; d) := \limsup_{\substack{y \to z \\ t \downarrow 0}} \frac{g(y + td) - g(y)}{t}.$$

Definition 12.4 Let $g : \Gamma \to \mathbb{R}$ be *Lipschitz* near $z \in \Gamma$. The *Clarke generalized subdifferential* of g at $z \in \Gamma$ is given as

$$\partial^c g(z) := \{\xi \in \Omega^* : g^\circ(z; d) \geq \langle \xi, d \rangle,\ \ \forall d \in \Omega\}.$$

Definition 12.5 [13] A function $g : \Gamma \to \mathbb{R}$ is said to be *approximate convex* at $z_\circ \in \Gamma$, if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $z, y \in B(z_\circ; \delta) \cap \Gamma$, one has

$$g(y) - g(z) \geq \langle \xi, y - z \rangle - \varepsilon \|y - z\|, \quad \forall \xi \in \partial^c g(z).$$

The following notions of generalized approximate convexity are from Bhatia et al. [3].

Definition 12.6 A function $g : \Gamma \to \mathbb{R}$ is said to be *approximate pseudoconvex of type I* at $z_\circ \in \Gamma$, if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $z, y \in B(z_\circ; \delta) \cap \Gamma$, and if

$$\langle \xi, y - z \rangle \geq 0, \text{ for some } \xi \in \partial^c g(z),$$

then

$$g(y) - g(z) \geq -\varepsilon \|y - z\|.$$

Definition 12.7 A function $g : \Gamma \to \mathbb{R}$ is said to be *approximate pseudoconvex of type II (or strictly approximate pseudoconvex of type II)* at $z_\circ \in \Gamma$, if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $z, y \in B(z_\circ; \delta) \cap \Gamma$, and if

$$\langle \xi, y - z \rangle + \varepsilon \|y - z\| \geq 0, \text{ for some } \xi \in \partial^c g(z),$$

then

$$g(y) \geq (>) g(z).$$

Definition 12.8 A function $g : \Gamma \to \mathbb{R}$ is said to be *approximate quasiconvex of type I* at $z_\circ \in \Gamma$, if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $z, y \in B(z_\circ; \delta) \cap \Gamma$, and if

$$g(y) \leq g(z),$$

then

$$\langle \xi, y - z \rangle - \varepsilon \|y - z\| \leq 0, \ \forall \xi \in \partial^c g(z).$$

Definition 12.9 A function $g : \Gamma \to \mathbb{R}$ is said to be *approximate quasiconvex of type II (or strictly approximate quasiconvex of type II)* at $z_\circ \in \Gamma$, if for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $z, y \in B(z_\circ; \delta) \cap \Gamma$, and if

$$g(y) \leq (<) g(z) + \varepsilon \|y - z\|,$$

then

$$\langle \xi, y - z \rangle \leq 0, \ \forall \xi \in \partial^c g(z).$$

Definition 12.10 An interval-valued function $g : \Gamma \to \mathcal{I}$ is said to be an *approximate LU-pseudoconvex function of type I (or approximate LU-pseudoconvex func-*

tion of type II) at $z_\circ \in \Gamma$, if and only if the real-valued functions $g^L(z)$ and $g^U(z)$ are approximate pseudoconvex functions of type I (or approximate pseudoconvex functions of type II) at $z_\circ \in \Gamma$.

Definition 12.11 An interval-valued function $g : \Gamma \to \mathcal{I}$ is said to be a *strictly approximate LU-pseudoconvex function of type II* at $z_\circ \in \Gamma$, if and only if the real-valued functions $g^L(z)$ and $g^U(z)$ are approximate pseudoconvex functions of type II and at least one of the $g^L(z)$ and $g^U(z)$ is strictly approximate pseudoconvex function of type II at $z_\circ \in \Gamma$.

Definition 12.12 An interval-valued function $g : \Gamma \to \mathcal{I}$ is said to be an *approximate LU-quasiconvex function of type I (approximate LU-quasiconvex function of type II)* at $z_\circ \in \Gamma$, if and only if the real-valued functions $g^L(z)$ and $g^U(z)$ are approximate quasiconvex functions of type I (or approximate quasiconvex function of type II) at $z_\circ \in \Gamma$.

We consider the following nonsmooth interval-valued multiobjective programming problem:

$$\text{(NIVMPP)} \quad \text{Minimize} \quad \boldsymbol{g}(z) = (g_1(z), \ldots, g_p(z)),$$
$$\text{subject to} \quad z \in \Gamma,$$

where $g_i = [g_i^L, g_i^U] : \Gamma \to \mathcal{I}$, $i \in I := \{1, \ldots, p\}$ are locally Lipschitz interval-valued functions and Γ be a nonempty, closed, and convex subset of Ω.

The following notions of approximate LU-efficient solution are the adaptation of the notions of approximate efficient solution introduced by Mishra and Laha [22].

Let $\epsilon = (\varepsilon, \ldots, \varepsilon)$, a point $z_\circ \in \Gamma$ is said to be an approximate LU-efficient solution:

$(\text{ALUES})_1$, if and only if for any sufficiently small $\varepsilon > 0$, there does not exist $\delta > 0$ such that, for all $z \in B(z_\circ; \delta) \cap \Gamma$, $z \neq z_\circ$, one has

$$\boldsymbol{g}(z) \prec_{LU} \boldsymbol{g}(z_\circ) + \epsilon \|z - z_\circ\|.$$

$(\text{ALUES})_2$, if and only if for any sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in B(z_\circ; \delta) \cap \Gamma$, one has

$$\boldsymbol{g}(z) \nprec_{LU} \boldsymbol{g}(z_\circ) + \epsilon \|z - z_\circ\|.$$

$(\text{ALUES})_3$, if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in B(z_\circ; \delta) \cap \Gamma$, one has

$$\boldsymbol{g}(z) \nprec_{LU} \boldsymbol{g}(z_\circ) - \epsilon \|z - z_\circ\|.$$

For more details about approximate efficient solution, we refer to [14, 22].

From now onward, $\epsilon := (\varepsilon, \ldots, \varepsilon)$, unless otherwise specified.

