Chapter 4
Multi-agent Consensus with Quantization and Communication Delays

Recently, substantial consensus problems have been studied in many previous literature [1–6]. Due to the energy and bandwidth constraints of the communication channels, the transmitted information in the multi-agent network needs to be quantized. The study on control problems using quantized information has a long history [7]. Over the past few years, considerable effort has been devoted to studying the information quantization on feedback control systems [8–11].

How to realize a distributed consensus with quantization has drawn considerable attention [12–18]. In [16], a coding–decoding scheme was developed to solve the average consensus problem with quantized information. In [14, 19], under the condition that each uniform quantizer has infinite quantization levels, it was shown that the multi-agent network could achieve practical consensus.

In this chapter, we will discuss the multi-agent network consensus problem with communication quantization and time delays simultaneously. It is shown that consensus can be achieved for the network under communication quantization and delays under certain topology conditions. Different from Chap. 3, the consensus protocol proposed in this chapter only considers quantized transmitted information. Moreover, the protocol does not assume that the communication delay is the same between different neighboring agents.

4.1 Discrete-Time Case

In this section, the consensus problem of discrete-time multi-agent networks with quantized data and delays is studied. The remainder of this section is organized as follows. In Sect. 4.1.1, the discrete-time multi-agent network model with communication quantization and time delays is presented. In Sect. 4.1.2, the consensus analysis of the proposed protocol is presented in detail. Finally, a

numerical simulation is given to demonstrate the validity of the theoretical results in Sect. 4.1.3.

4.1.1 Model Description

Consider the following network model with discrete-time integrator agents with dynamics:

$$x_i(k+1) = x_i(k) + u_i(k), \quad i \in \mathcal{N}, \tag{4.1}$$

where $x_i(k) \in \mathbb{R}$ is the state of the agent i and $u_i(k)$ is called the protocol.

The goal is to design the protocol $u_i(k)$ yielding the consensus of the states, i.e.,

$$\lim_{k \to +\infty} x_i(k) = c, \quad \forall i \in \mathcal{N}, \tag{4.2}$$

where c is a constant.

Due to the communication bandwidth constraints in many real multi-agent networks, the agents can only use quantized information of the neighboring agents. The following consensus protocol

$$u_i(k) = \sum_{j \in \mathcal{N}_i} a_{ij}[q_\mu(x_j(k - \tau_{ij})) - x_i(k)], \quad i \in \mathcal{N},$$

will be studied in this section, i.e.,

$$x_i(k+1) = x_i(k) + \sum_{j \in \mathcal{N}_i} a_{ij}[q_\mu(x_j(k - \tau_{ij})) - x_i(k)], \quad i \in \mathcal{N}, \tag{4.3}$$

where τ_{ij} is a nonnegative integer representing the communication delays from agent j to agent i, and $q_\mu(\cdot)$ denotes one-parameter family of uniform quantizers which is defined by (1.8), i.e.,

$$q_\mu(x) = \begin{cases} \lfloor \frac{x}{\mu} \rfloor \mu, & x \geq 0, \\ -\lfloor \frac{-x}{\mu} \rfloor \mu, & x < 0. \end{cases} \tag{4.4}$$

In this section, we assume that time delays only exist when the information is transmitted from one agent to another, i.e., $\tau_{ii} = 0, \ i \in \mathcal{N}$. Moreover, the following assumption is proposed in this section.

Assumption 4.1 A is a stochastic matrix such that $a_{ii} > 0, \ i \in \mathcal{N}$, and $\mathcal{G}$ is strongly connected.

4.1.2 Main Results

We introduce the main notations here which will be used in this section. For arbitrary fixed $k_0 \in \mathbb{R}$, denote

- $\tau = \max\{\tau_{ij}, \ i, \ j \in \mathcal{N}\}; \ \Upsilon_{-\tau} = \{-\tau, -\tau+1, \cdots, 0\};$
- $\mathbb{Z}_\mu = \{l\mu, \ l \in \mathbb{Z}\}; \ X = \{\psi : \Upsilon_{-\tau} \longmapsto \mathbb{R}\};$
- $\overline{V}(k) = \max_{\theta \in \Upsilon_{-\tau}} \max_{i \in \mathcal{N}} \{q_\mu(x_i(k+\theta))\}; \ \overline{v}(k) = \min_{\theta \in \Upsilon_{-\tau}} \min_{i \in \mathcal{N}} \{q_\mu(x_i(k+\theta))\};$
- for any $b \in \mathbb{Z}_\mu$, $\Gamma_b(k) = \{i \in \mathcal{N} : \exists \theta \in \Upsilon_{-\tau}, \ q_\mu(x_i(k+\theta)) = b\}.$

For a set B with finite elements, $|B|$ denotes the cardinality of B, i.e., the number of the element in the set B.

In the following, we will study the consensus result of model (4.3). The initial conditions associated with (4.3) are given as $x_i(s) \in X$, $i \in \mathcal{N}$. Before the main theorem of this section be given, we here give two important lemmas first, which will be used in the proof of Theorem 4.4.

Lemma 4.2 *Suppose that $x(t)$ is the solution to (4.3). Under Assumption 4.1, for any finite communication delays τ_{ij}, $\overline{V}(k)$ is a non-increasing function for k, and $\overline{v}(k)$ is a non-decreasing function for k.*

Proof For $\forall i \in \mathcal{N}$, we have

$$
\begin{aligned}
x_i(k+1) &= x_i(k) + \sum_{j \in \mathcal{N}_i} a_{ij}(q_\mu(x_j(k-\tau_{ij})) - x_i(k)) \\
&\leq x_i(k) + \sum_{j \in \mathcal{N}_i} a_{ij}(\overline{V}(k) - x_i(k)) \\
&= \overline{V}(k) + a_{ii}(x_i(k) - \overline{V}(k)) \\
&< \overline{V}(k) + \mu.
\end{aligned}
\tag{4.5}
$$

Note that $q_\mu(x_i(k+1)) \in \mathbb{Z}_\mu$ and $\overline{V}(k) \in \mathbb{Z}_\mu$; then, we can obtain that

$$
q_\mu(x_i(k+1)) \leq \overline{V}(k),
\tag{4.6}
$$

which implies that $\overline{V}(k+1) \leq \overline{V}(k)$. Hence, $\overline{V}(k)$ is a non-increasing function for k. Similarly, it can be proved that $\overline{v}(k)$ is a non-decreasing function for k.

Lemma 4.3 *For arbitrary fixed $k_0 \in \mathbb{R}$, suppose $M = \overline{V}(k_0)$ and $m = \overline{v}(k_0)$. If $M \neq m$, we have the following conclusion:*

(i) If $M > 0$, then $|\Gamma_M(k)|$ is a non-increasing function for k, and $\Gamma_M(k) = \varnothing$ in finite time.

(ii) If $m < 0$, then $|\Gamma_m(k)|$ is a non-increasing function for t, and $\Gamma_m(k) = \varnothing$ in finite time.

Proof We only prove conclusion (i). Conclusion (ii) can be proved similarly, and hence the proof is omitted here. For $M = \overline{V}(k_0) > 0$ and arbitrary $k_1 \geq k_0$, it follows from Lemma 4.2 that

$$q_\mu(x_j(k_1 - \tau_{ij})) \leq M, \ \forall j \in \mathcal{N}_i. \tag{4.7}$$

If $x_i(k_1) < M$, we can deduce that

$$
\begin{aligned}
x_i(k_1 + 1) &= x_i(k_1) + \sum_{j \in \mathcal{N}_i} a_{ij}(q_\mu(x_j(k_1 - \tau_{ij})) - x_i(k_1)) \\
&\leq x_i(k_1) + \sum_{j \in \mathcal{N}_i} a_{ij}(M - x_i(k_1)) \\
&= M + a_{ii}(x_i(k_1) - M) \\
&< M,
\end{aligned}
\tag{4.8}
$$

which implies that

$$q_\mu(x_i(k_1 + 1)) < M. \tag{4.9}$$

The inequalities (4.8) and (4.9) imply that $i \notin \Gamma_M(k_1 + 1)$ if $i \notin \Gamma_M(k_1)$. Hence, $|\Gamma_M(k)|$ is a non-increasing function for $k \geq k_0$.

