Chapter 9
Mathematics Education Research: Impact on Classroom Practices

Yew Hoong Leong

Abstract The longstanding criticism against education research is: Has it made a difference to actual classroom practice? In this chapter, I present a case for the affirmative in the context of mathematics education research in Singapore – not merely by describing cases but also extracting common underlying features that contribute to impact. These examples include the now well-known 'model method', mathematics problem-solving and the concrete-pictorial-abstract instructional heuristic.

Keywords Singapore mathematics · Instructional practices · Mathematical problem-solving · Instructional materials

For this chapter, I begin with a reflection of a specific area of mathematics education research work that I have been engaged in over the last decade which I consider one of the most impactful in terms of how actual classroom practices have shifted as a result of our research involvement. This zoom-in to one sustained research project is not merely to provide concrete specificity to readers who might not be 'insiders' to the Singapore mathematics education research scene; I mean to use a case to illustrate some characteristics of local research that can lead to a better understanding of 'impactful mathematics education research' in Singapore. I then broaden the scope of inquiry to include other mathematics education research programmes that have been identified as impactful to classroom practices in Singapore.

Y. H. Leong (✉)
Mathematics & Mathematics Education (MME), National Institute of Education, Nanyang Technological University (NIE/NTU), Singapore, Singapore
e-mail: yewhoong.leong@nie.edu.sg

O. S. Tan et al. (eds.), *Singapore Math and Science Education Innovation*, Empowering Teaching and Learning through Policies and Practice: Singapore and International Perspectives 1, https://doi.org/10.1007/978-981-16-1357-9_9

9.1 Mathematical Problem-Solving

My research work in mathematical problem-solving (MPS) formally started when I was a member of a research team in the project that was entitled ‘Mathematical Problem Solving for Everyone’ (MProSE) in 2009. Our interest in MPS arose from a few motivations:

(1) As mathematicians and mathematics educators, we have a deep commitment to the disciplinarity of mathematics; and MPS is at the heart of this disciplinarity. To clarify, when we speak of MPS, we are – together with many international researchers in this area of work (e.g. Schroeder & Lester, 1989; Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005) – referring to the work of solving mathematics problems that are experienced as ‘problems’ to the solver. In other words, within the ambit of ‘problem’ – as we conceived it – is not included the common types of mathematics questions in textbooks and school tests that are deemed as routine and only-procedural for the students. To us, ‘problems’ are tasks that will pose some mental ‘blockade’ because the solution path is not so readily obvious to the students. An example of such a problem is as follows.

9.1.1 Phoney Russian Roulette

Two bullets are placed in two consecutive chambers of a six-chamber revolver. The cylinder is then spun. Two persons play a safe version of Russian Roulette. The first points the gun at his mobile phone and pulls the trigger. The shot is blank. Suppose you are the second person and it is now your turn to point the gun at your mobile phone and pull the trigger. Should you pull the trigger or spin the cylinder another time before pulling the trigger? [For a solution of this problem and its potential to encourage students in the work of MPS, see Toh, Quek, Leong, Dindyal, and Tay (2011).]

For most, this problem does not trigger a set of ready-to-use mathematical procedure to follow (or it may trigger an initially incorrect intuition, ‘Should spin’). A typical solver would then need to slow down, reread the question, draw a diagram to make sense, tap upon relevant mathematical concepts (in this case, likely to be about probability) and devise a strategy that would help advance the solution (and, if need be, loop back to repeat the process if one is ‘stuck’). It is this disposition of productive struggle towards devising one’s own solution strategy – instead of merely following a set of procedural steps – that approximates the work of doing mathematics within the discipline and which we desire more students in our schools to learn.

(2) But, MPS of the kind we described in (1) is relatively uncommon in mathematics classrooms. This is the case as described in numerous articles internationally (e.g. Stacey, 2005) and also locally (e.g. Ho & Hedberg, 2005). That MPS is so elusive in our schools despite many decades of related extensive research and

developmental work shows that regularising MPS in schools is an immensely challenging task. However, instead of discouraging us, the scale of the challenge is a source of motivation.

