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# Preface

Recently, complex networked systems, such as distributed robots and mobile sensor networks, have been widely studied due to their broad applications. One of the focuses of studying complex networked systems is on how collective behavior emerges as a result of local interactions among agents.

Most of the literatures concerning collective behavior of complex networks concentrate on the analysis of network models with perfect communication, which assumed that each agent can receive information from its neighbors timely and accurately. Unfortunately, such models cannot reflect the most real circumstances, as the information flow between two neighboring nodes is generally affected by many uncertain factors including limited communication capacity, network-induced time delays, communication noise, random packet loss, and so on. Moreover, in many realistic complex networked systems, due to the complexity of systems and external attacks (or disturbance), the failure inevitably occurs in nodes or links. Therefore, the aforementioned communication constraints should be considered in the design of control strategies or algorithms. In this book, we investigate the collective behavior of complex networked systems under imperfect communication. The following issues will be presented in detail: (a) the consensus of continuous-time multi-agent networks with communication delays; (b) the consensus of continuous-time multi-agent networks with quantization and time delays; (c) the consensus of discrete-time multi-agent networks with quantization and time delays; (d) the distributed event-triggered control approach for consensus of discrete-time/continuous-time multi-agent networks; (e) the bipartite consensus problem of cooperative-antagonistic multi-agent networks with communication delays; (f) the synchronization problem of general dynamical networks with time delays; (g) the consensus recovery approach to nonlinear multi-agent system under node failure.

This book aims to introduce some recent research work on the collective behavior of complex networked systems under imperfect communication. The book is organized as follows:

Chapter 1: This chapter begins with the background of complex networked systems. Subsequently, the organization of this book, some important definitions, useful lemmas, and some basic knowledge about graph theory are introduced.

Chapter 2: The consensus problem of networks is investigated under the constraint of directed information flow and arbitrary finite communication delays. It is shown that the consensus can be realized whatever the finite communication delays are. Furthermore, one well-informed leader is proved to be enough for the regulation of final states for all nodes, even if the external signal is very weak.

Chapter 3: The consensus problem of continuous-time multi-agent networks with quantization and communication delays is investigated. Two types of communication constraints are discussed in this chapter: (i) each agent can only exchange quantized data with its neighbors, and (ii) each agent can only obtain the delayed information from its neighbors. Solutions of the resulting system are defined in the Filippov sense. By nonsmooth analysis technique, the existence of the global Filippov solution to the resulting system is proved. For the consensus protocol which only considers the quantization effect, we prove that Filippov solutions converge to a practical consensus set in a finite time. For the consensus protocol considering quantization and time delays simultaneously, it is shown that Filippov solutions will converge to a practical consensus set asymptotically. In addition, based on the nonsmooth analysis, convergence results are derived for the proposed model with uniform quantizers. It is pointed out that the multi-agent network will achieve consensus asymptotically under the proposed distributed protocols.

Chapter 4: The consensus problem of multi-agent networks with communication quantization and time delays is investigated. Both discrete-time model and continuous-time model are considered. For the discrete-time model, we present that the multi-agent network with communication quantization and arbitrary communication delays can achieve consensus. For the continuous-time model, we show that the global Filippov solution exists and the consensus can be achieved under communication quantization and communication delays simultaneously. Furthermore, a new distributed event-triggered scheme is proposed for the considered multi-agent network model. It is shown that the multi-agent network achieves consensus asymptotically under the proposed distributed event-triggered protocols.

Chapter 5: The consensus problem of discrete-time multi-agent networks under event-triggered control strategy is considered. We discuss networks of single-integrator without delays under centralized event-triggered control and single-integrator with communication delays under distributed event-triggered control, respectively. For each consensus protocol, we prove that the multi-agent network will achieve consensus asymptotically. In addition, the effect of communication delays for the discrete-time event-triggered multi-agent consensus is also discussed. Furthermore, a self-triggered consensus algorithm is proposed in which a set of iterative procedures is given to compute the event-triggered instants. Significantly, the final consensus value is theoretically obtained even in the presence of event-based communication and distinct finite delays.

Chapter 6: The consensus problem for multi-agent networks with antagonistic interactions and communication delays is investigated. For undirected signed networks, we establish two dynamic models corresponding to linear and nonlinear coupling, respectively. Based on matrix theory, Lyapunov stability theory, and some other mathematical analysis, it is proved that all agents on signed networks can reach an agreement on consensus values which are the same in modulus but opposite in sign. Further, a bipartite consensus solution is given for linear coupling networks, and an explicit expression associating with bipartite consensus solution is provided for nonlinear coupling networks.

Chapter 7: The fixed-time consensus problem for multi-agent systems with structurally balanced signed graph is studied. A new class of fixed-time nonlinear consensus protocols is designed by employing the neighbors' information. By using the Lyapunov stability method, states of all agents can be guaranteed to reach agreement in a fixed time under our presented protocols, and the consensus values are the same in modulus but different in sign. Moreover, it is shown that the settling time is independent of the initial conditions, and it provides great convenience for estimating the convergence time by just knowing the graph topology and the information flow of the multi-agent systems. In addition, finite-time bipartite consensus problem of multi-agent systems with detail-balanced antagonistic interactions is investigated. To be specific, two valid protocols are designed and expressed in a unified form. Further, by taking advantage of some recent findings on network stability, we obtain theoretical results to guarantee that the states of all agents reach agreement in finite time under our proposed protocols.

Chapter 8: The globally exponential synchronization problem is considered for general dynamical networks. One quantity is extracted from the coupling matrix to characterize the synchronizability of the corresponding dynamical networks. The calculation of such a quantity is very convenient even for large-scale networks. The network topology is assumed to be directed and weakly connected, which implies that the coupling configuration matrix can be asymmetric, weighted, and reducible. By using the Lyapunov functional method and the Kronecker product technique, some criteria are obtained to guarantee the globally exponential synchronization of general dynamical networks.

Chapter 9: Under event-based mechanism, pinning cluster synchronization in an array of coupled neural networks is studied. A new event-triggered sampled-data transmission strategy, where only local and event-triggering states are utilized to update the broadcasting state of each agent, is proposed to realize cluster synchronization of the coupled neural networks. Furthermore, a self-triggered pinning cluster synchronization algorithm is proposed, and a set of iterative procedures is given to compute the event-triggered time instants. Hence, this will reduce the computational load significantly.

Chapter 10: The consensus recovery approach under node failure is studied. First, consensus analysis is given for nonlinear multi-agent networks with arbitrary communication topology, which fully utilizes the global information of the network structure. Before presenting the consensus recovery approach, a new network reduction approach is proposed to reduce the size of the networks. Subsequently, a consensus recovery approach is proposed to investigate the consensus of general nonlinear multi-agent networks with node failure. The objective of the consensus recovery is to remove the failure nodes of the networks meanwhile the consensus property is reserved.

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# Contents

# Symbols

| | |
|---|---|
| $N$ | Number of agents in a networked multi-agent system |
| $\mathcal{N}$ | $\{1, \ldots, N\}$ |
| $\mathbb{R}$ | The set of real numbers |
| $\mathbb{Z}$ | The set of integer numbers |
| $\mathbb{R}^n$ | $n$-dimensional Euclidean space |
| $\mathbb{R}^{m\times n}$ | The set of $m \times n$ dimensional matrices |
| $\varnothing$ | Empty set |
| $\in$ | Belongs to |
| $\triangleq$ | Defined as |
| $\inf(\cdot)$ | The greatest lower bound |
| $\sup(\cdot)$ | The smallest upper bound |
| $\max_i\{x_i\}$ | The maximum element of vector $x = (x_1, x_2, \ldots, x_n)$ |
| $\min_i\{x_i\}$ | The minimum element of vector $x = (x_1, x_2, \ldots, x_n)$ |
| $\|x\|$ | Euclidean norm of the vector $x$ |
| $\|A\|$ | Spectral norm of the matrix $A$ |
| $I_n$ | $n \times n$ identity matrix |
| $\mathbf{0}$ | The vector of appropriate dimension with all elements being 0 |
| $\mathbf{1}$ | The vector of appropriate dimension with all elements being 1 |
| $x^\top$ | Transpose of the vector $x$ |
| $A^\top$ | Transpose of the matrix $A$ |
| $A^{-1}$ | Inverse of the matrix $A$ |
| $\lambda_{\min}(A)$ | The minimum eigenvalue of the real symmetric matrix $A$ |
| $\lambda_{\max}(A)$ | The maximum eigenvalue of the real symmetric matrix $A$ |
| $\lambda_2(A)$ | The second largest eigenvalue of the real symmetric matrix $A$ |
| $\mathrm{diag}\{d_1, d_2, \ldots, d_n\}$ | $n \times n$ diagonal matrix with diagonal element being $d_1, d_2, \ldots, d_n$ |
| $A > B$ (or $<$) | $A - B$ is positive (or negative) definite |
| $A \geq B$ (or $\leq$) | $A - B$ is positive (or negative) semi-definite |

| | |
|---|---|
| $\lfloor\cdot\rfloor$ | Lower integer function |
| $\lvert\cdot\rvert$ | Absolute value function |
| $[a, b]$ | Closed interval with endpoints $a$ and $b$ |
| $\mathrm{sgn}(\cdot)$ | Sign function |
| $C([a, b], \mathbb{R})$ | The set of continuous functions $f : [a, b] \to \mathbb{R}$ |
| $f^{-1}(\cdot)$ | The inverse function of $f(\cdot)$ |
| $C(\mathbb{R}^n, \mathbb{R}^n)$ | The set of continuous functions $g : \mathbb{R}^n \to \mathbb{R}^n$ |
| $\otimes$ | Kronecker product |

# Chapter 1
# Introduction

## 1.1 Background

With the rapid development of modern technology, the world has entered the age of networks. Typical examples of networks include the World Wide Web, routes of airlines, biological networks, human relationships, and so on [1]. As a special kind of network, complex networked systems consisting of large groups of cooperating agents have made a significant impact on a broad range of applications including cooperative control of autonomous underwater vehicles (AUVs) [2], scheduling of automated highway systems [3], and congestion control in communication networks [4].

The study of complex networks can be traced back to Euler's celebrated solution of the Königsberg bridge problem in 1735, which is often regarded as the first true proof in the theory of networks. In the early 1960s, a random-graph model was proposed by Paul Erdös and Alfréd Rényi [5], which laid a solid foundation for modern network theory. Watts and Strogatz proposed a model of small-world networks in 1998 [6], after that Albert and Barabasi proposed a model of scale-free networks in 1999 based on preferential attachment [7]. These two works reveal small-world effect and scale-free property of the complex networks and the reasons for the above phenomena. Over the past two decades, complex dynamical networks have been widely exploited by researchers in various fields of physics [8], mathematics [9], engineering [10, 11], biology [12], and sociology [13].

What makes complex networked systems distinct from other kinds of systems is that they make it possible to deploy a large number of subsystems as a team to cooperatively carry out a prescribed task. Furthermore, the most striking feature that can be observed in complex networked systems is their ability to show collective behavior that cannot be well explained in terms of individual dynamics of each single node. Two significant kinds of cooperative behaviors are synchronization and consensus [9, 14–18], both of which mean that all agents reach an agreement on certain quantities of interest.



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The formal study of consensus dates back to 1974 [19], where a mathematical model was presented to describe how the group reaches agreement. Another interesting discovery is the collective behavior of a group of birds exhibited in foraging or flight, which is found by biologists in the observation of birds' flocking [20]. If attention is paid, one can find that consensus is a universal phenomenon in nature, such as the shoaling behavior of fish [21], the synchronous flashing of fireflies [22], the swarming behavior of insects [20, 23, 24], and herd behavior of land animals [25]. The key feature of consensus is how local communications and cooperations among agents, i.e., consensus protocols (or consensus algorithms), can lead to certain desirable global behavior [26–29]. Various models have been proposed to study the mechanism of multi-agent consensus problem [30–37]. In [38], the consensus problem was considered of a switched multi-agent system which composed of continuous-time and discrete-time subsystems. The authors in [39] investigated consensus problems of a class of second-order continuous-time multi-agent systems with time-delay and jointly-connected topologies. Literature [40] focused on the mean square practical leader-following consensus of second-order nonlinear multi-agent systems with noises and unmodeled dynamics.

Synchronization, as typical collective behavior and basic motion in nature, means that the difference among the states of any two different subsystems goes to zero as time goes to infinity or time goes to certain fixed value. Synchronization phenomena exist widely and can be found in different forms in nature and man-made systems, such as fireflies' synchronous flashing, attitude alignment, and the synchronized applause of audiences. To reveal the mechanism of synchronization of complex dynamical networks, a vast volume of work on synchronization has been done over the past few years. Before the appearance of small-world [6] and scale-free [7] network models, Wu in [41, 42] investigated synchronization of an array of linearly coupled systems and gave some effective synchronization criteria. In 1998, Pecora and Carroll [43] proposed the concept of master stability function as synchronization criterion, which revealed that synchronization highly depends on the coupling strategy or the topology of the network. In [14, 44–46], synchronization in small-world and scale-free networks was studied in detail. Over the past few years, different kinds of synchronization have been found and studied, such as complete synchronization [14, 41, 42, 47, 48], cluster synchronization [49–52], phase synchronization [53], lag synchronization [54, 55], and generalized synchronization [56].

In the literatures, most works on the consensus/synchronization of complex networks mainly focus on the analysis of network models with perfect communication, in which it is assumed that each agent can receive timely and accurate information from its neighbors. However, such models cannot reflect real circumstances, since the information flow between two neighboring nodes can always be affected by many uncertain factors including limited communication capacity, network induced time delays, communication noise, random packet loss, and so on. The aforementioned constraints should be considered in the design of control strategy or algorithms. Hence, it is desirable to formulate more realistic models to describe such complex dynamical networks under imperfect communication constraints and node failure. In this book, three kinds of specific imperfect communications and node failure will be investigated, and some detailed analysis of consensus/synchronization of complex dynamical networks will be presented.

## 1.2 Research Problems

The following three kinds of imperfect communication problems are considered in this book:

- **Quantization**: In real-world networked systems, the amount of information that can be reliably transmitted over the communication channels is always bounded. To comply with such a communication constraint, the signals in real-world systems are required to be quantized before transmission, and the number of quantization levels is closely related to the information transmitting capacity between the components of the system. For example, information such as data and codes in computers is stored digitally in the form of a finite number of bits and hence all the signals need to be quantized before they are processed by the computer. In this book, two kinds of quantizations in networks are considered. One is called communication quantization which is related to communication from one agent to another. The other is called input quantization which is related to processing of the information arriving at each own agent. One natural question is *how does the state of a networked systems evolve under quantization?*
- **Communication delays**: In many real complex networked systems, due to the remote location of agents or the unreliable communication medium (such as Internet), communication delays will occur during the information exchange between the agents and their neighbors. Generally, communication delays can have a negative effect on the stability and consensus/synchronization performance of the complex networks. Thus, it is important to investigate the effect of time delays on the coordinate performance of the complex networked systems and design the delay-tolerant communication protocol. Moreover, it would be very interesting to study the collective behavior of the complex networked systems simultaneously with communication delays and quantization.
- **Event-driven sampled data**: In complex networked systems, it is assumed that all information exchange between the agent and its neighbors is timely. However, the communication channels generally are unreliable and the communication capacity is limited in many real networks such as sensor networks. Moreover, the sensing ability of each agent is restricted in the networked systems. Thus, it is more practical to use sampled information transmission, i.e., the nodes of the network can only use the information at some particular time instants instead of employing the whole spectrum of information of their neighbors. Sampled-data control has been widely studied in many areas such as tracking problems and consensus problems. Unlike traditional time-driven sampled control approach (i.e., periodic sampling), event-triggered control means the control signals are

kept constant until a certain condition is violated and then the control signal is updated (or recomputed). Event-driven control is more similar to the way in which a human being behaves as a controller since his or her behavior is event-driven rather than time-driven when control manually. Thus, an interesting question arises, i.e., *is it possible to propose an effective distributed event-triggered communication protocol to realize expected collective behaviors*?

Traditional distributed communication protocols require that the agents exchange perfect information with their neighbors over the complex networked systems. This kind of information exchange can be an implicit property of complex networked systems. The objective of this book is to design efficient distributed protocols or algorithms for the complex networked systems with imperfect communication and node failure in order to comply with bandwidth limitation and tolerate communication delays and node failure. Specifically, the following problems concerning the collective behavior analysis of complex networked systems will be addressed and investigated in detail:

**Problem 1.** How does one model the multi-agent networks with arbitrary finite communication delays and directed information flow simultaneously [57]? Can consensus be realized no matter what kind of form the finite communication delays are? How to regulate all nodes' final state of the multi-agent networks, even when the external signal is very weak? These three questions will be addressed in Chap. 2.

**Problem 2.** How can we model the multi-agent consensus model with input quantization and communication delays simultaneously [58, 59]? Does there exist the global solution for the considered consensus model with discontinuous quantization function? How do quantization and communication delays affect the final consensus result? These three questions will be addressed in Chap. 3.

**Problem 3.** When the communication quantization and communication delays exist simultaneously in discrete-time multi-agent networks, can the complex networked system achieve consensus [60, 61]? For the continuous-time cases, does the global solution exist? Can the consensus of such a kind of multi-agent network be realized? These questions will be explored in Chap. 4.

**Problem 4.** Can the discrete-time and the continuous-time multi-agent networks with communication delays achieve consensus via non-periodic sampled information transmission [62, 63]? How to decide when should the information be transmitted for each agent? What effect does the communication delay have on the multi-agent networks with non-periodic sampling information? Chap. 5 will focus on these problems.

**Problem 5.** It can be found in many real multi-agent networks that the agents possess not only cooperative but also antagonistic interactions. Ensuring the desired performance of the cooperative-antagonistic multi-agent networks in the presence of communication constraints is an important task in many applications of real systems. How does one model the cooperative-antagonistic multi-agent networks with arbitrary finite communication delays [64–66]? How to deal with the difficulty stemmed from communication delays in cooperative-antagonistic multi-agent networks? What are the final consensus results for this kind of networks with communication delays? How to design the consensus protocol for cooperative-antagonistic multi-agent networks under the event-triggered control? Chap. 6 will focus on these problems.

**Problem 6.** Finite-time (or fixed-time) consensus problem has become a hot topic due to its wide applications. For the cooperative-antagonistic multi-agent networks, how to design finite-time (or fixed-time) bipartite consensus protocols [67, 68]? How to establish criteria to guarantee the bipartite agreement of all agents, and show the explicit expression of the settling time? Chap. 7 will focus on these problems.

**Problem 7.** It should be pointed out that many of the real-world networks are very large. A nature question is how to obtain synchronization criteria for large-scale directed dynamical networks? When energy constraint is imposed, how to design event-triggered sampled-data transmission strategy to realize expected synchronization behaviors [69, 70]? Chaps. 8 and 9 will discuss these synchronization problems.

**Problem 8.** The size of most real-world networks is very large, which would greatly increase the complexity and difficulty of the consensus analysis of the corresponding networks. Is it possible to greatly reduce the size of the networks, but reserve the consensus property [71]? In large-scale networks, is it possible to isolate (or remove) the failure nodes of the networks and meanwhile reserve the consensus property? Chap. 10 will focus on these problems.

### *1.2.1 Consensus and Practical Consensus*

Consider a multi-agent network $\mathcal{A}$ with $N$ agents. Let $x_i \in \mathbb{R}$ be the information state of the $i$th agent which may be position, velocity, decision variable, and so on, where $i \in \mathcal{N}$.

**Definition 1.1 (Consensus)** If for all $x_i(0) \in \mathbb{R}$, $i = 1, 2, \ldots, N$, $x_i(t)$ converges to some common equilibrium point $x^*$ (dependent on the initial values of some agents), as $t \to +\infty$, then we say that multi-agent network $\mathcal{A}$ solves a consensus problem asymptotically. The common value $x^*$ is called the group decision value.

Now, we give the definition of the distance from a point to a set and practical consensus which will be used in Chap. 3.

**Definition 1.2** The distance from a point $p \in \mathbb{R}$ to a set $\mathbb{U} \subseteq \mathbb{R}$ is defined as the minimum distance between the given point and the points on the set, i.e.,

$$dist(p, \mathbb{U}) = \min_{r \in \mathbb{U}} \{dist(p, r)\} = \min_{r \in \mathbb{U}} \{|p - r|\}.$$

**Definition 1.3** If for all $x_i(0) \in \mathbb{R}$, $i \in \mathcal{N}$, the distance of $x_i(t)$ to a set $\mathbb{U} \subseteq \mathbb{R}$ converges to 0 as $t \to +\infty$. Then, the set $\mathbb{U}$ is called *practical consensus set*.

## *1.2.2 General Model Description*

In this subsection, a brief introduction of the multi-agent consensus model [32] is presented, which requires that each agent receives timely and accurate information from its neighbors.

### 1.2.2.1 Continuous-time Multi-agent Consensus Model

The continuous-time multi-agent consensus model is as follows:

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)), \quad i \in \mathcal{N}, \tag{1.1}$$

where $x_i(t) \in \mathbb{R}^n$, $\mathcal{N} = \{1, 2, \ldots, N\}$, $N > 1$, $\mathcal{N}_i = \{j \mid a_{ij} > 0, j = 1, 2, \ldots, N\}$, and $a_{ij}$ is defined as follows:

- when $i$ is not equal to $j$:

  If there is a connection from node $j$ to node $i$, $a_{ij} > 0$;
  otherwise, $a_{ij} = 0$;

- when $i$ is equal to $j$: $a_{ii} = 0$, for all $i \in \mathcal{N}$.

Let $l_{ij} = -a_{ij}$ for $i \neq j$, and $l_{ii} = -\sum_{j=1, j \neq i}^{N} l_{ij}$. The continuous-time linear consensus protocol (1.1) can be written in matrix form as

$$\dot{x}(t) = -(L \otimes I_n)x(t), \tag{1.2}$$

where $L = (l_{ij})_{N \times N}$ is the graph Laplacian matrix and $x = [x_1^\top, \ldots, x_N^\top]^\top$.

#### 1.2.2.2 Discrete-time Multi-agent Consensus Model

A general discrete-time multi-agent consensus model can be constructed as follows:

$$x_i(k+1) = x_i(k) + \iota \sum_{j \in \mathcal{N}_i} \bar{a}_{ij}(x_j(k) - x_i(k)), \quad i \in \mathcal{N}, \tag{1.3}$$

where $x_i(k) \in \mathbb{R}^n$, the constant $\iota > 0$ denotes the step size; $\bar{a}_{ij}$ is defined as follows:

- when $j$ is not equal to $i$:

  If there is a connection from node $j$ to node $i$, $\bar{a}_{ij} > 0$;
  otherwise, $\bar{a}_{ij} = 0$;

- when $i$ is equal to $j$: $\bar{a}_{ii} = 0$, for all $i \in \mathcal{N}$.

$\bar{A} = (\bar{a}_{ij})_{N\times N}$ represents the topological structure of the system. Let $A = (a_{ij})_{N\times N}$ with $a_{ij} = \iota\bar{a}_{ij} \geq 0$ for $i \neq j$, and $a_{ii} = 1 - \sum_{j=1,\, j\neq i}^{N} a_{ij}$. Then, the dynamic of multi-agent networks can be written in a compact form as

$$x(k+1) = (A \otimes I_n)x(k). \tag{1.4}$$

**Proposition 1.4 ([72])** *System (1.4) solves a consensus problem if and only if*

*(1)* $\rho(A) = 1$, *where* $\rho(A)$ *is the spectral radius of* $A$*;*
*(2) 1 is an algebraically simple eigenvalue of* $A$*, and is the unique eigenvalue of maximum modulus;*
*(3)* $A\mathbf{1} = \mathbf{1}$, *where* $\mathbf{1} = (1,\ 1,\ \ldots,\ 1)^\top \in \mathbb{R}^N$*;*
*(4) There exists a nonnegative left eigenvector* $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^\top \in \mathbb{R}^N$ *of* $A$ *associated with eigenvalue 1 such that* $\xi^\top\mathbf{1} = 1$.

## 1.3 Mathematical Preliminaries

### *1.3.1 Matrices and Graphs*

A graph is an essential tool of the diagrammatical representation of the multi-agent networks. The set of vertices for the network are described as $\mathcal{V}$ , and the set of edges among these vertices are described as $\mathcal{E}$. The graph is denoted as $\mathcal{G}(\mathcal{V}, \mathcal{E})$. To distinguish graphs from digraphs (directed graph), we generally refer to graphs as *undirected graphs*.

A graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ containing $N$ vertices is said to have order $N$. Analogously, the size of a graph is the number of its edges $m$, i.e., the number of elements in set $\mathcal{E}$. An edge of $\mathcal{G}$ is denoted by $e_{ij} = (v_i,\ v_j)$, where $v_i$ and $v_j$ are called neighbors.

- *Self-loop*: If two vertices of an edge are the same, we call this edge a *self-loop*.
- *Directed graph*: A graph in which all the edges are directed from one vertex to another.
- *Digraph*: A *path* in a digraph is an ordered sequence of vertices such that the sequence of any two consecutive vertices is a directed edge of the digraph.
- *Connected graph*: A graph is *connected*, if there is a path between any pair of vertices.
- *Strongly connected graph*: A graph is *strongly connected*, if there is a directed path between every two different vertices.
- *Subgraph*: A *subgraph* of a graph $\mathcal{G}_1(\mathcal{V}_1, \mathcal{E}_1)$ is a graph $\mathcal{G}_2(\mathcal{V}_2, \mathcal{E}_2)$ such that $\mathcal{V}_2 \subseteq \mathcal{V}_1, \mathcal{E}_2 \subseteq \mathcal{E}_1$.
- *Directed tree*: A *directed tree* is a digraph with $n$ vertices and $n - 1$ edges with a root vertex such that there is a directed path from the root vertex to every other vertex.
- *Rooted spanning tree*: A *rooted spanning tree* of a graph is a subgraph which is a directed tree with the same vertex set.

In general, graphs are *weighted*, i.e., a positive weight is associated to each edge.

There is an intrinsic relationship between graph theory and matrix theory, which can help us to better understand the main concept of them.

- *Reducible*: A matrix is said to be reducible if it can be written as

$$P \cdot \begin{pmatrix} A_1 & A_3 \\ \mathcal{O} & A_2 \end{pmatrix} \cdot Q, \tag{1.5}$$

  where $P$ and $Q$ are permutation matrices, $A_1$ and $A_2$ are square matrices and $\mathcal{O}$ is a null matrix.
- *Irreducible*: An irreducible matrix is a matrix which is not reducible.
- *Adjacency matrix*: The adjacency matrix $A = [a_{ij}]$ of a (di)graph is a nonnegative matrix defined as $a_{ji} = \omega$ if and only if $(i,\ j)$ is an edge with weight $\omega$.
- *Out-degree*: The out-degree $d_o(v)$ of a vertex $v$ is the sum of the weights of edges emanating from $v$.
- *In-degree*: The in-degree $d_i(v)$ of a vertex $v$ is the sum of the weights of edges into $v$.
- *Balance graph*: A vertex is *balanced* if its out-degree is equal to its in-degree. A graph is *balanced* if all of its vertices are balanced.
- *Laplacian matrix*: The *Laplacian matrix* of a graph is a zero row sums nonnegative matrix $L$ denoted as $L = D - A$, where $A$ is the adjacency matrix and $D$ is the diagonal matrix of vertex in-degrees.

**Lemma 1.5 ([73])** *A network is strongly connected if and only if its Laplacian matrix is irreducible.*

**Lemma 1.6 ([73])** *For an irreducible matrix $A = (a_{ij})_{N\times N}$ with nonnegative off-diagonal elements, which satisfies the diffusive coupling condition $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$, we have the following propositions:*

- *If $\lambda$ is an eigenvalue of A and $\lambda \neq 0$, then* $\mathrm{Re}(\lambda) < 0$*;*
- *A has an eigenvalue 0 with multiplicity 1 and the right eigenvector* $[1, 1, \ldots, 1]^\top$*;*
- *Suppose that $\xi = [\xi_1, \xi_2, \ldots, \xi_N]^\top \in \mathbb{R}^N$ satisfying $\sum_{i=1}^{N} \xi_i = 1$ is the normalized left eigenvector of A corresponding to eigenvalue 0. Then, $\xi_i > 0$ for all $i = 1, 2, \ldots, N$. Furthermore, if A is symmetric, then we have $\xi_i = \frac{1}{N}$ for $i = 1, 2, \ldots, N$.*

## *1.3.2 Signed Graphs*

Let $G(V, \varepsilon, A)$ be an undirected signed graph, where $V = \{\nu_1, \nu_2, \ldots, \nu_N\}$ is the set of finite nodes, $\varepsilon \subseteq V \times V$ is the set of edges, $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix of $G$ with the elements $a_{ij}$, and $a_{ij} \neq 0 \iff (\nu_j, \nu_i) \in \varepsilon$. Since $a_{ij}$ can be positive or negative, the adjacency matrix $A$ uniquely corresponds to a signed graph. $G(A)$ is used to denote the signed graph corresponding to $A$ for simplicity, and assume that $G(A)$ has no self-loops, i.e., $a_{ii} = 0$.

- *Path*: Let *a path of* $G(A)$ be a sequence of edges in $\varepsilon$ of the form: $(\nu_{i_l}, \nu_{i_{l+1}}) \in \varepsilon$ for $l = 1, 2, \ldots, j-1$, where $\nu_{i_1}, \nu_{i_2}, \ldots, \nu_{i_j}$ are distinct vertices.
- *Connected*: We say that an undirected graph $G(A)$ is *connected* when any two vertices of $G(A)$ can be connected through paths.
- *Structurally Balanced*: A signed graph $G(A)$ is *structurally balanced* if it admits a bipartition of the nodes $V_1$, $V_2$, $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$, such that $a_{ij} \geq 0$, $\forall \nu_i, \nu_j \in V_q$, $(q \in \{1, 2\})$; and $a_{ij} \leq 0$, $\forall \nu_i \in V_q$, $\nu_j \in V_r$, $q \neq r$, $(q, r \in \{1, 2\})$. It is said structurally unbalanced otherwise.

**Definition 1.7** $\mathcal{D} = \{diag(\sigma) \mid \sigma = [\sigma_1, \sigma_2, \ldots, \sigma_N], \sigma_i \in \{\pm 1\}\}$ is a set of diagonal matrices, where

$$diag(\sigma) = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{bmatrix}.$$

In the sequel, we consider $\{\sigma_i, i = 1, 2, \ldots, N\}$ as defined in Definition 1.7 for a structurally balanced signed graph. By following [74], the Laplacian matrix

$L = (l_{ij})_{N\times N}$ for a signed graph $G(A)$ is defined with elements given in the form of

$$l_{ij} = \begin{cases} \sum_{k=1}^{N} |a_{ik}|, & j = i, \\ -a_{ij}, & j \neq i. \end{cases}$$

**Lemma 1.8 ([74])** *A connected signed graph $G(A)$ is structurally balanced if and only if one of the following equivalent conditions holds:*

*(1) all cycles of $G(A)$ are positive;*
*(2) $\exists D \in \mathcal{D}$ such that $DAD$ has all nonnegative entries.*

*Remark 1.9* This lemma can be proved in a special way. The adjacency matrix $A$ can be rewritten as $A = \begin{bmatrix} A_{11}^{+} & A_{12}^{-} \\ A_{12}^{-} & A_{22}^{+} \end{bmatrix}$, then let $D = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, we have $DAD \geq 0$. This proof is simple and explicit.

**Lemma 1.10 ([74])** *A connected signed graph $G(A)$ is structurally unbalanced if and only if one of the following equivalent conditions holds:*

*(1) one or more cycles of $G(A)$ are negative;*
*(2) $\nexists D \in \mathcal{D}$ such that $DAD$ has all nonnegative entries.*

**Lemma 1.11 ([74])** *Consider a connected signed graph $G(A)$. Let $\lambda_k(L), k = 1, 2, \ldots, N$ be the $k$-th smallest eigenvalue of the Laplacian matrix $L$. If $G(A)$ is structurally balanced, then $0 = \lambda_1(L) < \lambda_2(L) \leq \cdots \leq \lambda_N(L)$.*

**Lemma 1.12 ([75])** *If a directed signed graph $\mathcal{G}$ contains a rooted spanning tree, then there exists a proper invertible matrix $P$ satisfying $PP^\top = I$ such that the Laplacian matrix $\mathcal{L}$ can be depicted in the following Frobenius normal form:*

$$P^\top \mathcal{L} P = \begin{bmatrix} \mathcal{L}_{11} & 0 & \cdots & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{p1} & \mathcal{L}_{p2} & \cdots & \mathcal{L}_{pp} \end{bmatrix}, \tag{1.6}$$

*where $\mathcal{L}_{ii}, i = 1, 2, \ldots, p$, are irreducible matrices, and for any $1 < k \leq p$, there exists at least one $q < k$ such that $\mathcal{L}_{kq}$ is nonzero.*

### *1.3.3 Quantizer*

A quantizer is a device which converts a real-valued signal into a piecewise constant one taking on a finite or countable infinite set of values, i.e., a piecewise constant

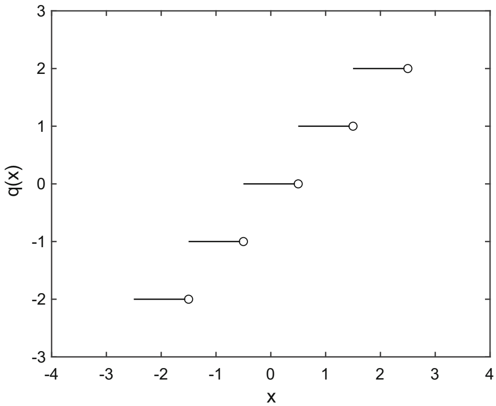


**Fig. 1.1** The first kind of uniform quantizer

function $q : \mathbb{R} \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite or countable infinite subset of $\mathbb{R}$ (see [76, 77]). Next, we introduce two kinds of uniform quantizers which will be used in Chaps. 2 and 3, respectively.

The first kind of uniform quantizer is defined as (see Fig. 1.1)

$$q(x) = \left\lfloor x + \frac{1}{2} \right\rfloor, \tag{1.7}$$

where $\lfloor \cdot \rfloor$ denote the lower integer function.

The second kind of uniform quantizer is defined as (see Fig. 1.2)

$$q(x) = \begin{cases} \lfloor x \rfloor, & x \geq 0, \\ -\lfloor -x \rfloor, & x < 0. \end{cases} \tag{1.8}$$

In this book, we will use the one-parameter family of quantizers $q_\mu(x) := \mu q(\frac{x}{\mu}),\ \mu > 0$.

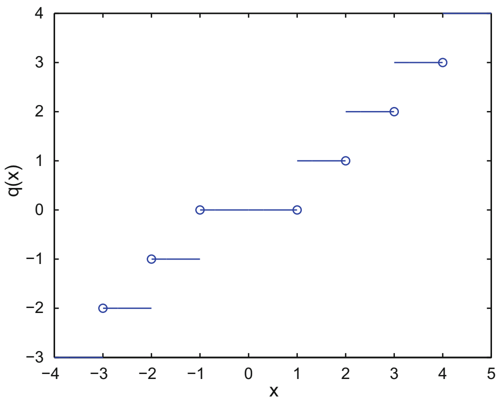


**Fig. 1.2** The second kind of uniform quantizer

### 1.3.4 Discontinuous Differential Equations

For differential equations with discontinuous right hand sides, we understand the solutions in terms of differential inclusions following Filippov [78].

**Definition 1.13** Let $I$ be an interval in the real line $\mathbb{R}$. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is absolutely continuous on $I$ if for every positive number $\epsilon$, there is a positive number $\delta$ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_k, y_k)$ of $I$ satisfies $\sum_k |y_k - x_k| < \delta$, then

$$\sum_k |f(y_k) - f(x_k)| < \epsilon. \tag{1.9}$$

Moreover, we call the function $\bar{f} = (f_1, f_2, \ldots, f_n) : I \subseteq \mathbb{R} \to \mathbb{R}^n$ is absolutely continuous on $I$ if every $f_i$, $i = 1, \ldots, n$ is absolutely continuous.

Now we introduce the concept of Filippov solution. Consider the following system:

$$\frac{dx(t)}{dt} = f(x(t)), \tag{1.10}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded.

**Definition 1.14** A set-valued map is defined as

$$\mathcal{K}(f(x)) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \bar{co}[f(B(x,\delta) \setminus N)], \tag{1.11}$$

where $\bar{co}(\Omega)$ is the closure of the convex hull of set $\Omega$, $B(x, \delta) = \{y : \|y - x\| \leq \delta\}$, and $\mu(N)$ is Lebesgue measure of set $N$.

**Definition 1.15 ([78])** A solution in the sense of Filippov of the Cauchy problem for Eq. (1.10) with initial condition $x(0) = x_0$ is an absolutely continuous function $x(t)$, $t \in [0, T]$, which satisfies $x(0) = x_0$ and differential inclusion:

$$\frac{dx}{dt} \in \mathcal{K}(f(x)), \quad a.e.\ t \in [0, T], \tag{1.12}$$

where $\mathcal{K}(f(x)) = (\mathcal{K}[f_1(x)], \ldots, \mathcal{K}[f_n(x)])$.

A property of Filippov differential inclusion $\mathcal{K}$ is presented in the following lemma:

**Lemma 1.16 ([79])** *Assume that $f, g : \mathbb{R}^m \to \mathbb{R}^n$ are locally bounded. Then,*

$$\mathcal{K}(f + g)(x) \subseteq \mathcal{K}(f)(x) + \mathcal{K}(g)(x). \tag{1.13}$$

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function and $S_h$ be the set of points where $h$ fails to be differentiable. Then,

- *Clarke generalized gradient* [80]: Clarke generalized gradient of $h$ at $x \in \mathbb{R}^n$ is the set $\partial_c h(x) = co\{\lim_{i \to +\infty} \nabla h(x^{(i)}) : x^{(i)} \to x,\, x^{(i)} \in \mathbb{R}^n,\, x^{(i)} \notin S \cup S_h\}$, where $co(\Omega)$ denotes the convex hull of set $\Omega$ and $S$ can be any set of zero measure.
- *Maximal solution* [80]: A Filippov solution to (1.10) is a maximal solution if it cannot be extended further in time.

**Definition 1.17 ([81])** $(\Omega, \mathcal{A})$ is a measurable space and $X$ is a complete separable metric space. Consider a set-valued map $F : \Omega \rightsquigarrow X$. A measurable map $f : \Omega \mapsto X$ satisfying

$$\forall \omega \in \Omega,\ f(\omega) \in F(\omega) \tag{1.14}$$

is called a measurable selection of $F$.

**Lemma 1.18 ([81] Measurable Selection)** *Let $X$ be a complete separable metric space, $(\Omega, \mathcal{A})$ a measurable space, and $F$ a measurable set-valued map from $\Omega$ to closed nonempty subsets of $X$. Then there exists a measurable selection of $F$.*

**Lemma 1.19 ([82] Chain Rule)** *If $V : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function and $\psi : \mathbb{R} \to \mathbb{R}^n$ is absolutely continuous, then for almost everywhere $(a.e.)$ $t$ there exists $p_0 \in \partial_c V(\psi(t))$ such that $\frac{d}{dt} V(\psi(t)) = p_0 \cdot \dot{\psi}(t)$.*

### *1.3.5 Some Lemmas*

**Lemma 1.20 ([83] Jensen Inequality)** *Assume that the vector function $\omega : [0, r] \longrightarrow \mathbb{R}^m$ is well defined for the following integrations. For any symmetric matrix $W \in \mathbb{R}^{m \times m}$ and scalar $r > 0$, one has*

$$r \int_0^r \omega^\top(s) W \omega(s) ds \geq \left( \int_0^r \omega(s) ds \right)^\top W \left( \int_0^r \omega(s) ds \right).$$

**Lemma 1.21 ([84])** *Consider the differential equation*

$$\dot{x}(t) = f(t, x_t).$$

*Suppose that $f$ is continuous and $f : \mathbb{R} \times C \to \mathbb{R}^n$ takes $\mathbb{R} \times$(bounded sets of $C$) into bounded sets of $\mathbb{R}^n$, and $u$, $v$, $w$: $\mathbb{R}^+ \to \mathbb{R}^+$ are continuous and strictly monotonically non-decreasing functions, $u(s)$, $v(s)$, $w(s)$ are positive for $s > 0$*

*with* $u(0) = v(0) = 0$. *If there exists a continuous functional* $V : \mathbb{R} \times C \to \mathbb{R}$ *such that*

$$u(\|x(t)\|) \le V(t, x(t)) \le v(\|x(t)\|),$$
$$\dot{V}(t, x(t)) \le -w(\|x(t)\|),$$

*where* $\dot{V}$ *is the derivative of* $V$ *along the solution of the above delayed differential equation, then the solution* $x = \mathbf{0}$ *of this equation is uniformly asymptotically stable.*

**Lemma 1.22 ([85])** *Let* $x(t)$ *be a solution to*

$$\dot{x} = g(x), \tag{1.15}$$

*where* $x(0) = x_0 \in \mathbb{R}^N$, *and let* $\Omega$ *be a bounded closed set. Suppose that there exists a continuous differentiable positive definite function* $V(x)$ *such that the derivative of* $V(t)$ *along the trajectories of system (1.15) satisfies* $\frac{dV}{dt} \le 0$. *Let* $E = \{x | \frac{dV}{dt} = 0, x \in \Omega\}$ *and* $M \subset E$ *be the biggest invariant set, then one has* $x(t) \to M$ *as* $t \to +\infty$.

**Lemma 1.23 ([86])** *If* $A = (a_{ij}) \in \mathbb{R}^{N \times N}$ *is an irreducible matrix satisfying* $a_{ij} = a_{ji} \ge 0$, *if* $i \ne j$, *and* $\sum_{j=1}^{N} a_{ij} = 0$, *for* $i = 1, 2, \ldots, N$. *For any* $\epsilon > 0$, *all eigenvalues of the matrix*

$$\overline{A} = \begin{pmatrix} a_{11} - \epsilon & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} \tag{1.16}$$

*are negative.*

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# Chapter 2
# Consensus Over Directed Static Networks with Arbitrary Finite Communication Delays

A particularly interesting aspect of the dynamics in complex networks is that certain types of globally collective behaviors emerge from local interactions among the nodes [1–5]. Such behavior arises ubiquitously in biological systems [6], ecosystems [7], and physical systems [8]. Moreover, there are many practical applications for the consensus of networks including cooperative robotics, formation flying of unmanned aerial vehicles [9], and coordinated control of land robots [10].

It can be observed that the consensus behavior is realized via the interconnections among the nodes [11–16]. However, due to the finite switching speed of amplifiers, time delays are ubiquitous at the moment of information exchanges among the nodes in many physical systems. The introduction of the communication delays will largely increase the complexity and difficulty of the consensus problem. In the literature [17], the consensus problem for systems with both diverse communication delays and diverse input delays was investigated. The unknown communication delays were considered in the high-order consensus problem for heterogeneous multi-agent systems [18]. Moreover, the directed information flow is another important challenge for the consensus problem. In the literature [19], the consensus problem for second-order multi-agent systems with inherent nonlinear dynamics under directed topologies was studied. In this chapter, we present results on the consensus problem in directed networks with arbitrary finite communication delays. By employing different techniques, we show that, under linear coupling as well as nonlinear coupling, consensus will be eventually realized for arbitrary finite communication delays. That is, the consensus behavior is robust against communication delays.

In the presence of communication delays, the final consensus state of the networked system is very hard to predict. For many physical, social, and biological systems, there is a common need to regulate the final behavior of large ensembles of interacting nodes [20–23]. However, it is very difficult and costly, if not impossible, to inform all the nodes about the objective state because of the limited communication abilities of individual nodes. Hence, new techniques are strongly



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required to make the regulation process much easier and cheaper for the complex network with arbitrary finite communication delays. After detailed analyses, we shall show that only one well-informed leader is enough for the success of consensus regulation in networked coupled systems with arbitrary finite delays. Also such navigational signal could be very weak. Moreover, the obtained results will be extended to complex networks with hierarchical structure. The derived results are beneficial for the better understanding of emergent behavior in networked coupled systems.

## 2.1 Linear Coupling

We first consider a set of $N$ linearly coupled identical nodes, with each node being an $n$-dimensional continuous dynamical system, in the following form

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t-\tau_{ij}) - x_i(t)), \quad i \in \mathcal{N}, \tag{2.1}$$

where $x_i(t) \in \mathbb{R}^n$ denotes the state of node $i$, and $\tau_{ij} > 0$ is the communication delay from node $j$ to node $i$ for $i \neq j$ and $\tau_{ii} = 0$. $A = (a_{ij})_{N\times N}$ is the adjacency matrix representing the network topology of the complex network, and $a_{ij}$ is defined as follows: if there exists information flow from node $j$ to node $i$, then $a_{ij} > 0$ $(i \neq j)$; $a_{ij} = 0$ otherwise, and the diagonal elements $a_{ii} = 0$ for $i \in \mathcal{N}$. The coupling network among the nodes is assumed to be strongly connected.

Let $\bar{A} = (\bar{a}_{ij})_{N\times N}$ be the Laplacian matrix with its elements defined as follows: $\bar{a}_{ij} = a_{ij}$ for $i \neq j$, and $\bar{a}_{ii} = -\sum_{j=1}^{N} a_{ij}$ for $i \in \mathcal{N}$. $\bar{A}$ is irreducible since the corresponding network is strongly connected.

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^\top$ be the normalized left eigenvector of $\bar{A}$ with respect to the zero eigenvalue satisfying $\max_i\{\xi_i\} = 1$. By the Perron–Frobenius theorem [24], one obtains that $\xi_i > 0$ for $i \in \mathcal{N}$.

Throughout this section, the consensus of the networked system (2.1) is said to be asymptotically realized if $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0, \forall i, j \in \mathcal{N}$.

### *2.1.1 The Case of Leaderless*

In this subsection, we study the consensus seeking of linear coupling system (2.1). The following theorem shows that the consensus of linear system (2.1) is robust against communication delays.

**Theorem 2.1** *Consider a linear coupled system* (2.1) *with a strongly connected graph* $\mathcal{G}$. *Whatever finite communication delays* $\tau_{ij}$ *are, the consensus is asymptotically reached for arbitrary initial conditions. That is,*

$$\lim_{t\to\infty} x_i(t) \to c, \quad \forall \ i \in \mathcal{N}, \tag{2.2}$$

*where* $c \in \mathbb{R}^n$ *is a constant vector.*

***Proof*** Since $\xi$ is the left eigenvalue of matrix $\bar{A}$ corresponding to eigenvalue zero, one has that $\xi^\top \bar{A} = \mathbf{0}$, which implies that

$$\xi_i \bar{a}_{ii} = -\sum_{j=1, j\neq i}^{N} \xi_j \bar{a}_{ji}. \tag{2.3}$$

Further because $\bar{a}_{ii} = -\sum_{j=1}^{N} a_{ij}$, we can obtain that

$$\sum_{j=1}^{N} \xi_i a_{ij} = \sum_{j=1}^{N} \xi_j a_{ji}, \ \text{ and } \ \sum_{i=1}^{N} \xi_j a_{ji} = \sum_{i=1}^{N} \xi_i a_{ij}. \tag{2.4}$$

Consider the following Lyapunov functional:

$$V(t) = V_1(t) + V_2(t), \tag{2.5}$$

where

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N} \xi_i x_i^\top(t) x_i(t), \tag{2.6}$$

and

$$V_2(t) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{t-\tau_{ji}}^{t} \xi_j a_{ji} x_i^\top(\theta) x_i(\theta) d\theta. \tag{2.7}$$

Differentiating the functional $V(t)$ along the trajectories of system (2.1) gives that

$$\begin{aligned} \dot{V}_1(t) &= \sum_{i=1}^{N} \xi_i x_i^\top(t) \dot{x}_i(t) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i a_{ij} [x_i^\top(t) x_j(t-\tau_{ji}) - x_i^\top(t) x_i(t)], \end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\dot{V}_2(t) &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_j a_{ji}[x_i^\top(t)x_i(t) - x_i^\top(t-\tau_{ji})x_i(t-\tau_{ji})] \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}x_i^\top(t)x_i(t) \\ &\quad -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}x_j^\top(t-\tau_{ij})x_j(t-\tau_{ij}). \end{aligned} \tag{2.9}$$

Therefore, by combining Eqs. (2.8) and (2.9), we obtain that

$$\begin{aligned}\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}\big[x_i^\top(t)x_i(t) - 2x_i^\top(t)x_j(t-\tau_{ij}) \\ &\quad +x_j^\top(t-\tau_{ij})x_j(t-\tau_{ij})\big] \\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}\big(x_i(t)-x_j(t-\tau_{ij})\big)^\top\big(x_i(t)-x_j(t-\tau_{ij})\big) \\ &\le 0. \end{aligned} \tag{2.10}$$

Hence, $V(t)$ is non-increasing. Together with $V(t) \ge 0$, it implies that $\lim_{t\to\infty} V(t)$ exists and is finite. Then, one can easily show the boundedness of $x_i(t)$ for $i \in \mathcal{N}$ by referring to the construction of $V(t)$. By referring to system (2.1), it can be concluded that $\dot{x}_i(t)$ is bounded for any $i \in \mathcal{N}$. Thus, we can conclude that $\ddot{V}(t)$ is also bounded by referring to the expression of $\dot{V}(t)$.

According to Barbalat's Lemma [25], we get that $\lim_{t\to\infty}\xi_i a_{ij}(x_i(t) - x_j(t-\tau_{ij}))^\top\big(x_i(t)-x_j(t-\tau_{ij})\big) = 0$, i.e., $\lim_{t\to\infty}(x_i(t)-x_j(t-\tau_{ij})) = \mathbf{0}$ if $a_{ij} > 0$. In addition, one can conclude that $\dot{x}_i(t) \to \mathbf{0}$ as $t \to \infty$ for $i \in \mathcal{N}$.

Since the network is strongly connected, for each pair of nodes $i, j \in \mathcal{N}$, one can find two constants $\tau_{ij}^*$ and $\tau_{ji}^*$ such that $x_i(t) \to x_j(t-\tau_{ij}^*)$ and $x_i(t-\tau_{ji}^*) \to x_j(t)$. In fact, the constants $\tau_{ij}^*$ and $\tau_{ji}^*$ are certain linear combinations of all communication delays $\tau_{ij}$. Hence, $x_i(t-\tau_{ij}^*-\tau_{ji}^*) \to x_i(t)$ for each $i \in \mathcal{N}$, which implies that $x_i(t)$ tends to be periodic with the constant period $\tau_{ij}^*+\tau_{ji}^*$. Noting the fact that $\dot{x}_i(t) \to \mathbf{0}$ as $t \to \infty$, we yield that $x_i(t)$ tends to a steady state $c_i \in \mathbb{R}^n$.

Since the matrix $A$ is irreducible, it follows that the largest invariant manifold of system (2.1) is $\mathcal{M} = \{x_1(t), x_2(t), \ldots, x_N(t) | x_1(t) = x_2(t) = \cdots = x_N(t)\}$. This implies that there exists a constant vector $c \in \mathbb{R}^n$ such that $c_i = c$ for each $i \in \mathcal{N}$. Hence, $x_i(t) \to c$ as $t \to \infty$ for $i \in \mathcal{N}$.

Therefore, regardless of the communication delay values and for arbitrary finite initial values, the consensus of the directed interconnected system (2.1) can be realized asymptotically.

### 2.1.2 The Case with One Well-Informed Leader

Let us now consider the regulation of networked coupled system (2.1). It has been shown in Sect. 2.1.1 that the consensus among nodes can be realized whatever the finite communication delays are. However, due to the injection of arbitrary finite communication delays, the final consensus state $c$ is very hard to predict. While in many physical, social, and biological systems, there are usually some needs to regulate the behavior of large ensembles of interconnected nodes [20, 26]. In many papers, it is assumed that all the nodes should be informed about the objective state, but such a regulation scheme is very difficult and expensive to implement.

In order to force the dynamics of the nodes onto a desired trajectory, we include here a well-informed leader. Such a well-informed leader exists in many natural processes [27], such as genetic regulatory networks and biological systems. In the following, we propose a much cheaper and easily implemented method, in which only one of the nodes is informed about the objective state to be reached.

Let the objective reference state be $x^*$, and the regulation of the linear system (2.1) is said to be successful if $x_i(t) \to x^*$ as $t \to \infty$ for any $i \in \mathcal{N}$. The first node with state $x_1(t)$ is chosen as the well-informed leader. Then the networked control system corresponding to (2.1) with leader $x_1$ can be written as

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t-\tau_{ij}) - x_i(t)) + u_i(t), \quad i \in \mathcal{N}, \tag{2.11}$$

where $u_i(t) = \begin{cases} -k(x_1(t) - x^*), & \text{for } i = 1; \\ \mathbf{0}, & \text{otherwise;} \end{cases}$ for $k > 0$. Let $e_i(t) = x_i(t) - x^*$, and we obtain the following regulated dynamical system:

$$\dot{e}_i(t) = \sum_{j=1}^{N} a_{ij}(e_j(t-\tau_{ij}) - e_i(t)) + u_i(t), \quad i \in \mathcal{N}. \tag{2.12}$$

The following theorem shows that one well-informed leader is sufficient for an efficient regulation of the networked system (2.11).

**Theorem 2.2** *Consider a controlled system* (2.11) *with a strongly connected graph* $\mathcal{G}$*. Whatever the values of the finite communication delays* $\tau_{ij}$ *are, the states of all nodes will be successfully controlled by the objective state* $x^*$*. That is,*

$$\lim_{t\to\infty} x_i(t) \to x^*, \quad \forall \; i \in \mathcal{N}, \tag{2.13}$$

*where* $x^* \in \mathbb{R}^n$ *is the objective state.*

***Proof*** Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^\top$ be the normalized left eigenvector of $\bar{A}$ with respect to the zero eigenvalue. Consider the Lyapunov–Krasovskii functional as $E(t) = E_1(t) + E_2(t)$ with $E_1(t) = \frac{1}{2}\sum_{i=1}^N \xi_i e_i^\top(t) e_i(t)$ and $E_2(t) = \frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \int_{t-\tau_{ji}}^t \xi_j a_{ji} e_i^\top(\theta) e_i(\theta) d\theta$. By some calculations, the derivative of the functional $E(t)$ along with the solution to system (2.12) can be obtained as

$$\begin{aligned}\dot{E}(t) = &-\frac{1}{2}\sum_{i=1}^N \sum_{j=1}^N \xi_i a_{ij} \big(e_i(t) - e_j(t-\tau_{ij})\big)^\top \\ &\times \big(e_i(t) - e_j(t-\tau_{ij})\big) - \xi_1 k e_1^\top(t) e_1(t).\end{aligned} \tag{2.14}$$

It is obvious that $\dot{E}(t) = 0$ if and only if $e_i(t) = e_j(t - \tau_{ij})$ for each pair of indexes $(i, j)$ satisfying $a_{ij} > 0$ and $e_1(t) = \mathbf{0}$. Hence, the set $\mathcal{S} = \{e_1(t) = \mathbf{0}, e_i(t) = e_j(t - \tau_{ij})$ for $(i, j)$ satisfying $a_{ij} > 0\}$ is the largest invariant set contained in $\dot{E}(t) = 0$ for system (2.12). Then by using the well-known invariance principle of functional differential equations [28], the orbit of system (2.12) converges asymptotically to the set $\mathcal{S}$. That is, $e_i(t) \to e_j(t-\tau_{ij})$ for each pair $(i, j)$ satisfying $a_{ij} > 0$ and $e_1(t) \to \mathbf{0}$ as $t \to \infty$. By a similar analysis as in Theorem 2.1, it follows that $e_i(t) \to e_j(t)$ for any $i$ and $j$, and further that $e_i(t) \to \mathbf{0}$ for $i \in \mathcal{N}$. Hence, all the nodes have been regulated to the objective state $x^*$ by only informing one of the nodes.

*Remark 2.3* The advantage of this scheme is that we do not need to inform all the nodes about the objective state. Instead, we proved that regulation process will be successfully implemented by only informing one of the nodes about the objective state, which will be spread efficiently via numerous local connections. It should be noted that any node can be chosen as the well-informed leader, and then the objective state will be realized. The “strongest" node with the highest out-degree should be a good choice to make the regulation process effective. The feedback strength $k$ is just required to be positive, i.e., the strength of the external signal can be very weak. Hence, the proposed regulation scheme is simple and cheap to implement.

## 2.2 Nonlinear Coupling

Now, we generalize the above approach to the class of nonlinearly coupled systems. Consider the following nonlinearly coupled system with directed information flow:

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(h(x_j(t - \tau_{ij})) - h(x_i(t))), \ i \in \mathcal{N}, \tag{2.15}$$

where $x_i(t) \in \mathbb{R}$ denotes the state of node $i$ at time $t$. Let $\tau = \max_{i,j}\{\tau_{ij}\}$. Throughout this section, the function $h(\cdot) : \mathbb{R} \to \mathbb{R}$ is assumed to be strictly increasing. Without loss of generality, we assume that $h(0) = 0$.

In the following theorem, we prove that the consensus of nonlinearly coupled system (2.15) is also quiet robust against the communication delays.

**Theorem 2.4** *Suppose that the graph $\mathcal{G}$ is strongly connected. Then, for nonlinear system* (2.15)*, the consensus can be realized globally for all initial conditions and arbitrary finite communication delays $\tau_{ij}$. That is,*

$$\lim_{t\to\infty} x_i(t) \to c, \ \ \forall \ i \in \mathcal{N}, \tag{2.16}$$

*where $c \in \mathbb{R}$ is a constant.*

***Proof*** Let $x(t) = [x_1^\top(t), x_2^\top(t), \ldots, x_N^\top(t)]^\top$, and consider the following Lyapunov–Krasovskii functional as

$$W(x(t)) = W_1(x(t)) + W_2(x(t)), \tag{2.17}$$

where

$$W_1(x(t)) = \sum_{i=1}^{N} \xi_i \int_0^{x_i(t)} h(s)ds$$

and

$$W_2(x(t)) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{t-\tau_{ij}}^{t} \xi_i a_{ij} h^2(x_j(\theta))d\theta.$$

Now, differentiating the functions $W_1(x(t))$ and $W_2(x(t))$ along the solution of system (2.15), it yields

$$\begin{aligned}\dot{W}_1(x(t)) &= \sum_{i=1}^{N}\xi_i h(x_i(t))\sum_{j=1}^{N}a_{ij}\big[h(x_j(t-\tau_{ij}))-h(x_i(t))\big]\\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}\big[2h(x_i(t))h(x_j(t-\tau_{ij}))-2h^2(x_i(t))\big],\end{aligned}$$

and from (2.4), it follows that

$$\begin{aligned}\dot{W}_2(x(t)) &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}[h^2(x_j(t))-h^2(x_j(t-\tau_{ij}))]\\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_j a_{ji}h^2(x_j(t))-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}h^2(x_j(t-\tau_{ij}))\\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}h^2(x_i(t))-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}h^2(x_j(t-\tau_{ij}))\\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}\big[h^2(x_i(t))-h^2(x_j(t-\tau_{ij}))\big].\end{aligned}$$

Therefore, we obtain that

$$\dot{W}(x(t)) = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i a_{ij}\cdot\big[h(x_i(t))-h(x_j(t-\tau_{ij}))\big]^2 \le 0. \tag{2.18}$$

Let $\mathcal{S}=\{x(t):\dot{W}(x(t))=0\}$. Since $\xi_i>0$ for $i\in\mathcal{N}$, it follows from (2.18) that $\mathcal{S}=\big\{x\in\mathcal{C}([t-\tau,t],\mathbb{R}^N): a_{ij}(h(x_i(t))-h(x_j(t-\tau_{ij})))=0\big\}$. It can be concluded that the set $\mathcal{S}$ is invariant with respect to system (2.15). By using the LaSalle invariance principle [28], we get that $x\to\mathcal{S}$ as $t\to+\infty$. Hence, for any ordered pair of subscripts $i$ and $j$ satisfying $a_{ij}\neq 0$, we have $h(x_i(t))-h(x_j(t-\tau_{ij}))\to 0$ as $t\to+\infty$. Since $h(\cdot)$ is strictly increasing with $h(0)=0$, we yield that $\lim_{t\to\infty}(x_i(t)-x_j(t-\tau_{ij}))=0$ when $a_{ij}\neq 0$.

Since the graph $\mathcal{G}$ is strongly connected, for any ordered pair of distinct nodes $i$ and $j$, one can find a directed path from node $i$ to node $j$ and simultaneously a directed path from node $j$ to node $i$. Hence, for each pair of nodes $i,j\in\mathcal{N}$, one can find two constants $\tau_{ij}^*$ and $\tau_{ji}^*$, which are certain linear combinations of all communication delays $\tau_{ij}$, such that $x_i(t)\to x_j(t-\tau_{ij}^*)$ and $x_i(t-\tau_{ji}^*)\to x_j(t)$.

Hence, $x_i(t - \tau_{ij}^* - \tau_{ji}^*) \to x_i(t)$ holds for each $i \in \mathcal{N}$, which implies that $x_i(t)$ tends to be periodic with the constant period $\tau_{ij}^* + \tau_{ji}^*$. It follows from (2.15) that $\dot{x}_i(t) \to 0$. Consequently, we obtain that $x_i(t)$ tends to a constant $c_i \in \mathbb{R}$ as $t \to \infty$.

According to the facts that $A$ is irreducible and $x_i(t) \to c_i$, we conclude that the largest invariant set of system (2.15) is $\mathcal{M} = \{x_1(t), x_2(t), \ldots, x_N(t) | x_1(t) = x_2(t) = \cdots = x_N(t)\}$. This implies that there exists a common constant $c$ such that $c_i = c \in \mathbb{R}$ for each $i \in \mathcal{N}$. Hence, $x_i(t) \to c$ as $t \to \infty$.

*Remark 2.5* If communication delays are not included (i.e., $\tau_{ij} = 0$ in (2.15)), nonlinearly coupled system (2.15) becomes the model as discussed in [29] and our result in Theorem 2.4 still holds. Therefore, Theorem 2.4 can be regarded as a generation of the nonlinear consensus problem without communication delays discussed in [29].

## 2.3 Hierarchical Structure

The above results hold under the assumption that the network structure is strongly connected. However, for many real life networks, from machines to government, this condition can hardly be satisfied. A typical example is the consensus decision-making among a group of people, in which underlings usually have few or no influence on their big bosses, while the bosses always have great influence on the underlings. In such systems, the individual nodes are divided into several levels and hence form a hierarchical structure. In this section, we study the networked coupled system with hierarchical topology.

Consider a networked coupled system with $N = \sum_{i=1}^{p} m_i$ nodes. The $N$ nodes are divided into $p$ different groups with $m_i$ nodes in the $i$-th group. The graph generated by the local connections of the nodes is assumed to have a rooted directed spanning tree [30]. In real life systems, this condition is not restrictive due to the ubiquitous existence of hierarchical structure.

Let $A$ be the coupling matrix of the networked coupled system with hierarchical structure in the form of $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ 0 & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{pp} \end{bmatrix}$ after certain permutations. Here the matrices $A_{qq} \in \mathbb{R}^{m_q \times m_q}$ are irreducible for $q = 1, 2, \ldots, p$. Due to the existence of rooted directed spanning trees, we obtain that for each $q$ $(q < p)$, there must exist a $\kappa > q$ such that $A_{q\kappa} \neq 0$.

The nodes denoted by the matrix $A_{pp}$ can be regarded as the leader group in the complex network. From Theorem 2.1, the consensus will be firstly realized in the $p$-th group due to the irreducibility of $A_{pp}$. Then the consensus state will be propagated to the nodes in the $(p-1)$-th group due to the existence of the nonzero matrix $A_{p-1,p}$ by using Theorem 2.2. By induction, we obtain that consensus of $N$ nodes will eventually be realized. A typical example of such consensus transmitted mechanism is the chain of President–Governor–Mayor in a governmental system.

Moreover, the regulation of such a networked coupled system with hierarchical structure can also be realized by choosing a node within the leader group as the well-informed leader.

## 2.4 Numerical Examples

In this section, numerical examples will be given to demonstrate the derived theoretical results. Throughout the examples, all communication delays are uniformly distributed in (0, 1). The initial conditions are also randomly chosen from $(-5, 5)$. It will be shown that the consensus process and the regulation are effective even for large-scale networks.

As the first example, networked coupled system with linear coupling (2.1) is considered. The connecting topology among the nodes is assumed to be a small-world directed network [31]. Opinion formation in small-world network [32] is simulated to see how the number of nodes and the communication delay affect the convergence time of reaching the consensus. Figure 2.1 shows that the consensus time increases with the increment of communication delay $\tau_{ij} = \tau$. We also studied how the consensus time changes as a function of the number of nodes. Figure 2.2 shows that the consensus time increases on the whole when the number of people increases. Furthermore, the consensus seeking and controlling of 1000 nodes small-world networks are respectively simulated in Figs. 2.3 and 2.4. In the simulations,

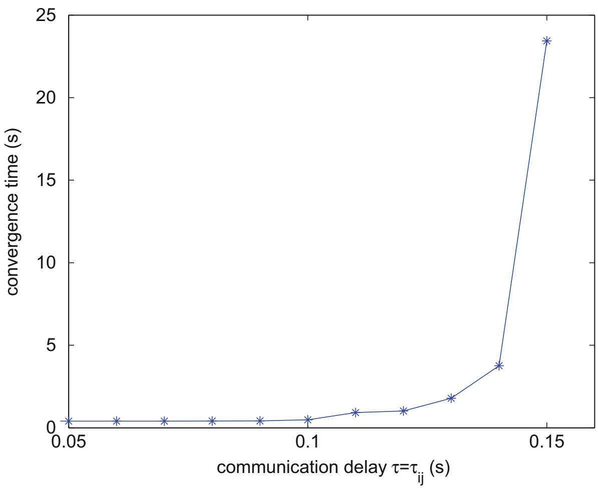


**Fig. 2.1** Convergence time versus communication delays for opinion formation in small-world network, which is generated by setting $N = 100$, $\bar{k} = 4$ and $\bar{p} = 0.01$ [31]

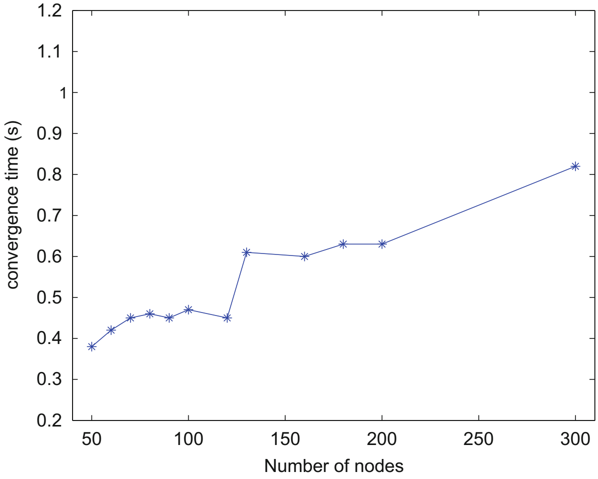


**Fig. 2.2** Convergence time versus the number of nodes for opinion formation in small-world network, which is generated by setting $\bar{k} = 4$ and $\bar{p} = 0.01$ [31]

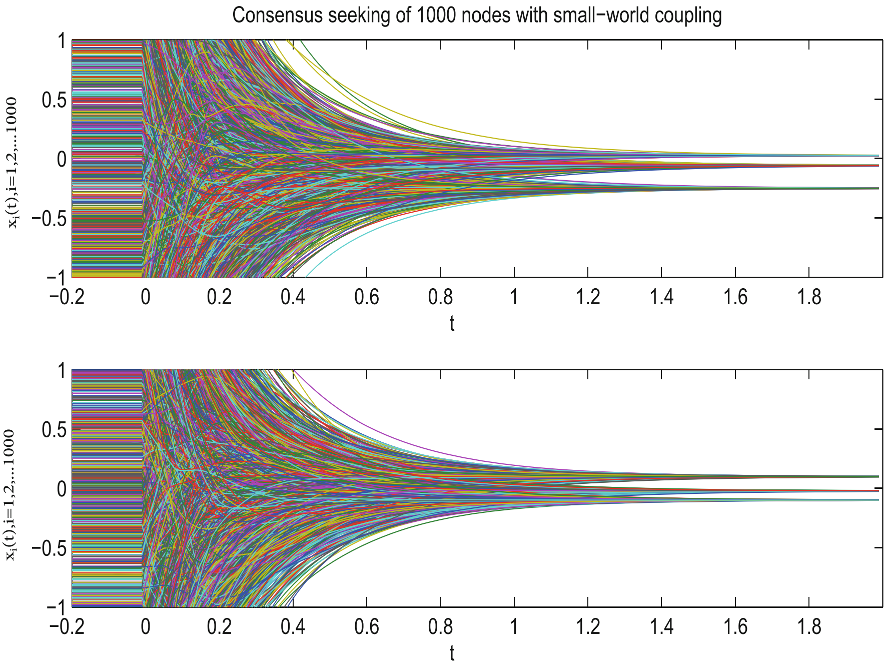


**Fig. 2.3** Consensus of 1000 nodes (three dimensions) with small-world coupling topology, which is generated by setting $\bar{k} = 4$ and $\bar{p} = 0.02$ [31]. Same initial conditions and different communication delays are used for two sub-figures

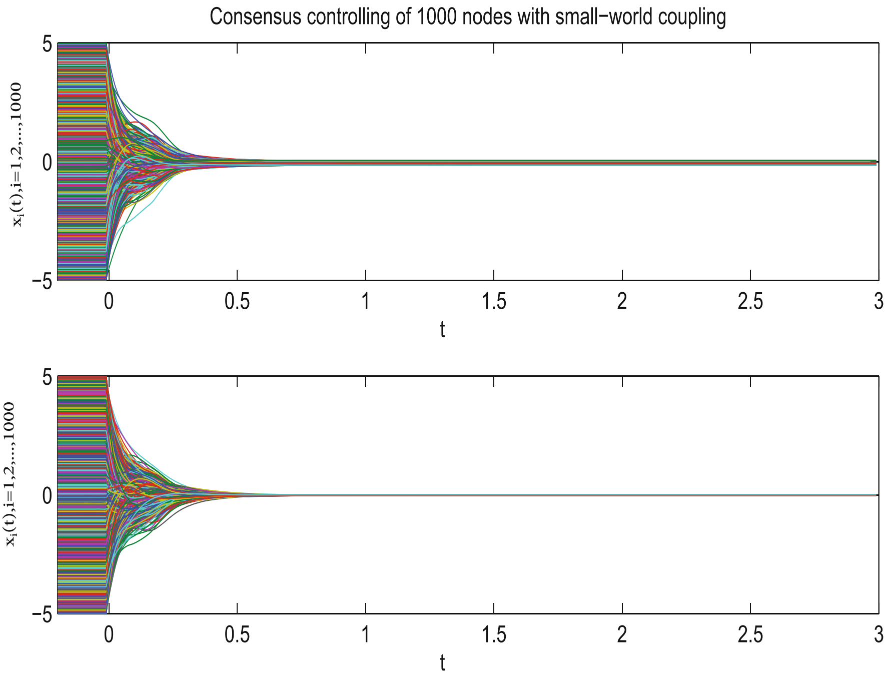


**Fig. 2.4** Consensus of 1000 nodes (three dimensions) with small-world coupling with one well-informed leader. Small-world network is generated by setting $\bar{k} = 4$ and $\bar{p} = 0.02$ [31]. The node with maximum out-degree 38 is controlled with a feedback gain 1. Initial conditions and communication delays are both different between two sub-figures

the initial degree of nodes and adding probability of directed edges are, respectively, chosen as $\bar{k} = 4$ and $\bar{p} = 0.02$ [31]. The dimension of each node is set to be $n = 3$. It follows from Theorem 2.1 that the consensus of these nodes will be realized. Our simulation results are shown in Fig. 2.3. It can be observed that even under the same initial conditions, the final agreement states could be distinct due to different communication delays. Hence, an external controller is needed if we want to force the final consensus state onto the original point $x^* = 0$. The node with maximum out-degree 38 is selected to be the well-informed leader with $\bar{\kappa} = 1$ (a relative weak and low-cost signal compared with the out-degree 38). Numerical results are depicted in Fig. 2.4, which clearly show the power of the proposed scheme.

For the second example, we consider the nonlinear coupled system (2.15). The nonlinear function is set as $h(x) = \alpha x + sin(x)$. It is obvious that $h(\cdot)$ is a strictly increasing function when $\alpha \geq 1$, but $h(\cdot)$ is not strictly increasing when $\alpha < 1$. A BA scale-free network [33] is used to describe the coupling structure of the networked system (2.15). The parameters for constructing the scale-free network are chosen as $m = m_0 = 3$. After the generation of the scale-free network, each directed edge is assigned a weighted value that is uniformly distributed in the interval $[1, 2]$. The dimension of each node is set to be 1. From Theorem 2.4, we conclude that the consensus of this nonlinearly coupled system can be realized if $\alpha \geq 1$. From Fig. 2.5, we can observe that the consensus is indeed successful when $\alpha = 2$. However, for $\alpha = 0.2$, the consensus cannot be guaranteed by Theorem 2.4 (see Fig. 2.6).

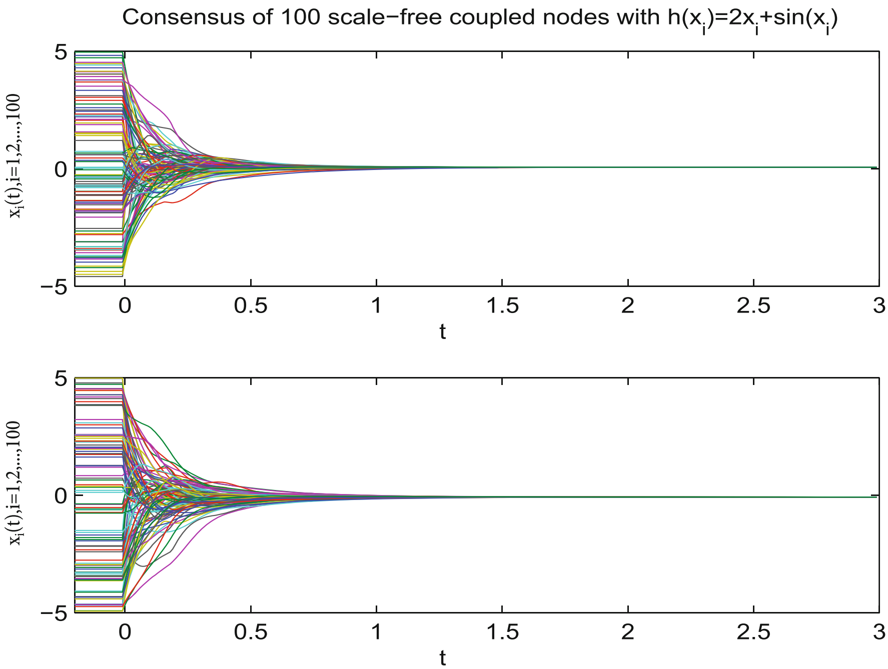


**Fig. 2.5** Consensus of nonlinearly scale-free coupled system with $\alpha = 2$. BA scale-free network composed of 100 nodes is obtained by taking $m = m_0 = 3$ [33]. The dimension of each agent is one

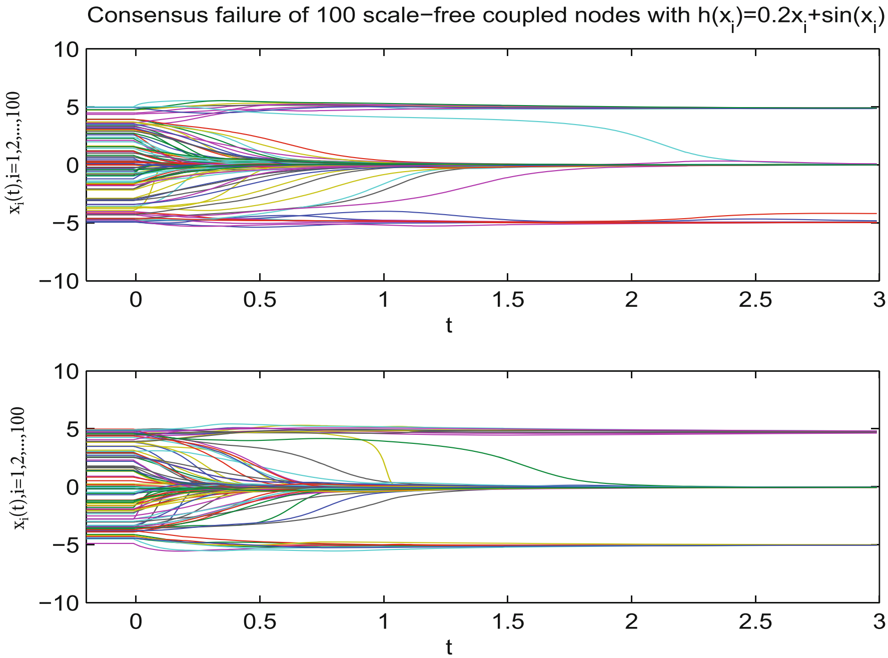


**Fig. 2.6** Consensus failure of nonlinearly scale-free coupled system with $\alpha = 0.2$. BA scale-free network composed of 100 nodes is obtained by setting $m = m_0 = 3$ [33]. The dimension of each agent is one

## 2.5 Summary

The consensus in complex networked system has been studied in this chapter under the constraint of directed information flow and arbitrary finite communication delays. We consider both linear coupling and nonlinear coupling. Compared with the existing results, our analyses and methods yield the following new results: (i) the information flow between each pair of nodes can be asymmetrical; (ii) communication delays can be arbitrarily finite and unknown; (iii) only one well-informed leader is sufficient to guarantee the successful regulation process; (iv) the external signal pinned to the leader can be very weak. Items (iii) and (iv) make the regulation process very easy and cheap to implement. The studied models are very close to real life, and the derived results would be valuable for the understanding of emergent and/or self-organized behaviors in social and biological systems.

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# Chapter 3
# Practical Consensus of Multi-agent Networks with Communication Constraints

Due to the limited bitrate of communication channels and limited bandwidth, communication constraints always exist in real-world systems which should be well considered in the design of control strategy or algorithm (see, e.g.,[1–8]). Two important communication constraints are signal quantization and time delay [9, 10]. Unlike the error-free information exchange, the signals in real-world systems are required to be quantized before transmission when high data rate is not available. On the other hand, consensus problems with quantization are also challenging and should be investigated.

In addition to quantization, another significant communication constraint in multi-agent networks is the time delay, which is usually caused by an agent waiting to send out messages via a busy channel, or by a signal processing and propagation [11–15]. In [11], under a new protocol and strongly connected network topology, consensus can be achieved for arbitrary finite time delays. In [13], a second-order consensus protocol for multi-agent systems with communication delay was proposed and it has been shown that consensus can be reached if the delays are small enough. The most important feature of the results obtained in [16] was that they do not impose restrictive conditions on the communication topologies and the communication time delays, and allow for arbitrary bounded nonuniform time delays. Recently, based on the delay-induced consensus protocol, the second-order consensus issue for multi-agent systems was discussed in [17].

In this chapter, the communication constraints are considered in the multi-agent consensus problem. Specifically, two types of communication constraints are discussed: (1) each agent can only exchange quantized data with its neighbors and (2) each agent can only obtain the delayed information from its neighbors. These two communication constraints can be frequently observed in many real multi-agent networks and lead to incomplete or inaccurate information of the node being available for its neighboring nodes. In Sect. 3.1, finite-time practical consensus of multi-agent networks with quantized data is studied. In Sect. 3.2, quantization and time delay are simultaneously investigated for continuous-time multi-agent network consensus problems.



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## 3.1 Practical Consensus with Quantized Data

In this section, consensus problem of continuous-time multi-agent networks with quantized data is studied. Ground work has been laid in [18, 19], which extended the quantized consensus model to the continuous-time case and quantized consensus results have been obtained for the network model. In [18], some mathematical difficulties resulting from the inherent discontinuity of the quantization function were analyzed in detail. In this section, we shall further extend the previous results of [18, 19] by using different methods.

The remainder of this section is organized as follows. In Sect. 3.1.1, the multi-agent network model with quantized data is presented. In Sect. 3.1.2, the consensus analysis of the proposed protocol is presented in detail. Finally, a numerical simulation is given to demonstrate the validity of the theoretical analysis in Sect. 3.1.3.

### *3.1.1 Model Description*

Consider the following multi-agent system with dynamics:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, \ldots, N,$$

where $x_i(t) \in \mathbb{R}$ is the state of the agent $i$, and $u_i(t)$ is called the consensus protocol. The following consensus protocol:

$$u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \ldots, N, \tag{3.1}$$

has been proposed in [20], which requires that each agent receives information from its neighbors timely and accurately.

Due to the communication bandwidth constraints in many real multi-agent networks, the agents can only use the quantized information of the neighboring agents. The following consensus protocol will be studied in this part:

$$\frac{dx_i(t)}{dt} = \sum_{j \in \mathcal{N}_i} a_{ij}[q_\mu(x_j(t)) - q_\mu(x_i(t))], \quad i = 1, \ldots, N, \tag{3.2}$$

where $q_\mu(z)$ denotes one-parameter family of uniform quantizers defined by $q_\mu(z) = \lfloor \frac{z}{\Delta\mu} + \frac{1}{2} \rfloor \mu$. Here, $\mu$ and $\Delta$ are called the *quantization parameter* and *sensitivity* of the quantizer, respectively. Moreover, if $x = (x_1, x_2, \ldots, x_N)^\top \in \mathbb{R}^N$, we denote $q_\mu(x) = (q_\mu(x_1), q_\mu(x_2), \ldots, q_\mu(x_N))^\top$.

We know that the system (3.2) may not have the global solution in the sense of Carathéodory due to the discontinuous of function $q_\mu(\cdot)$ [18]. Hence, we shall consider solutions in a more general sense, i.e., the Filippov solution of system (3.2). The concept of the Filippov solution to the differential equation (3.2) is provided in the following:

**Definition 3.1** An absolutely continuous function $x : [0, T) \to \mathbb{R}^N$ is a solution in the sense of Filippov for discontinuous system (3.2) if $x(t)$ satisfies that

$$\frac{dx_i(t)}{dt} \in \mathcal{K}\left[\sum_{j\in\mathcal{N}_i} a_{ij}(q_\mu(x_j(t)) - q_\mu(x_i(t)))\right], \quad i = 1, \ldots, N.$$

Based on Lemma 1.18, if $x(t)$ is a Filippov solution of system (3.2), then there exists a measurable function $\gamma(t) \in \mathcal{K}[q_\mu(x(t))]$ such that for almost all $t \in [0, T)$, the following equation is true:

$$\frac{dx_i(t)}{dt} = \sum_{j=1,\, j\neq i}^{N} a_{ij}(\gamma_j(t) - \gamma_i(t)), \quad i = 1, \ldots, N. \tag{3.3}$$

Any function $\gamma$ as in (3.3) is called an *output* function associated to the solution $x$.

*Remark 3.2* Due to the introduction of the quantization effect, complete consensus cannot be ensured by the proposed protocol, but only *practical consensus* can be achieved, as discussed in [18] and [19].

### *3.1.2 Finite-Time Practical Consensus Under Quantization*

To prove main results of this section, the following lemma is needed:

**Lemma 3.3** *Suppose $x(t)$ be a Filippov solution to (3.2). Note that $\mathcal{N} = \{1, \ldots, N\}$. Let $M(t) = \max_{i\in\mathcal{N}}\{x_i(t)\}$ and $m(t) = \min_{i\in\mathcal{N}}\{x_i(t)\}$. Then, $M(t)$ is a non-increasing function for $t$, and $m(t)$ is a non-decreasing function for $t$.*

***Proof*** The proof is similar to the one of Lemma 3 in [21]. We omit here.

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ be the normalized left eigenvector of Laplacian matrix with respect to the zero eigenvalue satisfying $\sum_{i=1}^{N} \xi_i = 1$. It can be obtained that $\xi_i > 0$ from Lemma 1.6.

