# Chapter 2
# Consensus Over Directed Static Networks with Arbitrary Finite Communication Delays

A particularly interesting aspect of the dynamics in complex networks is that certain types of globally collective behaviors emerge from local interactions among the nodes [1–5]. Such behavior arises ubiquitously in biological systems [6], ecosystems [7], and physical systems [8]. Moreover, there are many practical applications for the consensus of networks including cooperative robotics, formation flying of unmanned aerial vehicles [9], and coordinated control of land robots [10].

It can be observed that the consensus behavior is realized via the interconnections among the nodes [11–16]. However, due to the finite switching speed of amplifiers, time delays are ubiquitous at the moment of information exchanges among the nodes in many physical systems. The introduction of the communication delays will largely increase the complexity and difficulty of the consensus problem. In the literature [17], the consensus problem for systems with both diverse communication delays and diverse input delays was investigated. The unknown communication delays were considered in the high-order consensus problem for heterogeneous multi-agent systems [18]. Moreover, the directed information flow is another important challenge for the consensus problem. In the literature [19], the consensus problem for second-order multi-agent systems with inherent nonlinear dynamics under directed topologies was studied. In this chapter, we present results on the consensus problem in directed networks with arbitrary finite communication delays. By employing different techniques, we show that, under linear coupling as well as nonlinear coupling, consensus will be eventually realized for arbitrary finite communication delays. That is, the consensus behavior is robust against communication delays.

In the presence of communication delays, the final consensus state of the networked system is very hard to predict. For many physical, social, and biological systems, there is a common need to regulate the final behavior of large ensembles of interacting nodes [20–23]. However, it is very difficult and costly, if not impossible, to inform all the nodes about the objective state because of the limited communication abilities of individual nodes. Hence, new techniques are strongly



J. Lu et al., *Collective Behavior in Complex Networked Systems under Imperfect Communication*, https://doi.org/10.1007/978-981-16-1506-1_2

required to make the regulation process much easier and cheaper for the complex network with arbitrary finite communication delays. After detailed analyses, we shall show that only one well-informed leader is enough for the success of consensus regulation in networked coupled systems with arbitrary finite delays. Also such navigational signal could be very weak. Moreover, the obtained results will be extended to complex networks with hierarchical structure. The derived results are beneficial for the better understanding of emergent behavior in networked coupled systems.

## 2.1 Linear Coupling

We first consider a set of $N$ linearly coupled identical nodes, with each node being an $n$-dimensional continuous dynamical system, in the following form

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t-\tau_{ij}) - x_i(t)), \quad i \in \mathcal{N}, \tag{2.1}$$

where $x_i(t) \in \mathbb{R}^n$ denotes the state of node $i$, and $\tau_{ij} > 0$ is the communication delay from node $j$ to node $i$ for $i \neq j$ and $\tau_{ii} = 0$. $A = (a_{ij})_{N\times N}$ is the adjacency matrix representing the network topology of the complex network, and $a_{ij}$ is defined as follows: if there exists information flow from node $j$ to node $i$, then $a_{ij} > 0$ $(i \neq j)$; $a_{ij} = 0$ otherwise, and the diagonal elements $a_{ii} = 0$ for $i \in \mathcal{N}$. The coupling network among the nodes is assumed to be strongly connected.

Let $\bar{A} = (\bar{a}_{ij})_{N\times N}$ be the Laplacian matrix with its elements defined as follows: $\bar{a}_{ij} = a_{ij}$ for $i \neq j$, and $\bar{a}_{ii} = -\sum_{j=1}^{N} a_{ij}$ for $i \in \mathcal{N}$. $\bar{A}$ is irreducible since the corresponding network is strongly connected.

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^\top$ be the normalized left eigenvector of $\bar{A}$ with respect to the zero eigenvalue satisfying $\max_i\{\xi_i\} = 1$. By the Perron–Frobenius theorem [24], one obtains that $\xi_i > 0$ for $i \in \mathcal{N}$.

Throughout this section, the consensus of the networked system (2.1) is said to be asymptotically realized if $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \forall i, j \in \mathcal{N}$.

### *2.1.1 The Case of Leaderless*

In this subsection, we study the consensus seeking of linear coupling system (2.1). The following theorem shows that the consensus of linear system (2.1) is robust against communication delays.

