Chapter 6
Consensus of Networked Multi-agent Systems with Antagonistic Interactions and Communication Delays

A common feature of previous consensus results is the focus on cooperative systems [1–3]. The consensus of these systems is asymptotically achieved through collaboration, which is characterized by the diffusive coupling [4] and the non-negative weights among agents [5–8]. In many real-world cases, however, it is more reasonable to consider that some agents collaborate with each other, while others are competitive. Networks with antagonistic interactions are ubiquitous in real world [9], and it becomes a focus for studying in recent years [10–15]. Altafini [10] proved that bipartite consensus can be achieved over networks with antagonistic interactions. Furthermore, in [16], emergent behaviors were investigated over signed random dynamical networks. In [17], flocking behaviors were studied by using results about signed graph. In [18], the leader-following bipartite consensus issue for single-integrator multi-agent systems was investigated, where the signed digraph was considered to be structurally balanced and had a spanning tree.

To achieve the consensus, each node in a network has to transmit its state information to its neighbors via connections. However, because of physical and environmental limitations, communication constraints between connected nodes are unavoidable. As is well-known, the communication delay is one of the most universal communication constraints. Motivated by the aforementioned discussions, we investigate the consensus problem of signed networks with antagonistic interactions and communication delays in this chapter. To the best of our knowledge, only a few results have been done concerning such problem. Due to the difficulty that antagonistic interactions and communication delays need to be simultaneously considered, new techniques are required to deal with this problem. According to matrix theory, Lyapunov theorem, and some other mathematical analysis, we found that bipartite consensus can be achieved for those systems with communication delays. Furthermore, in order to obtain the final bipartite consensus solution, we construct an invariant function to study the relationship of the states of nodes and their initial states. Using some mathematical analysis skills, we provide the bipartite consensus solution with an explicit expression.

J. Lu et al., *Collective Behavior in Complex Networked Systems under Imperfect Communication*, https://doi.org/10.1007/978-981-16-1506-1_6

6.1 Continuous-Time Multi-agent Consensus

6.1.1 Linear Coupling

In this section, we consider a multi-agent system formed by N linearly coupled identical nodes, where each node's dynamic is described as follows:

$$\dot{x}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid [\mathrm{sgn}(a_{ij})x_j(t - \tau_{ij}) - x_i(t)],\ i \in \mathcal{N}, \tag{6.1}$$

where $x_i(t) \in \mathbb{R}^n$ is the state of node i at time t, and $\tau_{ij} > 0$ denotes the communication delay from v_j to v_i for $i \neq j$ and $\tau_{ii} = 0$. $A = [a_{ij}]_{N\times N}$ is the adjacency matrix of the network that is symmetric. Here it is assumed that there is no self-closed loop, which means that $a_{ii} = 0$.

Throughout this section, the bipartite consensus of dynamical system (6.1) is said to be realized if $\lim_{t\to\infty} x_i(t) = \alpha$ for $i \in \mathcal{V}_1$ and $\lim_{t\to\infty} x_i(t) = -\alpha$ for $i \in \mathcal{V}_2$.

Theorem 6.1 *Consider the networked multi-agent system (6.1) with a connected signed graph $G(A)$. The bipartite consensus can be asymptotically reached if $G(A)$ is structurally balanced. If instead $G(A)$ is structurally unbalanced, then* $\lim_{t\to\infty} x(t) = \mathbf{0}$.

Proof We first consider the case that $G(A)$ is structurally balanced. According to Lemma 1.8, one can obtain that $\exists D \in \mathcal{D}$ such that DAD has all nonnegative entries. Let $z(t) = Dx(t)$, we obtain that

$$z_i(t) = \sigma_i x_i(t)\ ,\ i \in \mathcal{N}. \tag{6.2}$$

Substituting (6.2) into (6.1) results in

$$\sigma_i \dot{z}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid [\mathrm{sgn}(a_{ij})\sigma_j z_j(t - \tau_{ij}) - \sigma_i z_i(t)]\ , i \in \mathcal{N}.$$

Since DAD is a nonnegative matrix, we have $\sigma_i \mathrm{sgn}(a_{ij})\sigma_j = 1$. Using $\sigma_i^2 = 1$, one can obtain the following equation:

$$\begin{aligned}\dot{z}_i(t) &= \sum_{j=1}^{N} \mid a_{ij} \mid [\sigma_i \mathrm{sgn}(a_{ij})\sigma_j z_j(t - \tau_{ij}) - \sigma_i^2 z_i(t)] \\ &= \sum_{j=1}^{N} \mid a_{ij} \mid [z_j(t - \tau_{ij}) - z_i(t)]\ ,\ i \in \mathcal{N}.\end{aligned} \tag{6.3}$$

Following [19], the consensus of networks system (6.3) is asymptotically reached. That is

$$\lim_{t\to\infty} z_i(t) \to \alpha,\ \forall i \in \mathcal{N}, \tag{6.4}$$

where $\alpha \in \mathbb{R}^n$ is a constant vector.

Hence, we can get that $\lim\limits_{t\to\infty} x_i(t) \to \sigma_i\alpha$ for $i \in \mathcal{N}$. Then, the bipartite consensus of system (6.1) can be reached if $G(A)$ is structurally balanced.

Next, we consider the case that $G(A)$ is structurally unbalanced. Following Lemma 1.10, we can conclude that $G(A)$ contains one or more negative cycles. For the sake of simplicity, let us first consider the simplest case of $G(A)$ with only one negative cycle. Without loss of generality, we assume that (ν_1, ν_2) belongs to the negative cycle and $a_{12} = a_{21} = a < 0$. According to Lemma 1.10, one can obtain that there is no $D \in \mathcal{D}$ such that DAD is a nonnegative matrix. However, for the subgraph $G(B)$, which denotes the rest part of $G(A)$ reducing the edge (ν_1, ν_2), it admits a bipartition of the nodes $\mathcal{V}_1$ and $\mathcal{V}_2$. Furthermore, one can find that $G(B)$ is connected and matrix B is irreducible. Now, we can make a hypothesis that nodes ν_1 and ν_2 simultaneously belong to $\mathcal{V}_1$ (or $\mathcal{V}_2$) and the rest nodes remain unchanged. Based on this hypothesis, we can choose $D_1 = diag(\sigma)$ with σ satisfying $\sigma_i = 1$ for $\nu_i \in \mathcal{V}_1$ and $\sigma_i = -1$ for $\nu_i \in \mathcal{V}_2$. Then $D_1AD_1 = A' = [a'_{ij}]_{N\times N}$ has exactly two negative elements, i.e., $a'_{12} = a'_{21} = a < 0$, and the rest elements are nonnegative. The following is a decomposition of the matrix A':

$$A' = A_{12} + A_{21} + B' \ , \tag{6.5}$$

where $A_{ij}, i, j \in \{1, 2\}$, denotes a matrix in which the element lied in the intersection of ith row and jth column is $a_{ij} \neq 0$ and others all are 0. $B' = [b'_{ij}]_{N\times N}$ is a nonnegative adjacency matrix. In order to clearly express the matrix B', we define a function as follows:

$$c(i, j) = \begin{cases} 0, & (i, j) = (1, 2)\ or\ (2, 1); \\ 1, & \text{otherwise}. \end{cases}$$

Hence, we get $b'_{ij} = c(i, j)|a_{ij}|$. It is easy to find that B' is irreducible. Let $\bar{B} = [\bar{b}_{ij}]_{N\times N}$ be the Laplacian matrix of $G(B')$, and its elements are defined as : $\bar{b}_{ij} = b'_{ij}\ (i \neq j)$, $\bar{b}_{ii} = -\sum_{j=1}^{N} b'_{ij}$. Therefore, $\xi = (1, 1, \cdots, 1)^\top$ is the left eigenvector of $\bar{B}$ corresponding to the zero eigenvalue, i.e., $\xi^\top \bar{B} = \mathbf{0}$, which implies that

$$\bar{b}_{ii} = -\sum_{j=1, j\neq i}^{N} \bar{b}_{ji}. \tag{6.6}$$

Further because $\bar{b}_{ii} = -\sum_{j=1}^{N} b'_{ij}$, one can obtain that

$$\sum_{j=1}^{N} b'_{ij} = \sum_{j=1}^{N} b'_{ji} \text{ and } \sum_{i=1}^{N} b'_{ji} = \sum_{i=1}^{N} b'_{ij}. \tag{6.7}$$

Let $z(t) = Dx(t)$, i.e., $z_i(t) = \sigma_i x_i(t)$ for any $i \in \mathcal{N}$, we have

$$\dot{z}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid [\sigma_i \sigma_j \text{sgn}(a_{ij}) z_j(t - \tau_{ij}) - z_i(t)]. \tag{6.8}$$

Consider the following Lyapunov functional for system (6.1):

$$V(t) = V_1(t) + V_2(t), \tag{6.9}$$

where

$$V_1(t) = \frac{1}{2} \sum_{i=1}^{N} x_i^\top(t) x_i(t), \tag{6.10}$$

and

$$V_2(t) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{t-\tau_{ji}}^{t} |a_{ji}| x_i^\top(\theta) x_i(\theta) d\theta. \tag{6.11}$$

Calculating the time derivative of $V_i(t)$ $(i = 1, 2)$ along the trajectories of system (6.1), we have

$$\begin{aligned}
\dot{V}_1(t) &= \sum_{i=1}^{N} x_i^\top(t) \dot{x}_i(t) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}| [x_i^\top(t) \text{sgn}(a_{ij}) x_j(t - \tau_{ij}) - x_i^\top(t) x_i(t)] \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}| [\sigma_i \sigma_j z_i^\top(t) \text{sgn}(a_{ij}) z_j(t - \tau_{ij}) - z_i^\top(t) z_i(t)] \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} b'_{ij} [z_i^\top(t) z_j(t - \tau_{ij}) - z_i^\top(t) z_i(t)] + a_{12} [z_1^\top(t) z_2(t - \tau_{12}) \\
&\quad + z_1^\top(t) z_1(t)] + a_{21} [z_2^\top(t) z_1(t - \tau_{21}) + z_2^\top(t) z_2(t)]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N}\sum_{j=1}^{N} b'_{ij}[z_i^\top(t)z_j(t-\tau_{ij}) - z_i^\top(t)z_i(t)] + \frac{1}{2}a[2z_1^\top(t)z_2(t-\tau_{12}) \\
&\quad + 2z_2^\top(t)z_1(t-\tau_{21}) + 2z_1^\top(t)z_1(t) + 2z_2^\top(t)z_2(t)],
\end{aligned}
\tag{6.12}
$$

and

$$
\begin{aligned}
\dot{V}_2(t) =& \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} |a_{ji}|[x_i^\top(t)x_i(t) - x_i^\top(t-\tau_{ji})x_i(t-\tau_{ji})] \\
=& \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b'_{ji}[x_i^\top(t)x_i(t) - x_i^\top(t-\tau_{ji})x_i(t-\tau_{ji})] \\
& + \frac{1}{2}a_{12}[x_2^\top(t-\tau_{12})x_2(t-\tau_{12}) - x_2^\top(t)x_2(t)] \\
& + \frac{1}{2}a_{21}[x_1^\top(t-\tau_{21})x_1(t-\tau_{21}) - x_1^\top(t)x_1(t)] \\
=& \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b'_{ij}z_i^\top(t)z_i(t) - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}z_j^\top(t-\tau_{ij})z_j(t-\tau_{ij}) \\
& + \frac{1}{2}a[z_2^\top(t-\tau_{12})z_2(t-\tau_{12} + z_1^\top(t-\tau_{21})z_1(t-\tau_{21}) \\
& - z_2^\top(t)z_2(t) - z_1^\top(t)z_1(t)].
\end{aligned}
\tag{6.13}
$$

Using Eqs. (6.12) and (6.13) gives that

$$
\begin{aligned}
\dot{V}(t) =& -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b'_{ij}[z_i^\top(t)z_i(t) - 2z_i^\top(t)z_j(t-\tau_{ij}) + z_j^\top(t-\tau_{ij})] \\
& + \frac{1}{2}a\{[2z_1^\top(t)z_2(t-\tau_{12}) + z_1^\top(t)z_1(t) + z_2^\top(t-\tau_{12})z_2(t-\tau_{12})] \\
& + [2z_2^\top(t)z_1(t-\tau_{21})] + z_2^\top(t)z_2(t) + z_1^\top(t-\tau_{21})z_1(t-\tau_{21})]\} \\
=& -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b'_{ij}[z_i - z_j(t-\tau_{ij})]^\top[z_i - z_j(t-\tau_{ij})] \\
& + \frac{1}{2}a\{[z_1(t) + z_2(t-\tau_{12})]^\top[z_1(t) + z_2(t-\tau_{12})] \\
& + [z_2(t) + z_1(t-\tau_{21})]^\top[z_2(t) + z_1(t-\tau_{21})]\} \\
\le& 0.
\end{aligned}
$$

Hence, $V(t)$ is non-increasing. Referring to the construction of $V(t)$, one has that $V(t) \geq 0$, which shows that $\lim_{t\to\infty} V(t)$ exists and is finite. Then, we can get the boundedness of $x_i(t)$ for $i \in \mathcal{N}$. Combining with the expression of $V(t)$, further one can easily show the boundedness of $\dot{x}_i(t)$ for $i \in \mathcal{N}$ by referring to system (6.1). Thus, $\dot{z}_i(t) = \sigma_i \dot{x}_i(t)$ is bounded, which implies $\ddot{V}(t)$ is also bounded.