**Theorem 3.4** *Consider multi-agent network (3.2) with a strongly connected graph $\mathcal{G}$. The initial conditions associated with (3.2) are given as $x_i(0)$, $(i = 1, 2, \ldots, N)$. Let $k = \lfloor \frac{\sum_{i=1}^{N} \xi_i x_i(0)}{\mu\Delta} + \frac{1}{2} \rfloor$. Then $x_i(t)$ will converge to the set $\mathcal{D} = [(k - \frac{1}{2})\mu\Delta,\ (k + \frac{1}{2})\mu\Delta]$ for any $i \in \mathcal{N}$ in a finite time.*

***Proof*** The proof of Theorem 3.4 is divided into two parts.

**Part (I)** We shall take three steps to prove that each agent in the network will converge to an interval $[(k - \frac{1}{2})\mu\Delta,\ (k + \frac{1}{2})\mu\Delta]$ in finite time.

Step 1. Consider the Lyapunov functional as

$$V(t) = \sum_{i=1}^{N} \xi_i \int_0^{x_i(t)} q_\mu(s) ds. \tag{3.4}$$

Note that $cq_\mu(c) \geq 0$ for any $c \in \mathbb{R}$, we have $V(t) \geq 0$.
In addition, for $p_i(s) = \int_0^s q_\mu(u) du$, we have $\partial p_i(s) = \{v \in \mathbb{R} : q_\mu^-(s) \leq v \leq q_\mu^+(s)\}$. Based on the chain rule (Lemma 1.19), $V(t)$ is differentiable for a.e. $t \geq 0$. Differentiating $V(t)$ along the solution of (3.3) gives that

$$\begin{aligned}\frac{dV(t)}{dt} &= \sum_{i=1}^{N} \xi_i \gamma_i(t) \sum_{j=1,\, j\neq i}^{N} a_{ij}[\gamma_j(t) - \gamma_i(t)] \\ &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} \xi_i a_{ij}[2\gamma_i(t)\gamma_j(t) - 2\gamma_i^2(t)]. \end{aligned}\tag{3.5}$$

Notice that

$$\begin{aligned}\sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} \xi_i a_{ij} \gamma_i^2(t) &= \sum_{i=1}^{N} (-a_{ii}) \xi_i \gamma_i^2(t) = \sum_{j=1}^{N} (-a_{jj}) \xi_j \gamma_j^2(t) \\ &= \sum_{j=1}^{N} \sum_{i=1,\, i\neq j}^{N} \xi_i a_{ij} \gamma_j^2(t) = \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} \xi_i a_{ij} \gamma_j^2(t), \end{aligned}\tag{3.6}$$

we have

$$\frac{dV(t)}{dt} = -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} \xi_i a_{ij} (\gamma_i(t) - \gamma_j(t))^2. \tag{3.7}$$

Step 2. Let $\Phi = \{x(t) \in \mathbb{R}^N : |\gamma_i(t) - \gamma_j(t)| < \frac{\mu}{N+1}, \forall i, j \in \mathcal{N}, i \neq j, a_{ij} \neq 0\}$. We claim that the agents in the network converge to the set $\Phi$ in finite time. Let $J = \{t \geq 0 : x(t) \notin \Phi\}$. For $x(t) \in \mathbb{R}^N$ and $t \in J$, there exist $i, j \in \{1, 2, \ldots, N\}$, $i \neq j$ and $a_{ij} \neq 0$ such that $|\gamma_i(t) - \gamma_j(t)| \geq \frac{\mu}{N+1}$. Hence, for $a.e.\, t \in J$, it holds that

$$\begin{aligned}\frac{dV(t)}{dt} &\leq -\frac{1}{2}\xi_i a_{ij}\left(\frac{\mu}{N+1}\right)^2\\ &\leq -\frac{1}{2}\zeta\mu^2, \end{aligned} \tag{3.8}$$

where $\zeta = \min_{1\leq i, j\leq N, a_{ij}>0}\{(\frac{1}{N+1})^2 a_{ij}\xi_i\}$. Further, we can obtain

$$0 \leq V(t) \leq V(0) - \frac{1}{2}\zeta\mu^2 t,\ t \geq 0. \tag{3.9}$$

It follows from $V(0) \geq 0$ that (3.9) holds if and only if $t \leq \frac{2V(0)}{\zeta\mu^2}$. Therefore, $x(t)$ will arrive to the set $\Phi$ in finite time.

Step 3. We shall prove that there exists $k \in \mathbb{Z}$ such that $x_i(t)$ will converge to the interval $[(k - \frac{1}{2})\mu\Delta, (k + \frac{1}{2})\mu\Delta]$ in finite time for every $i \in \mathcal{N}$.

Note that $\gamma_i(t) \in \mathcal{K}[q_\mu(x_i(t))]$ and $\gamma_j(t) \in \mathcal{K}[q_\mu(x_j(t)]$, we can get that there exists $k_{ij} \in \mathbb{Z}$ such that $x_i(t)$ and $x_j(t)$ belong to the interval $[(k_{ij} - \frac{1}{2})\mu\Delta, (k_{ij} + \frac{1}{2})\mu\Delta]$ if $|\gamma_i(t) - \gamma_j(t)| < \frac{\mu}{N+1}$. Hence, based on the proof of Step 2, there exists a $T_0 \geq 0$ such that $\forall i, j \in \mathcal{N}, i \neq j, a_{ij} \neq 0$, $x_i(T_0)$ and $x_j(T_0)$ belong to the interval $[(k_{ij} - \frac{1}{2})\mu\Delta, (k_{ij} + \frac{1}{2})\mu\Delta]$. Due to the network being strongly connected, there exists a $k \in \mathbb{Z}$ such that $k_{ij} = k$.

It follows from Lemma 3.3 that $x_i(t) \in [(k - \frac{1}{2})\mu\Delta, (k + \frac{1}{2})\mu\Delta]$ for $\forall i \in \mathcal{N}, t \geq T_0$.

**Part (II)** Estimate the value of $k$.

Up till now, we have proved the first part of the Theorem 3.4. Next, we shall give the value of $k$ which is shown to be dependent on the initial values of the multi-agent network. Let $\eta(t) = \sum_{i=1}^N \xi_i x_i(t)$. We can calculate the derivative of $\eta(t)$ as follows:

$$\begin{aligned}\dot{\eta}(t) &= \sum_{i=1}^N \xi_i \sum_{j=1, j\neq i}^N a_{ij}[\gamma_j(t) - \gamma_i(t)]\\ &= \sum_{i=1, i\neq j}^N \xi_i a_{ij} \sum_{j=1}^N \gamma_j(t) - \sum_{i=1}^N \xi_i\gamma_i(t) \sum_{j=1, j\neq i}^N a_{ij}\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{N} \xi_j a_{jj} \gamma_j(t) + \sum_{i=1}^{N} \xi_i a_{ii} \gamma_i(t) \\
&= 0.
\end{aligned}
$$

Due to $\dot{\eta}(t) = 0$ for a.e. $t \in [0, \infty)$ and the continuity of $\eta(t)$, it can be easily obtained that $\eta(t)$ is a constant. That is, $\eta(t) = \eta(0) = \sum_{i=1}^{N} \xi_i x_i(0)$.

Next, we need to estimate the value of $k$. Let $\mathcal{D} = [(k - \frac{1}{2})\mu\Delta,\ (k + \frac{1}{2})\mu\Delta]$, we have proved that there exists a $T_0$ such that $x_i(t) \in \mathcal{D},\ \ \forall t \geq T_0,\ \ \forall i \in \mathcal{N}$. It follows from $\sum_{i=1}^{N} \xi_i = 1$ that

$$
\sum_{i=1}^{N} \xi_i x_i(t) \in \mathcal{D}, \ \ \forall t \geq T_0.
$$

Thus,

$$
\sum_{i=1}^{N} \xi_i x_i(0) \in \left[ \left(k - \frac{1}{2}\right) \mu\Delta,\ \left(k + \frac{1}{2}\right) \mu\Delta \right].
$$

We consider the following two cases:

Case 1: If there exists a $k_0 \in \mathbb{Z}$ such that $\sum_{i=1}^{N} \xi_i x_i(0) = (k_0 - \frac{1}{2})\mu\Delta$, then $k = k_0$ or $k_0 - 1$. Since

$$
x_i(t) \in \mathcal{D}, \ \ \forall t \geq T_0, \ \ \forall i \in \mathcal{N},
$$

we have

$$
x_i(t) = \left(k_0 - \frac{1}{2}\right) \mu\Delta, \ \ \forall t \geq T_0.
$$

In this case, we can select $k = k_0$. That is,

$$
k = \frac{1}{\mu\Delta} \sum_{i=1}^{N} \xi_i x_i(0) + \frac{1}{2} = \left\lfloor \frac{1}{\mu\Delta} \sum_{i=1}^{N} \xi_i x_i(0) + \frac{1}{2} \right\rfloor.
$$

Case 2: If $\sum_{i=1}^{N} \xi_i x_i(0) \neq (k_0 - \frac{1}{2})\mu\Delta$ for any $k \in \mathbb{Z}$, then,

$$
\sum_{i=1}^{N} \xi_i x_i(0) \in \left( \left(k - \frac{1}{2}\right) \mu\Delta,\ \left(k + \frac{1}{2}\right) \mu\Delta \right).
$$

Hence,

$$k = \left\lfloor \frac{1}{\mu\varDelta} \sum_{i=1}^{N} \xi_i x_i(0) + \frac{1}{2} \right\rfloor.$$

Therefore, $k = \lfloor \frac{1}{\mu\varDelta} \sum_{i=1}^{N} \xi_i x_i(0) + \frac{1}{2} \rfloor$. This completes the proof of this theorem.

*Remark 3.5* As discussed in Theorem 2 and Proposition 4 of [18], the practical consensus results for model (3.2) with $\varDelta = 1$ are investigated via LaSalle invariance principle of differential inclusions. While in this section, using different methods, we extend the previous results from the following three aspects:

- We do not assume the network is undirected or balanced.
- We show that the Filippov solutions of (3.2) reach interval $\mathcal{D} = [(k - \frac{1}{2})\mu,\ (k + \frac{1}{2})\mu]$ in a finite time even if $x_{ave}(0) = \frac{1}{N} \sum_{i=1}^{N} x_i(0) = (k_0 + \frac{1}{2})\mu$ for some $k_0 \in \mathbb{Z}$.
- We present an explicit relationship between the practical consensus set and initial conditions.

**Corollary 3.6** *Consider multi-agent network (3.2) with a strongly connected graph $\mathcal{G}$. The initial conditions associated with (3.2) are given as $x_i(0)$, $(i = 1, 2, \ldots, N)$. Let $k = \lfloor \frac{1}{\mu\varDelta} \sum_{i=1}^{N} \xi_i x_i(0) + \frac{1}{2} \rfloor$. Then $x_i(t)$ will converge to the interval $\varOmega = [\sum_{i=1}^{N} \xi_i x_i(0) - \mu\varDelta,\ \sum_{i=1}^{N} \xi_i x_i(0) + \mu\varDelta]$ in a finite time.*

***Proof*** According to Theorem 3.4, $x_i(t)$ converges to the interval $[(k - \frac{1}{2})\mu\varDelta,\ (k + \frac{1}{2})\mu\varDelta]$ in a finite time, where $k = \lfloor \frac{1}{\mu\varDelta} \sum_{i=1}^{N} \xi_i x_i(0) + \frac{1}{2} \rfloor$. It follows from $\sum_{i=1}^{N} \xi_i x_i(0) - \frac{1}{2}\mu\varDelta \leq k\mu\varDelta \leq \sum_{i=1}^{N} \xi_i x_i(0) + \frac{1}{2}\mu\varDelta$ that $x(t)$ will converge to the interval $\varOmega = [\sum_{i=1}^{N} \xi_i x_i(0) - \mu\varDelta,\ \sum_{i=1}^{N} \xi_i x_i(0) + \mu\varDelta]$ in a finite time.

*Remark 3.7* From Corollary 3.6, how the initial condition of the agents and quantization parameter $\mu$ affect the practical consensus set $\varOmega$ can be observed explicitly. It is interesting to observe that the size of the practical consensus set can be made arbitrarily small by decreasing the quantization parameter $\mu$.

### *3.1.3 Numerical Example*

In this section, an example is given to illustrate the correctness of the theoretical results.

Consider multi-agent system (3.2) with five agents, where $\mu = 1$ and $\varDelta = 1$. The directed network topology is displayed in Fig. 3.1, and the weight of each edge is set as 1.

Figure 3.2 shows the state responses of (3.2) with the initial condition randomly chosen from $(-5, 5)$. It can be observed from Fig. 3.2 that the state of each agent converges to a practical consensus set in a finite time, which illustrates Theorem 3.4 very well.

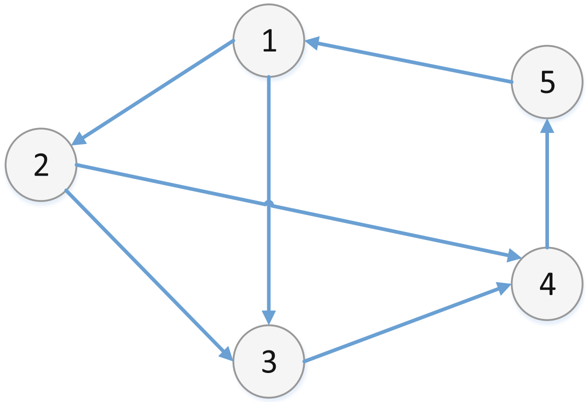


**Fig. 3.1** Network topology in example

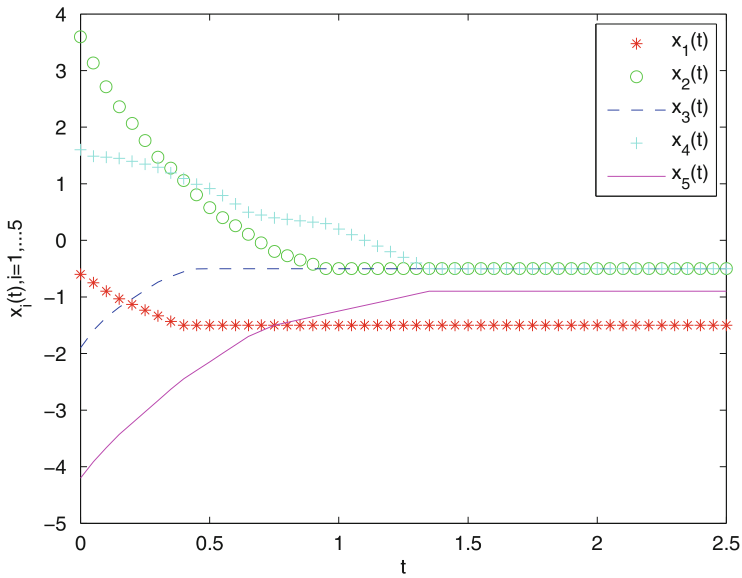


**Fig. 3.2** The states of the multi-agent network in example

## 3.2 Consensus with Hybrid Communication Constraints

In Sect. 3.1, multi-agent network consensus problem with quantization was studied. In this section, we investigate the consensus problem of multi-agent networks subject to quantization and time delays. It is worth noting that most of the aforementioned results about multi-agent consensus only consider one aspect of the communication constraints. Therefore, it is necessary to develop a new consensus protocol for multi-agent network with quantization and time delays. Moreover, from the viewpoint of both mathematics and engineering, it is worth noting that the quantization will lead to a system with time delays with no solutions in classical sense. Hence, considering solutions in a more general sense is necessary. This study focuses on solving these problems. The organization of the remaining part is given as follows. In Sect. 3.2.1, consensus protocol with quantization and time delays are formulated. In Sect. 3.2.2, the existence of Filippov solution is presented. In Sect. 3.2.3, consensus analysis of the proposed protocol is presented in detail. In Sect. 3.2.4, a numerical example is given to show the effectiveness of the theoretical results.

### *3.2.1 Model Description and Preliminaries*

Considering time delays as another very important communication constraint in the process of information exchange, we propose the following practical consensus protocol:

$$\frac{dx_i(t)}{dt} = \sum_{j\in\mathcal{N}_i} a_{ij}[q_\mu(x_j(t-\tau)) - q_\mu(x_i(t))], \quad i \in \mathcal{N}, \tag{3.10}$$

where $\tau$ is the communication delay from agent $j$ to agent $i$ and $q_\mu(z)$ denotes one-parameter family of uniform quantizers defined by $q_\mu(z) = \lfloor \frac{z}{\Delta\mu} + \frac{1}{2} \rfloor \mu$. Here $\mu$ and $\Delta$ are called the *quantization parameter* and *sensitivity* of the quantizer, respectively. For $x = (x_1, x_2, \ldots, x_N)^\top \in \mathbb{R}^N$, let $q_\mu(x) = (q_\mu(x_1), q_\mu(x_2), \ldots, q_\mu(x_N))^\top$. The initial conditions associated with (3.10) are given as

$$x_i(s) = \phi_i(s) \in \mathcal{C}([-\tau, 0], \mathbb{R}), \quad i \in \mathcal{N}.$$

*Remark 3.8* It should be pointed out that in many physical systems, time delay is ubiquitous at the moment of information exchanges among agents due to finite information transmission rate. Hence, it is more important and practical to consider time delay and quantization constraints simultaneously in real multi-agent networks. These two constraints are simultaneously considered in our model, which is more general than that of the previous results, such as [11, 18], and [19]. However, considering these two kinds of communication constraints together in (3.10) makes the consensus problem more difficult. Since the mathematical techniques, which were proposed and useful only for one of these two constraints, cannot be directly used for our model, relevant issues become more challenging.

*Remark 3.9* Generally, $\mu > 0$ is an adjustable parameter and $\Delta$ is a fixed parameter in the quantizer. Suppose $\mu = 1$, we can see that on the interval $[(k-\frac{1}{2})\Delta,\ (k+\frac{1}{2})\Delta)$ of length $\Delta$, the quantizer $q$ takes the value $k$. $\mathbb{R}$ is thereby divided into a number of quantization intervals, each with length $\varDelta$ and corresponding to a fixed value of $q(\cdot)$. For $\varDelta = 1$, the quantizer $q$ becomes the standard uniform round quantizer. For $\varDelta > 1$, the length of quantization interval will be larger than the standard uniform round quantizer, which means that the quantizer $q$ is less sensitive than the standard uniform round quantizer. For $\varDelta < 1$, the length of quantization interval will be smaller than the standard uniform round quantizer, which means that the quantizer $q$ is more sensitive than the standard uniform round quantizer. Hence, the quantizer $q$ is more general than the standard uniform round quantizer.

### 3.2.2 *The Existence of the Filippov Solution*

We know that system (3.10) may not have global solution in the sense of Carathéodory due to the discontinuity of the function $q(\cdot)$ [18]. Hence, we shall consider solutions in a more general sense, i.e., the Filippov solution of system (3.10). The concept of the Filippov solution to the differential equation (3.10) is given as follows:

**Definition 3.10** A function $x(t) : [-\tau, T) \to \mathbb{R}^N$ ($T$ might be $\infty$) is a solution in the sense of Filippov for the discontinuous system (3.10) on $[-\tau, T)$, if

1. $x(t)$ is continuous on $[-\tau, T)$ and absolutely continuous on $[0, T)$;
2. $x(t)$ satisfies that

$$\frac{dx_i(t)}{dt} \in \mathcal{K}\left[\sum_{j\in\mathcal{N}_i} a_{ij}(q_\mu(x_j(t-\tau)) - q_\mu(x_i(t)))\right], \quad i \in \mathcal{N}. \tag{3.11}$$

From Lemma 1.16, we have that

$$\begin{aligned}&\mathcal{K}\left[\sum_{j\in\mathcal{N}_i} a_{ij}(q_\mu(x_j(t-\tau)) - q_\mu(x_i(t)))\right]\\ &\subseteq \sum_{j\in\mathcal{N}_i} a_{ij}(\mathcal{K}[q_\mu(x_j(t-\tau))] - \mathcal{K}[q_\mu(x_i(t))]).\end{aligned} \tag{3.12}$$

According to Lemma 1.18, if $x(t)$ is the solution of system (3.10), then there exists a measurable function $\gamma(t) \in \mathcal{K}[q_\mu(x(t))]$ such that for a.e. $t \in [0, T)$, the following equation is true:

$$\frac{dx_i(t)}{dt} = \sum_{j\in\mathcal{N}_i} a_{ij}(\gamma_j(t-\tau) - \gamma_i(t)), \quad i \in \mathcal{N}. \tag{3.13}$$

Any function $\gamma$ as in (3.13) is called an *output function* associated to the solution $x$. Now, we shall present the definition of an initial value problem associated to (3.10).

**Definition 3.11** For any continuous function $\phi : [-\tau, 0] \to \mathbb{R}^N$ and any measurable selection $\psi : [-\tau, 0] \to \mathbb{R}^N$, such that $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$ for a.e. $s \in [-\tau, 0]$, an absolute continuous function $x(t) = x(t, \phi, \psi)$ is said to be a solution of the Cauchy problem for system (3.10) on $[0, T)$ with initial value $(\phi, \psi)$, if

$$\begin{cases} \dot{x}_i(t) = \sum_{j=1, j\neq i}^{N} a_{ij}(\gamma_j(t-\tau) - \gamma_i(t)) \\ \qquad for\ a.e.\ t \in [0, T), \quad i \in \mathcal{N}, \\ x(s) = \phi(s), \quad \forall s \in [-\tau, 0], \\ \gamma(s) = \psi(s) \quad a.e.\ s \in [-\tau, 0]. \end{cases} \tag{3.14}$$

Note that solution to the system (3.14) depends on the initial function $\phi$ and also on the selection of the output function $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$. Next, we shall study the existence of the global solution to the system (3.14).

**Theorem 3.12** *For any initial function $\phi$ and the selection of the output function $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$, there exists a global solution for system (3.14).*

***Proof*** The proof of Theorem 3.12 is divided into two parts:

**Part (I)** Existence of local solution.

Similar to the proof of Lemma 1 [21], one can conclude the existence of the solution defined on $[0, T)$ for system (3.14).

According to the theory of functional differential equations, a global solution can be guaranteed by the boundedness of the local solution (see, e.g., [22], p.46, Th. 3.2). Hence, we need to prove the boundedness of solution to system (3.14) in Part (II).

**Part (II)** The boundedness of the solution.

Suppose $x(t, \phi, \psi)$ is a solution of system (3.14). Denote $M(t) = \max_{i\in\mathcal{N}}\max_{\theta\in[-\tau,0]}\{x_i(t+\theta)\}$ and $m(t) = \min_{i\in\mathcal{N}}\min_{\theta\in[-\tau,0]}\{x_i(t+\theta)\}$. We claim that $M(t)$ is a non-increasing function for $t$ and $m(t)$ is a non-decreasing function for $t$. Next, we shall prove that $M(t)$ is a non-increasing function for $t$ by contradiction.

Suppose $M(t)$ is not a non-increasing function with respect to $t$, i.e., there exist $\bar{t}_0$ and $t_0$ such that $\bar{t}_0 > t_0 \geq 0$ and $M(\bar{t}_0) > M(t_0)$. Next, we will divide four steps to find the contradiction.

Step 1. Claim: there exists $t_0^* \in [t_0, \bar{t}_0)$ such that $M(t_0^*) = M(t_0)$ and $M(t) > M(t_0^*)$, $\forall t \in (t_0^*, \bar{t}_0]$.
Let $t_0^* = \sup(t \in [t_0, \bar{t}_0] : M(t) = M(t_0))$. Then, we have $M(t_0^*) = M(t_0)$ due to the continuity of the function $M(t)$. Next, we will prove $M(t) > M(t_0^*)$, $\forall t \in (t_0^*, \bar{t}_0]$ by contradiction. Suppose there exists $\tilde{t}_0 \in (t_0^*, \bar{t}_0]$ such that $M(\tilde{t}_0) \leq M(t_0^*)$. It follows from $M(\bar{t}_0) > M(t_0)$ and the intermediate value theorem of

continuous functions, there exists a $\bar{t}_1 \in (\tilde{t}_0, \bar{t}_0] \subseteq (t_0^*, \bar{t}_0]$ such that $M(\bar{t}_1) = M(t_0)$, which is contradictory with the definition of $t_0^*$.

Step 2. Claim: there exist $i_0 \in \mathcal{N}$ and $\delta > 0$ such that $M(t) = x_{i_0}(t + \theta(t))$ for $t \in [t_0^*, t_0^* + \delta)$, where $\theta(t) \in [-\tau, 0]$.

Let $H_i(t) = \max_{\theta \in [-\tau, 0]}\{x_i(t + \theta)\} = x_i(t + \theta(t))$. Then, using the definition of continuity, it can be easily proved that $H_i(t)$ is continuous with respect to $t$. It follows from $M(t) = \max_{i \in \mathcal{N}} \max_{\theta \in [-\tau, 0]}\{x_i(t + \theta)\}$ that $M(t) = \max_{i \in \mathcal{N}}\{H_i(t)\}$. Hence, for $i \in \mathcal{N}$, there exist $i_0 \in \mathcal{N}$ and $\delta > 0$ such that $M(t) = H_{i_0}(t)$ for $t \in [t_0^*, t_0^* + \delta)$, i.e., $M(t) = x_{i_0}(t + \theta(t))$ for $t \in [t_0^*, t_0^* + \delta)$, where $\theta(t) \in [-\tau, 0]$.

Step 3. Claim: $\theta(t_0^*) = 0$.

Assume by contradiction that $\theta(t_0^*) \neq 0$, then, we have $M(t_0^*) = x_{i_0}(t_0^* + \theta(t_0^*)) > x_{i_0}(t_0^*)$. Let $\varrho(t) = x_{i_0}(t_0^* + \theta(t_0^*)) - x_{i_0}(t)$. Then, we have that $\varrho(t_0^*) > 0$. According to the continuity of the function $\varrho(t)$, there exists $\delta_0 \in (0, \delta)$ such that $x_{i_0}(t_0^* + \theta(t_0^*)) > x_{i_0}(t)$ for $t \in (t_0^*, t_0^* + \delta_0)$. Hence, $M(t) \leq M(t_0^*)$ for $t \in (t_0^*, t_0^* + \delta_0)$, which is a contradiction with the Claim of Step 1. Thus, $\theta(t_0^*) = 0$.

Step 4. Claim: $M(t)$ is a non-increasing function for $t$. Here, two cases are divided as follows:

Case 1: If $x_{i_0}(t_0^*) \neq (k + \frac{1}{2})\mu\Delta$ for any $k \in \mathbb{Z}$, the continuity of $x_{i_0}(t)$ implies that there exists $\delta_1 < \delta_0$ such that $x_{i_0}(t) \neq (k + \frac{1}{2})\mu\Delta$ for any $k \in \mathbb{Z}$ and $\gamma_{i_0}(t) = \gamma_{i_0}(t_0^*) = q_\mu(x_{i_0}(t_0^*))$ for any $t \in (t_0^*, t_0^* + \delta_1)$. Due to $M(t_0^*) = x_{i_0}(t_0^*)$, we have $x_j(t_0^* - \tau) \leq x_{i_0}(t_0^*)$ for any $j \in \mathcal{N}$. Since $\gamma_j(t) \in \mathcal{K}[q_\mu(x_j(t))]$ and $q_\mu(\cdot)$ is a non-decreasing function, there exists a $\delta_2 \in (0, \delta_1)$ such that $\gamma_j(t - \tau) \leq \gamma_{i_0}(t)$ for any $t \in (t_0^*, t_0^* + \delta_2)$. As $\dot{x}_{i_0}(t) = \sum_{j=1, j \neq i_0}^N a_{ij}(\gamma_j(t - \tau) - \gamma_{i_0}(t))$, we get that $\dot{x}_{i_0}(t) \leq 0$ for a.e. $t \in (t_0^*, t_0^* + \delta_2)$. Additionally, since $x_{i_0}(t + \theta(t)) = M(t) > M(t_0^*) = x_{i_0}(t_0^*)$, $\forall t \in (t_0^*, t_0^* + \delta_2)$, there exists a $t_0^{**} \in (t_0^*, t_0^* + \delta_2)$ such that $x_{i_0}(t_0^{**}) > x_{i_0}(t_0^*)$. Then, there must exist a subset $\mathcal{I}_1$ of $(t_0^*, t_0^{**})$ ($\subseteq (t_0^*, t_0^* + \delta_2)$) such that $\mathcal{I}_1$ has a positive measure and $\dot{x}_{i_0}(t) > 0$ for a.e. $t \in \mathcal{I}_1$, which is contradict with $\dot{x}_{i_0}(t) \leq 0$ for a.e. $t \in (t_0^*, t_0^* + \delta_2)$.

Case 2: If $x_{i_0}(t_0^*) = (k + \frac{1}{2})\mu\Delta$ for some $k \in \mathbb{Z}$, according to the previous analysis, for any $\bar{\delta} < \delta_0$, there exists $t_1 \in (t_0^*, t_0^* + \bar{\delta}]$ such that $x_{i_0}(t_1) > x_{i_0}(t_0^*)$. Let $t_1^* = \sup(t \in [t_0^*, t_1] : x_{i_0}(t) = x_{i_0}(t_0^*))$. Due to the continuity of function $x_{i_0}(t)$, we have $t_0^* \leq t_1^* < t_1$ and $x_{i_0}(t_1^*) = x_{i_0}(t_0^*)$. Hence, for any $t \in (t_1^*, t_1]$, we have $x_{i_0}(t) \geq M(t_0^*)$ and $x_{i_0}(t) \neq (k + \frac{1}{2})\mu\Delta$ for any $k \in \mathbb{Z}$. Similar to the proof of Case 1, $\dot{x}_{i_0}(t) \leq 0$ for a.e. $t \in (t_1^*, t_1]$. Due to $x_{i_0}(t_1) > x_{i_0}(t_1^*)$, there must exist a subset $\mathcal{I}_2$ of $(t_1^*, t_1]$ such that $\mathcal{I}_2$ has a positive measure and $\dot{x}_{i_0}(t) > 0$ for a.e. $t \in \mathcal{I}_2$, which is contradict with $\dot{x}_{i_0}(t) \leq 0$ for a.e. $t \in (t_1^*, t_1]$. Therefore, $M(t)$ is a non-increasing function for $t$. Similarly, $m(t)$ can also be proved to be a non-decreasing function for $t$ using the same approach. Hence, for any $i \in \mathcal{N}$ one has $m(0) \leq x_i(t) \leq M(0)$, i.e., the solution $x(t)$ is bounded. This completes the proof of this theorem.

### 3.2.3 Practical Consensus Under Quantization and Time Delay

In this section, we shall study the consensus result with protocol (3.10). The initial conditions associated with (3.10) are given as $x_i(s) = \phi_i(s) \in \mathcal{C}([-\tau, 0], \mathbb{R})$, $i \in \mathcal{N}$. The Filippov solution of system (3.10) is defined in (3.14), and $\psi_j(s)$, $s \in [-\tau, 0]$ is the initial condition of measurable selection of $\gamma_j(s)$. Let

$$\eta(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0) + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_{-\tau}^{0}\psi_j(s)ds \tag{3.15}$$

and

$$A = \mu\left(\Delta - \frac{\tau}{N}\sum_{i=1}^{N} a_{ii}\right), \tag{3.16}$$

where $\mu$ and $\Delta$ are quantization parameter and sensitivity of the quantizer.

**Theorem 3.13** *Consider the multi-agent network (3.10) with communication topology that is defined by an undirected, connected graph G. Then, for any finite communication delay $\tau$, each agent in the network will converge to the set of $\Omega_1 = [(k-\frac{1}{2})\mu\Delta,\ (k+\frac{1}{2})\mu\Delta]$ asymptotically, where $k = \lfloor\frac{\eta(0)}{A}\rfloor$ or $\lfloor\frac{\eta(0)}{A}\rfloor + 1$, $\eta(0)$ and A are defined in (3.15) and (3.16), respectively.*

***Proof*** The proof of Theorem 3.13 is divided into two parts.

**Part (I)** We shall take three steps to prove that each agent in the network will converge to an interval $[(k-\frac{1}{2})\mu\Delta,\ (k+\frac{1}{2})\mu\Delta]$ asymptotically.

Step 1. We shall prove that for any $\epsilon > 0$ and $i \in \mathcal{N}$, there exists $T_0$, such that

$$|x_i(t+\vartheta) - x_i(t)| \leq \epsilon, \quad \forall t \geq T_0,\ \forall\vartheta \in [0,\ \tau]. \tag{3.17}$$

Consider the function

$$V(t) = V_1(t) + V_2(t), \tag{3.18}$$

where

$$V_1(t) = \sum_{i=1}^{N}\int_{0}^{x_i(t)} q_\mu(s)ds, \tag{3.19}$$

and

$$V_2(t) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1,\, j \neq i}^{N} \int_{t-\tau}^{t} a_{ij} \gamma_j^2(s) ds. \tag{3.20}$$

Note that $cq_\mu(c) \geq 0$ for any $c \in \mathbb{R}$, we have $V_1(t) \geq 0$ and $V_2(t) \geq 0$.
Notice that for $p_i(s) = \int_0^s q_\mu(u)du$, we have

$$\partial_c p_i(s) = \{v \in \mathbb{R} : q_\mu^-(s) \leq v \leq q_\mu^+(s)\}, \tag{3.21}$$

where $q_\mu^+(s)$ and $q_\mu^-(s)$ denote the right and left limits of the function $q_\mu$ at the point $s$. Based on the Lemma 1.19, $V_1(t)$ is differentiable for a.e. $t \geq 0$ and

$$\begin{aligned} \frac{dV_1(t)}{dt} &= \sum_{i=1}^{N} \gamma_i(t) \sum_{j=1,\, j \neq i}^{N} a_{ij}[\gamma_j(t-\tau) - \gamma_i(t)] \\ &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1,\, j \neq i}^{N} a_{ij}[2\gamma_i(t)\gamma_j(t-\tau) - 2\gamma_i^2(t)]. \end{aligned} \tag{3.22}$$

Since $\gamma_j(t) \in \mathcal{K}[q_\mu(x_j(t))]$, $\forall j \in \mathcal{N}$, we have $\gamma_j(t)$ is local integrable. Hence, $V_2(t)$ is differentiable for a.e. $t \geq 0$ and

$$\begin{aligned} \frac{dV_2(t)}{dt} &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1,\, j \neq i}^{N} a_{ij}[\gamma_j^2(t) - \gamma_j^2(t-\tau)] \\ &= \frac{1}{2} \sum_{j=1}^{N} \sum_{i=1,\, i \neq j}^{N} a_{ji}\gamma_i^2(t) - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1,\, j \neq i}^{N} a_{ij}\gamma_j^2(t-\tau) \\ &= \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1,\, j \neq i}^{N} a_{ij}[\gamma_i^2(t) - \gamma_j^2(t-\tau)]. \end{aligned} \tag{3.23}$$

Combining (3.22) and (3.23) gives that

$$\begin{aligned} \frac{dV(t)}{dt} &= \frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} \\ &= -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1,\, j \neq i}^{N} a_{ij}(\gamma_i(t) - \gamma_j(t-\tau))^2 \\ &\leq 0. \end{aligned} \tag{3.24}$$

Hence, $V(t)$ is non-increasing for $t$. Further, $V(t) \geq 0$ gives that $\lim_{t\to+\infty} V(t)$ exists. Let $\bar{a} = \max_{1\leq i<j\leq N,\, a_{ij}>0}\{a_{ij}\}$. Then, we have that for any $\epsilon > 0$ and $i,\ j \in \mathcal{N}$, there exists $T_0$ such that for $\forall t \geq T_0,\ \vartheta \in [0,\ \tau]$,

$$\begin{aligned}
\frac{\epsilon^2}{2N\bar{a}\tau} &\geq |V(t+\vartheta) - V(t)| \\
&= \left|\int_t^{t+\vartheta} \dot{V}(s)ds\right| \\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_t^{t+\vartheta}(\gamma_i(s) - \gamma_j(s-\tau))^2 ds,
\end{aligned} \tag{3.25}$$

which implies that for any $i \in \mathcal{N}$ and $t \geq T_0,\ \vartheta \in [0,\ \tau]$, we have

$$\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_t^{t+\vartheta}(\gamma_i(s) - \gamma_j(s-\tau))^2 ds \leq \frac{\epsilon^2}{N\bar{a}\tau}. \tag{3.26}$$

It follows from Jensen's inequality (see [23]) that

$$\begin{aligned}
\epsilon^2 &\geq \tau N \sum_{j=1,\, j\neq i}^{N} a_{ij}\bar{a}\int_t^{t+\vartheta}(\gamma_i(s) - \gamma_j(s-\tau))^2 ds \\
&\geq \vartheta N \sum_{j=1,\, j\neq i}^{N} a_{ij}\bar{a}\int_t^{t+\vartheta}(\gamma_i(s) - \gamma_j(s-\tau))^2 ds \\
&\geq N \sum_{j=1,\, j\neq i}^{N} a_{ij}^2\left(\int_t^{t+\vartheta}|\gamma_j(s-\tau) - \gamma_i(s)|ds\right)^2 \\
&\geq \left(\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_t^{t+\vartheta}|\gamma_j(s-\tau) - \gamma_i(s)|ds\right)^2 \\
&\geq \left|\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_t^{t+\vartheta}(\gamma_j(s-\tau) - \gamma_i(s))ds\right|^2.
\end{aligned} \tag{3.27}$$

Hence, for any $i \in \mathcal{N}$ and $t \geq T_0,\ \vartheta \in [0,\ \tau]$, one has

$$\left|\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_t^{t+\vartheta}(\gamma_j(s-\tau) - \gamma_i(s))ds\right| \leq \epsilon.$$

It follows from (3.14) that

$$
\begin{aligned}
|x_i(t+\vartheta) - x_i(t)| &= \left| \int_t^{t+\vartheta} \dot{x}_i(s)ds \right| \\
&= \left| \sum_{j=1,\, j\neq i}^{N} a_{ij} \int_t^{t+\vartheta} (\gamma_j(s-\tau) - \gamma_i(s))ds \right| \\
&\leq \epsilon. \qquad (3.28)
\end{aligned}
$$

Thus, for any $\epsilon > 0$ and $i \in \mathcal{N}$, there exists $T_0$ such that for $\forall t \geq T_0,\ \vartheta \in [0,\ \tau]$,

$$
|x_i(t+\vartheta) - x_i(t)| \leq \epsilon. \qquad (3.29)
$$

Step 2. Let $\Phi = \{x(t+\theta) \in \mathcal{C}([-\tau, 0];\ \mathbb{R}^N) : |\gamma_i(t) - \gamma_j(t-\tau)| < \frac{\mu}{N+1},\ \forall i,\ j \in \mathcal{N},\ i \neq j,\ a_{ij} \neq 0\}$. We claim that for arbitrary fixed $t_0 \geq 0$, there exists $\bar{t}_0 \geq t_0$ such that the agents in the network will go into the set of $\Phi$ at time $\bar{t}_0$.
Let $J = \{t \geq t_0 :\ x(t+\theta) \notin \Phi\}$. For $x(t+\theta) \in \mathcal{C}([-\tau, 0];\ \mathbb{R}^N)$ and $t \in J$, there exist $i,\ j \in \mathcal{N},\ i \neq j$ and $a_{ij} \neq 0$ such that

$$
|\gamma_i(t) - \gamma_j(t-\tau)| \geq \frac{\mu}{N+1}. \qquad (3.30)
$$

Hence, for $a.e.\, t \in J$, one has

$$
\begin{aligned}
\dot{V}(t) &\leq -\frac{1}{2} a_{ij} \left( \frac{\mu}{N+1} \right)^2 \\
&\leq -\frac{1}{2} \varsigma \mu^2, \qquad (3.31)
\end{aligned}
$$

where $\varsigma = \min\limits_{i,\, j \in \mathcal{N},\, a_{ij} > 0} \{(\frac{1}{N+1})^2 a_{ij}\}$. Next, we will prove the claim of Step 2 by contradiction.
Suppose that $t \in J$ for any $t \geq t_0$. Then, inequality (3.31) implies that

$$
V(t) - V(t_0) \leq -\frac{1}{2} \varsigma \mu^2 (t - t_0),\ t \geq t_0. \qquad (3.32)
$$

For $t > \frac{2V(t_0)}{\varsigma \mu^2} + t_0$, it follows from inequality (3.32) that $V(t) < 0$, which is a contradiction to the definition of $V(t)$. Therefore, for arbitrary $t_0 \geq 0$, there exists $\bar{t}_0 \geq t_0$ such that the agents in the network will go into the set of $\Phi$ at time $\bar{t}_0$.

Step 3. We shall prove that there exists $k \in \mathbb{Z}$ such that $x_i(t)$ converges to the set of $[(k-\frac{1}{2})\mu\Delta,\ (k+\frac{1}{2})\mu\Delta]$ asymptotically for every $i \in \mathcal{N}$.

It follows from $q_\mu(x_i(t)) = \lfloor \frac{x_i(t)}{\Delta\mu} + \frac{1}{2} \rfloor \mu$ and the definition of set value function $\mathcal{K}$ (defined in Sect. 1.3.4) that

$$\mathcal{K}[q_\mu(x_i(t))] = \begin{cases} k_0\mu, \text{ if } x_i(t) \in \left(\left(k_0 - \frac{1}{2}\right)\mu\Delta, \left(k_0 + \frac{1}{2}\right)\mu\Delta\right) \\ \qquad \text{for some } k_0 \in \mathbb{Z}, \\ [k_0\mu,\ (k_0+1)\mu], \text{ if } x_i(t) = \left(k_0 + \frac{1}{2}\right)\mu\Delta \\ \qquad \text{for some } k_0 \in \mathbb{Z}. \end{cases} \tag{3.33}$$

In Step 2, we have proved that for arbitrary fixed $t_0 \geq 0$, there exists $\bar{t}_0 \geq t_0$ such that the agents in the network will go into the set of $\Phi$ at time $\bar{t}_0$. It means that $\forall i,\ j \in \mathcal{N},\ i \neq j,\ a_{ij} \neq 0$,

$$|\gamma_i(\bar{t}_0) - \gamma_j(\bar{t}_0 - \tau)| < \frac{\mu}{N+1}. \tag{3.34}$$

Note that $\gamma_i(\bar{t}_0) \in \mathcal{K}[q_\mu(x_i(\bar{t}_0))]$ and $\gamma_j(\bar{t}_0 - \tau) \in \mathcal{K}[q_\mu(x_j(\bar{t}_0 - \tau))]$. Next, we will prove that **Claim I:** there exists $k_{ij} \in \mathbb{Z}$ such that

$$x_i(\bar{t}_0) \in \left[\left(k_{ij} - \frac{1}{2}\right)\mu\Delta,\ \left(k_{ij} + \frac{1}{2}\right)\mu\Delta\right] \tag{3.35}$$

and

$$x_j(\bar{t}_0 - \tau) \in \left[\left(k_{ij} - \frac{1}{2}\right)\mu\Delta,\ \left(k_{ij} + \frac{1}{2}\right)\mu\Delta\right]. \tag{3.36}$$

In order to show the relationship of $x_i(\bar{t}_0)$ and $x_j(\bar{t}_0 - \tau)$ for $i$ and $j$ satisfying $a_{ij} \neq 0$ and $|\gamma_i(\bar{t}_0) - \gamma_j(\bar{t}_0 - \tau)| < \frac{\mu}{N+1}$, we consider the following four cases, which cover all possibilities.

Case 1: There exist $k_1,\ k_2 \in \mathbb{Z}$ such that

$$\mathcal{K}[q_\mu(x_i(\bar{t}_0))] = k_1\mu \tag{3.37}$$

and

$$\mathcal{K}[q_\mu(x_j(\bar{t}_0 - \tau)] = k_2\mu. \tag{3.38}$$

Then we have $k_1 = k_2$, which implies that $x_i(\bar{t}_0)$ and $x_j(\bar{t}_0 - \tau)$ belong to the interval of $((k_1 - \frac{1}{2})\mu\Delta,\ (k_1 + \frac{1}{2})\mu\Delta)$.

Case 2: There exist $k_1,\ k_2 \in \mathbb{Z}$ such that

$$\mathcal{K}[q_\mu(x_i(\bar{t}_0))] = k_1\mu \tag{3.39}$$

and

$$\mathcal{K}[q_\mu(x_j(\bar{t}_0 - \tau)] = [k_2\mu,\ (k_2 + 1)\mu]. \tag{3.40}$$

It means that

$$x_i(\bar{t}_0) \in \left(\left(k_1 - \frac{1}{2}\right)\mu\Delta,\ \left(k_1 + \frac{1}{2}\right)\mu\Delta\right) \tag{3.41}$$

and

$$x_j(\bar{t}_0 - \tau) = \left(k_2 + \frac{1}{2}\right)\mu\Delta. \tag{3.42}$$

Hence, the inequality

$$|\gamma_i(\bar{t}_0) - \gamma_j(\bar{t}_0 - \tau)| < \frac{\mu}{N + 1} \tag{3.43}$$

implies that $k_1 = k_2$ or $k_1 = k_2 + 1$. Therefore, we can conclude that $x_i(\bar{t}_0)$ and $x_j(\bar{t}_0 - \tau)$ belong to the interval of $[(k_2 - \frac{1}{2})\mu\Delta,\ (k_2 + \frac{1}{2})\mu\Delta]$.

Case 3: There exist $k_1,\ k_2 \in \mathbb{Z}$ such that

$$\mathcal{K}[q_\mu(x_i(\bar{t}_0))] = [k_1\mu,\ (k_1 + 1)\mu] \tag{3.44}$$

and

$$\mathcal{K}[q_\mu(x_j(\bar{t}_0 - \tau)] = k_2\mu. \tag{3.45}$$

Following similar analysis of Case 2 gives that $x_i(\bar{t}_0)$ and $x_j(\bar{t}_0 - \tau)$ belong to the interval of $[(k_1 - \frac{1}{2})\mu\Delta,\ (k_1 + \frac{1}{2})\mu\Delta]$.

Case 4: There exist $k_1,\ k_2 \in \mathbb{Z}$ such that

$$\mathcal{K}[q_\mu(x_i(\bar{t}_0))] = [k_1\mu,\ (k_1 + 1)\mu] \tag{3.46}$$

and

$$\mathcal{K}[q_\mu(x_j(\bar{t}_0 - \tau)] = [k_2\mu,\ (k_2 + 1)\mu]. \tag{3.47}$$

It means that

$$x_i(\bar{t}_0) = \left(k_1 + \frac{1}{2}\right)\mu\Delta \tag{3.48}$$

and

$$x_j(\bar{t}_0 - \tau) = \left(k_2 + \frac{1}{2}\right)\mu\Delta. \tag{3.49}$$

The inequality

$$|\gamma_i(\bar{t}_0) - \gamma_j(\bar{t}_0 - \tau)| < \frac{\mu}{N+1} \tag{3.50}$$

implies that $k_1 = k_2$ or $|k_1 - k_2| = 1$. Hence, there exists $k_{ij}(= k_1 \, or \, k_2) \in \mathbb{Z}$ such that $x_i(\bar{t}_0)$ and $x_j(\bar{t}_0 - \tau)$ belong to the interval of $[(k_{ij} - \frac{1}{2})\mu\Delta, \ (k_{ij} + \frac{1}{2})\mu\Delta]$.
Therefore, we can conclude that there exists $k_{ij}(= k_1 \, or \, k_2) \in \mathbb{Z}$ such that $x_i(\bar{t}_0)$ and $x_j(\bar{t}_0 - \tau)$ belong to the interval of $[(k_{ij} - \frac{1}{2})\mu\Delta, \ (k_{ij} + \frac{1}{2})\mu\Delta]$.

Fix an agent $i_0 \in \mathcal{N}$. Consider the neighbors of agent $i_0$, i.e., $\mathcal{N}_{i_0}$. Without loss of generality, we assume $i_1 \in \mathcal{N}_{i_0}$. **Claim I** implies that

$$x_{i_0}(\bar{t}_0) \in \left[\left(k_{i_0 i_1} - \frac{1}{2}\right)\mu\Delta, \ \left(k_{i_0 i_1} + \frac{1}{2}\right)\mu\Delta\right] \tag{3.51}$$

and

$$x_{i_1}(\bar{t}_0 - \tau) \in \left[\left(k_{i_0 i_1} - \frac{1}{2}\right)\mu\Delta, \ \left(k_{i_0 i_1} + \frac{1}{2}\right)\mu\Delta\right]. \tag{3.52}$$

Let $k_{i_0 i_1} = k$. Next, we can consider the neighbors of agent $i_1$ similarly. Assume $i_2 \in \mathcal{N}_{i_1}$. **Claim I** implies that

$$x_{i_2}(\bar{t}_0) \in \left[\left(k_{i_2 i_1} - \frac{1}{2}\right)\mu\Delta, \ \left(k_{i_2 i_1} + \frac{1}{2}\right)\mu\Delta\right] \tag{3.53}$$

and

$$x_{i_1}(\bar{t}_0 - \tau) \in \left[\left(k_{i_2 i_1} - \frac{1}{2}\right)\mu\Delta, \ \left(k_{i_2 i_1} + \frac{1}{2}\right)\mu\Delta\right]. \tag{3.54}$$

Since the network is undirected and connected, we have that for any $i' \in \mathcal{N}$, the minimum number of nodes (exclude $i_0$) is no more than $N - 1$ for the path from $i_0$ to $i'$. According to the claim of Step 2, we can conclude that for any $i' \in \mathcal{N} \backslash \{i_0, \ i_1\}$, at least one of the following statement holds:

1. $|\gamma_{i_0}(\bar{t}_0) - \gamma_{i'}(\bar{t}_0 - \tau)| < \frac{N-1}{N+1}\mu$ and $|\gamma_{i_1}(\bar{t}_0 - \tau) - \gamma_{i'}(\bar{t}_0 - \tau)| < \frac{N}{N+1}\mu$;
2. $|\gamma_{i_0}(\bar{t}_0) - \gamma_{i'}(\bar{t}_0)| < \frac{N-1}{N+1}\mu$ and $|\gamma_{i_1}(\bar{t}_0 - \tau) - \gamma_{i'}(\bar{t}_0)| < \frac{N}{N+1}\mu$.

Hence, for arbitrary $t_0 \geq 0$ there exists $\bar{t}_0$ such that, for any $i' \in \mathcal{N}$, either $x_{i'}(\bar{t}_0)$ or $x_{i'}(\bar{t}_0 - \tau)$ belongs to the set of $\Omega_1 = [(k - \frac{1}{2})\mu\Delta,\ (k + \frac{1}{2})\mu\Delta]$.

Denote $M(t) = \max_{i \in \mathcal{N}} \max_{\theta \in [-\tau, 0]} \{x_i(t+\theta)\}$ and $m(t) = \min_{i \in \mathcal{N}} \min_{\theta \in [-\tau, 0]} \{x_i(t+\theta)\}$. It follows from the proof of Theorem 3.12 that $M(t)$ is a non-increasing function for $t$, and $m(t)$ is a non-decreasing function for $t$. Select $t_0 = T_0 + \tau$. According to the definition of $M(t)$ and $m(t)$, there exist $i_M$, $\theta_M$, and $i_m$, $\theta_m$ such that

$$M(\bar{t}_0) = x_{i_M}(\bar{t}_0 + \theta_M) \tag{3.55}$$

and

$$m(\bar{t}_0) = x_{i_m}(\bar{t}_0 + \theta_m). \tag{3.56}$$

Without loss of generality, we can assume that $x_{i_M}(\bar{t}_0)$ and $x_{i_m}(\bar{t}_0 - \tau)$ belongs to the set of $\Omega_1 = [(k - \frac{1}{2})\mu\Delta,\ (k + \frac{1}{2})\mu\Delta]$.

It follows from (3.29) that

$$M(\bar{t}_0) - x_{i_M}(\bar{t}_0) \leq \epsilon \tag{3.57}$$

and

$$x_{i_m}(\bar{t}_0) - m(\bar{t}_0) \leq \epsilon. \tag{3.58}$$

Hence,

$$x_{i_m}(\bar{t}_0 - \tau) - \epsilon \leq m(\bar{t}_0) \leq M(\bar{t}_0) \leq x_{i_M}(\bar{t}_0) + \epsilon.$$

The non-increasing property of $M(t)$ and the non-decreasing property of $m(t)$ imply that for any $t \geq \bar{t}_0$,

$$\begin{aligned} x_{i_m}(\bar{t}_0 - \tau) - \epsilon \leq m(\bar{t}_0) \leq m(t) \leq M(t) \\ \leq M(\bar{t}_0) \leq x_{i_M}(\bar{t}_0) + \epsilon. \end{aligned} \tag{3.59}$$

Hence, for any $\epsilon > 0$ and $i \in \mathcal{N}$, we have

$$dist(x_i(t),\ \Omega_1) \leq \epsilon,\ \ \forall t \geq \bar{t}_0. \tag{3.60}$$

That is, for every $i \in \mathcal{N}$, $x_i(t)$ converges to the interval of $[(k - \frac{1}{2})\mu\Delta,\ (k + \frac{1}{2})\mu\Delta]$ asymptotically.

**Part (II)** Estimate the value of $k$.

Next, we shall give the value of $k$ which depends on the initial values of the multi-agent network. Let

$$\eta(t) = \frac{1}{N}\left(\sum_{i=1}^{N} x_i(t) + \sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_{t-\tau}^{t}\gamma_j(s)ds\right). \tag{3.61}$$

By some calculations, we obtain

$$\begin{aligned}
\dot{\eta}(t) &= \sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N}\frac{1}{N}a_{ij}[\gamma_j(t-\tau)-\gamma_i(t)] - \sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N}\frac{1}{N}a_{ij}[\gamma_j(t-\tau)-\gamma_j(t)] \\
&= -\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N}\frac{1}{N}a_{ij}\gamma_i(t) + \sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N}\frac{1}{N}a_{ij}\gamma_j(t) \\
&= -\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N}\frac{1}{N}a_{ij}\gamma_i(t) + \sum_{j=1}^{N}\sum_{i=1,\, i\neq j}^{N}\frac{1}{N}a_{ji}\gamma_i(t) \\
&= 0.
\end{aligned} \tag{3.62}$$

Due to the fact that $\dot{\eta}(t) = 0$ for a.e. $t \in [0, \infty)$ and the continuity of $\eta(t)$, $\eta(t)$ in (3.61) is a constant. That is,

$$\begin{aligned}
\eta(t) &= \eta(0) \\
&= \frac{1}{N}\left(\sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_{-\tau}^{0}\gamma_j(s)ds\right) \\
&= \frac{1}{N}\sum_{i=1}^{N} x_i(0) + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_{-\tau}^{0}\psi_j(s)ds.
\end{aligned} \tag{3.63}$$

$\eta(t)$ in (3.63) will be used in the rest of the proof. To estimate the value of $k$, let

$$\Omega_1 = \left[\left(k-\frac{1}{2}\right)\mu\Delta,\ \left(k+\frac{1}{2}\right)\mu\Delta\right]. \tag{3.64}$$

In Step 3, we have proved that

$$\lim_{t\to+\infty} dist(x_i(t),\ \Omega_1) = 0, \qquad \forall i \in \mathcal{N}.$$

Next, we will prove that

$$\lim_{t\to+\infty} dist\left(\sum_{i=1}^{N}\frac{1}{N}x_i(t),\ \Omega_1\right) = 0. \tag{3.65}$$

It follows from

$$\lim_{t\to+\infty} dist(x_i(t),\ \Omega_1) = 0,\ \forall i \in \mathcal{N} \tag{3.66}$$

that for any $\epsilon > 0$, there exists $T_i > 0$ such that, when $t > T_i$,

$$dist(x_i(t),\ \Omega_1) < \epsilon. \tag{3.67}$$

Let $T = \max_{i\in\mathcal{N}}\{T_i\} + 1$, it can be obtained that, when $t > T$,

$$dist(x_i(t),\ \Omega_1) < \epsilon,\ \forall i \in \mathcal{N}. \tag{3.68}$$

Since $\Omega_1 = [(k-\frac{1}{2})\mu\Delta,\ (k+\frac{1}{2})\mu\Delta]$ is a closed set, for arbitrary $t > T$ there exists a point $c_t^i \in \Omega_1$ such that

$$dist(x_i(t),\ c_t^i) = |x_i(t) - c_t^i| < \epsilon. \tag{3.69}$$

Note that $\Omega_1$ is a convex set, we have $\frac{1}{N}\sum_{i=1}^{N} c_t^i \in \Omega_1$. Hence,

$$dist\left(\sum_{i=1}^{N}\frac{1}{N}x_i(t),\ \frac{1}{N}\sum_{i=1}^{N}c_t^i\right) = \frac{1}{N}\left|\sum_{i=1}^{N}x_i(t) - \sum_{i=1}^{N}c_t^i\right| < \epsilon. \tag{3.70}$$

Therefore, for any $\epsilon > 0$, there exists $T > 0$ such that, when $t > T$,

$$dist\left(\sum_{i=1}^{N}\frac{1}{N}x_i(t),\ \Omega_1\right) < \epsilon. \tag{3.71}$$

That is,

$$\lim_{t\to+\infty} dist\left(\sum_{i=1}^{N}\frac{1}{N}x_i(t),\ \Omega_1\right) = 0. \tag{3.72}$$

Note that for large enough $t$,

$$\gamma_i(t) \in [(k-1)\mu,\ (k+1)\mu], \qquad \forall i \in \mathcal{N}. \tag{3.73}$$

It follows from (3.72) and (3.73) that

$$\begin{aligned}
\eta(0) &= \lim_{t\to+\infty} \eta(t) \\
&= \lim_{t\to+\infty} \left( \sum_{i=1}^{N} \frac{1}{N} x_i(t) + \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1,\, j\neq i}^{N} a_{ij} \int_{t-\tau}^{t} \gamma_j(s)ds \right) \\
&\in \Bigg[ \left(k - \frac{1}{2}\right) \mu\varDelta + \sum_{i=1}^{N} \frac{1}{N} (-a_{ii}\tau)(k-1)\mu, \\
&\qquad \left(k + \frac{1}{2}\right) \mu\varDelta + \sum_{i=1}^{N} \frac{1}{N} (-a_{ii}\tau)(k+1)\mu \Bigg] \\
&= \left[ kA - \left(A - \frac{1}{2}\mu\varDelta\right),\ kA + \left(A - \frac{1}{2}\mu\varDelta\right) \right],
\end{aligned} \tag{3.74}$$

where $A = \mu(\varDelta - \frac{\tau}{N}\sum_{i=1}^{N} a_{ii})$. Then, one obtains

$$\frac{\eta(0)}{A} \in \left[ k - \frac{A - \frac{1}{2}\mu\varDelta}{A},\ k + \frac{A - \frac{1}{2}\mu\varDelta}{A} \right], \tag{3.75}$$

which together with $\frac{1}{2} < \frac{A - \frac{1}{2}\mu\varDelta}{A} < 1$ gives that

$$k \in \left\{ \left\lfloor \frac{\eta(0)}{A} \right\rfloor,\ \left\lfloor \frac{\eta(0)}{A} \right\rfloor + 1 \right\}. \tag{3.76}$$

This completes the proof of this theorem.

*Remark 3.14* Since Theorem 3.13 is obtained based on the Filippov solution of the time-delay system (3.10), we cannot completely determine the value of $k$. In fact, we can overcome this problem by alternatively choosing the practical consensus set $\varOmega_2 = [(k - \frac{1}{2})\mu\varDelta,\ (k + \frac{3}{2})\mu\varDelta]$, where $k = \lfloor \frac{\eta(0)}{A} \rfloor$.

According to Theorem 3.13, the multi-agent network (3.10) achieves practical consensus asymptotically. The practical consensus set depends on the initial condition of the system and also on the initial condition of the selected output function $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$. However, in some cases, we need to find a different form of practical consensus set which is not related to the selection of the output function. In Theorem 3.15, we shall investigate and solve this problem by using inequality techniques. Moreover, the explicit effect of time delay and quantization parameter on the practical consensus set will be obtained.

In Theorem 3.15, we assume that the initial functions associated with (3.10) are given as

$$x_i(s) = \phi_i(s) \in \mathcal{C}([-\tau, 0], \mathbb{R}), \quad \forall i \in \mathcal{N}. \tag{3.77}$$

Let

$$\zeta(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0) + \frac{1}{N\Delta}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_{-\tau}^{0}\phi_j(s)ds \tag{3.78}$$

and

$$\mathcal{D} = 1 - \frac{\tau}{N\Delta}\sum_{i=1}^{N} a_{ii}. \tag{3.79}$$

**Theorem 3.15** *Consider the multi-agent network (3.10) with communication topology that is defined by an undirected, connected graph G. For any finite communication delay $\tau$, the states of multi-agent network (3.10) will converge to the set of $\Omega = [\frac{\zeta(0)}{\mathcal{D}} - (2 - \frac{1}{\mathcal{D}})\mu\Delta,\ \frac{\zeta(0)}{\mathcal{D}} + (2 - \frac{1}{\mathcal{D}})\mu\Delta]$ asymptotically.*

***Proof*** By Theorem 3.13, one has that $x_i(t)$ will converge to the set of $[(k - \frac{1}{2})\mu\Delta,\ (k + \frac{1}{2})\mu\Delta]$ asymptotically for every $i \in \mathcal{N}$. It follows from (3.75) that

$$\frac{\eta(0)}{A} - \frac{B}{A} \leq k \leq \frac{\eta(0)}{A} + \frac{B}{A}, \tag{3.80}$$

where $A = \mu(\Delta - \frac{\tau}{N}\sum_{i=1}^{N} a_{ii})$ and $B = \mu(\frac{1}{2}\Delta - \frac{\tau}{N}\sum_{i=1}^{N} a_{ii})$.

According to the definition of the quantizer, for $t \in [-\tau,\ 0]$, we have

$$\begin{aligned} \gamma_j(t) &= \mu\left\lfloor \frac{\phi_j(t)}{\mu\Delta} + \frac{1}{2}\right\rfloor \\ &\leq \mu\left(\frac{\phi_j(t)}{\mu\Delta} + \frac{1}{2}\right) \\ &= \frac{\phi_j(t)}{\Delta} + \frac{1}{2}\mu. \end{aligned} \tag{3.81}$$

Similarly, to obtain the lower bound of $\gamma_j(t)$, $t \in [-\tau,\ 0]$, the problem is divided into two cases as presented in the following:

Case 1: If $\phi_j(t) \neq (k+\frac{1}{2})\mu\Delta$ for any $k \in \mathbb{Z}$, then,

$$\begin{aligned}\gamma_j(t) &= \mu \left\lfloor \frac{\phi_j(t)}{\mu\Delta} + \frac{1}{2} \right\rfloor \\ &\geq \mu\left(\frac{\phi_j(t)}{\mu\Delta} + \frac{1}{2} - 1\right) \\ &= \frac{\phi_j(t)}{\Delta} - \frac{1}{2}\mu. \end{aligned} \tag{3.82}$$

Case 2: If $\phi_j(t) = (k+\frac{1}{2})\mu\Delta$ for some $k \in \mathbb{Z}$, then,

$$\begin{aligned}\gamma_j(t) &\geq \mu \left(\left\lfloor \frac{\phi_j(t)}{\mu\Delta} + \frac{1}{2} \right\rfloor - 1\right) \\ &= \mu\left(\frac{\phi_j(t)}{\mu\Delta} + \frac{1}{2} - 1\right) \\ &= \frac{\phi_j(t)}{\Delta} - \frac{1}{2}\mu. \end{aligned} \tag{3.83}$$

Hence, for $t \in [-\tau,\ 0]$, we can obtain that

$$\gamma_j(t) \geq \frac{\phi_j(t)}{\Delta} - \frac{1}{2}\mu. \tag{3.84}$$

It follows from (3.63), (3.81), and (3.84) that

$$\begin{aligned}\eta(0) &\leq \frac{1}{N}\left[\sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_{-\tau}^{0}\left(\frac{\phi_j(s)}{\Delta} + \frac{1}{2}\mu\right) ds\right] \\ &= \zeta(0) - \frac{1}{2}\frac{\tau}{N}\sum_{i=1}^{N} a_{ii}\mu, \end{aligned} \tag{3.85}$$

and

$$\eta(0) \geq \frac{1}{N}\left[\sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_{-\tau}^{0}\left(\frac{\phi_j(s)}{\Delta} - \frac{1}{2}\mu\right)ds\right]$$
$$= \zeta(0) + \frac{1}{2}\frac{\tau}{N}\sum_{i=1}^{N} a_{ii}\mu, \tag{3.86}$$

where $\zeta(0) = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} x_i(0) + \dfrac{1}{N\Delta}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_{-\tau}^{0}\phi_j(s)ds.$

Then, we obtain

$$\left(k - \frac{1}{2}\right)\mu\Delta \geq \left(\frac{\eta(0)}{A} - \frac{B}{A} - \frac{1}{2}\right)\mu\Delta$$
$$\geq \left(\frac{\zeta(0)}{A} - \frac{B - \frac{1}{2}\frac{\tau}{N}\sum_{i=1}^{N} a_{ii}\mu}{A} - \frac{1}{2}\right)\mu\Delta$$
$$= \frac{\zeta(0)}{A}\mu\Delta - \frac{2B + A - \frac{\tau}{N}\sum_{i=1}^{N} a_{ii}\mu}{2A}\mu\Delta$$
$$= \frac{\zeta(0)}{\mathcal{D}} - \left(2 - \frac{1}{\mathcal{D}}\right)\mu\Delta, \tag{3.87}$$

and

$$\left(k + \frac{1}{2}\right)\mu\Delta \leq \left(\frac{\eta(0)}{A} + \frac{B}{A} + \frac{1}{2}\right)\mu\Delta$$
$$\leq \left(\frac{\zeta(0)}{A} + \frac{B - \frac{1}{2}\frac{\tau}{N}\sum_{i=1}^{N} a_{ii}\mu}{A} + \frac{1}{2}\right)\mu\Delta$$
$$= \frac{\zeta(0)}{A}\mu\Delta + \frac{2B + A - \frac{\tau}{N}\sum_{i=1}^{N} a_{ii}\mu}{2A}\mu\Delta$$
$$= \frac{\zeta(0)}{\mathcal{D}} + \left(2 - \frac{1}{\mathcal{D}}\right)\mu\Delta, \tag{3.88}$$

where $\mathcal{D} = 1 - \frac{\tau}{N\Delta}\sum_{i=1}^{N} a_{ii}$.

Therefore, we can conclude that all of the agents in the network (3.10) will converge to the practical consensus set $\Omega = [\frac{\zeta(0)}{\mathcal{D}} - (2 - \frac{1}{\mathcal{D}})\mu\Delta,\ \frac{\zeta(0)}{\mathcal{D}} + (2 - \frac{1}{\mathcal{D}})\mu\Delta]$.

*Remark 3.16* Different from Theorem 3.13, the practical consensus set in Theorem 3.15 is not related to the initial condition of the selected output function $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$. Instead, the way how time delay and quantization parameter $\mu$ affect the practical consensus set $\Omega$ is obtained explicitly. It is interesting to observe that the size of the practical consensus set can be made arbitrarily small by decreasing the quantization parameter $\mu$.

*Remark 3.17* In Theorems 3.13 and 3.15, it has been shown that the states of multi-agent network would reach a set of steady-states, which is a function of time delay, quantization parameter, and initial state of the system. In some particular situations, it would be more interesting to study the average consensus problem of the multi-agent system. With the effect of quantization and delay, the average consensus cannot be exactly realized in our protocol. In fact, our practical consensus set $\Omega$ can also be expressed as $\Omega^* = [\frac{1}{N}\sum_{i=1}^N x_i(0) - f(\mu, \Delta, \tau, x(0)), \frac{1}{N}\sum_{i=1}^N x_i(0) + g(\mu, \Delta, \tau, x(0))]$, where $f(\mu, \Delta, \tau, x(0)) = (2 - \frac{N\Delta}{N\Delta - \sum_{i=1}^N a_{ii}\tau})\mu\Delta - \frac{\tau\sum_{i=1}^N a_{ii}}{N\Delta - \sum_{i=1}^N a_{ii}\tau}(\frac{1}{N}\sum_{i=1}^N x_i(0)) - \frac{\sum_{i=1}^N\sum_{j=1,\, j\neq i}^N a_{ij}\int_{-\tau}^0 \phi_j(s)ds}{N\Delta - \sum_{i=1}^N a_{ii}\tau}$, and $g(\mu, \Delta, \tau, x(0)) = (2 - \frac{N\Delta}{N\Delta - \sum_{i=1}^N a_{ii}\tau})\mu\Delta + \frac{\tau\sum_{i=1}^N a_{ii}}{N\Delta - \sum_{i=1}^N a_{ii}\tau}(\frac{1}{N}\sum_{i=1}^N x_i(0)) + \frac{\sum_{i=1}^N\sum_{j=1,\, j\neq i}^N a_{ij}\int_{-\tau}^0 \phi_j(s)ds}{N\Delta - \sum_{i=1}^N a_{ii}\tau}$. Simple calculation gives that $g \geq -f$, which makes final consensus set meaningful. Moreover, according to the derived practical consensus set $\Omega^*$, one can observe that the functions $f$ and $g$ are not necessary to be positive functions. It implies that the value of $\frac{1}{N}\sum_{i=1}^N x_i(0)$ can be outside the practical consensus set $\Omega^*$.

*Remark 3.18* It is worthy to investigate the bounds of $f$, $g$ and how $f$ and $g$ change with respect to tunable parameters since they can show the size of the practical consensus set and how far is the consensus set from the average consensus value. The size of the practical consensus set is $g + f = (4 - \frac{2}{\mathcal{D}})\mu\Delta$, where $\mathcal{D} = 1 - \frac{\tau}{N\Delta}\sum_{i=1}^N a_{ii}$. Hence, we can observe that $g + f \to 4\mu\Delta$ as $\tau \to +\infty$ and $g + f \to 2\mu\Delta$ as $\tau \to 0$. Moreover, it follows from the expressions of $f$ and $g$ in Remark 3.17 that the values of $f$ and $g$ will become smaller by decreasing the quantization parameter $\mu$. Unfortunately, it is not easy to find exactly how $f$ and $g$ change with respect to $\tau$ since the values of $f$ and $g$ also depend on the initial value of the agents, i.e., $\phi_i(\theta)$, $\theta \in [-\tau, 0]$. Hence, it is interesting to estimate more precise consensus set in future.

### *3.2.4 Numerical Example*

In order to illustrate the effect of quantization parameter $\mu$ on the practical consensus set, a multi-agent network (3.10) with 20 agents is considered. The graph (Fig. 3.3) is generated by small-world algorithm, in which each node has 2 nearest neighbors and the rewiring probability of the edges is 0.5 (see [24]). Let initial conditions be randomly chosen from (0, 20).

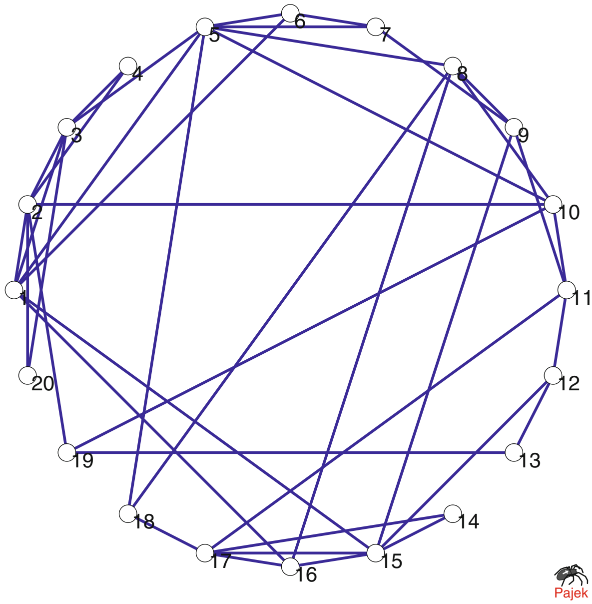


**Fig. 3.3** Network topology in the simulation example

Figures 3.4, 3.5, and 3.6 show the states responses of multi-agent network (3.10) with respect to $\mu = 5,\ 1$, and 0.2, respectively. It can be observed that for different values of $\mu$, the agents converge to different practical consensus sets. Interestingly,

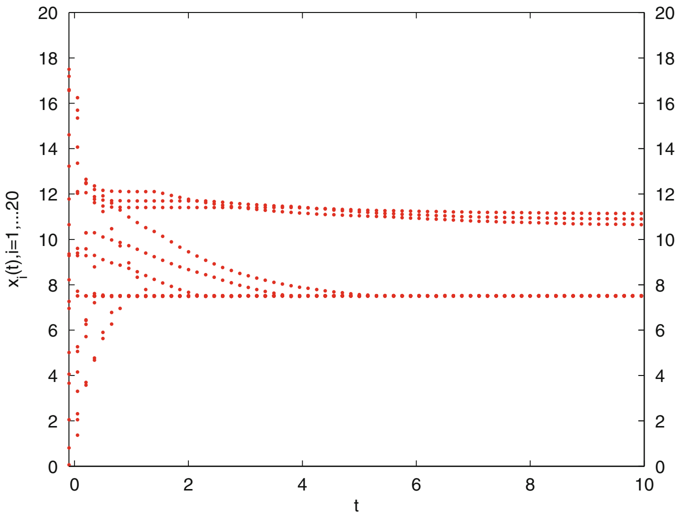


**Fig. 3.4** The states responses of the multi-agent networks with $\mu = 5$

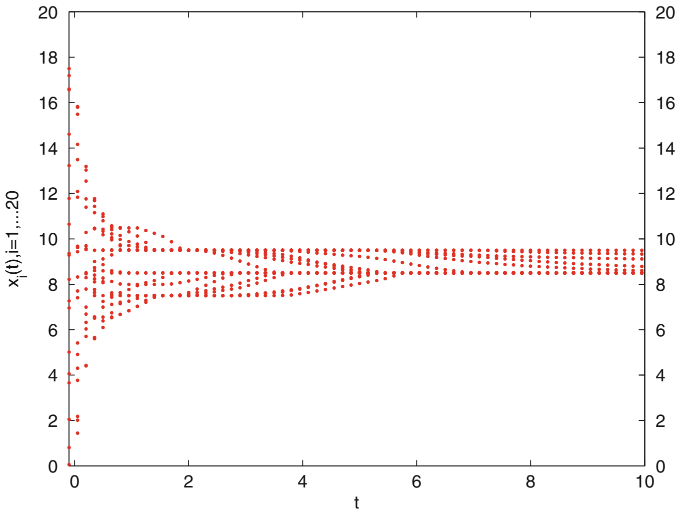


**Fig. 3.5** The states responses of the multi-agent networks with $\mu = 1$

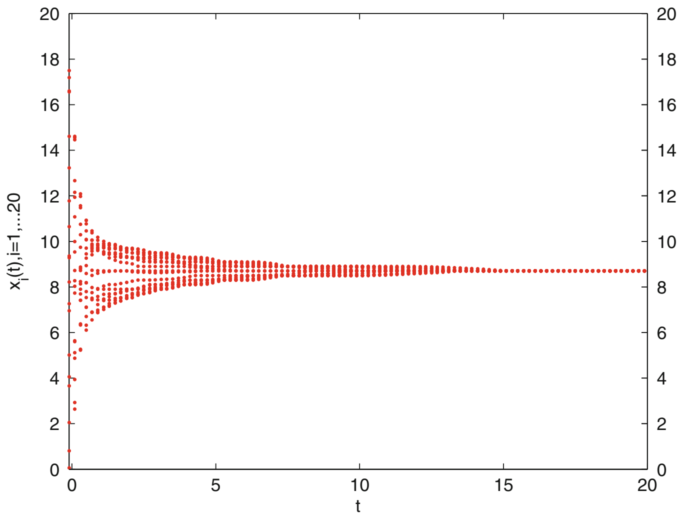


**Fig. 3.6** The states responses of the multi-agent networks with $\mu = 0.2$

we observe that the size of the practical consensus sets becomes smaller along with the decreasing of $\mu$. These three figures also verify Theorem 3.15 very well.

## *3.2.5 Discussions*

### 3.2.5.1 Can the "Time Delay" and "Quantization" Be Handled Independently in This Work?

1. The answer is NO. We shall discuss in the following sections.
2. For the individual case, we can deal with them as follows:

   - Without considering quantization in model (3.10), system (3.10) becomes

   $$\dot{x}(t) = Ax(t - \tau) - Dx(t). \tag{3.89}$$

   If $x(t)$ converges to some final value $\bar{x}$, then that final value is a consensus vector, because it satisfies

   $$A\bar{x} - D\bar{x} = 0. \tag{3.90}$$

   It remains to show that $x(t)$ does indeed converge in the time-delay system. This problem can be solved by using Laplace transforms and the final-value theorem;
   - Without considering time delay in (3.10), system (3.10) will become reference [18] considered the system

   $$\dot{x} = -Lq(x), \tag{3.91}$$

   where $q$ is a nonlinear, vector-valued quantizer. This is now a nonlinear robustness question. Reference [18] gives a rigorous treatment of this problem.

3. Next, we shall present the reasons why these two communication constraints cannot be dealt with independently.

   - We need to emphasize that these two communication constraints are mixed problems and cannot be treated as two separate independent issues. The main difficulty of this section lies in the discontinuity of function $q(\cdot)$ in a time delay system. We present a unified approach to handle these two communication constraints in the continuous-time consensus problem. This is one of our main contributions.
   - Due to the quantization discussed in this section, we cannot use Laplace transforms and the final-value theorem to treat time-delay system (3.10) (i.e., $\mathscr{L}(q(x(t)))$ is hard to compute). Moreover, the system (3.10) is continuous-time setting, and quantization function $q(\cdot)$ is not continuous. Hence, we

cannot use the Laplace transforms and the final-value theorem to change the system (3.10) into an algebra setting. Therefore, the Laplace transforms and the final-value theorem cannot be used to deal with the difficulty arising from time-delay system (3.10).

- In [18], the practical consensus problem for model (3.2) is investigated via the LaSalle invariance principle for differential inclusions. Since both time delay and quantization are considered *simultaneously* in this section and the LaSalle invariance principle for differential inclusions system with time delay is unknown, the technique used in [18] cannot be used in this section. Moreover, we present a new *practical consensus analysis* technique which is completely different from [18]. Moreover, we should mention that the results of reference [18] (with quantization only) have been further improved in Sect. 3.1 by using the new proofing technique.
- Due to the consideration of quantization together with time delay, we need to study the consensus of multi-agent system which is described by functional differential equation with discontinuous right hand side. Under this circumstance, there may not exist Carathéodory solution. Hence, we studied the solution of the multi-agent system in the Filippov sense and proved its existence, which we believe that it is theoretically important. A great deal of mathematics is used here to prove the existence of the solution and analyze its dynamical behavior. However, we would like to point out that the proof of some other papers on quantization seems simpler, since these papers studied their dynamical behavior by ignoring the existence of the solution. In this section, under the circumstance of the Filippov solution, we proved the convergence of the multi-agent system with quantization and delay, and found out its final consensus set. Moreover, by this theory, it would be interesting to explicitly present the relationship among the quantization parameter, time delay, and the convergence set.

#### 3.2.5.2 A New Result Stemming from Theorem 3.15

It follows from Theorem 3.15 that the final states of the networks will converge to the set $\Omega = [\frac{\zeta(0)}{\mathcal{D}} - (2 - \frac{1}{\mathcal{D}})\mu\Delta, \frac{\zeta(0)}{\mathcal{D}} + (2 - \frac{1}{\mathcal{D}})\mu\Delta]$. For $\Delta = 1$, as $\mu \to 0$, $q_\mu(x) \to x$ and set $\Omega$ will be reduced to one point

$$c' = \frac{\frac{1}{N}\sum_{i=1}^{N} x_i(0) + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\int_{-\tau}^{0} x_j(s)ds}{1 - \frac{1}{N}\sum_{i=1}^{N} a_{ii}\tau}. \tag{3.92}$$

Moreover, as $\mu \to 0$, the model (3.10) will degrade into

$$\frac{dx_i(t)}{dt} = \sum_{j\in\mathcal{N}_i} a_{ij}[x_j(t-\tau) - x_i(t)], \quad i = 1, \ldots, N. \tag{3.93}$$

It follows from Theorem 1 of [11] that model (3.93) can achieve consensus and the consensus value is $c$. However, the exact consensus value is not explicitly given in [11]. By the aforementioned analysis, we may conjecture that the final consensus value of model (3.93) is $c'$. In the following Proposition 3.19, using the proof method given in the Part II of Theorem 3.13, we can strictly prove that our conjecture is right. In the following, we shall present a more general result which can be seen as the supplementations of the Theorem 1 in [11]. The model proposed in [11] is as follows:

$$\frac{dx_i(t)}{dt} = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t-\tau_{ij}) - x_i(t)), \quad i = 1, \ldots, N. \tag{3.94}$$

**Proposition 3.19** *Consider the linear coupled system (3.94) with a strongly connected graph $\mathcal{G}$. Whatever finite communication delays $\tau_{ij}$ are, consensus is asymptotically reached for arbitrary initial conditions and the final consensus value is $c = \frac{\sum_{i=1}^N \xi_i x_i(0) + \sum_{i=1}^N \xi_i \sum_{j=1}^N a_{ij}\int_{-\tau_{ij}}^0 x_j(s)ds}{1+\sum_{i=1}^N \xi_i \sum_{j=1}^N a_{ij}\tau_{ij}}$, where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ is the normalized left eigenvector of matrix $L$ with respect to the zero eigenvalue.*

***Proof*** Let

$$\eta(t) = \sum_{i=1}^N \xi_i x_i(t) + \sum_{i=1}^N \xi_i \sum_{j=1}^N a_{ij}\int_{t-\tau_{ij}}^t x_j(s)ds. \tag{3.95}$$

Similar to the proof of (3.62), we can get $\dot{\eta}(t) = 0$. Thus, $\eta(t)$ is a constant.

We have known that the model (3.94) can achieve consensus and the consensus value is $c$ (see [11], Theorem 1).

Assume

$$\begin{aligned}\eta(t) &= d \\ &= \sum_{i=1}^N \xi_i x_i(0) + \sum_{i=1}^N \xi_i \sum_{j=1}^N a_{ij}\int_{-\tau_{ij}}^0 x_j(s)ds,\end{aligned} \tag{3.96}$$

then, we have

$$\begin{aligned} d &= \eta(t) \\ &= \lim_{t\to\infty} \eta(t) \\ &= \sum_{i=1}^N \xi_i c + \sum_{i=1}^N \xi_i \sum_{j=1}^N a_{ij}\tau_{ij} c \\ &= \left(1 + \sum_{i=1}^N \xi_i \sum_{j=1}^N a_{ij}\tau_{ij}\right) c. \end{aligned} \tag{3.97}$$

Therefore, we have

$$c = \frac{\sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \int_{-\tau_{ij}}^{0} x_j(s)ds}{1 + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij}\tau_{ij}}. \tag{3.98}$$

*Remark 3.20* In [11], it is mentioned that the final consensus state is hard to predict due to the injection of arbitrary finite communication delays although consensus can be achieved for multi-agent system (3.94). Here, Proposition 3.19 has solved this difficult problem and is an important extension for the results of [11].

## 3.3 Summary

In this chapter, we mainly addressed the consensus problem of continuous-time multi-agent networks where each agent can only obtain the quantized and delayed measurements of the states of its neighbors. Filippov solutions of the resulting system exist for any initial condition. We have proved that under certain network topology, the states of the multi-agent network which only considers quantization effect will converge to a practical consensus set in a finite time. For the multi-agent network model considering quantization and time delay simultaneously, it is shown that Filippov solutions converge to a practical consensus set asymptotically. Moreover, we also present how the initial states of the agents, time delay, and quantization parameter affect the final practical consensus set. The theoretical results have been well illustrated by two numerical examples.