Now, for interval-valued functions, we formulate the following approximate Minty and Stampacchia vector variational inequalities in terms of Clarke subdifferential:

(AMVI)$_1$ To find $z_\circ \in \Gamma$ such that, for any sufficiently small $\varepsilon > 0$, there does not exist $\delta > 0$ such that, for all $z \in B(z_\circ;\delta) \cap \Gamma$, $z \neq z_\circ$ and $\xi_i^L \in \partial^c g_i^L(z)$ and $\xi_i^U \in \partial^c g_i^U(z)$, $i \in I$, one has

$$(\langle \xi_1^L, z - z_\circ\rangle, \dots, \langle \xi_p^L, z - z_\circ\rangle) \leq \epsilon\|z - z_\circ\|,$$
$$(\langle \xi_1^U, z - z_\circ\rangle, \dots, \langle \xi_p^U, z - z_\circ\rangle) \leq \epsilon\|z - z_\circ\|.$$

(AMVI)$_2$ To find $z_\circ \in \Gamma$ such that, for any sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in B(z_\circ;\delta) \cap \Gamma$ and $\xi_i^L \in \partial^c g_i^L(z)$ and $\xi_i^U \in \partial^c g_i^U(z)$, $i \in I$, one has

$$(\langle \xi_1^L, z - z_\circ\rangle, \dots, \langle \xi_p^L, z - z_\circ\rangle) \nleq \epsilon\|z - z_\circ\|,$$
$$(\langle \xi_1^U, z - z_\circ\rangle, \dots, \langle \xi_p^U, z - z_\circ\rangle) \nleq \epsilon\|z - z_\circ\|.$$

(AMVI)$_3$ To find $z_\circ \in \Gamma$ such that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in B(z_\circ;\delta) \cap \Gamma$ and $\xi_i^L \in \partial^c g_i^L(z)$ and $\xi_i^U \in \partial^c g_i^U(z)$, $i \in I$, one has

$$(\langle \xi_1^L, z - z_\circ\rangle, \dots, \langle \xi_p^L, z - z_\circ\rangle) \nleq -\epsilon\|z - z_\circ\|,$$
$$(\langle \xi_1^U, z - z_\circ\rangle, \dots, \langle \xi_p^U, z - z_\circ\rangle) \nleq -\epsilon\|z - z_\circ\|.$$

(ASVI)$_1$ To find $z_\circ \in \Gamma$ such that, for any $\varepsilon > 0$ sufficiently small, there exist $z \in \Gamma$, $z \neq z_\circ$, $\zeta_i^L \in \partial^c g_i^L(z_\circ)$ and $\zeta_i^U \in \partial^c g_i^U(z_\circ)$, $i \in I$, such that

$$(\langle \zeta_1^L, z - z_\circ\rangle, \dots, \langle \zeta_p^L, z - z_\circ\rangle) \leq \epsilon\|z - z_\circ\|,$$
$$(\langle \zeta_1^U, z - z_\circ\rangle, \dots, \langle \zeta_p^U, z - z_\circ\rangle) \leq \epsilon\|z - z_\circ\|.$$

(ASVI)$_2$ To find $z_\circ \in \Gamma$ such that, for any sufficiently small $\varepsilon > 0$, for all $z \in \Gamma$, $\zeta_i^L \in \partial^c g_i^L(z_\circ)$ and $\zeta_i^U \in \partial^c g_i^U(z_\circ)$, $i \in I$, one has

$$(\langle \zeta_1^L, z - z_\circ\rangle, \dots, \langle \zeta_p^L, z - z_\circ\rangle) \nleq \epsilon\|z - z_\circ\|,$$
$$(\langle \zeta_1^U, z - z_\circ\rangle, \dots, \langle \zeta_p^U, z - z_\circ\rangle) \nleq \epsilon\|z - z_\circ\|.$$

(ASVI)$_3$ To find $z_\circ \in \Gamma$ such that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in B(z_\circ;\delta) \cap \Gamma$, $\zeta_i^L \in \partial^c g_i^L(z_\circ)$ and $\zeta_i^U \in \partial^c g_i^U(z_\circ)$, $i \in I$, one has

$$(\langle \zeta_1^L, z - z_\circ\rangle, \dots, \langle \zeta_p^L, z - z_\circ\rangle) \nleq -\epsilon\|z - z_\circ\|,$$
$$(\langle \zeta_1^U, z - z_\circ\rangle, \dots, \langle \zeta_p^U, z - z_\circ\rangle) \nleq -\epsilon\|z - z_\circ\|.$$

Remark 12.2 If each g_i, $i \in I$ is real-valued function, then the above vector variational inequalties coincide with the vector variational inequalities given in [14, 22].

12.3 Relationship Among (NIVMPP), (ASVI) and (AMVI)

In this section, we establish some relationships between the nonsmooth interval-valued multiobjective programming problem (NIVMPP) and approximate vector variational inequalities (AMVI), (ASVI) under generalized approximate LU-convexity.

The following theorem states the condition under which an approximate LU-efficient solution becomes a solution of approximate Minty variational inequality.

Theorem 12.3.1 *Let each $g_i^L, g_i^U : \Gamma \to \mathbb{R}$, $i \in I$ be locally Lipschitz functions. Then,*

1. *if each g_i, $i \in I$ is approximate LU-pseudoconvex of type II at $z_\circ \in \Gamma$ and $z_\circ$ is an (ALUES)$_1$ of the (NIVMPP), then $z_\circ$ also solves (AMVI)$_1$;*
2. *if each g_i, $i \in I$ is approximate LU-pseudoconvex of type II at $z_\circ \in \Gamma$ and $z_\circ$ is an (ALUES)$_2$ of the (NIVMPP), then $z_\circ$ also solves (AMVI)$_2$;*
3. *if each g_i, $i \in I$ is strictly approximate LU-pseudoconvex of type II at $z_\circ \in \Gamma$ and $z_\circ$ is an (ALUES)$_3$ of the (NIVMPP), then $z_\circ$ also solves (AMVI)$_3$.*