(3) This does not mean that we underestimate the multifaceted challenges of such a task. But we think it is vitally important that we identify clearly (and hence train our focus) on the key gap in this enterprise. We agree with Schoenfeld (2007, p. 539):

> That body of research—for details and summary, see Lester (1994) and Schoenfeld (1985, 1992)—was robust and has stood the test of time. It represented significant progress on issues of problem solving, but it also left some very important issues unresolved. … The theory had been worked out; all that needed to be done was the (hard and unglamorous) work of following through in practical terms.

In other words, there is a substantial and reliable corpus about MPS in terms of frameworks to analyse an individual's attempt at MPS; but there is far less research on 'making it work' in a sustainable way in mathematics classrooms. This is the gap that we are motivated to fill: to develop a 'theory of action' (Argyris & Schon, 1978; Henrick, Cobb, & Jackson, 2015) that would translate theoretical ideas of MPS into workable instructional practices as routines in the classroom.

9.2 MProSE

MProSE was the embodiment of our motivations. The MProSE began with a cooperative school in Singapore that provided conducive conditions for success – in our case, it was a school that ostensibly specialised in mathematics and science. Also, it was a school that ran an 'Integrated Programme', which meant that they had a mathematics curriculum which covered Year 7 to Year 12 without the usual Year 10 major high-stakes examination (and the associated distribution of students to other senior high schools). Without the constraints of gearing students for a common nationwide examination, there is more room for insertion of other emphases, such as MPS, in their mathematics curriculum. MProSE adopted a design research stance in the project: the goal of the research was to iteratively refine the entire MPS setup within the school, along multiple intertwined aspects which will be elaborated later; concomitantly, the theory of action was adjusted to account for the findings we obtained at various junctures of the project.

We worked intensively with the first school for about 3 years. As it turned out, we were able to get quite far with the school on MPS: All the mathematics teachers participated in a 10-h professional development programme on basic MPS framework, to familiarise them with the language and practice of MPS; through Lesson Study cycles (Stepanek, Appel, Leong, Mangan, & Mitchell, 2007), we were able to discuss with the teachers using actual instructional experiences the ways in which MPS can be taught in the classrooms (and the issues that needed to be attended to); the school adopted an MPS module for all their Year 12 students – just like other elective modules offered in the school's instructional programme for students – consisting of the contents we developed with them throughout the duration of the project. In the process, we developed our theory of action for scaling up the teaching of MPS to more schools. The theory consists of three closely linked components: conjectures, strategies and programme.

9.3 MProSE Theory of Action

9.3.1 Conjectures

These are the overarching principles that guide our entire research and development work with respect to spreading the teaching of MPS to more schools:

C1. The work of sustaining and scaling the teaching of MPS is a social process that involves diffusion of instructional innovation (Quek, Leong, Tay, Toh, & Dindyal, 2012; Rogers, 2003). The process is carried out through the community in social units of increasingly larger grain sizes, beginning with success at a smaller social unit. This principle applies within school and across schools.

C2. The work of sustaining and scaling the teaching of MPS involves teacher buy-in at each stage of the diffusion process (Bobis, 2011; Leong et al., 2011). Buy-in requires sufficient knowledge of and proximal contact with the innovation. In our case, it involves teachers participating in the experience of solving mathematics problems and in observing/teaching MPS instruction in actual classrooms.

C3. The work of sustaining and scaling the teaching of MPS requires the persistent support of school leaders (Lemke & Sabelli, 2008; Leong, Kaur, & Kwon, 2017). This refers both to the temporal duration of support (i.e. willingness to wait out for a longer term for instructional changes to take effect) and to the investment of structural support in terms of setting aside regular time for continual teacher professional development.

9.3.2 Strategies

These strategies are consistent with the conjectures and at an actionable level of consideration:

S1. Build a coherent group of researchers who also take on the role of professional development facilitators. This point is hardly mentioned in the literature. The reality of multiple-sites research and the concomitant demands of resources in expertise and time mean that the work cannot be confined to one or two experts.

S2. Invest heavily in each school *initially*. The human factors and the need to take into account the contextual givens necessitate this heavy investment approach, at least to a point when 'success' is visible to teachers and leaders of the school.