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# Chapter 4
# Multi-agent Consensus with Quantization and Communication Delays

Recently, substantial consensus problems have been studied in many previous literature [1–6]. Due to the energy and bandwidth constraints of the communication channels, the transmitted information in the multi-agent network needs to be quantized. The study on control problems using quantized information has a long history [7]. Over the past few years, considerable effort has been devoted to studying the information quantization on feedback control systems [8–11].

How to realize a distributed consensus with quantization has drawn considerable attention [12–18]. In [16], a coding–decoding scheme was developed to solve the average consensus problem with quantized information. In [14, 19], under the condition that each uniform quantizer has infinite quantization levels, it was shown that the multi-agent network could achieve practical consensus.

In this chapter, we will discuss the multi-agent network consensus problem with communication quantization and time delays simultaneously. It is shown that consensus can be achieved for the network under communication quantization and delays under certain topology conditions. Different from Chap. 3, the consensus protocol proposed in this chapter only considers quantized transmitted information. Moreover, the protocol does not assume that the communication delay is the same between different neighboring agents.

## 4.1 Discrete-Time Case

In this section, the consensus problem of discrete-time multi-agent networks with quantized data and delays is studied. The remainder of this section is organized as follows. In Sect. 4.1.1, the discrete-time multi-agent network model with communication quantization and time delays is presented. In Sect. 4.1.2, the consensus analysis of the proposed protocol is presented in detail. Finally, a



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numerical simulation is given to demonstrate the validity of the theoretical results in Sect. 4.1.3.

### 4.1.1 Model Description

Consider the following network model with discrete-time integrator agents with dynamics:

$$x_i(k+1) = x_i(k) + u_i(k), \quad i \in \mathcal{N}, \tag{4.1}$$

where $x_i(k) \in \mathbb{R}$ is the state of the agent $i$ and $u_i(k)$ is called the protocol.

The goal is to design the protocol $u_i(k)$ yielding the consensus of the states, i.e.,

$$\lim_{k \to +\infty} x_i(k) = c, \quad \forall i \in \mathcal{N}, \tag{4.2}$$

where $c$ is a constant.

Due to the communication bandwidth constraints in many real multi-agent networks, the agents can only use quantized information of the neighboring agents. The following consensus protocol

$$u_i(k) = \sum_{j \in \mathcal{N}_i} a_{ij}[q_\mu(x_j(k - \tau_{ij})) - x_i(k)], \quad i \in \mathcal{N},$$

will be studied in this section, i.e.,

$$x_i(k+1) = x_i(k) + \sum_{j \in \mathcal{N}_i} a_{ij}[q_\mu(x_j(k - \tau_{ij})) - x_i(k)], \quad i \in \mathcal{N}, \tag{4.3}$$

where $\tau_{ij}$ is a nonnegative integer representing the communication delays from agent $j$ to agent $i$, and $q_\mu(\cdot)$ denotes one-parameter family of uniform quantizers which is defined by (1.8), i.e.,

$$q_\mu(x) = \begin{cases} \lfloor \frac{x}{\mu} \rfloor \mu, & x \geq 0, \\ -\lfloor \frac{-x}{\mu} \rfloor \mu, & x < 0. \end{cases} \tag{4.4}$$

In this section, we assume that time delays only exist when the information is transmitted from one agent to another, i.e., $\tau_{ii} = 0, \ i \in \mathcal{N}$. Moreover, the following assumption is proposed in this section.

**Assumption 4.1** $A$ is a stochastic matrix such that $a_{ii} > 0, \ i \in \mathcal{N}$, and $\mathcal{G}$ is strongly connected.

### 4.1.2 Main Results

We introduce the main notations here which will be used in this section. For arbitrary fixed $k_0 \in \mathbb{R}$, denote

- $\tau = \max\{\tau_{ij},\ \ i,\ j \in \mathcal{N}\};\ \ \Upsilon_{-\tau} = \{-\tau, -\tau+1, \cdots, 0\}$;
- $\mathbb{Z}_\mu = \{l\mu,\ l \in \mathbb{Z}\};\ \ X = \{\psi : \Upsilon_{-\tau} \longmapsto \mathbb{R}\}$;
- $\overline{V}(k) = \max_{\theta\in\Upsilon_{-\tau}} \max_{i\in\mathcal{N}}\{q_\mu(x_i(k+\theta))\};\ \overline{v}(k) = \min\limits_{\theta\in\Upsilon_{-\tau}} \min\limits_{i\in\mathcal{N}}\{q_\mu(x_i(k+\theta))\}$;
- for any $b \in \mathbb{Z}_\mu$, $\Gamma_b(k) = \{i \in \mathcal{N} : \exists\theta \in \Upsilon_{-\tau},\ q_\mu(x_i(k+\theta)) = b\}$.

For a set $B$ with finite elements, $|B|$ denotes the cardinality of $B$, i.e., the number of the element in the set $B$.

In the following, we will study the consensus result of model (4.3). The initial conditions associated with (4.3) are given as $x_i(s) \in X$, $i \in \mathcal{N}$. Before the main theorem of this section be given, we here give two important lemmas first, which will be used in the proof of Theorem 4.4.

**Lemma 4.2** *Suppose that $x(t)$ is the solution to (4.3). Under Assumption 4.1, for any finite communication delays $\tau_{ij}$, $\overline{V}(k)$ is a non-increasing function for $k$, and $\overline{v}(k)$ is a non-decreasing function for $k$.*

***Proof*** For $\forall i \in \mathcal{N}$, we have

$$\begin{aligned} x_i(k+1) &= x_i(k) + \sum_{j\in\mathcal{N}_i} a_{ij}(q_\mu(x_j(k-\tau_{ij})) - x_i(k)) \\ &\le x_i(k) + \sum_{j\in\mathcal{N}_i} a_{ij}(\overline{V}(k) - x_i(k)) \\ &= \overline{V}(k) + a_{ii}(x_i(k) - \overline{V}(k)) \\ &< \overline{V}(k) + \mu. \end{aligned} \tag{4.5}$$

Note that $q_\mu(x_i(k+1)) \in \mathbb{Z}_\mu$ and $\overline{V}(k) \in \mathbb{Z}_\mu$; then, we can obtain that

$$q_\mu(x_i(k+1)) \le \overline{V}(k), \tag{4.6}$$

which implies that $\overline{V}(k+1) \le \overline{V}(k)$. Hence, $\overline{V}(k)$ is a non-increasing function for $k$. Similarly, it can be proved that $\overline{v}(k)$ is a non-decreasing function for $k$.

**Lemma 4.3** *For arbitrary fixed $k_0 \in \mathbb{R}$, suppose $M = \overline{V}(k_0)$ and $m = \overline{v}(k_0)$. If $M \neq m$, we have the following conclusion:*

*(i) If $M > 0$, then $|\Gamma_M(k)|$ is a non-increasing function for $k$, and $\Gamma_M(k) = \varnothing$ in finite time.*
*(ii) If $m < 0$, then $|\Gamma_m(k)|$ is a non-increasing function for $t$, and $\Gamma_m(k) = \varnothing$ in finite time.*

***Proof*** We only prove conclusion (i). Conclusion (ii) can be proved similarly, and hence the proof is omitted here. For $M = \overline{V}(k_0) > 0$ and arbitrary $k_1 \geq k_0$, it follows from Lemma 4.2 that

$$q_\mu(x_j(k_1 - \tau_{ij})) \leq M, \ \forall j \in \mathcal{N}_i. \tag{4.7}$$

If $x_i(k_1) < M$, we can deduce that

$$\begin{aligned} x_i(k_1+1) &= x_i(k_1) + \sum_{j \in \mathcal{N}_i} a_{ij}(q_\mu(x_j(k_1 - \tau_{ij})) - x_i(k_1)) \\ &\leq x_i(k_1) + \sum_{j \in \mathcal{N}_i} a_{ij}(M - x_i(k_1)) \\ &= M + a_{ii}(x_i(k_1) - M) \\ &< M, \end{aligned} \tag{4.8}$$

which implies that

$$q_\mu(x_i(k_1+1)) < M. \tag{4.9}$$

The inequalities (4.8) and (4.9) imply that $i \notin \Gamma_M(k_1+1)$ if $i \notin \Gamma_M(k_1)$. Hence, $|\Gamma_M(k)|$ is a non-increasing function for $k \geq k_0$.

Next, we shall prove $\Gamma_M(k) = \varnothing$ in finite time, i.e., there exists a $\tilde{k}_0 > k_0$ such that $\Gamma_M(\tilde{k}_0) = \varnothing$.

According to $M \neq m$, there exist $j_1 \in \mathcal{N}$ and $\theta_1 \in \Upsilon_{-\tau}$ such that

$$q_\mu(x_{j_1}(k_0 + \theta_1)) = m < M. \tag{4.10}$$

Equations (4.7)–(4.10) imply that

$$q_\mu(x_{j_1}(k)) < M, \quad \forall k \geq k_0 + \theta_1. \tag{4.11}$$

Hence,

$$j_1 \notin \Gamma_M(k), \ \forall k \geq k_0 + \tau. \tag{4.12}$$

Let $\Lambda_{j_1} = \{l \in \mathcal{N} : j_1 \in \mathcal{N}_l\}$. For any $j_2 \in \Lambda_{j_1}$, we consider the following two cases:

**Case 1:** $j_2 \notin \Gamma_M(k_0)$.

Equations (4.8) and (4.9) imply that $j_2 \notin \Gamma_M(k), \ \forall k \geq k_0$.

**Case 2:** $j_2 \in \Gamma_M(k_0)$.

**Claim I** There exists a $k_2 > k_0$, such that $j_2 \notin \Gamma_M(k_2)$. Next, we shall prove **Claim I** by using a contradiction approach. If for any $k > k_0$, $j_2 \in \Gamma_M(k)$, we can obtain that

$$q_\mu(x_{j_2}(k)) = M, \quad \forall k \geq k_0. \tag{4.13}$$

Then, we have

$$\begin{aligned} x_{j_2}(k+1) &= x_{j_2}(k) + \sum_{j \in \mathcal{N}_{j_2}} a_{j_2 j}(q_\mu(x_j(k-\tau_{j_2 j})) - x_{j_2}(k)) \\ &\leq x_{j_2}(k) + a_{j_2 j_1}(q_\mu(x_{j_1}(k-\tau_{j_2 j_1})) - x_{j_2}(k)) \\ &\leq x_{j_2}(k) - a_{j_2 j_1}\mu. \end{aligned} \tag{4.14}$$

Hence, $q_\mu(x_{j_2}(k)) \leq x_{j_2}(k) < M$ in finite time, which contradicts with (4.13). Thus, **Claim I** holds, which means that there exists $k_2 > k_0$, such that for any $j_2 \in \Lambda_{j_1}$, it holds $j_2 \notin \Gamma_M(k)$, $\forall k \geq k_2$.

For any $j_2 \in \Lambda_{j_1}$, same procedure applies to the agents set $\Lambda_{j_2} = \{\tilde{l} \in \mathcal{N} : j_2 \in \mathcal{N}_{\tilde{l}}\}$. It can be obtained that there exists $k_3 > k_2$, such that for any $j_3 \in \Lambda_{j_2}$, $j_3 \notin \Gamma_M(k)$, $\forall k \geq k_3$.

Repeat the above procedure. Given that the network is strongly connected, it implies that there exists a $\tilde{k}_0 > k_0$ such that $\Gamma_M(k) = \varnothing$, $\forall k \geq \tilde{k}_0$. This completes the proof of Lemma 4.3.

**Theorem 4.4** *Under Assumption 4.1, for any finite communication delays $\tau_{ij}$, the multi-agent network (4.3) will asymptotically achieve consensus for arbitrary initial conditions. That is,*

$$\lim_{t \to +\infty} x_i(t) = c, \quad \forall i \in \mathcal{N}, \tag{4.15}$$

*where $c$ is a constant.*

***Proof*** The proof of Theorem 4.4 is divided into two steps.

*Step 1 We shall prove that for any fixed $k_0 \in \mathbb{R}$, there exists $\bar{k}_0 \geq k_0$ such that*

$$\overline{V}(\bar{k}_0) = \overline{v}(\bar{k}_0). \tag{4.16}$$

The following three cases are considered:

**Case 1:** $\overline{V}(k_0) \geq 0$ and $\overline{v}(k_0) \geq 0$.

- If $\overline{V}(k_0) = \overline{v}(k_0)$, select $\bar{k}_0 = k_0$.
- If $\overline{V}(k_0) \neq \overline{v}(k_0)$, it follows from Lemma 4.3 that there exists $k_1 > k_0$, such that $\Gamma_{\overline{V}(k_0)}(k_1) = \varnothing$, which implies that $\overline{V}(k_1) < \overline{V}(k_0)$.
- If $\overline{V}(k_1) = \overline{v}(k_1)$, select $\bar{k}_0 = k_1$.
- If $\overline{V}(k_1) \neq \overline{v}(k_1)$, there exists $k_2 \geq k_1$, such that $\overline{v}(k_2) < \overline{V}(k_2)$.

Repeat the above procedure, we can finally find a $\bar{k}_2 \in \mathbb{R}$, such that $\overline{V}(\bar{k}_2) = \overline{v}(\bar{k}_2)$. Select $\bar{k}_0 = \bar{k}_2$.

**Case 2:** $\overline{V}(k_0) \leq 0$ and $\overline{v}(k_0) \leq 0$. Similar to the procedure of **Case 1** (replace $\Gamma_{\overline{V}(k_0)}(k)$ by $\Gamma_{\overline{v}(k_0)}(k)$), we can find a $\bar{k}_0 \geq k_0$ such that

$$\overline{V}(\bar{k}_0) = \overline{v}(\bar{k}_0). \tag{4.17}$$

**Case 3:** $\overline{V}(k_0) > 0$ and $\overline{v}(k_0) < 0$.

According to Lemma 4.3, there exists $k_1 > k_0$ such that

$$\Gamma_{\overline{V}(k_0)}(k_1) = \varnothing, \text{ and } \Gamma_{\overline{v}(k_0)}(k_1) = \varnothing,$$

which implies that $\overline{V}(k_1) < \overline{V}(k_0)$ and $\overline{v}(k_1) > \overline{v}(k_0)$.

- If $\overline{V}(k_1) = \overline{v}(k_1)$, select $\bar{k}_0 = k_1$.
- If $\overline{V}(k_1) \neq \overline{v}(k_1)$, one of the following three subcases holds:

(1) $\overline{V}(k_1) = \overline{v}(k_1) = 0$; (2) $\overline{V}(k_1) > 0$ and $\overline{v}(k_1) \geq 0$; and (3) $\overline{V}(k_1) \leq 0$ and $\overline{v}(k_l) < 0$.

For subcase (1), choose $\bar{k}_0 = k_1$. Subcases (2) and (3) have been reduced to the **Cases 2** and **3**, respectively.

This completes *Step 1* of the proof, i.e., there exists $\bar{k}_0 \geq k_0$ such that

$$\overline{V}(\bar{k}_0) = \overline{v}(\bar{k}_0). \tag{4.18}$$

*Step 2 We shall prove that the multi-agent network (4.3) achieves consensus asymptotically.*

From (4.18) and Lemma 4.2, it can be obtained that

$$\overline{V}(k) = \overline{v}(k), \; k \geq \bar{k}_0, \tag{4.19}$$

which implies

$$q_\mu(x_i(k+\theta_1)) = q_\mu(x_j(k+\theta_2)), \; \forall i, \; j \in \mathcal{N}, \; \forall \theta_1, \; \theta_2 \in \Upsilon_{-\tau}, \; k \geq \bar{k}_0. \tag{4.20}$$

Let $c = q_\mu(x_i(\bar{k}_0))$. It follows from (4.20) that for any $i \in \mathcal{N}$ and $k \geq \bar{k}_0$, system (4.3) can be written as follows:

$$\begin{aligned} x_i(k+1) &= x_i(k) + \sum_{j \in \mathcal{N}_i} a_{ij}(q_\mu(x_j(k-\tau_{ij})) - x_i(k)) \\ &= x_i(k) + \sum_{j \in \mathcal{N}_i} a_{ij}(c - x_i(k)) \\ &= a_{ii} x_i(k) + (1 - a_{ii})c. \end{aligned} \tag{4.21}$$

From (4.21), we have

$$x_i(k+1) - x_i(k) = a_{ii}(x_i(k) - x_i(k-1)), \;\; k \geq \bar{k}_0 + 1, \tag{4.22}$$

which implies that

$$x_i(k+1) - x_i(k) = a_{ii}^{k-\bar{k}_0}(x_i(\bar{k}_0+1) - x_i(\bar{k}_0)), \quad k \geq \bar{k}_0 + 1. \tag{4.23}$$

Hence, it follows from Assumption 4.1 that

$$\lim_{k\to+\infty}(x_i(k+1) - x_i(k)) = \lim_{k\to+\infty} a_{ii}^{k-\bar{k}_0}(x_i(\bar{k}_0+1) - x_i(\bar{k}_0)) = 0. \tag{4.24}$$

It follows from (4.21) and (4.24) that there exists a constant $c \in \mathbb{R}$ such that

$$\lim_{k\to+\infty} x_i(k) = c, \quad \forall i \in \mathcal{N}. \tag{4.25}$$

### *4.1.3 Numerical Example*

In this section, an example is given to illustrate the correctness of the theoretical results.

Consider network (4.3) with the topology shown in Fig. 4.1. Assume that $\mu = 1$ and $\tau_{ij} = 1$, $\forall i \in \mathcal{N}$, $j \in \mathcal{N}_i$. The initial condition of network (4.3) is randomly chosen from $(-5, 5)$. Suppose the weight of each edge is set as $\frac{1}{4}$. The stochastic matrix $A$ is

$$A = \begin{pmatrix} \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}. \tag{4.26}$$

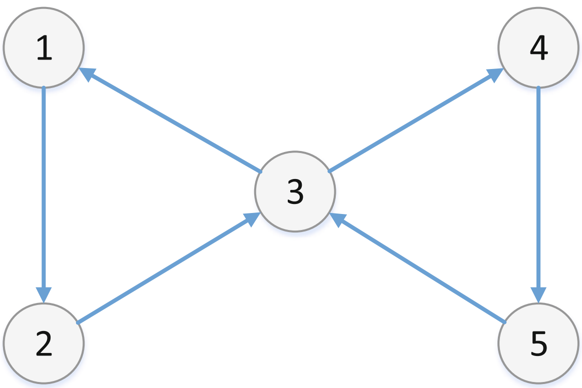


**Fig. 4.1** Network topology in example

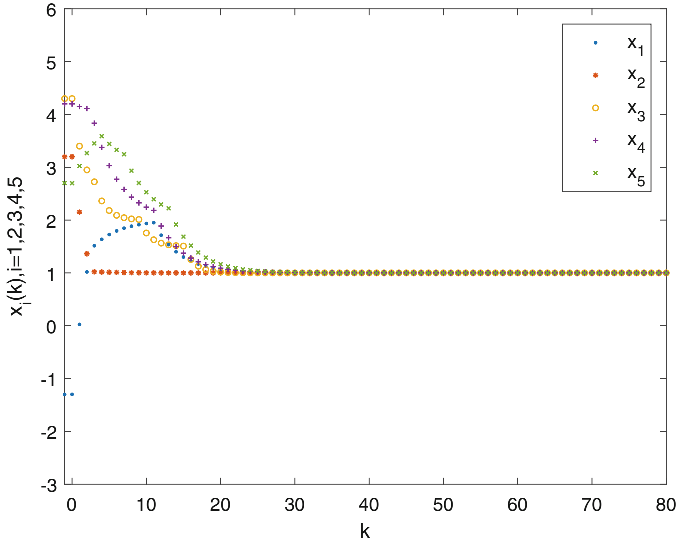


**Fig. 4.2** The state responses of the multi-agent system

The state responses of multi-agent networks (4.3) are shown in Fig. 4.2. It can be observed from Fig. 4.2 that the multi-agent network achieves consensus asymptotically, which illustrates Theorem 4.4 very well.

## 4.2 Continuous-Time Case

In Sect. 4.1, discrete-time multi-agent network consensus problem with quantization and time delays is studied. In this section, we shall investigate the corresponding continuous-time cases. The organization of the remaining part is given as follows. In Sect. 4.2.1, consensus protocol with quantization and time delays is formulated. In Sect. 4.2.2, the existence of the Filippov solution is presented. In Sect. 4.2.3, the consensus analysis of the proposed protocol is presented in detail. In Sect. 4.2.4, a numerical example is given to show the correctness of the theoretical results.

### *4.2.1 Model Description and Preliminaries*

In Chap. 3, the following multi-agent network model has been investigated:

$$\frac{dx_i(t)}{dt} = \sum_{j\in\mathcal{N}_i} a_{ij}[q_\mu(x_j(t-\tau)) - q_\mu(x_i(t))], \qquad i \in \mathcal{N},$$

where $\tau$ is the communication delays from agent $j$ to agent $i$.

In many real multi-agent networks, each agent can obtain its own precise information, which will not be effected by the limited communication bandwidth. Moreover, communication delays may be different between different neighboring agents. Hence, the following consensus protocol will be studied in this section:

$$\frac{dx_i(t)}{dt} = \sum_{j \in \mathcal{N}_i} a_{ij}[q_\mu(x_j(t - \tau_{ij})) - x_i(t)], \quad i \in \mathcal{N}, \tag{4.27}$$

where $\tau_{ij}$ is the communication delay from agent $j$ to agent $i$ and $q_\mu(z)$ is defined by (4.4). For $x = (x_1, x_2, \ldots, x_N)^\top \in \mathbb{R}^N$, let $q_\mu(x) = (q_\mu(x_1), q_\mu(x_2), \ldots, q_\mu(x_N))^\top$. The initial conditions associated with (4.27) are given as

$$x_i(s) = \phi_i(s) \in \mathcal{C}([-\tau, 0], \mathbb{R}), \quad i \in \mathcal{N}.$$

Different from the discrete-time cases, system (4.27) may not have global solution in the sense of $Carath\acute{e}odory$ due to the discontinuity of the function $q(\cdot)$. Hence, we need to prove the existence of the global Filippov solution to differential equation (4.27) as in Chap. 1.

### *4.2.2 The Existence of the Filippov Solution*

The concept of the Filippov solution to the differential equation (4.27) is given as follows.

**Definition 4.5** A function $x(t) : [-\tau, T) \to \mathbb{R}^N$ ($T$ might be $\infty$) is a solution in the sense of Filippov for the discontinuous system (4.27) on $[-\tau, T)$, if

1. $x(t)$ is continuous on $[-\tau, T)$ and absolutely continuous on $[0, T)$;
2. $x(t)$ satisfies that

$$\frac{dx_i(t)}{dt} \in \mathcal{K}[\sum_{j \in \mathcal{N}_i} a_{ij}(q_\mu(x_j(t - \tau_{ij})) - x_i(t))], \quad i \in \mathcal{N}. \tag{4.28}$$

It follows from Lemma 1.16 that

$$\begin{aligned} &\mathcal{K}[\sum_{j \in \mathcal{N}_i} a_{ij}(q_\mu(x_j(t - \tau_{ij})) - x_i(t))] \\ &\quad \subseteq \sum_{j \in \mathcal{N}_i} a_{ij}(\mathcal{K}[q_\mu(x_j(t - \tau_{ij}))] - x_i(t)). \end{aligned} \tag{4.29}$$

Similar to Chap. 3, if $x(t)$ is the solution of system (4.27), there exists the output function $\gamma(t) \in \mathcal{K}[q_\mu(x(t))]$ such that for a.e. $t \in [0, T)$, the following equation is true:

$$\frac{dx_i(t)}{dt} = \sum_{j \in \mathcal{N}_i} a_{ij}(\gamma_j(t - \tau_{ij}) - x_i(t)), \quad i \in \mathcal{N}. \tag{4.30}$$

**Definition 4.6** For any continuous function $\phi : [-\tau, 0] \to \mathbb{R}^N$ and any measurable selection $\psi : [-\tau, 0] \to \mathbb{R}^N$, such that $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$ for a.e. $s \in [-\tau, 0]$, an absolute continuous function $x(t) = x(t, \phi, \psi)$ is said to be a solution of the Cauchy problem for system (4.27) on $[0, T)$ with initial value $(\phi, \psi)$, if

$$\begin{cases} \dot{x}_i(t) = \sum\limits_{j=1,\, j \neq i}^{N} a_{ij}(\gamma_j(t - \tau_{ij}) - x_i(t)), \ for \ a.e. \ t \in [0, T), \quad i \in \mathcal{N}, \\ x(s) = \phi(s), \quad \forall s \in [-\tau, 0], \\ \gamma(s) = \psi(s) \quad a.e. \ s \in [-\tau, 0]. \end{cases} \tag{4.31}$$

Next, we shall study the existence of the global solution to the system (4.31).

**Lemma 4.7** *Suppose $x(\cdot)$ is a Filippov solution to (4.27). Let $M(t) = max_{i \in \mathcal{N}} max_{\theta \in [-\tau, 0]}\{x_i(t + \theta)\}$ and $m(t) = min_{i \in \mathcal{N}} min_{\theta \in [-\tau, 0]}\{x_i(t + \theta)\}$. Then, we have the following conclusion:*

*(i) If $M(t) \geq 0$, then $M(t)$ is a non-increasing function for $t$.*
*(ii) If $m(t) \leq 0$, then $m(t)$ is a non-decreasing function for $t$.*

***Proof*** We only prove the conclusion (i). (ii) can be proved similarly. For any fixed $t_0 \geq 0$, suppose $M(t_0) \geq 0$. Next, we will show that $M(t) \leq M(t_0)$ for any $t \geq t_0$ by contradiction.

Suppose there exist $\bar{t}_0$ and $t_0$ such that

$$M(\bar{t}_0) > M(t_0), \ \bar{t}_0 > t_0 \geq 0. \tag{4.32}$$

Similar to the proof of *Steps 1* and *2* of Theorem 3.12, it can be proved that there exist $i_0 \in \mathcal{N}$, $t_0^* \in [t_0, \bar{t}_0)$, and $\delta > 0$ such that

$$M(t_0) = M(t_0^*) = x_{i_0}(t_0^*), \tag{4.33}$$

and

$$M(t) > M(t_0^*), \ \forall t \in (t_0^*, \bar{t}_0], \tag{4.34}$$

and

$$M(t) = x_{i_0}(t + \theta(t)),\ \theta(t) \in [-\tau,\ 0],\ \forall t \in (t_0^*,\ t_0^* + \delta). \tag{4.35}$$

Let $\delta_1 = \min_{j \in \mathcal{N}_i}\{\tau_{ij}\}$ and $\delta_2 = \min\{\delta, \delta_1\}$. Since

$$x_{i_0}(t + \theta(t)) = M(t) > M(t_0^*) = x_{i_0}(t_0^*),\ \forall t \in (t_0^*,\ t_0^* + \delta_2), \tag{4.36}$$

there exists $t_1 \in (t_0^*,\ t_0^* + \delta_2]$ such that

$$x_{i_0}(t_1) > x_{i_0}(t_0^*). \tag{4.37}$$

Let $t_1^* = \sup\{t \in [t_0^*,\ t_1] : x_{i_0}(t) = x_{i_0}(t_0^*)\}$. Due to the continuity of function $x_{i_0}(t)$, we have

$$x_{i_0}(t_1^*) = x_{i_0}(t_0^*). \tag{4.38}$$

Hence, for any $t \in (t_1^*,\ t_1]$, we have

$$\begin{aligned} x_{i_0}(t) \geq M(t_0^*) &\geq \max_{j \in \mathcal{N}_i} \max_{t \in [t_1^*,\ t_1]}\{x_j(t - \tau_{ij})\} \\ &\geq \max_{j \in \mathcal{N}_i} \max_{t \in [t_1^*,\ t_1]}\{\gamma_j(t - \tau_{ij})\}. \end{aligned}$$

It follows from

$$\dot{x}_{i_0}(t) = \sum_{j=1,\ j \neq i_0}^{N} a_{ij}(\gamma_j(t - \tau) - x_{i_0}(t)),\ \ a.e.\ t \in (t_1^*,\ t_1], \tag{4.39}$$

that $\dot{x}_{i_0}(t) \leq 0,\ a.e.\ t \in (t_1^*,\ t_1]$. However, since $x_{i_0}(t_1) \geq x_{i_0}(t_1^*)$, there must exist a subset $\mathcal{I}_1$ of $(t_1^*,\ t_1]$ such that $\mathcal{I}_1$ has a positive measure and

$$\dot{x}_{i_0}(t) > 0,\ \ a.e. t \in \mathcal{I}_1, \tag{4.40}$$

which is contradictory with $\dot{x}_{i_0}(t) \leq 0$ for a.e. $t \in (t_1^*,\ t_1]$.

Therefore, $M(t)$ is a non-increasing function for $t$ if $M(t) \geq 0$. Similarly, $m(t)$ is a non-decreasing function for $t$ if $m(t) \leq 0$.

**Theorem 4.8** *For any initial function $\phi$ and the selection of the output function $\psi(s) \in \mathcal{K}[q_\mu(\phi(s))]$, there exists a global solution for the system (4.31).*

***Proof*** Similar to the proof of Theorem 3.12, the proof of Theorem 4.8 can also be divided into two parts:

**Part (I)** Existence of local solution

Similar to the proof of Lemma 1 in [20], one can conclude the existence of the solution defined on $[0,\ T)$ for system (4.31).

**Part (II)** The boundedness of the solution

Suppose $x(t, \phi, \psi)$ is a solution of system (4.31). Let $M(t) = \max_{i\in\mathcal{N}} \max_{\theta\in[-\tau,0]}\{x_i(t+\theta)\}$, and $m(t) = \min_{i\in\mathcal{N}} \min_{\theta\in[-\tau,0]}\{x_i(t+\theta)\}$. It follows from Lemma 4.7 that $M(t) \le \max\{M(0), 0\}$ and $m(t) \ge \min\{m(0), 0\}$. Hence, the solution $x(t)$ is bounded. According to the theory of functional differential equations [21], a global solution can be guaranteed by the boundedness of the local solution. This completes the proof of this theorem.

### 4.2.3 *Consensus Analysis Under Quantization and Time Delays*

In this section, we shall study the consensus result of the multi-agent system (4.27). We assume that the network topology is undirected in this section. The initial conditions associated with (4.27) are given as $x_i(s) = \phi_i(s) \in \mathcal{C}([-\tau, 0], \mathbb{R})$, $(i \in \mathcal{N})$. The Filippov solution of system (4.27) is defined in (4.31), and $\psi_j(s)$, $s \in [-\tau, 0]$ is the initial condition of measurable selection of $\gamma_j(s)$.

**Lemma 4.9** *Suppose $x(t)$ is a Filippov solution to (4.27). For any $\epsilon > 0$, let $\Phi = \{x(t+\theta) \in \mathcal{C}([-\tau, 0]; \mathbb{R}^N) : |\gamma_i(t) - \gamma_j(t-\tau_{ij})| < \frac{\epsilon}{2}, |x_i(t) - \gamma_j(t-\tau_{ij})| < \frac{\epsilon}{2}, \ \forall i \in \mathcal{N}, \ j \in \mathcal{N}_i\}$. Then, we have the following conclusion:*

*(i) There exists $T_0$, such that for any $i \in \mathcal{N}$,*

$$|x_i(t+\vartheta) - x_i(t)| \le \epsilon, \quad \forall t \ge T_0, \ \forall \vartheta \in [0, \tau]. \tag{4.41}$$

*(ii) For arbitrary fixed $t_0 \ge 0$, there exists $t_1 \ge t_0$ such that the agents in the network will go into the set of $\Phi$ at time $t_1$.*

***Proof*** Consider the function

$$V(t) = V_1(t) + V_2(t), \tag{4.42}$$

where

$$V_1(t) = \sum_{i=1}^{N} x_i^2(t) + \sum_{i=1}^{N} \int_0^{x_i(t)} q_\mu(s)ds, \tag{4.43}$$

and

$$V_2(t) = \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} \int_{t-\tau_{ij}}^{t} a_{ij}\gamma_j^2(s)ds. \tag{4.44}$$

Note that $cq_\mu(c) \ge 0$ for any $c \in \mathbb{R}$, and then we have $V_1(t) \ge 0$ and $V_2(t) \ge 0$.

Notice that for $p_i(s) = \int_0^s q_\mu(u)du$, and we have

$$\partial_c p_i(s) = \{v \in \mathbb{R} : q_\mu^-(s) \le v \le q_\mu^+(s)\}, \tag{4.45}$$

where $q_\mu^+(s)$ and $q_\mu^-(s)$ denote the right and left limits of the function $q_\mu$ at the point $s$. Based on Lemma 1.19, $V_1(t)$ is differentiable for a.e. $t \ge 0$ and

$$\begin{aligned}
\frac{dV_1(t)}{dt} &= \sum_{i=1}^{N} x_i(t) \sum_{j=1,\, j\neq i}^{N} a_{ij}[\gamma_j(t-\tau_{ij}) - x_i(t)] + \sum_{i=1}^{N} \gamma_i(t) \sum_{j=1,\, j\neq i}^{N} a_{ij} \\
&\quad \times[\gamma_j(t-\tau_{ij}) - x_i(t)] \\
&= \frac{1}{2}\sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_{ij}[2x_i(t)\gamma_j(t-\tau_{ij}) - 2x_i^2(t)] + \frac{1}{2}\sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_{ij} \\
&\quad \times[2\gamma_i(t)\gamma_j(t-\tau_{ij}) - 2\gamma_i(t)x_i(t)] \\
&\le \frac{1}{2}\sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_{ij}[2x_i(t)\gamma_j(t-\tau_{ij}) - 2x_i^2(t)] + \frac{1}{2}\sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_{ij} \\
&\quad \times[2\gamma_i(t)\gamma_j(t-\tau_{ij}) - 2\gamma_i^2(t)].
\end{aligned} \tag{4.46}$$

Since $\gamma_j(t) \in \mathcal{K}[q_\mu(x_j(t))]$, $\forall j \in \mathcal{N}$, we have that $\gamma_j(t)$ is locally integrable. Hence, $V_2(t)$ is differentiable for a.e. $t \ge 0$ and

$$\begin{aligned}
\frac{dV_2(t)}{dt} &= \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_{ij}[\gamma_j^2(t) - \gamma_j^2(t-\tau_{ij})] \\
&= \frac{1}{2}\sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_{ij}[2\gamma_i^2(t) - 2\gamma_j^2(t-\tau_{ij})] \\
&\le \frac{1}{2}\sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_{ij}[\gamma_i^2(t) - \gamma_j^2(t-\tau_{ij})] + \frac{1}{2}\sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_{ij}[x_i^2(t) \\
&\quad -\gamma_j^2(t-\tau_{ij})].
\end{aligned} \tag{4.47}$$

Combining (4.46) and (4.47) gives that

$$\begin{aligned}
\frac{dV(t)}{dt} &= \frac{dV_1(t)}{dt} + \frac{dV_2(t)}{dt} \\
&\le \frac{1}{2}\sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_{ij}[2\gamma_i(t)\gamma_j(t-\tau_{ij}) - \gamma_i^2(t) - \gamma_j^2(t-\tau_{ij})]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N} a_{ij}[2x_i(t)\gamma_j(t-\tau_{ij}) - x_i^2(t) - \gamma_j^2(t-\tau_{ij})] \\
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N} a_{ij}(\gamma_i(t) - \gamma_j(t-\tau_{ij}))^2 \\
&\quad -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N} a_{ij}(x_i(t) - \gamma_j(t-\tau_{ij}))^2 \\
&\leq 0.
\end{aligned} \tag{4.48}
$$

Hence, $V(t)$ is non-increasing for $t$. Together with $V(t) \geq 0$, it gives that $\lim_{t\to+\infty} V(t)$ exists. Let $\bar{a} = \max_{1\leq i<j\leq N,\, a_{ij}>0}\{a_{ij}\}$. Then, for any $\epsilon > 0$ and $i,\ j \in \mathcal{N}$, there exists $T_0$ such that $\forall t \geq T_0,\ \vartheta \in [0,\ \tau]$,

$$
\begin{aligned}
\frac{\epsilon^2}{2N\bar{a}\tau} &\geq |V(t+\vartheta) - V(t)| \\
&= |\int_t^{t+\vartheta} \dot{V}(s)ds| \\
&\geq \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N} a_{ij}\int_t^{t+\vartheta}(x_i(s) - \gamma_j(s-\tau_{ij}))^2 ds \\
&\quad +\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1,\,j\neq i}^{N} a_{ij}\int_t^{t+\vartheta}(\gamma_i(s) - \gamma_j(s-\tau_{ij}))^2 ds.
\end{aligned} \tag{4.49}
$$

Hence, for any $i \in \mathcal{N}$ and $t \geq T_0,\ \vartheta \in [0,\ \tau]$,

$$
\sum_{j=1,\,j\neq i}^{N} a_{ij}\int_t^{t+\vartheta}(x_i(s) - \gamma_j(s-\tau_{ij}))^2 ds \leq \frac{\epsilon^2}{N\bar{a}\tau}. \tag{4.50}
$$

It follows from Lemma 1.20 that

$$
\begin{aligned}
&\left|\sum_{j=1,\,j\neq i}^{N} a_{ij}\int_t^{t+\vartheta}(\gamma_j(s-\tau_{ij}) - x_i(s))ds\right|^2 \\
&\leq \left(\sum_{j=1,\,j\neq i}^{N} a_{ij}\int_t^{t+\vartheta}|\gamma_j(s-\tau_{ij}) - x_i(s)|ds\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq N \sum_{j=1,\ j\neq i}^{N} a_{ij}^2 \left(\int_t^{t+\vartheta} |\gamma_j(s-\tau_{ij}) - x_i(s)| ds\right)^2 \\
&\leq \tau N \sum_{j=1,\ j\neq i}^{N} a_{ij}\bar{a} \int_t^{t+\vartheta} (x_i(s) - \gamma_j(s-\tau_{ij}))^2 ds \\
&\leq \epsilon^2.
\end{aligned}
\tag{4.51}
$$

Hence, for any $i \in \mathcal{N}$ and $t \geq T_0,\ \vartheta \in [0,\ \tau]$,

$$
\left| \sum_{j=1,\ j\neq i}^{N} a_{ij} \int_t^{t+\vartheta} (\gamma_j(s-\tau) - x_i(s)) ds \right| \leq \epsilon.
$$

It follows from (4.31) that

$$
\begin{aligned}
|x_i(t+\vartheta) - x_i(t)| &= \left| \int_t^{t+\vartheta} \dot{x}_i(s) ds \right| \\
&= \left| \sum_{j=1,\ j\neq i}^{N} a_{ij} \int_t^{t+\vartheta} (\gamma_j(s-\tau) - x_i(s)) ds \right| \\
&\leq \epsilon.
\end{aligned}
\tag{4.52}
$$

Thus, for any $\epsilon > 0$ and $i \in \mathcal{N}$, there exists $T_0$ such that for $\forall t \geq T_0,\ \vartheta \in [0,\ \tau]$,

$$
|x_i(t+\vartheta) - x_i(t)| \leq \epsilon. \tag{4.53}
$$

Next, we will prove conclusion (ii). Let $J = \{t \geq t_0 : x(t+\theta) \notin \Phi\}$. For $x(t+\theta) \in \mathcal{C}([-\tau, 0]; \mathbb{R}^N)$ and $t \in J$, there exist $i,\ j \in \mathcal{N},\ i \neq j$ and $a_{ij} \neq 0$ such that

$$
|\gamma_i(t) - \gamma_j(t-\tau_{ij})| \geq \frac{\epsilon}{2}, \tag{4.54}
$$

or

$$
|x_i(t) - \gamma_j(t-\tau_{ij})| \geq \frac{\epsilon}{2}. \tag{4.55}
$$

Hence, for a.e. $t \in J$,

$$\begin{aligned}\dot{V}(t) &\leq -\frac{1}{8}\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}\epsilon^2 \\ &\leq -\frac{1}{8}\varsigma\epsilon^2,\end{aligned} \tag{4.56}$$

where $\varsigma = \sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N} a_{ij}$. Next, we will prove the claim by contradiction.

Suppose that $t \in J$ for any $t \geq t_0$. Then, inequality (4.56) implies that

$$V(t) - V(t_0) \leq -\frac{1}{8}\varsigma\epsilon^2(t - t_0),\ t \geq t_0. \tag{4.57}$$

For $t > \frac{8V(t_0)}{\varsigma\epsilon^2} + t_0$, it follows from inequality (4.57) that $V(t) < 0$, which is a contradiction to the definition of $V(t)$. Therefore, for arbitrary $t_0 \geq 0$, there exists $\bar{t}_0 \geq t_0$ such that the agents in the network will go into the set of $\Phi$ at time $\bar{t}_0$.

This completes the proof of this lemma.

**Theorem 4.10** *Consider the multi-agent network (4.27) with communication topology that is defined by an undirected, connected graph $G$. Then, for any finite communication delay $\tau_{ij}$, the multi-agent network will achieve consensus, i.e., there exists a constant $c$ such that*

$$\lim_{t\to+\infty} x_i(t) = c. \tag{4.58}$$

***Proof*** ***Step 1*** *We shall show some inequality to be used at later steps.*

For arbitrary $\epsilon > 0$ (without loss of generality, assume $\epsilon < \frac{\mu}{N}$), it follows from Lemma 4.9 that there exists $T_0 > 0$ such that for any $i \in \mathcal{N}$,

$$|x_i(t+\vartheta) - x_i(t)| \leq \epsilon, \quad \forall t \geq T_0,\ \forall\vartheta \in [0,\ \tau]. \tag{4.59}$$

Moreover, there exists $T_1 \geq T_0$ such that for any $i \in \mathcal{N}$ and $j \in \mathcal{N}_i$,

$$|\gamma_i(T_1) - \gamma_j(T_1 - \tau_{ij})| < \frac{\epsilon}{4}, \tag{4.60}$$

and

$$|x_i(T_1) - \gamma_j(T_1 - \tau_{ij})| < \frac{\epsilon}{4}. \tag{4.61}$$

It follows from (4.60) and (4.61) that

$$\begin{aligned}|x_i(T_1)-\gamma_i(T_1)| &\le |\gamma_i(T_1)-\gamma_j(T_1-\tau_{ij})|+|x_i(T_1)-\gamma_j(T_1-\tau_{ij})|\\ &< \frac{\epsilon}{4}+\frac{\epsilon}{4}=\frac{\epsilon}{2}.\end{aligned}\tag{4.62}$$

***Step 2*** *Fix $x_i(T_1)$*, and *without loss of generality, we assume $x_i(T_1)\ge 0$ (the proof is similar for $x_i(T_1)<0$). We shall prove that $|x_j(T_1)-x_i(T_1)|\le\epsilon,\ j\in\mathcal{N}_i$, by considering the following two cases:*

Case 1: $x_i(T_1)\ne k_0\mu,\ \forall k_0\in\mathbb{Z}$. Then,

$$\gamma_i(T_1)=q_\mu(x_i(T_1))=\lfloor\frac{x_i(t)}{\mu}\rfloor\mu.\tag{4.63}$$

If $\gamma_i(T_1)=0$, it is easy to see from (4.59) and (4.62) that $\gamma_i(T_1-\tau_{ji})=\gamma_i(T_1)=0$. Hence, it can be obtained that

$$\begin{aligned}|x_j(T_1)-x_i(T_1)| &\le |x_j(T_1)-\gamma_i(T_1-\tau_{ji})|+|\gamma_i(T_1-\tau_{ji})-\gamma_i(T_1)|\\ &\quad+|\gamma_i(T_1)-x_i(T_1)|\\ &\le \frac{\epsilon}{4}+0+\frac{\epsilon}{2}\\ &<\epsilon.\end{aligned}\tag{4.64}$$

If $\gamma_i(T_1)\ne 0$, we claim that for any $j\in\mathcal{N}_i$,

$$x_j(T_1)\ge\lfloor\frac{x_i(T_1)}{\mu}\rfloor\mu.\tag{4.65}$$

Otherwise, we have $x_j(T_1)<\lfloor\frac{x_i(T_1)}{\mu}\rfloor\mu$, which implies that

$$\gamma_j(T_1)\le(\lfloor\frac{x_i(T_1)}{\mu}\rfloor-1)\mu.\tag{4.66}$$

It follows from (4.59) that

$$\begin{aligned}x_j(T_1-\tau_{ij}) &\le x_j(T_1)+\epsilon\\ &\le(\lfloor\frac{x_i(T_1)}{\mu}\rfloor-1)\mu+2\epsilon\\ &<\lfloor\frac{x_i(T_1)}{\mu}\rfloor\mu,\end{aligned}\tag{4.67}$$

which implies $\gamma_j(T_1-\tau_{ij}) \le (\lfloor \frac{x_i(t)}{\mu} \rfloor - 1)\mu$. Hence, it can be obtained that

$$\begin{aligned} |x_i(T_1)-\gamma_j(T_1-\tau_{ij})| &\ge |\gamma_i(T_1)-\gamma_j(T_1-\tau_{ij})| - |x_i(T_1)-\gamma_i(T_1)| \\ &\ge \mu - \frac{\epsilon}{2} \\ &> \epsilon, \end{aligned} \tag{4.68}$$

which contradicts with (4.60). Hence,

$$x_j(T_1) \ge \lfloor \frac{x_i(t)}{\mu} \rfloor \mu. \tag{4.69}$$

The inequality $|x_j(T_1)-\gamma_i(T_1-\tau_{ji})| \le \frac{\epsilon}{4}$ implies that

$$\gamma_i(T_1-\tau_{ji}) \ge \lfloor \frac{x_i(t)}{\mu} \rfloor \mu - \frac{\epsilon}{4}. \tag{4.70}$$

It follows from (4.59) and (4.63) that

$$\gamma_i(T_1-\tau_{ji}) \le \lfloor \frac{x_i(t)}{\mu} \rfloor \mu. \tag{4.71}$$

Hence, we have

$$\begin{aligned} |x_j(T_1)-x_i(T_1)| &\le |x_j(T_1)-\gamma_i(T_1-\tau_{ji})| + |\gamma_i(T_1-\tau_{ji})-\gamma_i(T_1)| \\ &\quad + |\gamma_i(T_1)-x_i(T_1)| \\ &\le \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{4.72}$$

Case 2: There exists a $\bar{k}_0 \in \mathbb{Z}$ such that $x_i(T_1) = \bar{k}_0\mu$.
If $\bar{k}_0 = 0$, it follows from $|x_i(T_1)-x_i(T_1-\tau_{ji})| < \epsilon$ that

$$\gamma_i(T_1-\tau_{ji}) = \gamma_i(T_1) = 0. \tag{4.73}$$

Hence, we have

$$\begin{aligned} |x_j(T_1)-x_i(T_1)| &\le |x_j(T_1)-\gamma_i(T_1-\tau_{ji})| + |\gamma_i(T_1-\tau_{ji})-\gamma_i(T_1)| \\ &\quad + |\gamma_i(T_1)-x_i(T_1)| \\ &\le \frac{\epsilon}{4} + 0 + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned} \tag{4.74}$$

If $\bar{k}_0 \geq 1$, we claim that $x_j(T_1) \geq x_i(T_1)$. Otherwise, we have $\gamma_j(T_1) \leq (\bar{k}_0 - 1)\mu$. It follows from (4.59) and (4.62) that

$$\begin{aligned} x_j(T_1 - \tau_{ji}) &\leq x_j(T_1) + \epsilon \\ &\leq \gamma_j(T_1) + \epsilon + \epsilon \\ &\leq (\bar{k}_0 - 1)\mu + 2\epsilon, \end{aligned} \tag{4.75}$$

which implies that

$$\gamma_j(T_1 - \tau_{ji}) \leq (\bar{k}_0 - 1)\mu. \tag{4.76}$$

However, we can obtain from (4.61) that

$$\gamma_j(T_1 - \tau_{ji}) \geq x_i(T_1) - \frac{\epsilon}{4} \geq \bar{k}_0\mu - \frac{\epsilon}{4}, \tag{4.77}$$

which contradicts with (4.76). Hence, $x_j(T_1) \geq x_i(T_1)$.
If $x_j(T_1) = x_i(T_1) = \bar{k}_0\mu$, then

$$|x_j(T_1) - x_i(T_1)| = 0 < \epsilon. \tag{4.78}$$

If $x_j(T_1) > x_i(T_1) = \bar{k}_0\mu$, it can be easily obtained that $\gamma_j(T_1) = \bar{k}_0\mu$. Then,

$$|x_j(T_1) - x_i(T_1)| = |x_j(T_1) - \gamma_j(T_1)| \leq \epsilon. \tag{4.79}$$

In conclusion, we proved that for any $\epsilon > 0$, there exists $T_1 > 0$ such that for any fixed $x_i(T_1)$,

$$|x_j(T_1) - x_i(T_1)| \leq \epsilon, \ j \in \mathcal{N}_i. \tag{4.80}$$

***Step 3*** *We shall show that the multi-agent network (4.27) can achieve consensus.*
Since the network is connected, we can obtain from (4.80) that for any fixed $x_i(T_1)$ and $\forall j \in \mathcal{N}$,

$$|x_j(T_1) - x_i(T_1)| \leq (N - 1)\epsilon. \tag{4.81}$$

Denote $M(t) = \max\limits_{i \in \mathcal{N}} \max\limits_{\theta \in [-\tau, 0]} \{x_i(t + \theta)\}$ and $m(t) = \min\limits_{i \in \mathcal{N}} \min\limits_{\theta \in [-\tau, 0]} \{x_i(t + \theta)\}$. It follows from (4.59) and (4.81) that

$$|M(T_1) - m(T_1)| \leq (N + 1)\epsilon. \tag{4.82}$$

From (4.82), we can assume that $m(T_1)$ and $M(T_1)$ belong to the set of $\Omega_1 = ((k_0 - 1)\mu, \ (k_0 + 1)\mu), \ k_0 = \lfloor \frac{m(T_1)}{\mu} \rfloor$. Without loss of generality, we assume $k_0 \geq 0$ (the proof for $k_0 < 0$ is similar to the case $k_0 > 0$, which we omitted here). Next, we will prove that multi-agent networks achieve consensus by considering the following four cases.

Case 1: $\mu > M(T_1) \geq m(T_1) > -\mu$, i.e., $k_0 = 0$. From the definition of the quantizer function $q_\mu(\cdot)$ and output function $\gamma(\cdot)$, we have that

$$\gamma_i(T_1 + \theta_1) = \gamma_j(T_1 + \theta_2) = 0, \ \forall i, \ j \in \mathcal{N}, \ \forall \theta_1, \ \theta_2 \in [-\tau, \ 0]. \tag{4.83}$$

It follows from Lemma 4.9 that

$$\gamma_i(t) = 0, \ \forall i \in \mathcal{N}, \ t \geq T_1. \tag{4.84}$$

Hence, for $t \geq T_1$, the multi-agent network model is reduced to be

$$\frac{dx_i(t)}{dt} = -(1 - a_{ii})x_i(t), \quad i \in \mathcal{N}. \tag{4.85}$$

From (4.85), it is easy to find that all the agents will achieve consensus and the final consensus value is 0.

Case 2: $k_0 \geq 1$ and $M(T_1) \geq k_0\mu \geq m(T_1)$. From the definition of the quantizer function $q_\mu(\cdot)$ and output function $\gamma(\cdot)$, we can obtain that

$$\gamma_i(T_1 + \theta_1) \in [(k_0 - 1)\mu, \ k_0\mu], \ \ \forall i \in \mathcal{N}, \ \forall \theta_1 \in [-\tau, \ 0]. \tag{4.86}$$

It follows from Lemma 4.9 that

$$\gamma_i(t) \in [(k_0 - 1)\mu, \ k_0\mu], \ \ \forall i \in \mathcal{N}, \ \forall t \geq T_1. \tag{4.87}$$

If there exists $t_1 \geq T_1$ such that $x_i(t_1) < k_0\mu$, (4.27) implies that

$$x_i(t) < k_0\mu, \ \forall i \in \mathcal{N}, \ \forall t \geq t_1. \tag{4.88}$$

If $m(T_1) < k_0\mu$, clearly, there exist $i_m \in \mathcal{N}$ and $\theta_m \in [-\tau, \ 0]$ such that

$$m(T_1) = x_{i_m}(T_1 + \theta_m) \quad \text{and} \quad \gamma_{i_m}(T_1 + \theta_m) = (k_0 - 1)\mu. \tag{4.89}$$

For $t \geq T_1$, we obtain that

$$(k_0 - 1)\mu < x_{i_m}(t) < k_0\mu \quad \text{and} \quad \gamma_{i_m}(t) = (k_0 - 1)\mu. \tag{4.90}$$

For any $j$ such that $i_m \in \mathcal{N}_j$, we have

$$\frac{dx_j(t)}{dt} = \sum_{l \in \mathcal{N}_j} a_{jl}(\gamma_l(t - \tau_{jl}) - x_j(t)). \tag{4.91}$$

Hence, if $x_j(t_1) \geq k_0\mu$, there exists $t_2 \geq t_1$ such that $x_j(t_2) < k_0\mu$. Since the network is connected, it follows from (4.88) and (4.91) that there exists $T_2 \geq t_2$ such that

$$(k_0 - 1)\mu < x_i(t) < k_0\mu, \ \forall i \in \mathcal{N}, \ \forall t \geq T_2, \tag{4.92}$$

and Eq. (4.30) is reduced to be

$$\frac{dx_i(t)}{dt} = -(1 - a_{ii})(x_i(t) - (k_0 - 1)\mu), \quad i \in \mathcal{N}, \ t \geq T_2 + \tau. \tag{4.93}$$

Hence, the multi-agent network will achieve consensus, and the final consensus value is $(k_0 - 1)\mu$.

If $m(T_1) = k_0\mu$, by similar analyses, we can also obtain that the multi-agent network will achieve consensus and the final consensus value is $(k_0 - 1)\mu$ or $k_0\mu$.

Case 3: $M(T_1) \geq m(T_1) > k_0\mu$. In this case, we have

$$\gamma_i(T_1 + \theta_1) = k_0\mu, \ \ \forall i \in \mathcal{N}, \ \forall \theta_1 \in [-\tau, \ 0].$$

It follows from (4.30) and Lemma 4.9 that

$$(k_0 + 1)\mu > x_i(t) > k_0\mu \quad \text{and} \quad \gamma_i(t) = k_0\mu, \ \ \forall i \in \mathcal{N}, \quad \forall t \geq T_1.$$

Then, the system (4.30) is reduced to be

$$\frac{dx_i(t)}{dt} = -(1 - a_{ii})(x_i(t) - k_0\mu), \quad i \in \mathcal{N}. \tag{4.94}$$

It is easy to see that all the agents will achieve consensus and the final consensus value is $k_0\mu$.

Case 4: $k_0\mu > M(T_1) \geq m(T_1)$.

The analysis of this case is similar to Case 3, which is omitted here. In this case, the multi-agent network will achieve consensus and the final consensus value is $(k_0 - 1)\mu$.

In conclusion, the multi-agent network (4.27) achieves consensus asymptotically. This completes the proof of this theorem.

*Remark 4.11* Different from Chap. 3, we have shown that the multi-agent network can achieve complete consensus other than practical consensus in this section. However, it is difficult to estimate the final consensus state $c$ of model (4.27). It is an interesting problem to estimate how the final consensus value depends on the quantization and time delays in our future work.

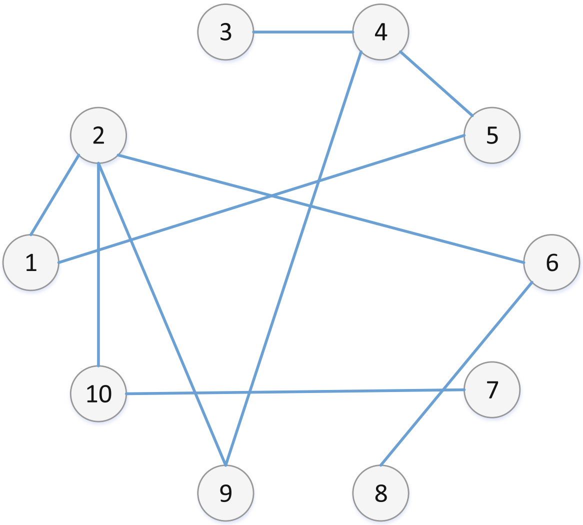


**Fig. 4.3** Network topology in the simulation example

### *4.2.4 Numerical Example*

Consider the multi-agent system (4.27) with communication quantization and time delays, where the network structure is shown in Fig. 4.3 with the weights on the connections. The graph (Fig. 4.3) is generated by the scale-free algorithm. Suppose that initial conditions are randomly chosen from (0, 10).

Figure 4.4 shows the state responses of multi-agent network (4.27) with respect to $\mu = 1$. It can be observed that the agents converge to a constant value, which illustrates Theorem 4.10 very well.

## 4.3 Summary

In this chapter, we mainly addressed the consensus problem of multi-agent networks where each agent can only obtain the quantized and delayed measurements of the states of its neighbors. Discrete-time formulation of the problem was studied first. We showed that the multi-agent network can achieve consensus for arbitrary finite communication delays. For the continuous-time cases, it was shown that Filippov solutions of the resulting system exist for any initial condition. We have proved that for the multi-agent network model considering quantization and time delays simultaneously, Filippov solutions of the resulting system converged to a constant value asymptotically under certain network topology. The theoretical results have been well illustrated by numerical examples.

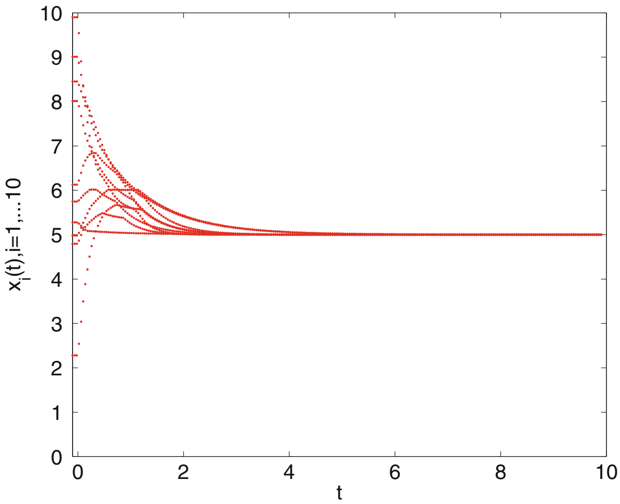


**Fig. 4.4** The states of the system in example

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# Chapter 5
# Event-Based Network Consensus with Communication Delays

Since the broad bandwidth of networks is unavailable in some cases, sampled control for multi-agent system is more coincident with the applications in our real life [1–3]. Under fixed undirected/directed interaction, two consensus algorithms for double-integrator dynamics within a uniform sampled-data setting were proposed in [4]. The results of reference [4] were extended to the dynamic network topology cases in [5]. Unlike time-driven control approach (i.e., periodic sampling), event-triggered control means that the control signals are kept constant until the certain condition is violated, and then the control signal will be updated (or recomputed). Some related results about event-triggered control have been reported in [6–14].

In [6], event-triggered control was applied to the continuous-time multi-agent consensus problem. Centralized and distributed triggering conditions that ensure the consensus of multi-agent network are presented, respectively. In [12], Seyboth et al. studied the multi-agent consensus under event-triggered control and three kinds of problems of networks of single-integrator agents with and without communication delays, and networks of double-integrator agents were discussed. It is worth noting that the framework in [6] and [12] about event-triggered multi-agent consensus only considered the continuous-time cases, and the network topology was assumed to be undirected or balanced.

In this chapter, the event-triggered cooperative control strategies are proposed for discrete-time/continuous-time directed multi-agent network. The distributed event-triggered controls with communication delays are analyzed. Under the assumption that the multi-agent network is strongly connected, consensus can be achieved for the proposed multi-agent networks.



J. Lu et al., *Collective Behavior in Complex Networked Systems under Imperfect Communication*, https://doi.org/10.1007/978-981-16-1506-1_5

## 5.1 Distributed Discrete-Time Event-Triggered Consensus with Delays

Time delay is a very important communication constraint in the process of information exchange and should be considered in the consensus protocol. In this section, we shall apply the event-triggered control to the problem of multi-agent consensus with communication delays. In particular, each agent decides when to transmit current state to its neighbors based on the received neighbors' information, latest broadcast information, and current state, i.e., only local information is used. The organization of the remaining part is given as follows. In Sect. 5.1.1, the problem formulation is presented. In Sect. 5.1.2, the consensus analysis of the proposed protocol is presented in detail. In Sect. 5.1.3, a numerical simulation example is given to show the effectiveness of the theoretical results.

We make the following assumption in this chapter:

**Assumption 5.1** $a_{ii} > 0$ for any $i \in \mathcal{N}$.

### *5.1.1 Model Description*

The distributed event-triggered cooperative control strategy requires that each agent can only use received neighbors' information and its own information to decide whether the event-triggered condition is satisfied.

In this section, we assume that each agent $i$ will broadcast its latest state to neighbors when the state measurement error of agent $i$ exceeds the prescribed level (i.e., the "event" occurs). Suppose that $k_0^i, k_1^i, \cdots, k_l^i, \cdots$ is the sequence of the event times of the agent $i$ which is defined based on the event-triggering condition. $\widehat{x}_i(k)$ denotes the latest broadcast state of agent $i$, which is given by $\widehat{x}_i(k) = x_i(k_l^i),\ k \in [k_l^i, k_{l+1}^i)$. Hence, the consensus model in this case is given by

$$x_i(k+1) = x_i(k) + \iota \sum_{j \in \mathcal{N}_i} \overline{a}_{ij}(\widehat{x}_j(k - \tau_{ij}) - \widehat{x}_i(k)), \tag{5.1}$$

where $\tau_{ij} > 0$ is the communication delay from agent $j$ to agent $i$, $\tau_{ii} = 0$. $\widehat{x}_j(k - \tau_{ij}) = x_j(k_{l'}^j),\ k - \tau_{ij} \in [k_{l'}^j, k_{l'+1}^j)$, and $\widehat{x}_i(k) = x_i(k_l^i),\ k \in [k_l^i, k_{l+1}^i)$.

Let $A = [a_{ij}]$ with $a_{ij} = \iota\bar{a}_{ij} \geq 0$ for $i \neq j$ and $a_{ii} = 1 - \sum_{j=1,\, j\neq i}^{N} a_{ij}$. The state measurement error of agent $i$ is defined as

$$e_i(k) = x_i(k) - \widehat{x}_i(k). \tag{5.2}$$

Then, the dynamics of multi-agent network (5.1) can be rewritten as

$$x_i(k+1) = e_i(k) + \sum_{j=1}^{N} a_{ij}\widehat{x}_j(k-\tau_{ij}), \quad i \in \mathcal{N}. \tag{5.3}$$

In this section, Assumption 5.1 is required.

### 5.1.2 Distributed Event-Triggered Approach

Denote $\tau = \max\{\tau_{ij},\ i = 1, \cdots, N,\ j \in \mathcal{N}_i\}$ and $X = \{\psi : \{-\tau, -\tau + 1, \cdots, -1, 0\} \to \mathbb{R}^N\}$, and suppose that the initial condition of the network is $\phi_i \in X, i \in \mathcal{N}$.

**Theorem 5.2** *Consider the multi-agent network (5.1), and assume that the communication graph $G$ is strongly connected. Then, for any finite communication delay $\tau_{ij}$, the network will achieve consensus asymptotically under the triggering condition given by*

$$e_i^2(k) > \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} a_{ij}(\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))^2, \quad i \in \mathcal{N}, \tag{5.4}$$

*where $0 < \sigma < 1$ is a constant. Moreover, the final consensus value is*

$$\frac{\sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij} \sum_{s=-\tau_{ij}}^{-1} x_j(s)}{1 + \sum_{i=1}^{N} \xi_i (1-a_{ii})\tau_{ij}}.$$

***Proof*** Consider the Lyapunov functional as

$$V(k) = V_1(k) + V_2(k), \tag{5.5}$$

where

$$V_1(k) = \sum_{i=1}^{N} \xi_i x_i^2(k), \tag{5.6}$$

and

$$V_2(k) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} \sum_{s=k-\tau_{ij}}^{k-1} \widehat{x}_j^2(s). \tag{5.7}$$

Differencing $V(k)$ along the solution of (5.3) gives that

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k), \tag{5.8}$$

where

$$\begin{aligned}
\Delta V_1(k) &= \sum_{i=1}^{N} \xi_i x_i^2(k+1) - \sum_{i=1}^{N} \xi_i x_i^2(k) \\
&= \sum_{i=1}^{N} \xi_i [e_i(k) + \sum_{j=1}^{N} a_{ij} \widehat{x}_j(k-\tau_{ij})]^2 - \sum_{i=1}^{N} \xi_i x_i^2(k) \\
&= \sum_{i=1}^{N} \xi_i [e_i^2(k) + \sum_{j=1,\, j\neq i}^{N} a_{ij}^2 \widehat{x}_j^2(k-\tau_{ij}) + a_{ii}^2 \widehat{x}_i^2(k) + 2 \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} \\
&\quad a_{ij} a_{il} \widehat{x}_j(k-\tau_{ij}) \widehat{x}_l(k-\tau_{il}) + 2 \sum_{j=1,\, j\neq i}^{N} a_{ij} a_{ii} \widehat{x}_j(k-\tau_{ij}) \widehat{x}_i(k) \\
&\quad + 2 a_{ii} \widehat{x}_i(k) e_i(k) + 2 \sum_{j=1,\, j\neq i}^{N} a_{ij} \widehat{x}_j(k-\tau_{ij}) e_i(k)] - \sum_{i=1}^{N} \xi_i [\widehat{x}_i^2(k) \\
&\quad + e_i^2(k) + 2 \widehat{x}_i(k) e_i(k)],
\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}
\Delta V_2(k) &= \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} [\sum_{k+1-\tau_{ij}}^{k} \widehat{x}_j^2(s) - \sum_{k-\tau_{ij}}^{k-1} \widehat{x}_j^2(s)] \\
&= \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} [\widehat{x}_j^2(k) - \widehat{x}_j^2(k-\tau_{ij})] \\
&= \frac{1}{2} \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} \sum_{l=1}^{N} a_{ij} a_{il} [\widehat{x}_j^2(k) - \widehat{x}_j^2(k-\tau_{ij}) + \widehat{x}_l^2(k) - \widehat{x}_l^2(k-\tau_{il})] \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} a_{ij}^2 (\widehat{x}_j^2(k) - \widehat{x}_j^2(k-\tau_{ij})) + \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} a_{ij} a_{il} (\widehat{x}_j^2(k) \\
&\quad - \widehat{x}_j^2(k-\tau_{ij}) + \widehat{x}_l^2(k) - \widehat{x}_l^2(k-\tau_{il})) + \sum_{j=1,\, j\neq i}^{N} a_{ij} a_{ii} (\widehat{x}_j^2(k) \\
&\quad - \widehat{x}_j^2(k-\tau_{ij}))].
\end{aligned} \tag{5.10}$$

Substituting (5.9) and (5.10) into (5.8), we can obtain that

$$\begin{aligned}
\Delta V(k) = &\sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} a_{ij}^2 \widehat{x}_j^2(k) + a_{ii}^2 \widehat{x}_i^2(k) + \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} a_{ij} a_{il} (\widehat{x}_j^2(k) \\
&+ \widehat{x}_l^2(k) - \widehat{x}_j^2(k-\tau_{ij}) - \widehat{x}_l^2(k-\tau_{il}) + 2a_{ij} a_{il} \widehat{x}_j(k-\tau_{ij}) \widehat{x}_l(k-\tau_{il})) \\
&+ \sum_{j=1,\, j\neq i}^{N} a_{ij} a_{ii} (\widehat{x}_j^2(k) - \widehat{x}_j^2(k-\tau_{ij}) + 2\widehat{x}_j(k-\tau_{ij}) \widehat{x}_i(k)) + 2a_{ii} \widehat{x}_i(k) \cdot \\
&e_i(k) + 2 \sum_{j=1,\, j\neq i}^{N} a_{ij} \widehat{x}_j(k-\tau_{ij}) e_i(k)] - \sum_{i=1}^{N} \xi_i [\widehat{x}_i^2(k) + 2\widehat{x}_i(k) e_i(k)] \\
= &\sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} a_{ij}^2 \widehat{x}_j^2(k) + \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} a_{ij} a_{il} (\widehat{x}_j^2(k) + \widehat{x}_l^2(k)) - \widehat{x}_i^2(k) \\
&+ \sum_{j=1,\, j\neq i}^{N} a_{ij} a_{ii} (\widehat{x}_j^2(k) + \widehat{x}_i^2(k)) - \sum_{j=1,\, j\neq i}^{N} a_{ij} a_{ii} (\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))^2 \\
&- \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} a_{ij} a_{il} (\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_l(k-\tau_{il}))^2] \\
&+ 2\sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} a_{ij} e_i(k) (\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))]. \qquad (5.11)
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} a_{ij}^2 \widehat{x}_j^2(k) + \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} a_{ij} a_{il} (\widehat{x}_j^2(k) + \widehat{x}_l^2(k)) \\
&+ \sum_{j=1,\, j\neq i}^{N} a_{ij} a_{ii} (\widehat{x}_j^2(k) + \widehat{x}_i^2(k)) - \widehat{x}_i^2(k)] \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} a_{ij}^2 \widehat{x}_j^2(k) + \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} a_{ij} a_{il} (\widehat{x}_j^2(k) + \widehat{x}_l^2(k)) \\
&+ \sum_{j<i}^{N} a_{ij} a_{ii} (\widehat{x}_j^2(k) + \widehat{x}_i^2(k)) + \sum_{l>i}^{N} a_{il} a_{ii} (\widehat{x}_l^2(k) + \widehat{x}_i^2(k)) - \widehat{x}_i^2(k)]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} a_{ij}^2 \widehat{x}_j^2(k) + \sum_{j=1}^{N} \sum_{l=1,\, l\neq j}^{N} a_{ij} a_{il} \widehat{x}_j^2(k) - \widehat{x}_i^2(k)] \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} \sum_{l=1}^{N} a_{ij} a_{il} \widehat{x}_j^2(k) - \widehat{x}_i^2(k)] \\
&= \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \widehat{x}_j^2(k) - \sum_{i=1}^{N} \xi_i \widehat{x}_i^2(k) \\
&= \sum_{j=1}^{N} \xi_j \widehat{x}_j^2(k) - \sum_{i=1}^{N} \xi_i \widehat{x}_i^2(k) \\
&= 0,
\end{aligned} \tag{5.12}
$$

and

$$
\begin{aligned}
&2\sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} a_{ij} e_i(k)(\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))] \\
&\quad \leq \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij} [\frac{1}{\alpha_i} e_i^2(k) + \alpha_i (\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))^2].
\end{aligned} \tag{5.13}
$$

Hence,

$$
\begin{aligned}
\Delta V(k) = &-\sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} a_{ij} a_{il} (\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_l(k-\tau_{il}))^2 \\
&+ \sum_{j=1,\, j\neq i}^{N} a_{ij} a_{ii} (\widehat{x}_i(k) - \widehat{x}_j(k-\tau_{ij}))^2] \\
&+2\sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} a_{ij} e_i(k) \cdot (\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))] \\
\leq &-\sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} a_{ij} a_{il} (\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j\neq i}^{N} a_{ij} \\
&\times (a_{ii} - \alpha_i)(\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))^2] + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} \frac{a_{ij}}{\alpha_i} e_i^2(k).
\end{aligned} \tag{5.14}
$$

Thus, a sufficient condition for $\Delta V(k) \le 0$ is given by

$$e_i^2(k) \le \frac{\sigma\alpha_i(a_{ii}-\alpha_i)}{1-a_{ii}} \sum_{j=1,\, j\ne i}^{N} a_{ij}(\widehat{x}_j(k-\tau_{ij})-\widehat{x}_i(k))^2. \tag{5.15}$$

Let $f(\alpha_i) = \frac{\alpha_i(a_{ii}-\alpha_i)}{1-a_{ii}}$. Then, we can easily obtain the maximum of $f(\alpha_i)$ by taking $\alpha = \frac{a_{ii}}{2}$, which makes (5.15) become

$$e_i^2(k) \le \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\ne i}^{N} a_{ij}(\widehat{x}_j(k-\tau_{ij})-\widehat{x}_i(k))^2. \tag{5.16}$$

Hence, we can choose the trigger condition

$$e_i^2(k) > \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\ne i}^{N} a_{ij}(\widehat{x}_j(k-\tau_{ij})-\widehat{x}_i(k))^2, \quad i \in \mathcal{N}. \tag{5.17}$$

Hence, under the trigger condition (5.17), we have $\Delta V(k) \le 0$, $\forall k \ge 0$. According to LaSalle's invariance principle, we can conclude that all agents in the network will converge to the maximal positively invariant set of the set $\Phi = \{x(k+\theta) \in X : \Delta V(k) = 0\}$ asymptotically. Note that $\Delta V(k) = 0$ if and only if $e_i(k) = 0$ and

$$\widehat{x}_j(k-\tau_{ij}) = \widehat{x}_i(k), \;\; \forall i,\; j \in \mathcal{N}_i. \tag{5.18}$$

Substituting (5.18) into (5.1) yields that

$$x_i(k+1) = x_i(k), \;\; \forall i \in \mathcal{N}. \tag{5.19}$$

Hence, we have

$$x_i(k) = \widehat{x}_i(k) = \widehat{x}_j(k-\tau_{ij}) = x_j(k-\tau_{ij}) = x_j(k), \;\; \forall j \in \mathcal{N}_i. \tag{5.20}$$

Since the topology of the network is strongly connected, we have

$$x_i(k) = x_j(k), \;\; k \ge -\tau_{ij}, \; \forall i, \; j \in \mathcal{N}. \tag{5.21}$$

Therefore, by LaSalle's invariance principle, we obtain

$$\lim_{k\to\infty}(x_j(k) - x_i(k)) = 0. \tag{5.22}$$

Next, the consensus value $c$ of the multi-agent network is shown below. One can prove that the value $c$ is dependent on the initial values of the multi-agent network. Let

$$\eta(k) = \sum_{i=1}^{N} \xi_i x_i(k) + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \sum_{s=k-\tau_{ij}}^{k-1} \widehat{x}_j(s). \tag{5.23}$$

We can calculate the difference of $\eta(k)$ as follows:

$$\begin{aligned}
\Delta\eta(k) &= \eta(k+1) - \eta(k) \\
&= \sum_{i=1}^{N} \xi_i (x_i(k+1) - x_i(k)) + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \Big( \sum_{s=k+1-\tau_{ij}}^{k} \widehat{x}_j(s) - \sum_{s=k-\tau_{ij}}^{k-1} \widehat{x}_j(s) \Big) \\
&= \sum_{i=1}^{N} \xi_i a_{ij} \sum_{j=1}^{N} (\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k)) + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} (\widehat{x}_j(k) - \widehat{x}_j(k-\tau_{ij})) \\
&= -\sum_{i=1}^{N} \xi_i a_{ij} \sum_{j=1}^{N} \widehat{x}_i(k) + \sum_{i=1}^{N} \xi_i a_{ij} \sum_{j=1}^{N} \widehat{x}_j(k) \\
&= -\sum_{i=1}^{N} \xi_i \widehat{x}_i(k) + \sum_{j=1}^{N} \xi_j \widehat{x}_j(k) \\
&= 0.
\end{aligned} \tag{5.24}$$

Due to $\Delta\eta(k) = 0$ for $k \geq 0$, it can be easily obtained that $\eta(k)$ is a constant; that is,

$$\begin{aligned}
\eta(k) &= \eta(0) \\
&= \sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \sum_{s=-\tau_{ij}}^{-1} \widehat{x}_j(s) \\
&= \sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \sum_{s=-\tau_{ij}}^{-1} x_j(s).
\end{aligned} \tag{5.25}$$

Hence,

$$\eta(0) = \lim_{k \to +\infty} \eta(k) = c + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \tau_{ij} c. \tag{5.26}$$

Therefore, we can conclude that

$$c = \frac{\sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \sum_{s=-\tau_{ij}}^{-1} x_j(s)}{1 + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \tau_{ij}}. \tag{5.27}$$

This completes the proof of this theorem.

*Remark 5.3* In Theorem 5.2, we have proved that the multi-agent network will achieve consensus under the event-triggered condition (5.17). For each agent $i$, the sequence of event times $0 = k_0^i < k_1^i < \cdots < k_l^i < \cdots$ is defined iteratively as

$$k_{l+1}^i = \inf\{k : k > k_l^i, g_i(k) > 0\}, \tag{5.28}$$

where

$$g_i(k) = (x_i(k) - x_i(k_l^i))^2 - \frac{\sigma a_{ii}^2}{4(1 - a_{ii})} \sum_{j=1,\, j \neq i}^{N} a_{ij} (\widehat{x}_j(k - \tau_{ij}) - x_i(k_l^i))^2. \tag{5.29}$$

*Remark 5.4* It should be emphasized that the event-triggered condition (5.17) is verified by each agent only based on its own and its neighboring agents' information, i.e., only local information is used to verify the event-triggered condition.

*Remark 5.5* It can be observed that the number of event-triggered times will be reduced as the step size $\iota$ decreases. This is an advantage to reduce the traffic loading of the communication channel. However, the convergence speed of the network will be decreased as the step size $\iota$ decreases. In practice, for achieving the consensus of the system (5.1) under the event-triggered control, there is often a tradeoff between the number of event-triggered and convergence speed.

*Remark 5.6* In [15], event-triggered average consensus control for discrete-time multi-agent model without communication delays is investigated. In this chapter, using different methods, we extend the previous results from the following three aspects:

- We do not assume the network to be undirected or balanced.
- The event-triggered condition given in Sect. 5.1 is not expressed in terms of the linear matrix inequality. Moreover, the event-triggered condition given in Sect. 5.1 does not use the final consensus information of the network.
- The event-triggered condition given in [15] is centralized. In Sect. 5.1, the distributed event-triggered condition is provided when the communication delays exist in multi-agent networks.

### 5.1.3 Numerical Example

Consider the multi-agent system (5.1) with 7 agents. The directed network topology is displayed in Fig. 5.1, and the weight of each edge is set as 1, i.e., $\bar{a}_{ij} = 1$. The step size $\iota$ and constant $\sigma$ are set to be $\frac{1}{5}$ and 0.9. Assume $\tau_{ij} = 1$. The initial conditions are randomly chosen from $[-5, 5]$.

Figure 5.2 shows the simulation result for the distributed event-triggered control for multi-agent network (5.1). We can see from Fig. 5.2 that the multi-agent system reaches consensus. It can be seen from Fig. 5.3 that the event is triggered totally 36, 39, 37, 36, 35 times, respectively, during the evolution of the agents' states.

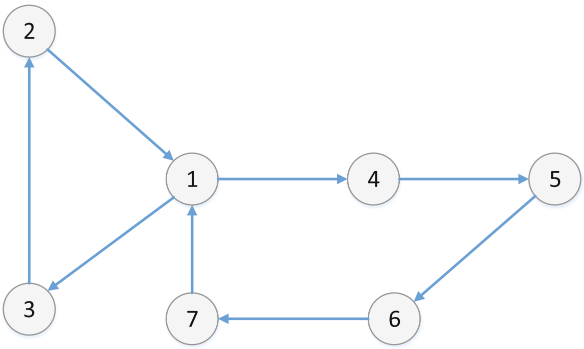


**Fig. 5.1** Network topology in example

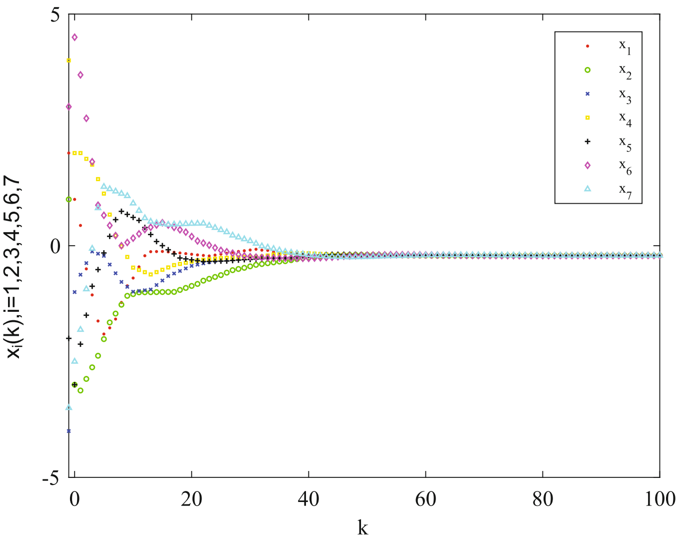


**Fig. 5.2** The states of the system in example

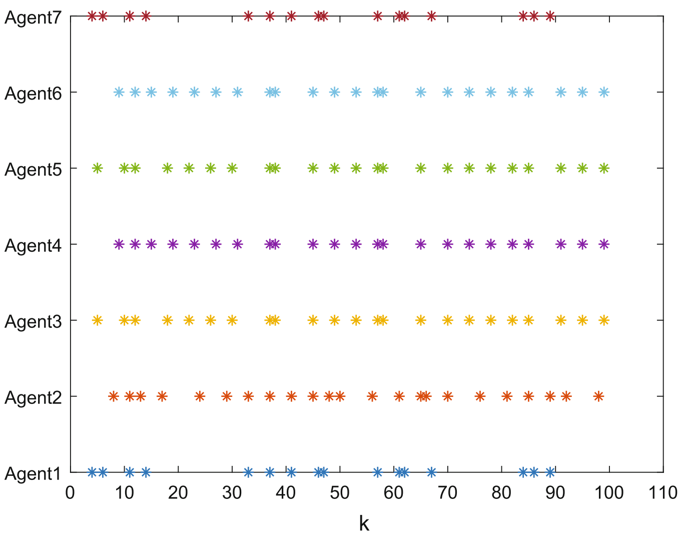


**Fig. 5.3** Event-triggered times in example

This example demonstrates that the event-triggered control approach is an effective method for the multi-agent network consensus problem.

Suppose that the consensus error of multi-agent network (5.1) is defined by

$$e(k) = \sqrt{\sum_{i=1}^{7} \sum_{j=1,\, j \neq i}^{7} (x_i(k) - x_j(k))^2}.$$

Select $\varepsilon = 0.001$, and the convergence time $k$ (select $T = 10$ in this simulation example) of the multi-agent network is defined to make

$$\frac{1}{T} \sum_{s=k}^{k+T} e(k) \leq \varepsilon.$$

In the simulation, the system runs for 200 iterations. In Table 5.1, [min, max] and average denote the minimum, the maximum and the average event-triggered times of the agents in the evolvement of the state of multi-agent networks. It can be seen from Table 5.1 that as the decreasing of the step size $\iota$, the number of event-triggered times will be reduced, but the convergence time of the agents will be increased.

**Table 5.1** The number of event-triggered and convergence time versus $\iota$

| $\iota$ | [min, max] | Average | Convergence time |
|---|---|---|---|
| $\frac{1}{3}$ | 121,132 | 126.6 | 58 |
| $\frac{1}{5}$ | 55,79 | 73 | 79 |
| $\frac{1}{8}$ | 37,46 | 43.8 | 99 |
| $\frac{1}{10}$ | 27,39 | 34 | 116 |
| $\frac{1}{20}$ | 15,19 | 17.7 | 161 |

## 5.2 Distributed Continuous-Time Event-Triggered Consensus with Delays

Different from previous work, this section will study the continuous-time multi-agent consensus problem with distinct communication delays under event-triggered control. The main contributions of this section can be listed as follows:

- Event-based consensus protocol considering distinct finite communication delays is proposed. Different from [16–18], our communication protocol does not require the agents continuously send their state to their neighboring agents. Distributed event-triggered condition is designed to achieve the multi-agent consensus.
- In [16, 19, 20], the final consensus value is hard to estimate due to the constraint of the event-triggered protocol. In this section, by distilling an invariant value of the multi-agent system, the final consensus value is theoretically obtained even in the presence of event-based communication and distinct communication delays.
- A novel synchronously event-triggered consensus protocol is proposed. Furthermore, it is shown that the Zeno behavior can be excluded under our proposed event-based protocol with communication delays. To avoid verifying the event-triggered condition continuously, a self-triggered algorithm is proposed for the multi-agent system with distinct communication delays.