According to Barbalat's Lemma ([20]), we can obtain that $\lim_{t\to\infty} b'_{ij}[z_i - z_j(t-\tau_{ij})]^\top[z_i - z_j(t-\tau_{ij})] = 0$, $\lim_{t\to\infty}[z_1(t)+z_2(t-\tau_{12})]^\top[z_1+z_2(t-\tau_{12})] = 0$, and $\lim_{t\to\infty}[z_2(t)+z_1(t-\tau_{21})]^\top[z_2(t)+z_1(t-\tau_{21})] = 0$, i.e., $\lim_{t\to\infty}[z_i - z_j(t-\tau_{ij})] = \mathbf{0}$, if $b'_{ij} > 0$, $\lim_{t\to\infty}[z_1 + z_2(t-\tau_{12})] = \mathbf{0}$, and $\lim_{t\to\infty}[z_2(t) + z_1(t-\tau_{21})] = \mathbf{0}$. Further considering the expression (6.8), we have

$$
\begin{aligned}
\dot{z}_1(t) = &\sum_{j=1}^{N} \mid a_{1j} \mid [\sigma_1\sigma_j \mathrm{sgn}(a_{1j}) z_j(t-\tau_{1j}) - z_1(t)] \\
&= a_{12}[z_1(t) + z_2(t-\tau_{12})] + \sum_{j=3}^{N} b'_{1j}[z_j(t-\tau_{1j}) - z_1(t)] \\
&\to \mathbf{0}, \text{ as } t \to \infty,
\end{aligned}
$$

$$
\begin{aligned}
\dot{z}_2(t) = &\sum_{j=1}^{N} \mid a_{2j} \mid [\sigma_2\sigma_j \mathrm{sgn}(a_{2j}) z_j(t-\tau_{2j}) - z_2(t)] \\
&= a_{21}[z_2(t) + z_1(t-\tau_{21})] + \sum_{j=3}^{N} b'_{2j}[z_j(t-\tau_{2j}) - z_2(t)] \\
&\to \mathbf{0}, \text{ as } t \to \infty,
\end{aligned}
$$

and

$$
\dot{z}_i(t) = \sum_{j=1}^{N} b'_{ij}[z_j(t-\tau_{ij}) - z_i(t)] \to \mathbf{0}, \text{ as } t \to \infty \text{ for } i \geq 3.
$$

Therefore, we get that $\lim_{t\to\infty} \dot{z}_i(t) \to \mathbf{0}$ for any $i \in \mathcal{N}$. Since the adjacency matrix B' is irreducible, one can obtain that $\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} z_2(t) = \cdots = \lim_{t\to\infty} z_N(t)$ by referring to $\lim_{t\to\infty} b'_{ij}[z_i - z_j(t-\tau_{ij})] = \mathbf{0}$. In addition, we can also get $\lim_{t\to\infty} z_1(t) = -\lim_{t\to\infty} z_2(t)$ by referring to $\lim_{t\to\infty}[z_2(t) + z_1(t-\tau_{21})] = \mathbf{0}$. Thus, the following continued equality can be concluded: $\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} z_2(t) = \cdots = \lim_{t\to\infty} z_N(t) = \mathbf{0}$. Therefore, we obtain that $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} \sigma_i z_i(t) = \mathbf{0}$ for $i \in \mathcal{N}$.

Now consider the case of $G(A)$ with m ($m \geq 2$) negative cycles. Referring to the above approach, we respectively select m negative cycles as follows: $(\nu_{i_1}, \nu_{j_1}), \cdots, (\nu_{i_m}, \nu_{j_m})$. There exists a diagonal matrix $D_2 \in \mathcal{D}$ such that

$$D_2 A D_2 = A_{i_1 j_1} + A_{j_1 i_1} + \cdots + A_{i_m j_m} + A_{j_m i_m} + B''.$$

Similarly we can get that $\lim_{t\to\infty} x_i(t) = \mathbf{0}$ for any $i \in \mathcal{N}$.

Remark 6.2 For the structurally unbalanced network, since the number of negative cycles is not the essential attribute of structurally unbalanced network, we just need to think about those structurally unbalanced networks with one negative cycle instead of all structurally unbalanced networks. This consideration reduces the difficulty of the problem.

Remark 6.3 There is a situation that needs to be considered. If a negative edge simultaneously belongs to two or more negative cycles, we should admit that the edge only belongs to one of those cycles and the rest negative cycles should be viewed positive cycles. Then we can still make a hypothesis that two vertices on this negative edge belong to $\mathcal{V}_1$ or $\mathcal{V}_2$. According to the proof progress of Theorem 6.1, it is obvious that the results of Theorem 6.1 still hold.

According to Theorem 6.1, if $G(A)$ is structurally balanced, the bipartite consensus can be asymptotically reached, and we have $\lim_{t\to\infty} x_i(t) = \alpha$ for $i \in \mathcal{V}_1$, $\lim_{t\to\infty} x_i(t) = -\alpha$ for $i \in \mathcal{V}_2$. Calculating the bipartite consensus value of $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^\top$ is not an easy task due to the existence of time delays. Here, the value of $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^\top$ will be obtained by an exact expression when the initial conditions of system (6.1) are given. We define $\mathbf{1} = (1, 1, \cdots, 1)^\top_{1\times N}$. The initial conditions about system (6.1) are provided as $x_i(s) = \sigma_i \varphi_i(s) \in C([-\tau, 0], \mathbb{R}^n)$, where $\tau = \max_{i,j}\{\tau_{ij}\}$. Hence, we have $z_i(s) = \varphi_i(s) \in C([-\tau, 0], \mathbb{R}^n)$. Let $\xi(t) = (\xi_1(t), \xi_2(t), \cdots, \xi_n(t))^\top$, where $\xi_r(t) = (1/N)(\sum_{i=1}^N z_{ir}(t) + \sum_{i=1}^N \sum_{j=1}^N |a_{ij}| \int_{t-\tau_{ij}}^t z_{jr}(s)ds)$, $r \in \{1, 2, \cdots, n\}$.

Theorem 6.4 *Consider a connected signed graph $G(A)$ that is structurally balanced. If $D \in \mathcal{D}$ renders DAD nonnegative, then the bipartite solution of (6.1) is* $\lim_{t\to\infty} x(t) = (D\mathbf{1}) \otimes [N\xi(0)/\sum_{i=1}^N (1 + \sum_{j=1}^N |a_{ij}|\tau_{ij})]$.

Proof Referring to the proof of Theorem 6.1, one can obtain that $\lim_{t\to\infty} z_i(t) = \alpha$ and $\lim_{t\to\infty} x(t) = (D\mathbf{1}) \otimes \alpha$. Using (6.3), we can obtain

$$\begin{aligned}\dot{\xi}_r(t) =& \frac{1}{N}\sum_{i=1}^N \sum_{j=1}^N |a_{ij}|[z_{jr}(t-\tau_{ij}) - z_{ir}(t)] \\ &+ \frac{1}{N}\sum_{i=1}^N \sum_{j=1}^N |a_{ij}|[z_{jr}(t) - z_{jr}(t-\tau_{ij})]\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|z_{ir}(t) + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|z_{jr}(t) \\
&= 0\,. \qquad (6.14)
\end{aligned}
$$

Therefore, $\xi_r(t)$ in (6.14) is a constant. That is,

$$
\begin{aligned}
\xi_r(t) &= \xi_r(0) \\
&= \frac{1}{N}\Big(\sum_{i=1}^{N} z_{ir}(0) + \sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|\int_{-\tau_{ij}}^{0}\varphi_{jr}(s)ds\Big). \qquad (6.15)
\end{aligned}
$$

Then, we can get

$$
\begin{aligned}
\xi_r(0) &= \lim_{t\to\infty}\xi_r(t) \\
&= \frac{1}{N}\Big(\sum_{i=1}^{N}\alpha_r + \sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|\tau_{ij}\alpha_r\Big) \\
&= \frac{\alpha_r}{N}\sum_{i=1}^{N}\Big(1 + \sum_{j=1}^{N}|a_{ij}|\tau_{ij}\Big)\,. \qquad (6.16)
\end{aligned}
$$

Hence, we have

$$
\alpha = \frac{N\xi(0)}{\sum_{i=1}^{N}(1+\sum_{j=1}^{N}|a_{ij}|\tau_{ij})}\,, \qquad (6.17)
$$

and

$$
\lim_{t\to\infty} x(t) = (D\mathbf{1}) \otimes \frac{N\xi(0)}{\sum_{i=1}^{N}(1+\sum_{j=1}^{N}|a_{ij}|\tau_{ij})}. \qquad (6.18)
$$

Remark 6.5 Referring to expression (6.17), for the case of network without communication delays, one can obtain that $\alpha = (1/N)\sum_{i=1}^{N}\varphi_i(0)$. This result is consistent with the one obtained in [10]. This shows that our results are more general. Moreover, we conclude that a bipartite consensus solution is not only associated with initial values of $x_i(t)$ but also closely related to communication delays and network structure.

Remark 6.6 According to Theorem 6.4, it is obvious that $\alpha \neq \mathbf{0}$ if $G(A)$ is structurally balanced unless $\xi_r(0) = \mathbf{0}$.

6.1.2 Nonlinear Coupling

In this subsection, we will investigate the multi-agent systems with nonlinear coupling. Consider the following multi-agent systems:

$$\dot{x}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid \{\mathrm{sgn}(a_{ij})h[x_j(t-\tau_{ij})] - h[x_i(t)]\},\ i \in \mathcal{N}, \tag{6.19}$$

where $x_i(t) \in \mathbb{R}$ is the state of node i. The function $h(\cdot) : \mathbb{R} \to \mathbb{R}$ is assumed to be odd and strictly monotone increasing, which implies $h(0) = 0$ and $h(-x) = -h(x)$. Further we assume that $h(\cdot)$ is unbounded.

Theorem 6.7 *Consider the nonlinear coupled system (6.19) with a connected signed graph $G(A)$. The bipartite consensus can be asymptotically reached if $G(A)$ is structurally balanced. If instead $G(A)$ is structurally unbalanced, then* $\lim_{t\to\infty} x(t) = \mathbf{0}$.

Proof Following Lemma 1.8, if $G(A)$ is structurally balanced, we can obtain that $\exists D \in \mathcal{D}$ such that DAD has all nonnegative entries. Let $Z(t) = Dx(t)$, i.e., $z_i(t) = \sigma_i x_i(t)$, one can easily get that

$$\dot{z}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid \{h[z_j(t-\tau_{ij})] - h[z_i(t)]\}\ ,\ i \in \mathcal{N}\ . \tag{6.20}$$

Following [19], we obtain that $\lim_{t\to\infty} z_i(t) \to \beta \in \mathbb{R}$ for any $i \in \mathcal{N}$, which shows that $\lim_{t\to\infty} x_i(t) \to \sigma_i\beta \in \mathbb{R}$ for $i \in \mathcal{N}$. Therefore, the bipartite consensus of system (6.19) can be reached if $G(A)$ is structurally balanced.

Next, we consider the case that $G(A)$ is structurally unbalanced. From Lemma 1.10, it follows that $G(A)$ contains one or more negative cycles. We first consider the case of $G(A)$ with only one negative cycle. The edge (ν_1, ν_2) is assumed to be a negative weighted edge belonging to the negative cycle, and $a_{12} = a_{21} = a < 0$. Choosing $D_1 = diag(\sigma)$ with σ satisfying $\sigma_i = 1$ for $\nu_i \in \mathcal{V}_1$ and $\sigma_i = -1$ for $\nu_i \in \mathcal{V}_2$, one can obtain that $D_1AD_1 = A'$ has exactly two negative elements, i.e., $a'_{12} = a'_{21} = a < 0$, and the rest elements are nonnegative. The decomposition of the matrix A' is shown as follows:

$$A' = A_{12} + A_{21} + B',$$

where the definitions of A_{12}, A_{21}, and B' are similar to the proof in Theorem 6.1.