Proof 1. On contrary assume that $z_\circ$ is an (ALUES)$_1$ of the (NIVMPP) but does not solves (AMVI)$_1$. Then, for some $\varepsilon > 0$ sufficiently small, there exists $\bar{\delta} > 0$, such that for all $z \in B(z_\circ; \bar{\delta}) \cap \Gamma$, $\xi_i^L \in \partial^c g_i^L(z)$ and $\xi_i^U \in \partial^c g_i^U(z)$, $i \in I$, we get

$$(\langle \xi_1^L, z - z_\circ \rangle, \ldots, \langle \xi_p^L, z - z_\circ \rangle) \leq \epsilon \|z - z_\circ\|,$$
$$(\langle \xi_1^U, z - z_\circ \rangle, \ldots, \langle \xi_p^U, z - z_\circ \rangle) \leq \epsilon \|z - z_\circ\|,$$

that is,

$$\begin{aligned} &\langle \xi_i^L, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| \geq 0 \\ &\langle \xi_i^U, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| \geq 0, \ \forall i \in I, i \neq j, \\ &\text{and} \\ &\langle \xi_j^L, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| > 0 \\ &\langle \xi_j^U, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| > 0, \ \text{for some } j \in I. \end{aligned} \tag{12.2}$$

Since, each g_i, $i \in I$ is approximate LU-pseudoconvex of type II at $z_\circ \in \Gamma$, it follows that each g_i^L and g_i^U, $i \in I$ are approximate pseudoconvex of type II. Hence, for all $\varepsilon > 0$, there exists $\hat{\delta} > 0$, such that, for all $z \in B(z_\circ, \hat{\delta}) \cap \Gamma$, if

$$\langle \xi_i^L, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| \geq 0, \ \text{for some } \xi_i^L \in \partial^c g_i^L(z), \ i \in I,$$

then

$$g_i^L(z) - g_i^L(z_\circ) \leq 0, \ \forall i \in I.$$

Similarly, if

$$\langle \xi_i^U, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| \geq 0, \ \text{for some } \xi_i^U \in \partial^c g_i^U(z), \ i \in I,$$

then

$$g_i^U(z) - g_i^U(z_\circ) \leq 0, \ \forall i \in I.$$

Let $\delta := \min\{\hat{\delta}, \bar{\delta}\}$, from (12.2) and the definition of approximate LU-pseudoconvexity of type II, we have

$$\boldsymbol{g}(z) - \boldsymbol{g}(z_\circ) \preceq_{LU} \mathbf{0} \prec_{LU} \epsilon \|z - z_\circ\|,$$

for all $z \in B(z_\circ; \delta) \cap \Gamma$, which contradicts our assumption.

2. On contrary assume that $z_\circ$ is an $(\text{ALUES})_2$ of the (NIVMPP) but does not solves $(\text{AMVI})_2$. Then, for some $\varepsilon > 0$, sufficiently small and for all $\bar{\delta} > 0$, there exists $z \in B(z_\circ; \bar{\delta}) \cap \Gamma$, $\xi_i^L \in \partial^c g_i^L(z)$ and $\xi_i^U \in \partial^c g_i^U(z)$, $i \in I$, we get

$$(\langle \xi_1^L, z - z_\circ \rangle, \ldots, \langle \xi_p^L, z - z_\circ \rangle) \leq \epsilon \|z - z_\circ\|,$$
$$(\langle \xi_1^U, z - z_\circ \rangle, \ldots, \langle \xi_p^U, z - z_\circ \rangle) \leq \epsilon \|z - z_\circ\|,$$

that is

$$\begin{aligned} &\langle \xi_i^L, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| \geq 0, \\ &\langle \xi_i^U, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| \geq 0, \ \forall i \in I, i \neq j, \\ &\text{and} \\ &\langle \xi_j^L, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| > 0, \\ &\langle \xi_j^U, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| > 0, \ \text{for some } j \in I. \end{aligned} \tag{12.3}$$

Since, each g_i, $i \in I$ is approximate LU-pseudoconvex of type II at $z_\circ \in \Gamma$, it follows that each g_i^L and g_i^U, $i \in I$ are approximate pseudoconvex of type II. Hence, for all $\varepsilon > 0$, there exists $\hat{\delta} > 0$, such that whenever $z \in B(z_\circ; \hat{\delta}) \cap \Gamma$ and if

$$\langle \xi_i^L, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| \geq 0, \ \text{for some } \xi_i^L \in \partial^c g_i^L(z), \ i \in I,$$

then

$$g_i^L(z) - g_i^L(z_\circ) \leq 0, \ \forall i \in I.$$

Similarly, if

$$\langle \xi_i^U, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| \geq 0, \ \text{for some } \xi_i^U \in \partial^c g_i^U(z), \ i \in I,$$

then

$$g_i^U(z) - g_i^U(z_\circ) \leq 0, \ \forall i \in I.$$

Let $\delta := \min\{\hat{\delta}, \bar{\delta}\}$, then from (12.3) and the definition of approximate LU-convexity of type II, one has

$$\boldsymbol{g}(z) - \boldsymbol{g}(z_\circ) \preceq_{LU} \mathbf{0} \prec_{LU} \epsilon \|z - z_\circ\|,$$

for some $z \in B(z_\circ; \delta) \cap \Gamma$, which contradicts our assumption.

3. On contrary assume that $z_\circ$ is an $(\text{ALUES})_3$ of the (NIVMPP) but does not solves $(\text{AMVI})_3$. Then, for some $\varepsilon > 0$ and for all $\bar{\delta} > 0$, one has

$$(\langle \xi_1^L, z - z_\circ \rangle, \ldots, \langle \xi_p^L, z - z_\circ \rangle) \leq -\epsilon \|z - z_\circ\| < \epsilon \|z - z_\circ\|,$$
$$(\langle \xi_1^U, z - z_\circ \rangle, \ldots, \langle \xi_p^U, z - z_\circ \rangle) \leq -\epsilon \|z - z_\circ\| < \epsilon \|z - z_\circ\|,$$

for all $z \in B(z_\circ; \bar{\delta}) \cap \Gamma$, $\xi_i^L \in \partial^c g_i^L(z)$ and $\xi_i^U \in \partial^c g_i^U(z)$,
that is,

$$\begin{aligned} &\langle \xi_i^L, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| > 0, \\ &\langle \xi_i^U, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| > 0, \ \forall i \in I. \end{aligned} \tag{12.4}$$