### *5.2.1 Model Description*

Consider a continuous-time multi-agent network with $N$ agents. The dynamics of each agent can be described by

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{N}, \tag{5.30}$$

where $x_i(t) \in \mathbb{R}$ is the state of the agent $i$, and $u_i(t)$ is called the consensus protocol.

Suppose that $t_1^i, t_2^i, \cdots, t_l^i, \cdots$ is the sequence of the event-triggered instants of the agent $i$ which will be defined based on the event-triggering condition. Considering time delay as another very important communication constraint in the process of information exchanging, we propose the following protocol:

$$u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(\widehat{x}_j(t - \tau_{ij}) - \widehat{x}_i(t)), \quad i \in \mathcal{N}, \tag{5.31}$$

where $\tau_{ij} > 0$ is the communication delay from agent $j$ to agent $i$, $\widehat{x}_j(t - \tau_{ij}) = x_j(t_{l'}^j)$, $t - \tau_{ij} \in [t_{l'}^j, t_{l'+1}^j)$, and $\widehat{x}_i(t) = x_i(t_l^i)$, $t \in [t_l^i, t_{l+1}^i)$. We assume that delays affect only the information that is actually being transmitted from one agent to another, i.e., $\tau_{ii} = 0$, and the communication delays $\tau_{ij} > 0$, $i \neq j$, can be distinct from each other.

The state measurement error of agent $i$ is defined as

$$e_i(t) = x_i(t) - \widehat{x}_i(t). \tag{5.32}$$

*Remark 5.7* Comparing with the previous event-triggered consensus work [6, 12, 21], the proposed consensus protocol does not assume the network topology to be undirected or balanced. Different from the pull-based communication protocol

$$u_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t_k^i) - x_i(t_k^i)), \quad \forall i \in \mathcal{N}, \ \ t \in [t_k^i, t_{k+1}^i), \tag{5.33}$$

proposed in [16–20], our communication protocol does not require that the agents $j \in \mathcal{N}_i$ send their states to agent $i$ at agent $i$'s event-triggered time instants. Moreover, the distinct communication delays are considered, and it makes our protocol more realistic compared with previous consensus models.

### 5.2.2 *Asynchronously Distributed Event-Triggered Approach*

In this section, we mainly consider the distributed event-triggered consensus for multi-agent network with communication delays. Denote $\tau = \max\{\tau_{ij}, i = 1, \cdots, N, j \in \mathcal{N}_i\}$. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ be the normalized left eigenvector of Laplacian matrix $L$ with respect to the eigenvalue 0 satisfying $\sum_{i=1}^{N} \xi_i = 1$. It can be obtained that $\xi_i > 0$ from the Perron–Frobenius theorem (see [22]). The initial conditions associated with (5.30) are given as $x_i(s) = \phi_i(s) \in \mathcal{C}([-\tau, 0], \mathbb{R})$, $i \in \mathcal{N}$.

**Theorem 5.8** *Consider the multi-agent network (5.30) with a strongly connected graph $\mathcal{G}$ and a control law (5.33). For any finite communication delay $\tau_{ij}$, the* *network will achieve consensus asymptotically under the triggering condition given by*

$$e_i^2(t) > \frac{\sigma}{4d_i} \sum_{j=1,\, j\neq i}^{N} a_{ij}(\widehat{x}_j(t-\tau_{ij}) - \widehat{x}_i(t))^2, \quad i \in \mathcal{N},$$

*and the difference of the inter-event time instants for each node is strictly positive, where* $0 < \sigma < 1$ *is a constant and* $d_i = \sum_{j\in\mathcal{N}_i} a_{ij}$. *Moreover, the final consensus value is*

$$\frac{\sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \int_{-\tau_{ij}}^{0} \phi_j(s)ds}{1 + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij}\tau_{ij}}.$$

***Proof*** The Lyapunov stability theory is applied to show that the multi-agent system (5.30) with distinct communication delays can achieve consensus under the proposed the event-triggered protocol.

Consider the Lyapunov functional as

$$V(t) = \sum_{i=1}^{N} \xi_i x_i^2(t) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} \int_{t-\tau_{ij}}^{t} a_{ij}\widehat{x}_j^2(s)ds. \tag{5.34}$$

Differentiating (Dini right-upper derivative) $V(t)$ along the solution of (5.30) gives that

$$\begin{aligned}
\dot{V}(t) = {} & 2\sum_{i=1}^{N} \xi_i x_i(t) \sum_{j=1,\, j\neq i}^{N} a_{ij}[\widehat{x}_j(t-\tau_{ij}) - \widehat{x}_i(t)] \\
& + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}[\widehat{x}_j^2(t) - \widehat{x}_j^2(t-\tau_{ij})] \\
= {} & 2\sum_{i=1}^{N} \xi_i (e_i(t) + \widehat{x}_i(t)) \sum_{j=1,\, j\neq i}^{N} a_{ij}[\widehat{x}_j(t-\tau_{ij}) - \widehat{x}_i(t)] \\
& + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}[\widehat{x}_j^2(t) - \widehat{x}_j^2(t-\tau_{ij})] \\
= {} & 2\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij} e_i(t)[\widehat{x}_j(t-\tau_{ij}) - \widehat{x}_i(t)]
\end{aligned}$$

$$+\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}a_{ij}[2\widehat{x}_i(t)\cdot\widehat{x}_j(t-\tau_{ij})-2\widehat{x}_i^2(t)]$$
$$+\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}a_{ij}[\widehat{x}_j^2(t)-\widehat{x}_j^2(t-\tau_{ij})]. \tag{5.35}$$

Note that

$$\begin{aligned}\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}a_{ij}\widehat{x}_j^2(t)&=\sum_{i=1}^{N}\sum_{j=1,\, j\neq i}^{N}\xi_i a_{ij}\widehat{x}_j^2(t)=\sum_{j=1}^{N}\sum_{i=1,\, i\neq j}^{N}\xi_i a_{ij}\widehat{x}_j^2(t)\\&=-\sum_{j=1}^{N}\xi_j a_{jj}\widehat{x}_j^2(t)=-\sum_{i=1}^{N}\xi_i a_{ii}\widehat{x}_i^2(t)\\&=\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}a_{ij}\widehat{x}_i^2(t).\end{aligned} \tag{5.36}$$

Substituting (5.36) into (5.35), we obtain that

$$\begin{aligned}\dot{V}(t)=&\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}a_{ij}[2\widehat{x}_i(t)\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i^2(t)-\widehat{x}_j^2(t)]\\&+2\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}a_{ij}e_i(t)[\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t)]\\&+\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}a_{ij}[\widehat{x}_j^2(t)-\widehat{x}_j^2(t-\tau_{ij})]\\=&-\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}a_{ij}[\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t)]^2\\&+2\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}a_{ij}e_i(t)[\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t)].\end{aligned} \tag{5.37}$$

Note that

$$2\sum_{i=1}^{N}\xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}e_i(t)[\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t)]$$
$$\leq \sum_{i=1}^{N}\xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}[\frac{1}{\alpha_i}e_i^2(t)+\alpha_i(\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t))^2].$$

Hence, we can obtain that

$$\begin{aligned}\dot{V}(t) \leq & -\sum_{i=1}^{N}\xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}[(\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t))^2-(\frac{1}{\alpha_i}e_i^2(t) \\ & +\alpha_i(\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t))^2)] \\ = & -\sum_{i=1}^{N}\xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}(1-\alpha_i)(\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t))^2+\sum_{i=1}^{N}\xi_i d_i\frac{1}{\alpha_i}e_i^2(t),\end{aligned} \tag{5.38}$$

where $d_i=\sum_{j\in\mathcal{N}_i}a_{ij}$. Thus, a sufficient condition to ensure $\dot{V}(t)\leq 0$ is that

$$e_i^2(t)\leq \frac{\sigma\alpha_i(1-\alpha_i)}{d_i}\sum_{j=1,\, j\neq i}^{N} a_{ij}(\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t))^2. \tag{5.39}$$

Let $f(\alpha_i)=\frac{\alpha_i(1-\alpha_i)}{d_i}$. Then, we can easily obtain the maximum of $f(\alpha_i)$ by taking $\alpha_i=\frac{1}{2}$, which makes (5.39) become

$$e_i^2(t)\leq \frac{\sigma}{4d_i}\sum_{j=1,\, j\neq i}^{N} a_{ij}(\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t))^2. \tag{5.40}$$

Hence, we can choose the trigger condition as follows:

$$e_i^2(t)> \frac{\sigma}{4d_i}\sum_{j=1,\, j\neq i}^{N} a_{ij}(\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t))^2,\quad i\in\mathcal{N}. \tag{5.41}$$

Under the trigger condition (5.41), we can conclude that for any $t \geq 0$,

$$\dot{V}(t) \leq -\frac{1}{2}\sum_{i=1}^{N}\xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}(1-\sigma)(\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t))^2 \\ \leq 0. \tag{5.42}$$

From (5.42), it is noted that $\dot{V}(t) = 0$ if and only if

$$|\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t)| = 0,\ \forall i,\ j \in \mathcal{N}_i, \tag{5.43}$$

which combined with (5.41) implies that $e_i(t) = 0$. Hence, by LaSalle's invariance principle, we have

$$\lim_{t\to\infty}(\widehat{x}_j(t-\tau_{ij})-\widehat{x}_i(t)) = 0,\ \forall i,\ j \in \mathcal{N}_i, \tag{5.44}$$

and

$$\lim_{t\to\infty} e_i(t) = 0, \qquad \forall i = 1, \ldots, N, \tag{5.45}$$

which combined with network model (5.30) implies that

$$\lim_{t\to\infty} \dot{x}_i(t) = 0, \qquad \forall i = 1, \ldots, N. \tag{5.46}$$

It follows from the mean value theorem that

$$\lim_{t\to\infty}(x_j(t-\tau_{ij})-x_j(t)) = 0. \tag{5.47}$$

Note that

$$|x_i(t)-x_j(t)| \leq |x_i(t)-\widehat{x}_i(t)| + |\widehat{x}_i(t)-\widehat{x}_j(t-\tau_{ij})| + |\widehat{x}_j(t-\tau_{ij}) \\ -x_j(t-\tau_{ij})| + |x_j(t-\tau_{ij})-x_j(t)|, \tag{5.48}$$

and $\mathcal{G}$ is strongly connected, and therefore we can obtain that

$$\lim_{t\to\infty}(x_j(t)-x_i(t)) = 0. \tag{5.49}$$

Therefore, the multi-agent system can realize consensus.

Next we show that $\forall i \in \mathcal{N}$, and the inter-event time $t_{k+1}^i - t_k^i$ is strictly larger than 0. For $t \in [t_k^i,\ t_{k+1}^i)$, we have

$$\dot{e}_i(t) = \frac{d}{dt}(x_i(t)-x_i(t_k^i)) = \dot{x}_i(t).$$

Hence,

$$|e_i(t)| = |\sum_{j=1,\, j\neq i}^{N} a_{ij} \int_{t_k^i}^{t} (\widehat{x}_j(s-\tau_{ij}) - x_i(t_k^i))ds|, \quad t \in [t_k^i,\, t_{k+1}^i).$$

Note that

$$\widehat{x}_j(t-\tau_{ij}) = x_j(t^j_{l(t-\tau_{ij})}),$$

where $l(t-\tau_{ij}) = \arg\max_{l\in\mathbb{N}}\{t_l^j | t_l^j \leq t-\tau_{ij}\}$. Let $\tau_{\min} = \min\{\tau_{ij},\, i \neq j\}$,

$$p_i(t) = \left(\sum_{j=1,\, j\neq i}^{N} a_{ij} \int_{t_k^i}^{t} (\widehat{x}_j(s-\tau_{ij}) - x_i(t_k^i))ds\right)^2, \quad t \in [t_k^i,\, t_{k+1}^i),$$

$$q_i(t) = \frac{\sigma}{4d_i} \sum_{j=1,\, j\neq i}^{N} a_{ij}(\widehat{x}_j(t-\tau_{ij}) - x_i(t_k^i))^2, \quad t \in [t_k^i,\, t_{k+1}^i),$$

and

$$\Delta_i(t) = p_i(t) - q_i(t). \tag{5.50}$$

For $t_0^i = 0$ and $t \in [0, \tau_{\min}]$, it can be observed that

$$\begin{aligned}\Delta_i(t) = &\left(\sum_{j=1,\, j\neq i}^{N} a_{ij} \int_{0}^{t} (\phi_j(s-\tau_{ij}) - x_i(0))ds\right)^2 \\ &- \frac{\sigma}{4d_i} \sum_{j=1,\, j\neq i}^{N} a_{ij}(\phi_j(t-\tau_{ij}) - x_i(0))^2,\end{aligned} \tag{5.51}$$

and

$$\Delta_i(0) = -\frac{\sigma}{4d_i} \sum_{j=1,\, j\neq i}^{N} a_{ij}(\phi_j(-\tau_{ij}) - x_i(0))^2 \leq 0.$$

- If $\Delta_i(0) < 0$, due to the continuity of the function $\Delta_i(t)$ for $t \in [0, \tau_{\min}]$, there exists a constant $\beta_1 > 0$, such that for any $t \in [0, \beta_1)$, $\Delta_i(t) < 0$. Hence, $t_1^i - t_0^i \geq \beta_1 > 0$.

- If $\Delta_i(0) = 0$, then $p_i(0) = 0$ and $q_i(0) = 0$. If there exists $\delta_1 \in (0, 1)$ such that $q_i(t) = 0$ for any $t \in [0, \delta_1)$. Then, we have $\Delta_i(t) = 0$ for any $t \in [0, \delta_1)$. Hence, $t_1^i - t_0^i \geq \delta_1 > 0$. If there is not $\delta_1 \in (0, 1)$ such that $q_i(t) = 0$ for any $t \in [0, \delta_1)$. We can choose a small enough constant $0 < \delta < 1$, such that $\Delta_i(t) \leq 0, \forall t \in [0, \delta)$. Hence, $t_1^i - t_0^i \geq \delta > 0$.

Similarly, for the event-triggered time instant satisfying $t_k^i \leq \tau$, we can prove one by one that the inter-event time satisfies $t_k^i - t_{k-1}^i > 0$. Next, we will consider the case $t_k^i > \tau$. For $t > t_k^i$, suppose that agent $i$ firstly receives the renewed information from some of its neighbors at time $\nu$. Set $\vartheta = \min\{\tau_{\min}, \nu\}$. For $t \in [t_k^i, t_k^i + \vartheta)$, the received state of agent $i$, $\widehat{x}_j(t - \tau_{ij}) = x_j(t^j_{l(t-\tau_{ij})})$, where $l(t - \tau_{ij}) = \arg\max_{l\in\mathbb{N}}\{t_l^j | t_l^j \leq t - \tau_{ij}\}$. It follows from $\Delta_i(t_k^i) \leq 0$ that the event is not triggered at time instant $t_k^i$. If $\Delta_i(t_k^i) = 0$, then $\Delta_i(t) = 0$ for any $t \in [t_k^i, t_k^i + \vartheta)$. If $\Delta_i(t_k^i) < 0$, we can obtain that for $t \in [t_k^i, t_k^i + \vartheta)$, $\Delta_i(t)$ is continuous since $\widehat{x}_j(t - \tau_{ij})$ is a constant. Hence, there exists a constant $\delta_2 \in [0, \vartheta)$, such that for any $t \in [t_k^i, t_k^i + \delta_2)$, $\Delta_i(t) < 0$. So, $t_{k+1}^i - t_k^i \geq \delta_2 > 0$.

Therefore, for each agent $i$, $i \in \mathcal{N}$, the inter-event time $t_{k+1}^i - t_k^i$ is strictly larger than 0.

At last, we estimate the final consensus value $c$ by constructing an invariant value of multi-agent system (5.30). Let

$$\eta(t) = \sum_{i=1}^{N} \xi_i x_i(t) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij} \int_{t-\tau_{ij}}^{t} \widehat{x}_j(s)ds. \tag{5.52}$$

Differentiating (Dini right derivative) $\eta(t)$ along the solution of (5.30), we obtain from (5.36) that

$$\begin{aligned}
\dot{\eta}(t) &= \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}[\widehat{x}_j(t-\tau_{ij}) - \widehat{x}_i(t)] - \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}[\widehat{x}_j(t-\tau_{ij}) \\
&\quad - \widehat{x}_j(t)] \\
&= -\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}\widehat{x}_i(t) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij}\widehat{x}_j(t) \\
&= 0.
\end{aligned} \tag{5.53}$$

Since $\dot{\eta}(t) = 0$ for $t \in [0, \infty)$ and the continuity of $\eta(t)$, $\eta(t)$ in (5.52) is a constant; that is,

$$\begin{aligned}\eta(t) &= \eta(0)\\ &= \xi_i(\sum_{i=1}^{N} x_i(0) + \sum_{i=1}^{N} \sum_{j=1,\, j\neq i}^{N} a_{ij} \int_{-\tau_{ij}}^{0} \widehat{x}_j(s)ds)\\ &= \sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} a_{ij} \int_{-\tau_{ij}}^{0} \phi_j(s)ds.\end{aligned}$$

Hence,

$$\eta(0) = \lim_{t\to+\infty} \eta(t) = c + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij}\tau_{ij}c. \tag{5.54}$$

Therefore, we can conclude that

$$c = \frac{\sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \int_{-\tau_{ij}}^{0} \phi_j(s)ds}{1 + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij}\tau_{ij}}.$$

This completes the proof.

*Remark 5.9* It can be observed that the final consensus value is not explicitly obtained for the event-triggered consensus protocol (5.33) proposed in [16–20]. Nevertheless, by constructing an invariant value $\eta(t)$ in (5.52) of the multi-agent system in this chapter, the final consensus value is theoretically obtained even in the presence of distinct finite communication delays. Moreover, since the directed topology and communication delays are considered in our protocol, average consensus cannot be reached.

*Remark 5.10* In Theorem 5.8, event-triggered condition (5.41) is verified by only using its received neighboring information, which means that only local information is used to verify the event-triggered condition. Furthermore, the inter-event times $\{t_{k+1}^i - t_k^i\}$ have been proved to be larger than 0. In order to give a positive lower bound of the inter-event time and avoid the Zeno behaviors, a synchronously event-triggered control method will be presented in the following section.

### 5.2.3 Synchronously Event-Triggered Control

We now present a synchronously event-triggered control for the multi-agent system (5.30). In Theorem 5.8, although it has been proved that the inter-event time is strictly larger than 0, it is hard to find a positive lower bound under the consideration of the communication delays. In this section, we aim to solve this difficult problem by designing a synchronously event-triggered protocol for the constant time delays, i.e., $\tau_{ij} = \tau$.

Notice that $\forall i = 1, \cdots, N, \ j \in \mathcal{N}_i, \bar{a} = \max_{1 \leq i < j \leq N, a_{ij} > 0}\{a_{ij}\}$. Define

$$g_i(t) = e_i^2(t) - \frac{\sigma}{4d_i} \sum_{j=1, j \neq i}^{N} a_{ij}(\widehat{x}_j(t-\tau) - \widehat{x}_i(t))^2.$$

Consider the synchronously event-triggered time sequence $t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$. Assume $t_0 = 0$, $t_1 = \max\{\tau, t^1\}$ and $t_k = t^k, \ k \geq 2$, where $t^k, \ k \geq 1$, is defined iteratively as

$$t^{k+1} = \inf\{t : t > t_k, \ \max_{i \in \mathcal{N}}\{g_i(t)\} > 0\}. \tag{5.55}$$

It follows from Theorem 5.8 that the multi-agent network (5.30) will achieve consensus under the synchronously event-triggered time sequence.

Note that

$$\dot{e}_i(t) = \frac{d}{dt}(x_i(t) - x_i(t_k)) = \dot{x}_i(t).$$

Hence, one can get

$$e_i(t) = \sum_{j=1, j \neq i}^{N} a_{ij} \int_{t_k}^{t} (\widehat{x}_j(s-\tau) - x_i(t_k))ds.$$

It is obvious that $t_1 \geq \tau$. For $t \in [t_1, t_1 + \tau)$, we have

$$\widehat{x}_j(t-\tau) = x_j(t_0), \tag{5.56}$$

and

$$\begin{aligned} g_i(t) &= [\sum_{j=1, j \neq i}^{N} a_{ij}(x_j(t_0) - x_i(t_1))]^2(t-t_1)^2 - \frac{\sigma}{4d_i} \sum_{j=1, j \neq i}^{N} a_{ij}(x_j(t_0) \\ &\quad - x_i(t_1))^2 \\ &\leq N \sum_{j=1, j \neq i}^{N} a_{ij}^2(x_j(t_0) - x_i(t_1))^2(t-t_1)^2 - \frac{\sigma}{4d_i} \sum_{j=1, j \neq i}^{N} a_{ij}(x_j(t_0) \\ &\quad - x_i(t_1))^2 \\ &\leq [N\bar{a}(t-t_1)^2 - \frac{\sigma}{4d_i}][\sum_{j=1, j \neq i}^{N} a_{ij}(x_j(t_0) - x_i(t_1))^2]. \end{aligned} \tag{5.57}$$

Hence, a necessary condition for $g_i(t) > 0$ is that

$$N\bar{a}(t - t_1)^2 - \frac{\sigma}{4d_i} > 0, \tag{5.58}$$

i.e., $t - t_1 > \sqrt[2]{\frac{\sigma}{4d_i N\bar{a}}}$. If we require $\tau \leq \min_{i \in \mathcal{N}} \left\{ \sqrt[2]{\frac{\sigma}{4d_i N\bar{a}}} \right\}$, then we have $t_2 - t_1 \geq \tau$.

For $t \in [t_k,\ t_{k+1}]$, similar to the discussions presented as above, we have

$$t_{k+1} - t_k \geq \tau.$$

Hence, a non-trivial inter-transmission time $\tau$ is obtained.

The aforementioned synchronously event-triggered multi-agent consensus is thus summarized as follows.

**Theorem 5.11** *Consider the multi-agent network (5.30) with control law (5.33). If communication delay $\tau \leq \min_{i \in \mathcal{N}} \left\{ \sqrt[2]{\frac{\sigma}{4d_i N\bar{a}}} \right\}$, the network achieves consensus asymptotically with the synchronous event time instants $t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots$ iteratively determined by $t_0 = 0$, $t_1 = max\{\tau,\ t^1\}$, and $t_k = t^k$, $k \geq 2$, where $t^k$ is defined in (5.55). Final consensus value is*

$$\frac{\sum_{i=1}^{N} \xi_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} a_{ij} \int_{-\tau}^{0} x_j(s)ds}{1 + \sum_{i=1}^{N} \xi_i \sum_{j=1, j \neq i}^{N} a_{ij}\tau}.$$

*Moreover, the Zeno behavior can be excluded since the lower bound of the inter-event time is $\tau$.*

*Remark 5.12* Based on above analysis, it can be found that under the synchronously event-triggered protocol, the Zeno behavior can be excluded for the multi-agent network with distinct communication delays. Unfortunately, the protocol requires all agents to be triggered simultaneously. It can be noted that, in many real networks, the agents use multi-channels to exchange different kinds of information with their neighboring agents [23]. The synchronously event-triggered protocol can be applied to such kinds of networks. Here, we can assume that neighboring agents can communicate via two different channels. One channel is used to transmit the state information, and another one is used to inform agents the occurrence of the events. It is still challenging for the considered model to design an asynchronously distributed event-triggered mechanism to realize consensus and meanwhile to avoid the Zeno behavior.

In the following, we shall propose a self-triggered consensus algorithm to achieve multi-agent consensus based on Theorem 5.11. Self-triggered algorithm means that the agents can predict next triggered time instant $t_{k+1}$ based on the information at time $t_k$. The advantage of the self-triggered algorithm lies that all agents are not required to verify the event-triggered condition continuously and hence could save more energy for the multi-agent system. Recall that

$$e_i^2(t) = \left( \sum_{j=1,\, j\neq i}^{N} a_{ij} \int_{t_k}^{t} (x_j(t_{l(s-\tau)}) - x_i(t_k))ds \right)^2,$$

and

$$q_i(t) = \frac{\sigma}{4d_i} \sum_{j=1,\, j\neq i}^{N} a_{ij}(x_j(t_{l(t-\tau)}) - x_i(t_k))^2,$$

where

$$l(t-\tau) = \arg\max_{l\in\mathbb{N}}\{t_l | t_l \leq t-\tau\}. \tag{5.59}$$

Let

$$\lambda_i(t) = \sum_{j=1,\, j\neq i}^{N} a_{ij}(x_j(t_{l(t-\tau)}) - x_i(t_k)). \tag{5.60}$$

Based on Theorem 5.11, we develop the following triggering strategy to find $t_{k+1}$. Note that once there exists an agent $l \in \mathcal{N}$, which finds self-triggering time instant $t_{k+1}^l$, the algorithm will stop and $t_{k+1} = t_{k+1}^l$. For agent $i$, we propose Algorithm 1.

**Algorithm 1** Self-triggered algorithm

*Step 1.* For each agent $i \in \mathcal{N}$, set $\Lambda = 0$ and $s = t_k$.

*Step 2.* Solving the equation $[\Lambda + \lambda_i(s)(t-s)]^2 - q_i(s) = 0$, we can obtain the solution $\varsigma_{k+1}^i$.

*Step 3.* For $t \geq s$, if agent $i$ does not receive the renewed information from its neighbors until $t = \varsigma_{k+1}^i$, then, set $t_{k+1}^i = \varsigma_{k+1}^i$ and stop the algorithm.

*Step 4.* If agent $i$ firstly receive the renewed information from some of its neighbors at time $t^0 < \varsigma_{k+1}^i$, set $\Lambda = \Lambda + \lambda_i(s)(t^0 - s)$. Update $s = t^0$ and go to *Step 2*.

Different from the previous self-triggered algorithm [16], the communication delay is considered in our algorithm, and it brings some difficulties but beneficial for conforming realistic situations. More importantly, one can observe that the Zeno behavior can be avoided by using this algorithm.

### 5.2.4 Numerical Example

In this section, an example is given to illustrate the effectiveness of the proposed event-triggered consensus protocol. Consider a multi-agent network with five agents, and its Laplacian matrix is given by

$$L = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

The dynamics of the agents are given in (5.30). Set $\sigma = 0.9$ and the communication delays $\tau_{ij} = 0.04$, $\forall j \in \mathcal{N}_i$. The initial states of the system are chosen as $\phi_1(s) = -4.3$, $\phi_2(s) = 1$, $\phi_3(s) = -1.4$, $\phi_4(s) = 2.4$, and $\phi_5(s) = -2$, $\forall s \in [-0.04, 0]$. We consider the evolvement of the agents under the asynchronously distributed event-triggered protocol in Theorem 5.8 and the synchronously event-triggered protocol in Theorem 5.11, respectively.

It can be seen from Figs. 5.4 and 5.6 that the multi-agent networks can achieve consensus under asynchronously and synchronously event-triggered communication protocol. Moreover, based on Theorems 5.8 and 5.11, the final consensus value

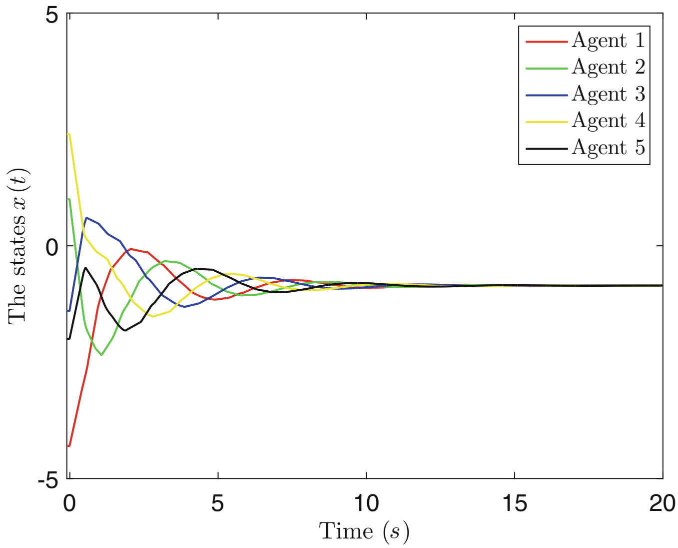


**Fig. 5.4** The states of the system under asynchronously event-triggered communication protocol in Example 5.2.4

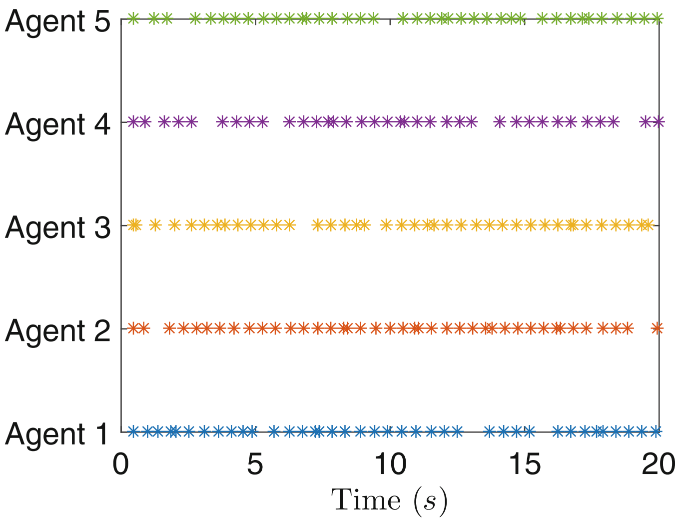


**Fig. 5.5** Event-triggered times in Example 5.2.4

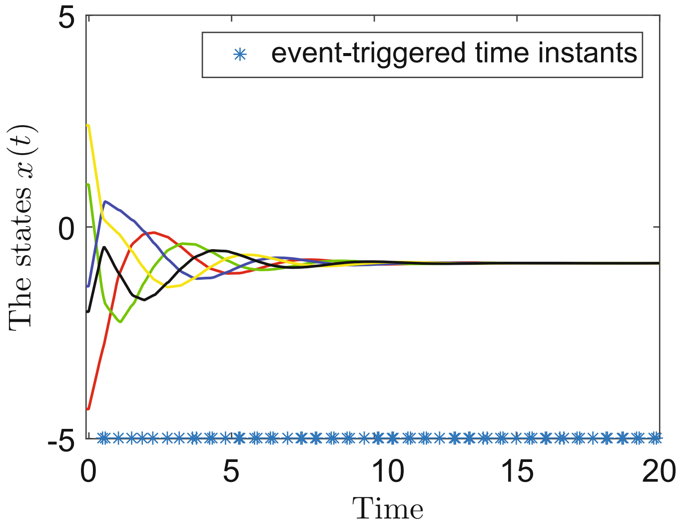


**Fig. 5.6** The states of the system under synchronously event-triggered communication protocol in Example 5.2.4

of the system should be

$$\frac{\sum_{i=1}^{N}\xi_i x_i(0)+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}a_{ij}\int_{-\tau_{ij}}^{0}\phi_j(s)ds}{1+\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}a_{ij}\tau_{ij}}=-0.86,$$

which can also be seen from Figs. 5.4 and 5.6. The individual event time instants under asynchronously and synchronously event-triggered protocols are shown in Figs. 5.5 and 5.6, respectively. One can also conclude from the simulation example that the event-based strategy in this chapter can significantly decrease the information transmission during the consensus process of the multi-agent system with distinct communication delays.

## 5.3 Summary

In this chapter, we have investigated the discrete-time and continuous-time multi-agent consensus problems where each agent transmits its current state to its neighbors only when a certain "event" occurs. The network topology of the multi-agent system is directed. Distributed event-triggered conditions have been established. Under the proposed distributed event-triggered protocols, it has been proved that consensus can be achieved for the discrete-time/continuous-time multi-agent network with communication delays. The theoretical results are well illustrated by two numerical examples.

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# Chapter 6
# Consensus of Networked Multi-agent Systems with Antagonistic Interactions and Communication Delays

A common feature of previous consensus results is the focus on cooperative systems [1–3]. The consensus of these systems is asymptotically achieved through collaboration, which is characterized by the diffusive coupling [4] and the non-negative weights among agents [5–8]. In many real-world cases, however, it is more reasonable to consider that some agents collaborate with each other, while others are competitive. Networks with antagonistic interactions are ubiquitous in real world [9], and it becomes a focus for studying in recent years [10–15]. Altafini [10] proved that bipartite consensus can be achieved over networks with antagonistic interactions. Furthermore, in [16], emergent behaviors were investigated over signed random dynamical networks. In [17], flocking behaviors were studied by using results about signed graph. In [18], the leader-following bipartite consensus issue for single-integrator multi-agent systems was investigated, where the signed digraph was considered to be structurally balanced and had a spanning tree.

To achieve the consensus, each node in a network has to transmit its state information to its neighbors via connections. However, because of physical and environmental limitations, communication constraints between connected nodes are unavoidable. As is well-known, the communication delay is one of the most universal communication constraints. Motivated by the aforementioned discussions, we investigate the consensus problem of signed networks with antagonistic interactions and communication delays in this chapter. To the best of our knowledge, only a few results have been done concerning such problem. Due to the difficulty that antagonistic interactions and communication delays need to be simultaneously considered, new techniques are required to deal with this problem. According to matrix theory, Lyapunov theorem, and some other mathematical analysis, we found that bipartite consensus can be achieved for those systems with communication delays. Furthermore, in order to obtain the final bipartite consensus solution, we construct an invariant function to study the relationship of the states of nodes and their initial states. Using some mathematical analysis skills, we provide the bipartite consensus solution with an explicit expression.



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# 6.1 Continuous-Time Multi-agent Consensus

## *6.1.1 Linear Coupling*

In this section, we consider a multi-agent system formed by $N$ linearly coupled identical nodes, where each node's dynamic is described as follows:

$$\dot{x}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid [\mathrm{sgn}(a_{ij})x_j(t - \tau_{ij}) - x_i(t)],\ i \in \mathcal{N}, \tag{6.1}$$

where $x_i(t) \in \mathbb{R}^n$ is the state of node $i$ at time $t$, and $\tau_{ij} > 0$ denotes the communication delay from $v_j$ to $v_i$ for $i \neq j$ and $\tau_{ii} = 0$. $A = [a_{ij}]_{N\times N}$ is the adjacency matrix of the network that is symmetric. Here it is assumed that there is no self-closed loop, which means that $a_{ii} = 0$.

Throughout this section, the bipartite consensus of dynamical system (6.1) is said to be realized if $\lim_{t\to\infty} x_i(t) = \alpha$ for $i \in \mathcal{V}_1$ and $\lim_{t\to\infty} x_i(t) = -\alpha$ for $i \in \mathcal{V}_2$.

**Theorem 6.1** *Consider the networked multi-agent system (6.1) with a connected signed graph $G(A)$. The bipartite consensus can be asymptotically reached if $G(A)$ is structurally balanced. If instead $G(A)$ is structurally unbalanced, then* $\lim_{t\to\infty} x(t) = \mathbf{0}$.

***Proof*** We first consider the case that $G(A)$ is structurally balanced. According to Lemma 1.8, one can obtain that $\exists D \in \mathcal{D}$ such that $DAD$ has all nonnegative entries. Let $z(t) = Dx(t)$, we obtain that

$$z_i(t) = \sigma_i x_i(t)\ ,\ i \in \mathcal{N}. \tag{6.2}$$

Substituting (6.2) into (6.1) results in

$$\sigma_i \dot{z}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid [\mathrm{sgn}(a_{ij})\sigma_j z_j(t - \tau_{ij}) - \sigma_i z_i(t)]\ , i \in \mathcal{N}.$$

Since $DAD$ is a nonnegative matrix, we have $\sigma_i \mathrm{sgn}(a_{ij})\sigma_j = 1$. Using $\sigma_i^2 = 1$, one can obtain the following equation:

$$\begin{aligned}\dot{z}_i(t) &= \sum_{j=1}^{N} \mid a_{ij} \mid [\sigma_i \mathrm{sgn}(a_{ij})\sigma_j z_j(t - \tau_{ij}) - \sigma_i^2 z_i(t)] \\ &= \sum_{j=1}^{N} \mid a_{ij} \mid [z_j(t - \tau_{ij}) - z_i(t)]\ ,\ i \in \mathcal{N}.\end{aligned} \tag{6.3}$$

Following [19], the consensus of networks system (6.3) is asymptotically reached. That is

$$\lim_{t\to\infty} z_i(t) \to \alpha,\ \forall i \in \mathcal{N}, \tag{6.4}$$

where $\alpha \in \mathbb{R}^n$ is a constant vector.

Hence, we can get that $\lim_{t\to\infty} x_i(t) \to \sigma_i\alpha$ for $i \in \mathcal{N}$. Then, the bipartite consensus of system (6.1) can be reached if $G(A)$ is structurally balanced.

Next, we consider the case that $G(A)$ is structurally unbalanced. Following Lemma 1.10, we can conclude that $G(A)$ contains one or more negative cycles. For the sake of simplicity, let us first consider the simplest case of $G(A)$ with only one negative cycle. Without loss of generality, we assume that $(\nu_1, \nu_2)$ belongs to the negative cycle and $a_{12} = a_{21} = a < 0$. According to Lemma 1.10, one can obtain that there is no $D \in \mathcal{D}$ such that $DAD$ is a nonnegative matrix. However, for the subgraph $G(B)$, which denotes the rest part of $G(A)$ reducing the edge $(\nu_1, \nu_2)$, it admits a bipartition of the nodes $\mathcal{V}_1$ and $\mathcal{V}_2$. Furthermore, one can find that $G(B)$ is connected and matrix $B$ is irreducible. Now, we can make a hypothesis that nodes $\nu_1$ and $\nu_2$ simultaneously belong to $\mathcal{V}_1$ ( or $\mathcal{V}_2$ ) and the rest nodes remain unchanged. Based on this hypothesis, we can choose $D_1 = diag(\sigma)$ with $\sigma$ satisfying $\sigma_i = 1$ for $\nu_i \in \mathcal{V}_1$ and $\sigma_i = -1$ for $\nu_i \in \mathcal{V}_2$. Then $D_1AD_1 = A' = [a'_{ij}]_{N\times N}$ has exactly two negative elements, i.e., $a'_{12} = a'_{21} = a < 0$, and the rest elements are nonnegative. The following is a decomposition of the matrix $A'$:

$$A' = A_{12} + A_{21} + B'\ , \tag{6.5}$$

where $A_{ij}, i, j \in \{1, 2\}$, denotes a matrix in which the element lied in the intersection of $i$th row and $j$th column is $a_{ij} \neq 0$ and others all are 0. $B' = [b'_{ij}]_{N\times N}$ is a nonnegative adjacency matrix. In order to clearly express the matrix $B'$, we define a function as follows:

$$c(i, j) = \begin{cases} 0, & (i, j) = (1, 2)\ or\ (2, 1); \\ 1, & \text{otherwise}. \end{cases}$$

Hence, we get $b'_{ij} = c(i, j)|a_{ij}|$. It is easy to find that $B'$ is irreducible. Let $\bar{B} = [\bar{b}_{ij}]_{N\times N}$ be the Laplacian matrix of $G(B')$, and its elements are defined as : $\bar{b}_{ij} = b'_{ij}\ (i \neq j)$, $\bar{b}_{ii} = -\sum_{j=1}^{N} b'_{ij}$. Therefore, $\xi = (1, 1, \cdots, 1)^\top$ is the left eigenvector of $\bar{B}$ corresponding to the zero eigenvalue, i.e., $\xi^\top \bar{B} = \mathbf{0}$ , which implies that

$$\bar{b}_{ii} = -\sum_{j=1, j\neq i}^{N} \bar{b}_{ji}. \tag{6.6}$$

Further because $\bar{b}_{ii} = -\sum_{j=1}^{N} b'_{ij}$, one can obtain that

$$\sum_{j=1}^{N} b'_{ij} = \sum_{j=1}^{N} b'_{ji} \quad \text{and} \quad \sum_{i=1}^{N} b'_{ji} = \sum_{i=1}^{N} b'_{ij}. \tag{6.7}$$

Let $z(t) = Dx(t)$, i.e., $z_i(t) = \sigma_i x_i(t)$ for any $i \in \mathcal{N}$, we have

$$\dot{z}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid [\sigma_i \sigma_j \text{sgn}(a_{ij}) z_j(t - \tau_{ij}) - z_i(t)]. \tag{6.8}$$

Consider the following Lyapunov functional for system (6.1):

$$V(t) = V_1(t) + V_2(t), \tag{6.9}$$

where

$$V_1(t) = \frac{1}{2} \sum_{i=1}^{N} x_i^\top(t) x_i(t), \tag{6.10}$$

and

$$V_2(t) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{t-\tau_{ji}}^{t} |a_{ji}| x_i^\top(\theta) x_i(\theta) d\theta. \tag{6.11}$$

Calculating the time derivative of $V_i(t)$ $(i = 1, 2)$ along the trajectories of system (6.1), we have

$$\begin{aligned}
\dot{V}_1(t) &= \sum_{i=1}^{N} x_i^\top(t) \dot{x}_i(t) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}| [x_i^\top(t) \text{sgn}(a_{ij}) x_j(t - \tau_{ij}) - x_i^\top(t) x_i(t)] \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}| [\sigma_i \sigma_j z_i^\top(t) \text{sgn}(a_{ij}) z_j(t - \tau_{ij}) - z_i^\top(t) z_i(t)] \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} b'_{ij} [z_i^\top(t) z_j(t - \tau_{ij}) - z_i^\top(t) z_i(t)] + a_{12} [z_1^\top(t) z_2(t - \tau_{12}) \\
&\quad + z_1^\top(t) z_1(t)] + a_{21} [z_2^\top(t) z_1(t - \tau_{21}) + z_2^\top(t) z_2(t)]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N}\sum_{j=1}^{N} b'_{ij}[z_i^\top(t)z_j(t-\tau_{ij}) - z_i^\top(t)z_i(t)] + \frac{1}{2}a[2z_1^\top(t)z_2(t-\tau_{12}) \\
&\quad + 2z_2^\top(t)z_1(t-\tau_{21}) + 2z_1^\top(t)z_1(t) + 2z_2^\top(t)z_2(t)],
\end{aligned} \tag{6.12}
$$

and

$$
\begin{aligned}
\dot{V}_2(t) =& \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} |a_{ji}|[x_i^\top(t)x_i(t) - x_i^\top(t-\tau_{ji})x_i(t-\tau_{ji})] \\
=& \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b'_{ji}[x_i^\top(t)x_i(t) - x_i^\top(t-\tau_{ji})x_i(t-\tau_{ji})] \\
&+ \frac{1}{2}a_{12}[x_2^\top(t-\tau_{12})x_2(t-\tau_{12}) - x_2^\top(t)x_2(t)] \\
&+ \frac{1}{2}a_{21}[x_1^\top(t-\tau_{21})x_1(t-\tau_{21}) - x_1^\top(t)x_1(t)] \\
=& \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b'_{ij}z_i^\top(t)z_i(t) - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}z_j^\top(t-\tau_{ij})z_j(t-\tau_{ij}) \\
&+ \frac{1}{2}a[z_2^\top(t-\tau_{12})z_2(t-\tau_{12} + z_1^\top(t-\tau_{21})z_1(t-\tau_{21}) \\
&- z_2^\top(t)z_2(t) - z_1^\top(t)z_1(t)].
\end{aligned} \tag{6.13}
$$

Using Eqs. (6.12) and (6.13) gives that

$$
\begin{aligned}
\dot{V}(t) =& -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b'_{ij}[z_i^\top(t)z_i(t) - 2z_i^\top(t)z_j(t-\tau_{ij}) + z_j^\top(t-\tau_{ij})] \\
&+ \frac{1}{2}a\{[2z_1^\top(t)z_2(t-\tau_{12}) + z_1^\top(t)z_1(t) + z_2^\top(t-\tau_{12})z_2(t-\tau_{12})] \\
&+ [2z_2^\top(t)z_1(t-\tau_{21})] + z_2^\top(t)z_2(t) + z_1^\top(t-\tau_{21})z_1(t-\tau_{21})]\} \\
=& -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b'_{ij}[z_i - z_j(t-\tau_{ij})]^\top[z_i - z_j(t-\tau_{ij})] \\
&+ \frac{1}{2}a\{[z_1(t) + z_2(t-\tau_{12})]^\top[z_1(t) + z_2(t-\tau_{12})] \\
&+ [z_2(t) + z_1(t-\tau_{21})]^\top[z_2(t) + z_1(t-\tau_{21})]\} \\
\le& 0.
\end{aligned}
$$

Hence, $V(t)$ is non-increasing. Referring to the construction of $V(t)$, one has that $V(t) \geq 0$, which shows that $\lim_{t\to\infty} V(t)$ exists and is finite. Then, we can get the boundedness of $x_i(t)$ for $i \in \mathcal{N}$. Combining with the expression of $V(t)$, further one can easily show the boundedness of $\dot{x}_i(t)$ for $i \in \mathcal{N}$ by referring to system (6.1). Thus, $\dot{z}_i(t) = \sigma_i \dot{x}_i(t)$ is bounded, which implies $\ddot{V}(t)$ is also bounded.

According to Barbalat's Lemma ([20]), we can obtain that $\lim_{t\to\infty} b'_{ij}[z_i - z_j(t - \tau_{ij})]^\top [z_i - z_j(t-\tau_{ij})] = 0$, $\lim_{t\to\infty}[z_1(t)+z_2(t-\tau_{12})]^\top[z_1+z_2(t-\tau_{12})] = 0$, and $\lim_{t\to\infty}[z_2(t)+z_1(t-\tau_{21})]^\top[z_2(t)+z_1(t-\tau_{21})] = 0$, i.e., $\lim_{t\to\infty}[z_i - z_j(t-\tau_{ij})] = \mathbf{0}$, if $b'_{ij} > 0$,$\lim_{t\to\infty}[z_1 + z_2(t-\tau_{12})] = \mathbf{0}$, and $\lim_{t\to\infty}[z_2(t) + z_1(t-\tau_{21})] = \mathbf{0}$. Further considering the expression (6.8), we have

$$
\begin{aligned}
\dot{z}_1(t) =& \sum_{j=1}^{N} | a_{1j} | [\sigma_1\sigma_j \mathrm{sgn}(a_{1j}) z_j(t-\tau_{1j}) - z_1(t)] \\
=& a_{12}[z_1(t) + z_2(t-\tau_{12})] + \sum_{j=3}^{N} b'_{1j}[z_j(t-\tau_{1j}) - z_1(t)] \\
\to& \mathbf{0}, \text{ as } t \to \infty,
\end{aligned}
$$

$$
\begin{aligned}
\dot{z}_2(t) =& \sum_{j=1}^{N} | a_{2j} | [\sigma_2\sigma_j \mathrm{sgn}(a_{2j}) z_j(t-\tau_{2j}) - z_2(t)] \\
=& a_{21}[z_2(t) + z_1(t-\tau_{21})] + \sum_{j=3}^{N} b'_{2j}[z_j(t-\tau_{2j}) - z_2(t)] \\
\to& \mathbf{0}, \text{ as } t \to \infty,
\end{aligned}
$$

and

$$
\dot{z}_i(t) = \sum_{j=1}^{N} b'_{ij}[z_j(t-\tau_{ij}) - z_i(t)] \to \mathbf{0}, \text{ as } t \to \infty \text{ for } i \geq 3.
$$

Therefore, we get that $\lim_{t\to\infty} \dot{z}_i(t) \to \mathbf{0}$ for any $i \in \mathcal{N}$. Since the adjacency matrix $B'$ is irreducible, one can obtain that $\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} z_2(t) = \cdots = \lim_{t\to\infty} z_N(t)$ by referring to $\lim_{t\to\infty} b'_{ij}[z_i - z_j(t-\tau_{ij})] = \mathbf{0}$. In addition, we can also get $\lim_{t\to\infty} z_1(t) = -\lim_{t\to\infty} z_2(t)$ by referring to $\lim_{t\to\infty}[z_2(t) + z_1(t-\tau_{21})] = \mathbf{0}$. Thus, the following continued equality can be concluded: $\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} z_2(t) = \cdots = \lim_{t\to\infty} z_N(t) = \mathbf{0}$. Therefore, we obtain that $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} \sigma_i z_i(t) = \mathbf{0}$ for $i \in \mathcal{N}$.

Now consider the case of $G(A)$ with $m$ ($m \geq 2$) negative cycles. Referring to the above approach, we respectively select $m$ negative cycles as follows: $(\nu_{i_1}, \nu_{j_1}), \cdots, (\nu_{i_m}, \nu_{j_m})$. There exists a diagonal matrix $D_2 \in \mathcal{D}$ such that

$$D_2 A D_2 = A_{i_1 j_1} + A_{j_1 i_1} + \cdots + A_{i_m j_m} + A_{j_m i_m} + B''.$$

Similarly we can get that $\lim_{t\to\infty} x_i(t) = \mathbf{0}$ for any $i \in \mathcal{N}$.

*Remark 6.2* For the structurally unbalanced network, since the number of negative cycles is not the essential attribute of structurally unbalanced network, we just need to think about those structurally unbalanced networks with one negative cycle instead of all structurally unbalanced networks. This consideration reduces the difficulty of the problem.

*Remark 6.3* There is a situation that needs to be considered. If a negative edge simultaneously belongs to two or more negative cycles, we should admit that the edge only belongs to one of those cycles and the rest negative cycles should be viewed positive cycles. Then we can still make a hypothesis that two vertices on this negative edge belong to $\mathcal{V}_1$ or $\mathcal{V}_2$. According to the proof progress of Theorem 6.1, it is obvious that the results of Theorem 6.1 still hold.

According to Theorem 6.1, if $G(A)$ is structurally balanced, the bipartite consensus can be asymptotically reached, and we have $\lim_{t\to\infty} x_i(t) = \alpha$ for $i \in \mathcal{V}_1$, $\lim_{t\to\infty} x_i(t) = -\alpha$ for $i \in \mathcal{V}_2$. Calculating the bipartite consensus value of $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^\top$ is not an easy task due to the existence of time delays. Here, the value of $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^\top$ will be obtained by an exact expression when the initial conditions of system (6.1) are given. We define $\mathbf{1} = (1, 1, \cdots, 1)^\top_{1\times N}$. The initial conditions about system (6.1) are provided as $x_i(s) = \sigma_i \varphi_i(s) \in C([-\tau, 0], \mathbb{R}^n)$, where $\tau = \max_{i,j}\{\tau_{ij}\}$. Hence, we have $z_i(s) = \varphi_i(s) \in C([-\tau, 0], \mathbb{R}^n)$. Let $\xi(t) = (\xi_1(t), \xi_2(t), \cdots, \xi_n(t))^\top$, where $\xi_r(t) = (1/N)(\sum_{i=1}^N z_{ir}(t) + \sum_{i=1}^N \sum_{j=1}^N |a_{ij}| \int_{t-\tau_{ij}}^t z_{jr}(s)ds)$, $r \in \{1, 2, \cdots, n\}$.

**Theorem 6.4** *Consider a connected signed graph $G(A)$ that is structurally balanced. If $D \in \mathcal{D}$ renders $DAD$ nonnegative, then the bipartite solution of (6.1) is* $\lim_{t\to\infty} x(t) = (D\mathbf{1}) \otimes [N\xi(0)/\sum_{i=1}^N (1 + \sum_{j=1}^N |a_{ij}|\tau_{ij})]$.

***Proof*** Referring to the proof of Theorem 6.1, one can obtain that $\lim_{t\to\infty} z_i(t) = \alpha$ and $\lim_{t\to\infty} x(t) = (D\mathbf{1}) \otimes \alpha$. Using (6.3), we can obtain

$$\begin{aligned}\dot{\xi}_r(t) =& \frac{1}{N}\sum_{i=1}^N \sum_{j=1}^N |a_{ij}|[z_{jr}(t-\tau_{ij}) - z_{ir}(t)] \\ &+ \frac{1}{N}\sum_{i=1}^N \sum_{j=1}^N |a_{ij}|[z_{jr}(t) - z_{jr}(t-\tau_{ij})]\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|z_{ir}(t) + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|z_{jr}(t) \\
&= 0\,.
\end{aligned} \tag{6.14}
$$

Therefore, $\xi_r(t)$ in (6.14) is a constant. That is,

$$
\begin{aligned}
\xi_r(t) =& \xi_r(0) \\
=& \frac{1}{N}(\sum_{i=1}^{N} z_{ir}(0) + \sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|\int_{-\tau_{ij}}^{0}\varphi_{jr}(s)ds).
\end{aligned} \tag{6.15}
$$

Then, we can get

$$
\begin{aligned}
\xi_r(0) = & \lim_{t\to\infty}\xi_r(t) \\
=& \frac{1}{N}(\sum_{i=1}^{N}\alpha_r + \sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|\tau_{ij}\alpha_r) \\
=& \frac{\alpha_r}{N}\sum_{i=1}^{N}(1 + \sum_{j=1}^{N}|a_{ij}|\tau_{ij})\,.
\end{aligned} \tag{6.16}
$$

Hence, we have

$$
\alpha = \frac{N\xi(0)}{\sum_{i=1}^{N}(1 + \sum_{j=1}^{N}|a_{ij}|\tau_{ij})}\,, \tag{6.17}
$$

and

$$
\lim_{t\to\infty} x(t) = (D\mathbf{1}) \otimes \frac{N\xi(0)}{\sum_{i=1}^{N}(1 + \sum_{j=1}^{N}|a_{ij}|\tau_{ij})}. \tag{6.18}
$$

*Remark 6.5* Referring to expression (6.17), for the case of network without communication delays, one can obtain that $\alpha = (1/N)\sum_{i=1}^{N}\varphi_i(0)$. This result is consistent with the one obtained in [10]. This shows that our results are more general. Moreover, we conclude that a bipartite consensus solution is not only associated with initial values of $x_i(t)$ but also closely related to communication delays and network structure.

*Remark 6.6* According to Theorem 6.4, it is obvious that $\alpha \neq \mathbf{0}$ if $G(A)$ is structurally balanced unless $\xi_r(0) = \mathbf{0}$.

### 6.1.2 Nonlinear Coupling

In this subsection, we will investigate the multi-agent systems with nonlinear coupling. Consider the following multi-agent systems:

$$\dot{x}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid \{\text{sgn}(a_{ij})h[x_j(t-\tau_{ij})] - h[x_i(t)]\},\ i \in \mathcal{N}, \tag{6.19}$$

where $x_i(t) \in \mathbb{R}$ is the state of node $i$. The function $h(\cdot) : \mathbb{R} \to \mathbb{R}$ is assumed to be odd and strictly monotone increasing, which implies $h(0) = 0$ and $h(-x) = -h(x)$. Further we assume that $h(\cdot)$ is unbounded.

**Theorem 6.7** *Consider the nonlinear coupled system (6.19) with a connected signed graph $G(A)$. The bipartite consensus can be asymptotically reached if $G(A)$ is structurally balanced. If instead $G(A)$ is structurally unbalanced, then* $\lim_{t\to\infty} x(t) = \mathbf{0}$.

***Proof*** Following Lemma 1.8, if $G(A)$ is structurally balanced, we can obtain that $\exists D \in \mathcal{D}$ such that $DAD$ has all nonnegative entries. Let $Z(t) = Dx(t)$, i.e., $z_i(t) = \sigma_i x_i(t)$, one can easily get that

$$\dot{z}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid \{h[z_j(t-\tau_{ij})] - h[z_i(t)]\}\ ,\ i \in \mathcal{N}\ . \tag{6.20}$$

Following [19], we obtain that $\lim_{t\to\infty} z_i(t) \to \beta \in \mathbb{R}$ for any $i \in \mathcal{N}$, which shows that $\lim_{t\to\infty} x_i(t) \to \sigma_i\beta \in \mathbb{R}$ for $i \in \mathcal{N}$. Therefore, the bipartite consensus of system (6.19) can be reached if $G(A)$ is structurally balanced.

Next, we consider the case that $G(A)$ is structurally unbalanced. From Lemma 1.10, it follows that $G(A)$ contains one or more negative cycles. We first consider the case of $G(A)$ with only one negative cycle. The edge $(\nu_1, \nu_2)$ is assumed to be a negative weighted edge belonging to the negative cycle, and $a_{12} = a_{21} = a < 0$. Choosing $D_1 = diag(\sigma)$ with $\sigma$ satisfying $\sigma_i = 1$ for $\nu_i \in \mathcal{V}_1$ and $\sigma_i = -1$ for $\nu_i \in \mathcal{V}_2$, one can obtain that $D_1AD_1 = A'$ has exactly two negative elements, i.e., $a'_{12} = a'_{21} = a < 0$, and the rest elements are nonnegative. The decomposition of the matrix $A'$ is shown as follows:

$$A' = A_{12} + A_{21} + B',$$

where the definitions of $A_{12}$, $A_{21}$, and $B'$ are similar to the proof in Theorem 6.1.

Let $Z(t) = Dx(t)$, i.e., $z_i(t) = \sigma_i x_i(t)$ for any $i \in \mathcal{N}$, we have

$$\dot{z}_i(t) = \sum_{j=1}^{N} \mid a_{ij} \mid \{\sigma_i\sigma_j \text{sgn}(a_{ij})h[z_j(t-\tau_{ij})] - h[z_i(t)]\}. \tag{6.21}$$

Consider the following Lyapunov–Krasovskii functional for system (6.19)

$$W(x(t)) = W_1(x(t)) + W_2(x(t)) ,$$

where

$$W_1(x(t)) = \sum_{i=1}^{N} \int_0^{x_i(t)} h(s)ds ,$$

and

$$W_2(x(t)) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{t-\tau_{ij}}^{t} |a_{ij}| h^2[x_j(s)]ds .$$

Calculating the time derivative of $W_i(t)(i = 1, 2)$ along the trajectories of system (6.19), we have

$$\begin{aligned}\dot{W}_1(x(t)) =& \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b'_{ij} \{2h[z_i(t)]h[z_j(t-\tau_{ij})] - 2h^2[z_i(t)]\} \\ &+ a\{h[z_1(t)]h[z_2(t-\tau_{12})] + h^2[z_1(t)] \\ &+ h[z_2(t)]h[z_1(t-\tau_{21})] + h^2[z_2(t)] ,\end{aligned} \tag{6.22}$$

and

$$\begin{aligned}\dot{W}_2(x(t)) =& \frac{1}{2} \sum_{i=1}^{N} b'_{ij} \sum_{j=1}^{N} h^2[z_i(t)] - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b_{ij} h^2[z_j(t-\tau_{ij})] \\ &+ \frac{1}{2} a\{h^2[z_2(t-\tau_{12})] - h^2[z_2(t)] \\ &+ h^2[z_1(t-\tau_{21})] - h^2[z_1(t)]\} .\end{aligned} \tag{6.23}$$

Using Eqs. (6.22) and (6.23), we get that

$$\begin{aligned}\dot{W}(x(t)) =& - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} b'_{ij} \{h[z_i(t)] - h[z_j(t-\tau_{ij})]\}^2 \\ &+ \frac{1}{2} a\{h[z_1(t)] + h[z_2(t-\tau_{12})]\}^2 \\ &+ \frac{1}{2} a\{h[z_2(t)] + h[z_1(t-\tau_{21})]\}^2 \\ \leq& 0 .\end{aligned} \tag{6.24}$$

Let $S = \{x(t) \mid \dot{W}(x(t)) = 0\}$. Then it follows from Eq. (6.24) that $S = \{x \in C([t-\tau, t], \mathbb{R}^N) \mid b'_{ij}\{h[z_i(t)] - h[z_j(t-\tau_{ij})]\} = 0,\ h[z_1(t)] + h[z_2(t-\tau_{12})] = 0,$ and $h[z_2(t)] + h[z_1(t-\tau_{21})] = 0\}$. Combining with the property of $h(\cdot)$, we can get that the set $S$ is an invariant set with respect to system (6.21). According to the LaSalle invariance principle [21], one can easily show that $x(t) \to S$ as $t \to \infty$. Thus, we have $\lim_{t\to\infty}\{h[z_i(t)] - h[z_j(t-\tau_{ij})]\} = 0$ for $b'_{ij} > 0$, $\lim_{t\to\infty}\{h[z_1(t)]+h[z_2(t-\tau_{12})]\} = 0$, and $\lim_{t\to\infty}\{h[z_2(t)]+h[z_1(t-\tau_{21})]\} = 0$. Hence, we have $\lim_{t\to\infty} \dot{z}_i(t) = 0$. In addition, since $h(\cdot)$ is unbounded and strictly increasing with $h(0) = 0$, we get that $\lim_{t\to\infty}[z_i(t) - z_j(t-\tau_{ij})] = 0$ when $b'_{ij} > 0$ and $\lim_{t\to\infty}[z_1(t) - z_2(t-\tau_{12})] = 0$. According to the fact that $B$ is irreducible, we conclude that $z_1(t) = z_2(t) = \cdots = z_N(t)$ as $t \to \infty$. It follows from $\lim_{t\to\infty}[z_1(t) - z_2(t-\tau_{12})] = 0$ that $z_1(t) = -z_2(t)$ as $t \to \infty$. Hence, the following equality can be concluded: $\lim_{t\to\infty} z_1(t) = \lim_{t\to\infty} z_2(t) = \cdots = \lim_{t\to\infty} z_N(t) = 0$. Therefore, we obtain that $\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} \sigma_i z_i(t) = 0$ for $i \in \mathcal{N}$.

The result still holds for the case of $G(A)$ with two or more negative cycles. The proof is similar to the case of linear coupling and is omitted for simplicity.

Similar to the case of linear coupling, we can give the bipartite solution of (6.19). The initial conditions about system (6.19) are provided as $x_i(s) = \sigma_i\psi_i(s) \in C([-\tau, 0], \mathbb{R})$. Hence, we have $z_i(s) = \psi_i(s) \in C([-\tau, 0], \mathbb{R})$. Let $\zeta(0) = (1/N)(\sum_{i=1}^N \psi_i(0) + \sum_{i=1}^N \sum_{j=1}^N |a_{ij}| \int_{-\tau_{ij}}^0 h(\psi_j(s))ds)$.

**Theorem 6.8** *Consider a connected signed graph $G(A)$ that is structurally balanced. If $D \in \mathcal{D}$ renders $DAD$ nonnegative, then the bipartite solution of (6.19) is* $\lim_{t\to\infty} x(t) = \beta D\mathbf{1}$, *where $\beta \in \mathbb{R}$ meets a relational expression as follows:*

$$\beta + h(\beta) \sum_{i=1}^N \sum_{j=1}^N (|a_{ij}|\tau_{ij}) - N \times \zeta(0) = 0. \tag{6.25}$$

***Proof*** Referring to the proof of Theorem 6.4, one can get the bipartite solution of (6.19) similarly. Let

$$\zeta(t) = (1/N)(\sum_{i=1}^N z_i(t) + \sum_{i=1}^N \sum_{j=1}^N |a_{ij}| \int_{t-\tau_{ij}}^t h(z_j(s))ds)\ . \tag{6.26}$$

Combining with (6.20), we have $\dot{\zeta}(t) = 0$, which implies $\zeta(t) = \zeta(0)$. From $\zeta(0) = \lim_{t\to\infty} \zeta(t)$, we can get the expression (6.25).

*Remark 6.9* In this theorem, although the value of $\beta$ cannot be given by an explicit expression, we can get a numerical solution by iterative algorithm from (6.25). In numerical examples, Example 6.10 gives a numerical solution to (6.25) for $h(x) = x + 0.5\sin(x)$, which illustrates the computational feasibility.

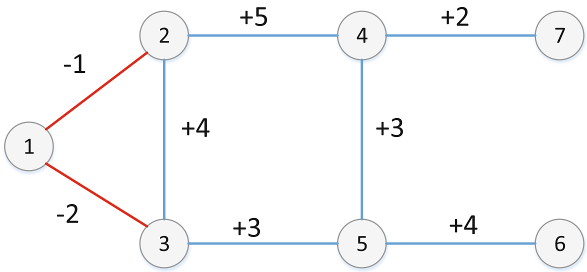


**Fig. 6.1** Structurally balanced signed undirected connectivity graphs with seven nodes

### *6.1.3 Numerical Examples*

In this subsection, numerical examples will be provided to demonstrate the effectiveness of our theoretical results.

*Example 6.10* Now, we will give an example to illustrate the correctness of our main results. Consider the structurally balanced graph of Fig. 6.1.
For systems (6.1) and (6.19), all nonzero communication delays are listed as follows: $\tau_{12} = 0.1$, $\tau_{13} = 0.3$, $\tau_{21} = 0.15$, $\tau_{23} = 0.2$, $\tau_{24} = 0.1$, $\tau_{31} = 0.11$, $\tau_{32} = 0.16$, $\tau_{35} = 0.23$, $\tau_{42} = 0.1$, $\tau_{45} = 0.2$, $\tau_{47} = 0.12$, $\tau_{53} = 0.1$, $\tau_{54} = 0.15$, $\tau_{56} = 0.24$, $\tau_{65} = 0.25$, and $\tau_{74} = 0.15$, and the initial states are chosen as $x_1(s) = 1$, $x_2(s) = 2$, $x_3(s) = 3$, $x_4(s) = 4$, $x_5(s) = 5$, $x_6(s) = -3$, and $x_7(s) = -5$, $\forall s \in [-0.3, 0]$. Let $D = diag\{-1, 1, 1, 1, 1, 1, 1\}$. Then, we have $z(t) = [-1, 2, 3, 4, 5, -3, -5]$, $\forall s \in [-0.3, 0]$. Further, we define that $h(s) = s + 0.5\sin(s)$. According to Theorem 6.4, one can easily conclude that $\alpha = 1.44$. Following Theorem 6.8, one can get the numerical solution $\beta = 1.42$ by iterative algorithm. Numerical results are depicted in Figs. 6.2 and 6.3, which verify our theoretical results very well.

*Example 6.11* Now let us consider a more general network topology with 100 nodes and signed weight edges. Here two simple signed networks with 100 nodes are constructed, where one is structurally balanced and another one is structurally unbalanced. The network with structurally balanced coupling is constructed as follows: we present 20 identical circular networks with 5 nodes, whose 5 nodes are numbered 1, 2, 3, 4, 5, respectively, and their adjacency matrix is $A = [a_{ij}]_{5\times 5}$, where $a_{ij}$ is chosen from $(-10, 0)$ or $(0, 10)$. Now the first two edges of the circular network are defined as negative edges and others are not negative edges, i.e. $a_{12} < 0$, $a_{23} < 0$, and the rest elements are non-negative Here these circular networks are arranged in a sequence. A connected graph with 100 nodes and structurally balanced coupling can be obtained by stochastic interconnections among the 3rd, 4th, and 5th nodes of adjacent circular networks. Similarly the network with structurally unbalanced coupling can be obtained according to the above method when the first three edges of pentagon are defined as negative edges and other steps are the same.

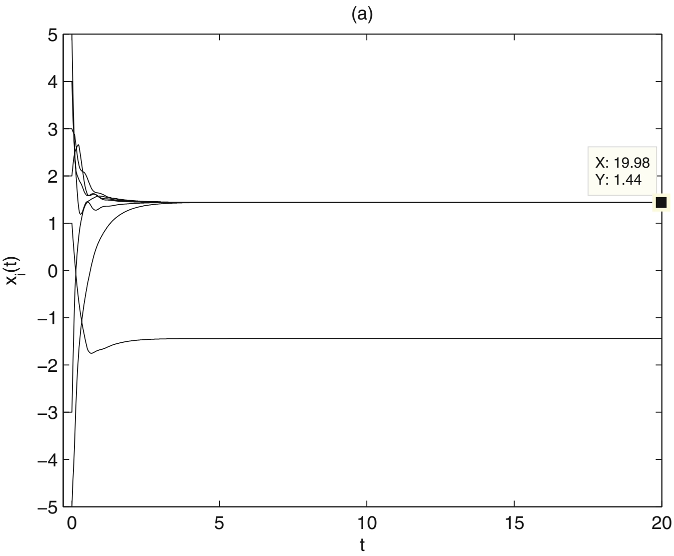


**Fig. 6.2** The bipartite consensus on multi-agent system (6.1) with structurally balanced graph and linear coupling in Example 6.10

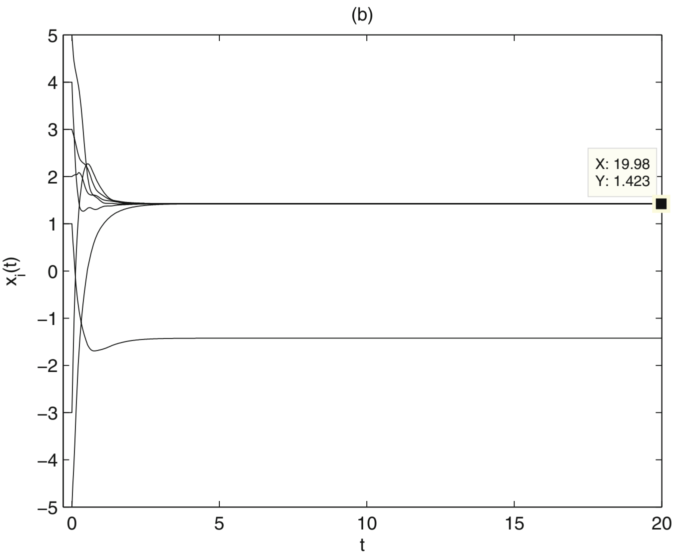


**Fig. 6.3** The bipartite consensus on multi-agent system (6.19) with structurally balanced graph and nonlinear coupling in Example 6.10

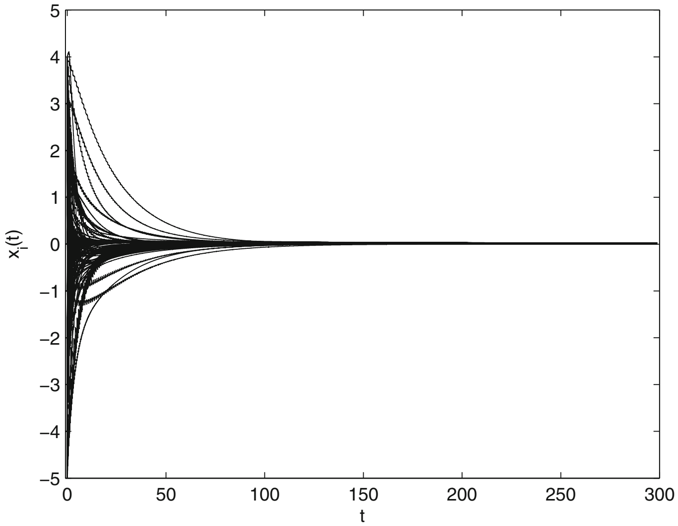


**Fig. 6.4** The states of system (6.1) with structurally unbalanced graph and linear coupling in Example 6.11

All communication delays of systems (6.1) and (6.19) are uniformly distributed in $(0, 1)$. Figure 6.4 shows that the consensus of system (6.1) can be achieved for $a_{ij} \in (-10, 10)$ and $\tau_{ij} \in (0, 1)$. Figure 6.5 shows that the consensus of system (6.19) can also be achieved for the above conditions. Throughout this example, the nonlinear function $h(x) = x + 0.5sin(x)$ is not changed. Figures 6.6 and 6.7 show that the bipartite consensus of systems (6.1) and (6.19) can be asymptotically reached, respectively.

## 6.2 Discrete-Time Multi-agent Consensus

### *6.2.1 Distributed Event-Based Bipartite Consensus*

Consider a discrete-time multi-agent network with the dynamics described by

$$x_i(k+1) = x_i(k) + u_i(k), \quad i \in \mathcal{N}, \tag{6.27}$$

where $x_i(k) \in \mathbb{R}$ is the state of the agent $i$, and $u_i(k)$ is called the consensus protocol.

In this section, we assume that the protocol $u_i(k)$ is based on the event-triggered information transmission. The event-triggered time sequence of the agent $i$ is given by $t_1^i, t_2^i, \cdots, t_l^i, \cdots$. At each triggering time $t_l^i$, the agent $i$ will transmit the

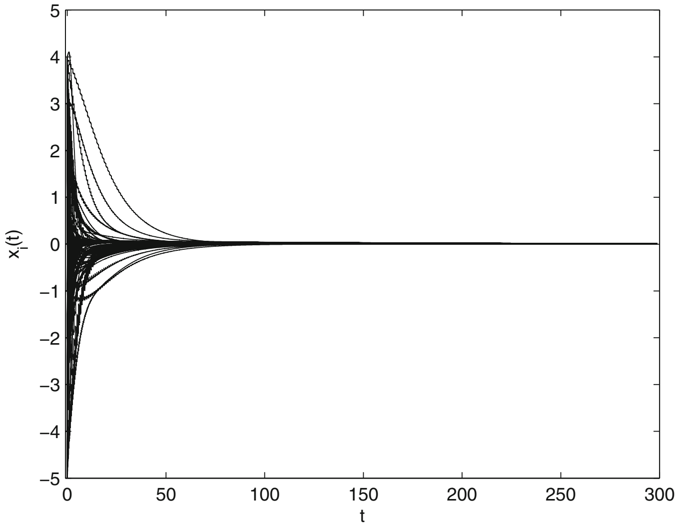


**Fig. 6.5** The states of system (6.19) with structurally unbalanced graph and nonlinear coupling in Example 6.11

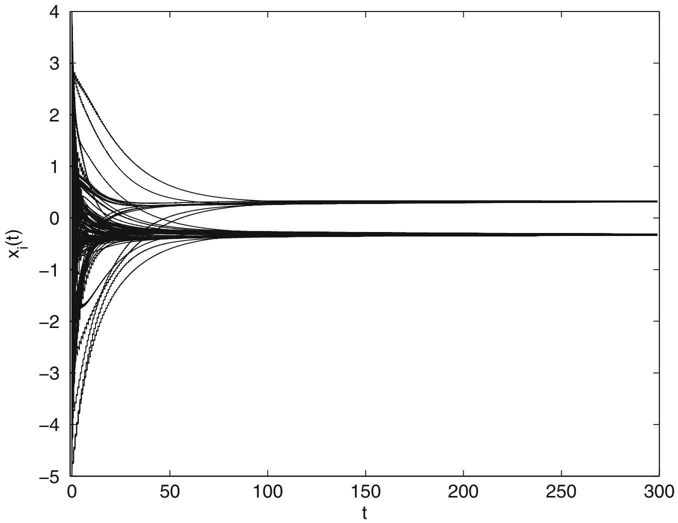


**Fig. 6.6** The bipartite consensus on multi-agent system (6.1) with structurally balanced graph and linear coupling in Example 6.11

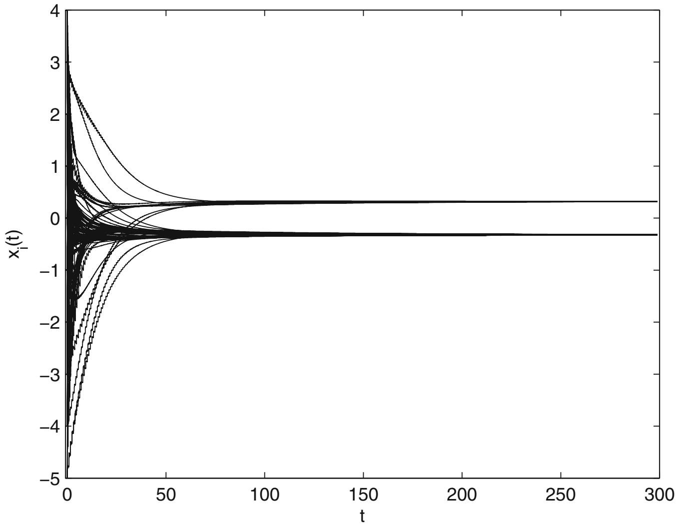


**Fig. 6.7** The bipartite consensus on multi-agent system (6.19) with structurally balanced graph and nonlinear coupling in Example 6.11

state information to its neighboring agents. Considering time delays during the information transmission, the consensus protocol is proposed as follows:

$$u_i(k) = \sum_{j\in\mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k)), \quad i \in \mathcal{N}, \tag{6.28}$$

where $\tau_{ij} > 0$ denotes the communication delay from agent $j$ to $i$, $\widehat{x}_j(k-\tau_{ij}) = x_j(k^j_{l'})$, $k-\tau_{ij} \in [k^j_{l'}, k^j_{l'+1})$, and $\widehat{x}_i(k) = x_i(k^i_l)$, $k \in [k^i_l, k^i_{l+1})$. It is assumed in this section that $\tau_{ii} = 0$, i.e., delays exist only in the information that is actually being transmitted between two different agents. The state measurement error of agent $i$ is defined as

$$e_i(k) = x_i(k) - \widehat{x}_i(k). \tag{6.29}$$

Denote $\tau = \max\{\tau_{ij},\ i,\ j \in \mathcal{N}\}$. The initial conditions associated with (6.27) are given as $x_i(s)$, $s = -\tau, \cdots, -1, 0$.

In this subsection, we will give the distributed event-based bipartite consensus criteria for the considered signed network model. We always assume that the network topology of the signed digraph is strongly connected in this section. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ be the normalized left eigenvector of matrix $|A|$ with respect to the eigenvalue 1. From Lemma 1.6, we obtain that $\sum_{i=1}^{N} \xi_i = 1$ and $\xi_i > 0$.

**Theorem 6.12** *Consider the multi-agent system (6.27) with arbitrary finite communication delay $\tau_{ij}$ under control law (6.28). If the first triggering time is $t_1^i = 0$, and agent $i$, $i \in \mathcal{N}$, determines the triggering time sequence $t_l^i|_{l=2}^{\infty}$ by*

$$\inf\left\{k > t_{l-1}^i : \ e_i^2(k) > \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{x}_i(k) - sgn(a_{ij})\widehat{x}_j(k-\tau_{ij}))^2\right\},$$

*where $0 < \sigma < 1$ is a constant. Then, we can obtain the following results:*

*(i) System (6.27) can achieve bipartite consensus asymptotically if the signed digraph $\mathcal{G}$ is structurally balanced. Moreover, the consensus value of the network is*

$$\frac{\sum_{i=1}^N \xi_i d_i x_i(0) + \sum_{i=1}^N \xi_i \sum_{j=1,\, j\neq i}^N |a_{ij}| \sum_{s=-\tau_{ij}}^{-1} d_j x_j(s)}{1 + \sum_{i=1}^N \xi_i \sum_{j=1,\, j\neq i}^N |a_{ij}|\tau_{ij}}.$$

*(ii) If the signed digraph $\mathcal{G}$ is structurally unbalanced, then the system (6.27) can achieve consensus and the final consensus value is 0, i.e., $\lim_{k\to+\infty} x_i(k) = 0,\ \forall i \in \mathcal{N}$.*

***Proof*** We take two steps for the remaining part of the proof.

Step 1: According to Lemma 1.8, if the network structure is balanced, there exists $D = \{d_1, \cdots, d_N\} \in \mathcal{D}$, such that $DAD$ is a stochastic matrix. Since $DAD$ has all nonnegative entries, one can get $d_i sgn(a_{ij}) d_j \geq 0$. Denote $y_i(k) = d_i x_i(k)$ and $\widehat{y}_i(k) = d_i \widehat{x}_i(k)$. Note that $\tau_{ii} = 0$, $\forall i \in \mathcal{N}$, then we can obtain that

$$y_i(k+1) = y_i(k) + \sum_{j\in\mathcal{N}_i} |a_{ij}|(d_i sgn(a_{ij}) d_j \widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)), \quad i \in \mathcal{N}, \tag{6.30}$$

i.e.,

$$\begin{aligned} y_i(k+1) &= y_i(k) + \sum_{j\in\mathcal{N}_i} |a_{ij}|(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\ &= y_i(k) + \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)), \quad i \in \mathcal{N}. \end{aligned} \tag{6.31}$$

Let $E_i(k) = d_i e_i(k)$. Referring to Theorem 1 in [22], the consensus of system (6.31) can be asymptotically reached under the event-triggered condition

$$E_i^2(k) > \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k))^2$$

$$
\begin{aligned}
&= \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{y}_j^2(k-\tau_{ij}) + \widehat{y}_i^2(k) - 2\widehat{y}_j(k-\tau_{ij})\widehat{y}_i(k)) \\
&= \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{x}_j^2(k-\tau_{ij}) + \widehat{x}_i^2(k) \\
&\quad - 2sgn(a_{ij})\widehat{x}_j(k-\tau_{ij})\widehat{x}_i(k)), \\
&= \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))^2, \qquad i \in \mathcal{N}.
\end{aligned} \tag{6.32}
$$

That is, $\lim_{k\to+\infty} d_i x_i(k) = c$, where $c$ is a constant value. Note that $E_i^2(k) = e_i^2(k)$. Hence, the event-triggered condition (6.32) can be rewritten as

$$
e_i^2(k) > \frac{\sigma a_{ii}^2}{4(1-|a_{ii}|)} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(k-\tau_{ij}) - \widehat{x}_i(k))^2, \qquad i \in \mathcal{N}. \tag{6.33}
$$

Therefore, under the event-triggered condition (6.33), the bipartite consensus of system (6.27) can be asymptotically reached if $\mathcal{G}$ is structurally balanced.

Next, the bipartite consensus value $c$ of the multi-agent networks is shown below. Let $\eta(k) = \sum_{i=1}^{N} \xi_i y_i(k) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \sum_{k-\tau_{ij}}^{k-1} \widehat{y}_j(s)$. Substituting (6.31) into $\eta(k+1)$, we can calculate the difference of $\eta(k)$ as follows:

$$
\begin{aligned}
\Delta\eta(k) &= \eta(k+1) - \eta(k) \\
&= \sum_{i=1}^{N} \xi_i (y_i(k+1) - y_i(k)) \\
&\quad + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}|\Big(\sum_{k+1-\tau_{ij}}^{k} \widehat{y}_j(s) - \sum_{k-\tau_{ij}}^{k-1} \widehat{y}_j(s)\Big) \\
&= \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\
&\quad + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{y}_j(k) - \widehat{y}_j(k-\tau_{ij})).
\end{aligned} \tag{6.34}
$$

Note that the row sum of matrix $|A|$ is 1 and $\{\xi_1, \xi_2, \ldots, \xi_N\}$ is the normalized left eigenvector of matrix $|A|$ with respect to the eigenvalue 1, we have

$$\sum_{j=1}^{N} |a_{ij}| = 1 \text{ and } \sum_{i=1}^{N} \xi_i |a_{ij}| = \xi_j.$$

Hence, we can obtain that

$$\begin{aligned}
\Delta\eta(k) &= -\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \widehat{y}_i(k) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \widehat{y}_j(k) \\
&= -\sum_{i=1}^{N} \xi_i \widehat{y}_i(k) \sum_{j=1,\, j\neq i}^{N} |a_{ij}| + \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} |a_{ij}| \widehat{y}_j(k) - \sum_{i=1}^{N} \xi_i a_{ii} \widehat{y}_i(k) \\
&= -\sum_{i=1}^{N} \xi_i (1 - a_{ii}) \widehat{y}_i(k) + \sum_{i=1}^{N} \xi_i |a_{ij}| \sum_{j=1}^{N} \widehat{y}_j(k) - \sum_{j=1}^{N} \xi_j a_{jj} \widehat{y}_j(k) \\
&= -\sum_{i=1}^{N} \xi_i (1 - a_{ii}) \widehat{y}_i(k) + \sum_{j=1}^{N} \xi_j (1 - a_{jj}) \widehat{y}_j(k) \\
&= 0.
\end{aligned} \tag{6.35}$$

Due to $\Delta\eta(k) = 0$ for $k \geq 0$, it can be easily obtained that $\eta(k)$ is a constant. That is,

$$\begin{aligned}
\eta(k) = \eta(0) &= \sum_{i=1}^{N} \xi_i y_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \sum_{-\tau_{ij}}^{-1} \widehat{y}_j(s) \\
&= \sum_{i=1}^{N} \xi_i y_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \sum_{-\tau_{ij}}^{-1} y_j(s).
\end{aligned}$$

Hence,

$$\eta(0) = \lim_{k\to+\infty} \eta(k) = c + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}|\tau_{ij} c.$$

Therefore, we can conclude that

$$c = \frac{\sum_{i=1}^{N} \xi_i d_i x_i(0) + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| \sum_{s=-\tau_{ij}}^{-1} d_j x_j(s)}{1 + \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}|\tau_{ij}}. \tag{6.36}$$

This completes the proof of this step.