**Theorem 2.1** *Consider a linear coupled system (2.1) with a strongly connected graph $\mathcal{G}$. Whatever finite communication delays $\tau_{ij}$ are, the consensus is asymptotically reached for arbitrary initial conditions. That is,*

$$\lim_{t\to\infty} x_i(t) \to c, \quad \forall\ i \in \mathcal{N}, \tag{2.2}$$

*where $c \in \mathbb{R}^n$ is a constant vector.*

***Proof*** Since $\xi$ is the left eigenvalue of matrix $\bar{A}$ corresponding to eigenvalue zero, one has that $\xi^\top \bar{A} = \mathbf{0}$, which implies that

$$\xi_i \bar{a}_{ii} = -\sum_{j=1, j\neq i}^{N} \xi_j \bar{a}_{ji}. \tag{2.3}$$

Further because $\bar{a}_{ii} = -\sum\limits_{j=1}^{N} a_{ij}$, we can obtain that

$$\sum_{j=1}^{N} \xi_i a_{ij} = \sum_{j=1}^{N} \xi_j a_{ji}, \quad \text{and} \quad \sum_{i=1}^{N} \xi_j a_{ji} = \sum_{i=1}^{N} \xi_i a_{ij}. \tag{2.4}$$

Consider the following Lyapunov functional:

$$V(t) = V_1(t) + V_2(t), \tag{2.5}$$

where

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N} \xi_i x_i^\top(t) x_i(t), \tag{2.6}$$

and

$$V_2(t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \int_{t-\tau_{ji}}^{t} \xi_j a_{ji} x_i^\top(\theta) x_i(\theta) d\theta. \tag{2.7}$$

Differentiating the functional $V(t)$ along the trajectories of system (2.1) gives that

$$\begin{aligned}\dot{V}_1(t) &= \sum_{i=1}^{N} \xi_i x_i^\top(t) \dot{x}_i(t) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} [x_i^\top(t) x_j(t-\tau_{ji}) - x_i^\top(t) x_i(t)],\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\dot{V}_2(t) &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_j a_{ji}[x_i^\top(t)x_i(t) - x_i^\top(t-\tau_{ji})x_i(t-\tau_{ji})] \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}x_i^\top(t)x_i(t) \\ &\quad -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}x_j^\top(t-\tau_{ij})x_j(t-\tau_{ij}).\end{aligned} \tag{2.9}$$

Therefore, by combining Eqs. (2.8) and (2.9), we obtain that

$$\begin{aligned}\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}\big[x_i^\top(t)x_i(t) - 2x_i^\top(t)x_j(t-\tau_{ij}) \\ &\quad + x_j^\top(t-\tau_{ij})x_j(t-\tau_{ij})\big] \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}\big(x_i(t) - x_j(t-\tau_{ij})\big)^\top\big(x_i(t) - x_j(t-\tau_{ij})\big) \\ &\le 0.\end{aligned} \tag{2.10}$$

Hence, $V(t)$ is non-increasing. Together with $V(t) \ge 0$, it implies that $\lim_{t\to\infty} V(t)$ exists and is finite. Then, one can easily show the boundedness of $x_i(t)$ for $i \in \mathcal{N}$ by referring to the construction of $V(t)$. By referring to system (2.1), it can be concluded that $\dot{x}_i(t)$ is bounded for any $i \in \mathcal{N}$. Thus, we can conclude that $\ddot{V}(t)$ is also bounded by referring to the expression of $\dot{V}(t)$.

According to Barbalat's Lemma [25], we get that $\lim_{t\to\infty} \xi_i a_{ij}(x_i(t) - x_j(t-\tau_{ij}))^\top\big(x_i(t) - x_j(t-\tau_{ij})\big) = 0$, i.e., $\lim_{t\to\infty}(x_i(t) - x_j(t-\tau_{ij})) = \mathbf{0}$ if $a_{ij} > 0$. In addition, one can conclude that $\dot{x}_i(t) \to \mathbf{0}$ as $t \to \infty$ for $i \in \mathcal{N}$.

Since the network is strongly connected, for each pair of nodes $i, j \in \mathcal{N}$, one can find two constants $\tau_{ij}^*$ and $\tau_{ji}^*$ such that $x_i(t) \to x_j(t-\tau_{ij}^*)$ and $x_i(t-\tau_{ji}^*) \to x_j(t)$. In fact, the constants $\tau_{ij}^*$ and $\tau_{ji}^*$ are certain linear combinations of all communication delays $\tau_{ij}$. Hence, $x_i(t-\tau_{ij}^*-\tau_{ji}^*) \to x_i(t)$ for each $i \in \mathcal{N}$, which implies that $x_i(t)$ tends to be periodic with the constant period $\tau_{ij}^*+\tau_{ji}^*$. Noting the fact that $\dot{x}_i(t) \to \mathbf{0}$ as $t \to \infty$, we yield that $x_i(t)$ tends to a steady state $c_i \in \mathbb{R}^n$.