Let $Z(t) = Dx(t)$, i.e., $z_i(t) = \sigma_i x_i(t)$ for any $i \in \mathcal{N}$, we have

$$\dot{z}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid \{\sigma_i\sigma_j \mathrm{sgn}(a_{ij})h[z_j(t-\tau_{ij})] - h[z_i(t)]\}. \tag{6.21}$$

Consider the following Lyapunov–Krasovskii functional for system (6.19)

$$W(x(t)) = W_1(x(t)) + W_2(x(t)) ,$$

where

$$W_1(x(t)) = \sum_{i=1}^{N} \int_0^{x_i(t)} h(s)ds ,$$

and

$$W_2(x(t)) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{t-\tau_{ij}}^{t} |a_{ij}| h^2[x_j(s)]ds .$$

Calculating the time derivative of $W_i(t)(i = 1, 2)$ along the trajectories of system (6.19), we have

$$\begin{aligned} \dot{W}_1(x(t)) =& \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b'_{ij} \{2h[z_i(t)]h[z_j(t-\tau_{ij})] - 2h^2[z_i(t)]\} \\ &+ a\{h[z_1(t)]h[z_2(t-\tau_{12})] + h^2[z_1(t)] \\ &+ h[z_2(t)]h[z_1(t-\tau_{21})] + h^2[z_2(t)] , \end{aligned} \tag{6.22}$$

and

$$\begin{aligned} \dot{W}_2(x(t)) =& \frac{1}{2} \sum_{i=1}^{N} b'_{ij} \sum_{j=1}^{N} h^2[z_i(t)] - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} h^2[z_j(t-\tau_{ij})] \\ &+ \frac{1}{2} a\{h^2[z_2(t-\tau_{12})] - h^2[z_2(t)] \\ &+ h^2[z_1(t-\tau_{21})] - h^2[z_1(t)]\} . \end{aligned} \tag{6.23}$$

Using Eqs. (6.22) and (6.23), we get that

$$\begin{aligned} \dot{W}(x(t)) =& -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b'_{ij} \{h[z_i(t)] - h[z_j(t-\tau_{ij})]\}^2 \\ &+ \frac{1}{2} a\{h[z_1(t)] + h[z_2(t-\tau_{12})]\}^2 \\ &+ \frac{1}{2} a\{h[z_2(t)] + h[z_1(t-\tau_{21})]\}^2 \\ \leq& 0 . \end{aligned} \tag{6.24}$$

Let $S = \{x(t) \mid \dot{W}(x(t)) = 0\}$. Then it follows from Eq. (6.24) that $S = \{x \in C([t-\tau, t], \mathbb{R}^N) \mid b'_{ij}\{h[z_i(t)] - h[z_j(t-\tau_{ij})]\} = 0,\ h[z_1(t)] + h[z_2(t-\tau_{12})] = 0,\ \text{and}\ h[z_2(t)] + h[z_1(t-\tau_{21})] = 0\}$. Combining with the property of $h(\cdot)$, we can get that the set S is an invariant set with respect to system (6.21). According to the LaSalle invariance principle [21], one can easily show that $x(t) \to S$ as $t \to \infty$. Thus, we have $\lim_{t\to\infty}\{h[z_i(t)] - h[z_j(t-\tau_{ij})]\} = 0$ for $b'_{ij} > 0$, $\lim_{t\to\infty}\{h[z_1(t)]+h[z_2(t-\tau_{12})]\} = 0$, and $\lim_{t\to\infty}\{h[z_2(t)]+h[z_1(t-\tau_{21})]\} = 0$. Hence, we have $\lim_{t\to\infty} \dot{z}_i(t) = 0$. In addition, since $h(\cdot)$ is unbounded and strictly increasing with $h(0) = 0$, we get that $\lim_{t\to\infty}[z_i(t) - z_j(t-\tau_{ij})] = 0$ when $b'_{ij} > 0$ and $\lim_{t\to\infty}[z_1(t) - z_2(t-\tau_{12})] = 0$. According to the fact that B is irreducible, we conclude that $z_1(t) = z_2(t) = \cdots = z_N(t)$ as $t \to \infty$. It follows from $\lim_{t\to\infty}[z_1(t) - z_2(t-\tau_{12})] = 0$ that $z_1(t) = -z_2(t)$ as $t \to \infty$. Hence, the following equality can be concluded: $\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} z_2(t) = \cdots = \lim_{t\to\infty} z_N(t) = 0$. Therefore, we obtain that $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} \sigma_i z_i(t) = 0$ for $i \in \mathcal{N}$.

The result still holds for the case of $G(A)$ with two or more negative cycles. The proof is similar to the case of linear coupling and is omitted for simplicity.

Similar to the case of linear coupling, we can give the bipartite solution of (6.19). The initial conditions about system (6.19) are provided as $x_i(s) = \sigma_i\psi_i(s) \in C([-\tau, 0], \mathbb{R})$. Hence, we have $z_i(s) = \psi_i(s) \in C([-\tau, 0], \mathbb{R})$. Let $\zeta(0) = (1/N)(\sum_{i=1}^N \psi_i(0) + \sum_{i=1}^N \sum_{j=1}^N |a_{ij}| \int_{-\tau_{ij}}^0 h(\psi_j(s))ds)$.

Theorem 6.8 *Consider a connected signed graph $G(A)$ that is structurally balanced. If $D \in \mathcal{D}$ renders DAD nonnegative, then the bipartite solution of (6.19) is* $\lim_{t\to\infty} x(t) = \beta D\mathbf{1}$, *where $\beta \in \mathbb{R}$ meets a relational expression as follows:*

$$\beta + h(\beta)\sum_{i=1}^N\sum_{j=1}^N(|a_{ij}|\tau_{ij}) - N \times \zeta(0) = 0. \tag{6.25}$$

Proof Referring to the proof of Theorem 6.4, one can get the bipartite solution of (6.19) similarly. Let

$$\zeta(t) = (1/N)(\sum_{i=1}^N z_i(t) + \sum_{i=1}^N\sum_{j=1}^N |a_{ij}| \int_{t-\tau_{ij}}^t h(z_j(s))ds)\ . \tag{6.26}$$

Combining with (6.20), we have $\dot{\zeta}(t) = 0$, which implies $\zeta(t) = \zeta(0)$. From $\zeta(0) = \lim_{t\to\infty} \zeta(t)$, we can get the expression (6.25).

Remark 6.9 In this theorem, although the value of β cannot be given by an explicit expression, we can get a numerical solution by iterative algorithm from (6.25). In numerical examples, Example 6.10 gives a numerical solution to (6.25) for $h(x) = x + 0.5\sin(x)$, which illustrates the computational feasibility.

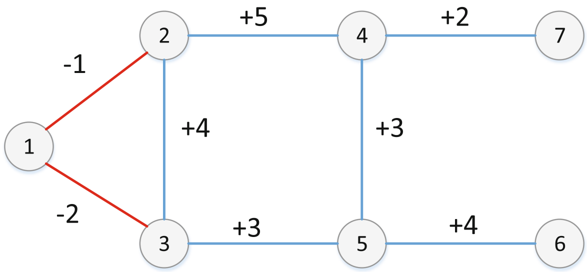

Fig. 6.1 Structurally balanced signed undirected connectivity graphs with seven nodes

6.1.3 Numerical Examples

In this subsection, numerical examples will be provided to demonstrate the effectiveness of our theoretical results.

Example 6.10 Now, we will give an example to illustrate the correctness of our main results. Consider the structurally balanced graph of Fig. 6.1.
For systems (6.1) and (6.19), all nonzero communication delays are listed as follows: $\tau_{12} = 0.1$, $\tau_{13} = 0.3$, $\tau_{21} = 0.15$, $\tau_{23} = 0.2$, $\tau_{24} = 0.1$, $\tau_{31} = 0.11$, $\tau_{32} = 0.16$, $\tau_{35} = 0.23$, $\tau_{42} = 0.1$, $\tau_{45} = 0.2$, $\tau_{47} = 0.12$, $\tau_{53} = 0.1$, $\tau_{54} = 0.15$, $\tau_{56} = 0.24$, $\tau_{65} = 0.25$, and $\tau_{74} = 0.15$, and the initial states are chosen as $x_1(s) = 1$, $x_2(s) = 2$, $x_3(s) = 3$, $x_4(s) = 4$, $x_5(s) = 5$, $x_6(s) = -3$, and $x_7(s) = -5$, $\forall s \in [-0.3, 0]$. Let $D = diag\{-1, 1, 1, 1, 1, 1, 1\}$. Then, we have $z(t) = [-1, 2, 3, 4, 5, -3, -5]$, $\forall s \in [-0.3, 0]$. Further, we define that $h(s) = s + 0.5\sin(s)$. According to Theorem 6.4, one can easily conclude that $\alpha = 1.44$. Following Theorem 6.8, one can get the numerical solution $\beta = 1.42$ by iterative algorithm. Numerical results are depicted in Figs. 6.2 and 6.3, which verify our theoretical results very well.

Example 6.11 Now let us consider a more general network topology with 100 nodes and signed weight edges. Here two simple signed networks with 100 nodes are constructed, where one is structurally balanced and another one is structurally unbalanced. The network with structurally balanced coupling is constructed as follows: we present 20 identical circular networks with 5 nodes, whose 5 nodes are numbered 1, 2, 3, 4, 5, respectively, and their adjacency matrix is $A = [a_{ij}]_{5\times5}$, where a_{ij} is chosen from $(-10, 0)$ or $(0, 10)$. Now the first two edges of the circular network are defined as negative edges and others are not negative edges, i.e. $a_{12} < 0$, $a_{23} < 0$, and the rest elements are non-negative Here these circular networks are arranged in a sequence. A connected graph with 100 nodes and structurally balanced coupling can be obtained by stochastic interconnections among the 3rd, 4th, and 5th nodes of adjacent circular networks. Similarly the network with structurally unbalanced coupling can be obtained according to the above method when the first three edges of pentagon are defined as negative edges and other steps are the same.

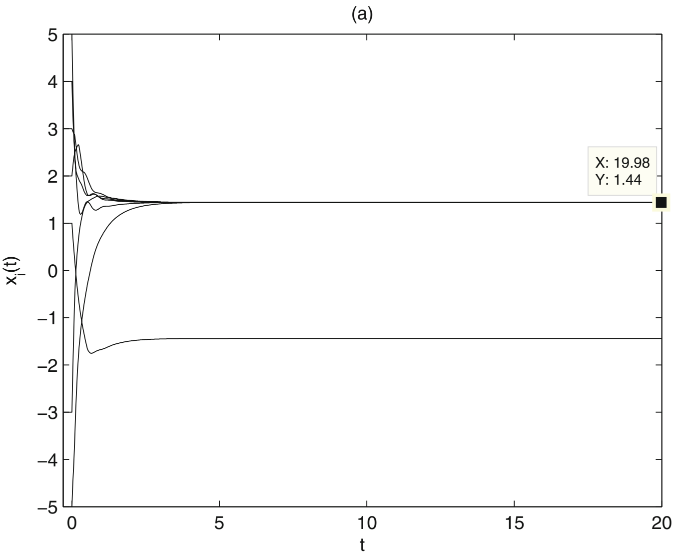

Fig. 6.2 The bipartite consensus on multi-agent system (6.1) with structurally balanced graph and linear coupling in Example 6.10

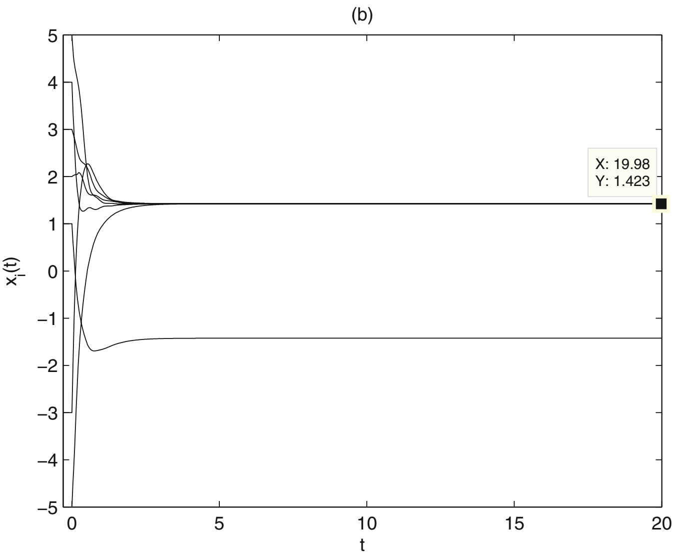

Fig. 6.3 The bipartite consensus on multi-agent system (6.19) with structurally balanced graph and nonlinear coupling in Example 6.10