Step 2: If the network structure is unbalanced, according to Lemma 1.10, there does not exist $D = \{d_1, \cdots, d_N\} \in \mathcal{D}$, such that $DAD$ is a stochastic matrix. For the sake of simplicity, the case of $\mathcal{G}$ with only one negative cycle is studied firstly. Here, we assume that this negative cycle contains an edge $a_{i_0 j_0} < 0$. Without loss of generality, we can assume that there exists $B = \{b_1, \cdots, b_N\} \in \mathcal{D}$, such that $BAB = [b_i a_{ij} b_j]_{N\times N}$ is a nonnegative matrix except the element $b_{i_0} a_{i_0 j_0} b_{j_0} < 0$. (If $\mathcal{G}$ contains $k$ ($k \geq 2$) negative cycles, there exists $D_l \in \mathcal{D}$ such that $D_l A D_l$ has exactly $l$ ($1 \leq l \leq k$) negative elements. The following proof for this case is similar to the case $k = 1$, and we omit it here due to space limit.) Denoting $y_i(k) = b_i x_i(k)$ and $\widehat{y}_i(k) = b_i \widehat{x}_i(k)$, then we can obtain that

$$y_i(k+1) = y_i(k) + \sum_{j\in\mathcal{N}_i} |a_{ij}|(b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)), \quad i \in \mathcal{N}. \tag{6.37}$$

Define the matrix $W = [w_{ij}]_{N\times N}$ as follows: $w_{i_0 j_0} = 0$, $w_{ii} = 1 - \sum_{j=1}^{N} w_{ij}$, $\forall i \in \mathcal{N}$, and $w_{ij} = b_i a_{ij} b_j$ otherwise. Let $E_i(k) = y_i(k) - \widehat{y}_i(k)$. Consider the Lyapunov functional as

$$V(k) = V_1(k) + V_2(k), \tag{6.38}$$

where

$$V_1(k) = \sum_{i=1}^{N} \xi_i y_i^2(k), \tag{6.39}$$

and

$$V_2(k) = \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} |a_{ij}| \sum_{s=k-\tau_{ij}}^{k-1} \widehat{y}_j^2(s). \tag{6.40}$$

Notice that $w_{i_0 i_0} = a_{i_0 i_0} + |a_{i_0 j_0}|$, and difference of $V(k)$ along the solution of (6.37) gives that

$$\begin{aligned}
\Delta V(k) \le & -\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 \\
& - \xi_{i_0} |a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}| (\widehat{y}_{j_0}(k-\tau_{i_0 j_0}) - \widehat{y}_l(k-\tau_{i_0 l}))^2 \\
& - \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} |a_{ij}| (a_{ii}-\alpha_i)(\widehat{y}_i(k) - b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}))^2 \\
& + \sum_{i=1}^{N} \xi_i (1-a_{ii}) \frac{1}{\alpha_i} E_i^2(k).
\end{aligned} \tag{6.41}$$

Actually,

$$\Delta V(k) = \Delta V_1(k) + \Delta V_2(k). \tag{6.42}$$

Note that for $i \in \mathcal{N}$, $\tau_{ii} = 0$, and $E_i(k) = y_i(k) - \widehat{y}_i(k)$, it holds that

$$\begin{aligned}
y_i(k+1) &= y_i(k) + \sum_{j\in\mathcal{N}_i} |a_{ij}| (b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\
&= y_i(k) + \sum_{j=1}^{N} |a_{ij}| (b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\
&= y_i(k) - \widehat{y}_i(k) + \sum_{j=1}^{N} |a_{ij}| b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij}),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{N} \xi_i y_i^2(k+1) &= \sum_{i=1}^{N} \xi_i [E_i(k) + \sum_{j=1}^{N} |a_{ij}| b_i sgn(a_{ij}) b_j \widehat{y}_j(k-\tau_{ij})]^2 \\
&= \sum_{i=1, i\neq i_0}^{N} \xi_i [E_i(k) + \sum_{j=1}^{N} w_{ij} \widehat{y}_j(k-\tau_{ij})]^2 + \xi_{i_0} [E_{i_0}(k) \\
&\quad + \sum_{j=1}^{N} w_{i_0 j} \widehat{y}_j(k-\tau_{i_0 j}) - |a_{i_0 j_0}| (\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))]^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\Delta V_1(k) &= \sum_{i=1}^{N} \xi_i y_i^2(k+1) - \sum_{i=1}^{N} \xi_i y_i^2(k) \\
&= \sum_{i=1}^{N} \xi_i [E_i(k) + \sum_{j=1}^{N} w_{ij} \widehat{y}_j(k-\tau_{ij})]^2 + \xi_{i_0}[a_{i_0 j_0}^2 (\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 \\
&\quad - 2|a_{i_0 j_0}|(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))(E_{i_0}(k) + \sum_{j=1}^{N} w_{i_0 j} \widehat{y}_j(k-\tau_{i_0 j}))] \\
&\quad - \sum_{i=1}^{N} \xi_i y_i^2(k) \\
&= \sum_{i=1}^{N} \xi_i [E_i^2(k) + \sum_{j=1,\, j\neq i}^{N} w_{ij}^2 \widehat{y}_j^2(k-\tau_{ij}) + w_{ii} \widehat{y}_i^2(k) \\
&\quad + 2 \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} \cdot \widehat{y}_j(k-\tau_{ij}) \widehat{y}_l(k-\tau_{ij}) \\
&\quad + 2 \sum_{j=1,\, j\neq i}^{N} w_{ij} w_{ii} \widehat{y}_j(k-\tau_{ij}) \widehat{y}_i(k) + 2 w_{ii} \widehat{y}_i(k) E_i(k) \\
&\quad + 2 \sum_{j=1,\, j\neq i}^{N} w_{ij} \widehat{y}_j(k-\tau_{ij}) E_i(k) - \sum_{i=1}^{N} \xi_i [\widehat{y}_i^2(k) \\
&\quad + E_i^2(k) + 2\widehat{y}_i(k) E_i(k)] + \xi_{i_0}[a_{i_0 j_0}^2 (\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 \\
&\quad - 2|a_{i_0 j_0}|(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))(E_{i_0}(k) + \sum_{j=1}^{N} w_{i_0 j} \widehat{y}_j(k-\tau_{i_0 j}))],
\end{aligned} \tag{6.43}$$

and

$$\begin{aligned}
\Delta V_2(k) &= \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} |a_{ij}| [\sum_{k+1-\tau_{ij}}^{k} \widehat{y}_j^2(s) - \sum_{k-\tau_{ij}}^{k-1} \widehat{y}_j^2(s)] \\
&= \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} |a_{ij}| [\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{ij})]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{N}\xi_i \sum_{j=1}^{N}\sum_{l=1}^{N}|a_{ij}||a_{il}|[\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{ij}) + \widehat{y}_l^2(k) - \widehat{y}_l^2(k-\tau_{il})] \\
&= \sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N} a_{ij}^2(\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{ij})) + \sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N}|a_{ij}||a_{il}| \\
&\quad (\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{ij}) + \widehat{y}_l^2(k) - \widehat{y}_l^2(k-\tau_{ij})) + \sum_{j=1,\, j\neq i}^{N}|a_{ij}||a_{ii}|(\widehat{y}_j^2(k) \\
&\quad - \widehat{y}_j^2(k-\tau_{ij}))] \\
&= \sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N} w_{ij}^2(\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{ij})) + \sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N} w_{ij}w_{il}(\widehat{y}_j^2(k) \\
&\quad - \widehat{y}_j^2(k-\tau_{ij}) + \widehat{y}_l^2(k) - \widehat{y}_l^2(k-\tau_{il})) + \sum_{j=1,\, j\neq i}^{N} w_{ij}w_{ii}(\widehat{y}_j^2(k) \\
&\quad - \widehat{y}_j^2(k-\tau_{ij}))] + \xi_{i_0}[a_{i_0j_0}^2(\widehat{y}_{j_0}^2(k) - \widehat{y}_{j_0}^2(k-\tau_{i_0j_0})) + |a_{i_0j_0}||a_{i_0i_0}|(\widehat{y}_{j_0}^2(k) \\
&\quad - \widehat{y}_{j_0}^2(k-\tau_{i_0j_0})) - |a_{i_0j_0}|\sum_{j=1,\, j\neq i_0}^{N} w_{i_0j}(\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{i_0j})) \\
&\quad + |a_{i_0j_0}|\sum_{l\neq i_0,\, l\neq j_0}^{N}|a_{i_0l}|(\widehat{y}_{j_0}^2(k) - \widehat{y}_{j_0}^2(k-\tau_{i_0j_0}) + \widehat{y}_l^2(k) - \widehat{y}_l^2(k-\tau_{i_0l}))].
\end{aligned}
\tag{6.44}
$$

Let

$$
\begin{aligned}
\Delta_1 = &\sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N} w_{ij}^2\widehat{y}_j^2(k-\tau_{ij}) + w_{ii}\widehat{y}_i^2(k) + 2\sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N} w_{ij}w_{il} \\
&\widehat{y}_j(k-\tau_{ij})\widehat{y}_l(k-\tau_{ij}) + 2\sum_{j=1,\, j\neq i}^{N} w_{ij}w_{ii}\widehat{y}_j(k-\tau_{ij})\widehat{y}_i(k) - \sum_{i=1}^{N}\xi_i\widehat{y}_i^2(k) \\
&+ \sum_{i=1}^{N}\xi_i[\sum_{j=1,\, j\neq i}^{N} w_{ij}^2(\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{ij})) + \sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N} w_{ij}w_{il}(\widehat{y}_j^2(k) \\
&- \widehat{y}_j^2(k-\tau_{ij}) + \widehat{y}_l^2(k) - \widehat{y}_l^2(k-\tau_{il})) + \sum_{j=1,\, j\neq i}^{N} w_{ij}w_{ii}(\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{ij}))]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \xi_i [ \sum_{j=1,\, j\neq i}^{N} w_{ij}^2 \widehat{y}_j^2(k) + w_{ii}^2 \widehat{y}_i^2(k) + \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j^2(k) + \widehat{y}_l^2(k) \\
&\quad -\widehat{y}_j^2(k-\tau_{ij}) - \widehat{y}_l^2(k-\tau_{il}) + 2\widehat{y}_j(k-\tau_{ij})\widehat{y}_l(k-\tau_{ij})) + \sum_{j=1,\, j\neq i}^{N} w_{ij} w_{ii} \\
&\quad \cdot(\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{ij}) + 2\widehat{y}_j(k-\tau_{ij})\widehat{y}_i(k))] - \sum_{i=1}^{N} \xi_i \widehat{y}_i^2(k) \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} w_{ij}^2 \widehat{y}_j^2(k) + \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j^2(k) + \widehat{y}_l^2(k)) + \sum_{j<i}^{N} w_{ij} w_{ii} \\
&\quad \cdot(\widehat{y}_j^2(k) + \widehat{y}_i^2(k)) + \sum_{l>i}^{N} w_{il} w_{ii} (\widehat{y}_l^2(k) + \widehat{y}_i^2(k)) - \widehat{y}_i^2(k)] \\
&\quad - \sum_{i=1}^{N} \xi_i [ \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} \\
&\quad w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j\neq i}^{N} w_{ij} w_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2] \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} w_{ij}^2 \widehat{y}_j^2(k) + \sum_{j=1}^{N} \sum_{l=1,\, l\neq j}^{N} w_{ij} w_{il} \widehat{y}_j^2(k) - \widehat{y}_i^2(k)] \\
&\quad - \sum_{i=1}^{N} \xi_i [ \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} \\
&\quad w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j\neq i}^{N} w_{ij} w_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2] \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} \sum_{l=1}^{N} w_{ij} w_{il} \widehat{y}_j^2(k) - \widehat{y}_i^2(k)] \\
&\quad - \sum_{i=1}^{N} \xi_i [ \sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) \\
&\quad -\widehat{y}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j\neq i}^{N} w_{ij} w_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2]
\end{aligned}
$$

$$= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} w_{ij} \widehat{y}_j^2(k) - \widehat{y}_i^2(k)] - \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij})$$

$$-\widehat{y}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j\neq i}^{N} w_{ij} w_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2], \tag{6.45}$$

$$\Delta_2 = \xi_{i_0} [a_{i_0 j_0}^2 (\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 - 2|a_{i_0 j_0}| (\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0})) \sum_{j=1}^{N} w_{i_0 j}$$

$$\cdot \widehat{y}_j(k-\tau_{i_0 j})] + \xi_{i_0} [a_{i_0 j_0}^2 (\widehat{y}_{j_0}^2(k) - \widehat{y}_{j_0}^2(k-\tau_{i_0 j_0})) + |a_{i_0 j_0}||a_{i_0 i_0}| (\widehat{y}_{j_0}^2(k)$$

$$-\widehat{y}_{j_0}^2(k-\tau_{i_0 j_0})) - |a_{i_0 j_0}| \sum_{j=1,\, j\neq i_0}^{N} w_{i_0 j} (\widehat{y}_j^2(k) - \widehat{y}_j^2(k-\tau_{i_0 j}))$$

$$+|a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}| (\widehat{y}_{j_0}^2(k) - \widehat{y}_{j_0}^2(k-\tau_{i_0 j_0}) + \widehat{y}_l^2(k) - \widehat{y}_l^2(k-\tau_{i_0 l}))]$$

$$= \xi_{i_0} [a_{i_0 j_0}^2 (\widehat{y}_{i_0}^2(k) + 2\widehat{y}_{i_0}(k)\widehat{y}_{j_0}(k-\tau_{i_0 j_0})) - 2|a_{i_0 j_0}| w_{i_0 i_0} \widehat{y}_{i_0}^2(k) - 2|a_{i_0 j_0}| w_{i_0 i_0}$$

$$\cdot \widehat{y}_{i_0}(k)\widehat{y}_{j_0}(k-\tau_{i_0 j_0}) + a_{i_0 j_0}^2 \widehat{y}_{j_0}^2(k) + |a_{i_0 j_0}||a_{i_0 i_0}| (\widehat{y}_{j_0}^2(k) + \widehat{y}_{i_0}^2(k)$$

$$-\widehat{y}_{j_0}^2(k-\tau_{i_0 j_0}) L - \widehat{y}_{i_0}^2(k)) - |a_{i_0 j_0}| \sum_{j=1,\, j\neq i_0}^{N} w_{i_0 j} (\widehat{y}_j^2(k) + \widehat{y}_{i_0}^2(k))$$

$$-|a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}| (\widehat{y}_{j_0}^2(k)$$

$$+\widehat{y}_l^2(k)) + |a_{i_0 j_0}| \sum_{j=1,\, j\neq i_0}^{N} w_{i_0 j} (-\widehat{y}_{i_0}^2(k) - \widehat{y}_j^2(k-\tau_{i_0 j})$$

$$-2\widehat{y}_{i_0}(k)\widehat{y}_j(k-\tau_{i_0 j}))$$

$$+|a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}| (-\widehat{y}_{j_0}^2(k-\tau_{i_0 j_0})$$

$$-\widehat{y}_l^2(k-\tau_{i_0 l}) - 2\widehat{y}_{j_0}(k-\tau_{i_0 j_0})\widehat{y}_l(k-\tau_{i_0 j})]$$

$$= \xi_{i_0} [a_{i_0 j_0}^2 \widehat{y}_{i_0}^2(k) - 2|a_{i_0 j_0}| w_{i_0 i_0} \widehat{y}_{i_0}^2(k) + a_{i_0 j_0}^2 \widehat{y}_{j_0}^2(k) + |a_{i_0 j_0}||a_{i_0 i_0}| (\widehat{y}_{j_0}^2(k)$$

$$+\widehat{y}_{i_0}^2(k)) + |a_{i_0 j_0}||a_{i_0 i_0}| (-\widehat{y}_{i_0}^2(k) - \widehat{y}_{j_0}^2(k-\tau_{i_0 j_0})$$

$$-2\widehat{y}_{i_0}(k)\widehat{y}_{j_0}(k-\tau_{i_0j_0})) - |a_{i_0j_0}| \sum_{j=1,\, j\neq i_0}^{N}$$

$$w_{i_0j}(\widehat{y}_j^2(k) + \widehat{y}_{i_0}^2(k)) + |a_{i_0j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0l}|(\widehat{y}_{j_0}^2(k) + \widehat{y}_l^2(k))$$

$$+|a_{i_0j_0}| \sum_{j=1,\, j\neq i_0}^{N} w_{i_0j} \cdot (\widehat{y}_{i_0}(k)$$

$$-\widehat{y}_j(k-\tau_{i_0j}))^2 - |a_{i_0j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0l}|(\widehat{y}_{j_0}(k-\tau_{i_0j_0}) - \widehat{y}_l(k-\tau_{i_0l}))^2]$$

$$= \xi_{i_0}[-|a_{i_0j_0}|\widehat{y}_{i_0}^2(k) + |a_{i_0j_0}|\widehat{y}_{j_0}^2(k) - |a_{i_0j_0}||a_{i_0i_0}|(\widehat{y}_{i_0}(k)$$

$$+\widehat{y}_{j_0}(k-\tau_{i_0j_0}))^2 + |a_{i_0j_0}|$$

$$\cdot \sum_{j=1,\, j\neq i_0}^{N} w_{i_0j}(\widehat{y}_{i_0}(k) - \widehat{y}_j(k-\tau_{i_0j}))^2 - |a_{i_0j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0l}|(\widehat{y}_{j_0}(k-\tau_{i_0j_0})$$

$$-\widehat{y}_l(k-\tau_{i_0l}))^2], \tag{6.46}$$

and

$$\varDelta_3 = \sum_{i=1}^{N} \xi_i[2w_{ii}\widehat{y}_i(k)E_i(k) + 2\sum_{j=1,\, j\neq i}^{N} w_{ij}\widehat{y}_j(k-\tau_{ij})E_i(k)] - 2\sum_{i=1}^{N} \xi_i\widehat{y}_i(k)$$

$$E_i(k) - 2|a_{i_0j_0}|(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0j_0}))E_{i_0}(k)$$

$$= 2\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} w_{ij}E_i(k)(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) - 2\xi_{i_0}|a_{i_0j_0}|E_{i_0}(k)(\widehat{y}_{i_0}(k)$$

$$+\widehat{y}_{j_0}(k-\tau_{i_0j_0}))$$

$$\leq \sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} w_{ij}[\frac{1}{\alpha_i}E_i^2(k) + \alpha_i(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k))^2] + \xi_{i_0}|a_{i_0j_0}|$$

$$[\frac{1}{\alpha_{i_0}}E_{i_0}^2(k) + \alpha_{i_0}(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0j_0}))^2]. \tag{6.47}$$

Substituting (6.45), (6.46), and (6.47) into (6.42), we can obtain that

$$
\begin{aligned}
\Delta V(k) &= \Delta_1 + \Delta_2 + \Delta_3 \\
&= \sum_{i=1}^{N} \xi_i [\sum_{j=1}^{N} w_{ij} \widehat{y}_j^2(k) - \widehat{y}_i^2(k)] + \xi_{i_0}[-|a_{i_0 j_0}| \widehat{y}_{i_0}^2(k) + |a_{i_0 j_0}| \widehat{y}_{j_0}^2(k)] \\
&\quad - \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 \\
&\quad + \sum_{j=1,\, j\neq i}^{N} w_{ij} w_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2] + \xi_{i_0}[-|a_{i_0 j_0}||a_{i_0 i_0}|(\widehat{y}_{i_0}(k) \\
&\quad + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 + |a_{i_0 j_0}| \sum_{j=1,\, j\neq i_0}^{N} w_{i_0 j}(\widehat{y}_{i_0}(k) - \widehat{y}_j(k-\tau_{i_0 j}))^2 \\
&\quad - |a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}|(\widehat{y}_{j_0}(k-\tau_{i_0 j_0}) - \widehat{y}_l(k-\tau_{i_0 l}))^2] \\
&\quad + 2\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} w_{ij} E_i(k)(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\
&\quad - 2\xi_{i_0}|a_{i_0 j_0}| E_{i_0}(k)(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0})) \\
&= \sum_{i=1}^{N} \xi_i \sum_{j=1}^{N} |a_{ij}| \widehat{y}_j^2(k) - \sum_{i=1}^{N} \xi_i \widehat{y}_i^2(k) - \sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} \cdot \\
&\quad (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2 + \sum_{j=1,\, j\neq i}^{N} w_{ij} a_{ii} (\widehat{y}_i(k) - \widehat{y}_j(k-\tau_{ij}))^2] \\
&\quad - \xi_{i_0}[|a_{i_0 j_0}||a_{i_0 i_0}|(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0}))^2 \\
&\quad + |a_{i_0 j_0}| \sum_{l\neq i_0,\, l\neq j_0}^{N} |a_{i_0 l}|(\widehat{y}_{j_0}(k-\tau_{i_0 j_0}) - \widehat{y}_l(k-\tau_{i_0 l}))^2] \\
&\quad + 2\sum_{i=1}^{N} \xi_i \sum_{j=1,\, j\neq i}^{N} w_{ij} E_i(k)(\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_i(k)) \\
&\quad - 2\xi_{i_0}|a_{i_0 j_0}| E_{i_0}(k)(\widehat{y}_{i_0}(k) + \widehat{y}_{j_0}(k-\tau_{i_0 j_0})) \\
&\leq -\sum_{i=1}^{N} \xi_i [\sum_{j=1,\, j\neq i}^{N} \sum_{l>j,\, l\neq i}^{N} w_{ij} w_{il} (\widehat{y}_j(k-\tau_{ij}) - \widehat{y}_l(k-\tau_{il}))^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1,\, j\neq i}^{N} w_{ij}(a_{ii}-\alpha_i)(\widehat{y}_i(k)-\widehat{y}_j(k-\tau_{ij}))^2] \\
&-\xi_{i_0}[|a_{i_0j_0}|(a_{i_0i_0}-\alpha_{i_0})(\widehat{y}_{i_0}(k)+\widehat{y}_{j_0}(k-\tau_{i_0j_0}))^2 \\
&+|a_{i_0j_0}|\sum_{l\neq i_0,\, l\neq j_0}^{N}|a_{i_0l}|(\widehat{y}_{j_0}(k-\tau_{i_0j_0})-\widehat{y}_l(k-\tau_{i_0l}))^2] \\
&+\xi_{i_0}|a_{i_0j_0}|\frac{1}{\alpha_{i_0}}E_{i_0}^2(k)+\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}w_{ij}\frac{1}{\alpha_i}E_i^2(k) \\
=&-\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}\sum_{l>j,\, l\neq i}^{N}w_{ij}w_{il}(\widehat{y}_j(k-\tau_{ij})-\widehat{y}_l(k-\tau_{il}))^2 \\
&-\xi_{i_0}|a_{i_0j_0}|\sum_{l\neq i_0,\, l\neq j_0}^{N}|a_{i_0l}|(\widehat{y}_{j_0}(k-\tau_{i_0j_0})-\widehat{y}_l(k-\tau_{i_0l}))^2 \\
&-\sum_{i=1}^{N}\xi_i\sum_{j=1,\, j\neq i}^{N}|a_{ij}|(a_{ii}-\alpha_i)(\widehat{y}_i(k)-b_isgn(a_{ij})b_j\widehat{y}_j(k-\tau_{ij}))^2 \\
&+\sum_{i=1}^{N}\xi_i(1-a_{ii})\frac{1}{\alpha_i}E_i^2(k), \qquad (6.48)
\end{aligned}
$$

which implies (6.41) holds.

Let $f(\alpha_i) = \frac{\alpha_i(a_{ii}-\alpha_i)}{1-a_{ii}}$. We aim to reduce the number of event-triggering time instants as much as possible when the parameter $\alpha_i$ is chosen. That is to say, the event-triggered condition needs to be more difficult to be satisfied when we select the parameter $\alpha_i$. To realize this objective, we choose $\alpha_i = \frac{a_{ii}}{2}$ such that $f(\alpha_i)$ can be maximized. Note that the event-triggered condition (6.32) can be rewritten as follows for $i \in \mathcal{N}$:

$$
\begin{aligned}
E_i^2(k) = e_i^2(k) &> \frac{\sigma a_{ii}^2}{4(1-a_{ii})}\sum_{j=1,\, j\neq i}^{N}|a_{ij}|(sgn(a_{ij})\widehat{x}_j(k-\tau_{ij})-\widehat{x}_i(k))^2 \\
&= \frac{\sigma a_{ii}^2}{4(1-a_{ii})}\sum_{j=1,\, j\neq i}^{N}|a_{ij}|(b_isgn(a_{ij})b_j\widehat{y}_j(k-\tau_{ij})-\widehat{y}_i(k))^2. \qquad (6.49)
\end{aligned}
$$

Under the event-triggered condition (6.49), we have that

$$\begin{aligned}
&-\sum_{i=1}^{N}\xi_i\sum_{j=1,\,j\neq i}^{N}|a_{ij}|(a_{ii}-\alpha_i)(\widehat{y}_i(k)-b_i sgn(a_{ij})b_j\widehat{y}_j(k-\tau_{ij}))^2\\
&\quad+\sum_{i=1}^{N}\xi_i(1-a_{ii})\frac{1}{\alpha_i}E_i^2(k)\\
&\leq-\sum_{i=1}^{N}\xi_i(a_{ii}-\frac{a_{ii}}{2})\frac{4(1-a_{ii})}{\sigma a_{ii}^2}e_i^2(k)+\sum_{i=1}^{N}\xi_i(1-a_{ii})\frac{2}{a_{ii}}E_i^2(k)\\
&=-\sum_{i=1}^{N}\xi_i\frac{2(1-a_{ii})}{a_{ii}}(\frac{1}{\sigma}-1)E_i^2(k). \qquad (6.50)
\end{aligned}$$

Note that $0<\sigma<1$. Hence, under the trigger condition (6.49), it holds that for $\forall k\geq 0$,

$$\begin{aligned}
\Delta V(k)\leq&-\sum_{i=1}^{N}\xi_i\sum_{j=1,\,j\neq i}^{N}\sum_{l>j,\,l\neq i}^{N}w_{ij}w_{il}(\widehat{y}_j(k-\tau_{ij})-\widehat{y}_l(k-\tau_{il})^2\\
&-\xi_{i_0}|a_{i_0j_0}|\cdot\sum_{l\neq i_0,\,l\neq j_0}^{N}|a_{i_0l}|(\widehat{y}_{j_0}(k-\tau_{i_0j_0})-\widehat{y}_l(k-\tau_{i_0l}))^2\\
&-\sum_{i=1}^{N}\xi_i\frac{2(1-a_{ii})}{a_{ii}}\cdot(\frac{1}{\sigma}-1)E_i^2(k)\\
\leq&0. \qquad (6.51)
\end{aligned}$$

According to LaSalle's invariance principle, all the agents in the network will converge to the maximal positively invariant set of $\Phi=\{\theta\in\Upsilon_{-\tau},\ x(k+\theta)\in X:\Delta V(k)=0\}$ asymptotically. Note that $\Delta V(k)=0$ if and only if $e_i(k)=0$,

$$\begin{aligned}
&-\sum_{i=1}^{N}\xi_i\sum_{j=1,\,j\neq i}^{N}\sum_{l>j,\,l\neq i}^{N}w_{ij}w_{il}(\widehat{y}_j(k-\tau_{ij})-\widehat{y}_l(k-\tau_{il})^2-\xi_{i_0}|a_{i_0j_0}|\sum_{l\neq i_0,\,l\neq j_0}^{N}\\
&|a_{i_0l}|(\widehat{y}_{j_0}(k-\tau_{i_0j_0})-\widehat{y}_l(k-\tau_{i_0l}))^2=0, \qquad (6.52)
\end{aligned}$$

and

$$\sum_{i=1}^{N}\xi_i\frac{a_{ii}}{2}\sum_{j=1,\,j\neq i}^{N}a_{ij}(b_i sgn(a_{ij})b_j\widehat{y}_j(k-\tau_{ij})-\widehat{y}_i(k))^2=0. \qquad (6.53)$$

Hence, $\Delta V(k) = 0$ if and only if $e_i(k) = 0$, and

$$\widehat{y}_j(k - \tau_{ij}) = sgn(a_{ij})\widehat{y}_i(k), \ \forall i, \ j \in \mathcal{N}_i, \tag{6.54}$$

or

$$\widehat{y}_j(k - \tau_{ij}) = \widehat{y}_i(k), \ if \ w_{ij} > 0, \tag{6.55}$$

$$\widehat{y}_{j_0}(k - \tau_{ij}) = -\widehat{y}_{i_0}(k). \tag{6.56}$$

Substituting (6.54) into (6.37) yields that

$$y_i(k+1) = y_i(k), \ \forall i \in \mathcal{N}. \tag{6.57}$$

Hence, we have

$$y_i(k) = \widehat{y}_i(k) = \widehat{y}_j(k - \tau_{ij}) = y_j(k - \tau_{ij}) = y_j(k), \ \forall w_{ij} > 0. \tag{6.58}$$

It follows that $\mathcal{G}$ is strongly connected, and it implies

$$y_i(k) = y_j(k), \ k \geq -\tau_{ij}, \ \forall i, \ j \in \mathcal{N}, \tag{6.59}$$

and

$$y_{i_0}(k) = -y_{j_0}(k), \ k \geq -\tau_{ij}. \tag{6.60}$$

Hence, $\Delta V(k) = 0$ if and only if $y_i(k) = 0, \ \forall i \in \mathcal{N}$. By LaSalle's invariance principle, we have

$$\lim_{k \to +\infty} y_1(k) = \lim_{k \to +\infty} y_2(k) = \cdots = \lim_{k \to +\infty} y_N(k) = 0,$$

which implies that

$$\lim_{k \to +\infty} x_1(k) = \lim_{k \to +\infty} x_2(k) = \cdots = \lim_{k \to +\infty} x_N(k) = 0.$$

*Remark 6.13* It can be observed that only the received neighboring states are used in the trigger condition (6.30). Hence, the event-based protocol proposed in this section is distributed. Zeno behavior is defined as an infinite number of triggering occurring in a finite-time interval, which should be avoided in the event-based consensus protocol. Nevertheless, the Zeno behavior can always be excluded in discrete-time multi-agent system since the maximum triggering number is the length of the finite-time interval.

### 6.2.2 Self-triggered Approach

In Theorem 6.12, we have proved that the proposed event-based protocol is effective to realize the bipartite consensus of the network model. However, the triggering condition needs to be continuously verified for each agent. In this section, we aim to solve this difficult problem by designing a self-triggered algorithm, i.e., the next update time is precomputed based on predictions using the received data. Under the proposed self-triggered algorithm, the signal remains unchanged until next triggering time of multi-agent networks. The appropriate equation for obtaining the triggering time guarantees desired levels of performance. Hence, self-triggered communication schemes for multi-agent networks can effectively reduce the communication costs.

Different from the event-triggered communication strategy, for self-triggered algorithm, the agent $i$ will predict next triggering time instant $t_{l+1}^i$ according to the information at time $t_l^i$. Next, we will give an algorithm to determine the time instant $t_{l+1}^i$.

Denote

$$l(k-\tau_{ij}) = \arg\max_{l\in\mathbb{N}}\{t_l^j \,|\, t_l^j \le k-\tau_{ij}\}. \tag{6.61}$$

Let

$$p_i(k) = \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t^j_{l(k-\tau_{ij})}) - x_i(t_l^i)), \tag{6.62}$$

and

$$q_i(k) = \frac{\sigma a_{ii}^2}{4(1-a_{ii})} \sum_{j=1,\, j\neq i}^{N} |a_{ij}|(\widehat{x}_i(t_l^i) - sgn(a_{ij})\widehat{x}_j(t^j_{l(k-\tau_{ij})}))^2. \tag{6.63}$$

For $k \in [t_l^i, t_{l+1}^i)$, recall that $e_i^2(k) = (x_i(k) - x_i(t_l^i))^2$. For the positive integer $m$, we have

$$\begin{aligned}
x_i(t_l^i + m) =& x_i(t_l^i + m - 1) + \sum_{j\in\mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t_l^i + m - 1 - \tau_{ij}) \\
& - \widehat{x}_i(t_l^i + m - 1)) \\
=& x_i(t_l^i + m - 2) + \sum_{j\in\mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t_l^i + m - 2 - \tau_{ij}) \\
& - \widehat{x}_i(t_l^i + m - 2)) + \sum_{j\in\mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t_l^i + m - 1 - \tau_{ij})
\end{aligned}$$

$$
\begin{aligned}
&- \widehat{x}_i(t_l^i + m - 1)) \\
=&\cdots\cdots \\
=&x_i(t_l^i) + \sum_{j \in \mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t_l^i - \tau_{ij}) - x_i(t_l^i)) + \cdots + \\
&+ \sum_{j \in \mathcal{N}_i} |a_{ij}|(sgn(a_{ij})\widehat{x}_j(t_l^i + m - 1 - \tau_{ij}) - \widehat{x}_i(t_l^i + m - 1)).
\end{aligned} \tag{6.64}
$$

To propose the self-triggered algorithm to find $t_{l+1}^i$, set $\Sigma = 0$ and $s = t_l^i$. The following two cases are considered:

Case 1: For $k > s$, if agent $i$ does not receive the renewed information from its neighbors, it follows from (6.64) that

$$
(x_i(k) - x_i(s))^2 = [\Sigma + p_i(s)(k - s)]^2, \qquad i \in \mathcal{N}. \tag{6.65}
$$

Solving the inequality $[\Sigma + p_i(s)(k - s)]^2 - q_i(s) > 0$, we can obtain that the minimum $k = \omega_l^i$ satisfying the above inequality. Hence, according to the event-triggered condition (6.33), the event-triggered time instant is is $t_{l+1}^i = \omega_l^i$ in this case.

Case 2: If agent $i$ firstly receives the renewed information from some of its neighbors at time $k^0 < \omega_l^i$, it follows from (6.64) that

$$
(x_i(k^0) - x_i(s))^2 = [p_i(s)(k^0 - s)]^2, \qquad i \in \mathcal{N}. \tag{6.66}
$$

Set $\Sigma = 0 + p_i(s)(k^0 - s)$. According to the event-triggered condition (6.33), we should update $s = k^0$ and then go back to Case 1.

Based on the above discussions, an efficient algorithm to find $t_{l+1}^i$, $\forall i \in \mathcal{N}$ can be summarized as follows.

---

**Algorithm 6.1** Self-triggered algorithm for system (6.27)

---

*Step 1.* For each agent $i \in \mathcal{N}$, set $\Sigma = 0$ and $s = t_l^i$.

*Step 2.* Solving the inequality $[\Sigma + p_i(s)(k - s)]^2 - q_i(s) > 0$, we can obtain the minimum $k = \omega_l^i$ such that the inequality holds.

*Step 3.* For $k \geq s$, if agent $i$ does not receive the renewed information from its neighbors until $k = \omega_l^i$, then set $t_{l+1}^i = \omega_l^i$ and stop the algorithm.

*Step 4.* If agent $i$ firstly receives the renewed information from some of its neighbors at time $k^0 < \omega_l^i$, set $\Sigma = \Sigma + p_i(s)(k^0 - s)$. Update $s = k^0$ and go to *Step 2*.

---

According to the above analysis, the following Theorem 6.14 can be obtained.

**Theorem 6.14** *Consider the multi-agent system (6.27) with arbitrary finite communication delay $\tau_{ij}$ under control law (6.28). If the first triggering time $t_1^i = 0$, agent $i$, $i \in \mathcal{N}$, determines the triggering time sequence $t_l^i|_{l=2}^{\infty}$ by self-triggered algorithm 6.1. Then, we can obtain the following results:*

*(i) System (6.27) can achieve bipartite consensus asymptotically if signed digraph $\mathcal{G}$ is structurally balanced.*
*(ii) If signed digraph $\mathcal{G}$ is structurally unbalanced, then the system (6.27) can achieve consensus and the final consensus value is 0, i.e., $\lim_{k \to +\infty} x_i(k) = 0, \forall i \in \mathcal{N}$.*

### 6.2.3 Numerical Example

*Example 6.15* Consider a signed multi-agent network with structurally balanced topology and structurally unbalanced topology, respectively (see Fig. 6.8). Set $\sigma = 0.9$ in event-triggered condition (6.30) and the distinct communication delays are as follows:

$$\Gamma = (\tau_{ij})_{6\times 6} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}.$$

According to Theorem 6.12, one can easily conclude that under the proposed event-triggered condition (6.30),

(i) the system with communication delays will achieve bipartite consensus when the network topology is shown in Fig. 6.8a;
(ii) the states of all the agents will converge to zero when the network topology is shown in Fig. 6.8b.

The evolvement of the agents under the event-triggered condition (6.30) is shown in Figs. 6.9 and 6.10, respectively. The numerical results in Fig. 6.9 show that the individual state of the multi-agent system converges to the bipartite constant limit that has the same modulus and different signs. The numerical results in Fig. 6.10 show that the individual state of the multi-agent system converges to zero. Figures 6.9 and 6.10 agree well with the proposed theoretical result.

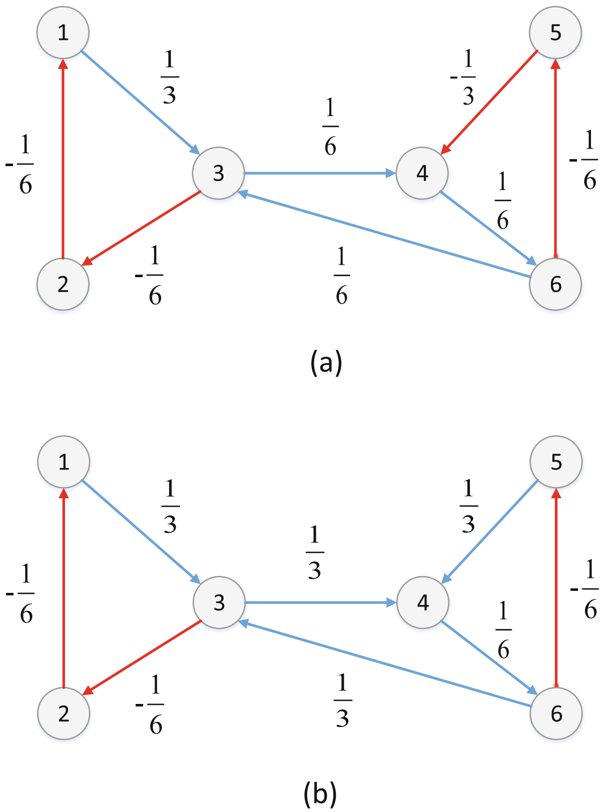


**Fig. 6.8** Network topology in Example 6.15. (**a**) structurally balanced. (**b**) Structurally unbalanced

The individual event time instants corresponding to Figs. 6.9 and 6.10 under the proposed event-triggered protocol are shown in Figs. 6.11 and 6.12, respectively. Table 6.1 illustrates the event-triggering frequency under two different network topologies. One can conclude from the simulation example that the event-based strategy in this chapter can significantly decrease the information transmission during the bipartite consensus process of the signed network model with distinct communication delays.

## 6.3 Summary

In this chapter, the bipartite consensus of continuous-time and discrete-time multi-agent system was studied. For the continuous-time model, according to Perron–Frobenius theorem and some other mathematical analysis, it was found that the bipartite consensus can be asymptotically reached if the strongly connected signed

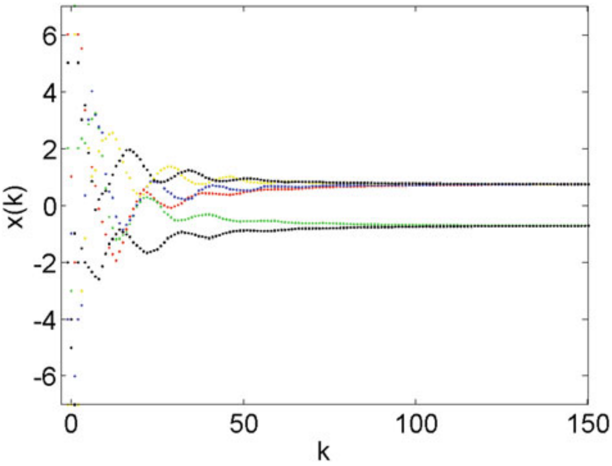


**Fig. 6.9** The states of multi-agent system (6.27) associated with signed diagraph with balanced structure in Fig. 6.8a

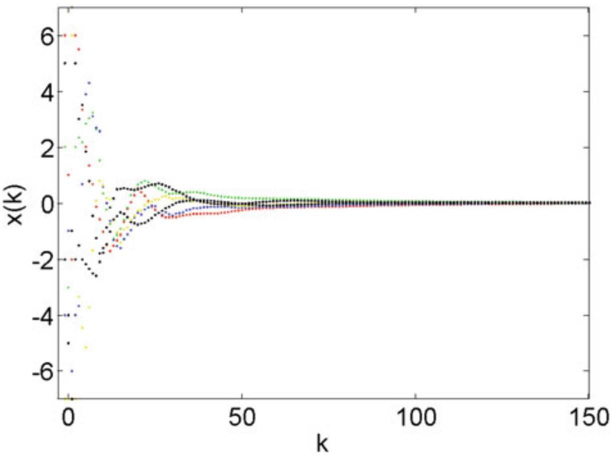


**Fig. 6.10** The states of multi-agent system (6.27) associated with signed digraph with unbalanced structure in Fig. 6.8b

digraph $\mathcal{G}$ is structurally balanced. For the discrete-time model, communication delays and event-based strategy were considered simultaneously. It is shown that under the proposed event-triggered condition the bipartite consensus can be asymptotically achieved if the network topology is structurally balanced, and all the agents converge to zero if the signed digraph is structurally unbalanced. Numerical examples were provided to demonstrate the effectiveness of our derived results.

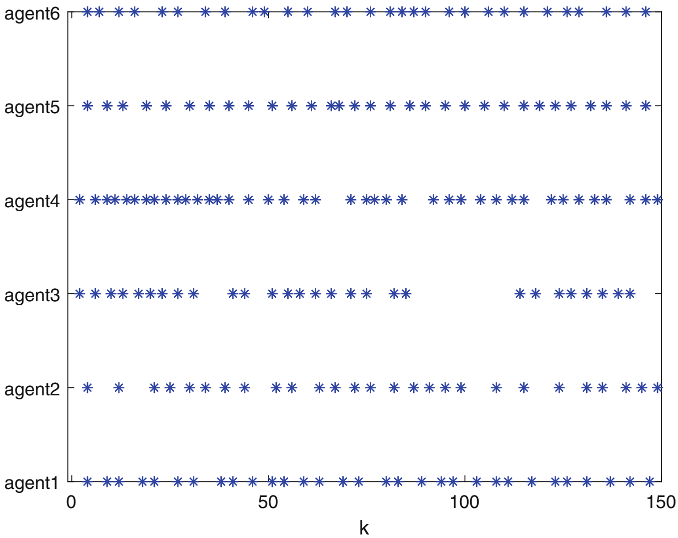


**Fig. 6.11** Event-trigger times associated with signed digraph in Fig. 6.8a

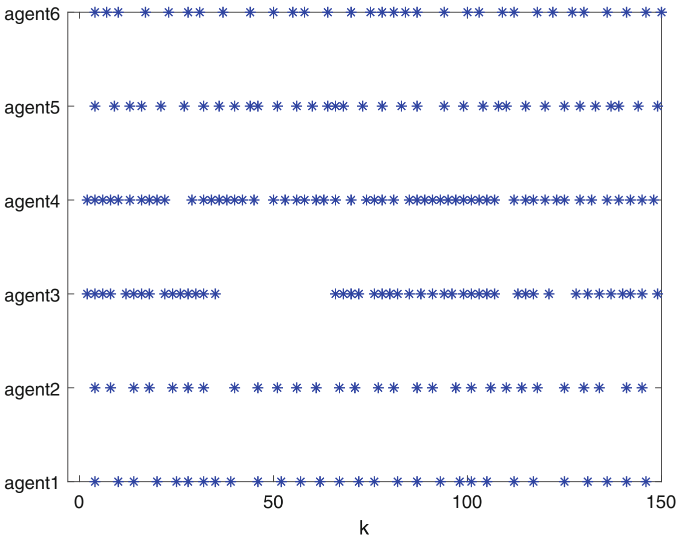


**Fig. 6.12** Event-trigger times associated with signed digraph in Fig. 6.8b

**Table 6.1** The total number of triggering over the total number of iterations in simulation under structurally balanced topology and structurally unbalanced topology, respectively

| Node | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Balanced | 26.6% | 25.3% | 21.3% | 28.6% | 26.0% | 26.0% |
| Unbalanced | 27.3% | 26.6% | 31.3% | 34% | 26.0% | 28.6% |

# References

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# Chapter 7
# Finite-Time and Fixed-Time Bipartite Consensus for Multi-agent Systems with Antagonistic Interactions

In the last decades, many typical consensus problems have been investigated, such as pinning consensus [1], impulsive consensus [2, 3], event-based consensus [4–9], minimum-energy consensus [10], and references therein [11–13]. Note that most results of consensus problem of multi-agent systems are only reached asymptotically. From the viewpoint of time optimization, it is one of the best control results that all states of the multi-agent systems are convergent in a finite time. Motivated by the discussion, a finite-time consensus problem has been studied to reach a high-speed convergence which can bring some good performances for the multi-agent systems. For single-integrator dynamics, Wang and Xiao [14] proposed a finite-time protocol for consensus of multi-agent systems with time-varying topologies. By applying the observer-based control algorithms, the finite-time consensus tracking in multi-agent systems was achieved in [15]. With the consideration that the global formation information was only available for a small number of agents, the authors in [16] addressed the finite-time formation control of multi-agent systems. For double-integrator dynamics, the finite-time consensus problem for leaderless and leader–follower multi-agent systems with external disturbances was discussed in literature [17]. Since the finite-time control theory has attracted much attention for researchers, there is a rich body of literature about finite-time control problems [18–21].

From the aspect of the convergence rate, consensus problems can be separated into asymptotic consensus problems and finite-time consensus problems. In many practical situations, consensus over a finite time is more efficient than asymptotic convergence because the finite-time consensus has a broader range of applications [22]. The study of finite-time consensus problems has drawn much attention from researchers because of their faster convergence and better robustness [23, 24]. Xiao and Wang [25] first proposed two finite-time protocols, one of them being $u_i(t) = \sum_{k\in\mathcal{N}_i} a_{ik}\mathrm{sgn}(x_k(t) - x_i(t))|x_k(t) - x_i(t)|^{\alpha}(0 < \alpha < 1)$, and this protocol can be used to solve the finite-time consensus problem of multi-agent systems with fixed undirected topology. Then, they [14] extended the finite-time



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consensus results to directed networks with time-varying topologies. The finite-time consensus for leader-following models was further discussed in [26], and homogeneous functions were used to prove the finite-time consensus. Using binary control protocols, Chen et al. in [18] studied the finite-time distributed consensus problem for multi-agent systems. Since the protocols are discontinuous, the concept of differential inclusion was used to deal with this problem [27]. For fractional-order multi-agent systems, the exponential finite-time consensus problem with a directed communication network was investigated in [28]. Du et al. considered the consensus tracking problem of multiple nonholonomic high-order chained-form systems in [29], and a finite-time observer-based distributed control strategy is proposed. In literature [30], Ning et al. firstly proposed a new distributed observer for each follower to estimate the leader state and the leader input in a prescribed nonholonomic chained-form dynamics. Due to some indispensable symmetric requirements, many results concerning the finite-time consensus were restricted to multi-agent systems with undirected graphs. However, the attraction–repulsion rule and coupled strength between two individuals in many practical networks are potentially not identical. Therefore, a concept named detail-balanced graph was proposed in [31] to describe more general models whose communication channels among agents are also bidirectional but different weights. Motivated by this, most of the finite-time consensus results can be extended to detail-balanced graphs, such as [32]. A general nonlinear finite-time consensus protocol was designed in [20], and the network topology was extended to detail-balanced directed graphs containing a rooted spanning tree.

Although many interesting finite-time consensus results have been obtained, the bound of the finite settling time heavily depends upon the initial conditions. This limits the practical applications, since the knowledge of initial states of agents in networks is unavailable in advance. Therefore, to overcome this drawback, Parsegov et al. [33] and Zuo et al. [34] have independently given a new concept called fixed-time stability for multi-agent systems, which can guarantee that the consensus in a finite time as well as the settling time is uniformly bounded for any initial conditions. This makes it possible to predetermine an accurate estimation for the settling time even though initial states of agents are unavailable in advance. Following this streamline of dealing with fixed-time consensus, many new results have been obtained, see [19, 35–37].

Recently, networks with antagonistic interactions were studied in [38, 39], and it becomes a focus for studying. As claimed in [40], bipartite consensus, where the sign of the edges of the graph can be negative or positive, can be realized over networks with antagonistic interactions. Thus, the resulting Laplacian graph is called signed Laplacian, which is distinguished from the normal Laplacian graph. The signed Laplacian graph is commonly used in bipartite consensus, cluster consensus, optimization control, affine formation control, and distance-based localization [41–43]. In [44], the results of bipartite consensus have been extended to higher-order multi-agent systems. Furthermore, the finite-time bipartite consensus, where all the agents can achieve the same value in modulus within a finite time, was investigated in [45]. Meng et al. proposed two consensus protocols and then gave some sufficient conditions to guarantee that all the agents can reach an agreement in finite time. Unfortunately, until now, few results of fixed-time consensus have been proposed for multi-agent systems with mixed cooperative and antagonistic interactions. It is challenging to generalize the fixed-time consensus to the networks with antagonistic interactions.

Motivated by the aforementioned discussions, in this chapter, we will consider a general nonlinear finite-time bipartite consensus and a new class of fixed-time bipartite consensus protocols for the multi-agent systems with structurally balanced signed graphs. By using the Lyapunov stability method, all agents can be guaranteed to reach bipartite consensus at some settling time.

## 7.1 Preliminaries

A vector function $g(z) = [g_1(z), g_2(z), \ldots, g_N(z)]^\top$ with $z \in \mathbb{R}^N$ is said to be homogeneous if there exist $\iota \in \mathbb{R}$ and $(l_1, l_2, \ldots, l_N)$ $(l_i > 0, i \in \mathcal{N})$ such that for any $\epsilon > 0$, $i \in \mathcal{N}$, $g_i(\epsilon^{l_1} z_1, \epsilon^{l_2} z_2, \ldots, \epsilon^{l_N} z_N) = \epsilon^{\iota + l_i} g_i(z)$. We also call $g(z)$ homogeneous with dilation $(l_1, l_2, \ldots, l_N)$ of degree $\iota$.

**Definition 7.1 ([46])** For a given $N$-dimensional system,

$$\dot{z}(t) = g(z), \quad z = (z_1, z_2, \ldots, z_N)^\top \in \mathbb{R}^N, \tag{7.1}$$

if $g(z) : D \to \mathbb{R}^N$, where $D \in \mathbb{R}^N$ is the domain of $g$, is homogeneous with dilation $(l_1, l_2, \ldots, l_N)$ of degree $\iota$, then we call system (7.1) an $\iota$ degree homogeneous system.

**Definition 7.2 ([46])** For a given $N$-dimensional system,

$$\dot{z}(t) = g(z) + \tilde{g}(z), \quad \tilde{g}(\mathbf{0}) = \mathbf{0}, \tag{7.2}$$

if $g(z) : D \to \mathbb{R}^N$ is homogeneous with dilation $(l_1, l_2, \ldots, l_N)$ of degree $\iota$ and $\tilde{g}(z)$ is a continuous vector function satisfying the condition

$$\lim_{\epsilon \to 0} \frac{\tilde{g}_i\left(\epsilon^{l_1} z_1, \epsilon^{l_2} z_2, \ldots, \epsilon^{l_N} z_N\right)}{\epsilon^{\iota + l_i}} = 0, \quad \forall z \neq \mathbf{0}, \quad i \in \mathcal{N}, \tag{7.3}$$

then we say that system (7.2) is an $\iota$ degree locally homogeneous system.

**Definition 7.3 ([47])** The equilibrium $z = \mathbf{0}$ of system (7.1) is said to be finite-time stable if the following conditions hold:

1. For any open neighborhood of the origin $\hat{U} \subseteq D$, $z = \mathbf{0}$ is asymptotically stable.
2. For any initial condition $z_0 \in D \backslash \{\mathbf{0}\}$, there exists a settling time $T > 0$ such that every solution $z(t, z_0)$ of system (7.1) satisfies $z(t, z_0) \in D \backslash \{\mathbf{0}\}$ for $t \in [0, T)$

and

$$\lim_{t \to T} z(t, z_0) = \mathbf{0}.$$

Furthermore, for any $t > T$, one has $z(t, z_0) = \mathbf{0}$.

In particular, if $z = \mathbf{0}$ is a finite-time stable point with $\hat{U} = D = \mathbb{R}^N$, then the equilibrium is said to be globally finite-time stable.

**Lemma 7.4 ([46])** *Suppose that system (7.1) is an $\iota$ degree homogeneous system and $z = \mathbf{0}$ is a stable equilibrium. If $\iota < 0$, then the equilibrium of system (7.1) is finite-time stable; furthermore, if Eq. (7.3) holds, then system (7.2) is locally finite-time stable.*

## 7.2 Finite-Time Bipartite Consensus

In this section, we will discuss the finite-time bipartite consensus problem for detail-balanced multi-agent networks with antagonistic interactions. First, consider the multi-agent systems with a structurally balanced signed graph $\mathcal{G}$, and assume that the agents have the following dynamics:

$$\dot{x}_i(t) = u_i(t), \quad i \in \mathcal{N}, \tag{7.4}$$

where $x_i(t) \in \mathbb{R}$ denotes the state of agent $i$, and $u_i(t) \in \mathbb{R}$ is the protocol to be designed.

### *7.2.1 Finite-Time Bipartite Consensus Protocol*

For the detail-balanced multi-agent systems (7.4) with cooperative and competitive coupling, we first design a bipartite consensus protocol as follows:

$$\begin{aligned} u_i(t) = \sum_{k=1}^{N} [& a_{ik} \mathrm{sgn}(x_k(t) - \mathrm{sgn}(a_{ik}) x_i(t)) \\ & \times \varphi(|x_k(t) - \mathrm{sgn}(a_{ik}) x_i(t)|)], \end{aligned} \tag{7.5}$$

where $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a continuous function satisfying the following assumption: $\varphi(x) = 0$ if and only if $x = 0$.

Under protocol (7.5), we now give the definition of finite-time bipartite consensus for system (7.4).

**Definition 7.5** System (7.4) achieves bipartite consensus if $\lim_{t\to+\infty}|x_i(t)| = c$ ($\forall i \in \mathcal{N}$). Moreover, if there exists a settling time $T_0 \in (0, +\infty)$ such that $\lim_{t\to T_0}|x_i(t)| = c$ and $|x_i(t)| = c$ ($\forall t \geq T_0$), then it achieves finite-time bipartite consensus.

Before investigating the finite-time bipartite consensus problems for multi-agent system (7.4) under protocol (7.5), we first introduce the concept of the order of function $\varphi(x)$.

**Definition 7.6** The order of $\varphi(x)$ is said to be $r$ if there exists a positive scalar $r$ satisfying $\varphi(x) = \lim_{x\to 0^+} O(x^r)$, i.e., $\varphi(x) = qx^r + o(x^r)$ around $x = 0$ for some positive constant $q$.

*Remark 7.7* This section aims to use the homogeneous function theory to study the bipartite consensus problems; thus, the function $\varphi$ with order $r$ is adopted to design protocol (7.5). Here, to make our results suitable for a wider range of systems, we adopt a function with order $r$ rather than the standard power function $\varphi(x) = x^r$. In addition, in contrast to the standard power function, the item $o(x^r)$ may increase the convergence speed in some special cases.

Consider the special case of $\varphi(x) = x^r$: If $r = 1$, then protocol (7.5) is described as $u_i(t) = \sum_{k=1}^{N} a_{ik}(x_k - \mathrm{sgn}(a_{ik})x_i)$, which is a consensus protocol to handle the bipartite consensus problem of multi-agent systems [40]. If $0 < r < 1$, then protocol (7.5) can be transformed into the finite-time bipartite consensus control protocol which has been studied in [48]. Furthermore, if $r = 0$, it is a discontinuous consensus protocol called the binary control protocol, and differential inclusion is used to investigate the binary consensus problem with the binary control protocol [18].

**Lemma 7.8** *Suppose that the structurally balanced signed graph $\mathcal{G}$ is strongly connected with a detail-balanced structure; then, under protocol (7.5), the following equation holds:*

$$\psi(t) := \left(1/\sum_{i=1}^{N}\xi_i\right)\sum_{i=1}^{N}\xi_i s_i x_i(t) \equiv \psi(0) \triangleq \psi^*. \tag{7.6}$$

***Proof*** By Lemma 1.8, since $\mathcal{G}$ is structurally balanced, we can conclude that there exists $S \in \mathcal{D}$ such that all the elements of $S\mathcal{A}S$ are nonnegative. Then, we have $s_i s_k a_{ik} = |a_{ik}|$, which implies $s_i s_k = \mathrm{sgn}(a_{ik})$. Combined with the fact that $\mathrm{sgn}(s_i) = s_i$, the following equation can be obtained:

$$\begin{aligned}
&a_{ik}\mathrm{sgn}(x_k - \mathrm{sgn}(a_{ik})x_i)\varphi(|x_k - \mathrm{sgn}(a_{ik})x_i|)\\
=&a_{ik}\mathrm{sgn}\big(s_k^2 x_k - s_i s_k x_i\big)\varphi\big(|s_k^2 x_k - s_i s_k x_i|\big)\\
=&a_{ik}\mathrm{sgn}(s_k)\mathrm{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - s_i x_i|)\\
=&a_{ik}s_k\mathrm{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - s_i x_i|).
\end{aligned} \tag{7.7}$$

Furthermore, we have

$$\begin{aligned}
&\xi_i s_i \sum_{k=1}^{N} a_{ik} s_k \operatorname{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - s_i x_i|) \\
&= \sum_{k=1}^{N} \xi_i s_i a_{ik} s_k \operatorname{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - s_i x_i|) \\
&= \sum_{k=1}^{N} |\xi_i a_{ik}| \operatorname{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - s_i x_i|).
\end{aligned} \tag{7.8}$$

Since $\mathcal{G}$ is detail-balanced with $\xi_i a_{ik} = \xi_k a_{ki}$, we can conclude that

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{k=1}^{N} |\xi_i a_{ik}| \operatorname{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - s_i x_i|) \\
&= \sum_{i=1}^{N}\sum_{k=1}^{N} |\xi_k a_{ki}| \operatorname{sgn}(s_i x_i - s_k x_k)\varphi(|s_i x_i - s_k x_k|) \\
&= -\sum_{i=1}^{N}\sum_{k=1}^{N} |\xi_k a_{ki}| \operatorname{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - s_i x_i|) \\
&= -\sum_{i=1}^{N}\sum_{k=1}^{N} |\xi_i a_{ik}| \operatorname{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - s_i x_i|),
\end{aligned}$$

which leads to $\sum_{i=1}^{N}\sum_{k=1}^{N} |\xi_i a_{ik}| \operatorname{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - s_i x_i|) = 0$. Then, calculating the derivative of $\psi(t)$ along the trajectories of system (7.4) gives

$$\begin{aligned}
\dot{\psi}(t) &= \left(1/\sum_{i=1}^{N}\xi_i\right)\sum_{i=1}^{N}\xi_i s_i \dot{x}(t) = \left(1/\sum_{i=1}^{N}\xi_i\right)\sum_{i=1}^{N}\xi_i s_i u_i(t) \\
&= (1/\sum_{i=1}^{N}\xi_i)\sum_{i=1}^{N}\sum_{k=1}^{N} |\xi_i a_{ik}| \operatorname{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - s_i x_i|) \\
&= 0.
\end{aligned} \tag{7.9}$$

The Proof of Lemma 7.8 is completed.

Here, $\psi^*$ is called the weighted signed-average.

Throughout this section, we always assume that the order of $\varphi$ is $r$.

**Theorem 7.9** *Suppose that the structurally balanced signed graph $\mathcal{G}$ is strongly connected with a detail-balanced structure. Then, under protocol (7.5), if $r \geq 0$, system (7.4) can achieve bipartite consensus. In particular, if $r \in (0, 1)$, system (7.4) can reach finite-time bipartite consensus.*

***Proof*** By Lemma 1.8, there exists a sign matrix $S \in \mathcal{D}$ such that $S\mathcal{A}S \geq 0$ for the structurally balanced graph $\mathcal{G}$. Let $z(t) = Sx(t)$ and set $z_0 = z(0) = Sx(0)$. We can obtain

$$z_i(t) = s_i x_i(t), \ i \in \mathcal{N}. \tag{7.10}$$

Substituting (7.10) into (7.4) with protocol (7.5) results in

$$\begin{aligned}\dot{z}_i(t) &= s_i \dot{x}_i(t) = s_i u_i(t)\\ &= \sum_{k=1}^{N} |a_{ik}| \text{sgn}(s_k x_k - s_i x_i)\varphi(|s_k x_k - k_i x_i|)\\ &= \sum_{k=1}^{N} |a_{ik}| \text{sgn}(z_k - z_i)\varphi(|z_k - z_i|).\end{aligned} \tag{7.11}$$

Let $e_i(t) = z_i(t) - \psi^*$. Lemma 7.8 guarantees that $\dot{e}_i = \dot{z}_i$. Set $e(t) = [e_1(t), e_2(t), \ldots, e_N(t)]^\top$ and $\xi = [\xi_1, \xi_2, \ldots, \xi_N]^\top$. We have that $\xi^\top \cdot e(t) = 0$. Consider the following Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} \xi_i e_i^2(t). \tag{7.12}$$

Obviously, $V(t)$ is a continuous positive definite function. Calculating the derivative of $V(t)$ along the trajectories yields

$$\begin{aligned}\frac{dV}{dt} &= \sum_{i=1}^{N} \xi_i e_i \sum_{k=1}^{N} |a_{ik}| \text{sgn}(e_k - e_i)\varphi(|e_k - e_i|)\\ &= \sum_{i=1}^{N}\sum_{k=1}^{N} e_i |\xi_i a_{ik}| \text{sgn}(e_k - e_i)\varphi(|e_k - e_i|)\\ &= -\sum_{i=1}^{N}\sum_{k=1}^{N} e_k |\xi_i a_{ik}| \text{sgn}(e_k - e_i)\varphi(|e_k - e_i|)\\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N} |\xi_i a_{ik}|(e_k - e_i)\text{sgn}(e_k - e_i)\varphi(|e_k - e_i|)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N}|\xi_i a_{ik}||e_k - e_i|\varphi(|e_k - e_i|) \\
&\leq 0. \qquad (7.13)
\end{aligned}
$$

If $\dot{V} = 0$, following $\xi^\top \cdot e(t) = 0$, one has $e(t) = \mathbf{0}$. Together with LaSalle's invariance principle (Lemma 1.22), one can conclude that $z_i(t)$ converges to $\psi^*$. $z_i(t) = s_i x_i(t)$, so one has $x_i(t) \to s_i\psi^*$ when $t \to +\infty$. The first conclusion of Theorem 7.9 is proved.

Next, we prove the case of $r \in (0, 1)$. According to Definition 7.6, one can conclude that there exists a constant $d$ such that every $x \in (0, d)$ satisfies $\varphi(x) = qx^r + o(x^r)$. Then, we can describe $\dot{e}_i(t)$ in the following form:

$$
\begin{aligned}
\dot{e}_i(t) &= q\sum_{k=1}^{N}|a_{ik}|\text{sgn}(e_k - e_i)|e_k - e_i|^r \\
&\quad + \sum_{k=1}^{N}|a_{ik}|\text{sgn}(e_k - e_i)o(|e_k - e_i|^r) \\
&= g_i + \tilde{g}_i, \quad i \in \mathcal{N}, \qquad (7.14)
\end{aligned}
$$

where $g_i = q\sum_{k=1}^{N}|a_{ik}|\text{sgn}(e_k - e_i)|e_k - e_i|^r$ and $\tilde{g}_i = \sum_{k=1}^{N}|a_{ik}|\text{sgn}(e_k - e_i)o(|e_k - e_i|^r)$. Let $(l_1, l_2, \ldots, l_N) = (1, 1, \ldots, 1)$. Furthermore, we have

$$
\begin{aligned}
&g_i\left(\epsilon^{l_1}x_1, \epsilon^{l_2}x_2, \ldots, \epsilon^{l_N}x_N\right) \\
&= g_i(\epsilon x_1, \epsilon x_2, \ldots, \epsilon x_N) \\
&= q\sum_{k=1}^{N}|a_{ik}|\text{sgn}(\epsilon e_k - \epsilon e_i)|\epsilon e_k - \epsilon e_i|^r \\
&= \epsilon^r q\sum_{k=1}^{N}|a_{ik}|\text{sgn}(e_k - e_i)|e_k - e_i|^r \\
&= \epsilon^{\iota+1}g_i, \qquad (7.15)
\end{aligned}
$$

where $\iota = r - 1$. Similarly, we can obtain that $\tilde{g}_i$ satisfies equation (7.3); thus, Eq. (7.14) is a homogeneous system of degree $\iota$. From Lemma 7.4 and the fact that $\iota < 0$, there exists a settling time $T_2$ such that system (7.14) converges to zero before $T_2$. Since system (7.14) is globally stable from the first conclusion, system (7.14) will converge to $(0, d)$ in time $T_1$. Let $T_0 = T_1 + T_2$, and then, system (7.14) will converge to zero in settling time $T_0$, which means that the finite-time bipartite consensus is obtained. The Proof of Theorem 7.9 is completed.

*Remark 7.10* Similarly to [32], if the signed graph $\mathcal{G}$ is structurally unbalanced, we can prove that the states of all agents of multi-agent system (7.4) reach zero in finite time by assuming the Lyapunov function as $V(t) = \frac{1}{2}\sum_{i=1}^{N} \xi_i x_i^2(t)$. For the detailed proof, one can refer to Theorem 7.9, and it is omitted here.

**Corollary 7.11** *Suppose that the structurally balanced signed graph $\mathcal{G}$ is connected with an undirected structure. Then, under protocol (7.5), if $r \geq 0$, system (7.4) can achieve bipartite consensus. In particular, if $r \in (0, 1)$, then system (7.4) can reach finite-time bipartite consensus.*

*Remark 7.12* Since an undirected graph $\mathcal{G}$ is always detail-balanced with $\xi_i \equiv 1$ for any $i \in \mathcal{N}$, the Proof of Corollary 7.11 is omitted here. When the signed graph $\mathcal{G}$ is structurally unbalanced and other conditions remain unchanged, we can also prove that the states of all agents for system (7.4) converge to zero in finite time.

### 7.2.2 Pinning Bipartite Consensus Protocol

Theorem 7.9 demonstrates that system (7.4) can reach finite-time bipartite consensus under protocol (7.5), and the agreement value is obtained as the weighted signed-average of the initial value. However, in many real-world applications, the final agreement is expected to be certain ideal value. To deal with this problem, pinning control is a helpful tool. In this part, we will investigate the finite-time bipartite consensus problem of multi-agent systems with a detail-balanced antagonistic interaction topology via a pinning controller. Consider the following multi-agent systems with pinning controller:

$$\dot{x_i} = \sum_{k=1}^{N} a_{ik}\text{sgn}(x_k - \text{sgn}(a_{ik})x_i)\varphi(|x_k - \text{sgn}(a_{ik})x_i|) - \eta_i, \tag{7.16}$$

where

$$\eta_i = \begin{cases} \text{sgn}(x_1 - c)\varphi(|x_1 - c|), & i = 1, \\ 0, & i = 2, 3, \ldots, N \end{cases}$$

is the pinning control protocol, and $c$ is the objective state, which all agents need to be forced to. Since the signed graph $\mathcal{G}$ is structurally balanced, let $\mathcal{V}_1 = \{1, 2, \ldots, j\}$ and $\mathcal{V}_2 = \{j+1, j+2, \ldots, N\}$. Additionally, we can choose $S \in \mathcal{D}$ with $s_i = 1, v_i \in \mathcal{V}_1$ and $s_i = -1, v_i \in \mathcal{V}_2$. Since $\mathcal{G}$ is strongly connected, we only need to control one agent, which is assumed to be $v_1$. The definition of finite-time pinning bipartite consensus is as follows.

**Definition 7.13** For system (7.4) under protocol (7.16), if $T = +\infty$ satisfies

$$\begin{cases} \lim_{t \to T} [x_i(t) - c] = 0, & i \in \mathcal{V}_1 \\ \lim_{t \to T} [x_i(t) + c] = 0, & i \in \mathcal{V}_2, \end{cases} \tag{7.17}$$

then system (7.4) reaches pinning bipartite consensus. Furthermore, if there exists a settling time $T \in (0, +\infty)$ such that $x_i(t) = c$ $(i \in \mathcal{V}_1)$ and $x_i(t) = -c$ $(i \in \mathcal{V}_2)$ hold for any $t \geq T$, then system (7.4) achieves finite-time pinning bipartite consensus.

**Theorem 7.14** *Suppose that the structurally balanced signed graph $\mathcal{G}$ is strongly connected with a detail-balanced structure; then, under protocol (7.16), if $r \geq 0$, system (7.4) can achieve bipartite consensus. In particular, if $r \in (0, 1)$, then system (7.4) can reach finite-time bipartite consensus.*

***Proof*** Let $z(t) = Sx(t)$ (respectively, $z_i(t) = s_i x_i(t)$). Calculating the derivative of $z(t)$ along the trajectories gives

$$\dot{z}_i(t) = \sum_{k=1}^{N} |a_{ik}| \text{sgn}(z_k - z_i) \varphi(|z_k - z_i|) - \eta_i. \tag{7.18}$$

Let $e_i(t) = z_i(t) - c$; then, we have

$$\begin{aligned} \dot{e}_i(t) &= \sum_{k=1}^{N} |a_{ik}| \text{sgn}(z_k - z_i) \varphi(|z_k - z_i|) - \eta_i \\ &= \sum_{k=1}^{N} |a_{ik}| \text{sgn}(e_k - e_i) \varphi(|e_k - e_i|) - \eta_i. \end{aligned} \tag{7.19}$$

The Lyapunov function $V(t) = \frac{1}{2} \sum_{i=1}^{N} \xi_i e_i^2(t)$ is considered here. Calculating its derivative yields

$$\begin{aligned} \frac{dV}{dt} &= \sum_{i=1}^{N} \xi_i e_i \left[ \sum_{k=1}^{N} |a_{ik}| \text{sgn}(e_k - e_i) \varphi(|e_k - e_i|) - \eta_i \right] \\ &= -\frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N} |\xi_i a_{ik}| |e_k - e_i| \varphi(|e_k - e_i|) - \sum_{i=1}^{N} \xi_i e_i \eta_i \\ &= -\frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N} |\xi_i a_{ik}| |e_k - e_i| \varphi(|e_k - e_i|) - \xi_1 |e_1| \varphi(|e_1|) \\ &\leq 0. \end{aligned} \tag{7.20}$$

Note that $\dot{V} = 0$ if and only if $e_k = e_i$ and $e_1 = 0$. Hence, we can obtain $e_i = 0$ for any $i \in \mathcal{N}$. From LaSalle's invariance principle (Lemma 1.22), $z_i(t)$ converges to $c$. The first conclusion of Theorem 7.14 is completed.

Now, we prove the case of $r \in (0, 1)$. Based on Definition 7.6, we know that there exists a positive constant $d$ such that for any $x \in (0, d)$, $\varphi(x) = qx^r + o(x^r)$ is satisfied. Then, $\dot{e}_i(t)$ can be reduced to the following form:

$$\begin{aligned}
\dot{e}_1(t) &= \sum_{k=1}^{N} a_{1k}\text{sgn}(e_k - e_1)\varphi(|e_k - e_1|) - \eta_1 \\
&= q\sum_{k=1}^{N} a_{1k}\text{sgn}(e_k - e_1)|e_k - e_1|^r - q\text{sgn}(e_1)|e_1|^r \\
&\quad + \text{sgn}(e_1)o(e_1)^r - \sum_{k=1}^{N} a_{1k}\text{sgn}(e_k - e_1)o(|e_k - e_1|^r) \\
&= g_1 + \tilde{g}_1,
\end{aligned} \tag{7.21}$$

where $g_1 = q\sum_{j=1}^{N} a_{1j}\text{sgn}(e_j - e_1)|e_j - e_1|^\beta - q\text{sgn}(e_1)|e_1|^\beta$ and $\tilde{g}_1 = \text{sgn}(e_1)o(e_1)^\beta - \sum_{j=1}^{N} a_{1j}\text{sgn}(e_j - e_1)o(|e_j - e_1|^\beta)$. Similarly, for $i = 2, 3, \ldots, N$,

$$\begin{aligned}
\dot{e}_i(t) &= \sum_{k=1}^{N} a_{ik}\text{sgn}(e_k - e_i)\varphi(|e_k - e_i|) - \eta_i \\
&= q\sum_{k=1}^{N} a_{ik}\text{sgn}(e_k - e_i)|e_k - e_i|^r \\
&\quad + \sum_{k=1}^{N} a_{ik}\text{sgn}(e_k - e_i)o(|e_k - e_i|^r) \\
&= g_i + \tilde{g}_i,
\end{aligned} \tag{7.22}$$

where $g_i = q\sum_{j=1}^{N} a_{ij}\text{sgn}(e_j - e_i)|e_j - e_i|^\beta$ and $\tilde{g}_i = \sum_{j=1}^{N} a_{ij}\text{sgn}(e_j - e_i)o(|e_j - e_i|^\beta)$. Similarly to the Proof of Theorem 7.9, system (7.22) is a homogeneous system with degree $\iota$. Thus, $z_i(t)$ converges to $c$ in finite time $T_0$. This proof is completed.

**Corollary 7.15** *Suppose that the structurally balanced signed graph $\mathcal{G}$ is connected with an undirected structure. Then, under protocol (7.16), if $r \geq 0$, then system (7.4) can achieve bipartite consensus. In particular, if $r \in (0, 1)$, then system (7.4) can reach finite-time bipartite consensus.*

In the following, we will consider more general network topology.

**Theorem 7.16** *Suppose that the structurally balanced signed graph $\mathcal{G}$ has a rooted spanning tree and each strongly connected component has a detail-balanced structure. Then, under protocol (7.5), if $r \geq 0$, system (7.4) can achieve bipartite consensus. In particular, if $r \in (0, 1)$, then system (7.4) can reach finite-time bipartite consensus.*

***Proof*** We only consider the case of $r \in (0, 1)$. The proof of bipartite consensus for $r > 0$ can be similarly derived.

Without loss of generality, based on Lemma 1.12, we assume that

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & 0 & \cdots & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{p1} & \mathcal{L}_{p2} & \cdots & \mathcal{L}_{pp} \end{bmatrix},$$

where $\mathcal{L}_{ii} \in \mathbb{R}^{m_i \times m_i}$, $i = 1, 2, \ldots, p$, are irreducible matrices, and for any $1 < k \leq p$, there exists at least one $q < k$ such that $\mathcal{L}_{kq}$ is nonzero. Then, we can conclude that every subgraph $\mathcal{G}_i$ corresponding to matrix $\mathcal{L}_{ii}$ is strongly connected. For arbitrary subgraph $\mathcal{G}_i$, the vertex $v_i \in \mathcal{G}_i$ is only influenced by the vertexes in $\mathcal{G}_j,\ j \leq i$.

For $\mathcal{G}_1$, we have

$$\dot{e}_i = \sum_{k \in \mathcal{G}_1} |a_{ik}| \mathrm{sgn}(e_k - e_i) \varphi(|e_k - e_i|) - \eta_i,\ i \leq m_1,$$

where $e_i$ was defined in Theorem 7.14. Since $\mathcal{G}_1$ is strongly connected, it follows from Theorem 7.14 that the vertexes in $\mathcal{G}_1$ will achieve finite-time bipartite consensus. Hence, there exists a constant $T_1$ such that $\lim_{t \to T_1} e_i = 0,\ i \in \mathcal{G}_1$.

For $i \in \mathcal{G}_2$, one has

$$\begin{aligned} \dot{e}_i = & \sum_{k \in \mathcal{G}_1} |a_{ik}| \mathrm{sgn}(e_k - e_i) \varphi(|e_k - e_i|) \\ & + \sum_{j \in \mathcal{G}_2} |a_{ij}| \mathrm{sgn}(e_j - e_i) \varphi(|e_j - e_i|). \end{aligned}$$

When $t \geq T_1$, $e_k(t) = 0,\ k \in \mathcal{G}_1$. Then, for $i \in \mathcal{G}_2$,

$$\begin{aligned} \dot{e}_i = & \sum_{j \in \mathcal{G}_2} |a_{ij}| \mathrm{sgn}(e_j - e_i) \varphi(|e_j - e_i|) \\ & - \mathrm{sgn}(e_i) \varphi(|e_i|) \sum_{k \in \mathcal{G}_1} |a_{ik}|,\ t \geq T_1. \end{aligned}$$

It follows from Lemma 1.11 that $\mathcal{L}_{21}$ is nonzero. Hence, there exists at least agent $i \in \mathcal{G}_2$ such that $\sum_{k\in\mathcal{G}_2} |a_{ik}| \neq 0$. Based on Theorem 7.9 and the fact that $\mathcal{G}_2$ is strongly connected, we can conclude that the vertexes in $\mathcal{G}_2$ will achieve finite-time bipartite consensus, where the item $\operatorname{sgn}(e_i)\varphi(|e_i|) \sum_{j\in\mathcal{G}_1} |a_{ij}|$ plays the same role as $\eta_i$ in (7.19). Hence, there exists $T_2 \geq T_1$ such that $\lim_{t\to T_2} e_i = 0,\ i \in \mathcal{G}_2$.

Hence, the finite-time bipartite consensus of subgraph $\mathcal{G}_1$ propagates to the second group. By mathematical induction, we can conclude that all agents in the system can reach finite-time bipartite consensus.

*Remark 7.17* Note that Theorem 7.16 is a general extension of Theorems 7.9 and 7.14. In the Proof of Theorem 7.16, we can observe the evolution of multi-agent systems: the strongly connected component that includes the root vertex will achieve bipartite consensus first, which then leads to the other connected components converging to the agreement state.

### *7.2.3 Numerical Examples*

In this subsection, we will give two numerical examples to illustrate our main results. Consider two structurally balanced signed graphs as shown in Fig. 7.1.

Here, the structurally balanced signed graph of Fig. 7.1a corresponds to the following adjacency matrix:

$$\mathcal{A}_1 = \begin{bmatrix} 0 & -2 & 3 \\ -4 & 0 & -3 \\ 2 & -1 & 0 \end{bmatrix},$$

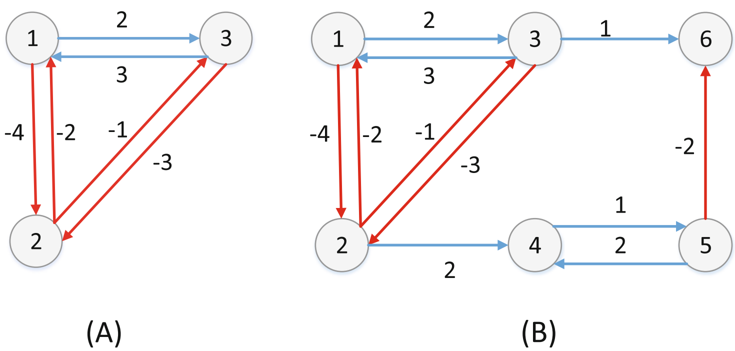


**Fig. 7.1** Structurally balanced graph: (**a**) strongly connected and (**b**) contains a rooted spanning tree

while the structurally balanced signed graph of Fig. 7.1b corresponds to the following adjacency matrix:

$$\mathcal{A}_2 = \begin{bmatrix} 0 & -2 & 3 & 0 & 0 & 0 \\ -4 & 0 & -3 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{bmatrix}.$$

Consider the adjacency matrix $\mathcal{A}_1$, and let $\xi = (\xi_1, \xi_2, \xi_3)^\top = (2, 1, 3)^\top$. We can verify that $\xi_i a_{ik} = \xi_k a_{ki} (i, k = 1, 2, 3)$; hence, $\mathcal{A}_1$ is strongly connected and detail-balanced. Similarly, for matrix $\mathcal{A}_2$ with the three strongly connected components $\{v_1, v_2, v_3\}$, $\{v_4, v_5\}$, and $\{v_6\}$, we can take $\xi_a = (2, 1, 3)^\top$, $\xi_b = (1, 2)^\top$, and $\xi_c = (1)$ to illustrate that the three connected components are detail-balanced.