Since the matrix $A$ is irreducible, it follows that the largest invariant manifold of system (2.1) is $\mathcal{M} = \{x_1(t), x_2(t), \ldots, x_N(t) | x_1(t) = x_2(t) = \cdots = x_N(t)\}$. This implies that there exists a constant vector $c \in \mathbb{R}^n$ such that $c_i = c$ for each $i \in \mathcal{N}$. Hence, $x_i(t) \to c$ as $t \to \infty$ for $i \in \mathcal{N}$.

Therefore, regardless of the communication delay values and for arbitrary finite initial values, the consensus of the directed interconnected system (2.1) can be realized asymptotically.

### 2.1.2 The Case with One Well-Informed Leader

Let us now consider the regulation of networked coupled system (2.1). It has been shown in Sect. 2.1.1 that the consensus among nodes can be realized whatever the finite communication delays are. However, due to the injection of arbitrary finite communication delays, the final consensus state $c$ is very hard to predict. While in many physical, social, and biological systems, there are usually some needs to regulate the behavior of large ensembles of interconnected nodes [20, 26]. In many papers, it is assumed that all the nodes should be informed about the objective state, but such a regulation scheme is very difficult and expensive to implement.

In order to force the dynamics of the nodes onto a desired trajectory, we include here a well-informed leader. Such a well-informed leader exists in many natural processes [27], such as genetic regulatory networks and biological systems. In the following, we propose a much cheaper and easily implemented method, in which only one of the nodes is informed about the objective state to be reached.

Let the objective reference state be $x^*$, and the regulation of the linear system (2.1) is said to be successful if $x_i(t) \to x^*$ as $t \to \infty$ for any $i \in \mathcal{N}$. The first node with state $x_1(t)$ is chosen as the well-informed leader. Then the networked control system corresponding to (2.1) with leader $x_1$ can be written as

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t-\tau_{ij}) - x_i(t)) + u_i(t), \quad i \in \mathcal{N}, \tag{2.11}$$

where $u_i(t) = \begin{cases} -k(x_1(t) - x^*), & \text{for } i = 1; \\ \mathbf{0}, & \text{otherwise;} \end{cases}$ for $k > 0$. Let $e_i(t) = x_i(t) - x^*$, and we obtain the following regulated dynamical system:

$$\dot{e}_i(t) = \sum_{j=1}^{N} a_{ij}(e_j(t-\tau_{ij}) - e_i(t)) + u_i(t), \quad i \in \mathcal{N}. \tag{2.12}$$

The following theorem shows that one well-informed leader is sufficient for an efficient regulation of the networked system (2.11).

**Theorem 2.2** *Consider a controlled system (2.11) with a strongly connected graph $\mathcal{G}$. Whatever the values of the finite communication delays $\tau_{ij}$ are, the states of all nodes will be successfully controlled by the objective state $x^*$. That is,*

$$\lim_{t\to\infty} x_i(t) \to x^*, \quad \forall\ i \in \mathcal{N}, \tag{2.13}$$

*where $x^* \in \mathbb{R}^n$ is the objective state.*

***Proof*** Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^\top$ be the normalized left eigenvector of $\bar{A}$ with respect to the zero eigenvalue. Consider the Lyapunov–Krasovskii functional as $E(t) = E_1(t) + E_2(t)$ with $E_1(t) = \frac{1}{2}\sum_{i=1}^{N} \xi_i e_i^\top(t)e_i(t)$ and $E_2(t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{t-\tau_{ji}}^{t} \xi_j a_{ji} e_i^\top(\theta)e_i(\theta)d\theta$. By some calculations, the derivative of the functional $E(t)$ along with the solution to system (2.12) can be obtained as

$$\begin{aligned}\dot{E}(t) = &-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}\big(e_i(t) - e_j(t-\tau_{ij})\big)^\top \\ &\times\big(e_i(t) - e_j(t-\tau_{ij})\big) - \xi_1 k e_1^\top(t)e_1(t).\end{aligned} \tag{2.14}$$

It is obvious that $\dot{E}(t) = 0$ if and only if $e_i(t) = e_j(t-\tau_{ij})$ for each pair of indexes $(i, j)$ satisfying $a_{ij} > 0$ and $e_1(t) = \mathbf{0}$. Hence, the set $\mathcal{S} = \{e_1(t) = \mathbf{0}, e_i(t) = e_j(t-\tau_{ij})$ for $(i, j)$ satisfying $a_{ij} > 0\}$ is the largest invariant set contained in $\dot{E}(t) = 0$ for system (2.12). Then by using the well-known invariance principle of functional differential equations [28], the orbit of system (2.12) converges asymptotically to the set $\mathcal{S}$. That is, $e_i(t) \to e_j(t-\tau_{ij})$ for each pair $(i, j)$ satisfying $a_{ij} > 0$ and $e_1(t) \to \mathbf{0}$ as $t \to \infty$. By a similar analysis as in Theorem 2.1, it follows that $e_i(t) \to e_j(t)$ for any $i$ and $j$, and further that $e_i(t) \to \mathbf{0}$ for $i \in \mathcal{N}$. Hence, all the nodes have been regulated to the objective state $x^*$ by only informing one of the nodes.