S3. Distinguish theoretical foundation from practical accommodation. The theoretical 'body of research' (Schoenfeld, 2007, and quoted above) on MPS is foundational and thus should form the non-negotiable basis of engagement with the schools. In terms of the basic framework on the key stages of MPS, we take it as well-tested, but there is nevertheless room for evidence-based peripheral refinements. Practical accommodations, however, refer to the tweaks that could be made to adapt to the local conditions of each school to increase the opportunities for success. These accommodations would not compromise on the theoretical grounds of the project.

S4. Leverage on the concrete instructional materials developed in the initial school. This is emphasised in other scaling-up research (e.g. Coburn, 2003; Tatar et al., 2008). Instead of discussing 'from scratch' about how to teach MPS, we used concrete instructional materials – such as actual mathematics problems, video segments of teaching MPS, assessment tools and lesson plans – refined from the initial school as a starting point to clarify goals and discuss adaptations.

9.3.3 Programme

In this section, I describe briefly the actual programme of engagement with the schools as a way to realise more specifically the strategies devised in the previous section:

P1. The first phase is for teachers to learn about MPS. We meet the teachers over a number of sessions that total some 10 h. All the mathematics teachers in the participating schools should be involved in this phase. Mathematical problems, such as the Phoney Russian Roulette Problem, which are mathematically rich in demonstrating various aspects of MPS will be introduced. The teachers will be given opportunities to solve problems and to learn our theoretical basis of MPS. In particular, we will cover Pólya's (Polya, 1945) four-stage model of Understand the Problem, Devise a Plan, Carry out the Plan, and Look Back and the four components of Schoenfeld (1985) for successful problem-solving, namely, cognitive resources, heuristics, belief system and control.

P2. The second phase is for teachers to learn to teach MPS. We will meet with each school to discuss the details of how the teachers intend to carry out the MPS module in their respective curriculum. During this phase, there will be intensive discussions on the suitability of the problems in the original set of materials given and how each problem can be tweaked or replaced for the students involved. There will also be opportunities to walk through with the teachers how some of these problems can be launched and scaffolded in the classroom.

P3. The third phase involves the embedding of MPS into the regular structure of the schools' mathematics curriculum. In this phase, the MPS module should be compulsory for the targeted students in the respective schools. Selected teachers who participated in the earlier two phases of professional development will teach the MPS module to the students. Experts will be assigned to each of the schools to hold regular discussions with the teachers with a view of tweaking elements of implementation.

P4. Further refinements in the mathematics problems and the way they will be used will be made for subsequent cohorts of students over the next few years. At this phase, the researchers should gradually retreat to the background and play an advisory role to the participating teachers.

9.4 MProSE Impact

We were guided by the explicated theory of action as we broadened MProSE design research work to four other Singapore schools. These four schools (labelled as A, B, C and D here) spanned the spectrum of Singapore secondary schools. After 4 years of work with these schools, I summarise the impact with respect to the adoption of MPS as follows.

The MPS in all the schools displayed a high degree of fidelity to the theoretical cornerstones of Polya's stages and Schoenfeld's framework (i.e. the theoretical foundation as delineated in Strategy S3), and yet each school differed in some local adaptations to suit their respective contexts (i.e. the practical accommodation mentioned in Strategy S3). As an example, Schools A, B and D implemented the MPS module in Year 7 but School C did so for Year 8.

As to the concrete instructional materials (i.e. Strategy S4), they were generally adopted by all the schools with minor modifications. The changes were in the set of problems used. Through their experience from detecting the level of their students' engagement with each problem over the years, they had selected different problems that were more suited to their students' profile. For example, the Phoney Russian Roulette Problem was highly recommended by School C as the students were readily engaged with the problem; however, teachers in School B (an all-girls school) noticed that the girls did not resonate well with revolvers.

