Chapter 7
Stochastic Delay Differential Equations

7.1 Introduction

Real biological systems are always exposed to influences that are not completely understood or not feasible to model explicitly, and therefore, there is an increasing need to extend the deterministic models to models that embrace more complex variations in the dynamics. A way of modeling these elements is by including stochastic influences or noise. A natural extension of a deterministic differential equations model is a system of stochastic differential equations (SDEs), where relevant parameters are modeled as suitable stochastic processes, or stochastic processes are added to the driving system equations. Therefore, stochastic delay differential equations (SDDEs) are crucial in ecology, epidemiology, and many other fields. SDDEs are also considered as a generalization of both deterministic delay differential equations (DDEs) and stochastic ordinary differential equations (SODEs). Some basic concepts about stochastic differential equations are discussed in [1–3]. The fundamental theory of existence and uniqueness of the solution of SDDEs has been studied by Mao [4] and Mohammed [5]. Some stability properties of numerical schemes of SDDEs are also studied in [6–8].

An important characteristic of environmental noise is its spectrum, which describes variance as a sum of sinusoidal waves of different frequencies. The spectrum of frequencies in noise is particularly important to the dynamics and persistence of systems [9]. However, Brownian motion with normally distributed errors is commonly used in the continuous differential models of dynamical systems. In this monograph, we consider white noise type. In white noise, the variance is the same at all frequencies. Therefore, this is the most thoroughly studied and applied form of noise. The reason for this is that, it is a simple and easily articulated model for noise. From an observational perspective, the random effect of Brownian motion is more visualized with normally distributed errors [1, 6].

In the literature, many numerical schemes for SDDEs have been investigated, such as Euler-type schemes [10, 11], drift-implicit Euler scheme [12, 13], Milstein

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schemes [14, 15], split-step schemes [16, 17], and multistep schemes [18]. The extension of numerical approaches for SODEs to SDDEs is non-trivial, particularly since the time-delays may induce instabilities in the basic SDDEs, while their corresponding SODEs are stable [12]. In addition, the presence of time-delays influences the convergence order and computational complexity of the numerical schemes [19]. In general, there is no analytical closed-form solution of the models considered in this dissertation, and we usually require numerical techniques to investigate the models quantitatively.

In this chapter, we briefly study qualitative features of SDDEs (see Sects. 7.2 and 7.3). We also introduce some numerical schemes for their approximate solutions. We investigate local and global errors; convergence and consistency of the scheme. We discuss strong discrete time approximations of solutions of non-autonomous SDDEs, including Euler and Taylor schemes and implicit schemes. The proposed schemes converge in a strong sense. The mean-square stability of the Milstein scheme is also discussed; see Sects. 7.4 and 7.5.

7.1.1 Preliminaries

Definition 7.1 ([20]) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{A}_t\}_{t\geq 0}$. A one-dimensional (standard) Brownian motion is a real-valued continuous $\{\mathcal{A}_t\}$-adapted process $\{W_t\}_{t\geq 0}$ satisfying the following properties:

1. $W(0) = 0$ a.s. (with probability 1).
2. For $0 \leq s < t \leq T$, the random variable given by the increment $W(t) - W(s)$ is normally distributed with mean zero and variance $t - s$; equivalently, $W(t) - W(s) \sim \sqrt{t-s}N(0, 1)$, where $N(0, 1)$ denotes a normally distributed random variable with zero mean and unit variance.
3. For $0 \leq s < t < u < v \leq T$, the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

Example 7.1 Let us consider the Hutchinson equation

$$\frac{dy(t)}{dt} = ry(t)\left(1 - \frac{y(t-\tau)}{K}\right). \tag{7.1}$$

Here, $r > 0$ is the intrinsic growth rate, $K > 0$ is the carrying capacity of the population, and time-delay τ is considered as hatching time. We can add a small random perturbation σdW, usually referred to as the noise term in Eq. (7.1), which then becomes

$$dy(t) = \left[ry(t)\left(1 - \frac{y(t-\tau)}{K}\right)\right]dt + \sigma dW. \tag{7.2}$$

In Eq. (7.2), the noise term does not include the dependent variable y, and hence, the equation is referred to as an SDDE with additive noise. However, it may be more natural to consider our extension from the Hutchinson equation by looking at the proportionate population change $\frac{dy(t)}{y(t)}$ and adding our stochastic term to this quantity. This gives us

$$\frac{dy(t)}{y(t)} = \Big[\big(1 - \frac{y(t-\tau)}{K}\big)\Big] dt. \tag{7.3}$$

Therefore, Eq. (7.3) becomes

$$\frac{dy(t)}{y(t)} = \Big[r\big(1 - \frac{y(t-\tau)}{K}\big)\Big] dt + \sigma dW. \tag{7.4}$$

Multiplying by $y(t)$ gives us the following SDDE with multiplicative noise:

$$dy = \Big[r\Big(1 - \frac{y(t-\tau)}{K}\Big)y(t)\Big] dt + \sigma y(t) dW. \tag{7.5}$$

This implies a more natural procedure, and we will only consider equations with multiplicative noise in this thesis. Figure 7.1 shows the effect of environmental fluctuations on a Hutchinson equation, such that $r = 0.15$ and $k = 1$. The figures at the top show simulation results for $\tau = 5.6$, which indicates that the population attains its steady state value of 1 regardless of the external noise. Hence, it fluctuates within the interval [0.95,1.15] as $\sigma^2 = 0.01$ (top-left), and as the intensities of white noise increases to $\sigma^2 = 0.05$, it fluctuates within [0.65,1.5] (top-right). When the magnitude of time-delay is increased to a threshold value $\tau = 11$ (periodic oscillations) and taking $\sigma^2 = 0.01$, the stochastic fluctuations disappears (bottom-left). As $\sigma^2 = 0.05$, we observe abrupt oscillation in population (bottom-right).

Remark 7.1 An important fact about the impact of environmental noise is that, it can suppress a potential population explosion [21]; see Fig. 7.2.

To illustrate this phenomenon, let us consider DDE with pure delay

$$\frac{dy}{dt} = \mu_1 y(t-\tau). \tag{7.6}$$

Equation (7.6) with multiplicative noise takes the form

$$dy = \mu_1 y(t-\tau)dt + \sigma y(t) dW. \tag{7.7}$$

As $\mu_1 > 0$, the solution of (7.6) increases exponentially to infinity as $t \to \infty$. However, Fig. 7.2 shows the effect of environmental fluctuations on (7.6), with $\mu_1 = 0.06$, $\tau = 0.4$, and $\sigma^2 = 0.16$.