Since, each g_i, $i \in I$ is strictly approximate LU-pseudoconvex of type II at $z_\circ \in \Gamma$, it follows that each g_i^L and g_i^U, $i \in I$ are approximate pseudoconvex of type II and atleast one of the g_i^L and g_i^U, $i \in I$ is strictly approximate pseudoconvex of type II at $z_\circ \in \Gamma$. Without loss of generality, assume that each g_i^L, $i \in I$ is strictly approximate pseudoconvex of type II. Hence, for all $\varepsilon > 0$, there exists $\hat{\delta} > 0$, such that whenever $z \in B(z_\circ; \hat{\delta}) \cap \Gamma$ and if

$$\langle \xi_i^L, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| \geq 0, \ \text{for some}\, \xi_i^L \in \partial^c g_i^L(z), \ i \in I,$$

then

$$g_i^L(z) - g_i^L(z_\circ) < 0, \ \forall i \in I.$$

Similarly, if

$$\langle \xi_i^U, z_\circ - z \rangle + \varepsilon \|z - z_\circ\| \geq 0, \ \text{for some}\, \xi_i^U \in \partial^c g_i^U(z), \ i \in I,$$

then

$$g_i^U(z) - g_i^U(z_\circ) \leq 0, \ \forall i \in I.$$

Let $\delta := \min\{\bar{\delta}, \hat{\delta}\}$, from (12.4) and the definition of strictly approximate LU-pseudo convexity of type II, we have

$$g_i(z) - g_i(z_\circ) \prec_{LU} 0, \ \forall i \in I, \tag{12.5}$$

for all $z \in B(z_\circ; \delta) \cap \Gamma$.
From (12.5), we can get an $\varepsilon > 0$ sufficiently small, such that

$$\boldsymbol{g}(z) - \boldsymbol{g}(z_\circ) \prec_{LU} -\epsilon \|z - z_\circ\|,$$

which contradicts our assumption. □

Theorem 12.3.2 *Let each* $g_i^L, g_i^U : \Gamma \to \mathbb{R}$, $i \in I$ *be locally Lipschitz functions. Then*

1. *if each g_i, $i \in I$ is approximate LU-quasiconvex of type II at $z_\circ \in \Gamma$ and $z_\circ$ solves (ASVI)$_1$, then $z_\circ$ is also an (ALUES)$_1$ of the (NIVMPP);*
2. *if each g_i, $i \in I$ is approximate LU-quasiconvex of type II at $z_\circ \in \Gamma$ and $z_\circ$ solves (ASVI)$_2$, then $z_\circ$ is also an (ALUES)$_2$ of the (NIVMPP);*
3. *if each g_i, $i \in I$ is approximate LU-pseudoconvex of type II at $z_\circ \in \Gamma$ and $z_\circ$ solves (ASVI)$_3$, then $z_\circ$ is also an (ALUES)$_3$ of the (NIVMPP).*

Proof 1. On contrary assume that $z_\circ$ is a solution of (ASVI)$_1$ but not an (ALUES)$_1$ of the (NIVMPP). Then, for some $\varepsilon > 0$, sufficiently small, there exists $\bar{\delta} > 0$, such that

$$\boldsymbol{g}(z) - \boldsymbol{g}(z_\circ) \prec_{LU} \epsilon \|z - z_\circ\|, \tag{12.6}$$

for all $z \in B(z_\circ; \bar{\delta}) \cap \Gamma$, $z \neq z_\circ$. Since, each g_i, $i \in I$ is approximate LU-quasiconvex of type II at $z_\circ$, it follows that each g_i^L and g_i^U, $i \in I$ are approximate quasiconvex of type II at $z_\circ$. Hence, for all $\varepsilon > 0$, there exists $\hat{\delta} > 0$, such that for all $z \in B(z_\circ; \hat{\delta}) \cap \Gamma$, if

$$g_i^L(z) \leq g_i^L(z_\circ) + \varepsilon \|z - z_\circ\|, \ \forall i \in I,$$

then

$$\langle \zeta_i^L, z - z_\circ \rangle \leq 0, \ \forall \zeta_i^L \in \partial^c g_i^L(z_\circ), \ i \in I.$$

Similarly, if

$$g_i^U(z) \leq g_i^U(z_\circ) + \varepsilon \|z - z_\circ\|, \ \forall i \in I,$$

then

$$\langle \zeta_i^U, z - z_\circ \rangle \leq 0, \ \forall \zeta_i^U \in \partial^c g_i^U(z_\circ), \ i \in I.$$

Let $\delta := \min\{\bar{\delta}, \hat{\delta}\}$, from (12.6) and the definition of approximate LU-quasiconvexity of type II, one has

$$\langle \zeta_i^L, z - z_\circ \rangle \leq 0 < \varepsilon \|z - z_\circ\|,$$
$$\langle \zeta_i^U, z - z_\circ \rangle \leq 0 < \varepsilon \|z - z_\circ\|,$$

for all $z \in B(z_\circ; \delta) \cap \Gamma$, $\zeta_i^L \in \partial^c g_i^L(z_\circ)$, $\zeta_i^U \in \partial^c g_i^U(z_\circ)$, $i \in I$, which contradicts our assumption.

2. Assume that $z_\circ$ is a solution of (ASVI)$_2$. Then, for any $\varepsilon > 0$ sufficiently small, for every $z \in \Gamma$, $\zeta_i^L \in \partial^c g_i^L(z_\circ)$ and $\zeta_i^U \in \partial^c g_i^U(z_\circ)$, $i \in I$, one has

$$(\langle \zeta_1^L, z - z_\circ \rangle, \ldots, \langle \zeta_p^L, z - z_\circ \rangle) \nleq \epsilon \|z - z_\circ\|,$$
$$(\langle \zeta_1^U, z - z_\circ \rangle, \ldots, \langle \zeta_p^U, z - z_\circ \rangle) \nleq \epsilon \|z - z_\circ\|,$$

that is,

$$\begin{aligned} (\langle \zeta_1^L, z - z_\circ \rangle, \ldots, \langle \zeta_p^L, z - z_\circ \rangle) \nleq 0, \\ (\langle \zeta_1^U, z - z_\circ \rangle, \ldots, \langle \zeta_p^U, z - z_\circ \rangle) \nleq 0. \end{aligned} \tag{12.7}$$