*Example 7.18* In this example, we illustrate the effectiveness of Theorem 7.9 with graph, Fig. 7.1a. Consider the initial states $x(0) = (8, 4, -2)^\top$ together with Eq. (7.6), leading to $\psi^* = (1/\sum_{i=1}^{3} \xi_i) \sum_{i=1}^{3} \xi_i s_i x_i(0) = 1$. The function $\varphi(x)$ is selected as $x^{\frac{1}{3}}$, $x^{\frac{1}{3}} + x$, and $x^{\frac{1}{3}} + |\sin(6x)|$. When $x \to 0^+$, $\varphi(x) = O(x^{\frac{1}{3}})$, i.e., the order of $\varphi(x)$ is $\frac{1}{3}$. The numerical results are listed in Figs. 7.2 and 7.3, which

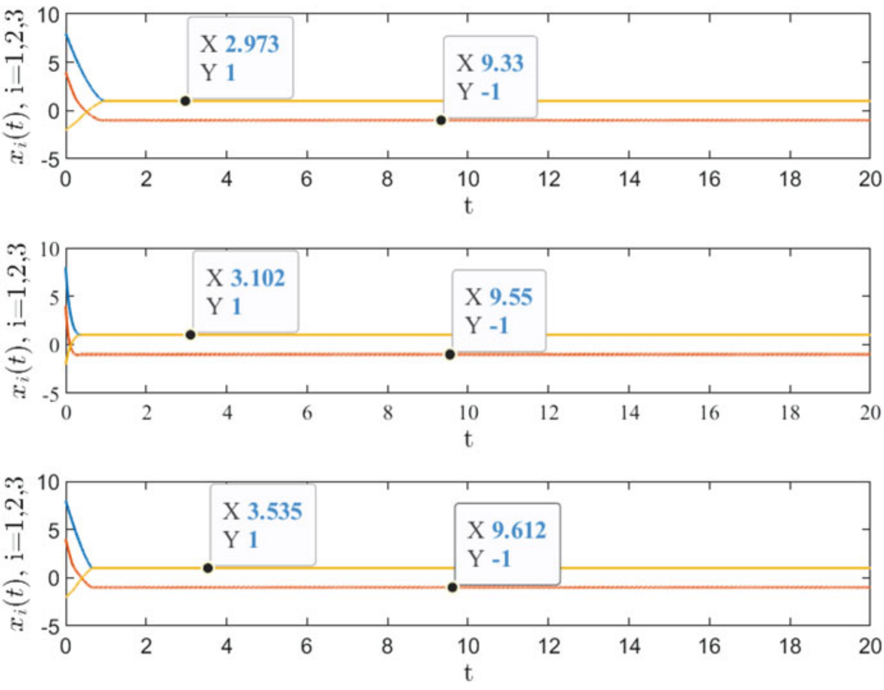


**Fig. 7.2** The agent state responses with protocol (7.5): from top to bottom, function $\varphi(x)$ corresponding to $x^{\frac{1}{3}}$, $x^{\frac{1}{3}} + x$, and $x^{\frac{1}{3}} + |\sin(6x)|$

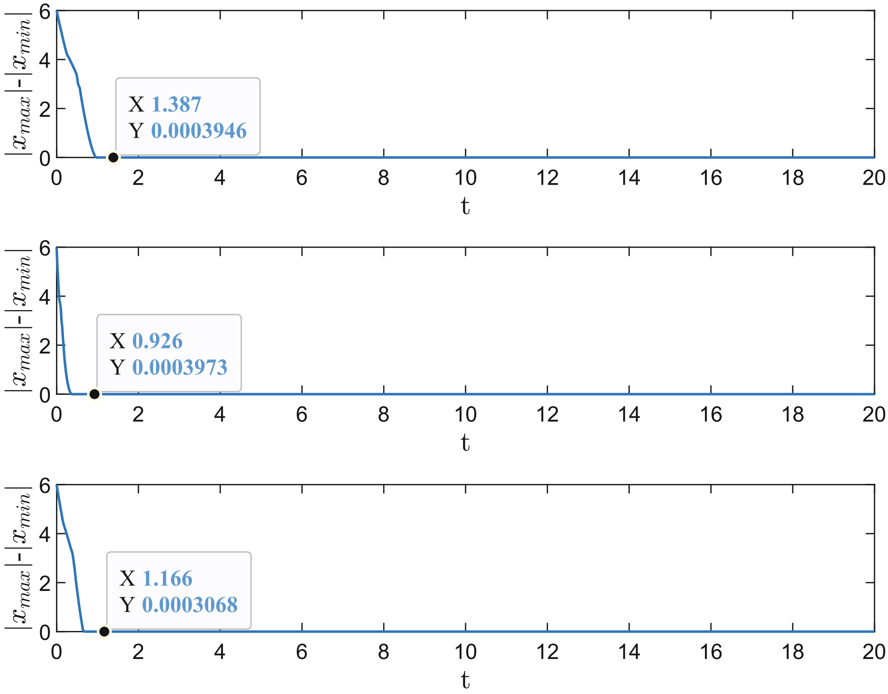


**Fig. 7.3** The responses of $|x_{\max}| - |x_{\min}|$ with protocol (7.5) corresponding to $x^{\frac{1}{3}}$, $x^{\frac{1}{3}} + x$, and $x^{\frac{1}{3}} + |\sin(6x)|$

verify Theorem 7.9 very well. For the simulation of Theorem 7.14, the pinning value $c$ is chosen as $c = 2$, and the other parameters are the same as those in the simulation of Theorem 7.9. The effectiveness is illustrated in Fig. 7.4.

*Example 7.19* We consider the case of 6 vertexes. The topological structure can be seen in Fig. 7.1b, which contains a rooted spanning tree and is detail-balanced. The function $\varphi(x)$ is chosen to be the same as that in Example 7.18. Consider the initial value $x(0) = (16, 8, -4, 6, -12, 2)^{\top}$. From Theorem 7.16, we can obtain $x^* = (1/(2+1+3)) \times (16 \times 2 - 8 \times 1 + (-4) \times 2) = 2$, which is consistent with the results of the numerical simulations shown in Figs. 7.5 and 7.6. Figure 7.7 shows the results when the initial values are randomly uniformly selected from [0, 100]. It also illustrates the validity of Theorem 7.16.

## 7.3 Fixed-Time Bipartite Consensus

The bipartite consensus problem involves finding a dynamic protocol such that a group of agents in a signed network can reach agreement, regarding consensus values that are the same in modulus but different in sign. This section investigates

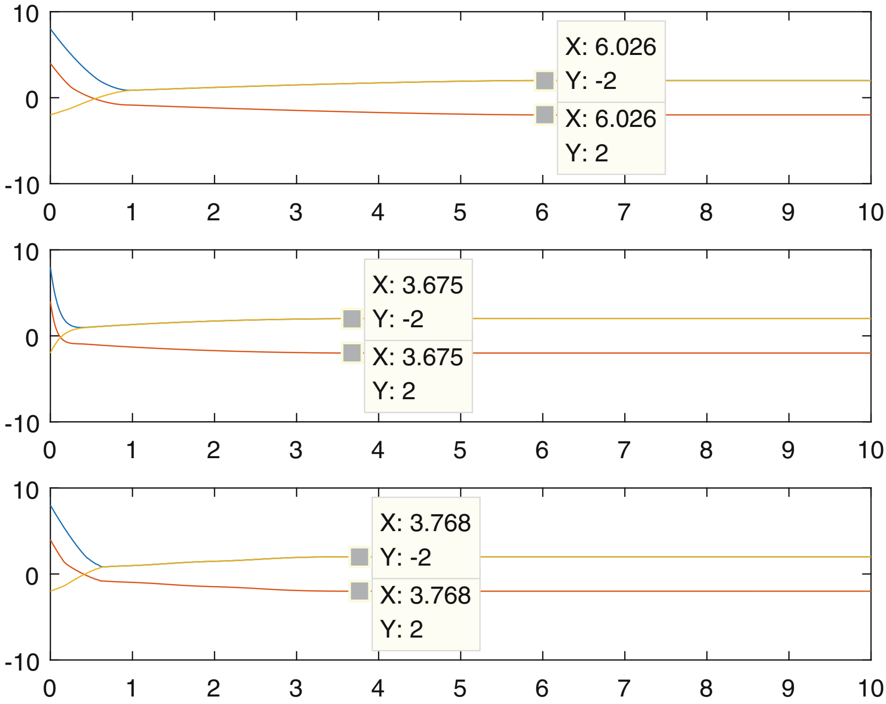


**Fig. 7.4** The agent state responses with protocol (7.16): from top to bottom, function $\varphi(x)$ corresponding to $x^{\frac{1}{3}}$, $x^{\frac{1}{3}} + x$, and $x^{\frac{1}{3}} + |\sin(6x)|$

fixed-time bipartite consensus protocols based on undirected information flow for structurally balanced signed graph.

Consider multi-agent systems with $N$ agents under a structurally balanced signed graph $G(A)$, and assume that the agents have the following dynamics:

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2 \ldots, N. \tag{7.23}$$

where $x_i(t) \in \mathbb{R}$ denotes the state of agent $i$, and $u_i(t) \in \mathbb{R}$ is the protocol to be designed. With the given protocol $u_i(t)$, we now give the definition of fixed-time bipartite consensus of the system in (7.23).

**Definition 7.20** It is said to achieve fixed-time bipartite consensus if, for $\forall x_i(0)$, $i \in \{1, 2, \ldots, N\}$, there exists a settling time $T \in (0, \infty)$, which is independent with initial condition such that

$$\begin{cases} \lim\limits_{t \to T} |x_i(t)| = c, \\ |x_i(t)| = c, \forall t \geq T, \end{cases} \tag{7.24}$$

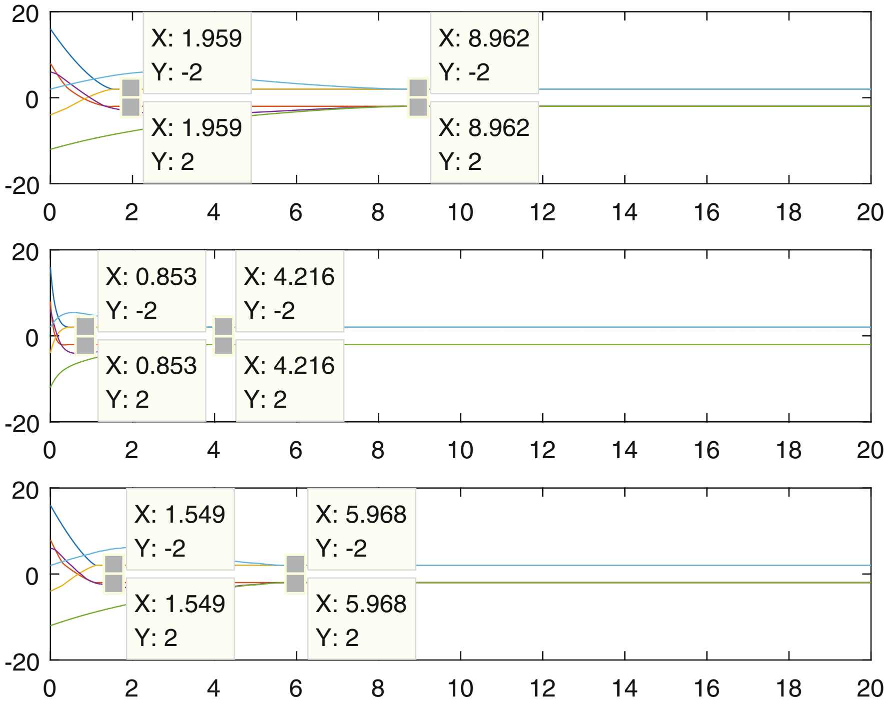


**Fig. 7.5** The agent state responses with signed graph ($B$): from top to bottom, function $\varphi(x)$ corresponding to $x^{\frac{1}{3}}$, $x^{\frac{1}{3}} + x$, and $x^{\frac{1}{3}} + |\sin(6x)|$

where $c \geq 0$ is the absolute value of the states about which all agents reach consensus in a fixed time. If $c = \frac{1}{N}\sum_{i=1}^{N}\sigma_i x_i(0)$, we say that it reaches signed-average consensus.

The problem considered in this section is formulated as follows.

**Problem 7.21** Given system (7.23) and a signed graph $G(A)$, we aim to design a distributed feedback control law $u_i$, such that the states of system (7.23) starting from any initial conditions reach fixed-time bipartite consensus, i.e., Eq. (7.24) is satisfied.

To solve Problem 7.21, some lemmas are given as follows.

**Lemma 7.22 ([34])** *Let* $\xi_1, \xi_2, \ldots, \xi_N \geq 0$ *and* $p \in (0, 1]$. *Then,*

$$\sum_{i=1}^{N}\xi_i^p \geq \left(\sum_{i=1}^{N}\xi_i\right)^p. \tag{7.25}$$

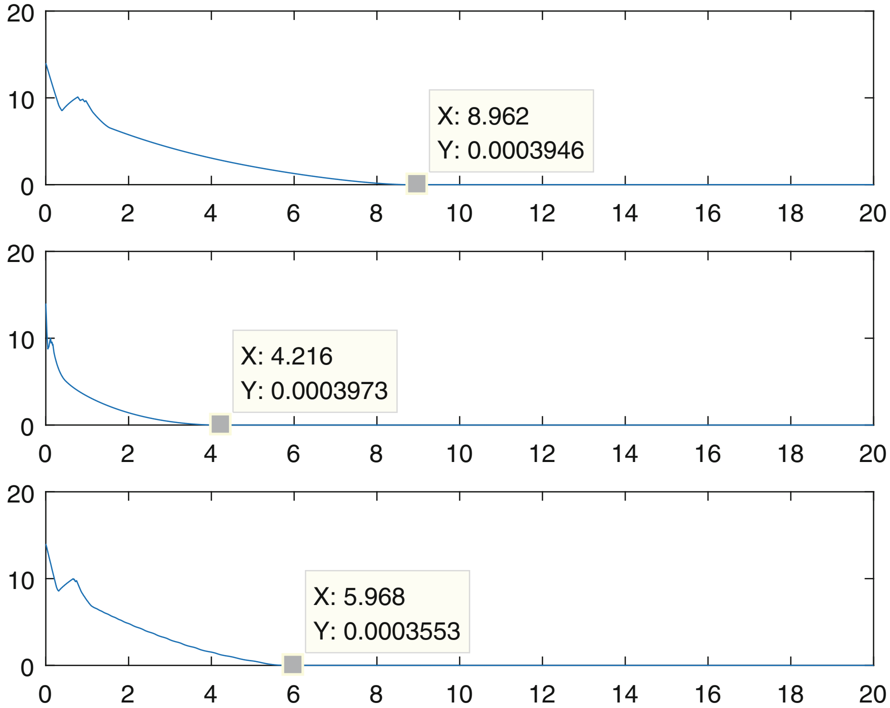


**Fig. 7.6** The responses of $|x_{\max}| - |x_{\min}|$ with signed graph $(B)$ corresponding to $x^{\frac{1}{3}}$, $x^{\frac{1}{3}} + x$, and $x^{\frac{1}{3}} + |\sin(6x)|$

**Lemma 7.23 ([34])** *Let* $\xi_1, \xi_2, \ldots, \xi_N \geq 0$ *and* $p \in (1, +\infty)$. *Then,*

$$\sum_{i=1}^{N} \xi_i^p \geq N^{1-p} \left( \sum_{i=1}^{N} \xi_i \right)^p . \tag{7.26}$$

In this section, we will develop a new class of fixed-time bipartite consensus protocols for a group of multi-agent systems with signed graph in Eq. (7.23). Before that and to make the idea clear, we first give a retrofit lemma as follows.

**Lemma 7.24 ([34])** *Consider a scalar system*

$$\dot{y} = -\alpha y^{2-\frac{p}{q}} - \beta y^{\frac{p}{q}}, \quad y(0) = y_0, \tag{7.27}$$

*where* $\alpha, \beta > 0$, *and* $p$ *and* $q$ *both are positive odd integers satisfying* $p < q$. *Then, the equilibrium of Eq. (7.27) is fixed-time stable, and the settling time is bounded by*

$$T \leq \frac{q\pi}{2\sqrt{\alpha\beta}(q-p)} . \tag{7.28}$$

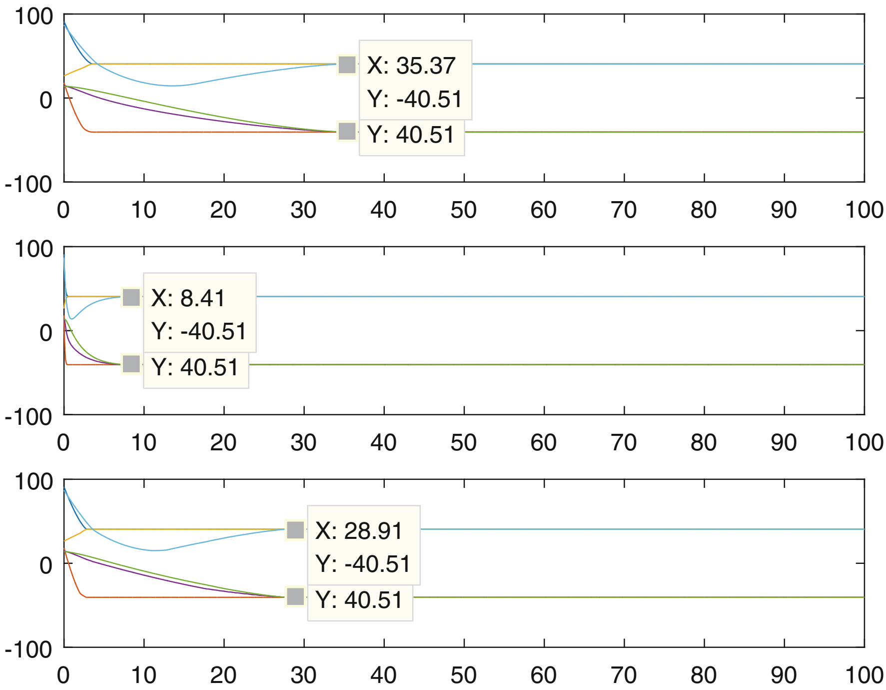


**Fig. 7.7** The agent state trajectories corresponding to $x^{\frac{1}{3}}$, $x^{\frac{1}{3}}+x$, and $x^{\frac{1}{3}}+|\sin(6x)|$. The initial values are randomly uniformly chosen from [0,100]

### *7.3.1 General Fixed-Time Bipartite Consensus*

With the above preparations for Problem (7.21), a protocol is firstly proposed as

$$u_i(t) = \alpha\left(\sum_{j=1}^{N} a_{ij}(x_j(t) - \mathrm{sign}(a_{ij})x_i(t))\right)^{2-\frac{p}{q}} + \beta\left(\sum_{j=1}^{N} a_{ij}(x_j(t) - \mathrm{sign}(a_{ij})x_i(t))\right)^{\frac{p}{q}}, \quad i = 1, 2, \ldots, N. \tag{7.29}$$

Let $x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^\top \in \mathbb{R}^N$. Obviously, the substitution of protocol (7.29) to system (7.23) leads to $\dot{x} = -Lx$ when $\alpha + \beta = 1$, $p = q$. The system $\dot{x} = -Lx$ was discussed in [40] for consensus of networks with antagonistic interactions, so it is the special case of our model. Now, we have the following result.

**Theorem 7.25** *Consider system (7.23) with structurally balanced signed graph $G(A)$, which is undirected and connected; then, Problem (7.21) can be solved by protocol (7.29) with $c > 0$, and the settling time $T$ is bounded by*

$$T \le \frac{q\pi N^{\frac{q-p}{4q}}}{2\sqrt{\alpha\beta}\lambda_2(L)(q-p)} . \tag{7.30}$$

***Proof*** Following Lemma 1.8, since $G(A)$ is structurally balanced, we can conclude that $\exists D \in \mathcal{D}$ such that $DAD$ has all nonnegative entries. Let $z(t) = Dx(t)$ and denote $z_0 = z(0) = Dx(0)$; then, we obtain that

$$z_i(t) = \sigma_i x_i(t) \ , \ i = 1, 2, \ldots, N. \tag{7.31}$$

Substituting (7.31) into (7.23) with protocol (7.29) results in

$$\dot{z}_i(t) = \sigma_i \dot{x}_i(t) = \sigma_i u_i(t). \tag{7.32}$$

Since $DAD \ge 0$, we have $\sigma_i\sigma_j a_{ij} = |a_{ij}|$, which implies $\sigma_i\sigma_j = \mathrm{sign}(a_{ij})$. Since $\sigma_i^{2-p/q} = \sigma_i^{p/q} = \sigma_i, \ \sigma_i^2 = 1$. Using these facts, one can obtain the following equation:

$$\begin{aligned}
&\dot{z}_i(t) \\
&= \sigma_i\alpha\left(\sum_{j=1}^N a_{ij}(x_j(t) - \mathrm{sign}(a_{ij})x_i(t))\right)^{2-\frac{p}{q}} \\
&\quad + \sigma_i\beta\left(\sum_{j=1}^N a_{ij}(x_j(t) - \mathrm{sign}(a_{ij})x_i(t))\right)^{\frac{p}{q}} \\
&= \alpha\left(\sum_{j=1}^N \sigma_i a_{ij}(\sigma_j^2 x_j(t) - \sigma_i\sigma_j x_i(t))\right)^{2-\frac{p}{q}} \\
&\quad + \beta\left(\sum_{j=1}^N \sigma_i a_{ij}\left(\sigma_j^2 x_j(t) - \sigma_i\sigma_j x_i(t)\right)\right)^{\frac{p}{q}} \\
&= \alpha\left(\sum_{j=1}^N \sigma_i\sigma_j a_{ij}(\sigma_j x_j(t) - \sigma_i x_i(t))\right)^{2-\frac{p}{q}}
\end{aligned} \tag{7.33}$$

$$
\begin{aligned}
&+\beta\left(\sum_{j=1}^{N}\sigma_i\sigma_j a_{ij}(\sigma_j x_j(t)-\sigma_i x_i(t))\right)^{\frac{p}{q}}\\
&=\alpha\left(\sum_{j=1}^{N}|a_{ij}|(z_j(t)-z_i(t))\right)^{2-\frac{p}{q}}+\beta\left(\sum_{j=1}^{N}|a_{ij}|(z_j(t)-z_i(t))\right)^{\frac{p}{q}}. \qquad (7.34)
\end{aligned}
$$

Consider the following Lyapunov function for (7.31):

$$
V(t)=\frac{1}{2}z(t)^{\top}\hat{L}z(t)=\frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|(z_j(t)-z_i(t))^2,
$$

where $\hat{L} = DLD$. Then, calculating the time derivative of $V(t)$ along the trajectories of system (7.31) gives that

$$
\begin{aligned}
\frac{dV}{dt}&=-\alpha\sum_{i=1}^{N}\left(\sum_{j=1}^{N}|a_{ij}|(z_j(t)-z_i(t))\right)^{\frac{3q-p}{q}}\\
&\quad-\beta\sum_{i=1}^{N}\left(\sum_{j=1}^{N}|a_{ij}|(z_j(t)-z_i(t))\right)^{\frac{p+q}{q}}\\
&=-\alpha\sum_{i=1}^{N}\left(\left(\sum_{j=1}^{N}|a_{ij}|(z_j(t)-z_i(t))\right)^2\right)^{\frac{3q-p}{2q}}\\
&\quad-\beta\sum_{i=1}^{N}\left(\left(\sum_{j=1}^{N}|a_{ij}|(z_j(t)-z_i(t))\right)^2\right)^{\frac{p+q}{2q}}\\
&\le-\alpha N^{\frac{p-q}{2q}}\left(\sum_{i=1}^{N}\left(\sum_{j=1}^{N}|a_{ij}|(z_j(t)-z_i(t))\right)^2\right)^{\frac{3q-p}{2q}}\\
&\quad-\beta\left(\sum_{i=1}^{N}\left(\sum_{j=1}^{N}|a_{ij}|(z_j(t)-z_i(t))\right)^2\right)^{\frac{p+q}{2q}}, \qquad (7.35)
\end{aligned}
$$

where Lemmas 7.22 and 7.23 are used in view of $(3q - p)/2q > 1$ and $0 < (p+q)/2q \leq 1$. The semi-positive definite property of $\hat{L}$ ensures that there exists a unique semi-positive definite matrix M such that $\hat{L} = M^\top M$ [49]. By Lemma 1.12, we have

$$\frac{\sum_{i=1}^N \left(\sum_{j=1}^N |a_{ij}|(z_j(t) - z_i(t))\right)^2}{V(t)} = \frac{2z(t)^\top M^\top M M^\top M z(t)}{z(t)^\top M^\top M z(t)}$$
$$= \frac{2z(t)^\top M^\top \hat{L}^\top M z(t)}{z(t)^\top M^\top M z(t)} \geq 2\lambda_2(\hat{L}) = 2\lambda_2(L). \tag{7.36}$$

Substituting (7.36) into (7.35), we have

$$\frac{dV}{dt} \leq -\alpha N^{\frac{p-q}{2q}} (2\lambda_2(L)V)^{\frac{3q-p}{2q}} - \beta(2\lambda_2(L)V)^{\frac{p+q}{2q}}$$
$$= -\left[\alpha N^{\frac{p-q}{2q}} (2\lambda_2(L)V)^{\frac{q-p}{q}} + \beta\right] (2\lambda_2(L)V)^{\frac{p+q}{2q}}. \tag{7.37}$$

If $V \neq 0$, then assume that $y(t) = \sqrt{2\lambda_2(L)V}$ is the solution to the following differential equation:

$$\dot{y}(t) \leq -\alpha N^{\frac{p-q}{2q}} \lambda_2(L) y(t)^{\frac{2q-p}{q}} - \beta\lambda_2(L) y(t)^{\frac{p}{q}}. \tag{7.38}$$

By Lemma 7.24 and comparison principle of differential equations [50], one has

$$\lim_{t\to T} V(t) = 0, \text{ and } V(t) = 0, \ \forall\, t \geq T,$$

where the settling time is given by

$$T = \frac{qN^{\frac{q-p}{4q}}}{\sqrt{\alpha\beta}\lambda_2(L)(q-p)} \tan^{-1}\left(N^{\frac{p-q}{4q}} \sqrt{\alpha/\beta}V(z_0)\right)$$
$$\leq \frac{q\pi N^{\frac{q-p}{4q}}}{2\sqrt{\alpha\beta}\lambda_2(L)(q-p)}, \tag{7.39}$$

which leads to

$$\lim_{t\to T} |z_j(t) - z_i(t)| = 0, \text{ and } z_j(t) = z_i(t), \ \forall t \geq T, \ \forall\, i, j.$$

Together with the fact $z_i(t) = \sigma_i x_i(t)$, Theorem 7.25 is proved.

*Remark 7.26* From Theorem 7.25, for multi-agent systems with structurally balanced signed graphs, one can see that protocol (7.29) not only solves the general bipartite consensus problem in a fixed time but also enables the settling time to be independent of the initial condition. Furthermore, from Eq. (7.30), the settling time depends on the design parameters and the algebraic connectivity of the signed graph. This overcomes the drawback of finite-time consensus that $T$ depends on the initial states of agents.

*Remark 7.27* Since multi-agent systems with nonnegative adjacency matrix can be viewed as a special case of structurally balanced signed graph, Theorem 7.25 can be considered as a significant extension of the fixed-time consensus results in, e.g., [34] in a more general way.

*Remark 7.28* If $G(A)$ is structurally unbalanced, we can prove that the states of all agents go to zero by constructing Lyapunov function $V(t) = \frac{1}{2}x(t)^\top Lx(t)$. The detailed proof can be referred to Theorem 7.25, and it is omitted here.

### 7.3.2 Signed-Average Fixed-Time Bipartite Consensus

To study the signed-average consensus, we next propose the following protocol:

$$\begin{aligned} u_i(t) &= \alpha \sum_{j=1}^{N} a_{ij}(x_j(t) - \operatorname{sign}(a_{ij})x_i(t))^{2-\frac{p}{q}} \\ &\quad + \beta \sum_{j=1}^{N} a_{ij}(x_j(t) - \operatorname{sign}(a_{ij})x_i(t))^{\frac{p}{q}}, \ i = 1, 2, \ldots, N. \end{aligned} \tag{7.40}$$

Firstly, we have the following lemma for the application of protocol (7.40). This lemma is similar with the lemma proposed in [43].

**Lemma 7.29** *Let $L_B = \left[l_{B,ij}\right] \in \mathbb{R}^{N\times N}$ and $L_C = \left[l_{C,ij}\right] \in \mathbb{R}^{N\times N}$ with elements given by*

$$l_{B,ij} = \begin{cases} \sum_{k=1}^{N} |a_{ik}|^{\frac{2q}{3q-p}}, & j = i, \\ -\, sign(a_{ij})|a_{ij}|^{\frac{2q}{3q-p}}, & j \neq i. \end{cases}$$

$$l_{C,ij} = \begin{cases} \sum_{k=1}^{N} |a_{ik}|^{\frac{2q}{p+q}}, & j = i, \\ -\, sign(a_{ij})|a_{ij}|^{\frac{2q}{p+q}}, & j \neq i. \end{cases}$$

*If $G(A)$ is structurally balanced, then $L_B$ and $L_C$ are semi-positive definite with eigenvalues given as $\lambda_N(L_B) \geq \cdots \geq \lambda_2(L_B) > \lambda_1(L_B) = 0$ and $\lambda_N(L_C) \geq \cdots \geq \lambda_2(L_C) > \lambda_1(L_C) = 0$.*

*Remark 7.30* This lemma can be established because $L_B$ and $L_C$ can be viewed as a Laplacian matrix of a connected graph which shares the same vertex set and edge set with $G(A)$, while with different edge weights (see, e.g., [51]).

With Lemma 7.29, we can give the following fixed-time signed-average consensus results.

**Theorem 7.31** *Consider system (7.23) with structurally balanced signed graph $G(A)$, which is undirected and connected; then, Problem (7.21) can be solved by the protocol (7.40) with $c = \frac{1}{N}\sum_{i=1}^{N}\sigma_i x_i(0)$, i.e., the signed-average is achieved, and the settling time $T$ is bounded by*

$$T \leq \frac{q\pi N^{\frac{q-p}{2q}}}{2\sqrt{\alpha\beta}\lambda(q-p)}, \tag{7.41}$$

*where $\lambda = \min\{\lambda_2(L_B), \lambda_2(L_C)\} > 0$.*

***Proof*** We first prove that $\phi(t) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i x_i(t)$ is time-invariant, i.e., $\phi(t) \equiv \phi(0) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i x_i(0)$. From Lemma 1.8, it follows that when $G(A)$ is structurally balanced, then $DAD$ is nonnegative. Since $A$ is symmetric, we can derive $\sigma_i\sigma_j = \text{sign}(a_{ij}) = \text{sign}(a_{ji})$. Regarding agents under protocol (7.29) and also noting $\sigma_i^2 = 1,\ \sigma_i^{2-p/q} = \sigma_i^{p/q} = \sigma_i$, we can obtain

$$\sigma_i a_{ij}(x_j(t) - \text{sign}(a_{ij})x_i(t))^{2-\frac{p}{q}} = -\sigma_j a_{ji}(x_i(t) - \text{sign}(a_{ji})x_j(t))^{2-\frac{p}{q}},$$

which can be further used to deduce

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i a_{ij}(x_j(t) - \text{sign}(a_{ij})x_i(t))^{2-\frac{p}{q}} \\
&= -\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_j a_{ji}(x_i(t) - \text{sign}(a_{ji})x_j(t))^{2-\frac{p}{q}} \\
&= -\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i a_{ij}(x_j(t) - \text{sign}(a_{ij})x_i(t))^{2-\frac{p}{q}}.
\end{aligned}$$

Then, $\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i a_{ij}(x_j(t) - \text{sign}(a_{ij})x_i(t))^{2-\frac{p}{q}} = 0$. Similarly, $\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i a_{ij}(x_j(t) - \text{sign}(a_{ij})x_i(t))^{\frac{p}{q}} = 0$. Then, calculating the time derivative of $\phi(t)$ along the trajectories of system gives that

$$\begin{aligned}\dot{\phi}(t) &= \frac{\alpha}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i a_{ij}(x_j(t)-\text{sign}(a_{ij})x_i(t))^{2-\frac{p}{q}} \\ &\quad + \frac{\beta}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_i a_{ij}(x_j(t)-\text{sign}(a_{ij})x_i(t))^{\frac{p}{q}} \\ &= 0.\end{aligned}$$

Let $\delta_i(t)=\sigma_i x_i(t)-\phi(0)$. Then, the group disagreement vector [51] can be written as $\delta(t)=[\delta_1(t),\delta_2(t).\ldots,\delta_N(t)]^\top$ and denote $\delta(0)=\delta_0$. Consider the following Lyapunov function:

$$V(t)=\frac{1}{2}\sum_{i=1}^{N}\delta_i^2(t).$$

Differentiating $V(t)$ along the trajectories yields

$$\begin{aligned}&\frac{dV}{dt} \\ &= \alpha\sum_{i=1}^{N}\delta_i(t)\sum_{j=1}^{N}\sigma_i a_{ij}\left(\sigma_j^2 x_j(t)-\sigma_i\sigma_j x_i(t)\right)^{2-\frac{p}{q}} \\ &\quad + \beta\sum_{i=1}^{N}\delta_i(t)\sum_{j=1}^{N}\sigma_i a_{ij}\left(\sigma_j^2 x_j(t)-\sigma_i\sigma_j x_i(t)\right)^{\frac{p}{q}} \\ &= \alpha\sum_{i=1}^{N}\delta_i(t)\sum_{j=1}^{N}|a_{ij}|(\delta_j(t)-\delta_i(t))^{2-\frac{p}{q}}+\beta\sum_{i=1}^{N}\delta_i(t)\sum_{j=1}^{N}|a_{ij}|(\delta_j(t)-\delta_i(t))^{\frac{p}{q}} \\ &= -\frac{1}{2}\alpha\sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|(\delta_j(t)-\delta_i(t))^{\frac{3q-p}{q}}-\frac{1}{2}\beta\sum_{i=1}^{N}\sum_{j=1}^{N}|a_{ij}|(\delta_j(t)-\delta_i(t))^{\frac{p+q}{q}} \\ &= -\frac{1}{2}\alpha\sum_{i=1}^{N}\sum_{j=1}^{N}\left[|a_{ij}^{\frac{2q}{3q-p}}|(\delta_j(t)-\delta_i(t))^2\right]^{\frac{3q-p}{2q}} \\ &\quad -\frac{1}{2}\beta\sum_{i=1}^{N}\sum_{j=1}^{N}\left[|a_{ij}^{\frac{2q}{p+q}}|(\delta_j(t)-\delta_i(t))^2\right]^{\frac{p+q}{2q}}\end{aligned}$$

$$\leq -\frac{1}{2}\alpha N^{\frac{p-q}{q}} \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}^{\frac{2q}{3q-p}}|(\delta_j(t) - \delta_i(t))^2 \right]^{\frac{3q-p}{2q}}$$
$$- \frac{1}{2}\beta \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}^{\frac{2q}{p+q}}|(\delta_j(t) - \delta_i(t))^2 \right]^{\frac{p+q}{2q}}, \tag{7.42}$$

where Lemmas 7.22 and 7.23 are inserted. Since $\mathbf{1_N^\top}\delta(\mathbf{t}) = \mathbf{0}$, Lemma 7.29 gives that

$$\begin{cases} \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}^{\frac{2q}{3q-p}}|(\delta_j(t) - \delta_i(t))^2 \geq 2\lambda_2(L_B)\delta(t)^\top\delta(t) \\ \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}^{\frac{2q}{p+q}}|(\delta_j(t) - \delta_i(t))^2 \geq 2\lambda_2(L_C)\delta(t)^\top\delta(t). \end{cases}$$

Since $\lambda = \min\{\lambda_2(L_B), \lambda_2(L_C)\} > 0$, then

$$\frac{dV}{dt} \leq -\frac{1}{2}\alpha N^{\frac{p-q}{q}}(4\lambda_2(L_B)V)^{\frac{3q-p}{2q}} - \frac{1}{2}\beta(4\lambda_2(L_C)V)^{\frac{p+q}{2q}}$$
$$\leq -\frac{1}{2}\left[\alpha N^{\frac{p-q}{q}}(4\lambda V)^{\frac{q-p}{q}} + \beta\right](4\lambda V)^{\frac{p+q}{2q}}. \tag{7.43}$$

If $V \neq 0$, then let $y(t) = \sqrt{4\lambda V}$, and we can get the following differential equation:

$$\dot{y}(t) \leq -\alpha N^{\frac{p-q}{q}}\lambda y(t)^{\frac{2q-p}{q}} - \beta\lambda y(t)^{\frac{p}{q}}. \tag{7.44}$$

Similarly, by Lemma 7.24 and comparison principle, we conclude that

$$\lim_{t\to T} \delta(t) = 0, \text{ and } \delta(t) = 0 \text{ for } \forall t \geq T,$$

where the settling time is given by

$$T = \frac{qN^{\frac{q-p}{2q}}}{\sqrt{\alpha\beta}\lambda(q-p)} \tan^{-1}\left(N^{\frac{p-q}{2q}}\sqrt{\alpha/\beta}V(\delta_0)\right)$$
$$\leq \frac{q\pi N^{\frac{q-p}{2q}}}{2\sqrt{\alpha\beta}\lambda(q-p)}, \tag{7.45}$$

which leads to

$$\lim_{t\to T} x_j(t) = \frac{1}{N}\sum_{i=1}^{N} \sigma_i x_i(0), \text{ and } x_j(t) = \frac{1}{N}\sum_{i=1}^{N} \sigma_i x_i(0), \ \forall t \geq T, \ \forall i, j.$$

Thus, protocol (7.40) solves the fixed-time signed-average consensus problem.

*Remark 7.32* From Theorem 7.31, when $G(A)$ is structurally unbalanced, we can also prove that states of all agents will reach zero by constructing Lyapunov function $V(t) = \frac{1}{2}\sum_{i=1}^{N} x_i^2(t)$. Also, it should be noted that we can determine the final consensus states of all agents, i.e., the signed-average of the initial values.

*Remark 7.33* From Theorems 7.25 and 7.31, when the signed graph $G(A)$ is structurally balanced, the null space of $DLD$ is spanned by $\mathbf{1}_N$. Hence, the finial states of agents are $D\mathbf{1}_N c$ for some $c \in \mathbb{R}$. When the signed graph $G(A)$ is structurally unbalanced, the null space of $L$ is spanned by $\mathbf{0}$. Hence, all of the agents' states would reach zero. From the aspect of consensus, the case of structurally unbalanced signed graph does not make sense, since all sates go to zero and the cooperation among the agents cannot be well displayed. So, Remarks 7.28 and 7.32 are given to explain the dynamical behavior of multi-agent systems with structurally unbalanced signed graph.

### 7.3.3 Numerical Examples

In this subsection, numerical examples will be given to illustrate the derived fixed-time bipartite consensus of our main results. Consider the structurally balanced signed graphs in Fig. 7.8.

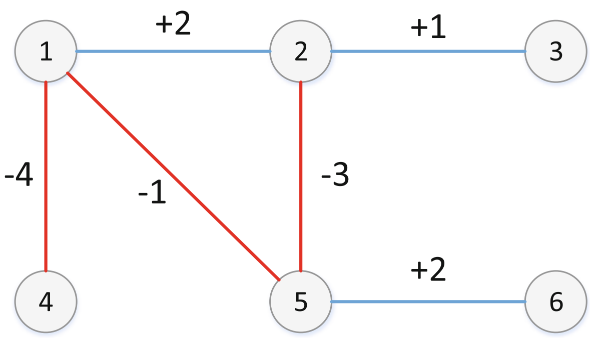


**Fig. 7.8** Structurally balanced signed graph

Consider the structurally balanced signed graph of Fig. 7.8 with the adjacency matrix:

$$A_1 = \begin{bmatrix} 0 & 2 & 0 & -4 & -1 & 0 \\ 2 & 0 & 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ -1 & -3 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}.$$

*Example 7.34* Now, we will illustrate the effectiveness of Theorem 7.25 with protocol (7.29). The initial states are randomly uniformly selected from [0, 10], and the parameters are selected as $\alpha = \beta = 2$, $p = 5$, and $q = 7$. Calculating the eigenvalues of the Laplacian matrix corresponding to the signed graph of Fig. 7.8, one can obtain $\lambda_1 = 0$, $\lambda_2 = 0.989$, $\lambda_3 = 1.39$, $\lambda_4 = 4.030$, $\lambda_5 = 9.128$, $\lambda_6 = 10.465$. Furthermore, by Eq. (7.30), we have $T \leq 3.159$. We take step length $\Delta t = 0.001$, and then numerical results are depicted in Figs. 7.9 and 7.10, which verify Theorem 7.25 very well.

*Example 7.35* Now, let us consider the structurally balanced signed graphs in Fig. 7.8 again. From Theorem 7.31, using protocol (7.40) leads to fundamentally a finite-time stability objective in the form of $\frac{1}{N}\sum_{i=1}^{N}\sigma_i x_i(0)$. Also, we can employ (7.41) to estimate the settling time. The initial states of agents are given as $[9, 6, 3, -1, 7, -2]$, and the parameters are chosen the same with Example 7.34. Then, we can calculate that the signed-average value is 2.333. This can be illustrated from Fig. 7.11. These two sub-figures show that the consensus under protocol

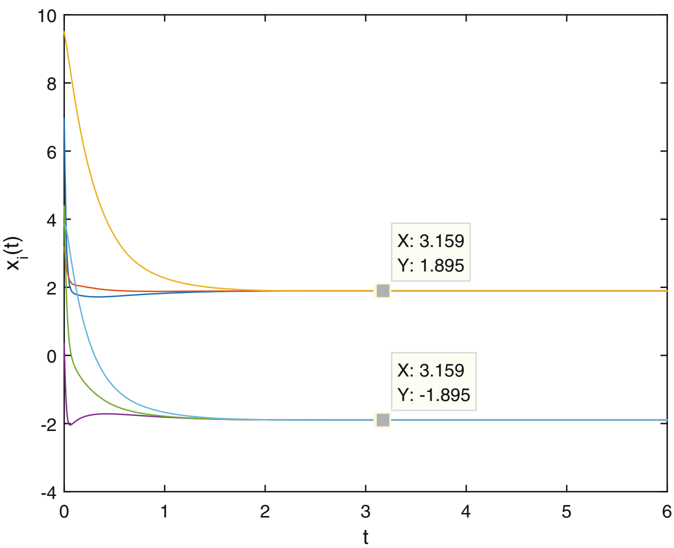


**Fig. 7.9** Fixed-time bipartite consensus of protocol (7.29) in Example 7.34

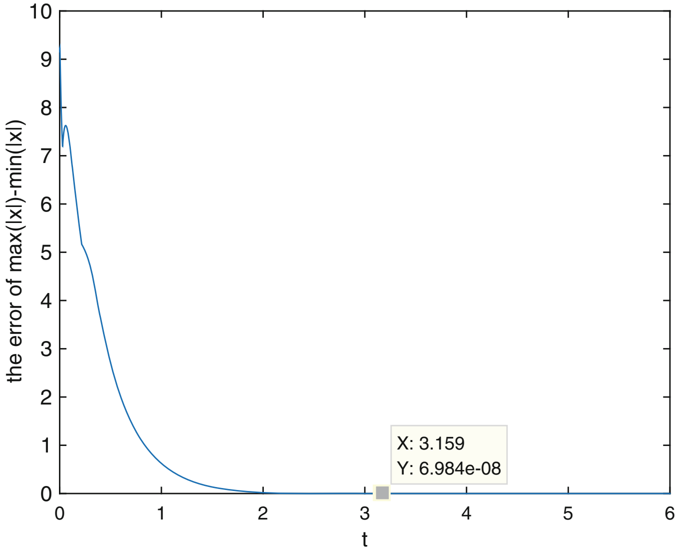


**Fig. 7.10** The error of $\max(|x|) - \min(|x|)$ in Example 7.34

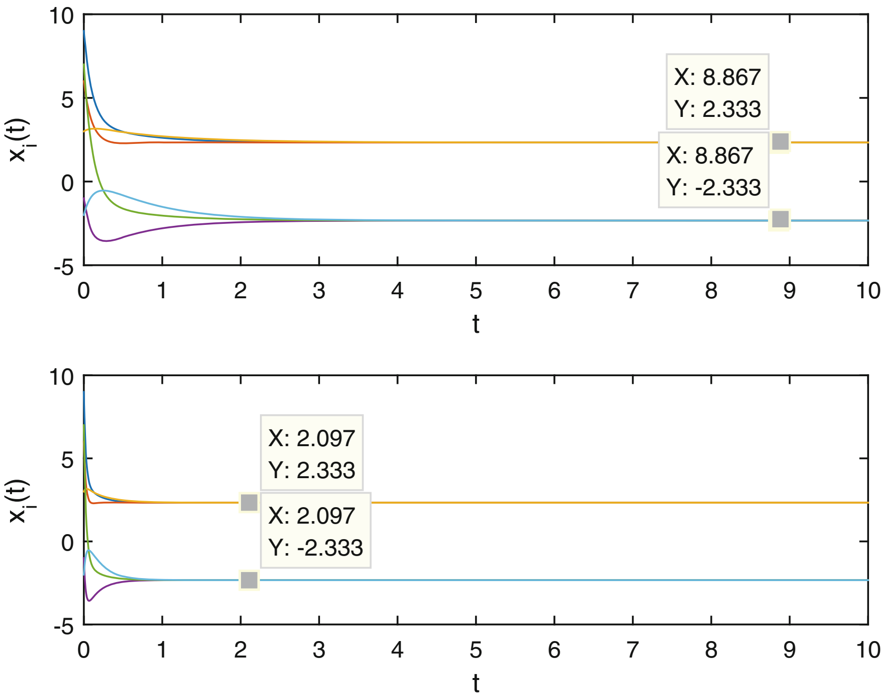


**Fig. 7.11** The initial states are selected as $[9, 6, 3, -1, 7, -2]$ under structurally balanced graph. Upper: the protocol of [40]. Lower: our protocol (7.40)

(7.40) in this chapter is much faster than the protocol of [40], i.e., $u_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t) - \text{sign}(a_{ij})x_i(t))$.

## 7.4 Summary

In this chapter, two new classes of fixed-time and finite-time bipartite consensus protocols were developed for the multi-agent systems with structurally balanced signed graph. With the Lyapunov analysis, it has been shown that fixed-time and finite-time bipartite consensus problems can be solved under the proposed protocols. Some criteria have been established to guarantee the bipartite agreement of all agents. Numerical examples were also given to demonstrate the effectiveness of our proposed consensus strategy.

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# Chapter 8
# Globally Exponential Synchronization and Synchronizability for General Dynamical Networks

Recently, there has been a growing interest in the study of synchronization of complex dynamical networks [1–8]. In [1], Wang and Chen introduced a uniform dynamical network model and investigated its synchronization in scale-free topology networks. In [3], Lü et al. studied chaos synchronization of time-varying complex dynamical networks. In [5], some sufficient conditions were derived for the globally exponential synchronization in arrays of coupled identical delayed neural networks. In [2], Wu showed that an array of coupled systems synchronizes for sufficiently large cooperative coupling if the underlying graph contains a spanning directed tree. Exponential synchronization of asymmetrically coupled dynamical networks has been studied in [9]. However, in [1, 3] and some other references, a common approach is to linearize the nonlinearly dynamical nodes around the synchronized state, and hence only local results were obtained. This scheme will fail if the underlying dynamical system is unknown or uncertain since linearization is very difficult in such cases. In [1, 3–5], the configuration coupling matrix was assumed to be symmetric and irreducible, which implies that the topology of the corresponding complex network is undirected and strongly connected. It is obviously not consistent with the realistic world. Moreover, only asymptotic or/and local results as well as small-scale network examples were discussed in most of the literatures. However, many of the real-world networks are very large. Hence, it is desirable to derive synchronization criteria for large-scale directed dynamical networks.

In order to overcome the aforementioned shortcomings arising from the local analysis and the constraints on the configuration coupling matrix, some alternative methods without using linearization are used in this chapter for the globally exponential synchronization of complex dynamical networks. The configuration coupling matrix is assumed to be asymmetric and reducible, which means that the structure of a network can be directed, weighted, or even weakly connected. Some sufficient conditions are derived for the globally exponential synchronization of dynamical networks by using the Lyapunov functional method and Kronecker



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product techniques. One quantitative measure is distilled from the network's coupling matrix to characterize the synchronizability of complex dynamical networks. The extraction of such a quantity can be easily and conveniently realized by using a normal computer, even for large-scale networks. Moreover, numerical simulations are given to show that our derived criteria can be easily used to make judgements on synchronization for large-scale dynamical networks. By referring to our criteria, it can be observed from simulations that directed small-world dynamical networks possess better synchronizability than undirected ones.

## 8.1 Preliminaries

We consider a complex dynamical network consisting of $N$ identical linearly coupled nodes being an $n$-dimensional neural network, which can be stable, periodic, almost-periodic, or even chaotic. The $i$th isolated node can be described by the following retarded functional differential equation:

$$\dot{x}_i(t) = -Cx_i(t) + B_1 f(x_i(t)) + B_2 f(x_i(t-\tau)) + I(t), \tag{8.1}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^\top$ is the state vector of the $i$th node at time $t$; $C = \mathrm{diag}\{c_1, c_2, \ldots, c_n\}$, with $c_k > 0$, denotes the rate with which the cell $k$ resets its potential to the resting state when isolated from other cells and inputs; $B_1 \in \mathbb{R}^{n\times n}$, $B_2 \in \mathbb{R}^{n\times n}$ represent the connection weight matrix and the delayed connection weight matrix, respectively; $f(x_i(t)) = [f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t))]^\top$, and $f_i(\cdot)$ $(i = 1, 2, \ldots, n)$ are activation functions; $I(t) = [I_1(t), I_2(t), \ldots, I_n(t)] \in \mathbb{R}^n$ is an external input vector.

The dynamical behavior of the complex dynamical network can be described by the following linearly coupled delayed differential equations [1]:

$$\begin{aligned} \dot{x}_i(t) = &-Cx_i(t) + B_1 f(x_i(t)) + B_2 f(x_i(t-\tau)) + I(t) \\ &+c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t),\ i = 1, 2, \ldots, N, \end{aligned} \tag{8.2}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^\top \in \mathbb{R}^n$ $(i = 1, 2, \ldots, N)$ is the state vector of the node $i$; $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ (satisfying $\gamma_i > 0$ for $i = 1, 2, \ldots, n$) is the diagonal inner coupling matrix between two connected nodes $i$ and $j$ $(i \neq j)$ at time $t$ for all $1 \leq i, j \leq N$; $c$ is the coupling strength; $A = (a_{ij})_{N\times N}$ is the negative Laplacian matrix representing the structure of the network, in which $a_{ij}$ is defined as follows: if there is a connection from node $j$ to node $i$ $(j \neq i)$, then $a_{ij} > 0$; otherwise, $a_{ij} = 0$. It means that the network topology could be directed and weighted. The diagonal entries of matrix $A$ are determined by the following diffusive coupling condition [10]:

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, \quad i = 1, 2, \ldots, N, \tag{8.3}$$

which implies that the general complex dynamical network (8.2) would be decoupled at the synchronized state.

*Remark 8.1* It should be noted that we are mainly concerned about the structure of the complex dynamical network model (8.2). The topology discussed in this chapter could be undirected, directed, or weakly connected containing rooted spanning tree.

Let $x(t) = (x_1^\top(t), x_2^\top(t), \ldots, x_N^\top(t))^\top$, $F(x(t)) = (f^\top(x_1(t)), f^\top(x_2(t)), \ldots, f^\top(x_N(t)))^\top$, and $\mathbf{I}(t) = (I^\top(t), I^\top(t), \ldots, I^\top(t))^\top$, then the general dynamical network (8.2) can be rewritten in the following Kronecker product form:

$$\begin{aligned}\dot{x}(t) = &-(I_N \otimes C)x(t) + (I_N \otimes B_1)F(x(t))\\ &+(I_N \otimes B_2)F(x(t-\tau)) + \mathbf{I}(t) + c(A \otimes \Gamma)x(t).\end{aligned} \tag{8.4}$$

For the activation functions $f_k(\cdot)$ of an isolated neural network and the coupling configuration matrix $A$, we have the following assumptions:

**Assumption 8.2** *$f_k(\cdot)$ $(k = 1, 2, \ldots, n)$ are globally Lipschitz continuous functions, i.e., there exist constants $l_k > 0$ $(k = 1, 2, \ldots, n)$ such that $|f_k(x_1) - f_k(x_2)| \leq l_k|x_1 - x_2|$ $(k = 1, 2, \ldots, n)$ hold for any $x_1, x_2 \in \mathbb{R}$. For convenience, denote $L = diag\{l_1, l_2, \ldots, l_n\}$.*

**Assumption 8.3** *The coupling configuration matrix $A$ is irreducible.*

**Assumption 8.4** *Real parts of eigenvalues of $A$ are all negative except an eigenvalue 0 with multiplicity 1.*

In order to derive our main results, the following lemmas and definitions are needed.

**Definition 8.5** The dynamical network (8.2) or (8.4) is said to be globally exponentially synchronized if there exist $\epsilon > 0$, $T > 0$, and $M > 0$ such that for any initial values $\phi_i(s)$ $(i = 1, 2, \ldots, N)$,

$$\|x_i(t) - x_j(t)\| \leq Me^{-\epsilon t}$$

hold for all $t > T$, and for any $i, j = 1, 2, \ldots, N$.

**Definition 8.6** For an $N \times N$ irreducible square matrix $A$ with non-negative off-diagonal elements, which satisfies the diffusive coupling condition (8.3), the quantity $\alpha(A)$ is defined as follows: Let $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^\top$ be the unique normalized left eigenvector of $A$ with respect to the eigenvalue zero satisfying $\sum_{k=1}^{N} \xi_k = 1$, and $\Xi = \text{diag}\{\xi_1, \xi_2, \ldots, \xi_N\}$. Then $\alpha(A)$ is defined to be the second largest eigenvalue of the symmetric matrix $\tilde{A} = \Xi A + A^\top \Xi$, that is, $\alpha(A) = \lambda_2(\Xi A + A^\top \Xi)$.

**Definition 8.7 ([11])** The ability that the structure of the network, which is represented by Laplacian matrix $A$, can guarantee the synchronization of dynamical network (8.2) is called the *synchronizability* of the network.

## 8.2 Synchronization Analysis

In this section, the globally exponential synchronization will be analyzed for general dynamical networks. The network topology can be undirected or directed, that is, the matrix $A$ can be either symmetric [4] or asymmetric. In the first subsection, we will investigate the globally exponential synchronization of the general dynamical network with irreducible coupling configuration matrix $A$. An explicit quantity will be distilled from the irreducible coupling matrix $A$ to characterize the synchronizability of the corresponding dynamical network. In the second subsection, a dynamical network with reducible coupling matrix $A$ will be studied.

### *8.2.1 Irreducible Case*

In this subsection, we shall discuss the synchronization condition of the dynamical network with irreducible coupling matrix $A$. By combining the Lyapunov functional method and Kronecker product techniques, some criteria ensuring the globally exponential synchronization of the dynamical network will be derived.

Suppose that $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^\top$ is the normalized left eigenvector of the configuration coupling matrix $A$ with respect to eigenvalue 0 satisfying $\sum_{i=1}^{N} \xi_i = 1$. Since the coupling configuration matrix $A$ is irreducible, according to the Lemma 1.6, we can conclude that $\xi_i > 0$ for $i = 1, 2, \ldots, N$. Let $\Xi = \text{diag}\{\xi_1, \xi_2, \ldots, \xi_N\} > 0$, and $W = \Xi - \xi\xi^\top$.

**Theorem 8.8** *Suppose that Assumptions 8.2 and 8.3 hold, and that there exist diagonal positive definite matrix $P$, positive definite matrix $Q$, and diagonal matrix $S \geq 0$, such that the following linear matrix inequality (LMI) is satisfied:*

$$\Upsilon = \begin{bmatrix} -2PC - c\gamma P\Gamma + 2L^\top SL & PB_1 & PB_2 \\ B_1^\top P & Q - 2S & 0 \\ B_2^\top P & 0 & -Q \end{bmatrix} < 0, \tag{8.5}$$

*where* $\gamma = -\alpha(A)/\lambda_{\max}(W)$, *and* $\alpha(A)$ *is defined in Definition 8.6. Then, the complex dynamical network (8.4) can reach globally exponential synchronization.*

***Proof*** It follows from LMI (8.5) that there exists a positive constant $\epsilon$, which may be very small, such that

$$\tilde{\Upsilon} = \begin{bmatrix} \epsilon P - 2PC - c\gamma P\Gamma + 2L^\top SL & PB_1 & PB_2 \\ B_1^\top P & e^{\epsilon\tau}Q - 2S & 0 \\ B_2^\top P & 0 & -Q \end{bmatrix} < 0. \tag{8.6}$$

Consider the following functional:

$$V(x) = e^{\epsilon t}x^\top(t)(W \otimes P)x(t) + \int_{t-\tau}^{t} e^{\epsilon(s+\tau)}F^\top(x(s))(W \otimes Q)F(x(s))ds. \tag{8.7}$$

Then, by calculating the derivative of the functional (8.7) along the trajectories of (8.4), one can obtain that

$$\begin{aligned} \dot{V}(x)|_{(8.4)} &= \epsilon e^{\epsilon t}x^\top(t)(W \otimes P)x(t) + 2e^{\epsilon t}x^\top(t)(W \otimes P)\dot{x}(t) + e^{\epsilon(t+\tau)}F^\top(x(t)) \\ &\quad \times (W \otimes Q)F(x(t)) - e^{\epsilon t}F^\top(x(t-\tau))(W \otimes Q)F(x(t-\tau)) \\ &= e^{\epsilon t}\Big[x^\top(t)(\epsilon W \otimes P)x(t) - 2x^\top(t)(W \otimes P)(I_N \otimes C)x(t) \\ &\quad +2x^\top(t)(W \otimes P)((I_N \otimes B_1)F(x(t)) + (I_N \otimes B_2)F(x(t-\tau))) \\ &\quad +2x^\top(t)(W \otimes P)\mathbf{I}(t) + 2x^\top(t)(W \otimes P)(cA \otimes \Gamma)x(t) \\ &\quad +e^{\epsilon\tau}F^\top(x(t))(W \otimes Q)F(x(t)) \\ &\quad -F^\top(x(t-\tau))(W \otimes Q)F(x(t-\tau))\Big]. \end{aligned}$$

Since $W = \Xi - \xi\xi^\top$, we have $w_{ij} = -\xi_i\xi_j$ for $i \neq j$, and $w_{ii} = \xi_i - \xi_i^2$. It follows from $\sum_{j=1}^N \xi_j = 1$ that $\sum_{j=1}^N w_{ij} = \xi_i - \sum_{j=1}^N \xi_i\xi_j = 0$. Hence, we can obtain that $(W \otimes P)\mathbf{I}(t) = \begin{bmatrix} w_{11}P & \cdots & w_{1N}P \\ \vdots & \ddots & \vdots \\ w_{N1}P & \cdots & w_{NN}P \end{bmatrix}\begin{bmatrix} I(t) \\ \vdots \\ I(t) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^N w_{1j}PI(t) \\ \vdots \\ \sum_{j=1}^N w_{Nj}PI(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.
Furthermore, we have $WA = (\Xi - \xi\xi^\top)A = \Xi A - \xi(\xi^\top A) = \Xi A$. By referring

to the structure of matrix $W$, we can obtain that

$$
\begin{aligned}
&2x^\top(t)(W\otimes PB_1)F(x(t))\\
&=\sum_{i=1}^{N}\sum_{j=1}^{N}\big[2w_{ij}x_i^\top(t)PB_1f(x_j)\big]\\
&=\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\big[-2\xi_i\xi_jx_i^\top(t)PB_1f(x_j)\big]+\sum_{i=1}^{N}\big[2\xi_i(1-\xi_i)x_i^\top(t)PB_1f(x_i)\big]\\
&=\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\big[-2\xi_i\xi_jx_i^\top(t)PB_1f(x_j)\big]\\
&\quad+\sum_{i=1}^{N}2\xi_i\left[\left(\sum_{j=1,j\neq i}^{N}\xi_j\right)x_i^\top(t)PB_1f(x_i)\right]\\
&=\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\xi_i\xi_j\big[x_i^\top(t)PB_1f(x_i)+x_i^\top(t)PB_1f(x_i)-x_i^\top(t)PB_1f(x_j)\\
&\quad-x_i^\top(t)PB_1f(x_j)\big](\text{by exchanging subscripts } i \text{ and } j)\\
&=\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\xi_i\xi_j\big[x_i^\top(t)PB_1f(x_i)+x_j^\top(t)PB_1f(x_j)-x_i^\top(t)PB_1f(x_j)\\
&\quad-x_j^\top(t)PB_1f(x_i)\big]\\
&=\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\xi_i\xi_j\big[(x_i(t)-x_j(t))^\top PB_1(f(x_i)-f(x_j))\big].
\end{aligned}
\tag{8.8}
$$

Similarly, we can calculate the remaining terms of $\dot V(x)|_{(8.4)}$, and further by adding a vanishing term $-x^\top(t)(W\otimes c\gamma P\Gamma)x(t)+x^\top(t)(W\otimes c\gamma P\Gamma)x(t)$, we can obtain that

$$
\begin{aligned}
&\dot V(x)|_{(8.4)}\\
&=\frac{1}{2}e^{\epsilon t}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\Bigg\{\xi_i\xi_j\Big[(x_i-x_j)^\top(\epsilon P-2PC-c\gamma P\Gamma)\\
&\quad\cdot(x_i-x_j)+2(x_i-x_j)^\top PB_1\big[f(x_i)-f(x_j)\big]
\end{aligned}
$$

$$
\begin{aligned}
&+2(x_i - x_j)^\top PB_2\big[f(x_i(t-\tau)) - f(x_j(t-\tau))\big] \\
&+[f(x_i) - f(x_j)]^\top e^{\epsilon\tau} Q\big[f(x_i) - f(x_j)\big] \\
&-[f(x_i(t-\tau)) - f(x_j(t-\tau))]^\top Q\big[f(x_i(t-\tau)) - f(x_j(t-\tau))\big]\Big]\Big\} \\
&+e^{\epsilon t} x^\top(t)\big[c(\varXi A + A^\top \varXi) \otimes P\varGamma + W \otimes c\gamma P\varGamma\big]x(t),
\end{aligned}
\tag{8.9}
$$

where $w_{ij} = -\xi_i\xi_j$ for $i \neq j$.

According to the Assumption 8.2, one has

$$
\begin{aligned}
&2(x_i - x_j)^\top L^\top SL(x_i - x_j) \\
&- 2(f(x_i) - f(x_j))^\top S(f(x_i) - f(x_j)) \geq 0.
\end{aligned}
\tag{8.10}
$$

By using the inequality (8.10), it follows from the equality (8.9) that

$$
\begin{aligned}
&\dot{V}(x)|_{(8.4)} \qquad\qquad (8.11)\\
&\leq \frac{1}{2} e^{\epsilon t} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \Big\{ \xi_i \xi_j \Big[ (x_i - x_j)^\top (\epsilon P - 2PC - c\gamma P\varGamma \\
&+2L^\top SL)(x_i - x_j) + 2(x_i - x_j)^\top PB_1\big[f(x_i) - f(x_j)\big] \\
&+2(x_i - x_j)^\top PB_2\big[f(x_i(t-\tau)) - f(x_j(t-\tau))\big] \\
&+[f(x_i) - f(x_j)]^\top (e^{\epsilon\tau} Q - 2S)\big[f(x_i) - f(x_j)\big] \\
&-[f(x_i(t-\tau)) - f(x_j(t-\tau))]^\top Q\big[f(x_i(t-\tau)) - f(x_j(t-\tau))\big]\Big]\Big\} \\
&+e^{\epsilon t} x^\top(t)\big[c(\varXi A + A^\top \varXi) \otimes P\varGamma + W \otimes c\gamma P\varGamma\big]x(t) \\
&= \frac{1}{2} e^{\epsilon t} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \xi_i \xi_j \cdot \eta_{ij}^\top(t) \tilde{\varUpsilon} \eta_{ij} + e^{\epsilon t} x^\top(t) \\
&\times\big[c(\varXi A + A^\top \varXi) \otimes P\varGamma + W \otimes c\gamma P\varGamma\big]x(t),
\end{aligned}
\tag{8.12}
$$

where $\eta_{ij}(t) = [(x_i - x_j)^\top, (f(x_i) - f(x_j))^\top, (f(x_i(t-\tau)) - f(x_j(t-\tau)))^\top]^\top$.

Let $\tilde{A} = \varXi A + A^\top \varXi$. Then, one can obtain that the elements of $\tilde{A}$ are $\tilde{a}_{ij} = \xi_i a_{ij} + \xi_j a_{ji}$. Since $\xi = [\xi_1, \xi_2, \ldots, \xi_N]^\top \in \mathbb{R}^N$ is the normalized left eigenvector of matrix $A$ with respect to eigenvalue zero satisfying $\sum_{i=1}^{N} \xi_i = 1$, it can be observed that $\tilde{A}$ is a zero row sum symmetric matrix with non-negative off-diagonal elements. The irreducibility of matrix $\tilde{A}$ can also be deduced from the irreducibility of matrix $A$. Hence, the eigenvalues of matrix $\tilde{A}$ can be arranged as follows: $0 = \lambda_1(\tilde{A}) > \lambda_2(\tilde{A}) \geq \lambda_3(\tilde{A}) \geq \cdots \geq \lambda_N(\tilde{A})$. By Definition 8.6, $\lambda_2(\tilde{A})$ is selected as $\alpha(A)$.

According to matrix decomposition theory [12], there exists a unitary matrix $U$ such that $\tilde{A} = U\Lambda U^\top$, where $\Lambda = \text{diag}\{0, \lambda_2(\tilde{A}), \ldots, \lambda_N(\tilde{A})\}$, and $U = [u_1, u_2, \ldots, u_N]$ with $u_1 = (\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \ldots, \frac{1}{\sqrt{N}})^\top$.

Let $y(t) = (U^\top \otimes I_n)x(t)$. Then, one has $x(t) = (U \otimes I_n)y(t)$. Let $y_i(t) \in \mathbb{R}^n$ $(i = 1, 2, \ldots, N)$ be such that $y(t) = [y_1^\top(t), y_2^\top(t), \ldots, y_N^\top(t)]^\top$. Then we have

$$
\begin{aligned}
& cx^\top(t)\big[(\Xi A + A^\top \Xi) \otimes P\Gamma\big]x(t) \\
&= cy^\top(t)(U^\top \otimes I_n)(\tilde{A} \otimes P\Gamma)(U \otimes I_n)y(t) \\
&= cy^\top(t)(U^\top \tilde{A} U \otimes P\Gamma)y(t) \\
&= cy^\top(t)(\Lambda \otimes P\Gamma)y(t) \\
&= c\sum_{i=1}^{N} \lambda_i(\tilde{A})y_i^\top(t)P\Gamma y_i(t) \\
&= c\sum_{i=2}^{N} \lambda_i(\tilde{A})y_i^\top(t)P\Gamma y_i(t) \\
&\leq c\lambda_2(\tilde{A})\sum_{i=2}^{N} y_i^\top(t)P\Gamma y_i(t) \\
&= c\alpha(A)\sum_{i=2}^{N} y_i^\top(t)P\Gamma y_i(t).
\end{aligned} \tag{8.13}
$$

By referring to the structure of matrix $W$, one can conclude that $W$ is an irreducible symmetric matrix with negative off-diagonal elements, and also $W$ satisfies the diffusive coupling condition (8.3). Hence, one can conclude that $\lambda_{\max}(W) > 0$. It follows from the construction of matrix $W$ that $W \cdot u_1 = (0, 0, \ldots, 0)^\top := \mathbf{0}_n \in \mathbb{R}^N$. Further, we can obtain that $U^\top W U = \begin{bmatrix} 0 & \mathbf{0}_n^\top \\ \mathbf{0}_n & \tilde{U}^\top W \tilde{U} \end{bmatrix}$, where $\tilde{U} = [u_2, u_3, \ldots, u_N]$ satisfying $\tilde{U}^\top \tilde{U} = I_{N-1}$. By utilizing the above analysis, one has that

$$
\begin{aligned}
& x^\top(t)(W \otimes c\gamma P\Gamma)x(t) \\
&= c\gamma y^\top(t)(U^\top W U \otimes P\Gamma)y(t) \\
&= c\gamma \tilde{y}^\top(t)(\tilde{U}^\top W \tilde{U} \otimes P\Gamma)\tilde{y}(t)
\end{aligned}
$$

$$\le c\gamma\lambda_{\max}(W)\tilde{y}^\top(t)(\tilde{U}^\top\tilde{U}\otimes P\Gamma)\tilde{y}(t)$$
$$= c\gamma\lambda_{\max}(W)\sum_{i=2}^{N} y_i^\top(t)P\Gamma y_i(t), \tag{8.14}$$

where $\tilde{y}(t) = [y_2^\top(t), \dots, y_N^\top(t)]^\top$.

By using (8.13), (8.14), and equality $\gamma = -\alpha(A)/\lambda_{\max}(W)$, it follows from (8.11) that

$$\begin{aligned}\dot{V}(x)|_{(8.4)} &\le \frac{1}{2}e^{\epsilon t}\sum_{i=1}^{N}\sum_{j=1, j\ne i}^{N}\xi_i\xi_j\cdot\eta_{ij}^\top(t)\tilde{\Upsilon}\eta_{ij} \\ &\quad + e^{\epsilon t}(c\cdot\alpha(A) + c\gamma\lambda_{\max}(W))\sum_{i=2}^{N} y_i^\top(t)P\Gamma y_i(t) \\ &= \frac{1}{2}e^{\epsilon t}\sum_{i=1}^{N}\sum_{j=1, j\ne i}^{N}\xi_i\xi_j\cdot\eta_{ij}^\top(t)\tilde{\Upsilon}\eta_{ij}.\end{aligned} \tag{8.15}$$

Using (8.6), inequality (8.11) gives that $\dot{V}(x)|_{(8.4)} \le 0$, which yields that $V(x(t)) \le V(x(0))$. This means that, $V(x(t))$ is bounded. Therefore, we can obtain that $e^{\epsilon t}x^\top(t)(W\otimes P)x(t)$ is also bounded, which implies

$$\begin{aligned}&\frac{1}{2}\xi_i\xi_j\lambda_{\min}(P)\|x_i(t) - x_j(t)\|^2 \\ &\le \frac{1}{2}\sum_{i=1, j=1}^{N}\xi_i\xi_j(x_i(t) - x_j(t))^\top P(x_i(t) - x_j(t)) \\ &= x^\top(t)(W\otimes P)x(t) = O(e^{-\epsilon t}).\end{aligned} \tag{8.16}$$

According to Definition 8.5, we can conclude that globally exponential synchronization of complex dynamical network (8.4) can be achieved under premise (8.5). The proof is completed.

*Remark 8.9* It should be noted that the functional $V(x)$ in (8.7) is not positive definite. From the equalities $x^\top(t)(W\otimes P)x(t) = \frac{1}{2}\sum_{i=1,j=1}^{N}\xi_i\xi_j(x_i(t) - x_j(t))^\top P(x_i(t) - x_j(t))$ and $F^\top(x(s))(W\otimes Q)F(x(s)) = \frac{1}{2}\sum_{i=1,j=1}^{N}\xi_i\xi_j \times (f(x_i(t)) - f(x_j(t)))^\top Q(f(x_i(t)) - f(x_j(t)))$, we can observe that $V(x)$ is a positive semi-definite functional. In fact, the functional $V(x)$ vanishes in the synchronization manifold $\mathcal{M} = \{x_1(t) = x_2(t) = \cdots = x_N(t)\}$.

*Remark 8.10* The derived result in Theorem 8.8 is delay-independent. When the time delay is small, such delay-independent result could be conservative. In this case, some techniques could be borrowed from [13] and [14] to derive some delay-dependent synchronization criteria.

By specifying slack variables of Theorem 8.8, we can obtain an algebraic criterion shown in Corollary 8.11. Comparing with linear matrix inequalities results in Theorem 8.8, the algebraic result shown in Corollary 8.11 is more conservative but easier to verify.

**Corollary 8.11** *Under Assumptions 8.2 and 8.3, directed complex dynamical network (8.4) can reach globally exponential synchronization if the following algebraic inequality is satisfied:*

$$c > \frac{\beta}{\gamma}, \tag{8.17}$$

*where* $\beta = \lambda_{\max}\big((-2C + 2L^\top L + B_1 B_1^\top + B_2 B_2^\top)\Gamma^{-1}\big)$ *and* $\gamma = -\alpha(A)/\lambda_{\max}(W)$ *with* $\alpha(A)$ *is given in Definition 8.6.*

***Proof*** By (8.17), we can obtain the following inequality:

$$-c\alpha(A)/\lambda_{\max}(W) > \lambda_{\max}\Big(\big(-2C + 2L^\top L + B_1 B_1^\top + B_2 B_2^\top\big)\Gamma^{-1}\Big), \tag{8.18}$$

which further implies that

$$\big(-c\alpha(A)/\lambda_{\max}(W)\big) * I_n > \big(-2C + 2L^\top L + B_1 B_1^\top + B_2 B_2^\top\big)\Gamma^{-1}. \tag{8.19}$$

Hence, we can obtain that

$$\big(-c\alpha(A)/\lambda_{\max}(W)\big) * \Gamma > -2C + 2L^\top L + B_1 B_1^\top + B_2 B_2^\top. \tag{8.20}$$

By referring to Schur complement [15], one can obtain that

$$\begin{bmatrix} -2C - c\gamma\Gamma + 2L^\top L & B_1 & B_2 \\ B_1^\top & -I_n & 0 \\ B_2^\top & 0 & -I_n \end{bmatrix} < 0. \tag{8.21}$$

Hence, LMI (8.5) is satisfied by taking $P = Q = S = I_n$. Proof of Corollary 8.11 is completed by using Theorem 8.8.

*Remark 8.12* It should be noted that $\beta$ in (8.17) is determined by the parameters of the individual node and inner coupling matrix $\Gamma$, and $\gamma$ in (8.17) completely depends on the structure of the complex network. Given the parameters of an individual node and the inner coupling matrix $\Gamma$, $\beta$ is fixed, and hence a large value of $\gamma$ implies that directed complex dynamical network (8.4) can synchronize with a small coupling strength $c$. Therefore, according to Corollary 8.11 and Definition 8.7, synchronizability of network (8.4) with respect to a specific coupling structure can be characterized by $\gamma$. The synchronizability of network structure is said to be *strong* if corresponding dynamical network (8.4) can synchronize with a *small* coupling strength $c$ [1].

If the complex dynamical network is assumed to be undirected, one can conclude that the Laplacian matrix $A$ is symmetrical. The normalized left eigenvector $\xi$ of matrix $A$ with respect to eigenvalue 0 can be explicitly obtained as $\xi = \frac{1}{N}[1, 1, \ldots, 1]^\top \in \mathbb{R}^N$. Then we have $\Xi = \frac{1}{N} I_N$, and $W = \frac{1}{N^2}\begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & \cdots & N-1 \end{bmatrix}$. In fact, $W$ can be regarded as the Laplacian matrix of the complete graph, and the eigenvalues of matrix $W$ are $\lambda_1(W) = \lambda_2(W) = \cdots = \lambda_{N-1}(W) = \frac{1}{N}$, and $\lambda_N(W) = 0$ [16]. Hence, $\lambda_{\max}(W) = \frac{1}{N}$. Moreover, since $\Xi = \frac{1}{N} I_N$ and $A$ is symmetrical, one can obtain that $\alpha(A) = \lambda_2(\Xi A + A^\top \Xi) = \frac{2}{N}\lambda_2(A)$, where $\lambda_2(A)$ is the second largest eigenvalue of matrix $A$. Therefore, for undirected network, the quantity $\gamma$ can be represented as $\gamma = -2\lambda_2(A)$. Then we can obtain the following theorem and corollary for undirected dynamical networks:

**Theorem 8.13** *Suppose that the coupling matrix $A$ is symmetrical and Assumptions 8.2 and 8.3 hold, and there exist diagonal positive definite matrix $P$, positive definite matrix $Q$, and diagonal matrix $S \geq 0$, such that the following linear matrix inequality is satisfied:*

$$\begin{bmatrix} -2PC - c\gamma P\Gamma + 2L^\top SL & PB_1 & PB_2 \\ B_1^\top P & Q - 2S & 0 \\ B_2^\top P & 0 & -Q \end{bmatrix} < 0, \tag{8.22}$$

*where $\gamma = -2\lambda_2(A)$. Then the undirected complex dynamical network (8.4) can reach globally exponential synchronization.*

**Corollary 8.14** *Under Assumptions 8.2 and 8.3, undirected complex dynamical network (8.4) can reach globally exponential synchronization if the following algebraic inequality is satisfied:*

$$c > \frac{\beta}{\gamma}, \tag{8.23}$$

*where $\beta = \lambda_{\max}\big((-2C + 2L^\top L + B_1 B_1^\top + B_2 B_2^\top)\Gamma^{-1}\big)$ and $\gamma = -2\lambda_2(A)$.*

*Remark 8.15* According to Corollaries 8.11 and 8.14, quantities $\frac{-\alpha(A)}{\lambda_{\max}(W)N}$ and $-2\lambda_2(A)$ have been, respectively, distilled to characterize the synchronizability of directed and undirected complex dynamical networks. Then, an obvious question to consider here is: *for directed and undirected networks, which possesses better* *synchronizability?* In Sect. 8.3, calculation of the quantity $\gamma$ shows that directed small-world networks possess better synchronizability than undirected ones.

*Remark 8.16* It should be noted that the size of LMI (8.5), which only depends on the dimension of single system ($n$), has nothing to do with the size of the network ($N$). The dimension of a single system is usually small (less than 100). Hence, it takes a short time to solve the corresponding LMI. The quantity $\gamma = -\alpha(A)/\lambda_{\max}(W)$, distilled from coupling matrix $A$, can be used to characterize the synchronizability of the corresponding dynamical network. In fact, the larger the $\gamma$ is, the easier the corresponding network can be synchronized. Therefore, the calculation of $\gamma$ is the key factor concerning whether our criteria are applicable to large-scale networks. Numerical examples will be given to show that it takes a reasonable amount of time to calculate $\gamma$ for large-scale networks. Hence, it can be concluded that our synchronization criterion is useful for large-scale networks.

### *8.2.2 Reducible Case*

In the following, we discuss the case that the configuration coupling matrix $A$ satisfies Assumption 8.4. Without loss of generality, the reducible matrix $A$ can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ 0 & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{pp} \end{bmatrix}, \tag{8.24}$$

where $A_{pp} \in \mathbb{R}^{m_p \times m_p}$ is irreducible, and $A_{qq} \in \mathbb{R}^{m_q \times m_q}$ for $q = 1, 2, \ldots, p-1$. Since $A$ satisfies Assumption 8.4, for each $q$, there must exist $k > q$ such that $A_{qk} \neq 0$ [2, 17]. In other words, the reverse of the graph generated by the configuration coupling matrix $A$ must contain a rooted spanning directed tree [2].

Let $N_q = \sum_{k=1}^{q} m_k$, and then $N = N_p$. Next, complex dynamical network (8.2) can be decomposed into $p$ coupled subsystems denoted by $\mathbb{S}_q = \{N_{q-1}+1, N_{q-1}+2, \ldots, N_q\}$ for $q = 1, 2, \ldots, p$, with $N_0 = 0$:

$$\begin{aligned} \mathbb{S}_p : \dot{x}_i(t) &= -Cx_i(t) + B_1 f(x_i(t)) + B_2 f(x_i(t-\tau)) + I(t) \\ &\quad + c \sum_{j \in \mathbb{S}_p} a_{ij} \Gamma x_j(t), \quad i \in \mathbb{S}_p \end{aligned} \tag{8.25}$$

and

$$\begin{aligned}\mathbb{S}_q : \dot{x}_i(t) &= -Cx_i(t) + B_1 f(x_i(t)) + B_2 f(x_i(t-\tau)) + I(t) \\ &\quad + c\sum_{j\in\mathbb{S}_q} a_{ij}\Gamma x_j(t) + c\sum_{\substack{r\in\mathbb{S}_k\\ k>q}} a_{ir}\Gamma x_r(t), \quad i\in\mathbb{S}_q, \\ q &= 1,2,\ldots,p-1. \end{aligned} \tag{8.26}$$

If the subsystems $\mathbb{S}_k$, $(k = q+1, q+2, \ldots, p)$ are exponentially globally synchronized with convergence rate $\epsilon$, then these subsystems are decoupled, and the final synchronized state $x^*(t)$ satisfies the following differential equation:

$$\dot{x}^*(t) = -Cx^*(t) + B_1 f(x^*(t)) + B_2 f(x^*(t-\tau)) + I(t) + O(e^{-\epsilon t}). \tag{8.27}$$

Therefore, according to the zero-sum rows, i.e., $\sum_{j\in\mathbb{S}_q} a_{ij} + \sum_{r\in\mathbb{S}_k, k>q} a_{ir} = 0$ for $i\in\mathbb{S}_q$, the next coupled subsystem $\mathbb{S}_q$ can be described by the following equation:

$$\begin{aligned}\dot{x}_i(t) &= -Cx_i(t) + B_1 f(x_i(t)) + B_2 f(x_i(t-\tau)) + I(t) \\ &\quad + c\sum_{j\in\mathbb{S}_q} a_{ij}\Gamma x_j(t) + c\sum_{r\in\mathbb{S}_k, k>q} a_{ir}\Gamma x_r(t) + O(e^{-\epsilon t}) \\ &= -Cx_i(t) + B_1 f(x_i(t)) + B_2 f(x_i(t-\tau)) + I(t) \\ &\quad + c\sum_{j\in\mathbb{S}_q} a_{ij}\Gamma x_j(t) + c\sum_{r\in\mathbb{S}_k, k>q} a_{ir}\Gamma x^*(t) + O(e^{-\epsilon t}) \\ &= -Cx_i(t) + B_1 f(x_i(t)) + B_2 f(x_i(t-\tau)) + I(t) \\ &\quad + c\sum_{j\in\mathbb{S}_q} a_{ij}\Gamma x_j(t) - c\sum_{j\in\mathbb{S}_q} a_{ij}\Gamma x^*(t) + O(e^{-\epsilon t}) \\ &= -Cx_i(t) + B_1 f(x_i(t)) + B_2 f(x_i(t-\tau)) + I(t) \\ &\quad + c\sum_{j\in\mathbb{S}_q} a_{ij}\Gamma (x_j(t) - x^*(t)) + O(e^{-\epsilon t}), \quad i\in\mathbb{S}_q \end{aligned} \tag{8.28}$$

and then

$$\begin{aligned}\dot{e}_i(t) &= -Ce_i(t) + B_1 g(e_i(t)) + B_2 g(e_i(t-\tau)) \\ &\quad + c\sum_{j\in\mathbb{S}_q} a_{ij}\Gamma e_j(t) + O(e^{-\epsilon t}), \quad i\in\mathbb{S}_q, \end{aligned} \tag{8.29}$$

where $e_i(t) = x_i(t) - x^*(t)$, $g(e_i(t)) = f(x_i(t)) - f(x^*(t))$, and $g(e_i(t-\tau)) = f(x_i(t-\tau)) - f(x^*(t-\tau))$ for $i\in\mathbb{S}_q$.

Hence, we can study the globally exponential synchronization of general dynamical network (8.2) by firstly investigating the synchronization of the subsystem $\mathbb{S}_p$ with irreducible coupling matrix $A_{pp}$, and secondly studying the stability of the above error subsystems (8.29) step by step. The following theorem will be given to show a synchronization criterion for complex dynamical networks with reducible coupling configuration matrix under this process.

By referring to the structure of matrix $A_{qq}$ ($q = 1, 2, \ldots, p-1$), matrix $A_{qq}$ can be decomposed uniquely as $A_{qq} = L_{qq} + D_{qq}$, where $L_{qq}$ is a zero row sum matrix and $D_{qq}$ is a diagonal matrix. Let $\xi^q = (\xi_{N_{q-1}+1}, \xi_{N_{q-1}+2}, \ldots, \xi_{N_q})^\top$ be the normalized left eigenvector of matrix $L_{qq}$ with respect to eigenvalue zero satisfying $\sum_{i\in\mathbb{S}_q} \xi_i = 1$. Denote $\Xi_q = \mathrm{diag}\{\xi_{N_{q-1}+1}, \xi_{N_{q-1}+2}, \ldots, \xi_{N_q}\}$, and $\tilde{A}_{qq} = \Xi_q A_{qq} + A_{qq}^\top \Xi_q$. Let $\beta(A_{qq}) = \lambda_{\max}(\tilde{A}_{qq})$ for $q = 1, 2, \ldots, p-1$. It has been proved in [2] and [18] that $\beta(A_{qq}) < 0$. Moreover, suppose that $\xi^p$ is the normalized left eigenvector of the configuration coupling matrix $A_{pp}$ with respect to eigenvalue 0. Let $\Xi_p = \mathrm{diag}\{\xi^p\} > 0$, and $W_{pp} = \Xi_p - \xi^p(\xi^p)^\top$.

**Theorem 8.17** *Suppose that Assumptions 8.2 and 8.4 hold. Then, complex dynamical network (8.4) with reducible coupling matrix (8.24) can be globally exponentially synchronized if there exist diagonal positive definite matrix $P$, positive definite matrix $Q$, and diagonal matrix $S \geq 0$, such that the following linear matrix inequality is satisfied:*

$$\Omega = \begin{bmatrix} -2PC - c\bar{\gamma}P\Gamma + 2L^\top SL & PB_1 & PB_2 \\ B_1^\top P & Q - 2S & 0 \\ B_2^\top P & 0 & -Q \end{bmatrix} < 0, \tag{8.30}$$

*where $\bar{\gamma} = \min\{-\alpha(A_{pp})/\lambda_{\max}(W_{pp}), -\beta(A_{p-1,p-1}), \ldots, -\beta(A_{11})\}$, function $\alpha(\cdot)$ is defined in Definition 8.6, and function $\beta(\cdot)$ is defined above.*

***Proof*** By using Theorem 8.8, it can be checked that LMI (8.30) can guarantee the globally exponential synchronization of the subsystem $\mathbb{S}_p$ with irreducible coupling matrix $A_{pp}$. Now, we are in the position to prove that LMI (8.30) can also ensure the globally exponential stability of the remaining $p-1$ error coupled subsystems (8.29).