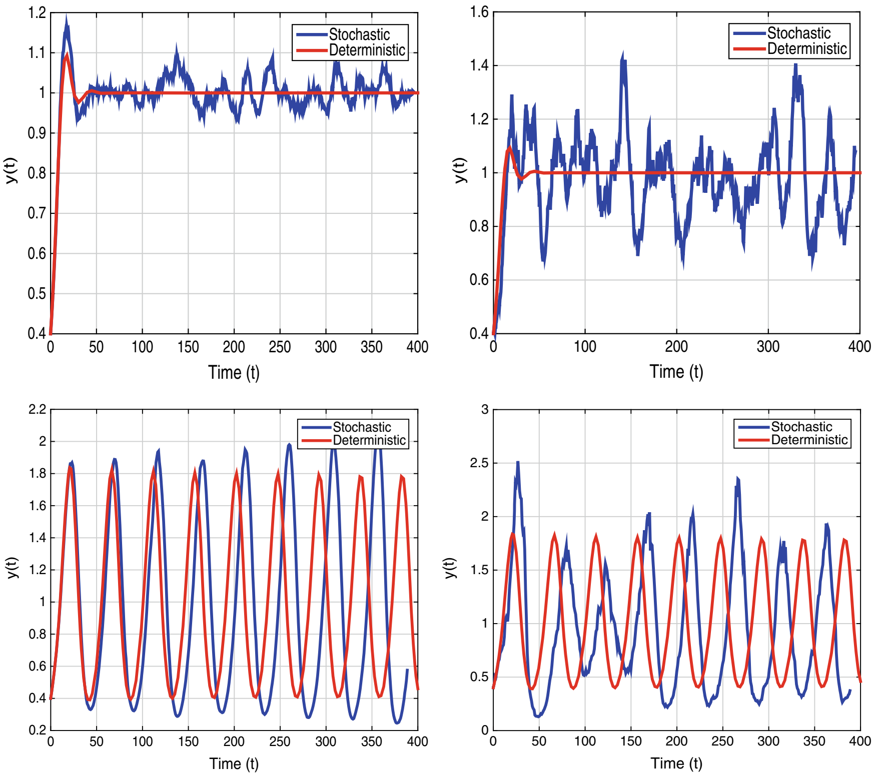

Fig. 7.1 Numerical simulations of deterministic Hutchinson DDE (7.1) and its corresponding SDDE (7.5) when $r = 0.15$ and $k = 1$

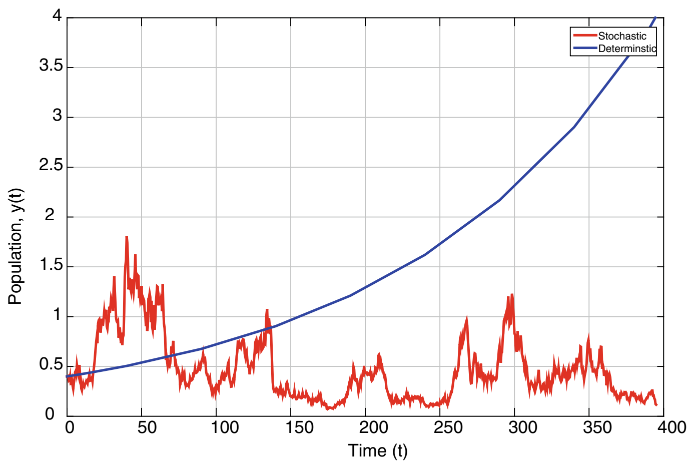

Fig. 7.2 How environmental Brownian noise suppresses explosions in population dynamics, described by $dy = \mu_1 y(t-\tau)dt + \sigma y(t)dW$ and its corresponding deterministic Eq. (7.6)

7.2 Existence and Uniqueness of Solutions for SDDEs

Let us consider d-dimensional SDDEs with r-dimensional standard Wiener processes on the filtered probability space $(\Omega, \mathcal{A}, \mathcal{A}_{t_0}, \mathbb{P})$. Therefore, we have equations of the form

$$d\mathbf{y}(t) = \underbrace{\mathbf{f}(t, \mathbf{y}(t), \mathbf{y}(\mathbf{t}-\tau))}_{\text{drift coefficient}}\, dt + \underbrace{\sum_{j=1}^{r} \mathbf{g}_j(t, \mathbf{y}(t), \mathbf{y}(t-\tau))\, d\mathbf{W}_j(t)}_{\text{diffusion coefficient}}, \quad t \in [0, T],$$

$$\mathbf{y}(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{7.8}$$

With one fixed delay τ, where $\psi(t)$ is an $\mathcal{A}_{t_0}$-measurable $C([-\tau, 0], \mathbb{R}^d)$-valued random variable. The drift coefficient $\mathbf{f} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and the diffusion coefficient $\mathbf{g}_j : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $j = 1, 2, \ldots, r$ are d-dimensional. Equation (7.8) can be formulated as

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t \mathbf{f}(s, \mathbf{y}(s), \mathbf{y}(s-\tau))ds + \sum_{j=1}^{r} \int_0^t \mathbf{g}(s, \mathbf{y}(s), \mathbf{y}(s-\tau))d\mathbf{W}_j(s), \tag{7.9}$$

for $t \in [0, T]$ and with $\mathbf{y}(t) = \psi(t)$ for $t \in [-\tau, 0]$.

Definition 7.2 (*Strong solution*) A d-dimensional stochastic process $\mathbf{y} = \{\mathbf{y}(t) : [-\tau, T]\}$ is called a strong solution of (7.8), if it has the following properties:

- $\{\mathbf{y}(\mathbf{t})\}$ is measurable, sample continuous process and $(\mathcal{A}_t)_{0 \le t \le T}$-adapted;
- Equations (7.8) and (7.9) hold for every $t \in [0, T]$ almost definitely.

Definition 7.3 (*Path-wise unique solution*) Let the set $\mathcal{X}$ denote some class of stochastic processes that solve (7.8). If any two processes $y^{(i)} = \{y^{(i)}(t), t \in [-\tau, T]\}$, $i = 1, 2$ from $\mathcal{X}$ with the same initial functions have the same path on $[0, T]$, almost definitely, i.e.,

$$\mathbb{P}(\sup_{0 \le t \le T} |y^{(1)}(t) - y^{(2)}(t)| > 0) = 0, \tag{7.10}$$

then the solution of (7.8) is path-wise unique within $\mathcal{X}$.