Since, each $g_i,\ i \in I$ is approximate LU-quasiconvex of type II at $z_\circ$, it follows that each g_i^L and $g_i^U,\ i \in I$ are approximate quasiconvex of type II at $z_\circ$. Hence, for all $\varepsilon > 0$, there exists $\hat{\delta} > 0$, such that for all $z \in B(z_\circ, \hat{\delta}) \cap \Gamma$, if

$$g_i^L(z) \leq g_i^L(z_\circ) + \varepsilon\|z - z_\circ\|,\ \forall i \in I,$$

then

$$\langle \zeta_i^L, z - z_\circ \rangle \leq 0,\ \forall \zeta_i^L \in \partial^c g_i^L(z_\circ),\ i \in I.$$

Similarly, if

$$g_i^U(z) \leq g_i^U(z_\circ) + \varepsilon\|z - z_\circ\|,\ \forall i \in I,$$

then

$$\langle \zeta_i^U, z - z_\circ \rangle \leq 0,\ \forall \zeta_i^U \in \partial^c g_i^U(z_\circ),\ i \in I.$$

From (12.7) and the definition of approximate LU-quasiconvexity of type II, it follows that

$$\boldsymbol{g}(z) - \boldsymbol{g}(z_\circ) \not\prec_{LU} \epsilon\|z - z_\circ\|,$$

for all $z \in B(z_\circ; \delta) \cap \Gamma,\ z \neq z_\circ$. Therefore, $z_\circ$ is an $(\text{ALUES})_2$ of the (NIVMPP).

3. On contrary assume that $z_\circ$ solves $(\text{ASVI})_3$ but not an $(\text{ALUES})_3$. Then, for some $\varepsilon > 0$, and for all $\bar{\delta} > 0$, there exists $z \in B(z_\circ; \bar{\delta}) \cap \Gamma$, such that

$$\boldsymbol{g}(z) - \boldsymbol{g}(z_\circ) \prec_{LU} -\epsilon\|z - z_\circ\|,$$

that is

$$\begin{aligned} g_i^L(z) - g_i^L(z_\circ) &< 0, \\ g_i^U(z) - g_i^U(z_\circ) &< 0,\ \forall i \in I. \end{aligned} \tag{12.8}$$

Since, each $g_i,\ i \in I$ is approximate LU-pseudoconvex of type II at $z_\circ$, it follows that each g_i^L and $g_i^U,\ i \in I$ are approximate pseudoconvex of type II at $z_\circ$. Hence, for all $\varepsilon > 0$, there exists $\hat{\delta} > 0$, such that for all $z \in B(z_\circ; \hat{\delta}) \cap \Gamma$, if

$$\langle \zeta_i^L, z - z_\circ \rangle + \varepsilon\|z - z_\circ\| \geq 0,\ \text{for some } \zeta_i^L \in \partial^c g_i^L(z_\circ),\ i \in I,$$

then

$$g_i^L(z) - g_i^L(z_\circ) \geq 0,\ \forall i \in I.$$

Similarly, if

$$\langle \zeta_i^U, z - z_\circ \rangle + \varepsilon\|z - z_\circ\| \geq 0,\ \text{for some } \zeta_i^U \in \partial^c g_i^U(z_\circ),\ i \in I,$$

then

$$g_i^U(z) - g_i^U(z_\circ) \geq 0,\ \forall i \in I.$$

Let $\delta := \min\{\hat{\delta}, \bar{\delta}\}$, from (12.8) and the definition of approximate LU-pseudoconvexity of type II, one has

$$\begin{aligned}&\langle \zeta_i^L, z - z_\circ \rangle < -\varepsilon \|z - z_\circ\|,\\&\langle \zeta_i^U, z - z_\circ \rangle < -\varepsilon \|z - z_\circ\|, \ \forall i \in I,\end{aligned} \tag{12.9}$$

for some $z \in B(z_\circ; \delta) \cap \Gamma$ and all $\zeta_i^L \in \partial^c g_i^L(z_\circ)$, $\zeta_i^U \in \partial^c g_i^U(z_\circ)$, $i \in I$, which contradicts our assumption. □

The following corollary is a direct consequence of Theorems 12.3.1 and 12.3.2.

Corollary 12.1 *Let each $g_i^L, g_i^U : \Gamma \to \mathbb{R}$, $i \in I$ be locally Lipschitz functions. Then,*

1. *if each g_i, $i \in I$ is approximate LU-quasiconvex of type II and approximate LU-pseudoconvex of type II at $z_\circ \in \Gamma$. Let $z_\circ$ is a solution of (ASVI)$_1$, then $z_\circ$ is also a solution of (AMVI)$_1$.*
2. *if each g_i, $i \in I$ is approximate LU-quasiconvex of type II and approximate LU-pseudoconvex of type II at $z_\circ \in \Gamma$. Let $z_\circ$ is a solution of (ASVI)$_2$, then $z_\circ$ is also a solution of (AMVI)$_2$.*
3. *if each g_i, $i \in I$ is strictly approximate LU-pseudoconvex of type II at $z_\circ \in \Gamma$. Let $z_\circ$ is a solution of (ASVI)$_3$, then $z_\circ$ is also a solution of (AMVI)$_3$.*

Now, to illustrate the significance of Theorems 12.3.1, 12.3.2 and Corollary 12.1, we have the following example.