Next, we shall prove $\Gamma_M(k) = \varnothing$ in finite time, i.e., there exists a $\tilde{k}_0 > k_0$ such that $\Gamma_M(\tilde{k}_0) = \varnothing$.

According to $M \neq m$, there exist $j_1 \in \mathcal{N}$ and $\theta_1 \in \Upsilon_{-\tau}$ such that

$$q_\mu(x_{j_1}(k_0 + \theta_1)) = m < M. \tag{4.10}$$

Equations (4.7)–(4.10) imply that

$$q_\mu(x_{j_1}(k)) < M, \quad \forall k \geq k_0 + \theta_1. \tag{4.11}$$

Hence,

$$j_1 \notin \Gamma_M(k), \ \forall k \geq k_0 + \tau. \tag{4.12}$$

Let $\Lambda_{j_1} = \{l \in \mathcal{N} : j_1 \in \mathcal{N}_l\}$. For any $j_2 \in \Lambda_{j_1}$, we consider the following two cases:

Case 1: $j_2 \notin \Gamma_M(k_0)$.
   Equations (4.8) and (4.9) imply that $j_2 \notin \Gamma_M(k), \ \forall k \geq k_0$.
Case 2: $j_2 \in \Gamma_M(k_0)$.

Claim I There exists a $k_2 > k_0$, such that $j_2 \notin \Gamma_M(k_2)$. Next, we shall prove **Claim I** by using a contradiction approach. If for any $k > k_0$, $j_2 \in \Gamma_M(k)$, we can obtain

that

$$q_\mu(x_{j_2}(k)) = M, \quad \forall k \geq k_0. \tag{4.13}$$

Then, we have

$$
\begin{aligned}
x_{j_2}(k+1) &= x_{j_2}(k) + \sum_{j \in \mathcal{N}_{j_2}} a_{j_2 j}(q_\mu(x_j(k - \tau_{j_2 j})) - x_{j_2}(k)) \\
&\leq x_{j_2}(k) + a_{j_2 j_1}(q_\mu(x_{j_1}(k - \tau_{j_2 j_1})) - x_{j_2}(k)) \\
&\leq x_{j_2}(k) - a_{j_2 j_1}\mu.
\end{aligned}
\tag{4.14}
$$

Hence, $q_\mu(x_{j_2}(k)) \leq x_{j_2}(k) < M$ in finite time, which contradicts with (4.13). Thus, **Claim I** holds, which means that there exists $k_2 > k_0$, such that for any $j_2 \in \Lambda_{j_1}$, it holds $j_2 \notin \Gamma_M(k), \forall k \geq k_2$.

For any $j_2 \in \Lambda_{j_1}$, same procedure applies to the agents set $\Lambda_{j_2} = \{\tilde{l} \in \mathcal{N} : j_2 \in \mathcal{N}_{\tilde{l}}\}$. It can be obtained that there exists $k_3 > k_2$, such that for any $j_3 \in \Lambda_{j_2}$, $j_3 \notin \Gamma_M(k), \forall k \geq k_3$.

Repeat the above procedure. Given that the network is strongly connected, it implies that there exists a $\tilde{k}_0 > k_0$ such that $\Gamma_M(k) = \varnothing, \forall k \geq \tilde{k}_0$. This completes the proof of Lemma 4.3.

Theorem 4.4 *Under Assumption 4.1, for any finite communication delays τ_{ij}, the multi-agent network (4.3) will asymptotically achieve consensus for arbitrary initial conditions. That is,*

$$\lim_{t \to +\infty} x_i(t) = c, \quad \forall i \in \mathcal{N}, \tag{4.15}$$

where c is a constant.

Proof The proof of Theorem 4.4 is divided into two steps.

Step 1 We shall prove that for any fixed $k_0 \in \mathbb{R}$, there exists $\bar{k}_0 \geq k_0$ such that

$$\overline{V}(\bar{k}_0) = \overline{v}(\bar{k}_0). \tag{4.16}$$

The following three cases are considered:

Case 1: $\overline{V}(k_0) \geq 0$ and $\overline{v}(k_0) \geq 0$.

- If $\overline{V}(k_0) = \overline{v}(k_0)$, select $\bar{k}_0 = k_0$.
- If $\overline{V}(k_0) \neq \overline{v}(k_0)$, it follows from Lemma 4.3 that there exists $k_1 > k_0$, such that $\Gamma_{\overline{V}(k_0)}(k_1) = \varnothing$, which implies that $\overline{V}(k_1) < \overline{V}(k_0)$.
- If $\overline{V}(k_1) = \overline{v}(k_1)$, select $\bar{k}_0 = k_1$.
- If $\overline{V}(k_1) \neq \overline{v}(k_1)$, there exists $k_2 \geq k_1$, such that $\overline{v}(k_2) < \overline{V}(k_2)$.

  Repeat the above procedure, we can finally find a $\bar{k}_2 \in \mathbb{R}$, such that $\overline{V}(\bar{k}_2) = \overline{v}(\bar{k}_2)$. Select $\bar{k}_0 = \bar{k}_2$.

Case 2: $\overline{V}(k_0) \leq 0$ and $\overline{v}(k_0) \leq 0$. Similar to the procedure of **Case 1** (replace $\Gamma_{\overline{V}(k_0)}(k)$ by $\Gamma_{\overline{v}(k_0)}(k)$), we can find a $\bar{k}_0 \geq k_0$ such that

$$\overline{V}(\bar{k}_0) = \overline{v}(\bar{k}_0). \tag{4.17}$$

Case 3: $\overline{V}(k_0) > 0$ and $\overline{v}(k_0) < 0$.

According to Lemma 4.3, there exists $k_1 > k_0$ such that

$$\Gamma_{\overline{V}(k_0)}(k_1) = \varnothing, \text{ and } \Gamma_{\overline{v}(k_0)}(k_1) = \varnothing,$$

which implies that $\overline{V}(k_1) < \overline{V}(k_0)$ and $\overline{v}(k_1) > \overline{v}(k_0)$.

- If $\overline{V}(k_1) = \overline{v}(k_1)$, select $\bar{k}_0 = k_1$.
- If $\overline{V}(k_1) \neq \overline{v}(k_1)$, one of the following three subcases holds:

(1) $\overline{V}(k_1) = \overline{v}(k_1) = 0$; (2) $\overline{V}(k_1) > 0$ and $\overline{v}(k_1) \geq 0$; and (3) $\overline{V}(k_1) \leq 0$ and $\overline{v}(k_l) < 0$.

For subcase (1), choose $\bar{k}_0 = k_1$. Subcases (2) and (3) have been reduced to the **Cases 2** and **3**, respectively.

This completes *Step 1* of the proof, i.e., there exists $\bar{k}_0 \geq k_0$ such that

$$\overline{V}(\bar{k}_0) = \overline{v}(\bar{k}_0). \tag{4.18}$$

Step 2 We shall prove that the multi-agent network (4.3) achieves consensus asymptotically.