*Remark 2.3* The advantage of this scheme is that we do not need to inform all the nodes about the objective state. Instead, we proved that regulation process will be successfully implemented by only informing one of the nodes about the objective state, which will be spread efficiently via numerous local connections. It should be noted that any node can be chosen as the well-informed leader, and then the objective state will be realized. The “strongest" node with the highest out-degree should be a good choice to make the regulation process effective. The feedback strength $k$ is just required to be positive, i.e., the strength of the external signal can be very weak. Hence, the proposed regulation scheme is simple and cheap to implement.

## 2.2 Nonlinear Coupling

Now, we generalize the above approach to the class of nonlinearly coupled systems. Consider the following nonlinearly coupled system with directed information flow:

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(h(x_j(t-\tau_{ij})) - h(x_i(t))),\ i \in \mathcal{N}, \tag{2.15}$$

where $x_i(t) \in \mathbb{R}$ denotes the state of node $i$ at time $t$. Let $\tau = \max_{i,j}\{\tau_{ij}\}$. Throughout this section, the function $h(\cdot) : \mathbb{R} \to \mathbb{R}$ is assumed to be strictly increasing. Without loss of generality, we assume that $h(0) = 0$.

In the following theorem, we prove that the consensus of nonlinearly coupled system (2.15) is also quiet robust against the communication delays.

**Theorem 2.4** *Suppose that the graph $\mathcal{G}$ is strongly connected. Then, for nonlinear system* (2.15)*, the consensus can be realized globally for all initial conditions and arbitrary finite communication delays $\tau_{ij}$. That is,*

$$\lim_{t\to\infty} x_i(t) \to c,\ \ \forall\ i \in \mathcal{N}, \tag{2.16}$$

*where $c \in \mathbb{R}$ is a constant.*

***Proof*** Let $x(t) = [x_1^\top(t), x_2^\top(t), \ldots, x_N^\top(t)]^\top$, and consider the following Lyapunov–Krasovskii functional as

$$W(x(t)) = W_1(x(t)) + W_2(x(t)), \tag{2.17}$$

where

$$W_1(x(t)) = \sum_{i=1}^{N} \xi_i \int_0^{x_i(t)} h(s)ds$$

and

$$W_2(x(t)) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{t-\tau_{ij}}^{t} \xi_i a_{ij} h^2(x_j(\theta))d\theta.$$

Now, differentiating the functions $W_1(x(t))$ and $W_2(x(t))$ along the solution of system (2.15), it yields

$$\begin{aligned}\dot{W}_1(x(t)) &= \sum_{i=1}^{N} \xi_i h(x_i(t)) \sum_{j=1}^{N} a_{ij}\big[h(x_j(t-\tau_{ij})) - h(x_i(t))\big] \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij}\big[2h(x_i(t))h(x_j(t-\tau_{ij})) - 2h^2(x_i(t))\big],\end{aligned}$$

and from (2.4), it follows that

$$\begin{aligned}\dot{W}_2(x(t)) &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij}[h^2(x_j(t)) - h^2(x_j(t-\tau_{ij}))] \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_j a_{ji} h^2(x_j(t)) - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} h^2(x_j(t-\tau_{ij})) \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} h^2(x_i(t)) - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} h^2(x_j(t-\tau_{ij})) \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij}\big[h^2(x_i(t)) - h^2(x_j(t-\tau_{ij}))\big].\end{aligned}$$

Therefore, we obtain that

$$\dot{W}(x(t)) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} \cdot \big[h(x_i(t)) - h(x_j(t-\tau_{ij}))\big]^2 \leq 0. \tag{2.18}$$

Let $\mathcal{S} = \{x(t) : \dot{W}(x(t)) = 0\}$. Since $\xi_i > 0$ for $i \in \mathcal{N}$, it follows from (2.18) that $\mathcal{S} = \big\{x \in \mathcal{C}([t-\tau, t], \mathbb{R}^N) : a_{ij}(h(x_i(t)) - h(x_j(t-\tau_{ij}))) = 0\big\}$. It can be concluded that the set $\mathcal{S}$ is invariant with respect to system (2.15). By using the LaSalle invariance principle [28], we get that $x \to \mathcal{S}$ as $t \to +\infty$. Hence, for any ordered pair of subscripts $i$ and $j$ satisfying $a_{ij} \neq 0$, we have $h(x_i(t)) - h(x_j(t - \tau_{ij})) \to 0$ as $t \to +\infty$. Since $h(\cdot)$ is strictly increasing with $h(0) = 0$, we yield that $\lim_{t\to\infty}(x_i(t) - x_j(t-\tau_{ij})) = 0$ when $a_{ij} \neq 0$.