Example 12.1 Consider the following nonsmooth interval-valued multiobjective programming problem

$$\begin{aligned}\text{(P)} \qquad & \text{Minimize} \quad \boldsymbol{g}(z) = (g_1(z), g_2(z))\\ & \text{subject to} \quad z \in \Gamma \subseteq \mathbb{R},\end{aligned}$$

where $\Gamma = [-1, 1]$ and $g_1, g_2 : \Gamma \to \mathcal{I}$ are defined as

$$g_1^L(z) = \begin{cases} z^3 + z, & z \geq 0,\\ 2z, & z < 0,\end{cases} \quad g_1^U(z) = \begin{cases} z^3 + 2z, & z \geq 0,\\ z, & z < 0,\end{cases}$$

and

$$g_2^L(z) = \begin{cases} z - z^2, & z \geq 0,\\ 2z, & z < 0,\end{cases} \qquad g_2^U(z) = \begin{cases} z + 1, & z \geq 0,\\ 2z + e^z, & z < 0.\end{cases}$$

The Clarke generalized subdifferentials of g_1 and g_2 are given by

$$\partial^c g_1^L(z) = \begin{cases} 3z^2 + 1, & z > 0,\\ [1, 2], & z = 0,\\ 2, & z < 0,\end{cases} \qquad \partial^c g_1^U(z) = \begin{cases} 3z^2 + 2, & z > 0,\\ [1, 2], & z = 0,\\ 1, & z < 0,\end{cases}$$

and

$$\partial^c g_2^L(z) = \begin{cases} 1-2z, & z>0, \\ [1,2], & z=0, \\ 2, & z<0, \end{cases} \qquad \partial^c g_2^U(z) = \begin{cases} 1, & z>0, \\ [1,3], & z=0, \\ 2+e^z, & z<0, \end{cases}$$

For any $0<\varepsilon<1$, let $\delta = \frac{1}{10}$, such that for all $z, y \in B(0;\delta)\cap\Gamma$, $\xi_1^L \in \partial^c g_1^L(z)$, $\xi_1^U \in \partial^c g_1^U(z)$, $\xi_2^L \in \partial^c g_2^L(z)$ and $\xi_2^U \in \partial^c g_2^U(z)$, one has

$$\langle \xi_1^L, y-z\rangle + \varepsilon\|y-z\| = \begin{cases} (3z^2+1)(y-z)+\varepsilon\|y-z\| > 0, & z>0, y>0, y-z>0; \\ (3z^2+1)(y-z)+\varepsilon\|y-z\| < 0, & z>0, y>0, y-z<0; \\ (3z^2+1)(y-z)+\varepsilon\|y-z\| < 0, & z>0, y\le 0; \\ 2(y-z)+\varepsilon\|y-z\| > 0, & z<0, y\ge 0; \\ 2(y-z)+\varepsilon\|y-z\| > 0, & z<0, y<0, y-z>0; \\ 2(y-z)+\varepsilon\|y-z\| < 0, & z<0, y<0, y-z<0; \\ k_1(y-z)+\varepsilon\|y-z\| > 0, & z=0, y>0, k_1\in[1,2]; \\ k_1(y-z)+\varepsilon\|y-z\| < 0, & z=0, y<0, k_1\in[1,2], \end{cases}$$

$$\langle \xi_1^U, y-z\rangle + \varepsilon\|y-z\| = \begin{cases} (3z^2+2)(y-z)+\varepsilon\|y-z\| > 0, & z>0, y>0, y-z>0; \\ (3z^2+2)(y-z)+\varepsilon\|y-z\| < 0, & z>0, y>0, y-z<0; \\ (3z^2+2)(y-z)+\varepsilon\|y-z\| < 0, & z>0, y\le 0; \\ (y-z)+\varepsilon\|y-z\| > 0, & z<0, y<0, y-z>0; \\ (y-z)+\varepsilon\|y-z\| < 0, & z<0, y<0, y-z<0; \\ (y-z)+\varepsilon\|y-z\| > 0, & z<0, y\ge 0; \\ k_2(y-z)+\varepsilon\|y-z\| > 0, & z=0, y>0, k_2\in[1,2]; \\ k_2(y-z)+\varepsilon\|y-z\| < 0, & z=0, y<0, k_2\in[1,2], \end{cases}$$

$$\langle \xi_2^L, y-z\rangle + \varepsilon\|y-z\| = \begin{cases} (1-2z)(y-z)+\varepsilon\|y-z\| > 0, & z>0, y>0, y-z>0; \\ (1-2z)(y-z)+\varepsilon\|y-z\| < 0, & z>0, y>0, y-z<0; \\ (1-2z)(y-z)+\varepsilon\|y-z\| < 0, & z>0, y\le 0; \\ 2(y-z)+\varepsilon\|y-z\| > 0, & z<0, y<0, y-z>0; \\ 2(y-z)+\varepsilon\|y-z\| < 0, & z<0, y<0, y-z<0; \\ 2(y-z)+\varepsilon\|y-z\| > 0, & z<0, y\ge 0; \\ t_1(y-z)+\varepsilon\|y-z\| > 0, & z=0, y>0, t_1\in[1,2]; \\ t_1(y-z)+\varepsilon\|y-z\| < 0, & z=0, y<0, t_2\in[1,2]; \end{cases}$$

and

$$\langle \xi_2^U, y-z\rangle + \varepsilon\|y-z\| = \begin{cases} (y-z)+\varepsilon\|y-z\| > 0, & z>0, y>0, y-z>0; \\ (y-z)+\varepsilon\|y-z\| < 0, & z>0, y>0, y-z<0; \\ (y-z)+\varepsilon\|y-z\| < 0, & z>0, y\le 0; \\ (2+e^z)(y-z)+\varepsilon\|y-z\| > 0, & z<0, y<0, y-z>0; \\ (2+e^z)(y-z)+\varepsilon\|y-z\| < 0, & z<0, y<0, y-z<0; \\ (2+e^z)(y-z)+\varepsilon\|y-z\| > 0, & z<0, y\ge 0; \\ t_2(y-z)+\varepsilon\|y-z\| > 0, & z=0, y>0, t_2\in[1,3]; \\ t_2(y-z)+\varepsilon\|y-z\| < 0, & z=0, y<0, t_2\in[1,3]. \end{cases}$$