From (4.18) and Lemma 4.2, it can be obtained that

$$\overline{V}(k) = \overline{v}(k), \ k \geq \bar{k}_0, \tag{4.19}$$

which implies

$$q_\mu(x_i(k+\theta_1)) = q_\mu(x_j(k+\theta_2)), \ \forall i, \ j \in \mathcal{N}, \ \forall \theta_1, \ \theta_2 \in \Upsilon_{-\tau}, \ k \geq \bar{k}_0. \tag{4.20}$$

Let $c = q_\mu(x_i(\bar{k}_0))$. It follows from (4.20) that for any $i \in \mathcal{N}$ and $k \geq \bar{k}_0$, system (4.3) can be written as follows:

$$\begin{aligned} x_i(k+1) &= x_i(k) + \sum_{j \in \mathcal{N}_i} a_{ij}(q_\mu(x_j(k-\tau_{ij})) - x_i(k)) \\ &= x_i(k) + \sum_{j \in \mathcal{N}_i} a_{ij}(c - x_i(k)) \\ &= a_{ii}x_i(k) + (1 - a_{ii})c. \end{aligned} \tag{4.21}$$

From (4.21), we have

$$x_i(k+1) - x_i(k) = a_{ii}(x_i(k) - x_i(k-1)), \ \ k \geq \bar{k}_0 + 1, \tag{4.22}$$

which implies that

$$x_i(k+1) - x_i(k) = a_{ii}^{k-\bar{k}_0}(x_i(\bar{k}_0+1) - x_i(\bar{k}_0)), \quad k \geq \bar{k}_0 + 1. \quad (4.23)$$

Hence, it follows from Assumption 4.1 that

$$\lim_{k \to +\infty}(x_i(k+1) - x_i(k)) = \lim_{k \to +\infty} a_{ii}^{k-\bar{k}_0}(x_i(\bar{k}_0+1) - x_i(\bar{k}_0)) = 0. \quad (4.24)$$

It follows from (4.21) and (4.24) that there exists a constant $c \in \mathbb{R}$ such that

$$\lim_{k \to +\infty} x_i(k) = c, \quad \forall i \in \mathcal{N}. \quad (4.25)$$

4.1.3   Numerical Example

In this section, an example is given to illustrate the correctness of the theoretical results.

Consider network (4.3) with the topology shown in Fig. 4.1. Assume that $\mu = 1$ and $\tau_{ij} = 1$, $\forall i \in \mathcal{N}$, $j \in \mathcal{N}_i$. The initial condition of network (4.3) is randomly chosen from $(-5, 5)$. Suppose the weight of each edge is set as $\frac{1}{4}$. The stochastic matrix A is

$$A = \begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}. \quad (4.26)$$
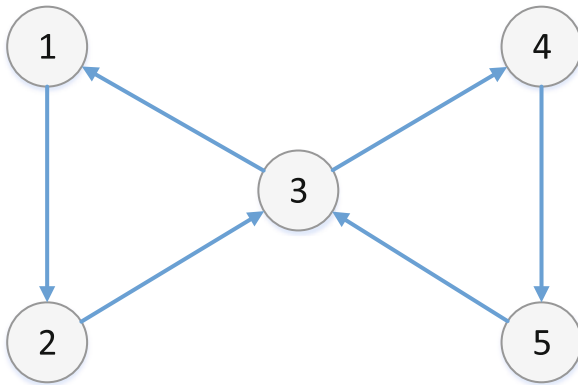

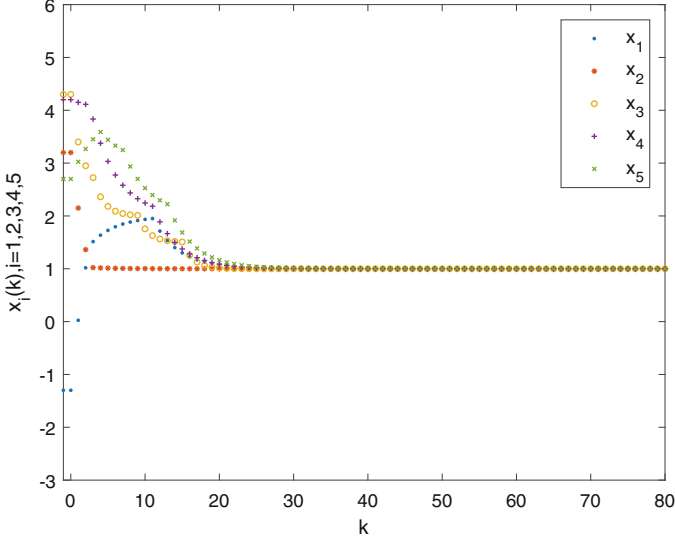**Fig. 4.1** Network topology in example

Fig. 4.2 The state responses of the multi-agent system

The state responses of multi-agent networks (4.3) are shown in Fig. 4.2. It can be observed from Fig. 4.2 that the multi-agent network achieves consensus asymptotically, which illustrates Theorem 4.4 very well.

4.2   Continuous-Time Case

In Sect. 4.1, discrete-time multi-agent network consensus problem with quantization and time delays is studied. In this section, we shall investigate the corresponding continuous-time cases. The organization of the remaining part is given as follows. In Sect. 4.2.1, consensus protocol with quantization and time delays is formulated. In Sect. 4.2.2, the existence of the Filippov solution is presented. In Sect. 4.2.3, the consensus analysis of the proposed protocol is presented in detail. In Sect. 4.2.4, a numerical example is given to show the correctness of the theoretical results.

4.2.1   Model Description and Preliminaries

In Chap. 3, the following multi-agent network model has been investigated:

$$\frac{dx_i(t)}{dt} = \sum_{j \in \mathcal{N}_i} a_{ij}[q_\mu(x_j(t - \tau)) - q_\mu(x_i(t))], \quad i \in \mathcal{N},$$

where τ is the communication delays from agent j to agent i.

In many real multi-agent networks, each agent can obtain its own precise information, which will not be effected by the limited communication bandwidth. Moreover, communication delays may be different between different neighboring agents. Hence, the following consensus protocol will be studied in this section:

$$\frac{dx_i(t)}{dt} = \sum_{j \in \mathcal{N}_i} a_{ij}[q_\mu(x_j(t - \tau_{ij})) - x_i(t)], \quad i \in \mathcal{N}, \tag{4.27}$$

where τ_{ij} is the communication delay from agent j to agent i and $q_\mu(z)$ is defined by (4.4). For $x = (x_1, x_2, \ldots, x_N)^\top \in \mathbb{R}^N$, let $q_\mu(x) = (q_\mu(x_1), q_\mu(x_2), \ldots, q_\mu(x_N))^\top$. The initial conditions associated with (4.27) are given as

$$x_i(s) = \phi_i(s) \in \mathcal{C}([-\tau, 0], \mathbb{R}), \quad i \in \mathcal{N}.$$

Different from the discrete-time cases, system (4.27) may not have global solution in the sense of *Carathéodory* due to the discontinuity of the function $q(\cdot)$. Hence, we need to prove the existence of the global Filippov solution to differential equation (4.27) as in Chap. 1.

4.2.2 The Existence of the Filippov Solution

The concept of the Filippov solution to the differential equation (4.27) is given as follows.