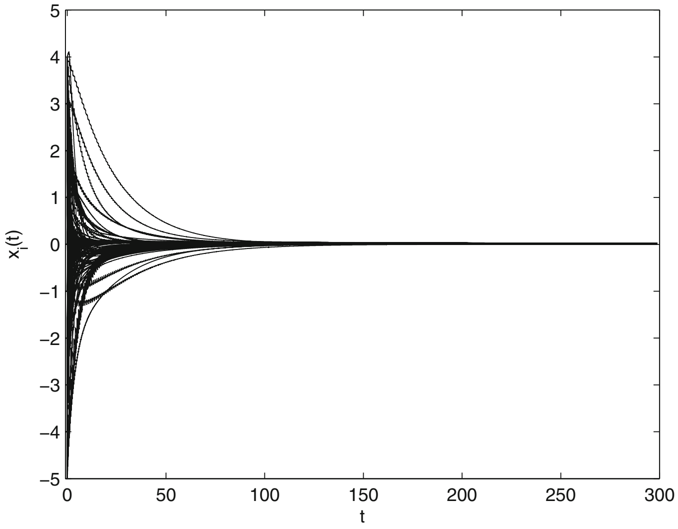

Fig. 6.4 The states of system (6.1) with structurally unbalanced graph and linear coupling in Example 6.11

All communication delays of systems (6.1) and (6.19) are uniformly distributed in (0, 1). Figure 6.4 shows that the consensus of system (6.1) can be achieved for $a_{ij} \in (-10, 10)$ and $\tau_{ij} \in (0, 1)$. Figure 6.5 shows that the consensus of system (6.19) can also be achieved for the above conditions. Throughout this example, the nonlinear function $h(x) = x + 0.5sin(x)$ is not changed. Figures 6.6 and 6.7 show that the bipartite consensus of systems (6.1) and (6.19) can be asymptotically reached, respectively.

6.2 Discrete-Time Multi-agent Consensus

6.2.1 Distributed Event-Based Bipartite Consensus

Consider a discrete-time multi-agent network with the dynamics described by

$$x_i(k+1) = x_i(k) + u_i(k), \quad i \in \mathcal{N}, \tag{6.27}$$

where $x_i(k) \in \mathbb{R}$ is the state of the agent i, and $u_i(k)$ is called the consensus protocol.

In this section, we assume that the protocol $u_i(k)$ is based on the event-triggered information transmission. The event-triggered time sequence of the agent i is given by $t_1^i, t_2^i, \cdots, t_l^i, \cdots$. At each triggering time t_l^i, the agent i will transmit the

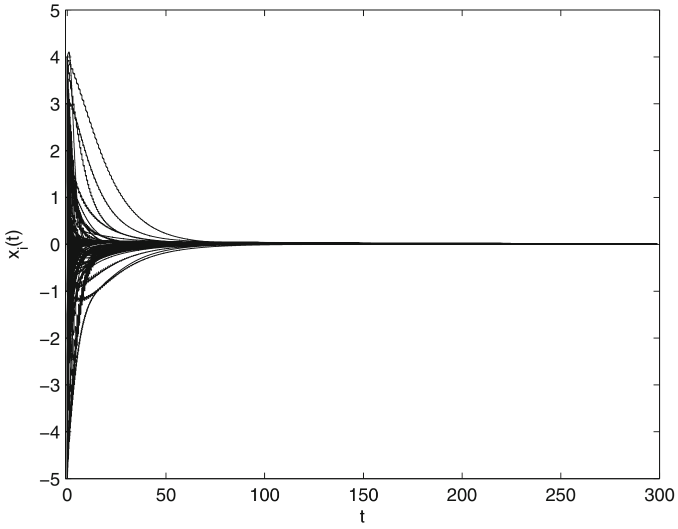

Fig. 6.5 The states of system (6.19) with structurally unbalanced graph and nonlinear coupling in Example 6.11

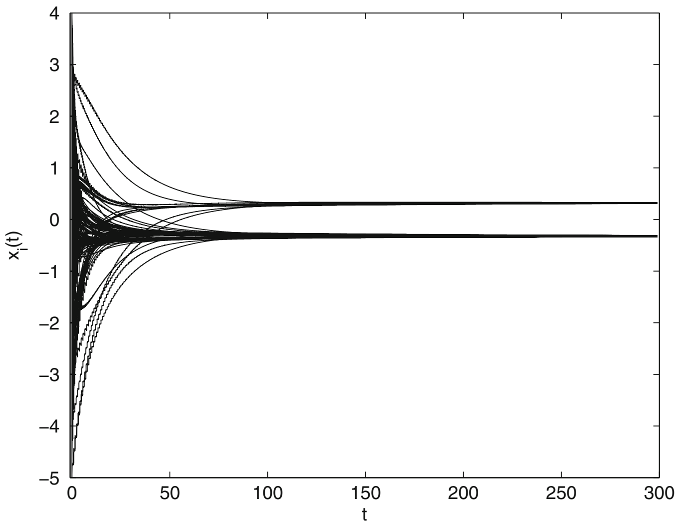

Fig. 6.6 The bipartite consensus on multi-agent system (6.1) with structurally balanced graph and linear coupling in Example 6.11

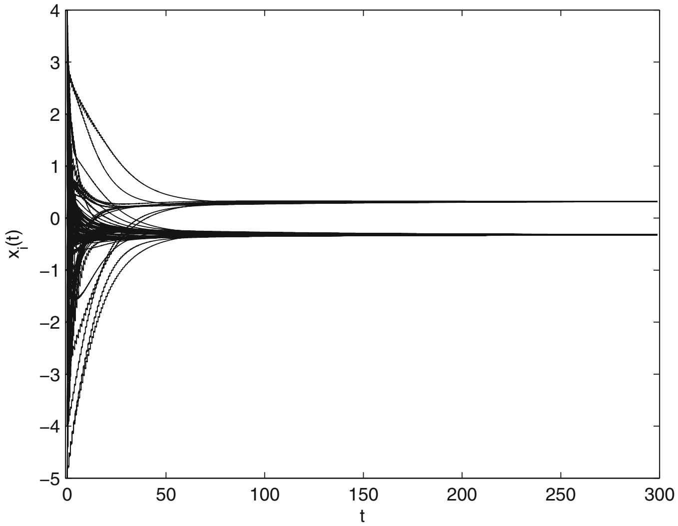

Fig. 6.7 The bipartite consensus on multi-agent system (6.19) with structurally balanced graph and nonlinear coupling in Example 6.11

state information to its neighboring agents. Considering time delays during the information transmission, the consensus protocol is proposed as follows:

$$u_i(k) = \sum_{j\in\mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k)), \quad i \in \mathcal{N}, \tag{6.28}$$

where $\tau_{ij} > 0$ denotes the communication delay from agent j to i, $\widehat{x}_j(k-\tau_{ij}) = x_j(k^j_{l'})$, $k-\tau_{ij} \in [k^j_{l'}, k^j_{l'+1})$, and $\widehat{x}_i(k) = x_i(k^i_l)$, $k \in [k^i_l, k^i_{l+1})$. It is assumed in this section that $\tau_{ii} = 0$, i.e., delays exist only in the information that is actually being transmitted between two different agents. The state measurement error of agent i is defined as

$$e_i(k) = x_i(k) - \widehat{x}_i(k). \tag{6.29}$$

Denote $\tau = \max\{\tau_{ij},\ i,\ j \in \mathcal{N}\}$. The initial conditions associated with (6.27) are given as $x_i(s)$, $s = -\tau, \cdots, -1, 0$.

In this subsection, we will give the distributed event-based bipartite consensus criteria for the considered signed network model. We always assume that the network topology of the signed digraph is strongly connected in this section. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ be the normalized left eigenvector of matrix $|A|$ with respect to the eigenvalue 1. From Lemma 1.6, we obtain that $\sum_{i=1}^{N} \xi_i = 1$ and $\xi_i > 0$.

Theorem 6.12 *Consider the multi-agent system (6.27) with arbitrary finite communication delay τ_{ij} under control law (6.28). If the first triggering time is $t_1^i = 0$, and agent $i,\ i \in \mathcal{N}$, determines the triggering time sequence $t_l^i|_{l=2}^{\infty}$ by*

$$\inf\left\{k > t_{l-1}^i:\ e_i^2(k) > \frac{\sigma a_{ii}^2}{4(1-a_{ii})}\sum_{j=1,\ j\neq i}^{N}|a_{ij}|(\widehat{x}_i(k) - sgn(a_{ij})\widehat{x}_j(k-\tau_{ij}))^2\right\},$$

where $0 < \sigma < 1$ is a constant. Then, we can obtain the following results:

(i) System (6.27) can achieve bipartite consensus asymptotically if the signed digraph $\mathcal{G}$ is structurally balanced. Moreover, the consensus value of the network is

$$\frac{\sum_{i=1}^{N}\xi_i d_i x_i(0) + \sum_{i=1}^{N}\xi_i\sum_{j=1,\ j\neq i}^{N}|a_{ij}|\sum_{s=-\tau_{ij}}^{-1}d_j x_j(s)}{1+\sum_{i=1}^{N}\xi_i\sum_{j=1,\ j\neq i}^{N}|a_{ij}|\tau_{ij}}.$$

(ii) If the signed digraph $\mathcal{G}$ is structurally unbalanced, then the system (6.27) can achieve consensus and the final consensus value is 0, i.e., $\lim_{k\to+\infty} x_i(k) = 0,\ \forall i \in \mathcal{N}$.

Proof We take two steps for the remaining part of the proof.

Step 1: According to Lemma 1.8, if the network structure is balanced, there exists $D = \{d_1, \cdots, d_N\} \in \mathcal{D}$, such that DAD is a stochastic matrix. Since DAD has all nonnegative entries, one can get $d_i sgn(a_{ij})d_j \geq 0$. Denote $y_i(k) = d_i x_i(k)$ and $\widehat{y}_i(k) = d_i\widehat{x}_i(k)$. Note that $\tau_{ii} = 0,\ \forall i \in \mathcal{N}$, then we can obtain that

$$y_i(k+1) = y_i(k)+\sum_{j\in\mathcal{N}_i}|a_{ij}|(d_i sgn(a_{ij})d_j\widehat{y}_j(k-\tau_{ij})-\widehat{y}_i(k)),\quad i \in \mathcal{N}, \tag{6.30}$$

i.e.,

$$\begin{aligned} y_i(k+1) &= y_i(k) + \sum_{j\in\mathcal{N}_i}|a_{ij}|(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\ &= y_i(k) + \sum_{j=1,\ j\neq i}^{N}|a_{ij}|(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)),\quad i \in \mathcal{N}. \end{aligned} \tag{6.31}$$

Let $E_i(k) = d_i e_i(k)$. Referring to Theorem 1 in [22], the consensus of system (6.31) can be asymptotically reached under the event-triggered condition

$$E_i^2(k) > \frac{\sigma a_{ii}^2}{4(1-a_{ii})}\sum_{j=1,\ j\neq i}^{N}|a_{ij}|(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k))^2$$

$$
\begin{aligned}
&= \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{y}_j^2(k-\tau_{ij}) + \widehat{y}_i^2(k) - 2\widehat{y}_j(k-\tau_{ij})\widehat{y}_i(k)) \\
&= \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{x}_j^2(k-\tau_{ij}) + \widehat{x}_i^2(k) \\
&\quad - 2sgn(a_{ij})\widehat{x}_j(k-\tau_{ij})\widehat{x}_i(k)), \\
&= \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))^2, \qquad i \in \mathcal{N}.
\end{aligned}
\tag{6.32}
$$

That is, $\lim_{k\to+\infty} d_i x_i(k) = c$, where c is a constant value. Note that $E_i^2(k) = e_i^2(k)$. Hence, the event-triggered condition (6.32) can be rewritten as

$$
e_i^2(k) > \frac{\sigma a_{ii}^2}{4(1-|a_{ii}|)} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))^2, \qquad i \in \mathcal{N}.
\tag{6.33}
$$

Therefore, under the event-triggered condition (6.33), the bipartite consensus of system (6.27) can be asymptotically reached if $\mathcal{G}$ is structurally balanced.