Since the graph $\mathcal{G}$ is strongly connected, for any ordered pair of distinct nodes $i$ and $j$, one can find a directed path from node $i$ to node $j$ and simultaneously a directed path from node $j$ to node $i$. Hence, for each pair of nodes $i, j \in \mathcal{N}$, one can find two constants $\tau_{ij}^*$ and $\tau_{ji}^*$, which are certain linear combinations of all communication delays $\tau_{ij}$, such that $x_i(t) \to x_j(t - \tau_{ij}^*)$ and $x_i(t - \tau_{ji}^*) \to x_j(t)$.

Hence, $x_i(t - \tau_{ij}^* - \tau_{ji}^*) \to x_i(t)$ holds for each $i \in \mathcal{N}$, which implies that $x_i(t)$ tends to be periodic with the constant period $\tau_{ij}^* + \tau_{ji}^*$. It follows from (2.15) that $\dot{x}_i(t) \to 0$. Consequently, we obtain that $x_i(t)$ tends to a constant $c_i \in \mathbb{R}$ as $t \to \infty$.

According to the facts that $A$ is irreducible and $x_i(t) \to c_i$, we conclude that the largest invariant set of system (2.15) is $\mathcal{M} = \{x_1(t), x_2(t), \ldots, x_N(t) | x_1(t) = x_2(t) = \cdots = x_N(t)\}$. This implies that there exists a common constant $c$ such that $c_i = c \in \mathbb{R}$ for each $i \in \mathcal{N}$. Hence, $x_i(t) \to c$ as $t \to \infty$.

*Remark 2.5* If communication delays are not included (i.e., $\tau_{ij} = 0$ in (2.15)), nonlinearly coupled system (2.15) becomes the model as discussed in [29] and our result in Theorem 2.4 still holds. Therefore, Theorem 2.4 can be regarded as a generation of the nonlinear consensus problem without communication delays discussed in [29].

## 2.3 Hierarchical Structure

The above results hold under the assumption that the network structure is strongly connected. However, for many real life networks, from machines to government, this condition can hardly be satisfied. A typical example is the consensus decision-making among a group of people, in which underlings usually have few or no influence on their big bosses, while the bosses always have great influence on the underlings. In such systems, the individual nodes are divided into several levels and hence form a hierarchical structure. In this section, we study the networked coupled system with hierarchical topology.

Consider a networked coupled system with $N = \sum_{i=1}^{p} m_i$ nodes. The $N$ nodes are divided into $p$ different groups with $m_i$ nodes in the $i$-th group. The graph generated by the local connections of the nodes is assumed to have a rooted directed spanning tree [30]. In real life systems, this condition is not restrictive due to the ubiquitous existence of hierarchical structure.

Let $A$ be the coupling matrix of the networked coupled system with hierarchical structure in the form of $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ 0 & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{pp} \end{bmatrix}$ after certain permutations. Here the matrices $A_{qq} \in \mathbb{R}^{m_q \times m_q}$ are irreducible for $q = 1, 2, \ldots, p$. Due to the existence of rooted directed spanning trees, we obtain that for each $q$ $(q < p)$, there must exist a $\kappa > q$ such that $A_{q\kappa} \neq 0$.

The nodes denoted by the matrix $A_{pp}$ can be regarded as the leader group in the complex network. From Theorem 2.1, the consensus will be firstly realized in the $p$-th group due to the irreducibility of $A_{pp}$. Then the consensus state will be propagated to the nodes in the $(p-1)$-th group due to the existence of the nonzero matrix $A_{p-1,p}$ by using Theorem 2.2. By induction, we obtain that consensus of $N$ nodes will eventually be realized. A typical example of such consensus transmitted mechanism is the chain of President–Governor–Mayor in a governmental system.

Moreover, the regulation of such a networked coupled system with hierarchical structure can also be realized by choosing a node within the leader group as the well-informed leader.

## 2.4 Numerical Examples

In this section, numerical examples will be given to demonstrate the derived theoretical results. Throughout the examples, all communication delays are uniformly distributed in (0, 1). The initial conditions are also randomly chosen from (−5, 5). It will be shown that the consensus process and the regulation are effective even for large-scale networks.