Definition 4.5 A function $x(t) : [-\tau, T) \to \mathbb{R}^N$ (T might be ∞) is a solution in the sense of Filippov for the discontinuous system (4.27) on $[-\tau, T)$, if

1. $x(t)$ is continuous on $[-\tau, T)$ and absolutely continuous on $[0, T)$;
2. $x(t)$ satisfies that

$$\frac{dx_i(t)}{dt} \in \mathcal{K}[\sum_{j \in \mathcal{N}_i} a_{ij}(q_\mu(x_j(t - \tau_{ij})) - x_i(t))], \quad i \in \mathcal{N}. \tag{4.28}$$

It follows from Lemma 1.16 that

$$\mathcal{K}[\sum_{j \in \mathcal{N}_i} a_{ij}(q_\mu(x_j(t - \tau_{ij})) - x_i(t))]$$

$$\subseteq \sum_{j \in \mathcal{N}_i} a_{ij}(\mathcal{K}[q_\mu(x_j(t - \tau_{ij}))] - x_i(t)). \tag{4.29}$$

Similar to Chap. 3, if $x(t)$ is the solution of system (4.27), there exists the output function $\gamma(t) \in \mathcal{K}[q_\mu(x(t))]$ such that for a.e. $t \in [0, T)$, the following equation is true:

$$\frac{dx_i(t)}{dt} = \sum_{j \in \mathcal{N}_i} a_{ij}(\gamma_j(t - \tau_{ij}) - x_i(t)), \quad i \in \mathcal{N}. \tag{4.30}$$

Definition 4.6 For any continuous function $\phi : [-\tau, 0] \to \mathbb{R}^N$ and any measurable selection $\psi : [-\tau, 0] \to \mathbb{R}^N$, such that $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$ for a.e. $s \in [-\tau, 0]$, an absolute continuous function $x(t) = x(t, \phi, \psi)$ is said to be a solution of the Cauchy problem for system (4.27) on $[0, T)$ with initial value (ϕ, ψ), if

$$\begin{cases} \dot{x}_i(t) = \displaystyle\sum_{j=1, \, j \neq i}^{N} a_{ij}(\gamma_j(t - \tau_{ij}) - x_i(t)), \; for \; a.e. \; t \in [0, T), \quad i \in \mathcal{N}, \\[2mm] x(s) = \phi(s), \quad \forall s \in [-\tau, 0], \\[2mm] \gamma(s) = \psi(s) \;\; a.e. \; s \in [-\tau, 0]. \end{cases} \tag{4.31}$$

Next, we shall study the existence of the global solution to the system (4.31).

Lemma 4.7 *Suppose $x(\cdot)$ is a Filippov solution to (4.27). Let $M(t) = max_{i \in \mathcal{N}} max_{\theta \in [-\tau, 0]} \{x_i(t + \theta)\}$ and $m(t) = min_{i \in \mathcal{N}} min_{\theta \in [-\tau, 0]} \{x_i(t + \theta)\}$. Then, we have the following conclusion:*

(i) If $M(t) \geq 0$, then $M(t)$ is a non-increasing function for t.
(ii) If $m(t) \leq 0$, then $m(t)$ is a non-decreasing function for t.

Proof We only prove the conclusion (i). (ii) can be proved similarly. For any fixed $t_0 \geq 0$, suppose $M(t_0) \geq 0$. Next, we will show that $M(t) \leq M(t_0)$ for any $t \geq t_0$ by contradiction.

Suppose there exist $\bar{t}_0$ and t_0 such that

$$M(\bar{t}_0) > M(t_0), \; \bar{t}_0 > t_0 \geq 0. \tag{4.32}$$

Similar to the proof of *Steps 1* and *2* of Theorem 3.12, it can be proved that there exist $i_0 \in \mathcal{N}$, $t_0^* \in [t_0, \bar{t}_0)$, and $\delta > 0$ such that

$$M(t_0) = M(t_0^*) = x_{i_0}(t_0^*), \tag{4.33}$$

and

$$M(t) > M(t_0^*), \; \forall t \in (t_0^*, \bar{t}_0], \tag{4.34}$$

and

$$M(t) = x_{i_0}(t + \theta(t)), \ \theta(t) \in [-\tau, \ 0], \ \forall t \in (t_0^*, \ t_0^* + \delta). \tag{4.35}$$

Let $\delta_1 = \min_{j \in \mathcal{N}_i}\{\tau_{ij}\}$ and $\delta_2 = \min\{\delta, \delta_1\}$. Since

$$x_{i_0}(t + \theta(t)) = M(t) > M(t_0^*) = x_{i_0}(t_0^*), \ \forall t \in (t_0^*, \ t_0^* + \delta_2), \tag{4.36}$$

there exists $t_1 \in (t_0^*, \ t_0^* + \delta_2]$ such that

$$x_{i_0}(t_1) > x_{i_0}(t_0^*). \tag{4.37}$$

Let $t_1^* = \sup\{t \in [t_0^*, \ t_1] : \ x_{i_0}(t) = x_{i_0}(t_0^*)\}$. Due to the continuity of function $x_{i_0}(t)$, we have

$$x_{i_0}(t_1^*) = x_{i_0}(t_0^*). \tag{4.38}$$

Hence, for any $t \in (t_1^*, \ t_1]$, we have

$$x_{i_0}(t) \geq M(t_0^*) \geq \max_{j \in \mathcal{N}_i}\max_{t \in [t_1^*, \ t_1]}\{x_j(t - \tau_{ij})\}$$

$$\geq \max_{j \in \mathcal{N}_i}\max_{t \in [t_1^*, \ t_1]}\{\gamma_j(t - \tau_{ij})\}.$$

It follows from

$$\dot{x}_{i_0}(t) = \sum_{j=1, \ j \neq i_0}^{N} a_{ij}(\gamma_j(t - \tau) - x_{i_0}(t)), \ a.e. \ t \in (t_1^*, \ t_1], \tag{4.39}$$

that $\dot{x}_{i_0}(t) \leq 0$, $a.e. \ t \in (t_1^*, \ t_1]$. However, since $x_{i_0}(t_1) \geq x_{i_0}(t_1^*)$, there must exist a subset $\mathcal{I}_1$ of $(t_1^*, \ t_1]$ such that $\mathcal{I}_1$ has a positive measure and

$$\dot{x}_{i_0}(t) > 0, \ a.e. \ t \in \mathcal{I}_1, \tag{4.40}$$

which is contradictory with $\dot{x}_{i_0}(t) \leq 0$ for a.e. $t \in (t_1^*, \ t_1]$.

Therefore, $M(t)$ is a non-increasing function for t if $M(t) \geq 0$. Similarly, $m(t)$ is a non-decreasing function for t if $m(t) \leq 0$. $\blacksquare$

Theorem 4.8 *For any initial function ϕ and the selection of the output function $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$, there exists a global solution for the system (4.31).*

Proof Similar to the proof of Theorem 3.12, the proof of Theorem 4.8 can also be divided into two parts:

Part (I) Existence of local solution

Similar to the proof of Lemma 1 in [20], one can conclude the existence of the solution defined on $[0, \ T)$ for system (4.31).

Part (II) The boundedness of the solution

Suppose $x(t, \phi, \psi)$ is a solution of system (4.31). Let $M(t) = \max_{i \in \mathcal{N}} \max_{\theta \in [-\tau, 0]} \{x_i(t + \theta)\}$, and $m(t) = \min_{i \in \mathcal{N}} \min_{\theta \in [-\tau, 0]} \{x_i(t + \theta)\}$. It follows from Lemma 4.7 that $M(t) \leq \max\{M(0), 0\}$ and $m(t) \geq \min\{m(0), 0\}$. Hence, the solution $x(t)$ is bounded. According to the theory of functional differential equations [21], a global solution can be guaranteed by the boundedness of the local solution. This completes the proof of this theorem.

4.2.3   Consensus Analysis Under Quantization and Time Delays

In this section, we shall study the consensus result of the multi-agent system (4.27). We assume that the network topology is undirected in this section. The initial conditions associated with (4.27) are given as $x_i(s) = \phi_i(s) \in \mathcal{C}([-\tau, 0], \mathbb{R})$, ($i \in \mathcal{N}$). The Filippov solution of system (4.27) is defined in (4.31), and $\psi_j(s)$, $s \in [-\tau, 0]$ is the initial condition of measurable selection of $\gamma_j(s)$.