Across the schools, we did not witness a fast growth in terms of the number of teachers involved in the actual teaching the MPS module. Nevertheless, there was a sizeable core of teachers in every school who remained since the start of MProSE within their schools to provide stability through the years of development of the module. In addition, these teachers had developed deepened appreciation of MPS and the teaching of MPS. This can be interpreted as a consequence of Strategies S1 and S2. The 'deepening' was along different dimensions in different schools due to different emphases in each school. For example, in School A, the deepening resulted in the identification of critical instructional skills that they needed to attend to for successful teaching of MPS; in School B, the deepening had more to do with the teachers' growth in the usefulness of MPS for themselves and for their students' learning of mathematics. Such deepening contributed to the growth of teacher capacity for the teaching of MPS.

There was also a long-term commitment to MPS instruction which reflected not merely a one-off buy-in by the leadership at the start of the project, but a process of ongoing buy-in throughout the project duration. This was evidenced by the moves taken by all the schools to make MPS a mainstay in their mathematics instructional programme. Factors that contributed to this renewal of buy-in included visibility of success, entrenchment of structures – such as a permanent place of the MPS module in the curriculum – and sunk-in investment of resources.

9.5 Reflections of MProSE and Zooming Out from It

As mentioned at the start of this chapter, the purpose of zooming-in to a particular project is not merely to illustrate a concrete case of mathematics education research that had significant impact on instructional practices; it also provides us with an opportunity to reflect upon characteristics of impactful education research. I summarise my reflections along the following categories.

9.5.1 Intersecting Domains of the Project

The main focus of the project should lie within the intersection of these domains of pursuit: research, policy, practice and disciplinarity. This is the case for MProSE. In terms of research, as mentioned in the earlier paragraphs, although basic research in MPS is well-developed and extensive for several decades now, the 'applied research' – as in, translating the theoretical ideas of earlier research into workable implements in the schools – is scarce and thus provides the impetus for authentic inquiry. In this regard, design research holds promise.

But, the research agenda should also be in line with the emphases of policy. As shown in Fig. 9.1, MPS remains at the heart (diagrammatically, it is also the case) of the Singapore mathematics curriculum framework. This has been so since the pentagonal model was first crafted in the late 1980s (for an in-depth discussion on how mathematics education has evolved in Singapore, see Chap. 7). This sustained policy commitment to MPS not only provides an official endorsement to studies on MPS, but it also locates MProSE as a piece of research whose proposed impact goes beyond the immediate context of the research schools to the mathematics curriculum of the whole of Singapore. Not only so, the policy stamp adds legitimacy to teachers' involvement to the project as they would want to be participating in studies that are aligned to the intended curricular goals of schooling.

This leads to the domain of practice. Authentic research inquiry and alignment to policy objectives are not sufficient to motivate teachers' commitment to the aims of

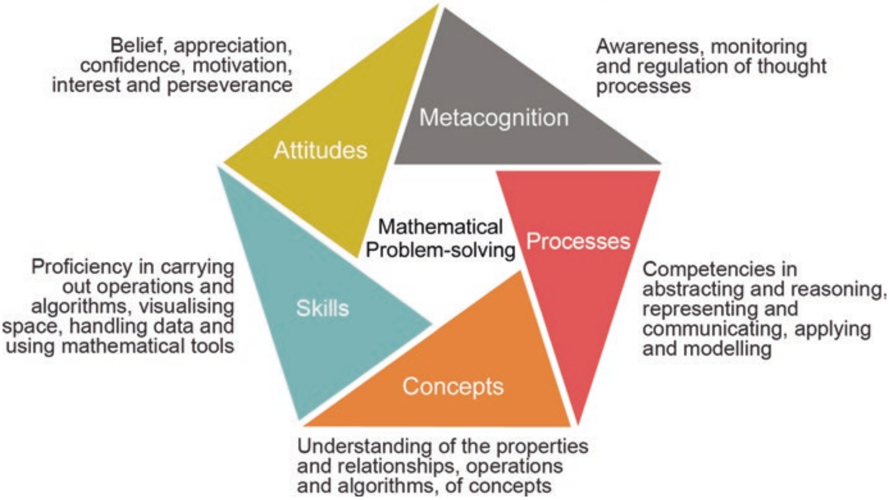

Fig. 9.1 The Singapore mathematics curriculum framework (MOE, 2019)

the project. For initial and continual buy-in, there is a requirement also for alignment to the aspirations of practice – as in, the project's focus is on an area where teachers can identify as an area they remain dissatisfied about in their current practice and thus desire for improvement. Teachers who participated in MProSE knew the challenges involved in teaching MPS in their classrooms; but they were also persuaded that it was a worthwhile goal because they wanted students to acquire the dispositions and skills of problem-solving. This gap between their intention and the actuality provided the motivation to take part in the project.