Also,

$$g_1^L(y)-g_1^L(z)=\begin{cases}(y-z)(y^2+zy+z^2+1), & z>0, y>0, y-z>0;\\ y^3+y-2z, & z<0, y>0;\\ 2(y-z), & z<0, y<0, y-z>0;\\ y^3+y, & z=0, y>0,\end{cases}$$
$$>0,$$

$$g_1^U(y)-g_1^U(z)=\begin{cases}(y-z)(y^2+zy+z^2+2), & z>0, y>0, y-z>0;\\ y^3+2y-z, & z<0, y>0;\\ y-z, & z<0, y<0, y-z>0;\\ y^3+2y, & z=0, y>0,\end{cases}$$
$$>0,$$

$$g_2^L(y)-g_2^L(z)=\begin{cases}(y-z)(1-z-y), & z>0, y>0, y-z>0;\\ y-y^2-2z, & z<0, y>0;\\ 2(y-z), & z<0, y<0, y-z>0;\\ y-y^2, & z=0, y>0,\end{cases}$$
$$>0,$$

and

$$g_2^U(y)-g_2^U(z)=\begin{cases}y-z, & z>0, y>0, y-z>0;\\ y+1-2z-e^z, & z<0, y>0;\\ 2(y-z)+e^y-e^z, & z<0, y<0, y-z>0;\\ y, & z=0, y>0,\end{cases}$$
$$>0.$$

Hence, $g_1=[g_1^L, g_1^U]$ and $g_2=[g_2^L, g_2^U]$ are approximate LU-pseudoconvex of type II at $z_\circ=0$.
Evidently, $z_\circ=0$, solves (ASVI)$_3$. Since, for any $z>0$, $z\in B(z_\circ;\delta)\cap\Gamma$, $\zeta_1^L\in\partial^c g_1^L(z_\circ)$, $\zeta_1^U\in\partial^c g_1^U(z_\circ)$, $\zeta_2^L\in\partial^c g_2^L(z_\circ)$ and $\zeta_2^U\in\partial^c g_2^U(z_\circ)$, we have

$$\begin{aligned}\langle\zeta_1^L, z-z_\circ\rangle+\varepsilon\|z-z_\circ\| &= k_1z+\varepsilon\|z\|>0,\ k_1\in[1,2],\\ \langle\zeta_1^U, z-z_\circ\rangle+\varepsilon\|z-z_\circ\| &= k_2z+\varepsilon\|z\|>0,\ k_2\in[1,2],\\ \langle\zeta_2^L, z-z_\circ\rangle+\varepsilon\|z-z_\circ\| &= t_1z+\varepsilon\|z\|>0,\ t_1\in[1,2],\\ \text{and } \langle\zeta_2^U, z-z_\circ\rangle+\varepsilon\|z-z_\circ\| &= t_2z+\varepsilon\|z\|>0,\ t_2\in[1,3],\end{aligned}$$

that is

$$(\langle \zeta_1^L, z - z_\circ \rangle, \langle \zeta_2^L, z - z_\circ \rangle) \nleq -\epsilon \|z - z_\circ\|,$$
$$(\langle \zeta_1^U, z - z_\circ \rangle, \langle \zeta_2^U, z - z_\circ \rangle) \nleq -\epsilon \|z - z_\circ\|.$$

Moreover, $z_\circ = 0$ is an $(\text{ALUES})_3$ of the problem (P). Since, for any $\varepsilon > 0$, let $\delta = \frac{1}{2}$, such that for all $z > 0$, $z \in B(z_\circ; \delta) \cap \Gamma$, we have

$$g_1^L(z) - g_1^L(z_\circ) + \varepsilon \|z - z_\circ\| = z^3 + z + \varepsilon \|z\| > 0,$$
$$g_1^U(z) - g_1^U(z_\circ) + \varepsilon \|z - z_\circ\| = z^3 + 2z + \varepsilon \|z\| > 0,$$
$$g_2^L(z) - g_2^L(z_\circ) + \varepsilon \|z - z_\circ\| = z - z^2 + \varepsilon \|z\| > 0,$$
$$g_2^U(z) - g_2^U(z_\circ) + \varepsilon \|z - z_\circ\| = z + \varepsilon \|z\| > 0,$$

that is

$$\boldsymbol{g}(z) - \boldsymbol{g}(z_\circ) + \epsilon \|z - z_\circ\| \not<_{LU} \mathbf{0}.$$

Furthermore, $z_\circ = 0$ solves $(\text{AMVI})_3$. Since, for any $\varepsilon > 0$, sufficiently small, let $\delta = \frac{1}{2}$, such that for all $z > 0$, $z \in B(z_\circ; \delta) \cap \Gamma$, $\xi_1^L \in \partial^c g_1(z)$, $\xi_1^U \in \partial^c g_1^U(z)$, $\xi_2^L \in \partial^c g_2^L(z)$ and $\xi_2^U \in \partial^c g_2^U(z)$, we have

$$\langle \xi_1^L, z - z_\circ \rangle + \varepsilon \|z - z_\circ\| = 3z^3 + z + \varepsilon \|z\| > 0,$$
$$\langle \xi_1^U, z - z_\circ \rangle + \varepsilon \|z - z_\circ\| = 3z^3 + 2z + \varepsilon \|z\| > 0,$$
$$\langle \xi_2^L, z - z_\circ \rangle + \varepsilon \|z - z_\circ\| = 1 - 2z + \varepsilon \|z\| > 0,$$
$$\langle \xi_2^U, z - z_\circ \rangle + \varepsilon \|z - z_\circ\| = z + \varepsilon \|z\| > 0,$$

that is

$$(\langle \xi_1^L, z - z_\circ \rangle, \langle \xi_2^L, z - z_\circ \rangle) \nleq -\epsilon \|z - z_\circ\|,$$
$$(\langle \xi_1^U, z - z_\circ \rangle, \langle \xi_2^U, z - z_\circ \rangle) \nleq -\epsilon \|z - z_\circ\|.$$

12.4 Conclusions

In this chapter, we have considered a class of nonsmooth interval-valued multiobjective programming problems (NIVMPP) and certain classes of approximate Minty and Stampacchia vector variational inequalities; namely, $(\text{AMVI})_1$, $(\text{AMVI})_2$, $(\text{AMVI})_3$, $(\text{ASVI})_1$, $(\text{ASVI})_2$, and $(\text{ASVI})_3$. We have established the equivalence among the solutions of these vector variational inequalities and the approximate LU-efficient solutions; namely, $(\text{ALUES})_1$, $(\text{ALUES})_2$, $(\text{ALUES})_3$ of the nonsmooth interval-valued multiobjective programming problem (NIVMPP). The numerical example has been given to justify the significance of these results. The results of the chapter extend and unify the corresponding results of [14, 22, 23, 30, 33] to a more general class of nonsmooth optimization problems, namely, nonsmooth interval-valued multiobjective programming problem (NIVMPP).