Lemma 4.9 *Suppose $x(t)$ is a Filippov solution to (4.27). For any $\epsilon > 0$, let $\Phi = \{x(t + \theta) \in \mathcal{C}([-\tau, 0]; \mathbb{R}^N) : |\gamma_i(t) - \gamma_j(t - \tau_{ij})| < \frac{\epsilon}{2}, |x_i(t) - \gamma_j(t - \tau_{ij})| < \frac{\epsilon}{2}, \forall i \in \mathcal{N}, j \in \mathcal{N}_i\}$. Then, we have the following conclusion:*

(i) There exists T_0, such that for any $i \in \mathcal{N}$,

$$|x_i(t + \vartheta) - x_i(t)| \leq \epsilon, \quad \forall t \geq T_0, \forall \vartheta \in [0, \tau]. \tag{4.41}$$

(ii) For arbitrary fixed $t_0 \geq 0$, there exists $t_1 \geq t_0$ such that the agents in the network will go into the set of Φ at time t_1.

Proof Consider the function

$$V(t) = V_1(t) + V_2(t), \tag{4.42}$$

where

$$V_1(t) = \sum_{i=1}^{N} x_i^2(t) + \sum_{i=1}^{N} \int_0^{x_i(t)} q_\mu(s) ds, \tag{4.43}$$

and

$$V_2(t) = \sum_{i=1}^{N} \sum_{j=1, \, j \neq i}^{N} \int_{t - \tau_{ij}}^{t} a_{ij} \gamma_j^2(s) ds. \tag{4.44}$$

Note that $c q_\mu(c) \geq 0$ for any $c \in \mathbb{R}$, and then we have $V_1(t) \geq 0$ and $V_2(t) \geq 0$.

Notice that for $p_i(s) = \int_0^s q_\mu(u)du$, and we have

$$\partial_c p_i(s) = \{v \in \mathbb{R} : q_\mu^-(s) \le v \le q_\mu^+(s)\}, \tag{4.45}$$

where $q_\mu^+(s)$ and $q_\mu^-(s)$ denote the right and left limits of the function q_μ at the point s. Based on Lemma 1.19, $V_1(t)$ is differentiable for a.e. $t \ge 0$ and

$$
\begin{aligned}
\frac{dV_1(t)}{dt} &= \sum_{i=1}^{N} x_i(t) \sum_{j=1, j \ne i}^{N} a_{ij}[\gamma_j(t - \tau_{ij}) - x_i(t)] + \sum_{i=1}^{N} \gamma_i(t) \sum_{j=1, j \ne i}^{N} a_{ij} \\
&\quad \times [\gamma_j(t - \tau_{ij}) - x_i(t)] \\
&= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} a_{ij}[2x_i(t)\gamma_j(t - \tau_{ij}) - 2x_i^2(t)] + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} a_{ij} \\
&\quad \times [2\gamma_i(t)\gamma_j(t - \tau_{ij}) - 2\gamma_i(t)x_i(t)] \\
&\le \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} a_{ij}[2x_i(t)\gamma_j(t - \tau_{ij}) - 2x_i^2(t)] + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} a_{ij} \\
&\quad \times [2\gamma_i(t)\gamma_j(t - \tau_{ij}) - 2\gamma_i^2(t)].
\end{aligned} \tag{4.46}
$$

Since $\gamma_j(t) \in \mathcal{K}[q_\mu(x_j(t))]$, $\forall j \in \mathcal{N}$, we have that $\gamma_j(t)$ is locally integrable. Hence, $V_2(t)$ is differentiable for a.e. $t \ge 0$ and

$$
\begin{aligned}
\frac{dV_2(t)}{dt} &= \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} a_{ij}[\gamma_j^2(t) - \gamma_j^2(t - \tau_{ij})] \\
&= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} a_{ij}[2\gamma_i^2(t) - 2\gamma_j^2(t - \tau_{ij})] \\
&\le \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} a_{ij}[\gamma_i^2(t) - \gamma_j^2(t - \tau_{ij})] + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} a_{ij}[x_i^2(t) \\
&\quad - \gamma_j^2(t - \tau_{ij})].
\end{aligned} \tag{4.47}
$$

Combining (4.46) and (4.47) gives that

$$
\begin{aligned}
\frac{dV(t)}{dt} &= \frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} \\
&\le \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} a_{ij}[2\gamma_i(t)\gamma_j(t - \tau_{ij}) - \gamma_i^2(t) - \gamma_j^2(t - \tau_{ij})]
\end{aligned}
$$

$$+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N}a_{ij}[2x_i(t)\gamma_j(t-\tau_{ij})-x_i^2(t)-\gamma_j^2(t-\tau_{ij})]$$

$$=-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N}a_{ij}(\gamma_i(t)-\gamma_j(t-\tau_{ij}))^2$$

$$-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N}a_{ij}(x_i(t)-\gamma_j(t-\tau_{ij}))^2$$

$$\leq 0. \tag{4.48}$$

Hence, $V(t)$ is non-increasing for t. Together with $V(t) \geq 0$, it gives that $\lim_{t\to+\infty} V(t)$ exists. Let $\bar{a} = \max_{1\leq i<j\leq N,\,a_{ij}>0}\{a_{ij}\}$. Then, for any $\epsilon > 0$ and $i, j \in \mathcal{N}$, there exists T_0 such that $\forall t \geq T_0$, $\vartheta \in [0, \tau]$,

$$\frac{\epsilon^2}{2N\bar{a}\tau} \geq |V(t+\vartheta)-V(t)|$$

$$= |\int_t^{t+\vartheta} \dot{V}(s)ds|$$

$$\geq \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N}a_{ij}\int_t^{t+\vartheta}(x_i(s)-\gamma_j(s-\tau_{ij}))^2ds$$

$$+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N}a_{ij}\int_t^{t+\vartheta}(\gamma_i(s)-\gamma_j(s-\tau_{ij}))^2ds. \tag{4.49}$$

Hence, for any $i \in \mathcal{N}$ and $t \geq T_0$, $\vartheta \in [0, \tau]$,

$$\sum_{j=1,\,j\neq i}^{N}a_{ij}\int_t^{t+\vartheta}(x_i(s)-\gamma_j(s-\tau_{ij}))^2ds \leq \frac{\epsilon^2}{N\bar{a}\tau}. \tag{4.50}$$

It follows from Lemma 1.20 that

$$\left|\sum_{j=1,\,j\neq i}^{N}a_{ij}\int_t^{t+\vartheta}(\gamma_j(s-\tau_{ij})-x_i(s))ds\right|^2$$

$$\leq \left(\sum_{j=1,\,j\neq i}^{N}a_{ij}\int_t^{t+\vartheta}|\gamma_j(s-\tau_{ij})-x_i(s)|ds\right)^2$$

$$\leq N \sum_{j=1,\, j\neq i}^{N} a_{ij}^2 \left(\int_t^{t+\vartheta} |\gamma_j(s-\tau_{ij}) - x_i(s)| ds \right)^2$$

$$\leq \tau N \sum_{j=1,\, j\neq i}^{N} a_{ij}\bar{a} \int_t^{t+\vartheta} (x_i(s) - \gamma_j(s-\tau_{ij}))^2 ds$$

$$\leq \epsilon^2. \tag{4.51}$$

Hence, for any $i \in \mathcal{N}$ and $t \geq T_0$, $\vartheta \in [0, \tau]$,

$$\left| \sum_{j=1,\, j\neq i}^{N} a_{ij} \int_t^{t+\vartheta} (\gamma_j(s-\tau) - x_i(s)) ds \right| \leq \epsilon.$$