As the first example, networked coupled system with linear coupling (2.1) is considered. The connecting topology among the nodes is assumed to be a small-world directed network [31]. Opinion formation in small-world network [32] is simulated to see how the number of nodes and the communication delay affect the convergence time of reaching the consensus. Figure 2.1 shows that the consensus time increases with the increment of communication delay $\tau_{ij} = \tau$. We also studied how the consensus time changes as a function of the number of nodes. Figure 2.2 shows that the consensus time increases on the whole when the number of people increases. Furthermore, the consensus seeking and controlling of 1000 nodes small-world networks are respectively simulated in Figs. 2.3 and 2.4. In the simulations,

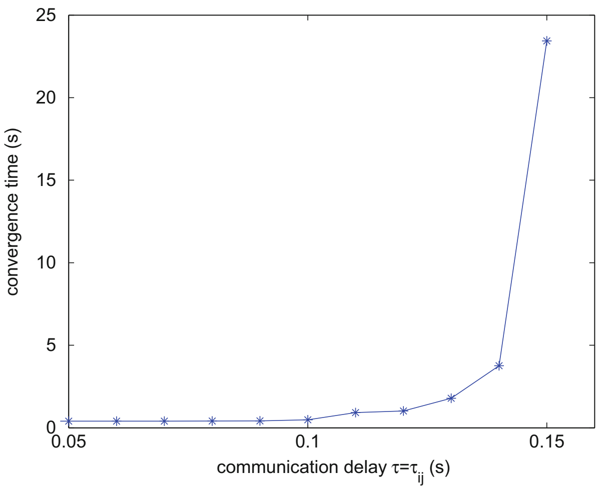


**Fig. 2.1** Convergence time versus communication delays for opinion formation in small-world network, which is generated by setting $N = 100$, $\bar{k} = 4$ and $\bar{p} = 0.01$ [31]

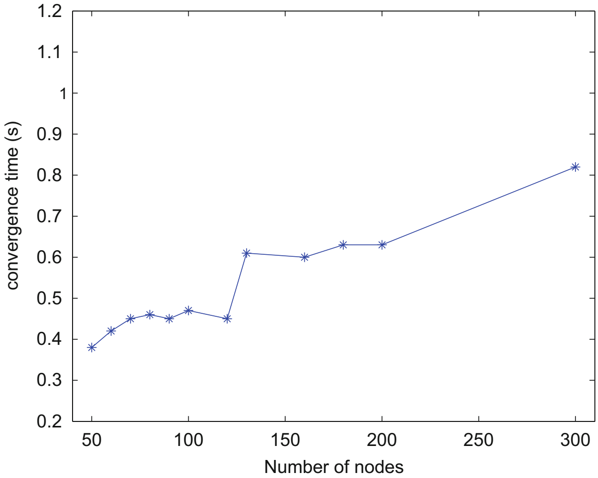


**Fig. 2.2** Convergence time versus the number of nodes for opinion formation in small-world network, which is generated by setting $\bar{k} = 4$ and $\bar{p} = 0.01$ [31]

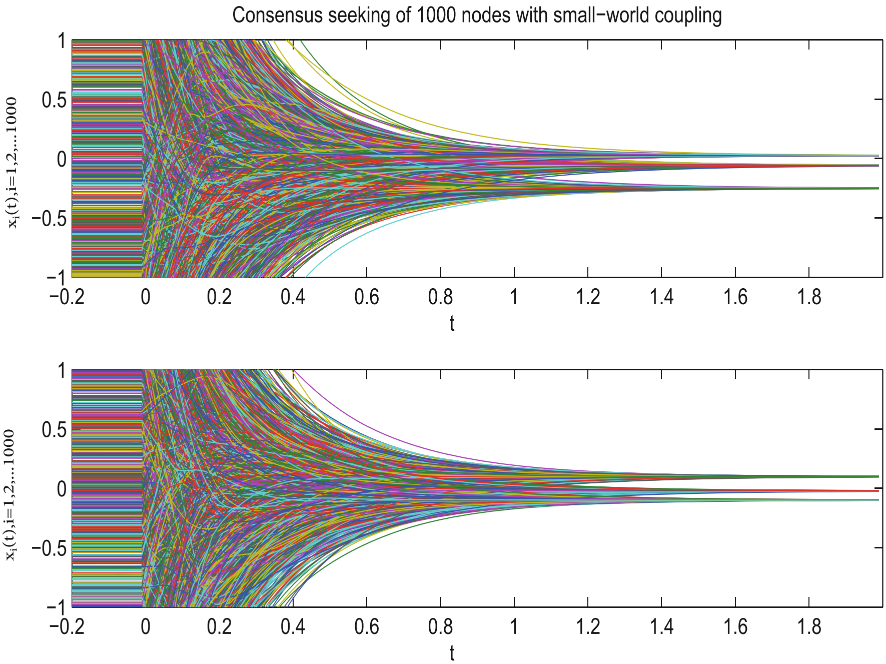


**Fig. 2.3** Consensus of 1000 nodes (three dimensions) with small-world coupling topology, which is generated by setting $\bar{k} = 4$ and $\bar{p} = 0.02$ [31]. Same initial conditions and different communication delays are used for two sub-figures