It follows from LMI (8.30) that there exists a positive constant $\eta$ ($\eta < \epsilon$), such that

$$\tilde{\Omega} = \begin{bmatrix} \eta P - 2PC - c\bar{\gamma}P\Gamma + 2L^\top SL & PB_1 & PB_2 \\ B_1^\top P & e^{\eta\tau}Q - 2S & 0 \\ B_2^\top P & 0 & -Q \end{bmatrix} < 0. \tag{8.31}$$

For subsystem $\mathbb{S}_p$ (8.29), consider the following Lyapunov functional:

$$V(x) = e^{\eta t} \sum_{i \in \mathbb{S}_q} \xi_i e_i^\top(t) P e_i(t) + \sum_{i \in \mathbb{S}_q} \int_{t-\tau}^{t} e^{\eta(s+\tau)} \xi_i g^\top(e_i(s)) Q g(e_i(s)) ds. \tag{8.32}$$

Then, calculating the derivative of functional (8.32) along the trajectories of (8.29) gives that

$$\begin{aligned}\dot{V}(t) = e^{\eta t} \sum_{i \in \mathbb{S}_q} \xi_i \Big\{ & e_i^\top(t) \eta P e_i(t) - 2 e_i^\top(t) P C e_i(t) + 2 e_i^\top(t) P B_1 g(e_i(t)) \\ & + 2 e_i^\top(t) P B_2 g(e_i(t-\tau)) + e^{\eta\tau} g^\top(e_i(t)) Q g(e_i(t)) \\ & - g^\top(e_i(t-\tau)) Q g(e_i(t-\tau)) \Big\} \\ & + e^{\eta t} \sum_{i \in \mathbb{S}_q} \xi_i \left[ 2 e_i^\top(t) P c \sum_{j \in \mathbb{S}_q} a_{ij} \Gamma e_j(t) \right] + O(e^{-(\epsilon-\eta)t}). \end{aligned} \tag{8.33}$$

By inequality (8.10), we have

$$2 e_i^\top(t) L^\top S L e_i(t) - 2 g^\top(e_i(t)) S g(e_i(t)) \geq 0. \tag{8.34}$$

Moreover, since $\beta(A_{qq}) < 0$, $0 < \xi_i \leq 1$ for $i \in \mathbb{S}_q$, and $\beta(A_{qq}) \leq -\bar{\gamma}$, we can obtain the following inequality by some algebraic calculations:

$$\begin{aligned} & \sum_{i \in \mathbb{S}_q} \xi_i \big[ 2 e_i^\top(t) P c \sum_{j \in \mathbb{S}_q} a_{ij} \Gamma e_j(t) \big] \\ = & \sum_{i \in \mathbb{S}_q} \sum_{j \in \mathbb{S}_q} 2 c \xi_i a_{ij} e_i^\top(t) P \Gamma e_j(t) \\ = & \sum_{k=1}^{n} c p_k \gamma_k \big( e^k(t) \big)^\top \big( \Xi_q A_{qq} + A_{qq}^\top \Xi_q \big) e^k(t) \\ \leq & \, \beta(A_{qq}) \sum_{k=1}^{n} c p_k \gamma_k \big( e^k(t) \big)^\top e^k(t) \\ = & \sum_{i \in \mathbb{S}_q} c \cdot \beta(A_{qq}) e_i^\top(t) P \Gamma e_i(t) \\ \leq & - \sum_{i \in \mathbb{S}_q} \xi_i \cdot c \cdot \bar{\gamma} e_i^\top(t) P \Gamma e_i(t), \end{aligned} \tag{8.35}$$

where $e^k(t) = [e^k_{N_{q-1}+1}, e^k_{N_{q-1}+2}, \ldots, e^k_{N_q}]^\top$.

Substituting inequalities (8.34) and (8.35) into (8.33) implies that

$$\begin{aligned}\dot{V}(t) \leq\ & e^{\eta t} \sum_{i \in \mathbb{S}_q} \xi_i \Big\{ e_i^\top(t)\big[\eta P - 2PC - c\bar{\gamma} P\Gamma + 2L^\top SL\big] e_i(t) \\ & +2e_i^\top(t)PB_1 g(e_i(t)) \\ & +2e_i^\top(t)PB_2 g(e_i(t-\tau)) + e^{\eta\tau} g^\top(e_i(t)) Q g(e_i(t)) \\ & -g^\top(e_i(t-\tau)) Q g(e_i(t-\tau)) \Big\} \\ & +O(e^{-(\epsilon-\eta)t}) \\ =\ & e^{\eta t} \sum_{i \in \mathbb{S}_q} \xi_i \cdot \zeta_i^\top(t) \tilde{\Omega} \zeta_i + O(e^{-(\epsilon-\eta)t}), \end{aligned} \tag{8.36}$$

where $\zeta_i(t) = [e_i^\top(t), g^\top(e_i(t)), g^\top(e_i(t-\tau))]^\top$.

The combination of (8.31) and (8.36) implies that $V(t)$ is bounded by $V(0)$. Similar to the discussion in Theorem 8.8, one can obtain that the $q$th error system (8.28) is globally exponentially stable. In other words, the $q$th subsystem (8.28) can be globally exponentially synchronized with the foregoing $(q+1)$th, $(q+2)$th,..., $p$th subsystems at the state $x^*(t)$. Therefore, the whole general complex dynamical network (8.2) with reducible and asymmetric configuration coupling matrix $A$ in the form of (8.24) is globally exponentially synchronized. The proof is completed.

*Remark 8.18* For the synchronization of the $q$th ($q = 1, 2, \ldots, p-1$) subsystem $\mathbb{S}_q$, we can learn from the deduction of Eq. (8.28) that it has nothing to do with the concrete value of $a_{ir}$ (where $i \in \mathbb{S}_q, r \in \mathbb{S}_k$ with $k > q$) if the sum value $\sum_{\substack{r \in \mathbb{S}_k \\ k>q}} a_{ir}$ of these terms is fixed. In other words, under the assumptions that the sum $\sum_{\substack{r \in \mathbb{S}_k \\ k>q}} a_{ir}$ is fixed and the remaining entries of matrix $A$ are also fixed, the synchronization condition of subsystem $\mathbb{S}_q$ (8.28) will remain no matter how $a_{ir}$ changes. From Theorem 8.17, we can also observe that the synchronization criterion of subsystem $\mathbb{S}_q$ is only related to the matrices $A_{kk}$ ($k = 1, 2, \ldots, p$) and the dynamical behavior of the isolated systems.

*Remark 8.19* The complex networks in the real-world could be very large, hence the LMU obtained in this chapter may also be very big. However, it can be efficiently solved by using the LMI Toolbox in Matlab, since the LMI solvers used in the LMI Lab is based on the interior-point optimization techniques, which has been developed as a powerful tool for solving very large-scale linear programming problem [15].

## 8.3 Numerical Examples

In this section, two numerical examples including small-world and scale-free networks will be given to illustrate the validity of our theoretical results for large-scale networks. In the first example, we consider the small-world coupled dynamical network. A complex dynamical network with scale-free coupling structure will be simulated as the second example. In both examples, a chaotic neural network is chosen as the isolated node of the network, which can be described by the following equation [19]:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + I(t), \tag{8.37}$$

with $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 1+\frac{\pi}{4} & 20 \\ 0.1 & 1+\frac{\pi}{4} \end{pmatrix}$, $B = \begin{pmatrix} -\frac{1.3\pi\sqrt{2}}{4} & 0.1 \\ 0.1 & -\frac{1.3\pi\sqrt{2}}{4} \end{pmatrix}$, $I(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $\tau = 0.9$, where $x(t) = (x_1(t), x_2(t))^\top$ is the state vector of the isolated node in the coupled network. Activation functions $f_i(s) = \frac{1}{2}(|s+1| - |s-1|)$ for $i = 1, 2$, and hence Assumption 8.2 is satisfied with $L = \text{diag}\{1, 1\}$. This neural network model (8.37) is chaotic as shown in Fig. 8.1, with initial values $x_1(s) = 3,\ \ x_2(s) = -1,\ \forall s \in [-0.9, 0]$.

The complex dynamical network model (8.2) is used in the following two examples. The inner coupling matrix $\Gamma$ and coupling strength $c$ are, respectively, chosen as $\Gamma = \text{diag}\{2, 1\}$ and $c = 1$. Since the coupling configuration matrix is assumed to be asymmetric in the numerical examples, the results in [3–6] cannot be used to judge whether the synchronization of the dynamical network can be achieved. Throughout the following simulations, a normal PC with Intel Core 2 Quad Q6600 @ 2.40 GHz and 1.96 GB memory is used.

*Example 8.20* In this example, we consider small-world networks. When the small-world networks are generated, the coupling strength $a_{ij}$ for each edge is defined as follows: if there is a connection from node $j$ to node $i$ ($i \neq j$), then $a_{ij} = 1$; otherwise, $a_{ij} = 0$.

By Theorems 8.8 and 8.13, the quantity $\gamma$ obtained from the coupling matrix $A$ can be used to determine the synchronizability of dynamical networks. The larger the $\gamma$ is, the better is the synchrony of the dynamical network. In the upper sub-figures of Figs. 8.2 and 8.3, we represent the evolution of the quantity $\gamma$ as a function of the number of nodes $N$ and adding probability $p$. Each $\gamma$ is obtained by averaging the results of 10 runs. Due to the randomness of small-world networks, the standard deviation of each 10-run is also plotted in Figs. 8.2 and 8.3 to show that obtained data are centralized and reasonable. As the average degree is increased, the quantity $\gamma$ is observed to increase for both undirected and directed networks, which further implies enhancement of synchronizability. Moreover, the results shown in Figs. 8.2 and 8.3 imply that the synchronizability of directed small-world networks is better than that of undirected ones.

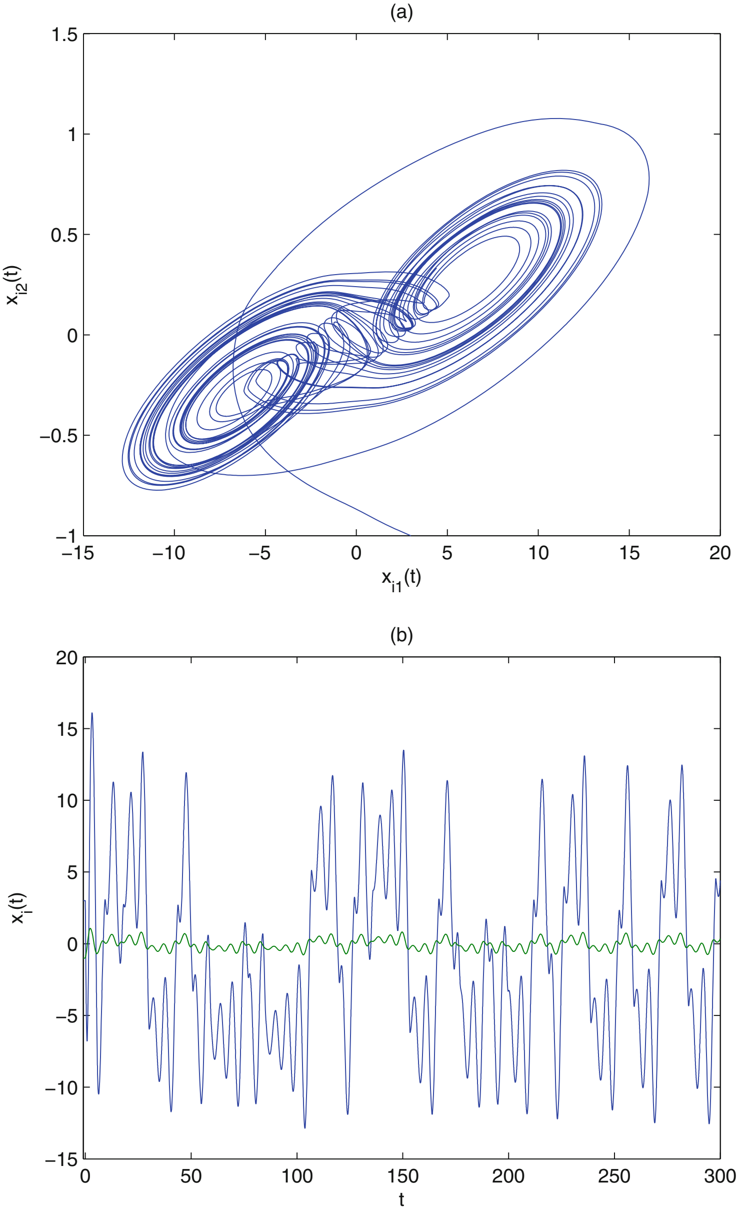



**Fig. 8.1** The state of neural network model (8.37): (**a**) Chaotic attractor; (**b**) Time series

The calculation time of $\gamma$, which determines the availability of our criteria for large-scale networks, is of great importance. Hence, average running time (seconds) for each $\gamma$ is also plotted in the lower sub-figures of Figs. 8.2 and 8.3. Running time of undirected networks is much smaller than that of directed networks. This is

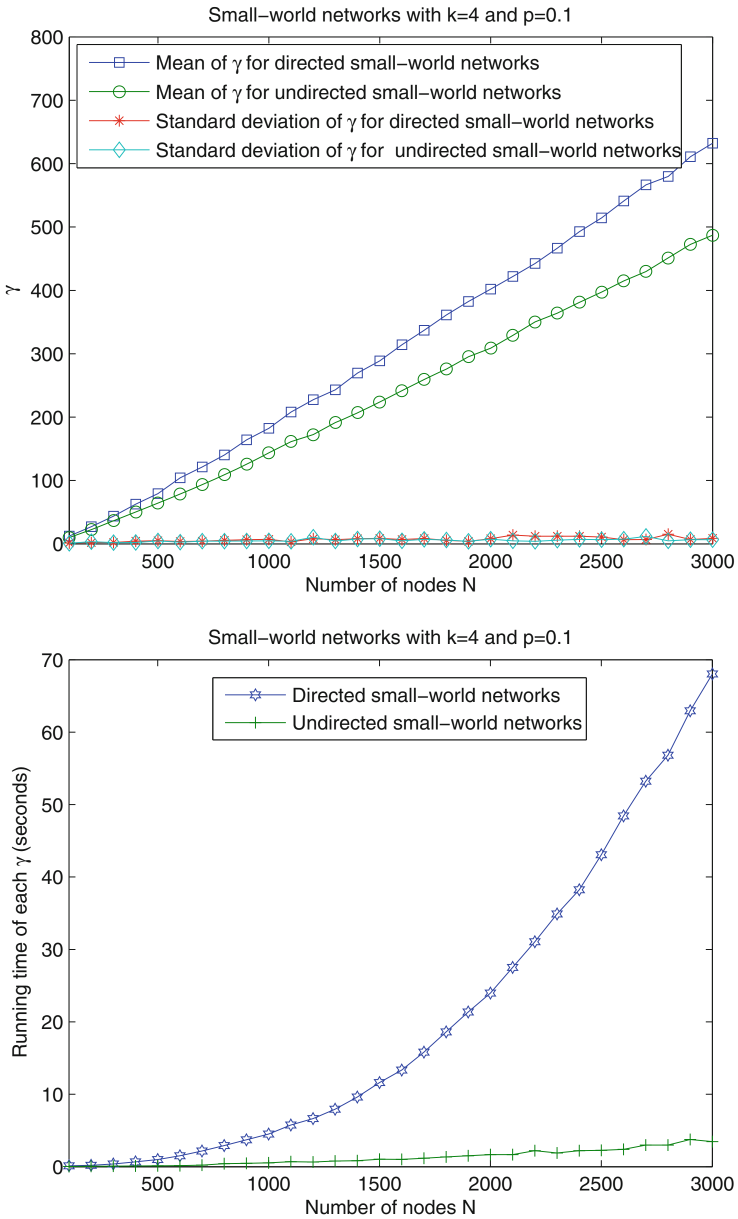
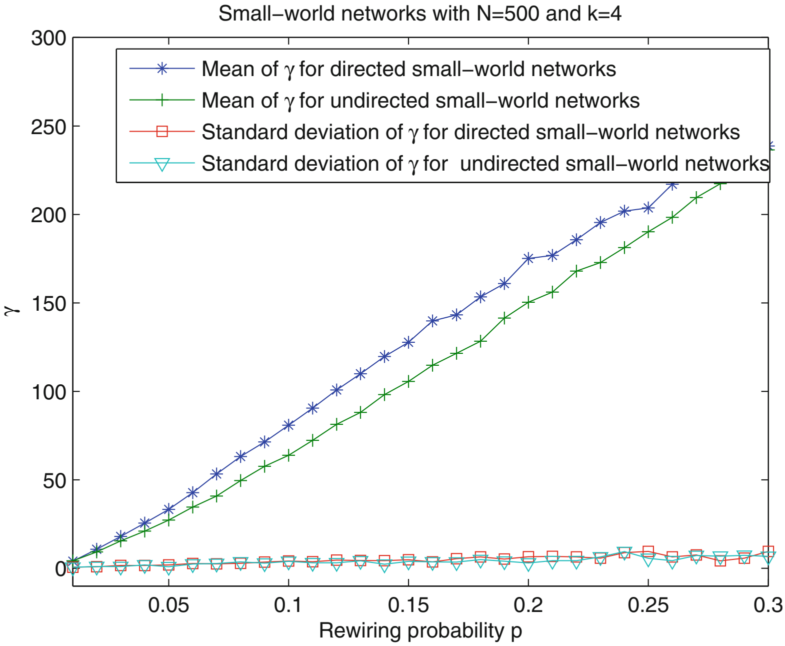



**Fig. 8.2** (Up). $\gamma$ versus Number of nodes; (Down). Program running time (seconds) versus Number of nodes



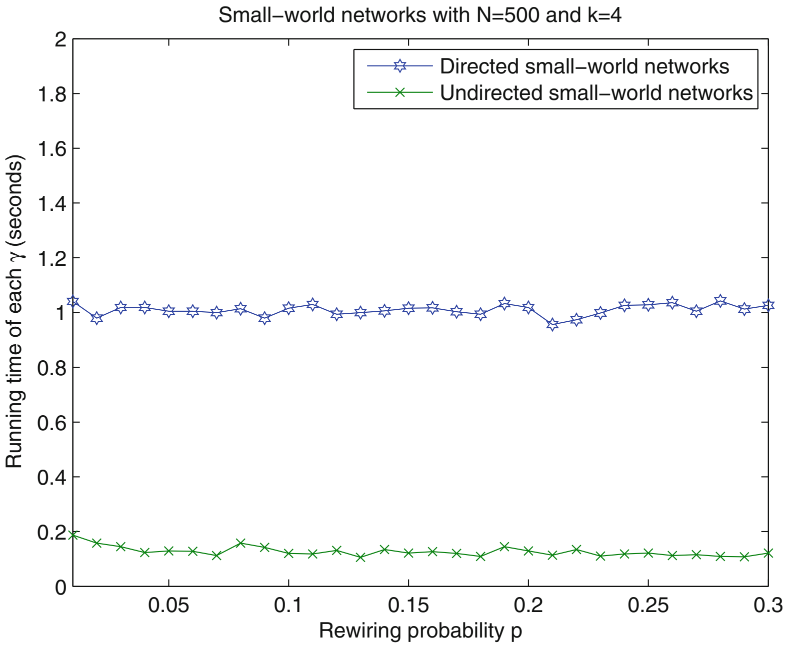




**Fig. 8.3** (Up). $\gamma$ versus Rewiring probability; (Down). Program running time (seconds) versus Rewiring probability

because $\gamma$ of undirected network is only determined by the second largest eigenvalue and the left eigenvector $\xi$, hence the undirected network does not need to be numerically solved. As we can observe, it takes only less than 70ṡ to generate a 3000-node directed small-world network and calculate the corresponding $\gamma$. By substituting the obtained $\gamma$ to the sufficient synchronization criteria derived in Sect. 8.2, we can judge whether the dynamical network can be synchronized. It means that our criteria are available for large-scale networks.

By constructing a directed 500 nodes small-world network with $k = 4$ and adding probability 0.02, we obtain that $\gamma = 11.3646$. Further, by using Matlab LMI Toolbox, we can find a feasible solution to the LMI (8.5) as follows: $P = \text{diag}\{0.1367, 0.8836\}$, $Q = \begin{bmatrix} 0.8923 & -0.2068 \\ -0.2068 & 1.9342 \end{bmatrix}$, and $S = \text{diag}\{0.7569, 4.2246\}$. Therefore, according to Theorem 8.8, we can conclude that the complex dynamical network with small-world coupling is globally exponentially synchronized. Figure 8.4 shows synchronization behavior of the directed small-world dynamical network, initial values of which are pseudo-random numbers uniformly distributed in [-10, 10].

*Example 8.21* BA Scale-free network is taken into account in this example [20]. The coupling matrix $A = [a_{ij}]$ for generated scale-free network is defined as follows: if there is a connection from node $j$ to node $i$ ($i \neq j$), then $a_{ij}$ will

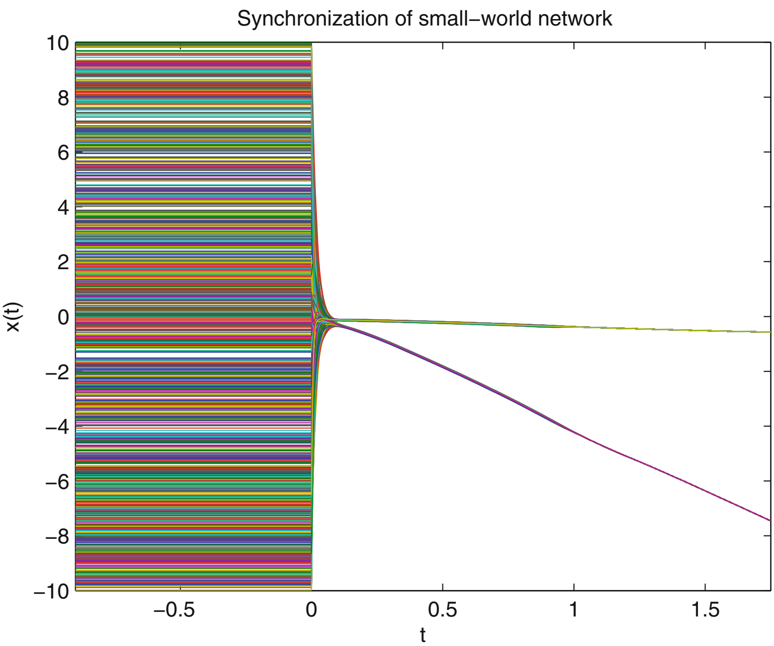



**Fig. 8.4** Synchronization of 500-node small-world network with rewiring probability 0.02

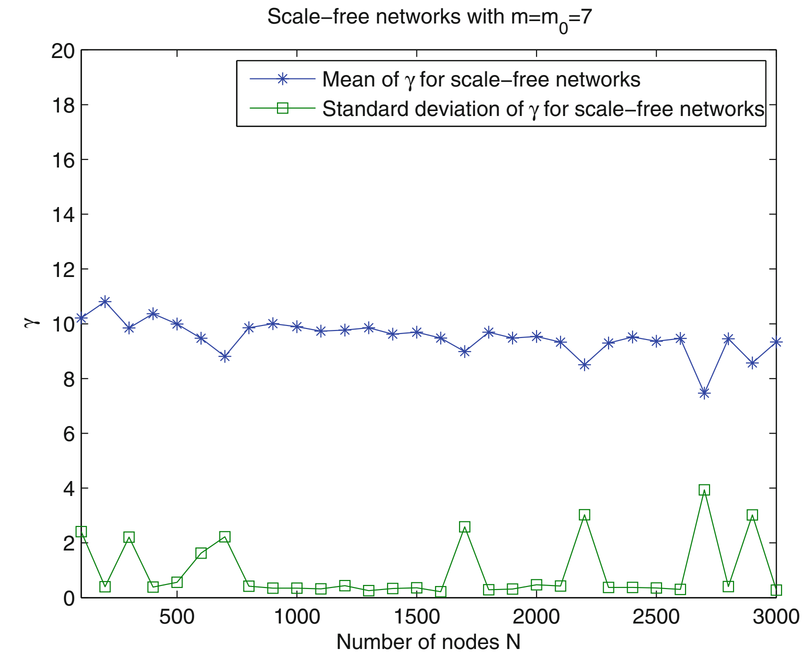
be randomly assigned a number which is uniformly distributed between 1 and 2; otherwise, $a_{ij} = 0$. Of course, the asymmetric matrix $A$ satisfies the diffusive coupling condition (8.3).

In Figs. 8.5, 8.6, and 8.7, each $\gamma$ is obtained by averaging the results of 10 runs. Due to the randomness of scale-free networks, the standard deviation of each 10-run is also plotted to show that obtained data are centralized and reasonable. In Fig. 8.5, we represent the quantity $\gamma$ of the scale-free network and corresponding running time as a function of size of network $N$. It can be observed that for BA scale-free networks, the quantity $\gamma$ is almost free from the scale of the network when average degree (m) is fixed. In Fig. 8.6, the quantity $\gamma$ is plotted against the number of edges added each time step m (equal to average degree). One can observe that synchronizability of BA scale-free network is improved as m (i.e., average degree) increases. Figure 8.7 is given to compare synchronizability of small-world and scale-free networks under the constraint of identical average degree. The results shown in Fig. 8.7 lead to the conclusion that synchronization of the dynamical network is enhanced as the heterogeneity of the degree distribution is decreased. This argument is consistent with that of Ref. [21].

With the selection of $N = 500$ and $m = m_0 = 7$, a scale-free network can be generated. By calculation, we obtain that $\gamma = 10.6640$. Then, a feasible solution to linear matrix inequality (8.5) can be obtained as follows by referring to Matlab LMI Toolbox: $P = \text{diag}\{3.5971, 27.0464\}$, $Q = \begin{bmatrix} 23.8274 & -5.8084 \\ -5.8084 & 58.0917 \end{bmatrix}$, and $S = \text{diag}\{19.5356,\ 120.7142\}$. Hence, it can be concluded from Theorem 8.8 that the scale-free coupled dynamical network can be globally exponentially synchronized. By taking uniformly distributed pseudo-random numbers in $[-10, 10]$, Fig. 8.8 displays the synchronization behavior of the directed scale-free dynamical network.

## 8.4 Summary

In this chapter, we study the exponential synchronization behavior of a complex dynamical network. One quantity is distilled from the coupling matrix to characterize the synchronizability of corresponding dynamical networks. The calculation of such a quantity is very convenient even for large-scale networks. The coupling configuration matrix is not assumed to be symmetric or/and irreducible. Some sufficient conditions are proposed to guarantee the globally exponential synchronization of the network by introducing the left eigenvector to the construction of Lyapunov functional. The criteria obtained in this chapter are expressed in terms of LMIs, which can be checked effectively by resorting to recently developed algorithms. In addition, two numerical examples including small-world and scale-free networks are given to demonstrate that our theoretical results are available for large-scale networks.



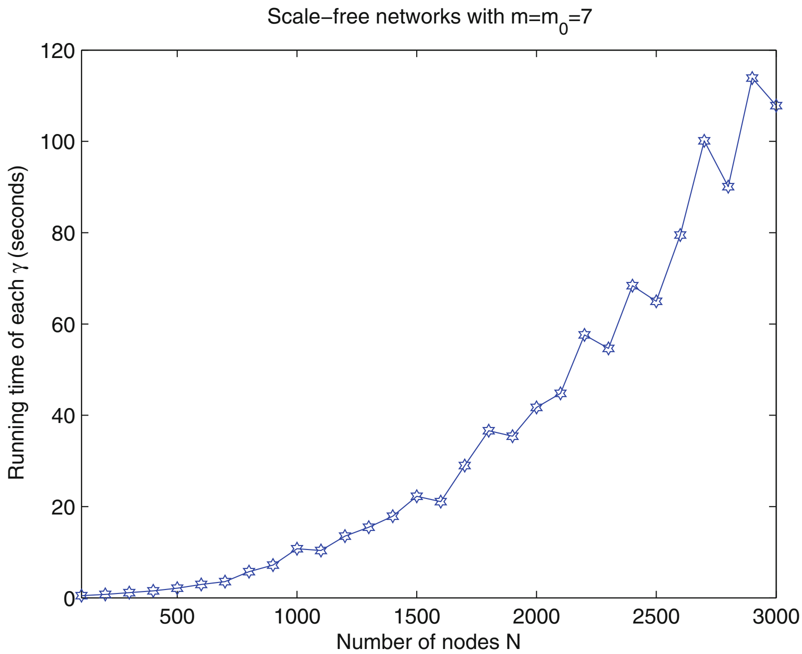




**Fig. 8.5** (Up). $\gamma$ versus Number of nodes; (Down). Program running time (seconds) versus Number of nodes

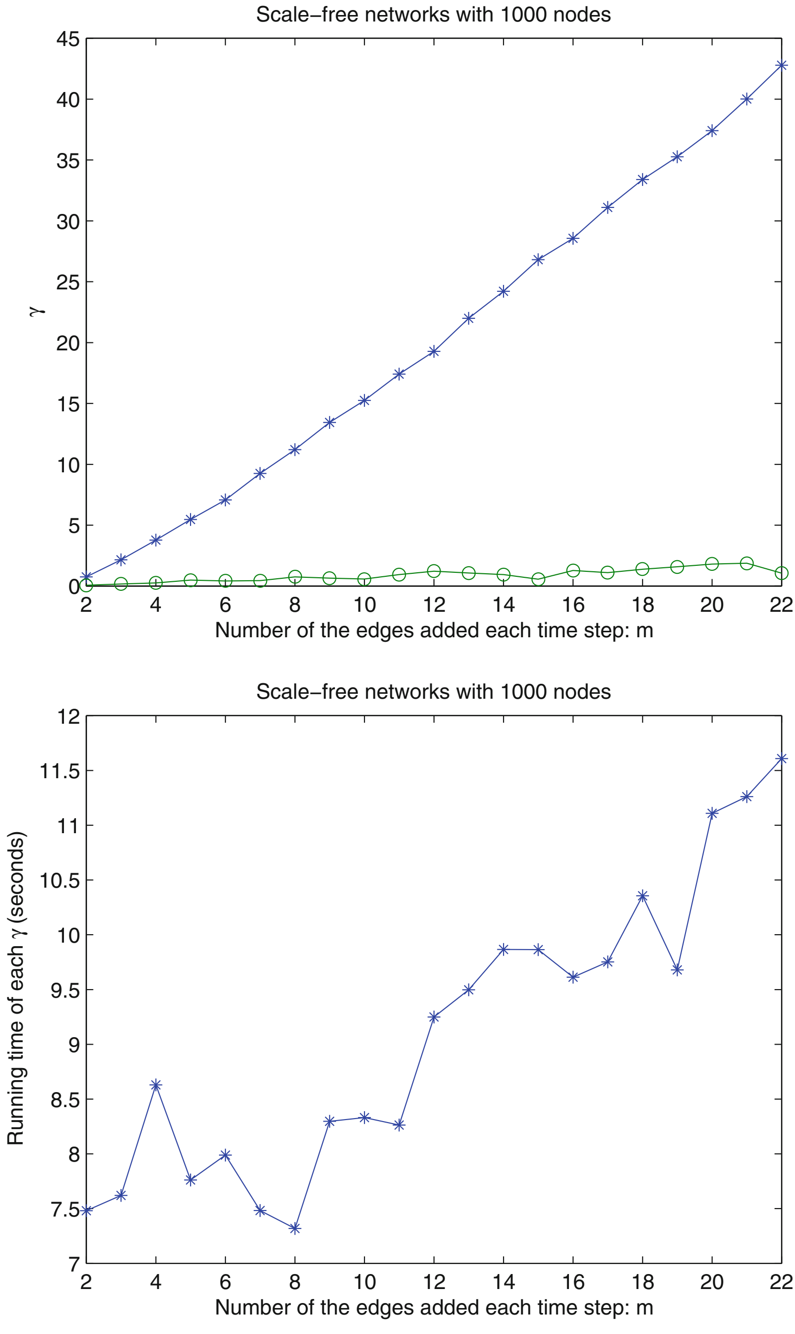
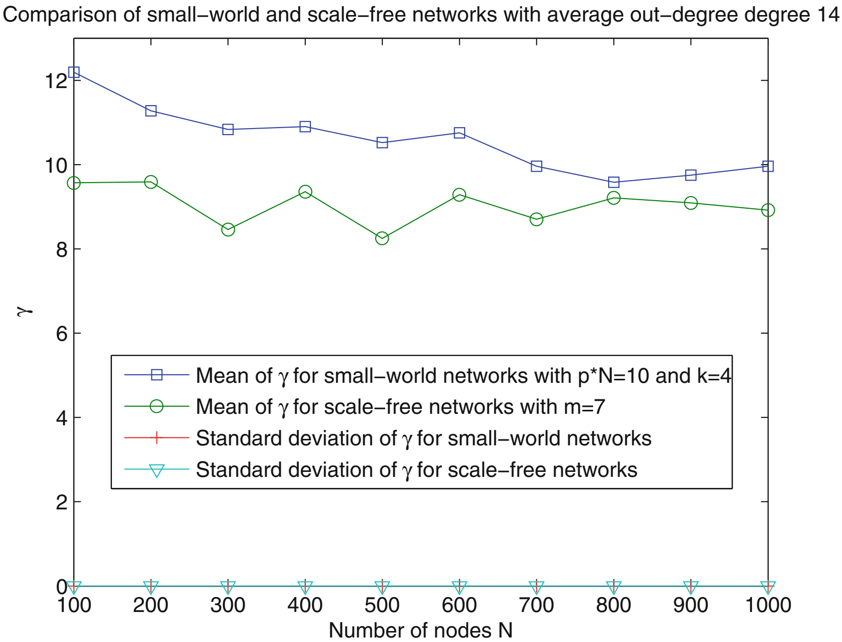



**Fig. 8.6** (Up). $\gamma$ versus m (Number of edges added each time); (Down). Program running time (seconds) versus m (Number of edges added each time)



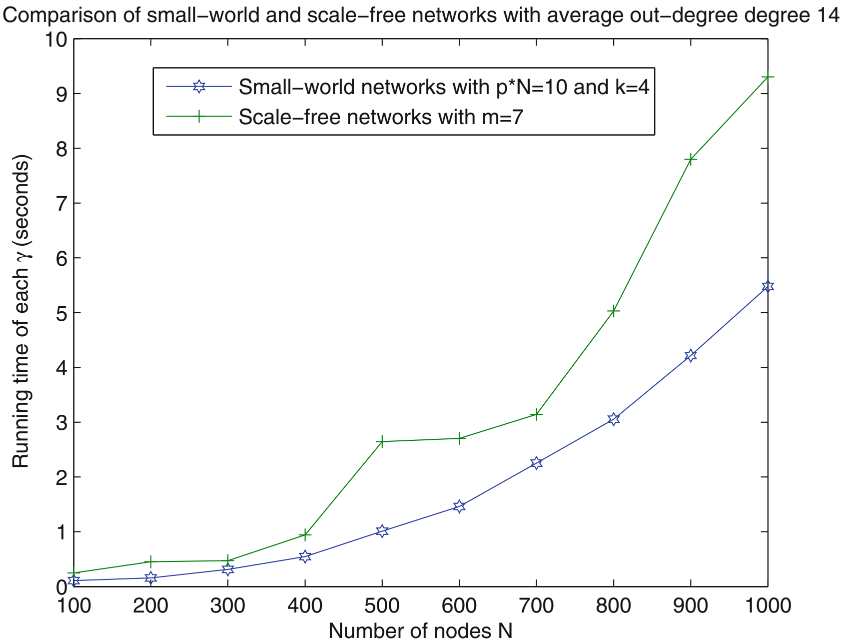




**Fig. 8.7** Comparison of small-world networks and scale-free networks with fixed average degree 14: (Up). $\gamma$ versus Number of nodes; (Down). Program running time (seconds) versus Number of nodes

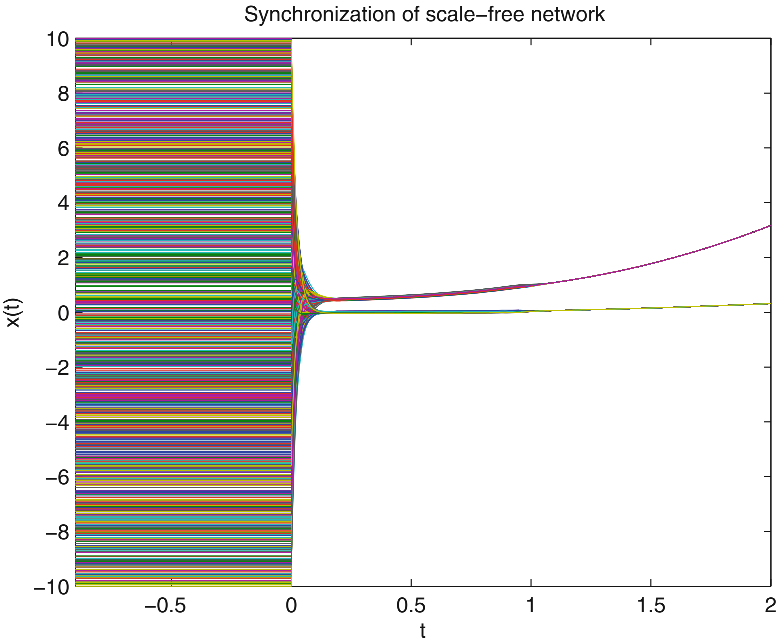



**Fig. 8.8** Synchronization of 500-node scale-free network with $m = m_0 = 7$

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# Chapter 9
# Pinning Cluster Synchronization in an Array of Coupled Neural Networks Under Event-Based Mechanism

As a special kind of synchronization phenomenon, cluster synchronization means that the synchronization occurs within one group, but there is no synchronization among different groups. Recently, cluster synchronization of complex networks has received increasing attention because of its applications in biological science and communication engineering [1–4]. In [5], different mechanisms (identical and non-identical self-dynamics) leading to cluster synchronization in diffusively coupled networks were discussed in detail. In [6], a pinning control algorithm was proposed to achieve leader-following consensus in a network of agents with nonlinear second-order dynamics. In [7], pinning control strategies were proposed to steer a dynamical network to an expected cluster synchronization pattern. The study of cluster synchronization phenomena is significant to the theoretical research on brain science and related practical applications [3, 8, 9]. Hence, in this chapter, cluster synchronization problem of coupled neural networks will be studied. In the past few years, some works have been devoted to studying cluster synchronization of coupled neural networks [3, 8]. In [3], by constructing a special coupling matrix, several sufficient criteria for cluster synchronization in an array of coupled neural networks were derived. The main results of [3] were further extended to the stochastic delayed neural networks in [8]. It should be noted that all previous mentioned works were based on continuous-time state information transmission.

In networked environment, successful and efficient communication among nodes is the key factor for dynamical systems to achieve desired collective behaviors. Due to the limited bandwidth of the communication channel among the nodes, it is necessary to save energy as much as possible. Recently, a novel communication protocol, namely event-triggered control (see, e.g., [10–13]), was developed to provide an effective methodology that satisfies the energy constraints of the system. Collective behaviors (e.g., synchronization and consensus) of complex

 
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network under event-triggered communication mechanism have become a hot research subject [14–17]. Some interesting works on synchronization of complex networks under event-triggered communication strategy were shown in [14, 18, 19]. Effective event-triggered conditions were designed to achieve the synchronization of the considered network models. In [15], state estimation of a class of complex networks was investigated under event-based information transmission. In [16], a new distributed event-triggered mechanism for pinning control synchronization of complex networks was presented. Unfortunately, the exclusion of Zeno behavior was not strictly proved in [16].

Motivated by above statements, we aim to study the pinning cluster synchronization of coupled neural networks by a novel event-triggered mechanism in this chapter. Under event-triggered mechanism, some controllers will be pinned to certain selected nodes in coupled neural networks to realize expected cluster synchronization. The main difficulties of this chapter are how to propose distributed event-triggered schemes to realize expected cluster synchronization and meanwhile exclude the Zeno behavior.

## 9.1 Preliminaries and Problem Formulation

In this subsection, we will first give some basic definitions and lemmas and then present the coupled neural networks model under event-triggered mechanism.

**Definition 9.1** A network with $N$ nodes is said to realize cluster synchronization, if the $N$ nodes are split into several clusters $G_1, G_2, \ldots, G_k$, such as $\{G_1 = (m_0+1, m_0+2, \ldots, m_1),\ G_2 = (m_1+1,\ m_1+2, \ldots,\ m_2), \ldots,\ G_k = (m_{k-1}+1,\ m_{k-1}+2, \ldots, m_k),\ m_0 = 0,\ m_k = N,\ m_{j-1} < m_j,\ j = 1,\ 2, \ldots, k\}$ such that

- the nodes in the same cluster synchronize with each other, i.e., for the states $x_i(t)$ and $x_j(t)$ of arbitrary nodes $i$ and $j$ in the same cluster, $\lim_{t\to+\infty} \|x_i(t)-x_j(t)\| = 0$ holds;
- the nodes in the different clusters do not synchronize, i.e., for the states $x_i(t)$ and $x_j(t)$ of arbitrary nodes $i$ and $j$ in different clusters, $\lim_{t\to+\infty} \|x_i(t) - x_j(t)\| \neq 0$ holds.

**Definition 9.2 ([7])** Consider $A = (a_{ij}) \in \mathbb{R}^{N\times N}$. If

- $a_{ij} \geq 0$, for $i \neq j$, and $a_{ii} = -\sum_{j=1,\, j\neq i}^{N} a_{ij} = -\sum_{j=1,\, j\neq i}^{N} a_{ji}$, $i = 1,\ 2, \ldots,\ N$;
- $A$ is irreducible.

Then we say $A \in \mathbf{A}_1$.

**Definition 9.3 ([7])** For a $N \times N$ matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix}, \tag{9.1}$$

with $A_{ii} \in \mathbb{R}^{(m_i - m_{i-1}) \times (m_i - m_{i-1})}$, $A_{ij} \in \mathbb{R}^{(m_i - m_{i-1}) \times (m_j - m_{j-1})}$, $i, j = 1, 2, \ldots, k$, if each block $A_{ij}$ is a zero row sum matrix, then we say $A \in M_1(k)$. Furthermore, if $A_{ii} \in \mathbf{A}_1$, $i = 1, 2, \ldots, k$, then we say $A \in M_2(k)$.

**Lemma 9.4 ([7])** *For a matrix $B \in \mathbb{R}^{p \times q}$, denote $\alpha(B) = \frac{1}{2} max\{p, q\} \cdot max_{i,j}\{|b_{ij}|\}$, then*

$$x^\top B y \leq \alpha(B)\big(x^\top x + y^\top y\big) \tag{9.2}$$

*holds for all $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$.*

In this chapter, the following assumptions are imposed.

**Assumption 9.5** *Function $f(\cdot)$ is Lipschitz continuous, i.e., there exists a positive constant L, such that*

$$||f(x) - f(y)|| \leq L||x - y|| \quad \forall x, y \in \mathbb{R}^n. \tag{9.3}$$

**Assumption 9.6** *Coupling matrix A satisfies $A \in M_2(k)$.*

The dynamic of linearly coupled neural networks under pinning control can be described as:

$$\frac{dx_i(t)}{dt} = -Cx_i(t) + Bf(x_i(t)) + I(t) + \sum_{j=1}^{N} a_{ij} \Gamma x_j(t) + u_i(t), \; i \in \mathcal{N}, \tag{9.4}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^\top \in \mathbb{R}^n$ is the state vector of the $i$th network at time $t$; $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$, with $c_i > 0$ denoting the rate with which the cell $i$ resets its potential to the resting when being isolated from other cells and inputs; $B = (b_{rj})_{n \times n}$ represents the connection weight matrix; $\Gamma$ represents the inner-coupling matrix; for simplicity, we assume that $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ in this chapter; $f$: $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the activation function with $f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t)))^\top$; $I = (I_1(t), I_2(t), \ldots, I_n(t))^\top \in \mathbb{R}^n$ is an external input vector; $u_i(t)$ is the pinning controller which will be given in the following part; $A = (a_{rj})_{n \times n}$ is the matrix representing the coupling structure of the coupled neural networks. It can be observed that cooperative and competitive couplings, which can characterize the real world better, are adopted in this chapter to describe the topology structure of the coupled neural networks.

In this chapter, the event-triggered information transmission mechanism will be adopted to investigate the cluster synchronization problem of coupled neural networks (9.4). Suppose that $t^i_{k_1}, t^i_{k_2}, \ldots, t^i_{k_d}, \ldots,$ is the sequence of the event times instants of the agent $i$ which will be defined based on the event-triggering condition. At time $t^i_{k_d}$, $d \in \mathbb{N}$, node $i$ and its neighboring nodes measure their own states, respectively, denoted by $x_i(t^i_{k_d})$ and $x_j(t^i_{k_d}),\ j \in \mathcal{N}_i$.

Suppose we want to control network (9.4) onto certain desired inhomogeneous state defined by $x_{m_0+1}(t), \ldots, x_{m_1}(t) \rightarrow s_1(t), x_{m_1+1}(t), \ldots, x_{m_2}(t) \rightarrow s_2(t), \ldots, x_{m_{k-1}+1}(t), \ldots, x_{m_k}(t) \rightarrow s_k(t)$. That is, $\mathcal{M} = (s_1(t), \ldots, s_1(t), \ldots, s_k(t), \ldots s_k(t)) \subset \mathbb{R}^{n\times N}$ is the desired cluster synchronization pattern under pinning control. Moreover, $s_l(t)$ can be an equilibrium point, a periodic orbit, or a chaotic attractor in the phase space satisfying

$$\dot{s}_l(t) = -Cs_l(t) + Bf(s_l(t)) + I(t), \quad l = 1, \ldots, k. \tag{9.5}$$

Without loss of generality, to achieve the goal of cluster synchronization, we apply the pinning control strategy on the nodes set $J = \{m_1, m_2, \ldots, m_k\}$. Hence, the following neural network model with event-triggered diffusive coupling will be studied in this chapter: for $t \in [t^i_{k_d}, t^i_{k_{d+1}})$,

$$\begin{cases} \dot{x}_i(t) = -Cx_i(t) + Bf(x_i(t)) + I(t) + \sum_{j=1}^{N} a_{ij}\Gamma x_j\left(t^i_{k_d}\right) \\ \qquad\qquad -\epsilon_l \Gamma(x_i\left(t^i_{k_d}\right) - s_l\left(t^i_{k_d}\right)), \quad i = m_l, \ l = 1, 2, \ldots, k, \\ \dot{x}_i(t) = -Cx_i(t) + Bf(x_i(t)) + I(t) + \sum_{j=1}^{N} a_{ij}\Gamma x_j\left(t^i_{k_d}\right), \quad i \neq m_l, \end{cases} \tag{9.6}$$

where $\epsilon_l > 0$, and $t^i_{k_d}$ is determined by the event-triggered condition which will be given in the following theorem. Define the error variables as

$$\delta x_i(t) = x_i(t) - s_l(t), \quad l = 1, 2, \ldots, k; \quad i = m_{l-1} + 1, \ldots, m_l.$$

The coupled neural networks (9.6) are said to realize expected cluster synchronization pattern if and only if

$$\lim_{t\to+\infty} \sum_{l=1}^{k} \sum_{i=m_{l-1}+1}^{m_l} \|x_i(t) - s_l(t)\| = 0.$$

*Remark 9.7* Compared with the previous work on cluster synchronization of coupled neural networks [3, 8, 20, 21], a new mechanism based on event-triggered sampling information is proposed in this chapter to realize the expected cluster synchronization. Moreover, we do not require the coupling matrix to be symmetric, and it makes our network model more realistic.

## 9.2 Pinning Cluster Synchronization Under Event-Triggered Mechanism

In this subsection, we will investigate the cluster synchronization of the considered network (9.6) under event-triggered mechanism. For $t \in [t^i_{k_d},\ t^i_{k_{d+1}})$, let

$$\kappa_i(t) = \begin{cases} \sum_{j=1}^{N} a_{ij}\Gamma\big(x_j\big(t^i_{k_d}\big) - x_j(t)\big) - \epsilon_l \Gamma\big(x_i\big(t^i_{k_d}\big) - s_l\big(t^i_{k_d}\big) - x_i(t) + s_l(t)\big), \\ \quad \text{if } i = m_l,\ l = 1,\ 2, \ldots, k, \\ \sum_{j=1}^{N} a_{ij}\Gamma\big(x_j\big(t^i_{k_d}\big) - x_j(t)\big), \ \ \text{if } i \neq m_l,\ l = 1,\ 2, \ldots, k. \end{cases}$$

We will use $\kappa_i(t)$ to give the event-triggered condition in the following theorem.

**Theorem 9.8** *Consider network (9.6) under event-triggered mechanism. Suppose Assumptions 9.5–9.6 are satisfied and there exists diagonal positive definite matrix $P$ such that*

$$\mathbf{H} + \mathbf{A} - \mathbf{\Xi} < 0, \tag{9.7}$$

*where $\mathbf{A} = \widetilde{A} \otimes (P\Gamma)$ with $\widetilde{A} = \frac{1}{2}(A + A^\top)$, $\mathbf{\Xi} = \Xi \otimes (P\Gamma)$ with $\Xi = diag\{0, \ldots, \epsilon_1, \ldots, 0, \ldots, \epsilon_k\}$, $\mathbf{H} = I_N \otimes H$ with $H = -PC + (\frac{\beta_1}{2}\lambda_{max}(PBB^\top P) + \frac{1}{2\beta_1}L^2)I_n + \frac{\beta_1}{2}P\Gamma\Gamma P$, and arbitrary positive constant $\beta_1$. Then, under the event-triggered condition*

$$||\kappa_i(t)|| > \beta exp(-\gamma t), \tag{9.8}$$

*where $\beta > 0$ is a constant which can be chosen arbitrarily, and $0 < \gamma < \dfrac{\lambda_{max}(\mathbf{H} + \mathbf{A} - \mathbf{\Xi})}{-2\lambda_{max}(P)}$. The network will achieve the desired cluster synchronization exponentially. Moreover, the difference of the inter-event time instant for each node is lower bounded by a positive common instant.*

***Proof*** Since $A \in M_1(k)$, we have

$$\begin{aligned} \sum_{j=1}^{N} a_{ij}\Gamma x_j\big(t^i_{k_d}\big) &= \sum_{l=1}^{k}\sum_{j=m_{l-1}+1}^{m_l} a_{ij}\Gamma\Big(x_j\big(t^i_{k_d}\big) - s_l\big(t^i_{k_d}\big) + s_l\big(t^i_{k_d}\big)\Big) \\ &= \sum_{l=1}^{k}\sum_{j=m_{l-1}+1}^{m_l} a_{ij}\Gamma\delta x_j\big(t^i_{k_d}\big) + \sum_{l=1}^{k}\sum_{j=m_{l-1}+1}^{m_l} a_{ij}\Gamma s_l\big(t^i_{k_d}\big) \\ &= \sum_{j=1}^{N} a_{ij}\Gamma\delta x_j\big(t^i_{k_d}\big) \end{aligned} \tag{9.9}$$

Therefore, we can obtain the error system as follows:

$$\begin{cases} \delta \dot{x}_i(t) = -C(x_i(t) - s_l(t)) + B(f(x_i(t)) - f(s_l(t))) + \sum_{j=1}^{N} a_{ij}\Gamma \delta x_j\big(t_{k_d}^i\big) \\ \qquad\quad - \epsilon_l \Gamma \delta x_i\big(t_{k_d}^i\big), \qquad\qquad i = m_l,\ l = 1,\ 2, \ldots,\ k, \\ \delta \dot{x}_i(t) = -C(x_i(t) - s_l(t)) + B(f(x_i(t)) - f(s_l(t))) \\ \qquad\quad + \sum_{j=1}^{N} a_{ij}\Gamma \delta x_j\big(t_{k_d}^i\big),\ \ i \neq m_l. \end{cases} \tag{9.10}$$

Define the Lyapunov function as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} \delta x_i^{\top}(t) P \delta x_i(t), \tag{9.11}$$

where $P$ is a positive diagonal matrix.

Differentiating the function $V(t)$ along the trajectories of system (9.10), we have

$$\begin{aligned}
&\dot{V}(t) \\
&= \sum_{l=1}^{k}\sum_{i=m_l} \delta x_i^{\top}(t) P[-C(x_i(t) - s_l(t)) + Bf(x_i(t)) - Bf(s_l(t)) \\
&\quad + \sum_{j=1}^{N} a_{ij}\Gamma \delta x_j\big(t_{k_d}^i\big) - \epsilon_l \Gamma \delta x_i\big(t_{k_d}^i\big)\big] + \sum_{l=1}^{k}\sum_{i=m_{l-1}+1}^{m_l-1} \delta x_i^{\top}(t) P\big[-C(x_i(t) \\
&\quad - s_l(t)) + Bf(x_i(t)) - Bf(s_l(t)) + \sum_{j=1}^{N} a_{ij}\Gamma \delta x_j\big(t_{k_d}^i\big)\big] \\
&= \sum_{i=1}^{N} \delta x_i^{\top}(t) P\big[-C\delta x_i(t) + B(f(x_i(t)) - Bf(s_l(t))) + \sum_{j=1}^{N} a_{ij}\Gamma x_j\big(t_{k_d}^i\big)\big] \\
&\quad - \epsilon_l \sum_{l=1}^{k}\sum_{i=m_l} \delta x_i^{\top}(t) P \Gamma \delta x_i\big(t_{k_d}^i\big) \\
&= -\sum_{i=1}^{N} \delta x_i^{\top}(t) P C \delta x_i(t) + \sum_{i=1}^{N} \delta x_i^{\top}(t) P B(f(x_i(t)) - f(s_l(t))) \\
&\quad + \sum_{l=1}^{k}\sum_{i=m_{l-1}+1}^{m_l-1} \delta x_i^{\top}(t) P \Gamma \sum_{j=1}^{N} a_{ij}(x_j\big(t_{k_d}^i\big) - x_j(t)) + \sum_{l=1}^{k}\sum_{i=m_l} \delta x_i^{\top}(t) P \Gamma
\end{aligned}$$

$$\times \left[ \sum_{j=1}^{N} a_{ij}(x_j(t_{k_d}^i) - x_j(t)) - \epsilon_l (x_i(t_{k_d}^i) - s_l(t_{k_d}^i) - x_i(t) + s_l(t)) \right]$$

$$+ \sum_{i=1}^{N} \delta x_i^\top(t) P \Gamma \sum_{j=1}^{N} a_{ij} x_j(t) - \sum_{l=1}^{k} \sum_{i=m_l} \epsilon_l \delta x_i^\top(t) P \Gamma \delta x_i(t). \tag{9.12}$$

It follows from Assumption 9.5 that

$$\begin{aligned} &\delta x_i^\top(t) P B(f(x_i(t)) - f(s_l(t))) \\ \leq &\frac{\beta_1}{2} \delta x_i^\top(t) P B B^\top P \delta x_i(t) + \frac{1}{2\beta_1}(f(x_i(t)) \\ &- f(s_l(t)))^\top (f(x_i(t)) - f(s_l(t))) \\ \leq &\frac{\beta_1}{2} \lambda_{\max}(P B B^\top P) \delta x_i^\top(t) \delta x_i(t) \\ &+ \frac{1}{2\beta_1} L^2 \delta x_i^\top(t) \delta x_i(t), \end{aligned} \tag{9.13}$$

where $\beta_1$ is a positive constant, which can be chosen arbitrarily. Furthermore, one can obtain

$$\begin{aligned} &\delta x_i^\top(t) P \left[ \sum_{j=1}^{N} a_{ij} \Gamma(x_j(t_{k_d}^i) - x_j(t)) - \epsilon_l \Gamma(x_i(t_{k_d}^i) - s_l(t_{k_d}^i) - x_i(t) + s_l(t)) \right] \\ \leq &\frac{\beta_1}{2} \delta x_i^\top(t) P^\top P \delta x_i(t) + \frac{1}{2\beta_1} \left[ \sum_{j=1}^{N} a_{ij} \Gamma(x_j(t_{k_d}^i) - x_j(t)) - \epsilon_l \Gamma(x_i(t_{k_d}^i) \right. \\ &\left. - s_l(t_{k_d}^i) - x_i(t) + s_l(t)) \right]^\top \left[ \sum_{j=1}^{N} a_{ij} \Gamma(x_j(t_{k_d}^i) - x_j(t)) - \epsilon_l \Gamma(x_i(t_{k_d}^i) \right. \\ &\left. - s_l(t_{k_d}^i) - x_i(t) + s_l(t)) \right], \end{aligned} \tag{9.14}$$

and

$$\begin{aligned}
&\delta x_i^\top(t)P\sum_{j=1}^{N}a_{ij}\Gamma\big(x_j\big(t_{k_d}^i\big)-x_j(t)\big)\\
&\le \frac{\beta_1}{2}\delta x_i^\top(t)P^\top P\delta x_i(t)+\frac{1}{2\beta_1}\left[\sum_{j=1}^{N}a_{ij}\Gamma\big(x_j\big(t_{k_d}^i\big)-x_j(t)\big)\right]^\top\\
&\quad\times\left[\sum_{j=1}^{N}a_{ij}\Gamma\big(x_j\big(t_{k_d}^i\big)-x_j(t)\big)\right].
\end{aligned}\tag{9.15}$$

Similar to (9.9), we have

$$\begin{aligned}
\sum_{j=1}^{N}a_{ij}\Gamma x_j(t)&=\sum_{l=1}^{k}\sum_{j=m_{l-1}+1}^{m_l}a_{ij}\Gamma(x_j(t)-s_l(t)+s_l(t))\\
&=\sum_{l=1}^{k}\sum_{j=m_{l-1}+1}^{m_l}a_{ij}\Gamma\delta x_j(t)+\sum_{l=1}^{k}\sum_{j=m_{l-1}+1}^{m_l}a_{ij}\Gamma s_l(t)\\
&=\sum_{j=1}^{N}a_{ij}\Gamma\delta x_j(t).
\end{aligned}\tag{9.16}$$

Substituting (9.13)–(9.16) into (9.12), one has

$$\begin{aligned}
\dot V(t)\le&-\sum_{i=1}^{N}\delta x_i^\top(t)PC\delta x_i(t)+\frac{\beta_1}{2}\lambda_{\max}\big(PBB^\top P\big)\sum_{i=1}^{N}\delta x_i^\top(t)\delta x_i(t)\\
&+\frac{1}{2\beta_1}L^2\sum_{i=1}^{N}\delta x_i^\top(t)\delta x_i(t)+\frac{\beta_1}{2}\sum_{l=1}^{k}\sum_{i=m_l}\delta x_i^\top(t)P^\top P\delta x_i(t)+\frac{\beta_1}{2}\sum_{l=1}^{k}\\
&\sum_{i=m_{l-1}+1}^{m_l-1}\delta x_i^\top(t)P^\top P\delta x_i(t)+\frac{1}{2\beta_1}\sum_{l=1}^{k}\sum_{i=m_{l-1}+1}^{m_l-1}\left[\sum_{j=1}^{N}a_{ij}\Gamma\Big(x_j\big(t_{k_d}^i\big)\right.\\
&\left.-x_j(t)\Big)\right]^\top\left[\sum_{j=1}^{N}a_{ij}\Gamma\big(x_j\big(t_{k_d}^i\big)-x_j(t)\big)\right]\\
&+\frac{1}{2\beta_1}\sum_{l=1}^{k}\sum_{i=m_l}\left[\sum_{j=1}^{N}a_{ij}\Gamma\Big(\Big(x_j\big(t_{k_d}^i\big)\right.
\end{aligned}$$

$$
\begin{aligned}
&-x_j(t)\Big) - \epsilon_l \Gamma\Big(x_i\big(t^i_{k_d}\big) - s_l\big(t^i_{k_d}\big) - x_i(t) + s_l(t)\Big)\Big]^\top \Big[\sum_{j=1}^{N} a_{ij}\Gamma((x_j\big(t^i_{k_d}\big) \\
&-x_j(t)) - \epsilon_l \Gamma(x_i\big(t^i_{k_d}\big) - s_l\big(t^i_{k_d}\big) - x_i(t) + s_l(t))\Big] \\
&+\sum_{i=1}^{N} \delta x_i^\top(t) P\Gamma \sum_{j=1}^{N} a_{ij}\delta x_j(t) - \sum_{l=1}^{k}\sum_{i=m_l} \epsilon_l \delta x_i^\top(t) P\Gamma \delta x_i(t) \\
=&\sum_{i=1}^{N} \delta x_i^\top(t)\Big[-PC + (\frac{\beta_1}{2}\lambda_{\max}(PBB^\top P) + \frac{1}{2\beta_1}L^2)I_n + \frac{\beta_1}{2}P^\top P\Big]\delta x_i(t) \\
&-\sum_{l=1}^{k}\sum_{i=m_l} \epsilon_l \delta x_i^\top(t) P\Gamma \delta x_i(t) + \sum_{i=1}^{N} \delta x_i^\top(t) P\Gamma \sum_{j=1}^{N} a_{ij}\delta x_j(t) + \frac{1}{2\beta_1}\cdot \\
&\sum_{l=1}^{k}\sum_{i=m_{l-1}+1}^{m_l-1}\Big[\sum_{j=1}^{N} a_{ij}\Gamma(x_j\big(t^i_{k_d}\big) - x_j(t))\Big]^\top \Big[\sum_{j=1}^{N} a_{ij}\Gamma(x_j\big(t^i_{k_d}\big) - x_j(t))\Big] \\
&+\frac{1}{2\beta_1}\sum_{l=1}^{k}\sum_{i=m_l}\Big[\sum_{j=1}^{N} a_{ij}\Gamma(x_j\big(t^i_{k_d}\big) - x_j(t)) - \epsilon_l\Gamma(x_i\big(t^i_{k_d}\big) - s_l\big(t^i_{k_d}\big) \\
&-x_i(t) + s_l(t))\Big]^\top \Big[\sum_{j=1}^{N} a_{ij}\Gamma(x_j\big(t^i_{k_d}\big) - x_j(t)) - \epsilon_l\Gamma(x_i\big(t^i_{k_d}\big) \\
&-s_l\big(t^i_{k_i}\big) - x_i(t) + s_l(t))\Big].
\end{aligned} \tag{9.17}
$$

Let $\delta x(t) = (\delta x_1^\top(t),\ \delta x_2^\top(t), \ldots,\ \delta x_N^\top(t))^\top$. Substituting $\kappa_i(t)$ into (9.17), it gives that

$$
\dot V(t) \le \delta x^\top(t)(\mathbf{H} + \mathbf{A} - \boldsymbol{\Xi})\delta x(t) + \frac{1}{2\beta_1}\sum_{i=1}^{N} k_i^\top(t)k_i(t). \tag{9.18}
$$

It follows from $\mathbf{H} + \mathbf{A} - \boldsymbol{\Xi} < 0$ and event-triggered condition (9.8) that

$$
\dot V(t) \le -\alpha V(t) + \frac{\beta^2}{2\beta_1}\exp(-2\gamma t), \tag{9.19}
$$

where $\alpha = \dfrac{\lambda_{\max}(\mathbf{H} + \mathbf{A} - \boldsymbol{\Xi})}{-\lambda_{\max}(P)}$.

Let $\beta_2 = \frac{\beta^2}{2\beta_1(\alpha-2\gamma)}$. From (9.19) and the condition $2\gamma < \alpha$, one has

$$\begin{aligned} V(t) &\leq \frac{\beta^2 \exp(-\alpha t)}{2\beta_1(\alpha-2\gamma)}(\exp((\alpha-2\gamma)t)-1) \\ &\leq \frac{\beta^2}{2\beta_1(\alpha-2\gamma)}(\exp(-2\gamma t)-\exp(-\alpha t)) \\ &\leq \frac{\beta^2}{2\beta_1(\alpha-2\gamma)}\exp(-\min\{2\gamma,\alpha\}\cdot t) \\ &= \beta_2 \exp(-2\gamma t), \end{aligned} \tag{9.20}$$

which implies that $V(t)$ converges to 0 exponentially.

Therefore,

$$\lim_{t\to+\infty}\sum_{l=1}^{k}\sum_{i=m_{l-1}+1}^{m_l}\|x_i(t)-s_l(t)\|=0,$$

i.e., network (9.6) realizes expected cluster synchronization.

Next, we shall show that under the event-triggered condition (9.8), $\forall i \in \mathcal{N}$, the inter-event time $t^i_{k_{d+1}} - t^i_{k_d}$ is lower bounded by a positive constant, i.e., the coupled neural network can avoid the Zeno behavior.

To simplify the proof, let

$$\epsilon_{\hat{i}} = \begin{cases} \epsilon_l, & \text{if } i = m_l,\ l = 1,\ 2,\dots,k, \\ 0, & \text{if } i \neq m_l,\ l = 1,\ 2,\dots,k, \end{cases} \tag{9.21}$$

and

$$s_{\hat{i}} = \begin{cases} s_l, & \text{if } i = m_l,\ l = 1,\ 2,\dots,k, \\ 0, & \text{if } i \neq m_l,\ l = 1,\ 2,\dots,k. \end{cases} \tag{9.22}$$

Then, we have

$$\begin{aligned} \dot{\delta x}_i(t) = &-C\delta x_i(t) + B(f(x_i(t)) - f(s_l(t))) + \kappa_i(t) + \sum_{j=1}^{N} a_{ij}\Gamma\delta x_j(t) \\ &-\epsilon_{\hat{i}}\Gamma\delta x_i(t). \end{aligned} \tag{9.23}$$

Hence,

$$
\begin{aligned}
&||\dot{\delta} x_i(t)|| \\
\leq\ & ||C||||\delta x_i(t)|| + L||B||||\delta x_i(t)|| + ||\kappa_i(t)|| + \sum_{j=1}^{N} |a_{ij}|||\Gamma||||\delta x_j(t)|| \\
& +\epsilon_{\hat{i}}||\Gamma||||\delta x_i(t)|| \\
\leq\ & \left( ||C|| + L||B|| + \sum_{j=1}^{N} |a_{ij}|||\Gamma|| + \epsilon_{\hat{i}}||\Gamma|| \right) \sqrt{\frac{\beta_2}{\lambda_{\min}(P)}} \exp\left( -\frac{1}{2} \times 2\gamma t \right) \\
& +\beta \exp(-\gamma t) \\
\leq\ & 2\max \left\{ \left( ||C|| + L||B|| + \sum_{j=1}^{N} |a_{ij}|||\Gamma|| + \epsilon_{\hat{i}}||\Gamma|| \right) \sqrt{\frac{\beta_2}{\lambda_{\min}(P)}},\ \beta \right\} \cdot \exp(-\gamma t) \\
=\ & \xi_i \exp(-\gamma t), \qquad (9.24)
\end{aligned}
$$

where $\xi_i = 2\max\{(||C|| + L||B|| + \sum_{j=1}^{N} |a_{ij}|||\Gamma|| + \epsilon_{\hat{i}}||\Gamma||)\sqrt{\frac{\beta_2}{\lambda_{\min}(P)}},\ \beta\}$.

Furthermore, one can obtain that

$$
\begin{aligned}
&\kappa_i(t) \\
=\ & \sum_{j=1}^{N} a_{ij}\Gamma(x_j(t_{k_d}^i) - x_j(t)) - \epsilon_{\hat{i}}\Gamma(x_i(t_{k_d}^i) - s_{\hat{i}}(t_{k_d}^i) - x_i(t) + s_{\hat{i}}(t)) \\
=\ & \sum_{j=1}^{N} a_{ij}\Gamma(\delta x_j(t_{k_d}^i) - \delta x_j(t)) - \epsilon_{\hat{i}}\Gamma(\delta x_i(t_{k_d}^i) - \delta x_i(t)) \\
=\ & \sum_{j=1, j\neq i}^{N} a_{ij}\Gamma(\delta x_j(t_{k_d}^i) - \delta x_j(t)) + (a_{ii} - \epsilon_{\hat{i}})\Gamma(\delta x_i(t_{k_d}^i) - \delta x_i(t)) \\
\leq\ & \sum_{j=1, j\neq i}^{N} |a_{ij}|||\Gamma|| \cdot || \int_{t_{k_d}^i}^{t} \dot{\delta} x_j(s)ds|| + |a_{ii} - \epsilon_{\hat{i}}| \cdot ||\Gamma|| \cdot || \int_{t_{k_d}^i}^{t} \dot{\delta} x_i(s)ds|| \\
\leq\ & \sum_{j=1, j\neq i}^{N} |a_{ij}|||\Gamma|| \cdot \int_{t_{k_d}^i}^{t} \xi_j \exp(-\gamma s)ds + |a_{ii} - \epsilon_{\hat{i}}| \cdot ||\Gamma||
\end{aligned}
$$

$$\times \int_{t^i_{k_d}}^{t} \xi_i \exp(-\gamma s) ds$$
$$\leq \left( \sum_{j=1, j\neq i}^{N} |a_{ij}|\xi_j + |a_{ii} - \epsilon_{\hat{i}}|\xi_i \right) ||\Gamma|| \exp\big( -\gamma t^i_{k_d}\big)\big(t - t^i_{k_d}\big). \tag{9.25}$$

Due to the continuity of $\kappa_i(t)$ between event-triggered time instants $t^i_{k_d}$ and $t^i_{k_{d+1}} (> t^i_{k_d})$, the event-triggered time instant $t^i_{k_{d+1}}$ should satisfy $\kappa_i(t^{i-}_{k_{d+1}}) = \beta\exp(-\gamma t^i_{k_{d+1}})$. Hence, (9.25) implies that in order to ensure event-triggered condition (9.8) satisfied after instant $t^i_{k_d}$, it is necessary to require the time instant $t^i_{k_{d+1}} (> t^i_{k_d})$ satisfying

$$\left( \sum_{j=1, j\neq i}^{N} |a_{ij}|\xi_j + |a_{ii} - \epsilon_{\hat{i}}|\xi_i \right) ||\Gamma|| \exp\big( -\gamma t^i_{k_d}\big)\big(t^i_{k_{d+1}} - t^i_{k_d}\big) = \beta \exp\big( -\gamma t^i_{k_i+1}\big), \tag{9.26}$$

i.e.,

$$\left( \sum_{j=1, j\neq i}^{N} |a_{ij}|\xi_j + |a_{ii} - \epsilon_{\hat{i}}|\xi_i \right) ||\Gamma|| \big(t^i_{k_{d+1}} - t^i_{k_d}\big) = \beta \exp\big( -\gamma \big(t^i_{k_{d+1}} - t^i_{k_d}\big)\big). \tag{9.27}$$

Therefore, the inter-event time $\{t^i_{k_{d+1}} - t^i_{k_d}\}$ between two trigger time instants of the node $i$ is lower bounded by

$$\sup \left\{ \tau^i_D \geq 0 : \left( \sum_{j=1, j\neq i}^{N} |a_{ij}|\xi_j + |a_{ii} - \epsilon_{\hat{i}}|\xi_i \right) ||\Gamma|| \tau^i_D \leq \beta \exp\big( -\gamma \tau^i_D \big) \right\}. \tag{9.28}$$

This completes the proof.

*Remark 9.9* In Theorem 9.8, the event-triggered condition (9.8) is verified only by using the neighboring states, which means that only local information is used to verify the event-trigger conditions. In [14] and [16], synchronization of complex networks under event-triggered control was investigated and effective event-triggered conditions were designed to achieve network synchronization. Compared with event-based work [14, 16], our event-triggered scheme is strictly proved that the inter-event time has a positive lower bound, i.e., the Zeno behavior can be excluded.

**Table 9.1** Coupling matrix construction method

| Coupling matrix construction method |
|---|
| *Step 1.* Select a matrix $$G = \begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1k} \\ G_{21} & G_{22} & \cdots & G_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ G_{k1} & G_{k2} & \cdots & G_{kk} \end{pmatrix} \in M_2(k), \quad (9.29)$$ with $G_{ii} \in \mathbb{R}^{(m_i - m_{i-1}) \times (m_i - m_{i-1})}$, $G_{ij} \in \mathbb{R}^{(m_i - m_{i-1}) \times (m_j - m_{j-1})}$, $i, j = 1, 2, \ldots, k$ ; |
| *Step 2.* Let $\epsilon_l^0 > 0$ be a constant. Choose $\epsilon_l = \vartheta_l \epsilon_l^0$, $\Xi_l^0 = diag\{0, \ldots, 0, \epsilon_l^0\} \in \mathbb{R}^{m_l \times m_l}$, $l = 1, 2, \ldots, k$ and coupling matrix $$A = \begin{pmatrix} \vartheta_1 G_{11} & G_{12} & \cdots & G_{1k} \\ G_{21} & \vartheta_2 G_{22} & \cdots & G_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ G_{k1} & G_{k2} & \cdots & \vartheta_k G_{kk} \end{pmatrix}; \quad (9.30)$$ |
| *Step 3.* Let $$\vartheta_l \geq \frac{2(k-1)\max_{u \neq v}\{\alpha(\widetilde{G}_{uv} \otimes P\Gamma)\} + \lambda_{\max}(H)}{-\lambda_{max}[(\widetilde{G}_{ll} - \Xi_l^0) \otimes P\Gamma]}, \ l = 1, 2, \ldots, k, \quad (9.31)$$ where $\widetilde{G}_{uv} = \frac{1}{2}(G_{uv} + G_{uv}^\top)$, $u \neq v$, $u, v = 1, 2, \ldots, k$. |

To make Theorem 9.8 more applicable, we can construct coupling matrix $A$ such that $\mathbf{H} + \mathbf{A} - \mathbf{\Xi} < 0$. In Table 9.1, applying the results of [7] to system (9.6), a coupling matrix construction method is given to realize the cluster synchronization of the coupled neural networks.

## 9.3 Pinning Cluster Synchronization Under Self-triggered Mechanism

To avoid continuous communication among agents, a self-triggered algorithm based on Theorem 9.8 is proposed. The self-triggered algorithm means that each node $i$ in the network can predict next triggered time instant $t_{k_{d+1}}^i$ based on the current state information and received information at time $t^i_{k_d}$. The advantage of the self-triggered algorithm lies in that the nodes are not required to verify the event-triggered condition continuously and hence could save more energy for the considered coupled neural networks.

Inspired by the work of [22], we develop the following self-triggered strategy. In the following, we will give some analysis for the main idea of the self-triggered strategy. Suppose that $t \in [t^i_{k_d}, t^i_{k_{d+1}})$ and node $j$ is the neighbor of the node $i$. Before the next event-triggered time instant of node $j$, we can obtain that

$$x_j(t) = x_j\big(t^i_{k_d}\big) + \int_{t^i_{k_d}}^{t} (-Cx_j(s) + Bf(x_j(s)) + I(s))ds \\ +\big(t - t^i_{k_d}\big)\left[\sum_{l=1}^{N} a_{jl}\Gamma x_l\big(t^j_{k_j(t)}\big) - \epsilon_{\hat{j}}\Gamma\Big(x_j\big(t^j_{k_j(t)}\big) - s_{\hat{j}}\big(t^j_{k_j(t)}\big)\Big)\right], \quad (9.32)$$

and

$$\delta x_j(t) = \delta x_j\big(t^i_{k_d}\big) + \int_{t^i_{k_d}}^{t} (-C\delta x_j(s) + B(f(x_j(t)) - f(s_{\hat{j}}(t))))ds \\ + \big(t - t^i_{k_d}\big)\left[\sum_{l=1}^{N} a_{jl}\Gamma \delta x_l\big(t^j_{k_j(t)}\big) - \epsilon_{\hat{j}}\Gamma\Big(x_j\Big(t^j_{k_j(t)}\Big) - s_{\hat{j}}\Big(t^j_{k_j(t)}\Big)\Big)\right], \quad (9.33)$$

where $\epsilon_{\hat{j}}$ and $s_{\hat{j}}$ are defined in (9.21) and (9.22), respectively, and

$$k_j(t) = \arg\max_{r\in\mathbb{N}}\big\{t^j_r | t^j_r \le t\big\}. \quad (9.34)$$

Hence, it holds that

$$||\delta x_j(t) - \delta x_j\big(t^i_{k_d}\big)|| \le (||C|| + ||B||L)\int_{t^i_{k_d}}^{t} ||\delta x_j(s) - \delta x_j\big(t^i_{k_d}\big)||ds \\ +\big(t - t^i_{k_d}\big)\Big[(||C|| + ||B||L)||\delta x_j\big(t^i_{k_d}\big)|| \\ +||\sum_{l=1}^{N} a_{jl}\Gamma \delta x_l\big(t^j_{k_j(t)}\big) - \epsilon_{\hat{j}}\Gamma\big(x_j\big(t^j_{k_j(t)}\big) \\ -s_{\hat{j}}\big(t^j_{k_j(t)}\big)\big)||\Big]. \quad (9.35)$$

It follows from (9.16) and (9.20) that

$$\sum_{l=1}^{N} a_{jl}\Gamma\delta x_l\big(t^i_{k_d}\big) = \sum_{l=1}^{N} a_{jl}\Gamma x_l\big(t^i_{k_d}\big), \tag{9.36}$$

and

$$||\delta x_j\big(t^i_{k_d}\big)|| \le \sqrt{\frac{\beta_2}{\lambda_{\min}(P)}}\exp\big(-\gamma t^i_{k_d}\big). \tag{9.37}$$

Let $\omega_1 = ||C|| + L||B||$ and

$$\omega^j_2 = \omega_1\sqrt{\frac{\beta_2}{\lambda_{\min}(P)}}\exp\big(-\gamma t^i_{k_d}\big) + ||\sum_{l=1}^{N} a_{jl}\Gamma x_l\big(t^j_{k_j(t)}\big) - \epsilon_{\hat{j}}\Gamma\big(x_j\big(t^j_{k_j(t)}\big) \\ -s_{\hat{j}}\big(t^j_{k_j(t)}\big)\big)||.$$

Then, we have

$$||\delta x_j(t) - \delta x_j\big(t^i_{k_d}\big)|| \le \omega_1\int_{t^i_{k_d}}^{t} ||\delta x_j(s) - \delta x_j\big(t^i_{k_d}\big)||ds + \omega^j_2\big(t - t^i_{k_d}\big). \tag{9.38}$$

It follows from the Grönwall inequality,

$$||\delta x_j(t) - \delta x_j\big(t^i_{k_d}\big)|| \le \frac{\omega^j_2}{\omega_1}\big(\exp\big(\omega_1\big(t - t^i_{k_d}\big)\big) - 1\big). \tag{9.39}$$

Hence, we have

$$\begin{aligned} ||\kappa_i(t)|| &= ||\sum_{j=1}^{N} a_{ij}\Gamma\big(\delta x_j\big(t^i_{k_d}\big) - \delta x_j(t)\big) - \epsilon_{\hat{i}}\Gamma\big(\delta x_i\big(t^i_{k_d}\big) - \delta x_i(t)\big)|| \\ &\le \left(\sum_{j=1}^{N} |a_{ij}|\omega^j_2 + \epsilon_{\hat{i}}\omega^i_2\right)\frac{||\Gamma||}{\omega_1}\big(\exp\big(\omega_1\big(t - t^i_{k_d}\big)\big) - 1\big). \end{aligned} \tag{9.40}$$

Combing with event-triggered condition (9.8), we propose the following self-triggered algorithm:

**Theorem 9.10** *Consider coupled neural networks (9.6). Suppose that Assumptions 9.5–9.6 are satisfied and condition (9.7) of Theorem 9.8 holds. Then, under the self-triggered Algorithm 9.1, the network can achieve cluster synchronization*

**Algorithm 9.1** Self-triggered algorithm

*Step 1.* For all $i = 1, 2, \cdots, N$, set $t^i_{k_1} = 0$.

*Step 2.* At time $t^i_{k_d}$, $d > 1$, solving the following equation to find $t^i_{k_{d+1}} = t^i_{k_d} + \varsigma^i_{k_d}$:

$$\sup\{\varsigma^i_{k_d} \geq 0 : \frac{(\sum_{j=1}^N |a_{ij}|\omega_2^j + \epsilon_{\hat{i}}\omega_2^i)||\Gamma||}{\omega_1}(\exp(\omega_1 \varsigma^i_{k_d}) - 1)$$
$$\leq \beta\exp(-\gamma(t^i_{k_d} + \varsigma^i_{k_d}))\}. \tag{9.41}$$

*Step 3.* If agent $i$ does not receive the renewed information from any of its neighbors during $(t^i_d, t^i_d + \varsigma^i_{k_d})$, node $i$ is triggered on time instant $t^i_{k_{d+1}} = t^i_{k_d} + \varsigma^i_{k_d}$.

*Step 4.* If agent $i$ receives the renewed information from its neighbor $j$ at time $t^0 < t^i_{k_{d+1}}$, compute the new value of $\omega_2^j$ and go to *Step 2*.

---

*asymptotically. Moreover, the lower bound of the inter-event time $\tau^i_{D'}$ is given as*

$$\sup\left\{\tau^i_{D'} \geq 0 : \frac{\left(\sum_{j=1}^N |a_{ij}|\zeta_j + \epsilon_{\hat{i}}\zeta_i\right)}{\omega_1}\left(exp\left(\omega_1 \tau^i_{D'}\right) - 1\right) \leq \beta exp\left(-\gamma\tau^i_{D'}\right)\right\}, \tag{9.42}$$

*where* $\zeta_i = (\omega_1 + \sum_{l=1}^N |a_{il}|||\Gamma|| + \epsilon_{\hat{i}})\sqrt{\frac{\beta_2}{\lambda_{\min}(P)}}$, $i = 1, 2, \ldots, N$.

***Proof*** Clearly, under the self-triggered Algorithm 9.1, one can get

$$||\kappa_i(t)|| \leq \beta\exp(-\gamma t).$$

Hence, from Theorem 9.8, coupled neural networks (9.6) with event-triggered mechanism can achieve desired cluster synchronization. Next, we shall show that under the self-triggered Algorithm 9.1, the inter-event interval of node $i$ is strictly positive and has a lower bound $\tau^i_{D'}$, which is given as (9.42). Note that

$$\omega_2^j = \omega_1\sqrt{\frac{\beta_2}{\lambda_{\min}(P)}}\exp\left(-\gamma t^i_{k_d}\right) + ||\sum_{l=1}^N a_{jl}\Gamma\delta x_l\left(t^i_{k_d}\right) - \epsilon_{\hat{j}}\left(x_j\left(t^j_{k_j(t)}\right) - s_{\hat{j}}\left(t^j_{k_j(t)}\right)\right)||$$
$$\leq \left(\omega_1 + \sum_{l=1}^N |a_{jl}|||\Gamma|| + \epsilon_{\hat{j}}\right)\sqrt{\frac{\beta_2}{\lambda_{\min}(P)}}\exp\left(-\gamma t^i_{k_d}\right). \tag{9.43}$$

Let $\zeta_i = (\omega_1 + \sum_{l=1}^{N} |a_{il}| \|\Gamma\| + \epsilon_{\hat{i}}) \sqrt{\frac{\beta_2}{\lambda_{\min}(P)}}$, $i = 1, 2, \ldots, N$. Hence, the sufficient condition to satisfy the event-triggered condition (9.8) is

$$\frac{\left(\sum_{j=1}^{N} |a_{ij}|\zeta_j + \epsilon_{\hat{i}}\zeta_i\right)\exp\left(-\gamma t_{k_d}^i\right)}{\omega_1}\left(\exp\left(\omega_1\left(t - t_{k_d}^i\right)\right) - 1\right) \leq \beta\exp(-\gamma t). \tag{9.44}$$

Therefore, for node $i$, under the self-triggered mechanism, the lower bound of inter-event time can be found as

$$\sup\left\{\tau_{D'}^i : \frac{\left(\sum_{j=1}^{N} |a_{ij}|\zeta_j + \epsilon_{\hat{i}}\zeta_i\right)}{\omega_1}\left(\exp\left(\omega_1\tau_{D'}^i\right) - 1\right) \leq \beta\exp\left(-\gamma\tau_{D'}^i\right)\right\}. \tag{9.45}$$

This completes the proof.