It follows from (4.31) that

$$|x_i(t+\vartheta) - x_i(t)| = \left| \int_t^{t+\vartheta} \dot{x}_i(s) ds \right|$$

$$= \left| \sum_{j=1,\, j\neq i}^{N} a_{ij} \int_t^{t+\vartheta} (\gamma_j(s-\tau) - x_i(s)) ds \right|$$

$$\leq \epsilon. \tag{4.52}$$

Thus, for any $\epsilon > 0$ and $i \in \mathcal{N}$, there exists T_0 such that for $\forall t \geq T_0$, $\vartheta \in [0, \tau]$,

$$|x_i(t+\vartheta) - x_i(t)| \leq \epsilon. \tag{4.53}$$

Next, we will prove conclusion (ii). Let $J = \{t \geq t_0 : x(t+\theta) \notin \Phi\}$. For $x(t+\theta) \in \mathcal{C}([-\tau, 0]; \mathbb{R}^N)$ and $t \in J$, there exist i, $j \in \mathcal{N}$, $i \neq j$ and $a_{ij} \neq 0$ such that

$$|\gamma_i(t) - \gamma_j(t-\tau_{ij})| \geq \frac{\epsilon}{2}, \tag{4.54}$$

or

$$|x_i(t) - \gamma_j(t-\tau_{ij})| \geq \frac{\epsilon}{2}. \tag{4.55}$$

Hence, for a.e. $t \in J$,

$$\dot{V}(t) \leq -\frac{1}{8} \sum_{i=1}^{N} \sum_{j=1, \, j \neq i}^{N} a_{ij} \epsilon^2$$

$$\leq -\frac{1}{8} \varsigma \epsilon^2, \tag{4.56}$$

where $\varsigma = \sum_{i=1}^{N} \sum_{j=1, \, j \neq i}^{N} a_{ij}$. Next, we will prove the claim by contradiction.

Suppose that $t \in J$ for any $t \geq t_0$. Then, inequality (4.56) implies that

$$V(t) - V(t_0) \leq -\frac{1}{8} \varsigma \epsilon^2 (t - t_0), \; t \geq t_0. \tag{4.57}$$

For $t > \frac{8V(t_0)}{\varsigma \epsilon^2} + t_0$, it follows from inequality (4.57) that $V(t) < 0$, which is a contradiction to the definition of $V(t)$. Therefore, for arbitrary $t_0 \geq 0$, there exists $\bar{t}_0 \geq t_0$ such that the agents in the network will go into the set of Φ at time $\bar{t}_0$.

This completes the proof of this lemma.

Theorem 4.10 *Consider the multi-agent network (4.27) with communication topology that is defined by an undirected, connected graph G. Then, for any finite communication delay τ_{ij}, the multi-agent network will achieve consensus, i.e., there exists a constant c such that*

$$\lim_{t \to +\infty} x_i(t) = c. \tag{4.58}$$

Proof **Step 1** *We shall show some inequality to be used at later steps.*

For arbitrary $\epsilon > 0$ (without loss of generality, assume $\epsilon < \frac{\mu}{N}$), it follows from Lemma 4.9 that there exists $T_0 > 0$ such that for any $i \in \mathcal{N}$,

$$|x_i(t + \vartheta) - x_i(t)| \leq \epsilon, \quad \forall t \geq T_0, \forall \vartheta \in [0, \tau]. \tag{4.59}$$

Moreover, there exists $T_1 \geq T_0$ such that for any $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$,

$$|\gamma_i(T_1) - \gamma_j(T_1 - \tau_{ij})| < \frac{\epsilon}{4}, \tag{4.60}$$

and

$$|x_i(T_1) - \gamma_j(T_1 - \tau_{ij})| < \frac{\epsilon}{4}. \tag{4.61}$$

It follows from (4.60) and (4.61) that

$$|x_i(T_1) - \gamma_i(T_1)| \leq |\gamma_i(T_1) - \gamma_j(T_1 - \tau_{ij})| + |x_i(T_1) - \gamma_j(T_1 - \tau_{ij})|$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}. \tag{4.62}$$

Step 2 *Fix $x_i(T_1)$, and without loss of generality, we assume $x_i(T_1) \geq 0$ (the proof is similar for $x_i(T_1) < 0$). We shall prove that $|x_j(T_1) - x_i(T_1)| \leq \epsilon$, $j \in \mathcal{N}_i$, by considering the following two cases:*

Case 1: $x_i(T_1) \neq k_0\mu$, $\forall k_0 \in \mathbb{Z}$. Then,

$$\gamma_i(T_1) = q_\mu(x_i(T_1)) = \lfloor \frac{x_i(t)}{\mu} \rfloor \mu. \tag{4.63}$$

If $\gamma_i(T_1) = 0$, it is easy to see from (4.59) and (4.62) that $\gamma_i(T_1 - \tau_{ji}) = \gamma_i(T_1) = 0$. Hence, it can be obtained that

$$|x_j(T_1) - x_i(T_1)| \leq |x_j(T_1) - \gamma_i(T_1 - \tau_{ji})| + |\gamma_i(T_1 - \tau_{ji}) - \gamma_i(T_1)|$$

$$+ |\gamma_i(T_1) - x_i(T_1)|$$

$$\leq \frac{\epsilon}{4} + 0 + \frac{\epsilon}{2}$$

$$< \epsilon. \tag{4.64}$$

If $\gamma_i(T_1) \neq 0$, we claim that for any $j \in \mathcal{N}_i$,

$$x_j(T_1) \geq \lfloor \frac{x_i(T_1)}{\mu} \rfloor \mu. \tag{4.65}$$

Otherwise, we have $x_j(T_1) < \lfloor \frac{x_i(T_1)}{\mu} \rfloor \mu$, which implies that

$$\gamma_j(T_1) \leq (\lfloor \frac{x_i(T_1)}{\mu} \rfloor - 1)\mu. \tag{4.66}$$

It follows from (4.59) that

$$x_j(T_1 - \tau_{ij}) \leq x_j(T_1) + \epsilon$$

$$\leq (\lfloor \frac{x_i(T_1)}{\mu} \rfloor - 1)\mu + 2\epsilon$$

$$< \lfloor \frac{x_i(T_1)}{\mu} \rfloor \mu, \tag{4.67}$$

which implies $\gamma_j(T_1 - \tau_{ij}) \leq (\lfloor \frac{x_i(t)}{\mu} \rfloor - 1)\mu$. Hence, it can be obtained that

$$
\begin{aligned}
|x_i(T_1) - \gamma_j(T_1 - \tau_{ij})| &\geq |\gamma_i(T_1) - \gamma_j(T_1 - \tau_{ij})| - |x_i(T_1) - \gamma_i(T_1)| \\
&\geq \mu - \frac{\epsilon}{2} \\
&> \epsilon,
\end{aligned} \tag{4.68}
$$

which contradicts with (4.60). Hence,

$$
x_j(T_1) \geq \lfloor \frac{x_i(t)}{\mu} \rfloor \mu. \tag{4.69}
$$

The inequality $|x_j(T_1) - \gamma_i(T_1 - \tau_{ji})| \leq \frac{\epsilon}{4}$ implies that

$$
\gamma_i(T_1 - \tau_{ji}) \geq \lfloor \frac{x_i(t)}{\mu} \rfloor \mu - \frac{\epsilon}{4}. \tag{4.70}
$$

It follows from (4.59) and (4.63) that

$$
\gamma_i(T_1 - \tau_{ji}) \leq \lfloor \frac{x_i(t)}{\mu} \rfloor \mu. \tag{4.71}
$$

Hence, we have

$$
\begin{aligned}
|x_j(T_1) - x_i(T_1)| &\leq |x_j(T_1) - \gamma_i(T_1 - \tau_{ji})| + |\gamma_i(T_1 - \tau_{ji}) - \gamma_i(T_1)| \\
&\quad + |\gamma_i(T_1) - x_i(T_1)| \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.
\end{aligned} \tag{4.72}
$$

Case 2:   There exists a $\bar{k}_0 \in \mathbb{Z}$ such that $x_i(T_1) = \bar{k}_0 \mu$.
If $\bar{k}_0 = 0$, it follows from $|x_i(T_1) - x_i(T_1 - \tau_{ji})| < \epsilon$ that

$$
\gamma_i(T_1 - \tau_{ji}) = \gamma_i(T_1) = 0. \tag{4.73}
$$

Hence, we have

$$
\begin{aligned}
|x_j(T_1) - x_i(T_1)| &\leq |x_j(T_1) - \gamma_i(T_1 - \tau_{ji})| + |\gamma_i(T_1 - \tau_{ji}) - \gamma_i(T_1)| \\
&\quad + |\gamma_i(T_1) - x_i(T_1)| \\
&\leq \frac{\epsilon}{4} + 0 + \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned} \tag{4.74}
$$

