Chapter 11
Fractional-Order Delay Differential Equations with Predator-Prey Systems

11.1 Introduction

Mathematical models using differential equations with integer order have proved valuable in understanding the dynamics of biological systems. However, most biological, physical, and engineering systems have long-range temporal memory [1–4] and/or long-range space interactions [5–7]. Modeling such systems using fractional-order differential equations is more advantageous than classical integer-order mathematical modeling, in which the effects of existence of time memory or long-range space interactions are neglected. Moreover, the fractional-order derivative is related to the whole space for a physical process, whereas the integer-order derivative describes the local properties of a certain position. Accordingly, the subject of fractional calculus (i.e., calculus of integral and derivatives of arbitrary order) has gained popularity and importance, mainly due to its demonstrated applications in numerous diverse and widespread fields of science and engineering. It has been successfully applied to system biology [3, 8–11], physics [12–15], chemistry and biochemistry [16], hydrology [17, 18], engineering [19, 20], medicine [21–23], and finance [24]. Examples of fractional-order systems in modeling and control can be found in [25–27]. In most cases, the *fractional-order differential equations* (FODEs) models seem more consistent with the real phenomena than integer-order models. This is because fractional derivatives and integrals enable the description of the memory and hereditary properties inherent in various materials and processes that exist within most biological systems.

In most biological systems time-lags or -delays exist intrinsically, such as predator-prey (PP) systems, where the predator needs time to mature [28–32]. Considerable attention has been given to study and investigate the different types of PP models due to their universal existence and importance. However, most such models have been either studied using integer-order equations with delays or using fractional order without delays [33, 34]. In this work, we combine the fractional-order with the delay terms in the model to describe the complex systems of PP interactions with memory effects. We also study the stability properties of such models.

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In [28], the authors addressed the existence and global stability of a periodic solution for a discrete PP system with the functional response and predator cannibalism, whereas a global analysis of Holling type II PP model with a constant prey refuge is presented in [30]. In [32], a Holling-Tanner PP model with time-delay is considered. By regarding the delay as the bifurcation parameter, the local asymptotic stability of the positive equilibrium is investigated. In [31], a delayed stage-structured PP model with non-monotone functional responses is proposed. It is assumed that immature individuals and mature individuals of the predator are divided by a fixed age and that immature predators do not have the ability to attack prey. In [29], the main feature is that the authors introduce time-delay and pulse into the PP (natural enemy-pest) model with age structure, exhibit a new modeling method that is applied to investigate impulsive DDEs, and give some reasonable suggestions for pest management. Next, we present fractional-order with time-delay in the system that allows greater degrees of freedom in the model and in describing systems with long-time memory, such as PP dynamics.

Despite these various applications of fractional calculus, there are some important challenges, such as numerical approximation and the physical interpretation, for the fractional derivative. Fractional differential equations are integro-differential equations and their numerical solution requires large computer memory and long runs of numerical simulations; this makes it very difficult to investigate the general properties of fractional dynamical systems. As a consequence, accurate approximation and a suitable numerical technique play an important role in identifying the solution behavior of such fractional equations and in exploring their applications (see, e.g., [35–37] and the references therein). Recently, an increasing number of investigators have been studying the qualitative properties and numerical solutions of fractional-order biological models [38, 39].

Motivated by the above, in this chapter, we suggest a fractional-order PP model with a feeding rate of delayed saturated form for the prey population. We study the qualitative behavior of the model using local and global stability of the equilibrium points and present conditions in the time-delay τ in which the model is stable. Hopf bifurcation analysis is also addressed and the results of simulation scenarios are presented. We also present suitable implicit schemes for the numerical treatments of such types of *fractional-order delay differential equations* (FODDEs). The organization of this chapter is as follows. In Sect. 11.2, we describe the model. In Sects. 11.3 and 11.4, we present local and global stability of equilibrium states, respectively. In Sect. 11.5, we provide an unconditional stable numerical method for FODDEs along with some numerical examples. Section 11.6 provides a brief discussion and concluding remarks about the obtained results.