More specifically, it is not merely work related to teaching that would draw teachers' interest in the project; it is also the fact that the project is about *mathematics* problem-solving. This is the disciplinarity aspect of the enterprise. Especially in Singapore secondary schools where teachers' professional identity is closely linked to the subject they teach, the effort to propose collaborative projects with schools should take into account this nearness to practice which must include the disciplinary distinctives of pedagogical considerations. MProSE fulfills this because it does not deal with generic problem-solving skills – and their problematic nature of not being easily translatable to specific problems within mathematics. [One can undergo a 'generic' problem-solving course and still be unable to solve mathematics problems.] Rather, it addresses MPS tools and skills which are directly applicable to mathematics problems that teachers would use in their classes.

For research projects that have the potential to impact the instructional work of teachers, mathematics education researchers need to craft a research programme that is aligned to policy, meet the needs of practice and close to the discipline-centric focus of mathematics teachers.

9.5.2 Strong Commitment to Teacher Professional Development

By teacher professional development (PD), I do not mean a mere one-off course conducted for teachers. [We certainly did this too in MProSE – as described under P1.] It includes a continual programme of PD which can concretely support the teachers' knowledge and implementation of MPS. This PD programme would need to be conditioned by the same domains highlighted in the preceding section – research, policy, practice and disciplinarity – as in, the PD work is brought within the ambit of design research and its associated rigours of retrial and refinements; the PD work has to align with policy mandates; the PD is geared towards addressing the needs of practice; and the PD emphasis must also attend to the gaining of relevant mathematical knowledge within the discipline.

Concretely, PD cannot stop at the boundary of the classroom, but must cross it – that is, PD work includes the study of instructional strategies that are actually workable in the classroom. This involves observation, discussion, refinement, retrial and further iterations – features that are now characteristic of Lesson Study (Lewis, 2002) and described in P2–P4. In fact, we have gone beyond emphasis of a single lesson (which is the emphasis of Lesson Study) into co-designing with teachers a whole unit of lessons. This commitment derives from an acknowledgement that a single lesson does not constitute sufficient temporal and content space to exemplify how MPS – and for this matter, other worthwhile instructional innovations – can be successively carried in classroom instruction. Moreover, teachers think and plan lessons in terms of coherence across lessons within the unit; as such, many find it initially hard to locate a MPS lesson coherently within the development trajectory of a unit of lessons. Through this joint work of redesigning units, teachers participate in a form of PD that affords the learning of different perspectives which are nonetheless relevant to the work of teaching mathematics in the classroom. We call this strategy of co-evolvement of instructional design and PD the Replacement Unit Strategy. Specific descriptions of this strategy can be found in Leong et al. (2016, 2016).

This has implications to mathematics education researchers themselves. To undertake the kind of PD work as described here, it is not just a matter of commitment; it means that it is insufficient that they be merely theoreticians. They will need to understand the workings of classrooms and effective instructional work well so as to guide teachers in the PD experience. The intersection of these expertise is rare in a single person. This accounts for the earlier recommendation of a pool of closely working researchers that, taken together, possess a range of relevant expertise, as mentioned under S1.

It is hard to imagine research having impact in schools if it does not have a comprehensive, continual and coherent strategy in teacher PD.

9.5.3 Development of Instructional Materials

Even with the most intensive and relevant PD programme, it is common that actual classroom implementation falls short of the shared goals of PD (Hill, 2009; Wallace, 2009). We can see this as a gap between the PD setting and the mathematics classroom. The space between the two domains in Fig. 9.2 is a diagrammatic representation of this gap which hinders impact.