If $\bar{k}_0 \geq 1$, we claim that $x_j(T_1) \geq x_i(T_1)$. Otherwise, we have $\gamma_j(T_1) \leq (\bar{k}_0 - 1)\mu$. It follows from (4.59) and (4.62) that

$$
\begin{aligned}
x_j(T_1 - \tau_{ji}) &\leq x_j(T_1) + \epsilon \\
&\leq \gamma_j(T_1) + \epsilon + \epsilon \\
&\leq (\bar{k}_0 - 1)\mu + 2\epsilon,
\end{aligned}
\tag{4.75}
$$

which implies that

$$
\gamma_j(T_1 - \tau_{ji}) \leq (\bar{k}_0 - 1)\mu.
\tag{4.76}
$$

However, we can obtain from (4.61) that

$$
\gamma_j(T_1 - \tau_{ji}) \geq x_i(T_1) - \frac{\epsilon}{4} \geq \bar{k}_0\mu - \frac{\epsilon}{4},
\tag{4.77}
$$

which contradicts with (4.76). Hence, $x_j(T_1) \geq x_i(T_1)$.
If $x_j(T_1) = x_i(T_1) = \bar{k}_0\mu$, then

$$
|x_j(T_1) - x_i(T_1)| = 0 < \epsilon.
\tag{4.78}
$$

If $x_j(T_1) > x_i(T_1) = \bar{k}_0\mu$, it can be easily obtained that $\gamma_j(T_1) = \bar{k}_0\mu$. Then,

$$
|x_j(T_1) - x_i(T_1)| = |x_j(T_1) - \gamma_j(T_1)| \leq \epsilon.
\tag{4.79}
$$

In conclusion, we proved that for any $\epsilon > 0$, there exists $T_1 > 0$ such that for any fixed $x_i(T_1)$,

$$
|x_j(T_1) - x_i(T_1)| \leq \epsilon, \ j \in \mathcal{N}_i.
\tag{4.80}
$$

Step 3 *We shall show that the multi-agent network (4.27) can achieve consensus.*
  Since the network is connected, we can obtain from (4.80) that for any fixed $x_i(T_1)$ and $\forall j \in \mathcal{N}$,

$$
|x_j(T_1) - x_i(T_1)| \leq (N-1)\epsilon.
\tag{4.81}
$$

Denote $M(t) = \max\limits_{i \in \mathcal{N}} \max\limits_{\theta \in [-\tau, 0]} \{x_i(t + \theta)\}$ and $m(t) = \min\limits_{i \in \mathcal{N}} \min\limits_{\theta \in [-\tau, 0]} \{x_i(t + \theta)\}$. It follows from (4.59) and (4.81) that

$$
|M(T_1) - m(T_1)| \leq (N+1)\epsilon.
\tag{4.82}
$$

From (4.82), we can assume that $m(T_1)$ and $M(T_1)$ belong to the set of $\Omega_1 = ((k_0 - 1)\mu, \ (k_0 + 1)\mu)$, $k_0 = \lfloor \frac{m(T_1)}{\mu} \rfloor$. Without loss of generality, we assume $k_0 \geq 0$ (the proof for $k_0 < 0$ is similar to the case $k_0 > 0$, which we omitted here). Next, we will

prove that multi-agent networks achieve consensus by considering the following four cases.

Case 1: $\mu > M(T_1) \geq m(T_1) > -\mu$, i.e., $k_0 = 0$. From the definition of the quantizer function $q_\mu(\cdot)$ and output function $\gamma(\cdot)$, we have that

$$\gamma_i(T_1 + \theta_1) = \gamma_j(T_1 + \theta_2) = 0, \ \forall i, \ j \in \mathcal{N}, \ \forall \theta_1, \ \theta_2 \in [-\tau, \ 0]. \tag{4.83}$$

It follows from Lemma 4.9 that

$$\gamma_i(t) = 0, \ \forall i \in \mathcal{N}, \ t \geq T_1. \tag{4.84}$$

Hence, for $t \geq T_1$, the multi-agent network model is reduced to be

$$\frac{dx_i(t)}{dt} = -(1 - a_{ii})x_i(t), \quad i \in \mathcal{N}. \tag{4.85}$$

From (4.85), it is easy to find that all the agents will achieve consensus and the final consensus value is 0.

Case 2: $k_0 \geq 1$ and $M(T_1) \geq k_0\mu \geq m(T_1)$. From the definition of the quantizer function $q_\mu(\cdot)$ and output function $\gamma(\cdot)$, we can obtain that

$$\gamma_i(T_1 + \theta_1) \in [(k_0 - 1)\mu, \ k_0\mu], \ \forall i \in \mathcal{N}, \ \forall \theta_1 \in [-\tau, \ 0]. \tag{4.86}$$

It follows from Lemma 4.9 that

$$\gamma_i(t) \in [(k_0 - 1)\mu, \ k_0\mu], \ \forall i \in \mathcal{N}, \ \forall t \geq T_1. \tag{4.87}$$

If there exists $t_1 \geq T_1$ such that $x_i(t_1) < k_0\mu$, (4.27) implies that

$$x_i(t) < k_0\mu, \ \forall i \in \mathcal{N}, \ \forall t \geq t_1. \tag{4.88}$$

If $m(T_1) < k_0\mu$, clearly, there exist $i_m \in \mathcal{N}$ and $\theta_m \in [-\tau, \ 0]$ such that

$$m(T_1) = x_{i_m}(T_1 + \theta_m) \quad \text{and} \quad \gamma_{i_m}(T_1 + \theta_m) = (k_0 - 1)\mu. \tag{4.89}$$

For $t \geq T_1$, we obtain that

$$(k_0 - 1)\mu < x_{i_m}(t) < k_0\mu \quad \text{and} \quad \gamma_{i_m}(t) = (k_0 - 1)\mu. \tag{4.90}$$

For any j such that $i_m \in \mathcal{N}_j$, we have

$$\frac{dx_j(t)}{dt} = \sum_{l \in \mathcal{N}_j} a_{jl}(\gamma_l(t - \tau_{jl}) - x_j(t)). \tag{4.91}$$

Hence, if $x_j(t_1) \geq k_0\mu$, there exists $t_2 \geq t_1$ such that $x_j(t_2) < k_0\mu$. Since the network is connected, it follows from (4.88) and (4.91) that there exists $T_2 \geq t_2$ such that

$$(k_0 - 1)\mu < x_i(t) < k_0\mu, \ \forall i \in \mathcal{N}, \ \forall t \geq T_2, \qquad (4.92)$$

and Eq. (4.30) is reduced to be

$$\frac{dx_i(t)}{dt} = -(1 - a_{ii})(x_i(t) - (k_0 - 1)\mu), \ \ i \in \mathcal{N}, \ t \geq T_2 + \tau. \quad (4.93)$$

Hence, the multi-agent network will achieve consensus, and the final consensus value is $(k_0 - 1)\mu$.

If $m(T_1) = k_0\mu$, by similar analyses, we can also obtain that the multi-agent network will achieve consensus and the final consensus value is $(k_0 - 1)\mu$ or $k_0\mu$.