Herein, we formulate the Lipschitz condition (L_1) and growth condition (L_2) to guarantee the existence of a unique solution of (7.8). Assuming that $|.|$ denotes the Euclidian norm, we have

(L_1) *Lipschitz condition*: There exists a constant $K \in (0, \infty)$, such that

$$|\mathbf{f}(t, x_1, y_1) - \mathbf{f}(t, x_2, y_2)| + |\mathbf{g}_1(t, x_1, y_1) - \mathbf{g}_1(t, x_2, y_2) + \cdots + |\mathbf{g}_r(t, x_1, y_1) - \mathbf{g}_r(t, x_2, y_2)| \le K(|x_2 - x_1| + |y_2 - y_1|),$$

for $t \in [0, T]$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$.

(L_2) Growth condition: There exists a constant $G \in (0, \infty)$, such that

$$|\mathbf{f}(t,x,y)|^2 + |\mathbf{g}_1(t,x,y)|^2 + \cdots + |\mathbf{g}_r(t,x,y)|^2 \leq G(1+|x|^2+|y|^2),$$

for $t \in [0,T]$ and $x, y \in \mathbb{R}^d$.

Let $C = C([-\tau,0],\mathbb{R}^d)$ be the Banach space of all d-dimensional continuous functions η on $[-\tau,0]$ equipped with the sup-norm $\|\eta\|_C = \sup_{s\in[-\tau,0]}|\eta(s)|$. For every function $\xi|[-\tau,T] \to \mathbb{R}^d$ and every $t \in [0,T]$, so that

$$\xi_t = \{a_t(s) := \xi(t+s), s \in [-\tau,0]\},$$

a function defined on $[-\tau,0]$, the segment of ξ at t. In the same manner, the segment-valued function $t \to \xi_t$ for $t \in [0,T]$ is obtained. Additionally, we denote $\mathcal{L}_2(\Omega, C, \mathcal{A}_0)$, the set of $\mathbb{R}^d$-valued continuous processes $\eta = \{\eta(s), s \in [-\tau,0]\}$ with $\eta(s)$ being $\mathcal{A}_0$-measurable for all $s \in [-\tau,0]$ and

$$\mathbb{E}\|\eta\|_C^2 = \mathbb{E}\sup_{s\in[-\tau,0]}|\eta(s)|^2 < \infty. \tag{7.11}$$

Note that the initial function ψ can be considered as a square integrable $C = C([-\tau,0],\mathbb{R}^d)$-valued random variable on $(\Omega, \mathcal{A}_0, \mathbb{P})$. Hence, the above assumptions lead to the following theorem:

Theorem 7.1 ([11]) *Assume that* (L_1) *and* (L_2) *hold, and* ψ *be in* $\mathcal{L}_2(\Omega, C, \mathcal{A}_0)$. *Then the SDDE (7.8), with initial segment* ψ*, has a path-wise unique strong solution* $\mathbf{y} = \{\mathbf{y}(t), t \in [-\tau,T]\}$ *in* $\mathcal{L}_2(\Omega, C, \mathcal{A}_0)$. *Moreover*

$$\mathbb{E}\sup_{t\in[-\tau,t]}|\mathbf{y}(t)|^2 < \infty, \tag{7.12}$$

and for each $t \in [0,T]$*, the segment* $\mathbf{y}_t = \{\mathbf{y}(t+s), s \in [-\tau,0]\}$ *is a* $C([-\tau,0],\mathbb{R}^d)$*-valued process having continuous paths. Additionally, if we have* $\mathbb{E}\|\psi\|_C^{2k} < \infty$ *for some* $k \geq 1$*, then*

$$\mathbb{E}\|\mathbf{y}_t\|_C^{2k} = \mathbb{E}\sup_{s\in[-\tau,0]}|\mathbf{y}(t+s)|^{2k} < \infty \tag{7.13}$$

and

$$\mathbb{E}\|\mathbf{y}_t\|_C^{2k} \leq C_k[1+\mathbb{E}\|\psi\|_C^{2k}]. \tag{7.14}$$

For the proof of the above theorem, refer to [5].

Consider $W(t)$ to be a one-dimensional Wiener process, an autonomous scalar stochastic delay differential equation of the form

$$\begin{aligned} dy(t) &= f(y(t), y(t-\tau))dt + g(y(t), y(t-\tau))dW(t), \quad t \in [0,T], \\ y(t) &= \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{7.15}$$

Equation (7.15) can be formulated as

$$y(t) = y(0) + \int_0^t f(y(s), y(s-\tau))ds + \int_0^t g(y(s), y(s-\tau))dW(s), \quad (7.16)$$

for $t \in [0, T]$ and with $y(t) = \psi(t)$ for $t \in [-\tau, 0]$. The second integral in (7.16) is a stochastic integral in the Itô sense. If it is taken as a Stratonovich integral, we will use notation of the form $\int_0^t g(s, y(s)) \circ dW(s)$. Let us consider $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $\psi : [-\tau, 0] \to \mathbb{R}$. Now, we introduce the following theorem for Eq. (7.15) [10, 22]:

Theorem 7.2 *Problem (7.15) has a unique strong solution, provided that the uniform Lipschitz condition and a linear growth bound are satisfied for both f and g.*

Example 7.2 Consider the stochastic delay differential equation

$$\begin{aligned} dy(t) &= \mu_1 y(t-\tau)dt + \sigma dW(t), \quad t \geq 0, \\ y(t) &= t+1, \quad t \in [-\tau, 0]. \end{aligned} \quad (7.17)$$

Assume that $\mu_1 = -1$ and $\tau = 1$; we can easily verify the conditions of Theorem 7.2. Thus, we solve (7.17) using Itô's formula in the interval [0,1], so that

$$y_1(t) = y(0) - \int_0^t s ds + \int_0^t \sigma dW(s) = 1 - \frac{t^2}{2} + \sigma W(t).$$

In the interval [1, 2], we have

$$\begin{aligned} y_2(t) &= y(1) + \sigma W(1) + \int_1^t (-1 + \frac{(s-1)^2}{2} + \sigma W(s-1))ds + \int_1^t \sigma dW(s) \\ &= \frac{(t-1)^3}{6} - t + \frac{3}{2} + \int_1^t \sigma W(s-1)ds + \sigma W(t). \end{aligned}$$