11.1.1 Preliminaries

The are different definitions of fractional order are Riemann-Liouville, Grunwald-Letnikov, Weyle, Marchaud, Jumarie, Hadamard, and Caputo sense. The Caputo frac-

tional operator provides flexibility to the physical, biological models to attain the different considerable dynamical behaviors and chance to know the better understanding for the model dynamics Moreover, the Caputo fractional-order system allows the local initial values to be included in a proper way, well understandable features of physical situations, frequently applied to tackle real-world problems.

Definition 11.1 ([40]) Caputo derivative of fractional-order α for a function $f(t)$ is defined by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^n(\tau) d\tau, \tag{11.1}$$

where $n-1 < \alpha < n \in \mathbb{Z}^+$, $\Gamma(\cdot)$ is the Gamma function.

The Laplace transform of Caputo fractional-order derivative is defined as follows:

$$\mathcal{L}\{D^{\alpha} f(t); s\} = s^{\alpha}\mathcal{F}(s) - \sum_{i=1}^{n-1} s^{\alpha-i-1} f^{(i)}(0) \tag{11.2}$$

where $\mathcal{F}(s) = \mathcal{L}\{f(t)\}$. In particular, when $f^{(i)}(0) = 0, i = 1, 2, \ldots, n-1$, then $\mathcal{L}\{D^{\alpha} f(t); s\} = s^{\alpha}\mathcal{F}(s)$.

Remark 11.1 For $0 < \alpha \leq 1$, the fractional-order derivative defined based on Caputo sense (see Definition 11.1), the memory effects in dynamical systems is described by using a convolution integral with power-law memory kernel. The memory kernel (time correlation function) decaying rate depends on fractional order α. The lower value of α corresponds to more slowly decaying long memory (time-correlation functions). Then, $\alpha \to 1$, the influence of memory decreases.

For given the following m-dimensional fractional-order system

$$\begin{aligned}
D^{\alpha_1} x_1(t) &= a_{11}x_1(t-\tau_{11}) + a_{12}x_2(t-\tau_{12}) + \cdots + a_{1m}x_m(t-\tau_{1m}) \\
D^{\alpha_2} x_2(t) &= a_{21}x_1(t-\tau_{21}) + a_{22}x_2(t-\tau_{22}) + \cdots + a_{2m}x_m(t-\tau_{2m}) \\
&\vdots \\
D^{\alpha_m} x_m(t) &= a_{m1}x_1(t-\tau_{m1}) + a_{m2}x_2(t-\tau_{m2}) + \cdots + a_{mm}x_m(t-\tau_{mm})
\end{aligned} \tag{11.3}$$

where $0 < \alpha_i < 1 \ \ (i = 1, 2, \ldots, m)$. The smooth initial conditions $x_i(t) = \psi_i(t),\ t \in [-\max_{i,j} \tau_{ij}, 0]$, $i, j = 1, 2, \ldots, m$. The state variables $x(t), x(t-\tau_{ij}) \in \mathbb{R}$. Taking Laplace transform for both sides of (11.3), yields