*Remark 9.11* To achieve cluster synchronization, network nodes need to exchange their state information with their neighbors via couplings. In the self-triggered algorithm, node $j$ triggered at time $t_{k_l}^j$ means that node $j$ renews its coupling value at time $t_{k_l}^j$ and sends $x_j(t_{k_l}^j)$ and $\bar{u}_j(t_{k_l}^j) = \sum_{l=1}^{N} a_{jl}\Gamma x_l(t_{k_l}^j) - \epsilon_{\hat{j}}\Gamma(x_j(t_{k_l}^j) - s_{\hat{j}}(t_{k_l}^j)$ to all its neighbors immediately. It can be found that under the proposed self-triggered algorithm, each node in the network does not need to verify the event-triggered condition at every time instant. Hence, the self-triggered algorithm can reduce the computational load and save more energy of the coupled neural networks. Moreover, it should be addressed that the self-triggered algorithm is distributed since for each node in the network, only the neighboring states are used to verify the event-triggered conditions during the steps of the algorithm.

## 9.4 Numerical Example

In this section, an example is given to illustrate the effectiveness of the theoretical results. Coupled neural networks with 10 nodes are selected for illustration since the coupling matrix and visualization of larger network cannot be well displayed.

*Example 9.12* Consider the following coupled cellular neural networks with event-triggered mechanism as follows:

$$\begin{cases} \dot{x}_i(t) = -Cx_i(t) + Bf(x_i(t)) + I(t) + \sum_{j=1}^{N} a_{ij}x_j\big(t_{k_d}^i\big) - \epsilon_l\big(x_i\big(t_{k_d}^i\big) \\ \qquad - s_l\big(t_{k_d}^i\big)\big), \qquad\qquad i = 1,\ 6. \\ \dot{x}_i(t) = -Cx_i(t) + Bf(x_i(t)) + I(t) + \sum_{j=1}^{N} a_{ij}x_j\big(t_{k_d}^i\big), \\ \qquad i = 2,\ 3,\ 4,\ 5,\ 7,\ 8,\ 9,\ 10, \end{cases} \tag{9.46}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^\top$, $\quad I(t) = \mathbf{0}$, $\quad \Gamma = I_3$, $C = I_3$, $f(x_i(t)) = (f(x_{i1}(t)), f(x_{i2}(t)), f(x_{i3}(t)))^\top$ with $f(s) = \frac{1}{2}(|s+1| - |s-1|)$, and

$$B = \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1 \end{pmatrix}.$$

Assume that the desired cluster synchronization states of system are $s_1(t)$ and $s_2(t)$, which satisfy

$$\dot{s}_i(t) = -Cs_i(t) + Bf(s_i(t)) + I(t), \quad i = 1,\ 2, \tag{9.47}$$

with initial values $s_1(0) = [-0.4,\ 1.3,\ 4.8]^\top$ and $s_2(0) = [2.3,\ -1,\ -2.8]^\top$. As indicated in [23], system (9.47) has a double-scrolling chaotic attractor (see Fig. 9.1). By simple computation, we can obtain the Lipschitz constant for system

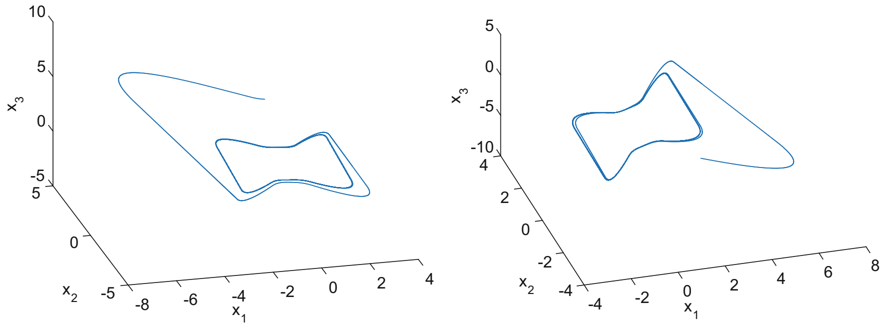


**Fig. 9.1** The desired cluster synchronization trajectories $s_1$ and $s_2$

(9.47) as 1. Using the method proposed in Table 9.1, we can construct the coupling matrix as follows:

1. Choose $\epsilon_l^0 = 3$, $\Xi_l^0 = diag\{\epsilon_l^0, 0, 0, 0, 0\} \in \mathbb{R}^{5\times5}$, $l = 1, 2$ and

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \in M_2(k), \tag{9.48}$$

where

$$G_{11} = G_{22} = G_{12} = G_{21} = \frac{1}{2}\begin{pmatrix} -4 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & 1 & -3 \end{pmatrix}, \ i, \ j = 1, \ 2. \tag{9.49}$$

2. Choose $\epsilon_l = \vartheta_l \epsilon_l^0$ and coupling matrix

$$A = \begin{pmatrix} \vartheta_1 G_{11} & G_{12} \\ G_{21} & \vartheta_2 G_{22} \end{pmatrix}. \tag{9.50}$$

3. Choose $\vartheta_1 = \vartheta_2 = 32$ to satisfy the inequality (9.31). Then, we can obtain the coupling matrix (9.50) which satisfies the requirement of Theorem 9.8.

Let $\beta = 50$ and $\gamma = 0.1$ in event-trigger condition (9.8). The initial state values of the system (9.46) are set as $x_i(0) = [1 + 0.2i, -1 + 0.2i, -2 + 0.2i]^\top$, $i = 1, 2, \ldots, 10$. The simulation step size is set as 0.001.

Figure 9.2 shows the state trajectories of system (9.46) under event-triggered condition (9.8). It can be seen that the expected cluster synchronization can be achieved under the proposed event-based information transmission. In Fig. 9.3, the individual event time instants of the nodes in coupled neural networks (9.46) are given. Figure 9.4 shows the state trajectories of system (9.46) under self-triggered algorithm. It is demonstrated from Fig. 9.4 that the coupled neural networks (9.46) can achieve the expected cluster synchronization under the proposed self-triggered algorithm.

To achieve the desired cluster synchronization of the coupled neural networks, there is a tradeoff between decreasing the number of event-triggering and verifying the event-triggering condition at every time instant for using these two proposed event-based schemes. Table 9.2 illustrates the event-triggering frequency under these two proposed event-based schemes. One can see from Table 9.2 that the

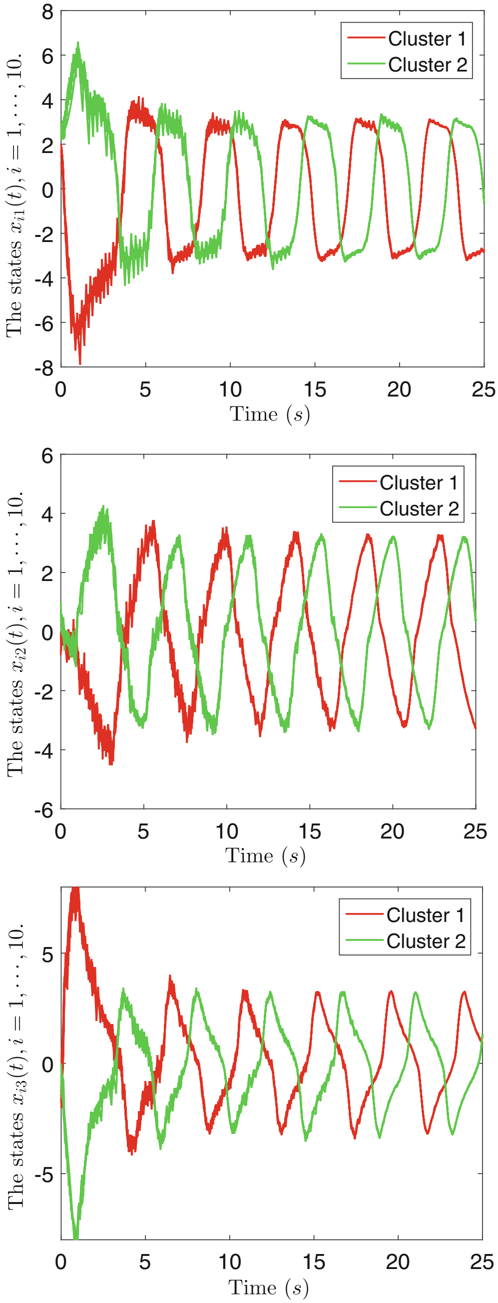


**Fig. 9.2** The state trajectories of system (9.46) under event-triggered condition (9.8)

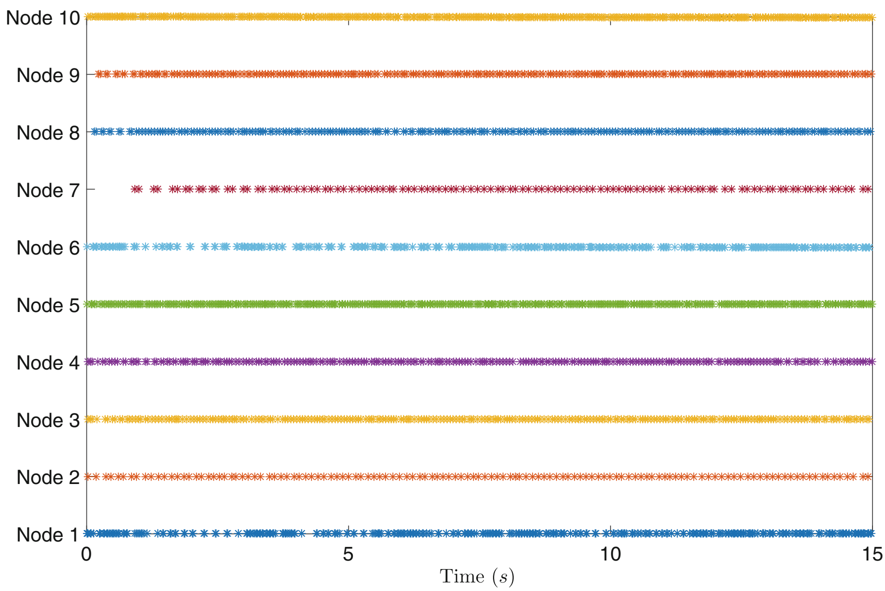


**Fig. 9.3** Event-triggered time instants in Example 9.12

event-triggered scheme proposed in Theorem 9.8 is more effective on decreasing the number of event-triggering than that of self-triggered algorithm. Nevertheless, the self-triggered algorithm can avoid verifying the event-triggered condition at each time instant, which can be seen from Table 9.3. Notice that we cannot compute precisely the percentage of event-triggering instants over communication instants in continuous-time system. For comparison purpose, we compute this percentage by using its discretization with step size 0.001.

## 9.5 Summary

In this chapter, we study the cluster synchronization of coupled neural networks under event-triggered mechanism. Two effective event-triggered schemes are proposed to realize expected cluster synchronization of coupled neural networks. Firstly, distributed event-triggered condition is designed and sufficient conditions to realize cluster synchronization are presented. Furthermore, a self-triggered algorithm, where each node computes its next triggering time independently without verifying the event-triggered condition at each instant, is designed. In addition, for both event-triggered schemes, it is shown that the event-triggered time sequences do not exhibit Zeno behavior. The theoretical results are well demonstrated by simulation example.

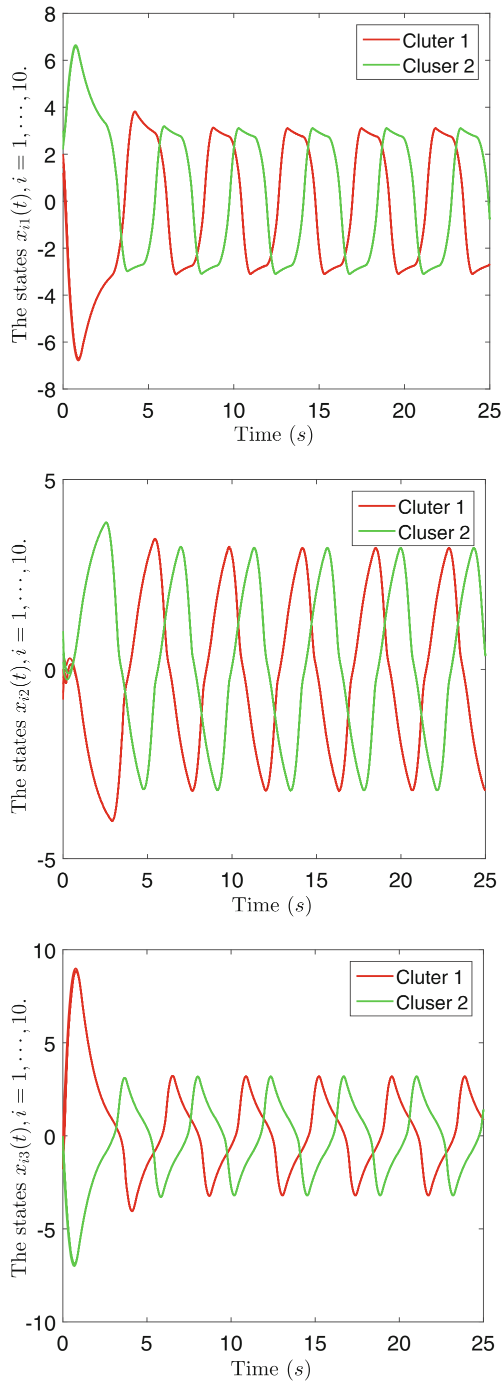


**Fig. 9.4** The state trajectories of system (9.46) under self-triggered algorithm

**Table 9.2** The total number of triggering over total number of iterations in simulation under two different schemes

| Node | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Event-triggered scheme | 2.75% | 0.84% | 1.68% | 1.64% | 2.52% |
| Self-triggered scheme | 14.10% | 4.10% | 6.56% | 6.55% | 8.75% |
| Node | 6 | 7 | 8 | 9 | 10 |
| Event-triggered scheme | 2.75% | 0.77% | 1.66% | 1.67% | 2.57% |
| Self-triggered scheme | 13.90% | 3.88% | 6.30% | 6.30% | 8.54% |

**Table 9.3** The total number of verifying triggering condition over total number of iterations in simulation under two different schemes

| Node | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Event-triggered scheme | 100% | 100% | 100% | 100% | 100% |
| Self-triggered scheme | 41.32% | 30.15% | 38.90% | 37.98% | 39.94% |
| Node | 6 | 7 | 8 | 9 | 10 |
| Event-triggered scheme | 100% | 100% | 100% | 100% | 100% |
| Self-triggered scheme | 42.42% | 30.16% | 38.98% | 38.95% | 39.94% |

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# Chapter 10
# Multi-agent Consensus Recovery Approach Under Node Failure

In the past few years, substantial consensus or synchronization problems have been studied in much previous literature [1–13]. With the proliferation of distributed multi-agent systems interacting through networks, the increasing complexity and higher safety demands of modern engineering systems highlight the significance of the reliability of the system. However, node (or link) failure is inevitable in real multi-agent systems due to unexpected external disturbance. Devastating consequence would follow the failure of the nodes in networked systems [14–16]. Some efforts have been devoted to studying the networked systems with link failures. In [17], an erasure model was studied: the network links fail independently in space (independently of each other) and in time (link failure events are temporarily independent). In [18], synchronization problems were investigated for complex dynamical networks subject to recoverable link failures.

In order to improve the reliability, a recovery scheme should be designed to maintain the performance of the system. However, there have been few studies devoted into this issue. This chapter focuses on studying this problem in the aspect of consensus. Efficient consensus recovery algorithm will be proposed to deal with the problems arising from the node failure in multi-agent consensus. Motivated by the above discussions, we aim to design a consensus recovery approach to compensate for the undesirable effects of the failure nodes, and the consensus result is reserved after recovery process.

## 10.1 Preliminaries

In the following, we give an important lemma, which will be used in the proof of Theorem 10.4.



J. Lu et al., *Collective Behavior in Complex Networked Systems under Imperfect Communication*, https://doi.org/10.1007/978-981-16-1506-1_10

**Lemma 10.1** *Consider an n-dimensional ordinary differential equation*

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \tag{10.1}$$

$x = (x_1, \ldots, x_n)^\top$, $f = (f_1, f_2, \ldots, f_n)^\top \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}^n)$. *Suppose that initial value problem (10.1) has a unique solution on* $[0, +\infty)$. *If there exists positive definite function* $V(x) \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ *satisfying* $V(\mathbf{0}) = 0$ *such that*

$$\dot{V}(x(t)) \mid_{(10.1)} = -W(x(t)) + a(t), \tag{10.2}$$

*where* $\lim_{t \to +\infty} a(t) = 0$, $W(x(t))$ *is a positive function, i.e.,* $W(x) \geq 0$ *and* $W(x) = 0$ *if and only if* $x = \mathbf{0}$, *then* $\lim_{t \to +\infty} x(t) = \mathbf{0}$.

***Proof*** Since $W(x(t))$ is a positive function, there exist $\varphi \in \mathcal{K}$ such that

$$W(x(t)) \geq \varphi(||x(t)||). \tag{10.3}$$

Arbitrarily given $\varepsilon > 0$, by the continuity of $V(x)$ and $V(\mathbf{0}) = 0$, there exists $\varepsilon_1$ such that

$$V(x) < \varepsilon \quad \text{if} \quad ||x|| < \varepsilon_1. \tag{10.4}$$

Let $\delta_1 = \frac{1}{2}\varphi(\varepsilon_1)$, there exists $T_1$, such that

$$|a(t)| < \delta_1 \quad \text{if} \quad t > T_1. \tag{10.5}$$

Consider the value of $V(x(T_1))$, if $V(x(T_1)) = V_0 > \varepsilon$, definitely, $||x(T_1)|| \geq \varepsilon_1$ holds.

We claim that there exists $T_2 > T_1$ such that $V(x(T_2)) = \varepsilon$. Let $T = T_2$. Assume that for any $t > T_1$, $V(x(t)) > \varepsilon$. Then, for $t > T_1$ and $\delta_1 = \frac{1}{2}\varphi(\varepsilon_1)$, we have

$$\dot{V}(x(t)) \leq -\varphi(\varepsilon_1) + \delta_1 \leq -\delta_1. \tag{10.6}$$

Integrating (10.6) from $T_1$ to $t$ yields that

$$\begin{aligned} \varepsilon - V(x(T_1)) &\leq V(x(t)) - V(x(T_1)) \\ &\leq -\delta_1(t - T_1) \longrightarrow -\infty \, (t \longrightarrow \infty), \end{aligned} \tag{10.7}$$

which is a contradiction. Hence, there exists $T_2 > T_1$ such that $V(x(T_2)) = \varepsilon$.

Now, we claim that $V(x(t)) \leq \varepsilon$ if $t > T_2$.

To prove this claim, we assume that there exists $T_3 > T_2$ such that $V(x(T_3)) > \varepsilon$. Because $V(x(t))$ is continuous on $[T_2, T_3]$, $V(x(t_0)) = \max_{t\in[T_2,T_3]} V(x(t))$ exists. We can easily conclude that $\dot{V}(x(t_0)) \geq 0$, which is contradictory to (10.6). Hence, $V(x(t)) \leq \varepsilon$ for $t > T_2$.

For $V(x(T_1)) = V_0 \leq \varepsilon$, using a similar method, we can prove that $V(x(t)) \leq \varepsilon$ for $t > T_1$. Then, choose $T = T_1$.

Summing up the above, we know that for any $\varepsilon > 0$, there exists $T > 0$ such that $V(x(t)) \leq \varepsilon$ for $t > T$, i.e., $\lim_{t\to+\infty} V(x(t)) = 0$. Hence, $\lim_{t\to+\infty} x(t) = \mathbf{0}$.

*Remark 10.2* It should be noted that we do not require the derivative of Lyapunov function $V(x)$ to be negative, which is a standard requirement for Lyapunov asymptotic stability theory. Lemma 10.1 can be used to deal with the consensus and synchronization problem of many dynamical systems and networks, and to ensure the final states of the whole system converge as $t \longrightarrow +\infty$.

## 10.2 Consensus Analysis of General Multi-agent Networks

In this section, we will study the nonlinear consensus protocol for multi-agent networks under a general communication structure, which can also be used to illustrate the correctness and efficiency of the network reduction approach. Consider the following nonlinear multi-agent networks model:

$$\frac{dx_i(t)}{dt} = \sum_{j\in\mathcal{N}_i} a_{ij}[f(x_j(t)) - f(x_i(t))], \quad i = 1, \ldots, N, \tag{10.8}$$

where $x_i(t) \in \mathbb{R}$ is the state of the agent $i$, $f(\cdot)$ is a nonlinear function with the same dimension of $x_i$ satisfying $f(0) = 0$.

**Assumption 10.3** *Throughout this chapter, one requires the nonlinear function $f(\cdot)$ to be continuous and strictly monotonically increasing on $\mathbb{R}$.*

For any communication structure, Laplacian matrix can always be written in the following form after certain permutations [19]. Without loss of generality, we assume the Laplacian matrix of model (10.8) to be $L$ in the form of

$$\begin{pmatrix} L_{11} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & L_{kk} & 0 & 0 & \cdots & 0 \\ L_{k+1,1} & \cdots & L_{k+1,k} & L_{k+1,k+1} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ L_{k+m,1} & \cdots & L_{k+m,k} & L_{k+m,k+1} & L_{k+m,k+2} & \cdots & L_{k+m,k+m} \end{pmatrix},$$

where $L_{ii}$ are irreducible square matrices and in each line, there exist at least one entry satisfying $L_{k+i,j} \neq 0$ $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, k+i-1)$.

In the following, we call the nodes belonging to the strongly connected components of the network which are represented by $L_{ii}$ $(i = 1, 2, \ldots, k)$, *leaders*. The others are called *followers*. Some detailed explanations will be given in Remark 10.5. As the main result of this section, the following theorem reveals a general consensus result for the nonlinear directed multi-agent network.

**Theorem 10.4** *Consider multi-agents network (10.8) with Laplacian matrix L. The following conclusions can be obtained:*

*(i) The leaders in $L_{ii}$ will achieve consensus separately;*
*(ii) The observed states of the followers will asymptotically converge to the convex combination of $\{f(\bar{x}_{\hat{\xi}_i}),\ i = 1, 2, \ldots, k\}$, where $\bar{x}_{\hat{\xi}_i}$ is the consensus value of the leaders in $L_{ii}$.*

***Proof*** Let $x = [x_1, \ldots, x_N]^\top \in \mathbb{R}^N$, $\overline{x}_1 = [x_1, \ldots, x_{r_1}]^\top$, $\overline{x}_2 = [x_{r_1+1}, \ldots, x_{r_2}]^\top$,
$\cdots$, $\overline{x}_{k+m} = [x_{r_{k+m-1}+1}, \ldots, x_N]^\top$. $F(\overline{x}_1) = [f(x_1), \ldots, f(x_{r_1})]^\top$, $F(\overline{x}_2) = [f(x_{r_1+1}), \ldots, f(x_{r_2})]^\top, \ldots, F(\overline{x}_{k+m}) = [f(x_{r_{k+m-1}+1}), \ldots, f(x_N)]^\top$. The dynamics of the multi-agents network (10.8) can be written as:

$$\dot{\overline{x}}_i = -L_{ii} F(\overline{x}_i), \quad i = 1, 2, \ldots, k, \tag{10.9}$$

and

$$\dot{\overline{x}}_{k+j} = -L_{k+j,1} F(\overline{x}_1) - \ldots - L_{k+j,k+j} F(\overline{x}_{k+j}), \ j = 1, 2, \ldots, m. \tag{10.10}$$

Firstly, we know that the solution of system (10.8) always exists on $[0, \infty)$ according to the properties of $f$ in Assumption 10.3. Meanwhile, the uniqueness of the solutions can be easily proved by using contradiction method. The detailed proof is omitted here.

Secondly, we will prove that the leaders in the same strongly connected component of the network will achieve consensus. Let $\{\xi_{r_{l-1}+1}, \ldots, \xi_{r_l}\}$ be the normalized left eigenvector of $L_{ll}$, $l = 1, 2, \ldots, k$, with respect to the zero eigenvalue. Define

$$x_{\hat{\xi}_l}(t) = \sum_{i=r_{l-1}+1}^{r_l} \xi_i x_i(t), \ \ \overline{x}_{\hat{\xi}_l}(t) = \mathbf{1} \otimes (x_{\hat{\xi}_l}(t)),$$

then we have

$$\begin{aligned}\dot{x}_{\hat{\xi}_l}(t) &= \sum_{i=r_{l-1}+1}^{r_l} \xi_i \dot{x}_i(t) \\ &= -\sum_{i=r_{l-1}+1}^{r_l} \xi_i \sum_{j=r_{l-1}+1}^{r_l} a_{ij} f(x_j(t)) \\ &= 0, \quad l = 1, 2, \ldots, k. \end{aligned} \tag{10.11}$$

Consider the Lyapunov function as

$$V_l(t) = \sum_{i=r_{l-1}+1}^{r_l} \xi_i \int_{x_{\hat{\xi}_l}}^{x_i(t)} (f(s) - f(x_{\hat{\xi}_l}))ds, \quad l = 1, 2, \ldots, k. \tag{10.12}$$

Clearly, $V_l(t) \geq 0$ and $V_l(t) = 0$ if and only if $x_i(t) = \overline{x}_{\hat{\xi}_l}$. Calculating the time derivative of $V_l(t)$ along the trajectories of (10.8) gives that

$$\begin{aligned}\dot{V}_l(t) &= \sum_{i=r_{l-1}+1}^{r_l} \xi_i (f(x_i(t)) - f(x_{\hat{\xi}_l})) \sum_{j=r_{l-1}+1}^{r_l} a_{ij} f(x_j(t)) \\ &= -F(\overline{x}_l(t))^\top B_l F(\overline{x}_l(t)), \end{aligned} \tag{10.13}$$

where $B_l = \frac{1}{2}(\Xi L_{ll} + L_{ll}^\top \Xi)$ and $\Xi = \text{diag}\{\xi_{r_{l-1}+1}, \ldots, \xi_{r_l}\}$. It is easy to prove that $B_l$ is irreducible, symmetric and with zero-row sum. Hence, $B_l$ is semi-positive definite and the eigenvalues of $B_l$ are $0 = \lambda_1(B_l) < \lambda_2(B_l) \leq \cdots \leq \lambda_{r_l - r_{l-1}}(B_l)$. Hence, $\dot{V}_l(t) \leq 0$ and $\dot{V}_l(t) = 0$ if and only if

$$F(\overline{x}_l(t)) \in span(\mathbf{1}),$$

i.e.,

$$f(x_{r_{l-1}+1}(t)) = f(x_{r_{l-1}+2}(t)) = \cdots = f(x_{r_l}(t)),$$

which equals to

$$x_{r_{l-1}+1}(t) = x_{r_{l-1}+2}(t) = \cdots = x_{r_l}(t) = \overline{x}_{\hat{\xi}_l}(t).$$

LaSalle's invariant principle gives that

$$x_i(t) \to x_{\hat{\xi}_l}, \;\; t \to \infty, \;\; i = r_{l-1} + 1, \, r_{l-1} + 2, \ldots, r_l, \;\; l = 1, 2, \ldots, k.$$

Finally, we will prove that the observed states of the followers will converge asymptotically to the convex combination of $\{f(\bar{x}_{\hat{\xi}_i}),\ i = 1,\ 2, \ldots,\ k\}$. Define

$$\begin{aligned}\overline{x}_{\hat{\xi}_{k+h}}(t) &= \left[\overline{x}_{\hat{\xi}_{k+h}}^{(r_{k+h-1}+1)}(t), \ldots, \overline{x}_{\hat{\xi}_{k+h}}^{(r_{k+h})}(t)\right]^{\top} \\ &= F^{-1}\left[-L_{k+h,k+h}^{-1}\sum_{i=1}^{k+h-1} L_{k+h,i}F(\overline{x}_{\hat{\xi}_i}(t))\right], \\ h &= 1, 2, \ldots, m. \end{aligned} \tag{10.14}$$

If $h = 1$, we have

$$-L_{k+1,k+1}^{-1}\sum_{i=1}^{k} L_{k+1,i} = L_{k+1,k+1}^{-1}[L_{k+1,k+1}\cdot \mathbf{1}] = \mathbf{1}. \tag{10.15}$$

If $h = 2$,

$$\begin{aligned}&\overline{x}_{\hat{\xi}_{k+2}}(t) \\ &= F^{-1}\left[-L_{k+2,k+2}^{-1}\sum_{i=1}^{k+1} L_{k+2,i}F(\overline{x}_{\hat{\xi}_i}(t))\right] \\ &= F^{-1}\left[-L_{k+2,k+2}^{-1}\sum_{i=1}^{k} L_{k+2,i}F(\overline{x}_{\hat{\xi}_i}(t)) - L_{k+2,k+2}^{-1}L_{k+2,k+1}\right. \\ &\quad \left.\times\left(-L_{k+1,k+1}^{-1}\sum_{i=1}^{k} L_{k+1,i}F(\overline{x}_{\hat{\xi}_i}(t))\right)\right] \\ &= F^{-1}\left[\sum_{i=1}^{k}\big(-L_{k+2,k+2}^{-1}L_{k+2,i} + L_{k+2,k+2}^{-1}L_{k+2,k+1}L_{k+1,k+1}^{-1}L_{k+1,i}\big)F(\overline{x}_{\hat{\xi}_i}(t))\right],\end{aligned} \tag{10.16}$$

and

$$\begin{aligned}&\left[\sum_{i=1}^{k}\big(-L_{k+2,k+2}^{-1}L_{k+2,i} + L_{k+2,k+2}^{-1}L_{k+2,k+1}L_{k+1,k+1}^{-1}L_{k+1,i}\big)\right]\mathbf{1} \\ &= -L_{k+2,k+2}^{-1}\left(\sum_{i=1}^{k} L_{k+2,i}\right)\cdot \mathbf{1} - \left[L_{k+2,k+2}^{-1}L_{k+2,k+1}L_{k+1,k+1}^{-1}\sum_{i=1}^{k}(-L_{k+2,i})\right]\cdot \mathbf{1}\end{aligned}$$

$$= \left[ -L_{k+2,k+2}^{-1} \left( \sum_{i=1}^{k} L_{k+2,i} \right) - L_{k+2,k+2}^{-1} L_{k+2,k+1} \right] \cdot \mathbf{1}$$
$$= L_{k+2,k+2}^{-1} L_{k+2,k+2} \mathbf{1}$$
$$= \mathbf{1}. \tag{10.17}$$

For $h$=1 and $h$=2, equalities (10.15)–(10.17) imply that $f(\overline{x}_{\hat{\xi}_{k+h}}^{(j)}(t)),\ j = r_{k+h-1} + 1,\ r_{k+h-1} + 2, \ldots,\ r_{k+h}$ are the convex combination of $f(\overline{x}_{\hat{\xi}_1}(t)), \ldots,\ f(\overline{x}_{\hat{\xi}_k}(t))$. For $h = 3,\ 4, \ldots,\ m$, by similar procedure, we can also obtain this conclusion. The detailed proof is omitted here.

Next, we will prove that $\overline{x}_{\hat{\xi}_{k+h}}(t)$ is the final state of $\overline{x}_{k+h}(t),\ h = 1,\ 2, \ldots m$.

Since there exists at least one entry satisfying $L_{k+h,j} \neq 0$ ($h = 1,\ 2, \ldots,\ m$, $j = 1,\ 2, \ldots,\ k + i - 1$), one can easily verify that $L_{k+h,k+h}$ is a $M$-matrix. By the property of $M$-matrix, there exists a positive definite diagonal matrix $D$ =diag$(d_{r_{k+h-1}+1},\ \ldots,\ d_{r_{k+h}})$ such that the matrix $\hat{L}_{k+h,k+h} = \frac{1}{2}(DL_{k+h,k+h} + L_{k+h,k+h}^{\top} D)$ is positive definite.

Define the Lyapunov functional as

$$V_{k+h}(t) = \sum_{i=r_{k+h-1}+1}^{r_{k+h}} d_i \int_{\overline{x}_{\hat{\xi}_{k+h}}^{(i)}}^{x_i(t)} (f(s) - f(x_{\hat{\xi}_{k+h}}(t)))ds,$$
$$h = 1,\ 2, \ldots,\ m. \tag{10.18}$$

Obviously, $V_{k+h}(t) \geq 0$ and $V_{k+h}(t) = 0$ if and only if $\overline{x}_{k+h}(t) = \overline{x}_{\hat{\xi}_{k+h}}$.

Choose sufficiently small positive constants $c_1^{(h)},\ c_2^{(h)}, \ldots,\ c_{k+h-1}^{(h)}$, such that $\pi_h = \lambda_{min}(\hat{L}_{k+h,k+h}) - \frac{1}{2} \sum_{i=1}^{k+h-1} c_i^{(h)} > 0$, then we can obtain

$$\begin{aligned}
&\dot{V}_{k+h}(t) \\
&= \sum_{i=r_{k+h-1}+1}^{r_{k+h}} d_i \Big( f(x_i(t)) - f\Big(\overline{x}_{\hat{\xi}_{k+h}}^{(i)}\Big)\Big) \sum_{j=1}^{r_{k+h}} a_{ij} f(x_j(t)) \\
&= -\Big(F(\overline{x}_{k+h}(t)) - F\Big(\overline{x}_{\hat{\xi}_{k+h}}\Big)\Big)^{\top} D \left( \sum_{i=1}^{k+h} L_{k+h,i} F(\overline{x}_i(t)) \right) \\
&= -\Big(F(\overline{x}_{k+h}(t)) - F\Big(\overline{x}_{\hat{\xi}_{k+h}}\Big)\Big)^{\top} DL_{k+h,k+h} [F(\overline{x}_{k+h}(t)) \\
&\quad - \left( -L_{k+h,k+h}^{-1} \sum_{i=1}^{k+h-1} L_{k+h,i} F(\overline{x}_i(t)) \right)]
\end{aligned}$$

$$= -\Big(F(\overline{x}_{k+h}(t)) - F\Big(\overline{x}_{\hat{\xi}_{k+h}}\Big)\Big)^{\top} DL_{k+h,k+h}\Big(F(\overline{x}_{k+h}(t)) - F\Big(\overline{x}_{\hat{\xi}_{k+h}}\Big)\Big)$$
$$-\Big(F(\overline{x}_{k+h}(t)) - F\Big(\overline{x}_{\hat{\xi}_{k+h}}\Big)\Big)^{\top} D\left[\sum_{i=1}^{k+h-1} L_{k+h,i}(F(\overline{x}_i(t)) - F\Big(\overline{x}_{\hat{\xi}_i}(t)\Big)\Big)\right]$$
$$\leq -A_h(t) + a^{(h)}(t), \quad h = 1, 2, \ldots, m, \tag{10.19}$$

where

$$A_h(t) = \pi_h(F(\overline{x}_{k+h}(t)) - F(\overline{x}_{\hat{\xi}_{k+h}}))^{\top}(F(\overline{x}_{k+h}(t)) - F(\overline{x}_{\hat{\xi}_{k+h}})),$$

$$a^{(h)}(t) = \sum_{i=1}^{k+h-1} a_i^{(h)}(t), \quad \text{and}$$

$$a_i^{(h)}(t) = \tfrac{1}{2c_i}\lambda_{\max}\big(L_{k+h,i}^{\top} D^{\top} DL_{k+h,i}\big)(F(\overline{x}_i(t)) - F\Big(\overline{x}_{\hat{\xi}_i}(t))\Big)^{\top}\Big(F(\overline{x}_i(t)) - F\Big(\overline{x}_{\hat{\xi}_i}(t)\Big)\Big).$$

Clearly, $A_h(t)$ is a positive definite function. For $h$=1, we have

$$a^{(1)}(t) = \sum_{i=1}^{k} a_i^{(1)}(t). \tag{10.20}$$

According to the proof of the first part,

$$\lim_{t\to+\infty} a_i^{(1)}(t) = 0. \tag{10.21}$$

By Lemma 10.1, one can obtain that

$$\lim_{t\to+\infty} \overline{x}_{k+1}(t) = \overline{x}_{\hat{\xi}_{k+1}}. \tag{10.22}$$

Similarly, we can prove one by one that

$$\lim_{t\to+\infty} \overline{x}_{k+h}(t) = \overline{x}_{\hat{\xi}_{k+h}}, \ h = 2, \ldots, m. \tag{10.23}$$

The proof is thus completed.

*Remark 10.5* It can be seen from Theorem 10.4 that the final states of the network are determined by the nodes in $L_{ii}$ ($i = 1, 2, \ldots, k$). Therefore, we call these nodes in $L_{ii}$, the leaders. Since the final states of the other nodes in the network are determined by the leaders, they are called followers. Moreover, an interesting result obtained in Theorem 10.4 is that the observed value of the followers converges asymptotically to the convex combination of observed consensus value of the leaders, while different followers may have different combination coefficients.

Related to previous results [20], when $k = 1$, we can obtain the condition to guarantee the consensus of the model (10.8). Moreover, the network cannot reach consensus if $k \geq 2$, i.e., the network does not contain a rooted spanning tree. Thus the following corollary can be obtained, which has been shown in [20].

**Corollary 10.6** *Under Assumption 10.3, multi-agent network (10.8) can realize consensus if and only if the directed communication topology of the network contains a rooted spanning tree.*

## 10.3 Consensus Recovery Approach

A general consensus result for multi-agents network (10.8) has been given in Sect. 10.2. However, in many realistic network systems, node failure is a common phenomenon and sometimes the failure nodes are non-repairable. Under this circumstance, these nodes should be removed from the network, but the most important property of the system (consensus here) should be reserved. From this aspect, it is necessary to propose a certain method to deal with such a node failure problem and compensate for the resultant undesirable behavior of networked multi-agent systems. In other words, an efficient method should be developed to keep the consensus property unchanged.

In this section, we will firstly present a novel network reduction method to reduce the size of the network, such that the consensus property of the large-size network can be obtained by studying the derived small-size network. This network reduction method can make the consensus analysis much easier and the computational cost much cheaper. Based on the network reduction algorithm, we will further discuss a consensus recovery method for multi-agent systems with node failure.

If node $p$ is removed from the network, one straightforward approach to ensure the consensus property unchanging is to make sure that the information of node $p$ is preserved by its neighbouring nodes. For example, consider a network structure as Fig. 10.1. Clearly, the states of nodes $q_j$ $(j = 1, 2, 3)$ are affected by the state of node $p$ directly and the states of nodes $I_i$ $(i = 1, 2)$ through intermediate node $p$. If node $p$ is removed, naturally, the connection strength between node $I_i$ and node $q_j$ should be increased (Fig. 10.1) and the initial value of node $q_j$ should be updated to reserve the consensus property of the network. Now, the remaining question is how to update the initial value of node $q_j$ and the connection strength between the node $I_i$ and node $q_j$. This problem will be solved based on Theorem 10.4.

Next, we propose a network reduction algorithm to greatly reduce the size of the network while reserving the consensus property.

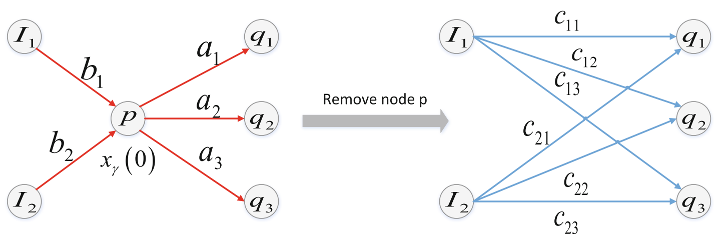


**Fig. 10.1** Remove node $p$ in the network

**Algorithm 10.1** For multi-agent network $\mathcal{A}$, supposing its original graph is $\mathcal{G}(\mathcal{V}, \mathcal{E})$. The following algorithm can be used to greatly reduce the size of the network and meanwhile keep the consensus property unchanged:

Step 1. (Reducing the size of the network)

(a) For node $p$ in network $\mathcal{A}$, suppose that there exist $m$ connections to the nodes $q_1, q_2, \ldots, q_m$ and $k$ connections from nodes $l_1, l_2, \ldots, l_k$ to node $p$. Suppose that the connection strength from node $p$ to node $q_j$ is $a_j$ and the connection strength from node $l_i$ to node $p$ is $b_i$. Removing node $p$ and its connected edges yields a new network $\mathcal{A}_p$. Keep the original initial value and coupling unchanged but the initial value of node $q_j$ is increased by $\frac{a_j}{\beta}x_p(0)$ and the coupling strength between node $l_i$ and node $q_j$ is increased by $\frac{b_i}{\beta}a_j$, where $i = 1, \ldots, k$, $j = 1, \ldots, m$, and $\beta = \sum_{i=1}^{k} b_i$. For example, in Fig. 10.1, $c_{ij} = \frac{b_i}{b_1+b_2}a_j$, $i = 1, 2$, $j = 1, 2, 3$.

(b) For the node with the self-loop, delete the self-loop.

(c) Repeat Step 1.(a) and (b) until no nodes can be reduced.

Step 2. (Rescaling the initial value of the reduced network)

After Step 1, the nodes in the reduced network can be split into two classes. The first class is composed of zero in-degree nodes. The nodes in the second class (may be empty set) are nodes with non-zero in-degree but zero out-degree. For each node with zero in-degree, suppose that its initial value in the reduced network is $\sum_{i=1}^{k} \xi_i x_i(0)$, then one can rescale the initial value of the node into $\frac{1}{\gamma}\sum_{i=1}^{k} \xi_i x_i(0)$, where $\gamma = \sum_{i=1}^{k} \xi_i$.

*Remark 10.7* Following the above-mentioned reduction process, it should be emphasized that only local information is used in the process of network size reduction. When node $p$ is deleted, one only needs to update the initial values of the out-neighboring nodes $l_i$ and connection strength between in-neighboring nodes $l_i$ and out-neighboring nodes $q_j$ $(i = 1, \ldots, k,\ j = 1, \ldots, m)$.

The logical relationship between the reduced small-size network and the original one is clearly presented in the following remark.

*Remark 10.8* For the final reduced network $\hat{\mathcal{A}}$, suppose that $r_i$ is the zero in-degree node with initial value $\hat{x}_{r_i}(0)$ and node $q_j$ is the zero out-degree node with the connection strength $a_{ij}$ from node $r_i$ to node $q_j$, where $i = 1, \ldots, k$, $j = 1, \ldots, m$. Then, the following facts about the multi-agent network $\mathcal{A}$ can be obtained from the reduced network $\hat{\mathcal{A}}$:

1. The set of nodes $\Delta_i = \{r_{i-1} + 1, \ldots, r_i\}$ $(i = 1, \ldots, k)$ are strongly connected components of graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$;
2. The nodes in these strongly connected components are the leaders of the network $\mathcal{A}$ and each leader in the network $\mathcal{A}$ must belong to one of the nodes set $\Delta_i$ $(i = 1, \ldots, k)$;
3. The leaders in the same strongly connected component will reach consensus and the consensus value $\bar{x}_{\hat{\xi}_i}$ is equal to $\hat{x}_{r_i}(0)$;
4. Nodes $q_j$, $j = 1, \ldots, m$ in the reduced network $\hat{\mathcal{A}}$ are the followers of the original network $\mathcal{A}$ and its final state is $f^{-1}[\frac{1}{\sum_{i=1}^{k} a_{ij}} \sum_{i=1}^{k} a_{ij} f(\bar{x}_{\hat{\xi}_i})]$;
5. Based on 3 and 4, one can conclude that:
   - if $k = 1$, i.e., there is only one leader in the final reduced network $\hat{\mathcal{A}}$, the multi-agent network can achieve consensus;
   - if $k \neq 1$ in the reduced network, the multi-agent network will achieve $k$ different consensus values which are, respectively, decided by leader sets $\Delta_i$, $i = 1, \ldots, k$. Further explanation will be given in Example 2.

Based on the network reduction method discussed above, we will propose a network recovery approach to solve the node failure problem.

Let $\{\xi_{r_{l-1}+1}, \ldots, \xi_{r_l}\}$ be the normalized left eigenvector of $L_{ll}$, $l = 1, 2, \ldots, k$, with respect to the zero eigenvalue. Suppose agent $p$ fails at time $t_0$ due to external disturbance (see Fig. 10.1). If $p$ is the leader, we want the network to preserve the information of $p$ in the network after removing $p$. Hence, we should adjust the local information of node $p$ as Step 1 of Algorithm 10.1. As for the followers, we only need to adjust the edge weighting information between the neighboring nodes to preserve the convex combination coefficients in the reduced network. Suppose that there exist $m$ connections to nodes $q_1, q_2, \ldots, q_m$ and $k$ connections from nodes $l_1, l_2, \ldots, l_k$ to node $p$. Suppose the connection strength from node $p$ to node $q_j$ is $a_j$ and the connection strength from node $l_i$ to node $p$ is $b_i$.

**Algorithm 10.2** For the multi-agent network $\mathcal{A}$, supposing its original graph is $\mathcal{G}(\mathcal{V}, \mathcal{E})$, the following algorithm can be used to remove the failure node of the network and keep the consensus property as much as possible:

Step 1. (Removing the failure node of the network)

Suppose $s$ nodes fail during the time interval $[t_0, t_1]$. For any node $p$ which fails at time $t_0^{'}$, there are two cases we should consider:

Case 1: Node $p$ is the follower. Removing node $p$ and its connected edges yields a new network $\mathcal{A}_p$. The coupling strength between node $l_i$ and node $q_j$ is increased by $\frac{b_i}{\beta}a_j$, where $i = 1, \ldots, k$, $j = 1, \ldots, m$, and $\beta = \sum_{i=1}^{k} b_i$. For example, in Fig. 10.1, $c_{ij} = \frac{b_i}{b_1+b_2}a_j$, $i = 1, 2$, $j = 1, 2, 3$.

Case 2: (a) Node $p$ is the leader. Adjust the local information of $p$ in the reduced network, i.e., the current state $x_{q_j}(t_0^{'})$ of node $q_j$ is increased by $\frac{a_j}{\beta}x_p(t_0^{'})$ and the coupling strength between node $l_i$ and node $q_j$ is increased by $\frac{b_i}{\beta}a_j$, where $i = 1, \ldots, k$, $j = 1, \ldots, m$, and $\beta = \sum_{i=1}^{k} b_i$. For example, in Fig. 10.1, $c_{ij} = \frac{b_i}{b_1+b_2}a_j$, $i = 1, 2$, $j = 1, 2, 3$.

(b) For the node with the self-loop, delete the self-loop.

Step 2. (Rescaling the current state of the reduced network)

Let $\{\xi_{r_{l-1}+1}, \ldots, \xi_{r_l}\}$ be the normalized left eigenvector of $L_{ll}$, $l = 1, 2, \ldots, k$, with respect to the zero eigenvalue. After Step 1, suppose $s_l$ nodes are removed from the leader group $L_{ll}$, $l = 1, 2, \ldots, k$. Without loss of generality, these nodes are assumed to be nodes $r_{l-1} + 1, \ldots, r_{l-1} + s_l$. For each remaining node in leader group $L_{ll}$, suppose that its current state is $x_c(t_1)$, then one can rescale the initial value of the node into $(1 - \sum_{j=1}^{s_l} \xi_{r_{l-1}+j})x_c(t_1)$.

We will prove the correctness of Algorithm 10.2 in the following theorem.

**Theorem 10.9** *Consider multi-agents network (10.8) with Laplacian matrix L as (10.3). Suppose* $s$ $(s \le N - 1)$ *nodes fail during the time interval* $[t_0, t_1]$. *Then, the consensus property of the network (10.8) is maintained if Algorithm 10.2 is implemented when the nodes fail.*

***Proof*** We firstly consider the failure node $p$ is the leader. Without loss of generality, we assume the label of node $p$ to be node 1, which belongs to the first leader group (Laplacian matrix is $L_{11}$) and will be removed in Algorithm 10.2. Assume also that the Laplacian matrix of the reduced network is $L'_{11}$. Suppose

$$L_{11} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r_1} \\ a_{21} & a_{22} & \cdots & a_{2r_1} \\ \vdots & \cdots & \cdots & \vdots \\ a_{r_1 1} & a_{r_1 2} & \cdots & a_{r_1 r_1} \end{pmatrix}.$$

According to Step 1 of Algorithm 10.2, we can find that

$$L'_{11} = \begin{pmatrix} a_{22} - \frac{a_{12}a_{21}}{a_{11}} & a_{23} - \frac{a_{13}a_{21}}{a_{11}} & \cdots & a_{2r_1} - \frac{a_{1r_1}a_{21}}{a_{11}} \\ a_{32} - \frac{a_{12}a_{31}}{a_{11}} & a_{33} - \frac{a_{13}a_{31}}{a_{11}} & \cdots & a_{3r_1} - \frac{a_{1r_1}a_{31}}{a_{11}} \\ \vdots & \cdots & \cdots & \vdots \\ a_{r_1 2} - \frac{a_{12}a_{r_1 1}}{a_{11}} & a_{r_1 3} - \frac{a_{13}a_{r_1 1}}{a_{11}} & \cdots & a_{r_1 r_1} - \frac{a_{1r_1}a_{r_1 1}}{a_{11}} \end{pmatrix}.$$

Let $\{\xi_1, \ldots, \xi_{r_1}\}$ be the normalized left eigenvector of $L_{11}$ and $\{\xi'_2, \ldots, \xi'_{r_1}\}$ is the normalized left eigenvector of $L'_{11}$. Denote $\Phi = (\xi_1, \ldots, \xi_{r_1})$ and $\Phi' = (\xi'_2, \ldots, \xi'_{r_1})$. Hence, we have

$$\Phi \cdot L_{11} = \mathbf{0} \quad \text{and} \quad \Phi' \cdot L'_{11} = \mathbf{0},$$

which together with $\sum_{i=1}^{r_1} \xi_i = 1$ and $\sum_{i=2}^{r_1} \xi'_i = 1$ imply that

$$\xi'_i = \frac{1}{1-\xi_1}\xi_i, \quad i = 2, \ldots, r_1. \tag{10.24}$$

It follows from Theorem 10.4 that the final consensus value of the first leader group is $\sum_{i=1}^{r_1} \xi_i x_i(0)$. The Eq. (10.11) implies that

$$\sum_{i=1}^{r_1} \xi_i x_i(t) = \sum_{i=1}^{r_1} \xi_i x_i(0), \;\; \forall t > 0. \tag{10.25}$$

If node $p$ fails at time $t_1$ and Step 1 of Algorithm 10.2 is implemented, we have

$$\sum_{i=2}^{r_1} \xi'_i x'_i(t) = \sum_{i=2}^{r_1} \xi'_i x'_i(t_1), \;\; \forall t > t_1. \tag{10.26}$$

Then, it can be obtained that

$$\begin{aligned}
\sum_{i=2}^{r_1} \xi_i' x_i'(t_1) &= \sum_{i=2}^{r_1} \xi_i' \Big( x_i(t_1) + \frac{a_{i1}}{a_{11}} x_1(t_1) \Big) \\
&= \frac{1}{1-\xi_1} \left( \sum_{i=2}^{r_1} \xi_i x_i(t_1) + \sum_{i=2}^{r_1} \xi_i \frac{a_{i1}}{a_{11}} x_1(t_1) \right) \\
&= \frac{1}{1-\xi_1} \left( \sum_{i=2}^{r_1} \xi_i x_i(t_1) + \xi_1 x_1(t_1) \right) \\
&= \frac{1}{1-\xi_1} \left( \sum_{i=1}^{r_1} \xi_i x_i(t_1) \right) \\
&= \frac{1}{1-\xi_1} \left( \sum_{i=1}^{r_1} \xi_i x_i(0) \right). \qquad (10.27)
\end{aligned}$$

Hence,

$$\sum_{i=1}^{r_1} \xi_i x_i(0) = (1-\xi_1) \sum_{i=2}^{r_1} \xi_i' x_i'(t_1) = (1-\xi_1) \sum_{i=2}^{r_1} \xi_i' x_i'(t). \qquad (10.28)$$

If $s$ nodes (assume they are $1, \ldots, s$) fail in the first leader group and their failure time is $t_i$, $i = 1, \ldots, s$, satisfying $t_0 \le t_1 \le t_2 \le \cdots \le t_s \le t_0^{'}$, repeat the above proof process and we can obtain that

$$\begin{aligned}
\sum_{i=1}^{r_1} \xi_i x_i(0) &= (1-\xi_1) \sum_{i=2}^{r_1} \xi_i' x_i'(t_1) \\
&= (1-\xi_1)\big(1-\xi_2'\big) \sum_{i=3}^{r_1} \xi_i^{(2)} x_i^{(2)}(t_2) \\
&= (1-\xi_1) \left( 1 - \frac{\xi_2}{1-\xi_1} \right) \sum_{i=3}^{r_1} \xi_i^{(2)} x_i^{(2)}(t_2) \\
&= (1-\xi_1-\xi_2) \sum_{i=3}^{r_1} \xi_i^{(2)} x_i^{(2)}(t_2) \\
&= \cdots\cdots \\
&= \left( 1 - \sum_{i=1}^{s} \xi_i \right) \sum_{i=s+1}^{r_1} \xi_i^{(s)} x_i^{(s)}(t_s) \sum_{i=1}^{r_1} \xi_i x_i(0) \\
&= \left( 1 - \sum_{i=1}^{s} \xi_i \right) \sum_{i=s+1}^{r_1} \xi_i^{(s)} x_i^{(s)}(t), \quad \forall t \ge t_s,
\end{aligned}$$

where $\{\xi_i^{(s)}, \xi_{i+1}^{(s)} \ldots, \xi_{r_1}^{(s)}\}$, $s = 2, \ldots; i = 2, \ldots, s+1$, is the normalized left eigenvector of $L_{11}^{(s)}$. Therefore, after implementing Step 2, the new network will have the same consensus property as the original one.

Next, we will study the case that the failure node $p$ is the follower. Without loss of generality, we assume the label of node $p$ to be node $r_k + 1$, which belongs to the first follower group $L_{k+1,k+1}$ and will be removed in Algorithm 10.2.

We have proved in Theorem 10.4 that the observed states of the followers will asymptotically converge to the convex combination of $\{f(\bar{x}_{\hat{\xi}_i}),\ i = 1, 2, \ldots, k\}$, where $\bar{x}_{\hat{\xi}_i}$ is the consensus value of the leaders in $L_{ii}$. Let

$$L_{k+1,k+1} = \begin{pmatrix} a_{r_k+1,r_k+1} & a_{r_k+1,r_k+2} & \cdots & a_{r_k+1,r_{k+1}} \\ a_{r_k+2,r_k+1} & a_{r_k+2,r_k+2} & \cdots & a_{r_k+2,r_{k+1}} \\ \vdots & \cdots & \cdots & \vdots \\ a_{r_{k+1},r_k+1} & a_{r_{k+1},r_k+2} & \cdots & a_{r_{k+1},r_{k+1}} \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{a_{r_k+2,r_k+1}}{a_{r_k+1,r_k+1}} & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -\frac{a_{r_{k+1},r_k+1}}{a_{r_k+1,r_k+1}} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and

$$Q_2 = \begin{pmatrix} 1 & -\frac{a_{r_k+1,r_k+2}}{a_{r_k+1,r_k+1}} & \cdots & -\frac{a_{r_k+1,r_{k+1}-1}}{a_{r_k+1,r_k+1}} & -\frac{a_{r_k+1,r_{k+1}}}{a_{r_k+1,r_k+1}} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Let $\overline{L}_{k+1,k+1} = Q_1 L_{k+1,k+1} Q_2$ and $\overline{L}_{k+1,i} = Q_1 L_{k+1,i} Q_2$, $i = 1, \cdots, k+1$. It can be seen that the matrix $L'_{k+1,k+1}$ and $L'_{k+1,i}$ can be obtained by removing the first row and column of the matrix $\overline{L}_{k+1,k+1}$ and $\overline{L}_{k+1,i}$, respectively.

In Theorem 10.4, we have proved that the observed state of the follower group $L_{k+1,k+1}$ can be explicitly expressed as

$$\begin{aligned} F\Big(\overline{x}_{\hat{\xi}_{k+1}}(t)\Big) &= \Big[f(\overline{x}_{\hat{\xi}_{k+1}}^{(r_k+1)}(t)), \ldots, f(\overline{x}_{\hat{\xi}_{k+1}}^{(r_{k+1})}(t))\Big]^\top \\ &= \left[-L_{k+1,k+1}^{-1} \sum_{i=1}^{k} L_{k+1,i} F(\overline{x}_{\hat{\xi}_i}(t))\right]. \end{aligned}$$

By some simple computation, we can easily see that after implementing Step 1 of the recovery algorithm, the final convex coefficient will be the same as the original ones, i.e., how leaders decide the final states of the follower is preserved.

## 10.4 Numerical Examples

Now, a simple example will be given to illustrate the consensus recovery process and also for better understanding of the proposed algorithm.

*Example 10.10* Consider a simple sensor network consisting of 8 sensors. Let $x_i(0),\ i = 1, \ldots, 8$, be the initial states of network. $f(x(t)) = [f(x_1(t)), \ldots, f(x_8(t))]^\top$ with $f(x_i) = x_i^3$. Suppose the initial values are set as $[-1, 3, 5, 2, 2, -3, 0, -2]^\top$. The network topology is displayed in Fig. 10.2, and the weight of each edge is set as 1. We can see that the network is balanced and the final consensus value is the average state of the network (see Fig. 10.5).

Suppose node $p$ fails at time $t_0 = 5$ (see Fig. 10.3), by using the consensus recovery approach, removing node $p$ gives the reduced graph (Fig. 10.4). The

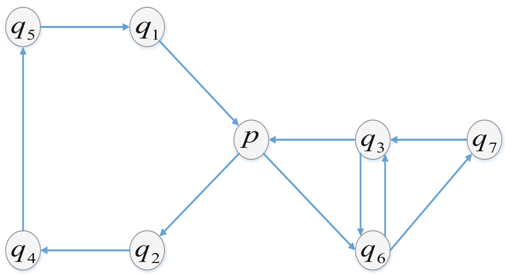


**Fig. 10.2** Original network structure

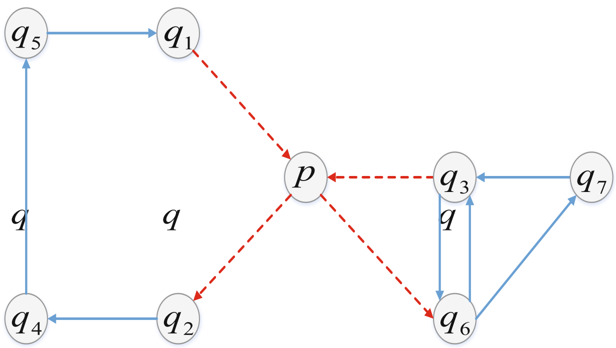


**Fig. 10.3** Node $p$ has failure due to external attack

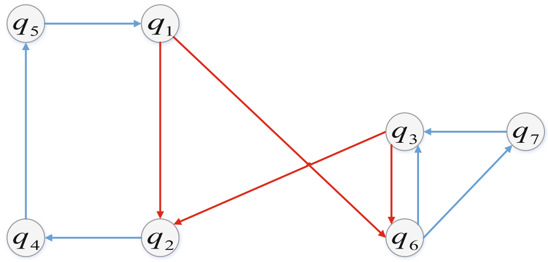


**Fig. 10.4** Remove node $p$ in the network

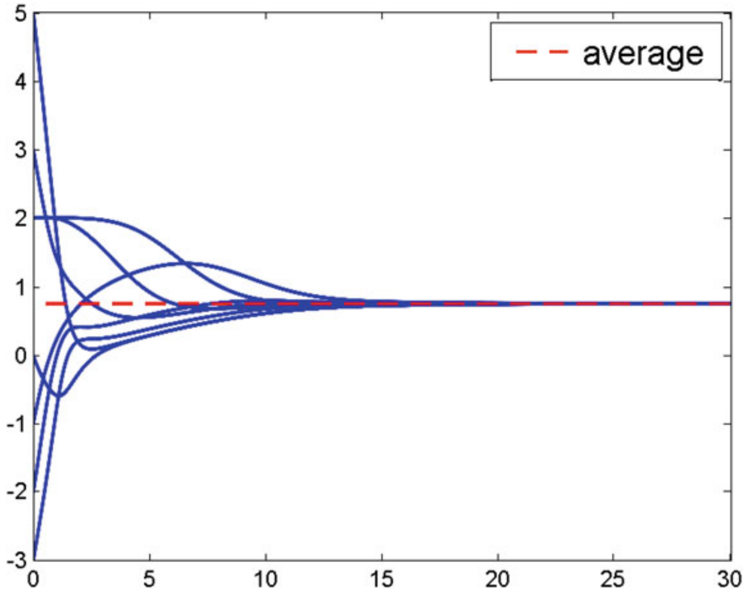


**Fig. 10.5** The states of the original network

corresponding state trajectories of system (10.8) are shown in Fig. 10.5. It can be seen from Fig. 10.6 that the network cannot achieve consensus since the connectivity of the network is destroyed by the failed node $p$. Figure 10.7 shows the evolution of the network when the recovery procedure is implemented for the failure node $p$. This simple example illustrates the correctness and efficiency of the proposed network recovery method very well.

In the following, an example with more nodes is given to illustrate the effectiveness of the theoretical results. In this example, a network with 30 nodes will be reduced to a smaller network with only 5 nodes. A network with 30 nodes is selected for illustration since the coupling matrix and visualization of a larger network cannot be well displayed.

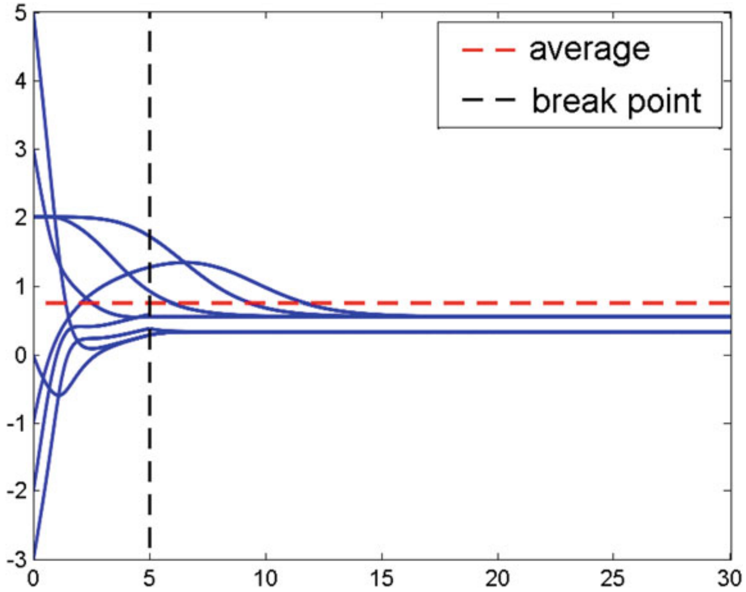


**Fig. 10.6** The states of the network when node $p$ fails at time $t_0 = 5$

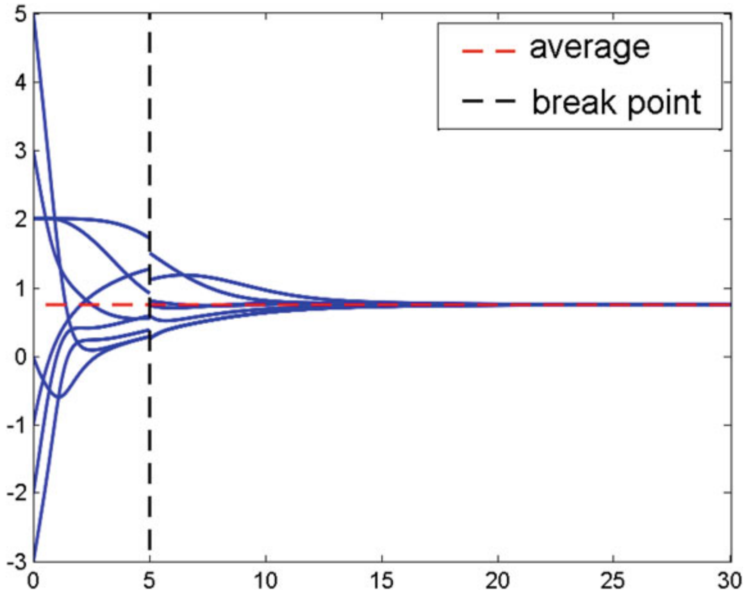


**Fig. 10.7** The states of the network when the recovery procedure is implemented

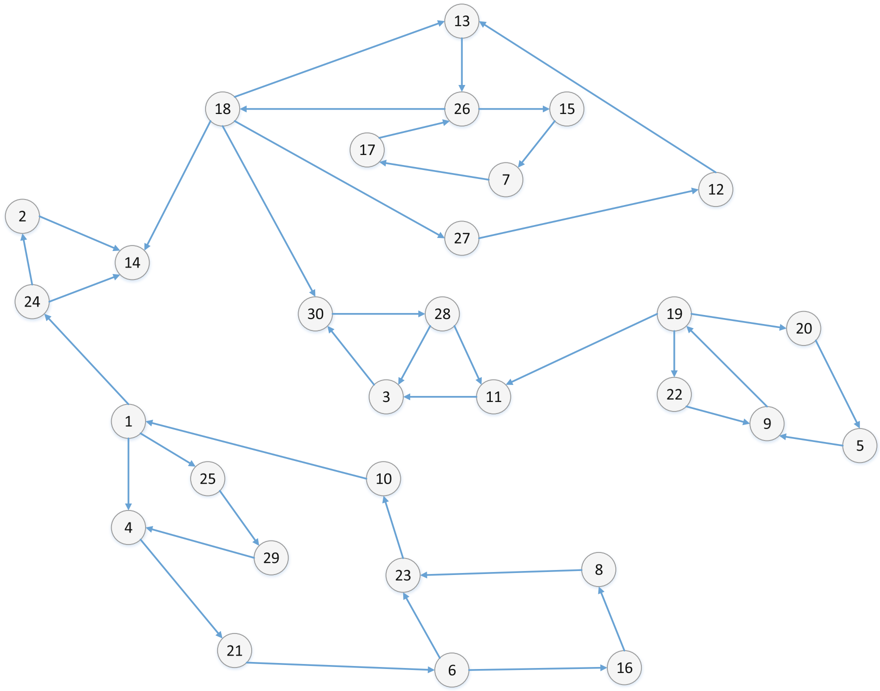


**Fig. 10.8** The original graph of network (10.8) with 30 nodes

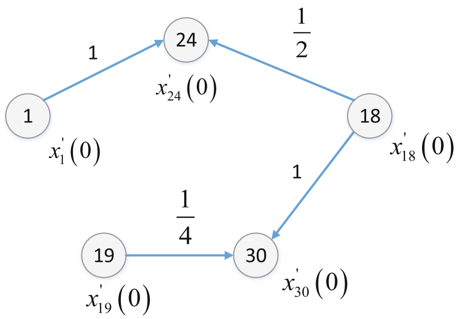


**Fig. 10.9** The reduced graph of network (10.8) with 30 nodes

*Example 10.11* Consider the multi-agent system (10.8) with 30 nodes (Fig. 10.8), and the nonlinear function $f$ is chosen as $f(x) = \tanh(x)$.

Firstly, based on Algorithm 10.1, we can reduce the original network into a smaller one. The reduced network contains only 5 nodes and the corresponding structure can be seen in Fig. 10.9. Corresponding to Remark 10.8, we can conclude that:

1. The nodes sets $\Delta_1 = \{1, 4, 6, 8, 10, 16, 21, 23, 25, 29\}$, $\Delta_2 = \{7, 12, 13, 15, 17, 18, 26, 27\}$, and $\Delta_3 = \{5, 9, 19, 20, 22\}$ are the leaders in the original network. After reduction, only leader nodes $1 \in \Delta_1$, $18 \in \Delta_2$, and $19 \in \Delta_3$ are still active in the reduced network;
2. The final consensus states of the nodes in the set $\Delta_i$, $i = 1, 2, 3$, are, respectively, $\hat{x}_1(0)$, $\hat{x}_{18}(0)$, and $\hat{x}_{19}(0)$, where $\hat{x}_i(0)$, $i = 1, 18, 19$, are the initial values of node $i$ in the reduced network. For instance, $\hat{x}_{19}(0) = \frac{1}{6}(x_5(0)+x_9(0)+x_{20}(0) + x_{22}(0)) + \frac{1}{3}x_{19}(0)$;
3. The nodes set $\Delta = \{2, 3, 11, 14, 24, 28, 30\}$ are the followers in the original network;
4. The final observed state $x(t)$ of nodes 24 and 30 are $\tanh^{-1}(\frac{2}{3}\tanh(x_{1'}(0)) + \frac{1}{3}\tanh(x_{18'}(0)))$ and $\tanh^{-1}(\frac{1}{5}\tanh(x_{19'}(0)) + \frac{4}{5}\tanh(x_{18'}(0)))$, respectively.

The state responses of the original network and the reduced network are, respectively, given in Figs. 10.10 and 10.11, which can well illustrate the correctness of the former conclusion.

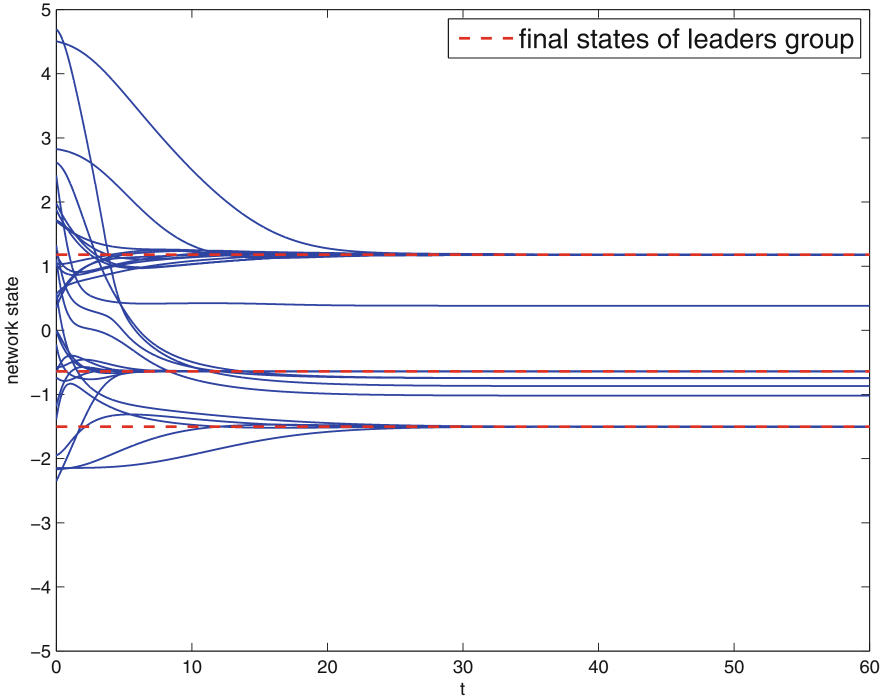


**Fig. 10.10** The states of the original network

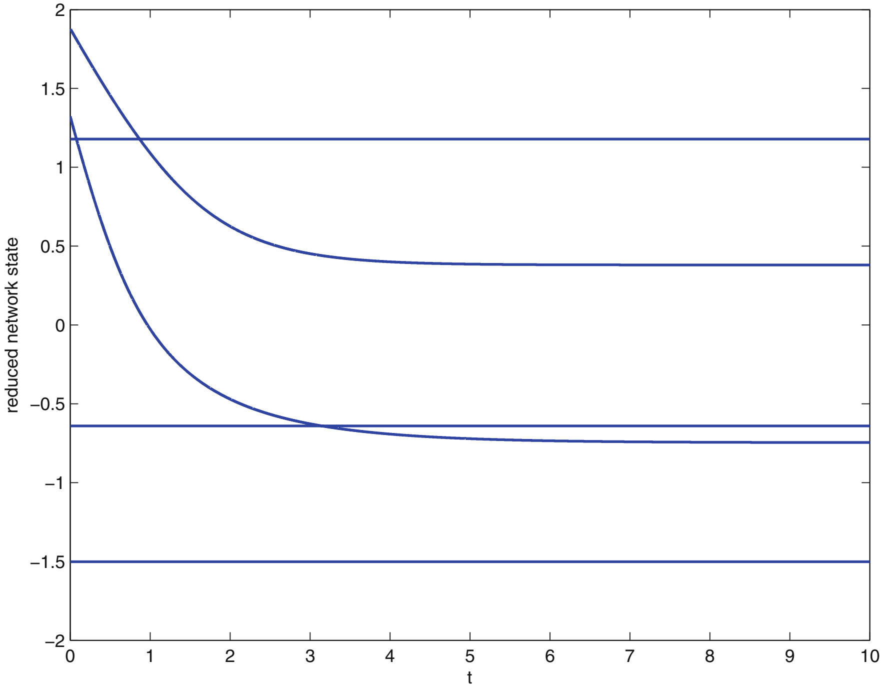


**Fig. 10.11** The states of the reduced network

To show the correctness and effectiveness of the consensus recovery approach, we assume that nodes 1, 2, and 10 fail at time $t_0 = 5$ and nodes 5, 22, and 28 fail at time $t_1 = 10$, respectively. If the consensus recovery is not applied, we can see from Fig. 10.12 that the network will not converge to the same final state as the original network in Fig. 10.10. Figure 10.13 shows the evolution of the same network when the proposed consensus recovery algorithm is implemented. It can be found that after implementing the consensus recovery algorithm, the network has the same final state as the original network.

## 10.5 Summary

Node failure problem is very normal in network multi-agent systems. In this chapter, a novel consensus recovery approach has been introduced to analyze the consensus of nonlinear coupled multi-agent networks with node failure. The node removing

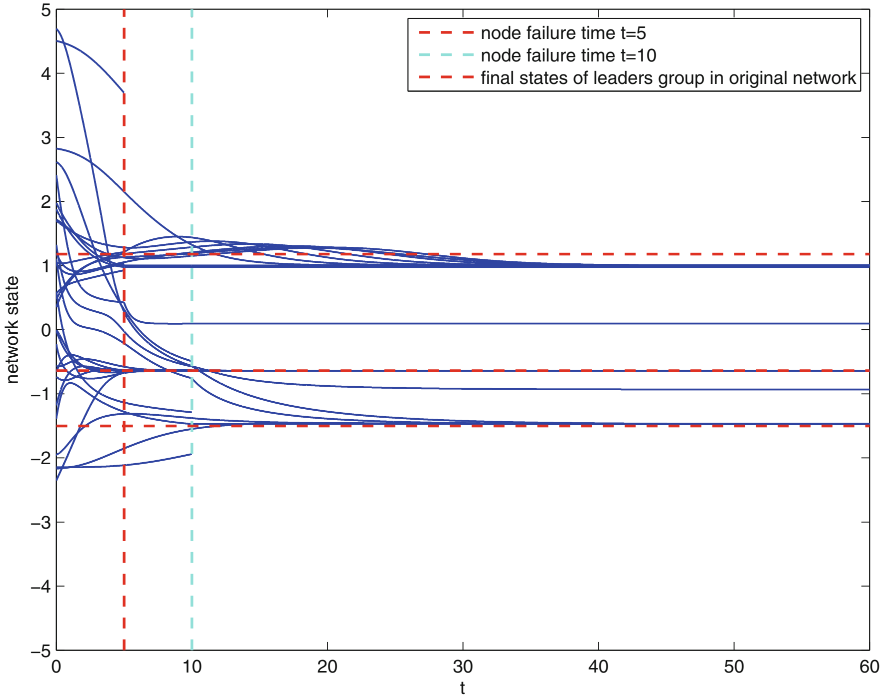


**Fig. 10.12** The states of the network when the recovery procedure is not implemented

process consists of two important operations including the updates of the edge weighs and initial values. In the process of removing nodes, only local information is used, and hence our proposed network reduction method is quite practical and easy to implement. After the consensus recovery operation, the consensus property is well reserved. Theoretical consensus analyses have also been presented in this chapter by fully utilizing the network structure. One important benefit of the analytical results is that it verifies the correctness of the consensus recovery approach. The theoretical results have been well illustrated by using numerical examples.

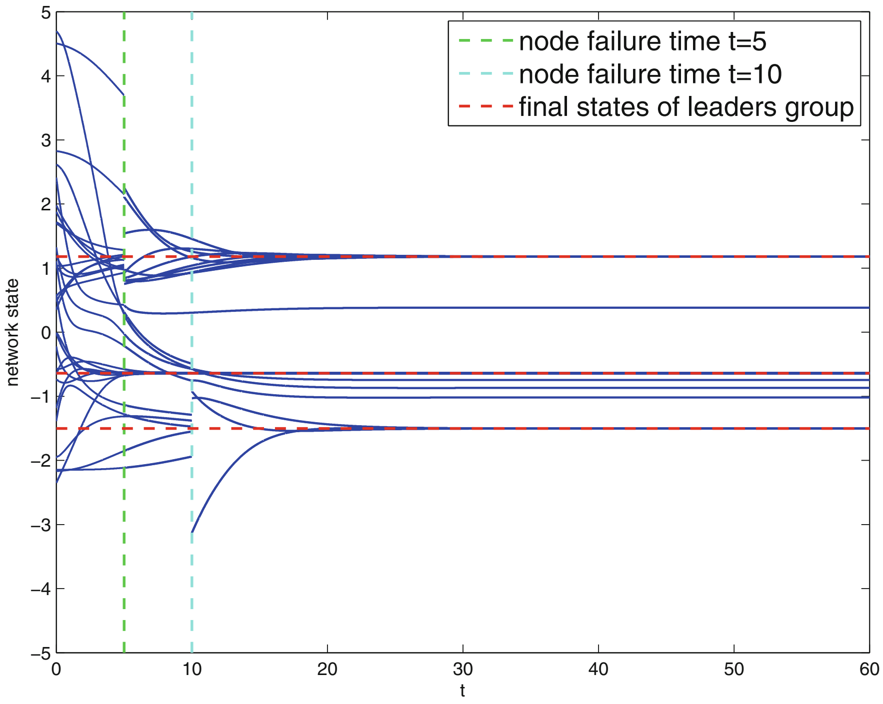


**Fig. 10.13** The states of the network when the recovery procedure is implemented

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# Chapter 11
# Conclusion and Future Work

In this chapter, some concluding remarks and related future work are given.

## 11.1 Conclusion

This book is mainly concerned with the complex networked systems under imperfect communication constraints. The research problems proposed in Sect. 1.2 have been respectively discussed in this book.

(a) Chapter 2 is mainly devoted to studying Problem 1.2. Under arbitrary finite communication delays, the consensus problem in directed static networks is studied. It has been proved that consensus can be realized whatever the finite communication delays are.

(b) Two types of imperfect communication, i.e., quantization and communication delays, are studied in Chaps. 3 and 4. In Chap. 3, Problem 1.2 is investigated in detail and a unified framework for continuous-time multi-agent consensus problem with quantization and communication delays is developed. By using the nonsmooth technique, practical consensus of the multi-agent systems is obtained. It is interesting to observe that the quantization parameter decides the size of the practical consensus set and communication delays only affect the center of the practical consensus set. Chapter 4 is devoted to studying Problem 1.2. Without considering input quantization, consensus problems with communication quantization and communication delays simultaneously are considered. Discrete-time protocol and continuous-time protocol are, respectively, discussed in Sects. 4.1 and 4.2. We have proved that under the connected network topology, the multi-agent network can achieve consensus.

(c) Chapter 5 is concerned with Problem 1.2. Compared with the traditional periodic sampled-data control or time-driven control method, the event-driven control method is more flexible and robust in some real multi-agent systems.



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In Chap. 5, discrete-time and continuous-time multi-agent consensus problems under event-triggered control are studied, respectively. Discrete-time consensus protocol with communication delays is discussed in Sect. 5.1. In Sect. 5.2, continuous-time event-triggered and self-triggered consensus protocols are proposed, respectively. It is also shown that the Zeno behavior can be excluded under our proposed event-based protocol with communication delays.

(d) Chapters 6 and 7 are concerned with Problem 1.2 and Problem 1.2, respectively. Continuous-time bipartite consensus and fixed-time/finite-time bipartite consensus problems in networks of agents with antagonistic interactions and communication delays are investigated. Effective consensus protocols are designed to realize expected collective behaviors. By the Lyapunov stability method and homogeneity of function analysis, the relation between the order of $\phi(x)$ and the speed of convergence is obtained.

(e) Chapters 8 and 9 are devoted to studying Problem 1.2. In Chap. 8, the exponential synchronization behavior of a general complex dynamical network is investigated. Some sufficient conditions are proposed to guarantee the globally exponential synchronization of the network. Moreover, one quantity is distilled from the coupling matrix to characterize the synchronizability of corresponding dynamical networks. Chapter 9 mainly studies the pinning cluster synchronization of coupled neural networks by a novel event-triggered mechanism. An effective distributed event-triggered scheme is proposed to realize expected cluster synchronization and meanwhile exclude the Zeno behavior. Under event-triggered mechanism, some controllers will be pinned to certain selected nodes in coupled neural networks to realize expected cluster synchronization.

(f) Chapter 10 is devoted to Problem 1.2. First, consensus analysis of nonlinear multi-agent network with arbitrary communication topology is given, which uses the global information of the network. For large scale multi-agent networks, to reduce the size of the networks and meanwhile conserve the consensus property, a new network reduction approach is proposed which only uses the local information of the network in the reduction process. Furthermore, based on the network reduction method discussed in this chapter, a novel consensus recovery approach is provided to improve the reliability of the network system and preserve the consensus property under node failure.

## 11.2 Future Work

Some related topics for future research are listed as follows:

(1) In this book, we only study the first-order multi-agent consensus problem under imperfect communications. In the future, we will extend the results of this book to the high-order multi-agent consensus problems. In particular, we will design new consensus protocols for higher-order multi-agent systems and investigate how quantization and communication delays affect the final

consensus results. Moreover, we will design event-triggered consensus protocol for higher-order multi-agent networks and derive distributed event-triggered conditions for reaching consensus.

(2) In Chaps. 3 and 4, the effect of two kinds of uniform quantization is analyzed in detail. In future work, we will focus on the effect of other different types of quantization, such as logarithm quantization, in the distributed complex networked systems. Moreover, in many real-world complex networks, individuals in the network are often able to build new links or suppress old ones among themselves as time goes on. Hence, collective dynamical behavior problems with quantization and communication delays under time-varying communication topology will be considered in future research.

(3) In Chaps. 6 and 7, bipartite consensus of multi-agent systems with antagonistic interactions and communication delays is studied. In the future, we will extend the results of this book to the bipartite synchronization problems. Furthermore, we will focus on more imperfect communication on the cooperative-competitive complex networks, such as quantization, data dropout, noise, etc.

(4) Chapters 8 and 9 mainly concentrate on the synchronization problem of complex dynamical networks with communication delays. We will further consider the collective behaviors of stochastic complex networks with communication delays in our future work. Furthermore, the application of network synchronization problem with communication delays also will be studied, such as distributed Kalman filtering for sensor networks, distributed fault diagnosis, etc.

(5) In Chap. 10, network reduction and recovery methods were proposed for multi-agent consensus with node failure. In the future, we will also consider the reverse problem of the network reduction process, i.e., what would happen if some new nodes were added to the multi-agent network? The application of network reduction and its converse process in some practical problems, such as optimal allocation of sensors network, distributed fault diagnosis and consensus maintenance, and so on, shall all be explored in the future work.