The perforated arrows in Fig. 9.2 show the areas in which links can be deliberately built in order to strengthen the opportunities to translate teacher learning in PD settings into classroom practices – and, hence, increase impact of PD work. Apart from working with teacher goals, which can be directly ‘carried’ into their instructional work in their mathematics classes, another area involved ‘concretisations’. These are objectifications of the innovation and design work which the researchers and teachers co-develop during PD settings. They are in the form of actual instructional materials which teachers can use as tools to realise the goals they bring into their teaching of mathematics. In MProSE as mentioned under S4, concretisations were in the form of actual mathematics problems, templates for students to work on these problems that would guide them along the stages and heuristics of Polya and representations on the whiteboards which teachers use to illustrate the stages of MPS. More can be said about the nature that would render such concretisations as effectively supportive of the innovation. But further discussions will necessarily bring us into the specifics of MProSE – which is not our purpose here, as MProSE is meant to help me illustrate the features that brought about impact. For more details about concretisations, the reader may refer to Leong et al. (2019).

Suffice for our current discussion is the emphasis on development of instructional materials that are suitable for actual use in the classrooms. The point is not merely that instructional materials be *provided* – many educational reform efforts both locally and elsewhere *provide* extensive curricular materials, but still fail in generating impact in the schools. Figure 9.2 draws our attention to the need to co-develop these materials that harness the buy-in and integration of teachers’ genuine goals in the process (see the triad on the left side of the figure). The bidirectionality of the perforated arrows also reminds us that this crafting of instructional materials is not a one-off work, but, consistent to the iterative nature of design experiments,

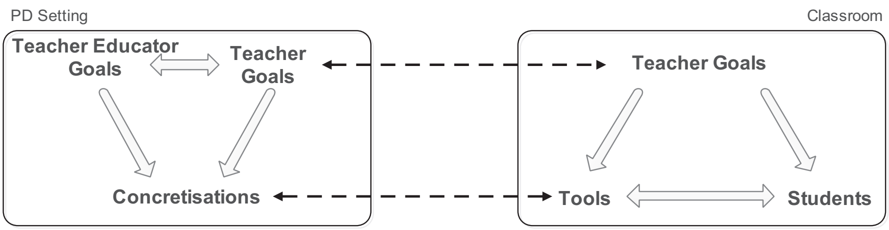

Fig. 9.2 A model of links between PD setting and the classroom, extracted from Leong, Tay, Toh, Quek, and Yap (2019)

involves an ongoing process of refinement that takes into account the use of the materials in actual classroom instruction.

9.5.4 Evidence of Success

This is in line with empirical research – claims will have to be substantiated with rigorous analysis of evidence. But in the case of research that is meant to lead to impact in schools, the evidence will need to be of a kind that persuades schools, particularly school leaders. It has to capture some form of 'success', as mentioned under S2. I do not think success in this case needs to be narrowly conceived – for example, to what statistical measures can substantiate. Evidence of success to schools can mean teachers' perception that research-informed innovations in their teaching lead to improvements in students' growth in certain aspects of mathematics and that this perception is similarly shared by the school leaders. In the case of MProSE, the teachers felt that the focus on MPS in their lessons provided both teachers and students with a common set of language tools to advance conversations about MPS, and they saw it as a positive development in their growth as mathematics teachers. This may explain the continual support of MProSE among the school leaders – to the extent that they were willing to commit resources (such as allocated curriculum hours and PD slots) permanently to the development of MPS expertise in the schools.

There are ingredients that can heighten the chance of success: (i) Start the research process with a school that is most conducive for success. This was described earlier as the best-case scenario approach to design research. (ii) Without compromising on the theoretical fundamentals, accommodate the research design to fit the contextual givens of the research school. This point was mentioned under S3. Instead of adopting a universalistic one-size-fits-all mindset, MProSE was flexible on matters that did not threaten the theoretical integrity of the research enterprise. This means that success can be better achieved within the local setting if we are prepared to tweak certain aspects of design to fit the particularistic context of schools and classrooms.

Evidence of success is important in Singapore schools because the education system here stresses high levels of accountability – at every level of the school structure. Teachers, heads of department and principals are expected to account for the investment of (extra) resources to particular projects, including research projects. Moreover, due to an open culture of change in schools (and, more broadly, the Singapore society), there is constant competition against other enterprises of change. Evidence of success provides the impetus and justification for staying with a particular innovation over the long term – which is essential for sustained impact.