$$
\begin{aligned}
s^{\alpha_1}X_1(s) - s^{\alpha_1-1}\psi_1(0) &= a_{11}e^{-s\tau_{11}}\left(X_1(s) + \int_{-\tau_{11}}^{0} e^{-st}\psi_1(t)dt\right) + a_{12}e^{-s\tau_{12}}\left(X_2(s)\right.\\
&\quad \left. + \int_{-\tau_{12}}^{0} e^{-st}\psi_2(t)dt\right) + \cdots + a_{1m}e^{-s\tau_{1m}}\left(X_m(s) + \int_{-\tau_{1m}}^{0} e^{-st}\psi_m(t)dt\right)\\
s^{\alpha_2}X_2(s) - s^{\alpha_2-1}\psi_2(0) &= a_{21}e^{-s\tau_{21}}\left(X_1(s) + \int_{-\tau_{21}}^{0} e^{-st}\psi_1(t)dt\right) + a_{22}e^{-s\tau_{22}}\left(X_2(s)\right.\\
&\quad \left. + \int_{-\tau_{22}}^{0} e^{-st}\psi_2(t)dt\right) + \cdots + a_{2m}e^{-s\tau_{2m}}\left(X_m(s) + \int_{-\tau_{2m}}^{0} e^{-st}\psi_m(t)dt\right) \qquad (11.4)\\
&\vdots\\
s^{\alpha_m}X_m(s) - s^{\alpha_m-1}\psi_m(0) &= a_{m1}e^{-s\tau_{m1}}\left(X_1(s) + \int_{-\tau_{m1}}^{0} e^{-st}\psi_1(t)dt\right) + a_{m2}e^{-s\tau_{m2}}\left(X_2(s)\right.\\
&\quad \left. + \int_{-\tau_{m2}}^{0} e^{-st}\psi_2(t)dt\right) + \cdots + a_{mm}e^{-s\tau_{mm}}\left(X_m(s) + \int_{-\tau_{mm}}^{0} e^{-st}\psi_m(t)dt\right)
\end{aligned}
$$

where $X_i(s)$ is the Laplace transform of $x_i(t)$. This system can be written in a matrix form

$$
\Delta(s)\cdot\begin{bmatrix} X_1(s)\\ X_2(s)\\ \vdots\\ X_m(s)\end{bmatrix} = \begin{bmatrix} b_1(s)\\ b_2(s)\\ \vdots\\ b_m(s)\end{bmatrix}. \qquad (11.5)
$$

We call $\Delta(s)$ a characteristic matrix of the system

$$
\Delta(s) = \begin{bmatrix} s^{\alpha_1} - a_{11}e^{-s\tau_{11}} & -a_{12}e^{-s\tau_{12}} & \dots & -a_{1m}e^{-s\tau_{1m}}\\ -a_{21}e^{-s\tau_{21}} & s^{\alpha_2} - a_{22}e^{-s\tau_{22}} & \dots & -a_{2m}e^{-s\tau_{2m}}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{m1}e^{-s\tau_{m1}} & -a_{m2}e^{-s\tau_{m2}} & \dots & s^{\alpha_m} - a_{mm}e^{-s\tau_{mm}}\end{bmatrix} \qquad (11.6)
$$

and

$$
\begin{aligned}
b_1(s) =& s^{\alpha_1-1}\psi_1(0) + a_{11}e^{-s\tau_{11}}\int_{-\tau_{11}}^{0} e^{-st}\psi_1(t)dt + a_{12}e^{-s\tau_{12}}\int_{-\tau_{12}}^{0} e^{-st}\psi_2(t)dt\\
&+ \cdots + a_{1m}e^{-s\tau_{1m}}\int_{-\tau_{1m}}^{0} e^{-st}\psi_m(t)dt\\
b_2(s) =& s^{\alpha_2-1}\psi_2(0) + a_{21}e^{-s\tau_{21}}\int_{-\tau_{21}}^{0} e^{-st}\psi_1(t)dt + a_{22}e^{-s\tau_{22}}\int_{-\tau_{22}}^{0} e^{-st}\psi_2(t)dt\\
&+ \cdots + a_{2m}e^{-s\tau_{2m}}\int_{-\tau_{2m}}^{0} e^{-st}\psi_m(t)dt\\
&\vdots\\
b_m(s) =& s^{\alpha_m-1}\psi_m(0) + a_{m1}e^{-s\tau_{m1}}\int_{-\tau_{m1}}^{0} e^{-st}\psi_1(t)dt + a_{m2}e^{-s\tau_{m2}}\int_{-\tau_{m2}}^{0} e^{-st}\psi_2(t)dt\\
&+ \cdots + a_{mm}e^{-s\tau_{mm}}\int_{-\tau_{mm}}^{0} e^{-st}\psi_m(t)dt.
\end{aligned}
$$

Remark 11.2 Zero solution of system (11.3) is Lyapunov asymptotically stable if all the roots of $\det(\Delta(s)) = 0$ have negative real parts.