Next, the bipartite consensus value c of the multi-agent networks is shown below. Let $\eta(k) = \sum_{i=1}^{N} \xi_i y_i(k) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \sum_{k-\tau_{ij}}^{k-1} \widehat{y}_j(s)$. Substituting (6.31) into $\eta(k+1)$, we can calculate the difference of $\eta(k)$ as follows:

$$
\begin{aligned}
\Delta\eta(k) &= \eta(k+1) - \eta(k) \\
&= \sum_{i=1}^{N} \xi_i (y_i(k+1) - y_i(k)) \\
&\quad + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\sum_{k+1-\tau_{ij}}^{k} \widehat{y}_j(s) - \sum_{k-\tau_{ij}}^{k-1} \widehat{y}_j(s)) \\
&= \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\
&\quad + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{y}_j(k) - \widehat{y}_j(k-\tau_{ij})).
\end{aligned}
\tag{6.34}
$$

Note that the row sum of matrix $|A|$ is 1 and $\{\xi_1, \xi_2, \ldots, \xi_N\}$ is the normalized left eigenvector of matrix $|A|$ with respect to the eigenvalue 1, we have

$$\sum_{j=1}^{N} |a_{ij}| = 1 \text{ and } \sum_{i=1}^{N} \xi_i |a_{ij}| = \xi_j.$$

Hence, we can obtain that

$$\begin{aligned}
\Delta\eta(k) &= -\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \widehat{y}_i(k) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \widehat{y}_j(k) \\
&= -\sum_{i=1}^{N} \xi_i \widehat{y}_i(k) \sum_{j=1,\, j\neq i}^{N} |a_{ij}| + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} |a_{ij}| \widehat{y}_j(k) - \sum_{i=1}^{N} \xi_i a_{ii} \widehat{y}_i(k) \\
&= -\sum_{i=1}^{N} \xi_i (1 - a_{ii}) \widehat{y}_i(k) + \sum_{i=1}^{N} \xi_i |a_{ij}| \sum_{j=1}^{N} \widehat{y}_j(k) - \sum_{j=1}^{N} \xi_j a_{jj} \widehat{y}_j(k) \\
&= -\sum_{i=1}^{N} \xi_i (1 - a_{ii}) \widehat{y}_i(k) + \sum_{j=1}^{N} \xi_j (1 - a_{jj}) \widehat{y}_j(k) \\
&= 0. \qquad (6.35)
\end{aligned}$$

Due to $\Delta\eta(k) = 0$ for $k \geq 0$, it can be easily obtained that $\eta(k)$ is a constant. That is,

$$\begin{aligned}
\eta(k) = \eta(0) &= \sum_{i=1}^{N} \xi_i y_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \sum_{-\tau_{ij}}^{-1} \widehat{y}_j(s) \\
&= \sum_{i=1}^{N} \xi_i y_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \sum_{-\tau_{ij}}^{-1} y_j(s).
\end{aligned}$$

Hence,

$$\eta(0) = \lim_{k \to +\infty} \eta(k) = c + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j \neq i}^{N} |a_{ij}| \tau_{ij} c.$$

Therefore, we can conclude that

$$c = \frac{\sum_{i=1}^{N} \xi_i d_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \sum_{s=-\tau_{ij}}^{-1} d_j x_j(s)}{1 + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \tau_{ij}}. \tag{6.36}$$

This completes the proof of this step.

Step 2: If the network structure is unbalanced, according to Lemma 1.10, there does not exist $D = \{d_1, \cdots, d_N\} \in \mathcal{D}$, such that DAD is a stochastic matrix. For the sake of simplicity, the case of $\mathcal{G}$ with only one negative cycle is studied firstly. Here, we assume that this negative cycle contains an edge $a_{i_0 j_0} < 0$. Without loss of generality, we can assume that there exists $B = \{b_1, \cdots, b_N\} \in \mathcal{D}$, such that $BAB = [b_i a_{ij} b_j]_{N\times N}$ is a nonnegative matrix except the element $b_{i_0} a_{i_0 j_0} b_{j_0} < 0$. (If $\mathcal{G}$ contains k ($k \geq 2$) negative cycles, there exists $D_l \in \mathcal{D}$ such that $D_l A D_l$ has exactly l ($1 \leq l \leq k$) negative elements. The following proof for this case is similar to the case $k = 1$, and we omit it here due to space limit.) Denoting $y_i(k) = b_i x_i(k)$ and $\widehat{y}_i(k) = b_i \widehat{x}_i(k)$, then we can obtain that

$$y_i(k+1) = y_i(k) + \sum_{j\in\mathcal{N}_i} |a_{ij}|(b_i sgn(a_{ij}) b_j \widehat{y}_j(k - \tau_{ij}) - \widehat{y}_i(k)), \quad i \in \mathcal{N}. \tag{6.37}$$

Define the matrix $W = [w_{ij}]_{N\times N}$ as follows: $w_{i_0 j_0} = 0$, $w_{ii} = 1 - \sum_{j=1}^{N} w_{ij}$, $\forall i \in \mathcal{N}$, and $w_{ij} = b_i a_{ij} b_j$ otherwise. Let $E_i(k) = y_i(k) - \widehat{y}_i(k)$. Consider the Lyapunov functional as

$$V(k) = V_1(k) + V_2(k), \tag{6.38}$$

where

$$V_1(k) = \sum_{i=1}^{N} \xi_i y_i^2(k), \tag{6.39}$$

and

$$V_2(k) = \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} |a_{ij}| \sum_{s=k-\tau_{ij}}^{k-1} \widehat{y}_j^2(s). \tag{6.40}$$

Notice that $w_{i_0 i_0} = a_{i_0 i_0} + |a_{i_0 j_0}|$, and difference of $V(k)$ along the solution of (6.37) gives that

$$\begin{aligned}\Delta V(k) \le & -\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 \\ & - \xi_{i_0} |a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}| (\widehat{y}_{j_0}(k-\tau_{i_0 j_0}) - \widehat{y}_l(k-\tau_{i_0 l}))^2 \\ & - \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| (a_{ii} - \alpha_i)(\widehat{y}_i(k) - b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}))^2 \\ & + \sum_{i=1}^{N} \xi_i (1-a_{ii}) \frac{1}{\alpha_i} E_i^2(k). \end{aligned} \tag{6.41}$$

Actually,

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k). \tag{6.42}$$

Note that for $i \in \mathcal{N}$, $\tau_{ii} = 0$, and $E_i(k) = y_i(k) - \widehat{y}_i(k)$, it holds that

$$\begin{aligned} y_i(k+1) &= y_i(k) + \sum_{j\in\mathcal{N}_i} |a_{ij}| (b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\ &= y_i(k) + \sum_{j=1}^{N} |a_{ij}| (b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\ &= y_i(k) - \widehat{y}_i(k) + \sum_{j=1}^{N} |a_{ij}| b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}), \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{N} \xi_i y_i^2(k+1) &= \sum_{i=1}^{N} \xi_i [E_i(k) + \sum_{j=1}^{N} |a_{ij}| b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij})]^2 \\ &= \sum_{i=1, i\neq i_0}^{N} \xi_i [E_i(k) + \sum_{j=1}^{N} w_{ij} \widehat{y}_j(k-\tau_{ij})]^2 + \xi_{i_0} [E_{i_0}(k) \\ &\quad + \sum_{j=1}^{N} w_{i_0 j} \widehat{y}_j(k-\tau_{i_0 j}) - |a_{i_0 j_0}| (\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))]^2. \end{aligned}$$

Hence, we have

$$
\begin{aligned}
\Delta V_1(k) =& \sum_{i=1}^{N} \xi_i y_i^2(k+1) - \sum_{i=1}^{N} \xi_i y_i^2(k) \\
=& \sum_{i=1}^{N} \xi_i [E_i(k) + \sum_{j=1}^{N} w_{ij} \widehat{y}_j(k-\tau_{ij})]^2 + \xi_{i_0}[a_{i_0 j_0}^2 (\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 \\
& - 2|a_{i_0 j_0}|(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))(E_{i_0}(k) + \sum_{j=1}^{N} w_{i_0 j} \widehat{y}_j(k-\tau_{i_0 j}))] \\
& - \sum_{i=1}^{N} \xi_i y_i^2(k) \\
=& \sum_{i=1}^{N} \xi_i [E_i^2(k) + \sum_{j=1,\, j\neq i}^{N} w_{ij}^2 \widehat{y}_j^2(k-\tau_{ij}) + w_{ii} \widehat{y}_i^2(k) \\
& + 2 \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} \cdot \widehat{y}_j(k-\tau_{ij}) \widehat{y}_l(k-\tau_{ij}) \\
& + 2 \sum_{j=1,\, j\neq i}^{N} w_{ij} w_{ii} \widehat{y}_j(k-\tau_{ij}) \widehat{y}_i(k) + 2 w_{ii} \widehat{y}_i(k) E_i(k) \\
& + 2 \sum_{j=1,\, j\neq i}^{N} w_{ij} \widehat{y}_j(k-\tau_{ij}) E_i(k) - \sum_{i=1}^{N} \xi_i [\widehat{y}_i^2(k) \\
& + E_i^2(k) + 2\widehat{y}_i(k) E_i(k)] + \xi_{i_0}[a_{i_0 j_0}^2 (\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 \\
& - 2|a_{i_0 j_0}|(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))(E_{i_0}(k) + \sum_{j=1}^{N} w_{i_0 j} \widehat{y}_j(k-\tau_{i_0 j}))],
\end{aligned}
\tag{6.43}
$$

and

$$
\begin{aligned}
\Delta V_2(k) =& \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} |a_{ij}| [\sum_{k+1-\tau_{ij}}^{k} \widehat{y}_j^2(s) - \sum_{k-\tau_{ij}}^{k-1} \widehat{y}_j^2(s)] \\
=& \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} |a_{ij}| [\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{ij})]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\sum_{l=1}^{N}|a_{ij}||a_{il}|[\widehat{y}_j^2(k)-\widehat{y}_j^2(k-\tau_{ij})+\widehat{y}_l^2(k)-\widehat{y}_l^2(k-\tau_{il})] \\
&= \sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N} a_{ij}^2(\widehat{y}_j^2(k)-\widehat{y}_j^2(k-\tau_{ij}))+\sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N}|a_{ij}||a_{il}| \\
&\quad (\widehat{y}_j^2(k)-\widehat{y}_j^2(k-\tau_{ij})+\widehat{y}_l^2(k)-\widehat{y}_l^2(k-\tau_{ij}))+\sum_{j=1,\, j\neq i}^{N}|a_{ij}||a_{ii}|(\widehat{y}_j^2(k) \\
&\quad -\widehat{y}_j^2(k-\tau_{ij}))] \\
&= \sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N} w_{ij}^2(\widehat{y}_j^2(k)-\widehat{y}_j^2(k-\tau_{ij}))+\sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N}w_{ij}w_{il}(\widehat{y}_j^2(k) \\
&\quad -\widehat{y}_j^2(k-\tau_{ij})+\widehat{y}_l^2(k)-\widehat{y}_l^2(k-\tau_{il}))+\sum_{j=1,\, j\neq i}^{N}w_{ij}w_{ii}(\widehat{y}_j^2(k) \\
&\quad -\widehat{y}_j^2(k-\tau_{ij}))]+\xi_{i_0}[a_{i_0j_0}^2(\widehat{y}_{j_0}^2(k)-\widehat{y}_{j_0}^2(k-\tau_{i_0j_0}))+|a_{i_0j_0}||a_{i_0i_0}|(\widehat{y}_{j_0}^2(k) \\
&\quad -\widehat{y}_{j_0}^2(k-\tau_{i_0j_0}))-|a_{i_0j_0}|\sum_{j=1,\, j\neq i_0}^{N}w_{i_0j}(\widehat{y}_j^2(k)-\widehat{y}_j^2(k-\tau_{i_0j})) \\
&\quad +|a_{i_0j_0}|\sum_{l\neq i_0,\, l\neq j_0}^{N}|a_{i_0l}|(\widehat{y}_{j_0}^2(k)-\widehat{y}_{j_0}^2(k-\tau_{i_0j_0})+\widehat{y}_l^2(k)-\widehat{y}_l^2(k-\tau_{i_0l}))].
\end{aligned}
\tag{6.44}
$$