Case 3: $M(T_1) \geq m(T_1) > k_0\mu$. In this case, we have

$$\gamma_i(T_1 + \theta_1) = k_0\mu, \ \forall i \in \mathcal{N}, \ \forall \theta_1 \in [-\tau, 0].$$

It follows from (4.30) and Lemma 4.9 that

$$(k_0 + 1)\mu > x_i(t) > k_0\mu \quad \text{and} \quad \gamma_i(t) = k_0\mu, \ \forall i \in \mathcal{N}, \quad \forall t \geq T_1.$$

Then, the system (4.30) is reduced to be

$$\frac{dx_i(t)}{dt} = -(1 - a_{ii})(x_i(t) - k_0\mu), \quad i \in \mathcal{N}. \qquad (4.94)$$

It is easy to see that all the agents will achieve consensus and the final consensus value is $k_0\mu$.

Case 4: $k_0\mu > M(T_1) \geq m(T_1)$.

The analysis of this case is similar to Case 3, which is omitted here. In this case, the multi-agent network will achieve consensus and the final consensus value is $(k_0 - 1)\mu$.

In conclusion, the multi-agent network (4.27) achieves consensus asymptotically. This completes the proof of this theorem.

Remark 4.11 Different from Chap. 3, we have shown that the multi-agent network can achieve complete consensus other than practical consensus in this section. However, it is difficult to estimate the final consensus state c of model (4.27). It is an interesting problem to estimate how the final consensus value depends on the quantization and time delays in our future work.
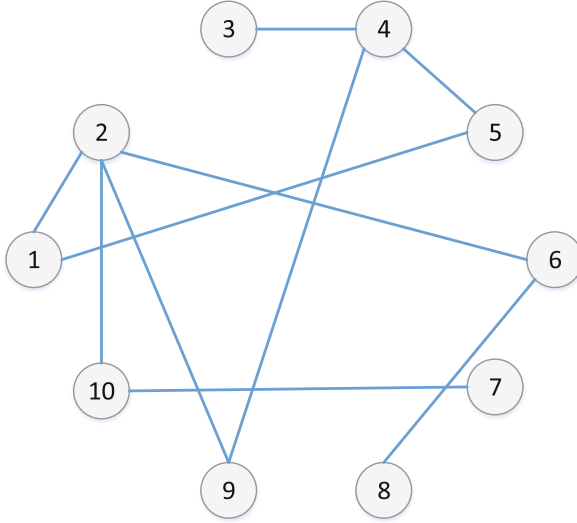
Fig. 4.3 Network topology in the simulation example

4.2.4 Numerical Example

Consider the multi-agent system (4.27) with communication quantization and time delays, where the network structure is shown in Fig. 4.3 with the weights on the connections. The graph (Fig. 4.3) is generated by the scale-free algorithm. Suppose that initial conditions are randomly chosen from (0, 10).

Figure 4.4 shows the state responses of multi-agent network (4.27) with respect to $\mu = 1$. It can be observed that the agents converge to a constant value, which illustrates Theorem 4.10 very well.

4.3 Summary

In this chapter, we mainly addressed the consensus problem of multi-agent networks where each agent can only obtain the quantized and delayed measurements of the states of its neighbors. Discrete-time formulation of the problem was studied first. We showed that the multi-agent network can achieve consensus for arbitrary finite communication delays. For the continuous-time cases, it was shown that Filippov solutions of the resulting system exist for any initial condition. We have proved that for the multi-agent network model considering quantization and time delays simultaneously, Filippov solutions of the resulting system converged to a constant value asymptotically under certain network topology. The theoretical results have been well illustrated by numerical examples.
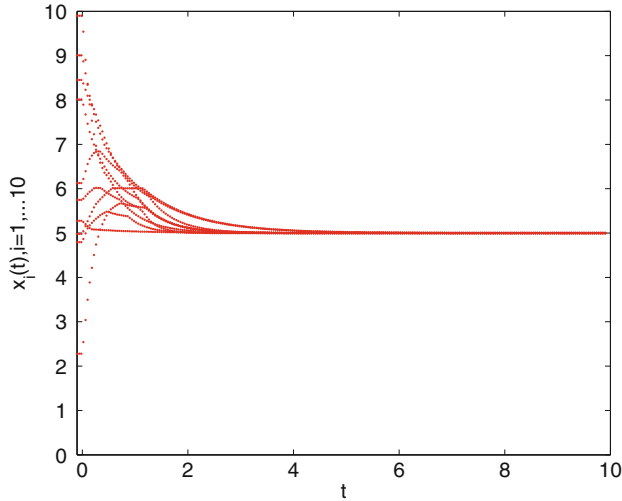
Fig. 4.4 The states of the system in example

References

1. Xie GM, Wang L. Consensus control for a class of networks of dynamic agents. Int J Robust Nonlinear Control IFAC-Affiliated J. 2007;17(10–11):941–959.
2. Chen C, Chen G, Guo L. Consensus of flocks under M-nearest-neighbor rules. J Syst Sci Complex. 2015;28(1):1–15.
3. Zhu JD, Tian YP, Kuang J. On the general consensus protocol of multi-agent systems with double-integrator dynamics. Linear Algebra Appl. 2009;431(5–7):701–715.
4. Chen Y, Lü JH, Yu XH, Lin ZL. Consensus of discrete-time second-order multi-agent systems based on infinite products of general stochastic matrices. SIAM J Control Optim. 2013;51(4):3274–3301.
5. Wang XL, Su HS, Wang XF, Chen GR. Nonnegative edge quasi-consensus of networked dynamical systems. IEEE Trans Circuits Syst II Exp Briefs 2017;64(3):304–308.
6. Zheng YS, Wang L. Consensus of switched multiagent systems. IEEE Trans Circuits Syst II Exp Briefs 2016;63(3):314–318.
7. Kalman RE. Nonlinear aspects of sampled-data control systems. In Proceedings of the symposium on nonlinear circuit theory;1956.
8. Liberzon D. Hybrid feedback stabilization of systems with quantized signals. Automatica 2003;39(9):1543–1554.
9. Liberzon D. Quantization, time delays, and nonlinear stabilization. IEEE Trans Autom Control 2006;51(7):1190–1195.
10. Gao HJ, Chen TW. A new approach to quantized feedback control systems. Automatica 2008;44(2):534–542.
11. Shen B, Wang ZD, Shu HS, Wei GL. Robust h_∞ finite-horizon filtering with randomly occurred nonlinearities and quantization effects. Automatica 2010;46(11):1743–1751.
12. Kashyap A, Basar T, Srikant R. Quantized consensus. Automatica 2007;43(7):1192–1203.
13. Aysal TC, Coates M, Rabbat M. Distributed average consensus with dithered quantization. IEEE Trans Signal Process. 2008;56(10):4905–4918.
14. Frasca P, Carli R, Fagnani F, Zampieri S. Average consensus on networks with quantized communication. Int J Robust Nonlinear Control 2009;19(16):1787–1816.

15. Kar S, Moura JMF. Distributed consensus algorithms in sensor networks: quantized data and random link failures. IEEE Trans Signal Process. 2010;58(3):1383–1400.
16. Li T, Fu MY, Xie LH, Zhang JF. Distributed consensus with limited communication data rate. IEEE Trans Autom Control 2011;56(2):279–292.
17. Nedic A, Olshevsky A, Ozdaglar A, Tsitsiklis JN. On distributed averaging algorithms and quantization effects. IEEE Trans Autom Control 2009;54(11):2506–2517.
18. Wen GH, Duan ZS, Yu WW, Chen GR. Consensus in multi-agent systems with communication constraints. Int J Robust Nonlinear Control 2012;22(2):170–182.
19. Carli R, Fagnani F, Frasca P, Zampieri S. A probabilistic analysis of the average consensus algorithm with quantized communication. IFAC Proc Vol. 2008;41(2):8062–8067.
20. Forti M, Nistri P, Papini D. Global exponential stability and global convergence in finite time of delayed neural networks with infinite gain. IEEE Trans Neural Netw. 2005;16(6):1449–1463.
21. Hale JK, Lunel SMV. Introduction to functional differential equations. Applied mathematical sciences. Berlin: Springer;1933.