Similarly, in the interval [2, 3], the solution is

$$\begin{aligned} y_3(t) &= -\frac{1}{3} - \int_2^t \Big(\frac{(t-2)^3}{6} - t + \frac{5}{2}\Big)ds + \int_1^2 \sigma W(s-1)ds + \sigma W(2) \\ &\quad + \int_2^t \int_1^{s_1-1} \sigma W(s-1)ds ds_1 + \int_2^t \sigma W(s-1)ds + \int_2^t \sigma dW(s) \\ &= \frac{8}{3} - \frac{(t-2)^4}{24} + \frac{t^2}{2} - \frac{5}{2}t + \int_1^2 \sigma W(s-1)ds + \int_2^t \int_1^{s_1-1} \sigma W(s-1)ds ds_1 \\ &\quad + \int_2^t \sigma W(s-1)ds + \sigma W(t). \end{aligned}$$

Note that $\int_0^t \sigma dW(s)$ is a martingale. Hence, $\mathbb{E}\Big(\int_0^t \sigma dW(s)\Big) = 0$. To find the mean function of $y(t)$, we can take the expectation of the solutions on their intervals as follows:

$$\mathbb{E}(y(t)) = \begin{cases} 1 - \frac{t^2}{2}, & t \in [0, 1]; \\ \frac{(t-1)^3}{6} - t + \frac{3}{2}, & t \in [1, 2]; \\ \frac{8}{3} - \frac{(t-2)^4}{24} + \frac{t^2}{2} - \frac{5}{2}t, & t \in [2, 3]. \end{cases}$$

Numerical methods for SDDEs are currently being actively studied and developed. Hence, they should be used carefully for deterministic DDEs and Stochastic Ordinary Differential Equations (SODEs).

7.3 Stability Criteria for SDDEs

There are at least three different types of stability for SDDEs [6]. Consider the following scalar SDDE with $W(t)$ being a one-dimensional Wiener process:

$$\begin{aligned} dy(t) &= f(t, y(t), y(t-\tau))dt + g(t, y(t), y(t-\tau))dW(t), \quad t \in [0, T], \\ y(t) &= \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{7.18}$$

Hence, Eq. (7.18) can be formulated as

$$y(t) = y(0) + \int_0^t f(s, y(s), y(s-\tau))ds + \int_0^t g(s, y(s), y(s-\tau))dW(s). \tag{7.19}$$

We are supposed to be concerned with the main ideas of the pth mean stability of the trivial solution of Eq. (7.19) with respect to perturbations in $\psi(.)$ (for $1 \le p < \infty$), and also with mean-square stability when $p = 2$.

Definition 7.4 ([23]) For some $p > 0$, the trivial solution of the SDDE (7.19) is called

- Locally stable in the pth mean, if for each $\epsilon > 0$, there exists a $\delta \ge 0$ such that $\mathbb{E}(|y(t; t_0, \psi)|^p) < \epsilon$ whenever $t \ge t_0$ and $\mathbb{E}(\sup_{t\in[t_0-\tau,t_0]} |\psi(t)|^p) < \delta$;
- Locally asymptotically stable in the pth mean if it is stable in the pth mean and if there exists a $\delta \ge 0$ such that whenever $\mathbb{E}(\sup_{t\in[t_0-\tau,t_0]|\psi(t)|^p}) < \delta$, then $\mathbb{E}(|y(t; t_0, \psi)|^p) \to 0$ for $t \to \infty$;
- Locally exponentially stable in the pth mean if it is stable in the pth mean and if there exists a $\delta \ge 0$ such that whenever $\mathbb{E}(\sup_{t\in[t_0-\tau,t_0]|\psi(t)|^p}) < \delta$, there exists some finite constant C and a $u^* > 0$ such that
  $\mathbb{E}(|y(t; t_0, \psi)|^p) \le C\mathbb{E}(\sup_{s\in[t_0-\tau,t_0]} |\psi(s)|^p) exp(-u^*(t - t_0))$ $(t_0 \le t < \infty)$.
  If δ is arbitrarily large, then the stability in the above, in each case, is global rather than local.

A different approach to stability for SDDEs, that of stochastic stability or stability in probability, is as follows:

- The trivial solution of the SDDE (7.19) is termed stochastically stable in probability if for each $e \in (0, 1)$ and $\epsilon > 0$, there exists a $\delta \equiv \delta(e, \epsilon) \geq 0$, such that

$$\mathbb{P}(|y(t; t_0, \psi)| \leq \epsilon \quad \text{for all} \quad t \geq t_0) \geq 1 - e,$$

whenever $t \geq t_0$ and $\sup_{t \in [t_0 - \tau, t_0]} |\psi(t)|^p < \delta$ with probability 1.

Certain stability conditions for SDDEs can be stated in terms of Lyapunov functionals, similar to the theorems for DDEs. Now, we present the Lyapunov theory approach for SDDEs. Let us consider a more general type for (7.8) with one delay. Thus, an Itô type SDDE is given by

$$\begin{aligned} d\mathbf{y}(t) &= \mathbf{f}(t, \mathbf{y}_t)dt + \mathbf{g}(t, \mathbf{y}_t)dW(t), \quad t \geq t_0, \\ \mathbf{y}_t(\theta) &= \mathbf{y}(t + \theta), \quad -\tau \leq \theta \leq 0, \\ \mathbf{f}(t, 0) &\equiv 0, \quad \mathbf{y}_{t_0} = \psi. \end{aligned} \tag{7.20}$$

Define $\mathbf{y}_t \in C_n$ by $\mathbf{y}_t(\theta) = \mathbf{y}(t + \theta)$ for $\theta \in [-\tau, 0]$, where $\psi \in C_n$, such that when we consider the existence and uniqueness of solutions, without loss of generality, the solution $\mathbf{y}_t = 0$ is an equilibrium.

Theorem 7.3 ([24]) *Suppose there is a continuous functional* $V : [t_0, \infty] \times C[-\tau, 0] \to \mathbb{R}$ *such that for any solution of (7.20), where* $\mathbf{y}_t(\theta) = \mathbf{y}(t + \theta)$ *such that* $-\tau \leq \theta \leq 0$, *the following inequalities hold, such that* C_i $i = 1, 2, 3$ *are positive constants:*

$$\begin{aligned} V(t, \mathbf{y}_t) &\geq C_1 |\mathbf{y}(t)|^2 \\ \mathbb{E}V(t, \mathbf{y}_t) &\leq C_2 \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\mathbf{y}(t + \theta)|^2, \end{aligned} \tag{7.21}$$

for arbitrary $t \geq t_0$, $s \geq t$

$$\mathbb{E}[V(s, \mathbf{y}_s) - V(t, \mathbf{y}_t)] \leq -C_3 \int_t^s \mathbb{E}|\mathbf{y}(h)|^2 dh. \tag{7.22}$$

Then, the trivial solution of (7.20) is asymptotically mean-square stable.