Let

$$
\begin{aligned}
\Delta_1 = &\sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N} w_{ij}^2\widehat{y}_j^2(k-\tau_{ij})+w_{ii}\widehat{y}_i^2(k)+2\sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N}w_{ij}w_{il} \\
&\widehat{y}_j(k-\tau_{ij})\widehat{y}_l(k-\tau_{ij})+2\sum_{j=1,\, j\neq i}^{N}w_{ij}w_{ii}\widehat{y}_j(k-\tau_{ij})\widehat{y}_i(k)-\sum_{i=1}^{N}\xi_i\widehat{y}_i^2(k) \\
&+\sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N} w_{ij}^2(\widehat{y}_j^2(k)-\widehat{y}_j^2(k-\tau_{ij}))+\sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N}w_{ij}w_{il}(\widehat{y}_j^2(k) \\
&-\widehat{y}_j^2(k-\tau_{ij})+\widehat{y}_l^2(k)-\widehat{y}_l^2(k-\tau_{il}))+\sum_{j=1,\, j\neq i}^{N}w_{ij}w_{ii}(\widehat{y}_j^2(k)-\widehat{y}_j^2(k-\tau_{ij}))]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j \neq i}^{N} w_{ij}^2 \widehat{y}_j^2(k) + w_{ii}^2 \widehat{y}_i^2(k) + \sum_{j=1,\, j \neq i}^{N} \sum_{l>j,\, l \neq i}^{N} w_{ij} w_{il} (\widehat{y}_j^2(k) + \widehat{y}_l^2(k) \\
&\quad - \widehat{y}_j^2(k-\tau_{ij}) - \widehat{y}_l^2(k-\tau_{il}) + 2\widehat{y}_j(k-\tau_{ij})\widehat{y}_l(k-\tau_{ij})) + \sum_{j=1,\, j \neq i}^{N} w_{ij} w_{ii} \\
&\quad \cdot (\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{ij}) + 2\widehat{y}_j(k-\tau_{ij})\widehat{y}_i(k))] - \sum_{i=1}^{N} \xi_i \widehat{y}_i^2(k) \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} w_{ij}^2 \widehat{y}_j^2(k) + \sum_{j=1,\, j \neq i}^{N} \sum_{l>j,\, l \neq i}^{N} w_{ij} w_{il} (\widehat{y}_j^2(k) + \widehat{y}_l^2(k)) + \sum_{j<i}^{N} w_{ij} w_{ii} \\
&\quad \cdot (\widehat{y}_j^2(k) + \widehat{y}_i^2(k)) + \sum_{l>i}^{N} w_{il} w_{ii} (\widehat{y}_l^2(k) + \widehat{y}_i^2(k)) - \widehat{y}_i^2(k)] \\
&\quad - \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j \neq i}^{N} \sum_{l>j,\, l \neq i}^{N} \\
&\quad w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j \neq i}^{N} w_{ij} w_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2] \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} w_{ij}^2 \widehat{y}_j^2(k) + \sum_{j=1}^{N} \sum_{l=1,\, l \neq j}^{N} w_{ij} w_{il} \widehat{y}_j^2(k) - \widehat{y}_i^2(k)] \\
&\quad - \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j \neq i}^{N} \sum_{l>j,\, l \neq i}^{N} \\
&\quad w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j \neq i}^{N} w_{ij} w_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2] \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} \sum_{l=1}^{N} w_{ij} w_{il} \widehat{y}_j^2(k) - \widehat{y}_i^2(k)] \\
&\quad - \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j \neq i}^{N} \sum_{l>j,\, l \neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) \\
&\quad - \widehat{y}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j \neq i}^{N} w_{ij} w_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2]
\end{aligned}
$$

$$= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} w_{ij} \widehat{y}_j^2(k) - \widehat{y}_i^2(k)] - \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j \neq i}^{N} \sum_{l>j,\, l \neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij})$$

$$-\widehat{y}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j\neq i}^{N} w_{ij} w_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2], \tag{6.45}$$

$$\Delta_2 = \xi_{i_0}[a_{i_0 j_0}^2 (\widehat{y}_{i_0}(k)+\widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 - 2|a_{i_0 j_0}|(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0})) \sum_{j=1}^{N} w_{i_0 j}$$

$$\cdot \widehat{y}_j(k-\tau_{i_0 j})] + \xi_{i_0}[a_{i_0 j_0}^2 (\widehat{y}_{j_0}^2(k) - \widehat{y}_{j_0}^2(k-\tau_{i_0 j_0})) + |a_{i_0 j_0}||a_{i_0 i_0}|(\widehat{y}_{j_0}^2(k)$$

$$-\widehat{y}_{j_0}^2(k-\tau_{i_0 j_0})) - |a_{i_0 j_0}| \sum_{j=1,\, j\neq i_0}^{N} w_{i_0 j}(\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{i_0 j}))$$

$$+|a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}|(\widehat{y}_{j_0}^2(k) - \widehat{y}_{j_0}^2(k-\tau_{i_0 j_0}) + \widehat{y}_l^2(k) - \widehat{y}_l^2(k-\tau_{i_0 l}))]$$

$$= \xi_{i_0}[a_{i_0 j_0}^2 (\widehat{y}_{i_0}^2(k) + 2\widehat{y}_{i_0}(k)\widehat{y}_{j_0}(k-\tau_{i_0 j_0})) - 2|a_{i_0 j_0}| w_{i_0 i_0} \widehat{y}_{i_0}^2(k) - 2|a_{i_0 j_0}| w_{i_0 i_0}$$

$$\cdot \widehat{y}_{i_0}(k)\widehat{y}_{j_0}(k-\tau_{i_0 j_0}) + a_{i_0 j_0}^2 \widehat{y}_{j_0}^2(k) + |a_{i_0 j_0}||a_{i_0 i_0}|(\widehat{y}_{j_0}^2(k) + \widehat{y}_{i_0}^2(k)$$

$$-\widehat{y}_{j_0}^2(k-\tau_{i_0 j_0})L - \widehat{y}_{i_0}^2(k)) - |a_{i_0 j_0}| \sum_{j=1,\, j\neq i_0}^{N} w_{i_0 j}(\widehat{y}_j^2(k) + \widehat{y}_{i_0}^2(k))$$

$$-|a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}|(\widehat{y}_{j_0}^2(k)$$

$$+\widehat{y}_l^2(k)) + |a_{i_0 j_0}| \sum_{j=1,\, j\neq i_0}^{N} w_{i_0 j}(-\widehat{y}_{i_0}^2(k) - \widehat{y}_j^2(k-\tau_{i_0 j})$$

$$-2\widehat{y}_{i_0}(k)\widehat{y}_j(k-\tau_{i_0 j}))$$

$$+|a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}|(-\widehat{y}_{j_0}^2(k-\tau_{i_0 j_0})$$

$$-\widehat{y}_l^2(k-\tau_{i_0 l}) - 2\widehat{y}_{j_0}(k-\tau_{i_0 j_0})\widehat{y}_l(k-\tau_{i_0 j})]$$

$$= \xi_{i_0}[a_{i_0 j_0}^2 \widehat{y}_{i_0}^2(k) - 2|a_{i_0 j_0}| w_{i_0 i_0} \widehat{y}_{i_0}^2(k) + a_{i_0 j_0}^2 \widehat{y}_{j_0}^2(k) + |a_{i_0 j_0}||a_{i_0 i_0}|(\widehat{y}_{j_0}^2(k)$$

$$+\widehat{y}_{i_0}^2(k)) + |a_{i_0 j_0}||a_{i_0 i_0}|(-\widehat{y}_{i_0}^2(k) - \widehat{y}_{j_0}^2(k-\tau_{i_0 j_0})$$

$$-2\widehat{y}_{i_0}(k)\widehat{y}_{j_0}(k-\tau_{i_0j_0})) - |a_{i_0j_0}| \sum_{j=1,\, j\neq i_0}^{N}$$

$$w_{i_0j}(\widehat{y}_j^2(k) + \widehat{y}_{i_0}^2(k)) + |a_{i_0j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0l}|(\widehat{y}_{j_0}^2(k) + \widehat{y}_l^2(k))$$

$$+|a_{i_0j_0}| \sum_{j=1,\, j\neq i_0}^{N} w_{i_0j} \cdot (\widehat{y}_{i_0}(k)$$

$$-\widehat{y}_j(k-\tau_{i_0j}))^2 - |a_{i_0j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0l}|(\widehat{y}_{j_0}(k-\tau_{i_0j_0}) - \widehat{y}_l(k-\tau_{i_0l}))^2]$$

$$= \xi_{i_0}[-|a_{i_0j_0}|\widehat{y}_{i_0}^2(k) + |a_{i_0j_0}|\widehat{y}_{j_0}^2(k) - |a_{i_0j_0}||a_{i_0i_0}|(\widehat{y}_{i_0}(k)$$

$$+\widehat{y}_{j_0}(k-\tau_{i_0j_0}))^2 + |a_{i_0j_0}|$$

$$\cdot \sum_{j=1,\, j\neq i_0}^{N} w_{i_0j}(\widehat{y}_{i_0}(k) - \widehat{y}_j(k-\tau_{i_0j}))^2 - |a_{i_0j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0l}|(\widehat{y}_{j_0}(k-\tau_{i_0j_0})$$

$$-\widehat{y}_l(k-\tau_{i_0l}))^2], \tag{6.46}$$

and

$$\Delta_3 = \sum_{i=1}^{N} \xi_i[2w_{ii}\widehat{y}_i(k)E_i(k) + 2\sum_{j=1,\, j\neq i}^{N} w_{ij}\widehat{y}_j(k-\tau_{ij})E_i(k)] - 2\sum_{i=1}^{N} \xi_i\widehat{y}_i(k)$$

$$E_i(k) - 2|a_{i_0j_0}|(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0j_0}))E_{i_0}(k)$$

$$= 2\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} w_{ij}E_i(k)(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) - 2\xi_{i_0}|a_{i_0j_0}|E_{i_0}(k)(\widehat{y}_{i_0}(k)$$

$$+\widehat{y}_{j_0}(k-\tau_{i_0j_0}))$$

$$\leq \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} w_{ij}[\frac{1}{\alpha_i}E_i^2(k) + \alpha_i(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k))^2] + \xi_{i_0}|a_{i_0j_0}|$$

$$[\frac{1}{\alpha_{i_0}}E_{i_0}^2(k) + \alpha_{i_0}(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0j_0}))^2]. \tag{6.47}$$

Substituting (6.45), (6.46), and (6.47) into (6.42), we can obtain that

$$
\begin{aligned}
\Delta V(k) &= \Delta_1 + \Delta_2 + \Delta_3 \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} w_{ij} \widehat{y}_j^2(k) - \widehat{y}_i^2(k)] + \xi_{i_0}[-|a_{i_0 j_0}| \widehat{y}_{i_0}^2(k) + |a_{i_0 j_0}| \widehat{y}_{j_0}^2(k)] \\
&\quad - \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 \\
&\quad + \sum_{j=1,\, j\neq i}^{N} w_{ij} w_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2] + \xi_{i_0}[-|a_{i_0 j_0}||a_{i_0 i_0}|(\widehat{y}_{i_0}(k) \\
&\quad + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 + |a_{i_0 j_0}| \sum_{j=1,\, j\neq i_0}^{N} w_{i_0 j} (\widehat{y}_{i_0}(k) - \widehat{y}_j(k-\tau_{i_0 j}))^2 \\
&\quad - |a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}| (\widehat{y}_{j_0}(k-\tau_{i_0 j_0}) - \widehat{y}_l(k-\tau_{i_0 l}))^2] \\
&\quad + 2\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} w_{ij} E_i(k)(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\
&\quad - 2\xi_{i_0} |a_{i_0 j_0}| E_{i_0}(k)(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0})) \\
&= \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} |a_{ij}| \widehat{y}_j^2(k) - \sum_{i=1}^{N} \xi_i \widehat{y}_i^2(k) - \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} \cdot \\
&\quad (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j\neq i}^{N} w_{ij} a_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2] \\
&\quad - \xi_{i_0}[|a_{i_0 j_0}||a_{i_0 i_0}|(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 \\
&\quad + |a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}| (\widehat{y}_{j_0}(k-\tau_{i_0 j_0}) - \widehat{y}_l(k-\tau_{i_0 l}))^2] \\
&\quad + 2\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} w_{ij} E_i(k)(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\
&\quad - 2\xi_{i_0} |a_{i_0 j_0}| E_{i_0}(k)(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0})) \\
&\leq -\sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1,\, j\neq i}^{N} w_{ij}(a_{ii}-\alpha_i)(\widehat{y}_i(k)-\widehat{y}_j(k-\tau_{ij}))^2] \\
&-\xi_{i_0}[|a_{i_0 j_0}|(a_{i_0 i_0}-\alpha_{i_0})(\widehat{y}_{i_0}(k)+\widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 \\
&+|a_{i_0 j_0}|\sum_{l\neq i_0,\, l\neq j_0}^{N}|a_{i_0 l}|(\widehat{y}_{j_0}(k-\tau_{i_0 j_0})-\widehat{y}_l(k-\tau_{i_0 l}))^2] \\
&+\xi_{i_0}|a_{i_0 j_0}|\frac{1}{\alpha_{i_0}}E_{i_0}^2(k)+\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N} w_{ij}\frac{1}{\alpha_i}E_i^2(k) \\
=&-\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N} w_{ij}w_{il}(\widehat{y}_j(k-\tau_{ij})-\widehat{y}_l(k-\tau_{il}))^2 \\
&-\xi_{i_0}|a_{i_0 j_0}|\sum_{l\neq i_0,\, l\neq j_0}^{N}|a_{i_0 l}|(\widehat{y}_{j_0}(k-\tau_{i_0 j_0})-\widehat{y}_l(k-\tau_{i_0 l}))^2 \\
&-\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}|a_{ij}|(a_{ii}-\alpha_i)(\widehat{y}_i(k)-b_i sgn(a_{ij})b_j\widehat{y}_j(k-\tau_{ij}))^2 \\
&+\sum_{i=1}^{N}\xi_i(1-a_{ii})\frac{1}{\alpha_i}E_i^2(k),
\end{aligned}
\tag{6.48}
$$

which implies (6.41) holds.