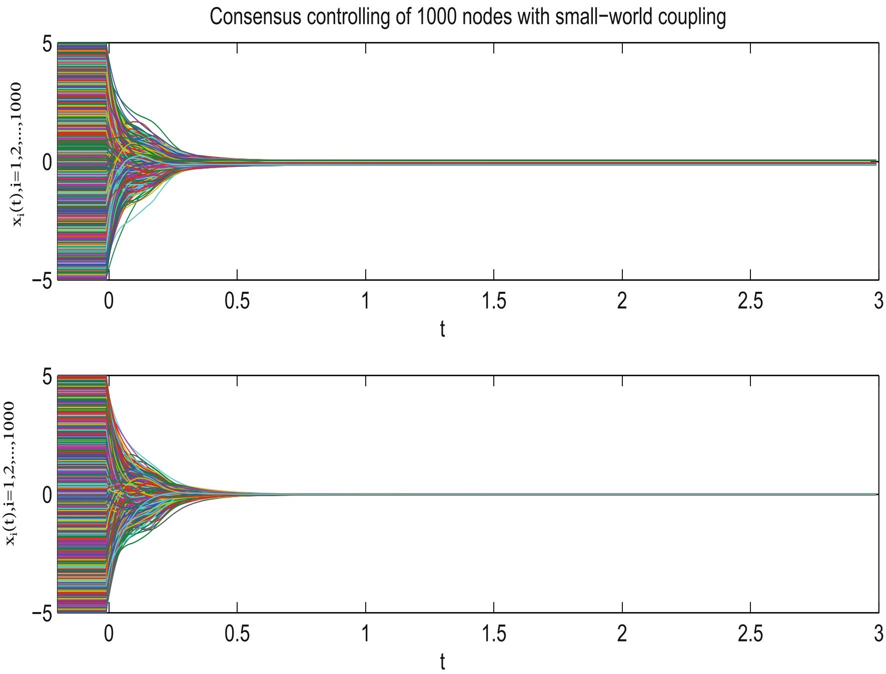


**Fig. 2.4** Consensus of 1000 nodes (three dimensions) with small-world coupling with one well-informed leader. Small-world network is generated by setting $\bar{k} = 4$ and $\bar{p} = 0.02$ [31]. The node with maximum out-degree 38 is controlled with a feedback gain 1. Initial conditions and communication delays are both different between two sub-figures

the initial degree of nodes and adding probability of directed edges are, respectively, chosen as $\bar{k} = 4$ and $\bar{p} = 0.02$ [31]. The dimension of each node is set to be $n = 3$. It follows from Theorem 2.1 that the consensus of these nodes will be realized. Our simulation results are shown in Fig. 2.3. It can be observed that even under the same initial conditions, the final agreement states could be distinct due to different communication delays. Hence, an external controller is needed if we want to force the final consensus state onto the original point $x^* = 0$. The node with maximum out-degree 38 is selected to be the well-informed leader with $\bar{\kappa} = 1$ (a relative weak and low-cost signal compared with the out-degree 38). Numerical results are depicted in Fig. 2.4, which clearly show the power of the proposed scheme.

For the second example, we consider the nonlinear coupled system (2.15). The nonlinear function is set as $h(x) = \alpha x + sin(x)$. It is obvious that $h(\cdot)$ is a strictly increasing function when $\alpha \geq 1$, but $h(\cdot)$ is not strictly increasing when $\alpha < 1$. A BA scale-free network [33] is used to describe the coupling structure of the networked system (2.15). The parameters for constructing the scale-free network are chosen as $m = m_0 = 3$. After the generation of the scale-free network, each directed edge is assigned a weighted value that is uniformly distributed in the interval $[1, 2]$. The dimension of each node is set to be 1. From Theorem 2.4, we conclude that the consensus of this nonlinearly coupled system can be realized if $\alpha \geq 1$. From Fig. 2.5, we can observe that the consensus is indeed successful when $\alpha = 2$. However, for $\alpha = 0.2$, the consensus cannot be guaranteed by Theorem 2.4 (see Fig. 2.6).