Example 7.3 Consider an SDDE of the form

$$dy(t) = -\mu_1 y(t - \tau)dt + \mu_2 y(t)dW(t), \quad t > t_0, \tag{7.23}$$

where μ_1, μ_2 are positive constants. Sufficient conditions for asymptotic mean-square stability of (7.23) are

$$0 < \mu_1 \tau < 1, \qquad \mu_1(1 - \mu_1 \tau) > \frac{\mu_2^2}{2}.$$

To prove this, consider the functional

$$V(\psi) = \Big[\psi(0) - \mu_1 \int_{-\tau}^{0} \psi(\theta)d\theta\Big]^2 + \mu_1^2 \int_{-\tau}^{0} ds \int_{s}^{0} \psi^2(\theta)d\theta. \tag{7.24}$$

Using Itô formula, we obtain

$$\begin{aligned} dV(y_t) &= 2\Big[y(t) - \mu_1 \int_{t-\tau}^{t} y(\theta)d\theta\Big](dy(t) - \mu_1 y(t)dt + \mu_1 y(t-\tau)dt) \\ &\quad + \Big[\mu_2^2 y^2(t) + \mu_1^2 \tau y^2(t) - \mu_1^2 \int_{t-\tau}^{t} y^2(\theta)d\theta\Big]dt, \\ &= 2\Big[y(t) - \mu_1 \int_{t-\tau}^{t} y(\theta)d\theta\Big](\mu_2 y(t)dW(t) - \mu_1 y(t)dt) \\ &\quad + \Big[\mu_2^2 y^2(t) + \mu_1^2 \tau y^2(t) - \mu_1^2 \int_{t-\tau}^{t} y^2(\theta)d\theta\Big]dt. \end{aligned}$$

Note that

$$2\mu_1^2 y(t) \int_{t-\tau}^{t} y(\theta)d\theta \le \mu_1^2\Big[\tau y^2(t) + \int_{t-\tau}^{t} y^2(\theta)d\theta\Big].$$

Hence

$$dV(y_t) \le 2\mu_2\Big[y(t) - \mu_1 \int_{t-\tau}^{t} y(\theta)d\theta\Big]y(t)dW(t) - [2\mu_1(1-\mu_2\tau) - \mu_2^2]y^2(t). \tag{7.25}$$

Integration of both parts of (7.25) from $s \in [t_0, t]$ to t, and then taking the expectation yields

$$\mathbb{E}[V(y_t) - V(y_s)] \le -[2\mu_1(1-\mu_1\tau) - \mu_2^2] \int_{s}^{t} \mathbb{E}y^2(h)dh. \tag{7.26}$$

From inequality (7.26), we have

$$\mathbb{E}V(y_t) \le \mathbb{E}V(y_{t_0}), \quad t \ge t_0. \tag{7.27}$$

Therefore,

$$\mathbb{E}\Big[y(t) - \mu_1 \int_{t-\tau}^{t} y(\theta)d\theta\Big]^2 \le \mathbb{E}V(y_{t_0}), \quad \int_{t_0}^{\infty} \mathbb{E}y^2(s)ds < \infty. \tag{7.28}$$

Inequalities (7.28) and condition $\mu_1\tau < 1$ imply mean-square stability, since

$$\sup_{t\ge t_0} \mathbb{E}y^2(t) \le C_1 \sup_{-\tau\le\theta\le 0} \mathbb{E}\psi^2(\theta). \tag{7.29}$$

Therefore, asymptotic mean-square stability is implied based on inequalities (7.28) and the fact that $\lim_{t\to\infty} \mathbb{E}y^2(t) = 0$.

Next, we introduce a numerical scheme for an autonomous SDDE.

7.4 Numerical Scheme for Autonomous SDDEs

Given a scalar autonomous SDDE of the form

$$\begin{aligned} dy(t) &= f(y(t), y(t-\tau))dt + g(y(t), y(t-\tau))dW(t), \quad t \in [0, T], \\ y(t) &= \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{7.30}$$

which can be formulated as

$$y(t) = y(0) + \int_0^t f(y(s), y(s-\tau))ds + \int_0^t g(y(s), y(s-\tau))dW(s), \tag{7.31}$$

for $t \in [0, T]$ and with $y(t) = \psi(t)$ for $t \in [-\tau, 0]$. The second integral in (7.31) is a stochastic integral in the It$\hat{o}$ sense.

We define mesh points with a uniform step on the interval $[0, T]$, so that $h = T/N$, $t_n = nh$, where $n = 0, \ldots, N$. We also assume that, for the given h, there is a corresponding integer m, where the time-delay can be expressed in terms of the stepsize as $\tau = mh$. For all indices $n - m \leq 0$, we have $\tilde{y}_{n-m} := \psi(t_n - \tau)$; otherwise, the numerical approximation of (7.30) takes the form

$$\tilde{y}_{n+1} = \tilde{y}_n + \phi(h, \tilde{y}_n, \tilde{y}_{n-m}, I_\phi), \quad n = 0, \ldots, N-1. \tag{7.32}$$

The increment function $\phi(h, \tilde{y}_n, \tilde{y}_{n-m}, I_\phi) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ includes a finite number of multiple It$\hat{o}$-integrals (see [25, 26]) of the form

$$I_{(j_1,\ldots,j_l),h} = \int_t^{t+h} \int_t^{s_l} \ldots \int_t^{s_2} dW^{j1}(s_1) \ldots dW^{j_{l-1}}(s_{l-1}) dW^{j_l}(s_l),$$

where $j_i \in \{0, 1\}$ and $dW^0(t) = dt$, and with $t = t_n$ for (7.32), we denote I_ϕ the collection of It$\hat{o}$-integrals required to compute the increment function ϕ.