Let $f(\alpha_i)=\frac{\alpha_i(a_{ii}-\alpha_i)}{1-a_{ii}}$. We aim to reduce the number of event-triggering time instants as much as possible when the parameter α_i is chosen. That is to say, the event-triggered condition needs to be more difficult to be satisfied when we select the parameter α_i. To realize this objective, we choose $\alpha_i=\frac{a_{ii}}{2}$ such that $f(\alpha_i)$ can be maximized. Note that the event-triggered condition (6.32) can be rewritten as follows for $i\in\mathcal{N}$:

$$
\begin{aligned}
E_i^2(k)=e_i^2(k)&>\frac{\sigma a_{ii}^2}{4(1-a_{ii})}\sum_{j=1,\, j\neq i}^{N}|a_{ij}|(sgn(a_{ij})\widehat{x}_j(k-\tau_{ij})-\widehat{x}_i(k))^2 \\
&=\frac{\sigma a_{ii}^2}{4(1-a_{ii})}\sum_{j=1,\, j\neq i}^{N}|a_{ij}|(b_i sgn(a_{ij})b_j\widehat{y}_j(k-\tau_{ij})-\widehat{y}_i(k))^2.
\end{aligned}
\tag{6.49}
$$

Under the event-triggered condition (6.49), we have that

$$
\begin{aligned}
& -\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(a_{ii}-\alpha_i)(\widehat{y}_i(k) - b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}))^2 \\
& \quad + \sum_{i=1}^{N} \xi_i (1-a_{ii}) \frac{1}{\alpha_i} E_i^2(k) \\
& \leq -\sum_{i=1}^{N} \xi_i (a_{ii} - \frac{a_{ii}}{2}) \frac{4(1-a_{ii})}{\sigma a_{ii}^2} e_i^2(k) + \sum_{i=1}^{N} \xi_i (1-a_{ii}) \frac{2}{a_{ii}} E_i^2(k) \\
& = -\sum_{i=1}^{N} \xi_i \frac{2(1-a_{ii})}{a_{ii}} (\frac{1}{\sigma} - 1) E_i^2(k).
\end{aligned} \tag{6.50}
$$

Note that $0 < \sigma < 1$. Hence, under the trigger condition (6.49), it holds that for $\forall k \geq 0$,

$$
\begin{aligned}
\Delta V(k) \leq & -\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 \\
& - \xi_{i_0} |a_{i_0 j_0}| \cdot \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}| (\widehat{y}_{j_0}(k-\tau_{i_0 j_0}) - \widehat{y}_l(k-\tau_{i_0 l}))^2 \\
& - \sum_{i=1}^{N} \xi_i \frac{2(1-a_{ii})}{a_{ii}} \cdot (\frac{1}{\sigma} - 1) E_i^2(k) \\
\leq & 0.
\end{aligned} \tag{6.51}
$$

According to LaSalle's invariance principle, all the agents in the network will converge to the maximal positively invariant set of $\Phi = \{\theta \in \Upsilon_{-\tau},\, x(k+\theta) \in X : \Delta V(k) = 0\}$ asymptotically. Note that $\Delta V(k) = 0$ if and only if $e_i(k) = 0$,

$$
\begin{aligned}
& -\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 - \xi_{i_0} |a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} \\
& |a_{i_0 l}| (\widehat{y}_{j_0}(k-\tau_{i_0 j_0}) - \widehat{y}_l(k-\tau_{i_0 l}))^2 = 0,
\end{aligned} \tag{6.52}
$$

and

$$
\sum_{i=1}^{N} \xi_i \frac{a_{ii}}{2} \sum_{j=1,\, j\neq i}^{N} a_{ij} (b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k))^2 = 0. \tag{6.53}
$$

Hence, $\Delta V(k) = 0$ if and only if $e_i(k) = 0$, and

$$\widehat{y}_j(k - \tau_{ij}) = sgn(a_{ij})\widehat{y}_i(k), \ \forall i, \ j \in \mathcal{N}_i, \tag{6.54}$$

or

$$\widehat{y}_j(k - \tau_{ij}) = \widehat{y}_i(k), \ if \ w_{ij} > 0, \tag{6.55}$$

$$\widehat{y}_{j_0}(k - \tau_{ij}) = -\widehat{y}_{i_0}(k). \tag{6.56}$$

Substituting (6.54) into (6.37) yields that

$$y_i(k+1) = y_i(k), \ \forall i \in \mathcal{N}. \tag{6.57}$$

Hence, we have

$$y_i(k) = \widehat{y}_i(k) = \widehat{y}_j(k - \tau_{ij}) = y_j(k - \tau_{ij}) = y_j(k), \ \forall w_{ij} > 0. \tag{6.58}$$

It follows that $\mathcal{G}$ is strongly connected, and it implies

$$y_i(k) = y_j(k), \ k \geq -\tau_{ij}, \ \forall i, \ j \in \mathcal{N}, \tag{6.59}$$

and

$$y_{i_0}(k) = -y_{j_0}(k), \ k \geq -\tau_{ij}. \tag{6.60}$$

Hence, $\Delta V(k) = 0$ if and only if $y_i(k) = 0, \ \forall i \in \mathcal{N}$. By LaSalle's invariance principle, we have

$$\lim_{k \to +\infty} y_1(k) = \lim_{k \to +\infty} y_2(k) = \cdots = \lim_{k \to +\infty} y_N(k) = 0,$$

which implies that

$$\lim_{k \to +\infty} x_1(k) = \lim_{k \to +\infty} x_2(k) = \cdots = \lim_{k \to +\infty} x_N(k) = 0.$$

Remark 6.13 It can be observed that only the received neighboring states are used in the trigger condition (6.30). Hence, the event-based protocol proposed in this section is distributed. Zeno behavior is defined as an infinite number of triggering occurring in a finite-time interval, which should be avoided in the event-based consensus protocol. Nevertheless, the Zeno behavior can always be excluded in discrete-time multi-agent system since the maximum triggering number is the length of the finite-time interval.

6.2.2 *Self-triggered Approach*

In Theorem 6.12, we have proved that the proposed event-based protocol is effective to realize the bipartite consensus of the network model. However, the triggering condition needs to be continuously verified for each agent. In this section, we aim to solve this difficult problem by designing a self-triggered algorithm, i.e., the next update time is precomputed based on predictions using the received data. Under the proposed self-triggered algorithm, the signal remains unchanged until next triggering time of multi-agent networks. The appropriate equation for obtaining the triggering time guarantees desired levels of performance. Hence, self-triggered communication schemes for multi-agent networks can effectively reduce the communication costs.

Different from the event-triggered communication strategy, for self-triggered algorithm, the agent i will predict next triggering time instant t^i_{l+1} according to the information at time t^i_l. Next, we will give an algorithm to determine the time instant t^i_{l+1}.

Denote

$$l(k-\tau_{ij}) = \arg\max_{l\in\mathbb{N}}\{t^j_l \,|\, t^j_l \le k-\tau_{ij}\}. \tag{6.61}$$

Let

$$p_i(k) = \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t^j_{l(k-\tau_{ij})}) - x_i(t^i_l)), \tag{6.62}$$

and

$$q_i(k) = \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{x}_i(t^i_l) - sgn(a_{ij})\widehat{x}_j(t^j_{l(k-\tau_{ij})}))^2. \tag{6.63}$$

For $k \in [t^i_l, t^i_{l+1})$, recall that $e_i^2(k) = (x_i(k) - x_i(t^i_l))^2$. For the positive integer m, we have

$$\begin{aligned} x_i(t^i_l + m) =& x_i(t^i_l + m - 1) + \sum_{j\in\mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t^i_l + m - 1 - \tau_{ij}) \\ & - \widehat{x}_i(t^i_l + m - 1)) \\ =& x_i(t^i_l + m - 2) + \sum_{j\in\mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t^i_l + m - 2 - \tau_{ij}) \\ & - \widehat{x}_i(t^i_l + m - 2)) + \sum_{j\in\mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t^i_l + m - 1 - \tau_{ij}) \end{aligned}$$

$$
\begin{aligned}
&\quad - \widehat{x}_i(t_l^i + m - 1)) \\
&= \cdots\cdots \\
&= x_i(t_l^i) + \sum_{j \in \mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t_l^i - \tau_{ij}) - x_i(t_l^i)) + \cdots + \\
&\quad + \sum_{j \in \mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t_l^i + m - 1 - \tau_{ij}) - \widehat{x}_i(t_l^i + m - 1)).
\end{aligned}
\tag{6.64}
$$

To propose the self-triggered algorithm to find t_{l+1}^i, set $\Sigma = 0$ and $s = t_l^i$. The following two cases are considered:

Case 1: For $k > s$, if agent i does not receive the renewed information from its neighbors, it follows from (6.64) that

$$(x_i(k) - x_i(s))^2 = [\Sigma + p_i(s)(k - s)]^2, \qquad i \in \mathcal{N}. \tag{6.65}$$

Solving the inequality $[\Sigma + p_i(s)(k - s)]^2 - q_i(s) > 0$, we can obtain that the minimum $k = \omega_l^i$ satisfying the above inequality. Hence, according to the event-triggered condition (6.33), the event-triggered time instant is is $t_{l+1}^i = \omega_l^i$ in this case.

Case 2: If agent i firstly receives the renewed information from some of its neighbors at time $k^0 < \omega_l^i$, it follows from (6.64) that

$$(x_i(k^0) - x_i(s))^2 = [p_i(s)(k^0 - s)]^2, \qquad i \in \mathcal{N}. \tag{6.66}$$

Set $\Sigma = 0 + p_i(s)(k^0 - s)$. According to the event-triggered condition (6.33), we should update $s = k^0$ and then go back to Case 1.

Based on the above discussions, an efficient algorithm to find t_{l+1}^i, $\forall i \in \mathcal{N}$ can be summarized as follows.

Algorithm 6.1 Self-triggered algorithm for system (6.27)

Step 1. For each agent $i \in \mathcal{N}$, set $\Sigma = 0$ and $s = t_l^i$.

Step 2. Solving the inequality $[\Sigma + p_i(s)(k - s)]^2 - q_i(s) > 0$, we can obtain the minimum $k = \omega_l^i$ such that the inequality holds.

Step 3. For $k \geq s$, if agent i does not receive the renewed information from its neighbors until $k = \omega_l^i$, then set $t_{l+1}^i = \omega_l^i$ and stop the algorithm.

Step 4. If agent i firstly receives the renewed information from some of its neighbors at time $k^0 < \omega_l^i$, set $\Sigma = \Sigma + p_i(s)(k^0 - s)$. Update $s = k^0$ and go to *Step 2*.

According to the above analysis, the following Theorem 6.14 can be obtained.

Theorem 6.14 *Consider the multi-agent system (6.27) with arbitrary finite communication delay* τ_{ij} *under control law (6.28). If the first triggering time* $t_1^i = 0$, *agent* i, $i \in \mathcal{N}$, *determines the triggering time sequence* $t_l^i|_{l=2}^{\infty}$ *by self-triggered algorithm 6.1. Then, we can obtain the following results:*

(i) System (6.27) can achieve bipartite consensus asymptotically if signed digraph $\mathcal{G}$ *is structurally balanced.*
(ii) If signed digraph $\mathcal{G}$ *is structurally unbalanced, then the system (6.27) can achieve consensus and the final consensus value is 0, i.e.,* $\lim_{k \to +\infty} x_i(k) = 0, \forall i \in \mathcal{N}$.