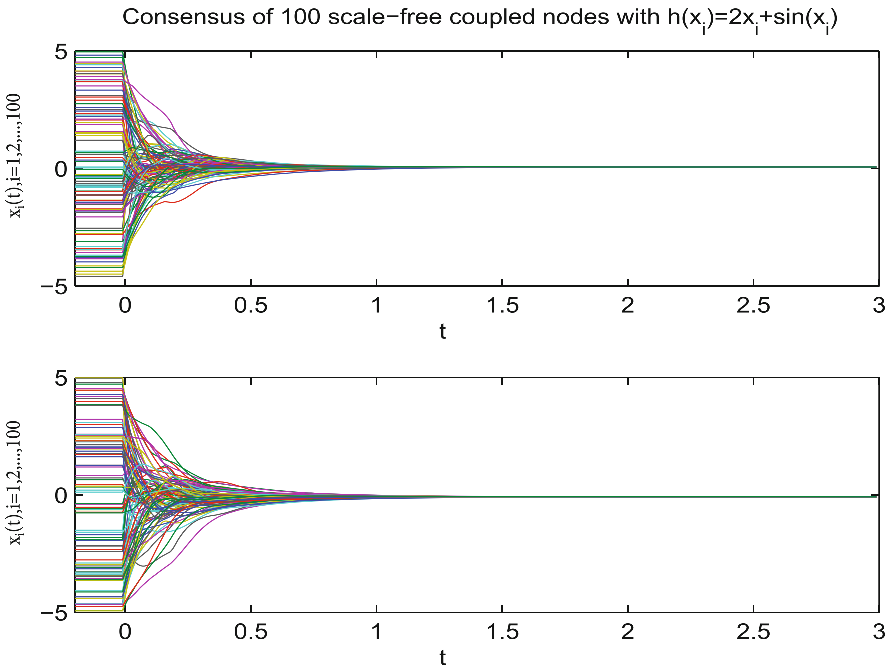


**Fig. 2.5** Consensus of nonlinearly scale-free coupled system with $\alpha = 2$. BA scale-free network composed of 100 nodes is obtained by taking $m = m_0 = 3$ [33]. The dimension of each agent is one

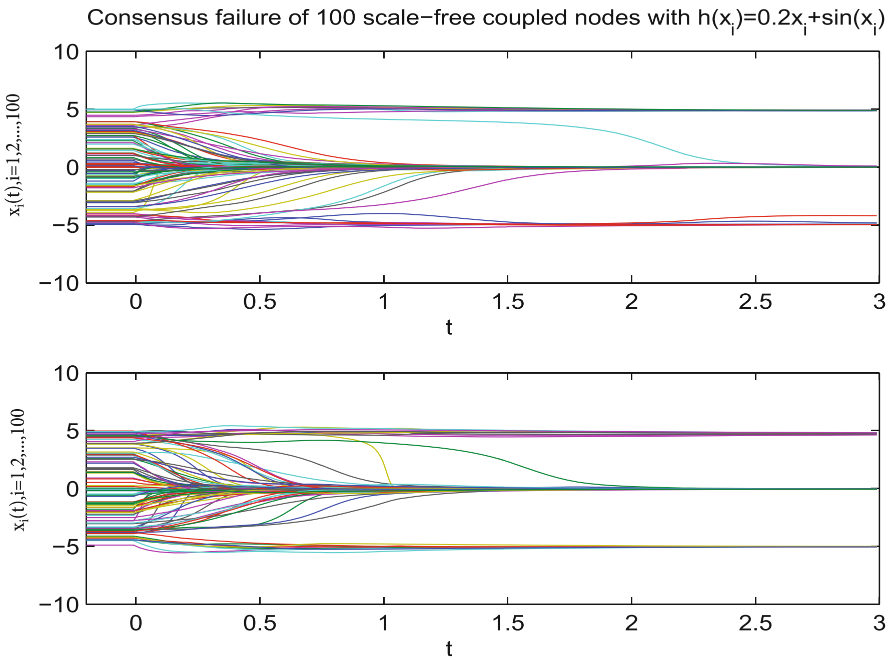


**Fig. 2.6** Consensus failure of nonlinearly scale-free coupled system with $\alpha = 0.2$. BA scale-free network composed of 100 nodes is obtained by setting $m = m_0 = 3$ [33]. The dimension of each agent is one

## 2.5 Summary

The consensus in complex networked system has been studied in this chapter under the constraint of directed information flow and arbitrary finite communication delays. We consider both linear coupling and nonlinear coupling. Compared with the existing results, our analyses and methods yield the following new results: (i) the information flow between each pair of nodes can be asymmetrical; (ii) communication delays can be arbitrarily finite and unknown; (iii) only one well-informed leader is sufficient to guarantee the successful regulation process; (iv) the external signal pinned to the leader can be very weak. Items (iii) and (iv) make the regulation process very easy and cheap to implement. The studied models are very close to real life, and the derived results would be valuable for the understanding of emergent and/or self-organized behaviors in social and biological systems.

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