To guarantee the existence of the numerical solution, some assumptions should be given to the increment function ϕ of (7.32: Suppose that there exist positive constants $V_1, V_2, and V_3$, such that for all $\kappa, \kappa^{'}, \omega, \omega^{'} \in \mathbb{R}$, we have

$$\begin{aligned} \left|\mathbb{E}\Big(\phi(h, \kappa, \omega, I_\phi) - \phi(h, \kappa^{'}, \omega^{'}, I_\phi)\Big)\right| &\leq V_1 h(|\kappa - \kappa^{'}| + |\omega - \omega^{'}|), \\ \mathbb{E}\Big(|\phi(h, \kappa, \omega, I_\phi) - \phi(h, \kappa^{'}, \omega^{'}, I_\phi)|^2\Big) &\leq V_2 h\Big(|\kappa - \kappa^{'}|^2 + |\omega - \omega^{'}|^2\Big), \end{aligned} \tag{7.33}$$

and

$$\mathbb{E}\Big(|\phi(h,\kappa,\omega,I_\phi)|^2\Big) \le V_3 h\Big(1+|\kappa|^2+|\omega|^2\Big). \tag{7.34}$$

Lemma 7.1 ([10]) *If the increment function ϕ in Eq. (7.32) satisfies condition (7.34), then $\mathbb{E}|\tilde{y}_n|^2 < \infty$ for all $n \le N$.*

Let $y(t_{n+1})$ be the exact solution of (7.30) at mesh point t_{n+1}. $\tilde{y}_{n+1}$ is the value of the approximate solution given by (7.32), and $\tilde{y}(t_{n+1})$ is the solution of (7.32) after just one step, so that

$$\tilde{y}(t_{n+1}) = y(t_n) + \phi(h, y(t_n), y(t_n-\tau), I_\phi).$$

Definition 7.5 (*Local and global errors*) The local error that occurs in one step of the above approximation $\{\tilde{y}_n\}$ is the sequence of random variables

$$\delta_{n+1} = y(t_{n+1}) - \tilde{y}(t_{n+1}), \quad n = 0,\ldots,N-1. \tag{7.35}$$

However, the global error is the amount of error that occurs in the use of a numerical approximation to solve a problem, which is the sequence of random variables

$$\epsilon_n := y(t_n) - \tilde{y}_n, \quad n = 1,\ldots,N. \tag{7.36}$$

Note that ϵ_n is $\mathcal{A}_{t_n}$-measurable since both $y(t_n)$ and $\tilde{y}_n$ are $\mathcal{A}_{t_n}$-measurable random variable, such that $\Big(\mathbb{E}|\epsilon_n|^2\Big)^{1/2}$ is the $\mathcal{L}^2$-norm of (7.36).

7.4.1 Convergence and Consistency

Definition 7.6 Assume that

$$\delta_{n+1} = y(t_{n+1}) - \tilde{y}(t_{n+1}), \quad n = 0,\ldots,N-1. \tag{7.37}$$

The numerical scheme (7.32) is said to be consistent with order p_1 in the mean and with order p_2 in the mean square if, with

$$p_2 \ge \frac{1}{2} \quad \text{and} \quad p_1 \ge p_2 + \frac{1}{2}, \tag{7.38}$$

the estimates

$$\max_{0\le n\le N-1} |\mathbb{E}(\delta_{n+1})| \le Ch^{p_1} \quad \text{as} \quad h \to 0, \tag{7.39}$$

and

$$\max_{0\leq n\leq N-1}\left(|\mathbb{E}(\delta_{n+1})|^2\right)^{1/2} \leq Ch^{p_2} \quad \text{as} \quad h \to 0, \tag{7.40}$$

hold, where constant C does not depend on h, but may depend on T, and on the initial data.

Therefore, we can now introduce the basic theorem about the convergence of method (7.32).

Theorem 7.4 ([10]) *Assume that the conditions of Theorem 7.1 are satisfied. Suppose that the method defined by Eq. (7.32) is consistent with order p_1 in the mean and order p_2 in the mean-square sense, such that p_1, p_2 fulfilling (7.38), and the increment function ϕ on Eq. (7.32) satisfies the estimates (7.33). Then, the approximation (7.32) for Eq. (7.30) is convergent in $\mathcal{L}^2$ (as $h \to 0$ with $\tau/h \in \mathbb{N}$) with order $p = p_2 - 1/2$. That is, convergent is in the mean-square sense, such that*

$$\max_{0\leq n\leq N-1}\left(|\mathbb{E}(\delta_{n+1})|^2\right)^{1/2} \leq Ch^{p} \quad \textit{as} \quad h \to 0, \tag{7.41}$$

Theorem 7.5 ([10]) *If the increment function ϕ of the approximation (7.32) satisfies the estimates (7.33), then the one-step method (7.32) is zero stable in the quadratic mean-square sense.*

Next, we extend our analysis to non-autonomous system of SDDEs (7.8).

7.5 Numerical Schemes for Non-autonomous SDDE

There are some specific discrete time approximations for (7.8). The simplest scheme, which is defined by stochastic difference equation, is represented by Euler approximation as

$$\tilde{\mathbf{y}}_{n+1} = \tilde{\mathbf{y}}_n + \mathbf{f}(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})h + \sum_{j=1}^{r} \mathbf{g}_j(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})\Delta W_n^j, \tag{7.42}$$

where $\tilde{\mathbf{y}} = \{\tilde{\mathbf{y}}(t), t \in [-\tau, T]\}$ is right continuous with left-hand limits, a discrete time approximation with stepsize h, such that for each $n \in \{1, \ldots, N\}$. The random variable $\tilde{\mathbf{y}}(t_n)$ is $\mathcal{A}_{t_n}$-measurable and $\tilde{\mathbf{y}}(t_{n+1})$ can be expressed as a function of $\tilde{\mathbf{y}}(t_{-m})$, $\tilde{\mathbf{y}}(t_{-m+1})$,..., $\tilde{\mathbf{y}}(t_n)$, discretization time t_n, and a finite number of $\mathcal{A}_{t_{n+1}}$-measurable random variable. With $\Delta W_n^j = W^j(t_{n+1}) - W^j(t_n)$, for $n = 0, 1, \ldots, N-1$ and $j = 0, 1, \ldots, r$. By more general assumptions, we can check that Euler approximation strongly converges with order $1/2$ [11].

7.5.1 Taylor Approximation

For stochastic differential equations, it is common that by application of the Wagner-Platen stochastic Taylor expansion [27], we can construct discrete time approximations that converge with a given order of strong convergence, which involve in each time step certain multiple integrals. For the general multi-dimensional case $d, r = 1, 2, \ldots$ the order-one strong Taylor approximation has the form

$$\begin{aligned}\tilde{\mathbf{y}}_{n+1} &= \tilde{\mathbf{y}}_n + \mathbf{f}(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})h + \sum_{j=1}^{r} \mathbf{g}_j(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})\Delta W_n^j \\ &+ \sum_{j_1, j_2=1}^{r} \sum_{i=1}^{d} g_{i,j_1}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \frac{\partial}{\partial \tilde{y}_n^i} g_{i,j_2}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW^{j_1}(s_2) dW^{j_2}(s_1) \\ &+ \sum_{j_1, j_2=1}^{r} \sum_{i_1=1}^{d} g_{i,j_1}(t_{n-m}, \tilde{y}_{n-m}, \tilde{y}_{n-2m}) \frac{\partial}{\partial \tilde{y}_{n-m}^i} g_{i,j_2}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \\ &\times \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW^{j_1}(s_2 - \tau) dW^{j_2}(s_1),\end{aligned} \tag{7.43}$$

for $n = 0, 1, \ldots, N-1$, $i = 1, 2, \ldots, d$. One can check that approximation (7.43) converges under suitable assumptions with strong-order-one Taylor approximation [11]. In the one-dimensional case, when $\tau = 0$, scheme (7.43) coincides with the well-known Milstein Scheme for SDEs. However, the time-delay in (7.43) generates an extra term, which describes a double Wiener integral that integrates an earlier segment of the Wiener path with respect to the actual Wiener path.