6.2.3 Numerical Example

Example 6.15 Consider a signed multi-agent network with structurally balanced topology and structurally unbalanced topology, respectively (see Fig. 6.8). Set $\sigma = 0.9$ in event-triggered condition (6.30) and the distinct communication delays are as follows:

$$\Gamma = (\tau_{ij})_{6\times 6} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

According to Theorem 6.12, one can easily conclude that under the proposed event-triggered condition (6.30),

(i) the system with communication delays will achieve bipartite consensus when the network topology is shown in Fig. 6.8a;
(ii) the states of all the agents will converge to zero when the network topology is shown in Fig. 6.8b.

The evolvement of the agents under the event-triggered condition (6.30) is shown in Figs. 6.9 and 6.10, respectively. The numerical results in Fig. 6.9 show that the individual state of the multi-agent system converges to the bipartite constant limit that has the same modulus and different signs. The numerical results in Fig. 6.10 show that the individual state of the multi-agent system converges to zero. Figures 6.9 and 6.10 agree well with the proposed theoretical result.

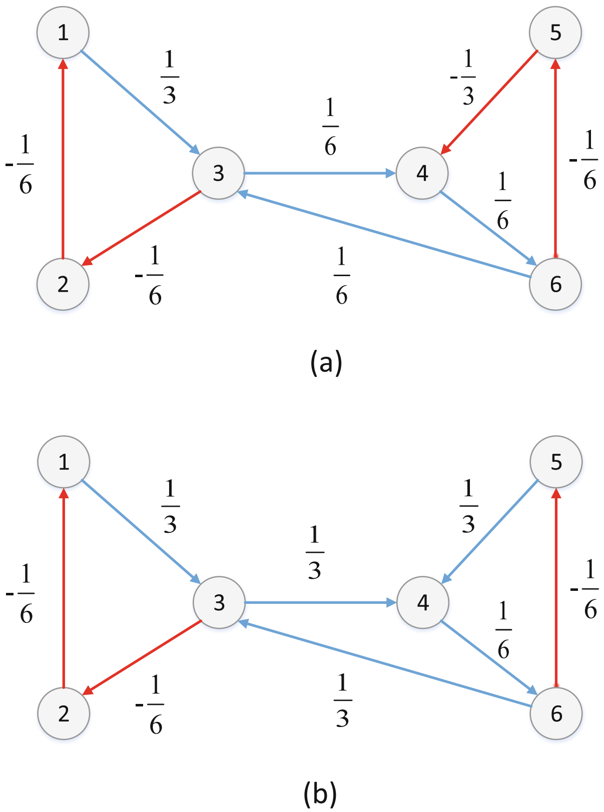

Fig. 6.8 Network topology in Example 6.15. (**a**) structurally balanced. (**b**) Structurally unbalanced

The individual event time instants corresponding to Figs. 6.9 and 6.10 under the proposed event-triggered protocol are shown in Figs. 6.11 and 6.12, respectively. Table 6.1 illustrates the event-triggering frequency under two different network topologies. One can conclude from the simulation example that the event-based strategy in this chapter can significantly decrease the information transmission during the bipartite consensus process of the signed network model with distinct communication delays.

6.3 Summary

In this chapter, the bipartite consensus of continuous-time and discrete-time multi-agent system was studied. For the continuous-time model, according to Perron–Frobenius theorem and some other mathematical analysis, it was found that the bipartite consensus can be asymptotically reached if the strongly connected signed

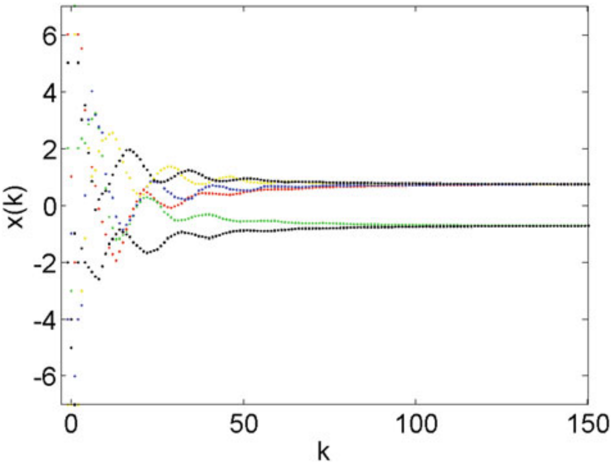

Fig. 6.9 The states of multi-agent system (6.27) associated with signed diagraph with balanced structure in Fig. 6.8a

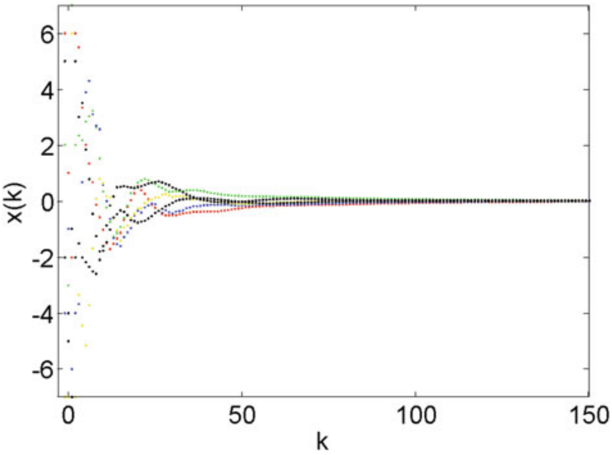

Fig. 6.10 The states of multi-agent system (6.27) associated with signed digraph with unbalanced structure in Fig. 6.8b

digraph $\mathcal{G}$ is structurally balanced. For the discrete-time model, communication delays and event-based strategy were considered simultaneously. It is shown that under the proposed event-triggered condition the bipartite consensus can be asymptotically achieved if the network topology is structurally balanced, and all the agents converge to zero if the signed digraph is structurally unbalanced. Numerical examples were provided to demonstrate the effectiveness of our derived results.

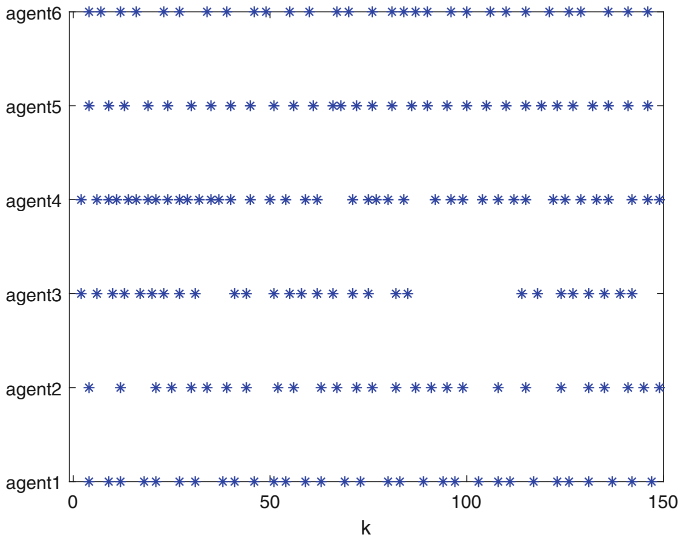

Fig. 6.11 Event-trigger times associated with signed digraph in Fig. 6.8a

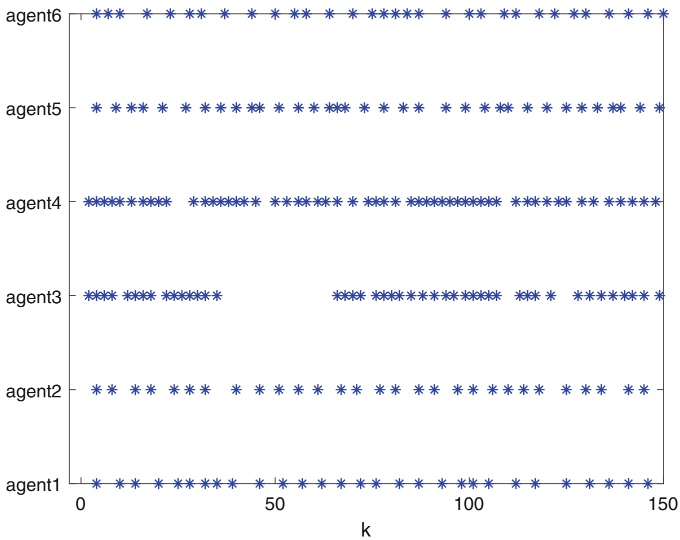

Fig. 6.12 Event-trigger times associated with signed digraph in Fig. 6.8b

Table 6.1 The total number of triggering over the total number of iterations in simulation under structurally balanced topology and structurally unbalanced topology, respectively

Node	1	2	3	4	5	6
Balanced	26.6%	25.3%	21.3%	28.6%	26.0%	26.0%
Unbalanced	27.3%	26.6%	31.3%	34%	26.0%	28.6%

References

1. Olfati-Saber R, Murray RM. Consensus problems in networks of agents with switching topology and time delays. IEEE Trans Autom Control 2004;49(9):1520–1533.
2. Ren W, Beard RW. Consensus seeking in multiagent systems under dynamically changing interaction topologies. IEEE Trans Autom Control 2005;50(5):655–661.
3. Xie GM, Wang L. Consensus control for a class of networks of dynamic agents. Int J Robust Nonlinear Control IFAC-Affiliated J. 2007;17(10-11):941–959.
4. Wu CW, Chua LO. Synchronization in an array of linearly coupled dynamical systems. IEEE Trans Circuits Syst I Fundam Theory Appl. 1995;42(8):430–447.
5. Fax JA, Murray RM. Information flow and cooperative control of vehicle formations. IEEE Trans Autom Control 2004;49(9):1465–1476.
6. Zhang HG, Feng T, Yang GH, Liang HJ. Distributed cooperative optimal control for multiagent systems on directed graphs: an inverse optimal approach. IEEE Trans Cybern. 2015;45(7):1315–1326.
7. Li ZK, Duan ZS, Lewis FL. Distributed robust consensus control of multi-agent systems with heterogeneousn matching uncertainties. Automatica 2014;50(3):883–889.
8. Liu W, Huang J. Cooperative global robust output regulation for nonlinear output feedback multiagent systems under directed switching networks. IEEE Trans Autom Control 2017;62(12):6339–6352.
9. Wasserman S, Faust K. Social network analysis: methods and applications. New York: Cambridge University Press. 1994, p. 8.
10. Altafini C. Consensus problems on networks with antagonistic interactions. IEEE Trans Autom Control 2013;58(4):935–946.
11. Wen GH, Wang H, Yu XH, Yu WW. Bipartite tracking consensus of linear multi-agent systems with a dynamic leader. IEEE Trans Circuits Syst II Exp Briefs 2018;65(9):1204–1208.
12. Xia WG, Cao M, Johansson KH. Structural balance and opinion separation in trust-mistrust social networks. IEEE Trans Control Netw Syst. 2017;3(1):45–56.
13. Meng, ZY, Shi, GD, Johansson KH, Cao M, Hong, YG. Behaviors of networks with antagonistic interactions and switching topologies. Automatica 2016;73:110–116.
14. Meng, DY. Bipartite containment tracking of signed networks. Automatica 2017;79:282–289.
15. Lu JQ, Guo X, Huang TW, Wang ZD. Consensus of signed networked multi-agent systems with nonlinear coupling and communication delays. Appl Math Comput. 2019;350:153–162.
16. Shi GD, Proutiere A, Johansson M, Baras JS, Johansson KH. Emergent behaviors over signed random dynamical networks: atate-flipping model. IEEE Trans Control Netw Syst. 2015;2(2):142–153.
17. Fan MC, Zhang HT, Wang MM. Bipartite flocking for multi-agent systems. Commun Nonlinear Sci Numer Simul. 2014;19(9):3313–3322.
18. Ma CQ, Xie LH. Necessary and sufficient conditions for leader-following bipartite consensus with measurement noise. IEEE Trans Syst Man Cybern Syst. 2020;50(5):1976–1981.
19. Lu JQ, Ho DWC, Kurths J. Consensus over directed static networks with arbitrary finite communication delays. Phys Rev E 2009;80(6):066121.
20. Popov VM, Georgescu R. Hyperstability of control systems. Berlin: Springer;1973.
21. Hale JK, Lunel SMV. Introduction to functional differential equations, vol. 99. New York: Springer;1993.
22. Li LL, Ho DWC , Xu SY. A distributed event-triggered scheme for discrete-time multi-agent consensus with communication delays. IET Control Theory Appl. 2014;8(10):830–837.