9.6 Other Mathematics Education Research Programmes that Are Impactful

I should think that when an international colleague thinks about Singapore mathematics education research, they would first highlight the Singapore 'model method' of teaching mathematics at the primary levels. Much has been written about this over the last few decades (e.g. Ng & Lee, 2009), and so I would not repeat the details here. It involves a method of transforming word problems in mathematics into diagrammatic form which looks like comparative rectangles (also known as 'models') that allow students to compare and manipulate these visually to aid in solving the problems. This method is seen as 'powerful' at the primary levels – it does not require the rigour of solving equations algebraically and yet can be easily adapted to solve a whole range of problems that are equivalent to linear equations. This method was introduced in Singapore in the 1980s; today, all primary schools in Singapore teach the method to their students at the upper primary levels – some as early as Year 8.

Interestingly, this project of diffusing the 'model method' to all primary schools in Singapore shares the characteristics of impactful research that I described in the preceding section: it cuts across multiple domains of research, policy, practice and disciplinarity; there was sustained professional development for teachers to gain proficiency in the method, especially in the first decade since its introduction; there is an abundance of materials on the model method, including commercially produced books; and the sense of success with the use of the method is strong – students who use the method feel empowered to solve a wide range of word problems. However, unlike MProSE which was essentially an innovation which was conceived and driven by researchers initially, the 'model method' was largely from the policy 'centre' – initiated by curriculum developers from the Ministry of Education and subsequently developed through research formulations and tweaks arising from requirements of practice.

Another initiative which has impact and that shares this characteristic of arising from the centre of policy generation and was supported by the four features I listed earlier is the concrete-pictorial-abstract (CPA) instructional heuristic (Leong, Ho, & Cheng, 2015). It has its roots in the enactive-iconic-symbolic sequence of Bruner (1966). The change in labels of each of the modes appears more an attempt at language simplification rather than conscious theory revision. Translated to the sequencing of lessons, it means beginning the concept-exploration phases with facilitating students' access through concrete experiences; this is followed by a representation of these experiences into pictorial or diagrammatic forms; these are in turn expressed into increasingly more 'abstract' forms that approximate the technical language and symbols of mathematics. Illustrations of how this progression can be made in actual mathematics topics within the Singapore syllabus can be found in Leong et al. (2010) and Leong et al. (2016).

The CPA approach appeared in Singapore primary mathematics textbooks in the early 1980s. But the formulation as a guiding principle of teaching only began in the official documents of the Ministry of Education in the early 1990s. It was also then extended to the lower secondary levels. Today, the CPA strategy is a well-known label among Singapore mathematics teachers of all levels (and many international scholars in mathematics education). It is common to read of lesson plans crafted by teachers – both novice and expert – that appeal to CPA as the underlying principle in the ordering of mathematical content.

9.7 Going Forward

Along with global trends in education research, there is currently an emphasis on scalability of research, which is associated with the increasing demands from society and funding agencies to link research to impact. As described in this chapter, the mathematics education research community in Singapore is in keeping with this trend. Striving for impactful education research should remain the enterprise for the future.

I end this chapter with a few thoughts on Singapore mathematics education research in the foreseeable future:

1. Strengthen collaborations with policymakers and practitioners in conceptualising and trialling of promising theoretical innovations. These tight links among the various stakeholders in the education landscape are critical to the alignment of educational goals in Singapore. It is by sustained efforts of working together that educational designs can meet the standards required by all parties and thus be embedded in the system.
2. Develop pedagogies that are particularly suited for impact within the targeted cultural context. I think the Singapore mathematics education community has reached a point of maturity where we should seek out ‘organic’ pedagogies that have emerged robust within our evolving cultural systems – instead of merely looking for pedagogies ‘out there’. This does not mean that we become insular to the broader international development of pedagogical theories. The work is in the careful syncretising of theoretical models – that upon closer scrutiny may be derivable from incompatible foundational traditions. The question of ‘cultural fit’ should become increasingly significant.

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