7.5.2 Implicit Strong Approximations

In practice, explicit schemes not only have smaller computational costs, but also have lower accuracy compared to implicit methods. It is sometimes recommended to use implicit schemes to have numerically stable approximate solutions for SDDEs, as in the case of stiff problem[1]

For the general multi-dimensional case (7.8), the family of implicit Euler approximations are

$$\tilde{\mathbf{y}}_{n+1} = \tilde{\mathbf{y}}_n + [\theta \mathbf{f}(t_{n+1}, \tilde{\mathbf{y}}_{n+1}, \tilde{\mathbf{y}}_{n-m+1}) + (1-\theta)\mathbf{f}(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})]h + \sum_{j=1}^{r} \mathbf{g}_j(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})\Delta W_n^j, \tag{7.44}$$

for $n = 0, 1, \ldots, N-1$, such that $\theta \in [0, 1]$ stands for the degree of implicitness. If $\theta = 0$, we have the explicit Euler approximation (7.42). For $\theta = 1$, we obtain the

[1] A stiff problem is defined as that in which the global accuracy of the numerical solution is determined by stability rather than local error, and implicit methods are more appropriate for it.

fully implicit Euler approximation. The approximation (7.44) converges with strong order 1\2 [13].

In the same manner, we can establish an order-one strong implicit Taylor approximation with

$$
\begin{aligned}
\tilde{\mathbf{y}}_{n+1} &= \tilde{\mathbf{y}}_n + [\theta \mathbf{f}(t_{n+1}, \tilde{\mathbf{y}}_{n+1}, \tilde{\mathbf{y}}_{n-m+1}) + (1-\theta)\mathbf{f}(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})]h + \sum_{j=1}^{r} \mathbf{g}_j(t_n, \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_{n-m})\Delta W_n^j \\
&+ \sum_{j_1, j_2=1}^{r} \sum_{i=1}^{d} g_{i,j_1}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \frac{\partial}{\partial \tilde{y}_n^i} g_{i,j_2}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW^{j_1}(s_2) dW^{j_2}(s_1) \\
&+ \sum_{j_1, j_2=1}^{r} \sum_{i=1}^{d} g_{i,j_1}(t_{n-m}, \tilde{y}_{n-m}, \tilde{y}_{n-2m}) \frac{\partial}{\partial \tilde{y}_{n-m}^i} g_{i,j_2}(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \\
&\times \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW^{j_1}(s_2 - \tau) dW^{j_2}(s_1),
\end{aligned}
\tag{7.45}
$$

Next, we will discuss in detail the mean-square stability of Milstein method since we have used this scheme in the numerical simulations for SDDEs models.

7.6 Milstein Scheme for SDDEs

In this section, we introduce the Milstein scheme for SDDEs and show that the numerical method is mean-square stable under suitable conditions.

Given the one-dimensional version of (7.8), $r = d = 1$, of the following form:

$$
\begin{aligned}
dy(t) &= f(t, y(t), y(t-\tau))dt + g(t, y(t), y(t-\tau))dW, \quad t \in [0, T], \\
y(t) &= \psi(t), \quad t \in [-\tau, 0].
\end{aligned}
\tag{7.46}
$$

The order one strong Taylor approximation for (7.46) the one-dimensional case is defined by

$$
\begin{aligned}
\tilde{y}_{n+1} &= \tilde{y}_n + f(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \int_{t_n}^{t_{n+1}} ds_1 + g(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \int_{t_n}^{t_{n+1}} dW(s_1) \\
&+ g(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \frac{\partial}{\partial \tilde{y}_n} g(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW(s_2) dW(s_1) \\
&+ g(t_{n-m}, \tilde{y}_{n-m}, \tilde{y}_{n-2m}) \frac{\partial}{\partial \tilde{y}_{n-m}} g(t_n, \tilde{y}_n, \tilde{y}_{n-m}) \\
&\times \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW(s_2 - \tau) dW(s_1).
\end{aligned}
\tag{7.47}
$$

Once we have the Taylor approximation, we can construct the Milstein scheme for (7.46)

$$\begin{aligned}\tilde{y}_{n+1} &= \tilde{y}_n + hf(t_n, \tilde{y}_n, \tilde{y}_{n-m}) + g(t_n, \tilde{y}_n, \tilde{y}_{n-m})\Delta W_n + \frac{1}{2} g(t_n, \tilde{y}_n, \tilde{y}_{n-m}) g'(t_n, \tilde{y}_n, \tilde{y}_{n-m})[(\Delta W_n)^2 - h] \\ &\quad + g(t_{n-m}, \tilde{y}_{n-m}, \tilde{y}_{n-2m}) \frac{\partial}{\partial \tilde{y}_{n-m}} g(t_n, \tilde{y}_n, \tilde{y}_{n-m}) I,\end{aligned} \tag{7.48}$$

where $I = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW(s_2 - \tau) dW(s_1)$.

7.6.1 Convergence and Mean-Square Stability of the Milstein Scheme

Consider the linear scalar SDDE of the form

$$\begin{aligned} dy(t) &= [\rho_0 y(t) + \rho_1 y(t-\tau)]dt + [\rho_2 y(t) + \rho_3 y(t-\tau)]dW(t), \quad t \in [0, T], \\ y(t) &= \psi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{7.49}$$

where $\rho_0, \rho_1, \rho_2, \rho_3 \in \mathbb{R}$, $W(t)$ is a one-dimensional standard Wiener process, and $\psi(t)$ is continuous and bounded function with $\mathbb{E}[\|\psi\|^2] < \infty$, where $\|\psi\| = \sup_{-\tau \le t \le 0} |\psi(t)|$.