Acknowledgements The first author is supported by the Science and Engineering Research Board, Department of Science and Technology, Government of India, through grant number "ECR/2016/001961." The authors would like to express their thanks to the anonymous reviewers for their very valuable comments and suggestions to improve the quality of the chapter.

References

1. Antczak, T.: Optimality conditions and duality results for nonsmooth vector optimization problems with the multiple interval-valued objective function. Acta Math. Sci. Ser. B **37**, 1133–1150 (2017)
2. Antczak, T.: Exactness property of the exact absolute value penalty function method for solving convex nondifferentiable interval-valued optimization problems. J. Optim. Theory Appl. **176**, 205–224 (2018)
3. Bhatia, D., Gupta, A., Arora, P.: Optimality via generalized approximate convexity and quasi-efficiency. Optim. Lett. **7**, 127–135 (2013)
4. Chinchuluun, A., Pardalos, P.M.: A survey of recent developments in multiobjective optimization. Ann. Oper. Res. **154**, 29–50 (2007)
5. Chinchuluun, A., Pardalos, P.M., Migdalas, A., Pitsoulis, A.: Pareto Optimality. Game Theory and Equilibria. Springer, Berlin (2008)
6. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
7. Dafermos, S.: Exchange price equilibria and variational inequalities. Math. Program. **46**, 391–402 (1990)
8. Ghosh, D.: Newton method to obtain efficient solutions of the optimization problems with interval-valued objective functions. J. Appl. Math. Comput. **53**, 709–731 (2017)
9. Ghosh, D., Chauhan, R.S., Mesiar, R., Debnath, A.K.: Generalized Hukuhara Gâteaux and Fréchet derivatives of interval-valued functions and their application in optimization with interval-valued functions. Inform. Sci. **510**, 317–340 (2020)
10. Giannessi, F.: Theorems of the alternative, quadratic programs and complementarity problems. In: Cottle, R.W., Giannessi, F., Lions, J.L. (eds.) Variational Inequalities and Complementarity Problems, pp. 151–186. Wiley, Chichester (1980)
11. Giannessi, F.: On Minty variational principle. In: Giannessi, F., Komlósi, S., Rapcśak, T. (eds.) New Trends in Mathematical Programming, pp. 93–99. Kluwer Academic Publishers, Dordrecht (1997)
12. Gupta, D., Mehra, A.: Two types of approximate saddle points. Numer. Funct. Anal. Optim. **29**, 532–550 (2008)
13. Gupta, A., Mehra, A., Bhatia, D.: Approximate convexity in vector optimisation. Bull. Aust. Math. Soc. **74**, 207–218 (2006)
14. Gupta, P., Mishra, S.K.: On Minty variational principle for nonsmooth vector optimization problems with generalized approximate convexity. Optimization **67**, 1157–1167 (2018)
15. Hanson, M.A., Mond, B.: Necessary and sufficient conditions in constrained optimization. Math. Program. **37**, 51–58 (1987)
16. Jayswal, A., Ahmad, I., Banerjee, J.: Nonsmooth interval-valued optimization and saddle-point optimality criteria. Bull. Malays. Math. Sci. Soc. **39**, 1391–1411 (2016)
17. Jayswal, A., Stancu-Minasian, I., Banerjee, J.: Optimality conditions and duality for interval-valued optimization problems using convexifactors. Rend. Circ. Mat. Palermo **65**, 17–32 (2016)
18. Jeyakumar, V., Mond, B.: On generalised convex mathematical programming. J. Austral. Math. Soc. Ser. B. **34**, 43–53 (1992)
19. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, London (1980)

20. Lee, G.M., Lee, K.B.: Vector variational inequalities for nondifferential convex vector optimization problems. J. Glob. Optim. **32**, 597–612 (2005)
21. Loridan, P.: ϵ-solutions in vector minimization problems. J. Optim. Theory Appl. **43**, 265–276 (1984)
22. Mishra, S.K., Laha, V.: On Minty variational principle for nonsmooth vector optimization problems with approximate convexity. Optim. Lett. **10**, 577–589 (2016)
23. Mishra, S.K., Upadhyay, B.B.: Some relations between vector variational inequality problems and nonsmooth vector optimization problems using quasi efficiency. Positivity **17**, 1071–1083 (2013)
24. Mishra, S.K., Upadhyay, B.B.: Pseudolinear Functions and Optimization. Chapman and Hall, CRC Press (2014)
25. Moore, R.E.: Interval Analysis. Prentice-Hall, Englewood Cliffs, New Jersey (1966)
26. Moore, R.E.: Methods and Applications of Interval Analysis. SIAM Studies in Applied Mathematics, Philadelphia (1979)
27. Upadhyay, B.B., Mishra, P.: On generalized Minty and Stampacchia vector variational-like inequalities and nonsmooth vector optimization problem involving higher order strong invexity. J. Sci. Res. **64**, 182–191 (2020)
28. Upadhyay, B.B., Mishra, P.: On vector variational inequalities and vector optimization problems. In: Soft Computing: Theories and Applications, pp. 257–267. Springer, Singapore (2020)
29. Upadhyay, B.B., Mishra, P., Mohapatra, R.N., Mishra, S.K.: On the applications of nonsmooth vector optimization problems to solve generalized vector variational inequalities using convexificators. Adv. Intell. Sys. Comput. https://doi.org/10.1007/978-3-030-21803-4_66
30. Upadhyay, B.B., Mohapatra, R.N., Mishra, S.K.: On relationships between vector variational inequality and nonsmooth vector optimization problems via strict minimizers. Adv. Nonlinear Var. Inequal. **20**, 1–12 (2017)
31. Wu, H.-C.: The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function. Eur. J. Oper. Res. **176**, 46–59 (2007)
32. Yang, X.Q.: Vector variational inequality and vector pseudolinear optimization. J. Optim. Theory Appl. **95**, 729–734 (1997)
33. Zhang, J., Zheng, Q., Ma, X., Li, L.: Relationships between interval-valued vector optimization problems and variational inequalities. Fuzzy Optim. Decis. Mak. **15**, 33–55 (2016)