Theorem 7.6 ([11]) *Suppose that*

$$\rho_0 < -|\rho_1| - \frac{(|\rho_2| + |\rho_3|)^2}{2}, \tag{7.50}$$

then the solution of (7.49) satisfies $\lim_{t\to\infty} \mathbb{E}[|y(t)|^2] = 0$, *i.e., the solution is mean-square stable.*

Using order one strong Taylor approximation formula to the linear one delay system (7.49), we have

$$\begin{aligned} y_{n+1} =& y_n + h(\rho_0 y_n + \rho_1 y_{n-m}) + (\rho_2 y_n + \rho_3 y_{n-m})\Delta W_n \\ &+ \rho_3(\rho_2 y_{n-m} + \rho_3 y_{n-2m}) I_1 + \rho_2(\rho_2 y_n + \rho_3 y_{n-m}) I_2, \end{aligned} \tag{7.51}$$

where y_n is an approximation to $y(t_n)$, such that $I_1 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW(t-\tau) dW(s)$, $I_2 = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW(t) dW(s)$. The convergence order of (7.51) can be obtained by Theorem 10.2 in [11], since the coefficients of (7.51) satisfy the Lipschitz condition and growth condition. Thus, the Milstein scheme (7.51) is strongly convergent of order 1.

Theorem 7.7 *The Milstein scheme (7.51) is mean-square stable, if condition (7.50) is satisfied [28].*

Proof By reorganizing the terms of (7.51), we get

$$
\begin{aligned}
y_{n+1} =&(1+\rho_0 h+\rho_2\Delta W_n)y_n+(\rho_1 h+\rho_3\Delta W_n)y_{n-m}+\rho_3(\rho_2 y_{n-m}+\rho_3 y_{n-2m})I_1\\
&+\rho_2(\rho_2 y_n+\rho_3 y_{n-m})I_2.
\end{aligned} \tag{7.52}
$$

Squaring both sides of (7.52), it follows from $2ab \le a^2+b^2$ ($\forall a, b \in \mathcal{R}$), we have

$$
\begin{aligned}
y_{n+1}^2 \le&(1+\rho_1 h+\rho_2\Delta W_n)^2 y_n^2+(\rho_1 h+\rho_3\Delta W_n)^2 y_{n-m}^2+\rho_2^2[(\rho_2^2+|\rho_2\rho_3|)y_n^2+(\rho_3^2+|\rho_2\rho_3|)y_{n-m}^2]I_2^2\\
&+\rho_3^2[(\rho_2^2+|\rho_2\rho_3|)y_{n-m}^2+(\rho_3^2+|\rho_2\rho_3|)y_{n-2m}^2]I_1^2+|1+\rho_0 h||\rho_1|h(y_n^2+y_{n-m}^2)\\
&+|\rho_2\rho_3|\Delta W_n^2(y_n^2+y_{n-m}^2)+2[(1+\rho_0 h)\rho_3+\rho_1\rho_2 h]\Delta W_n y_n y_{n-m}\\
&+2\rho_2\rho_3(\rho_2 y_n+\rho_3 y_{n-m})(\rho_2 y_{n-m}+\rho_3 y_{n-2m})I_1 I_2\\
&+2\rho_2(1+\rho_0 h+\rho_2\Delta_n)(\rho_2 y_n+\rho_3 y_{n-m})y_n I_2\\
&+2\rho_3(1+\rho_0 h+\rho_2\Delta_n)(\rho_2 y_{n-m}+\rho_3 y_{n-2m})y_n I_1\\
&+2\rho_2(\rho_1 h+\rho_3\Delta W_n)(\rho_2 y_n+\rho_3 y_{n-m})y_{n-m} I_2\\
&+2\rho_3(\rho_1 h+\rho_3\Delta W_n)(\rho_2 y_{n-m}+\rho_3 y_{n-2m})y_{n-m} I_1
\end{aligned} \tag{7.53}
$$

Assume that $x_n = \mathbb{E}[y_n^2]$, then take expectation for both sides of (7.53), which yields the following:

$$
x_{n+1} \le A_1 x_n + A_2 x_{n-m} + A_3 x_{n-2m}, \tag{7.54}
$$

where

$$
\begin{aligned}
A_1 &= (1+\rho_0 h)^2+\rho_2^2 h+|1+\rho_0 h||\rho_1|h+|\rho_2\rho_3|h+\frac{h^2}{2}\rho_2^2(\rho_2^2+|\rho_2\rho_3|),\\
A_2 &= \rho_1^2 h^2+\rho_3^2 h+|1+\rho_0 h||\rho_1|h+|\rho_2\rho_3|h+\frac{h^2}{2}\rho_2^2(\rho_3^2+|\rho_2\rho_3|)\\
&\quad+\frac{h^2}{2}\rho_3^2(\rho_2^2+|\rho_2\rho_3|),\quad A_3=\frac{h^2}{2}\rho_3^2(\rho_2^2+|\rho_2\rho_3|).
\end{aligned} \tag{7.55}
$$

Therefore

$$
\begin{aligned}
&(1+\rho_0 h)^2+\rho_1^2 h^2+(\rho_2^2+\rho_3^2+2|\rho_2\rho_3|)h+2|1+\rho_0 h||\rho_1|h\\
&\qquad+\frac{h^2}{2}(\rho_2^2+\rho_3^2)(|\rho_2|+|\rho_3|)^2<1.
\end{aligned} \tag{7.56}
$$

Consider

$$
\begin{aligned}
h_1 =&\frac{-[2\rho_0+2|\rho_1|+(|\rho_2|+|\rho_3|)^2]}{(|\rho_0|+|\rho_1|)^2+\frac{1}{2}(\rho_2^2+\rho_3^2)(|\rho_2|+|\rho_3|)^2}>0,\\
h_2 =&\min\{\frac{1}{|\rho_0|},\frac{-[2\rho_0+2|\rho_1|+(|\rho_2|+|\rho_3|)^2]}{(|\rho_0|+|\rho_1|)^2+\frac{1}{2}(\rho_2^2+\rho_3^2)(|\rho_2|+|\rho_3|)^2}\}>0,
\end{aligned} \tag{7.57}
$$

- If $h \in (0, h_1)$, inequality (7.56) holds;

- If $h \in (0, h_2)$, then $1 + \rho_0 h > 0$ (wider range of stable stepsize values) and inequality (7.56) holds;
- Let $h_0 = \max\{h_1, h_2\}$; thus, the Milstein scheme is MS-stable whenever $h \in (0, h_0)$.

□

7.7 Concluding Remarks

In this chapter, we have briefly introduced some features of SDDEs. We have also discussed some numerical schemes for SDDEs. Convergence and consistency of such schemes have been investigated as well. The mean-square stability of the Milstein scheme has been discussed and the obtained result shows that the method preserves the stability property of a class of linear scalar SDDEs. In this monograph, we adopted the above discussed Milstein scheme for solving different examples and models of SDDEs; See Appendix C.

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