11.2 Fractional Delayed Predator-Prey Model

Lotka [41] in 1925 and Volterra [42] in 1926 introduced the first PP model. After that, many more complicated but realistic PP models, with various forms of "functional responses," have been formulated by ecologists and mathematicians. One of the most popular PP models was introduced by Freedman [43] in 1980, which has the Michaelis-Menten type or Holling type-II functional response $\frac{\beta x(t)y(t)}{1+\sigma x(t)}$, where $x(t)$ and $y(t)$ are the population densities of the prey and predator, respectively. β (units: 1/time) is the feeding rate, or the maximal predator per-capita consumption rate, i.e., the maximum number of preys that can be eaten by a predator in each time unit, while σ (units: 1/prey) is a positive constant that describes the effects of capture rate. The prey host population is assumed to have logistic growth $rx(t)\left(1-\frac{x(t)}{K}\right)$ with carrying capacity K (in a closed community) and a specific growth rate constant r. Then, the interactions between prey and predator with time-delay τ in the saturation term takes the form

$$\begin{aligned} Dx(t) &= rx(t)\left(1-\frac{x(t)}{K}\right)-\frac{\beta x(t)y(t-\tau)}{1+\sigma x(t)}, \\ Dy(t) &= \frac{\beta x(t)y(t-\tau)}{1+\sigma x(t)}-ay(t), \end{aligned} \tag{11.7}$$

where τ is regarded as reaction time of the predations and a is a positive real number. In system (11.7), it is assumed that each individual predator has the same ability to feed on prey.

In this chapter, we extend the derivatives of (11.7) to an arbitrary order to investigate the combination of both fractional-order formulation and time-delay in the same model. Therefore, the model becomes

$$\begin{aligned} D^{\alpha}x(t) &= rx(t)\left(1-\frac{x(t)}{K}\right)-\frac{\beta x(t)y(t-\tau)}{1+\sigma x(t)} \\ D^{\alpha}y(t) &= \frac{\beta x(t)y(t-\tau)}{1+\sigma x(t)}-ay(t), \quad 0<\alpha\leq 1,\ t\geq 0 \end{aligned} \tag{11.8}$$

with initial conditions $x(0) > 0$ and $y(t) = \psi(t) > 0$ when $t \in [-\tau, 0]$, where $\psi(t)$ is a smooth function. Next, we study the impact of the fractional-order and time-delay τ in the dynamics of the model.

By choosing the delay τ as a bifurcation parameter, when it crosses some critical values τ^*, a Hopf bifurcation about the stability of interior equilibrium in (11.8) can

occur, as we shall discuss in the next section. The fractional order grants the model a greater degree of freedom and consistency with real interactions due to its ability to provide an exact description of the non-linear phenomena.

11.3 Local Stability Analysis and Hopf Bifurcation

Consider the fractional-order systems of the form

$$\begin{aligned} D^{\alpha}x(t) &= f_1(x, y),\ D^{\alpha}y(t) = f_2(x, y),\ \alpha \in (0, 1], \\ x(0) &= x_0,\quad y(0) = y_0 \end{aligned} \tag{11.9}$$

with an equilibrium point (x_e, y_e). Therefore,

Lemma 11.1 *The equilibrium point (x_e, y_e) of the fractional differential system (11.9) is locally asymptotically stable if and only if all eigenvalues λ_i of the Jacobian matrix*

$$J = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix},$$

evaluated at the equilibrium point (x_e, y_e), satisfy the condition that $|arg(\lambda_i)| > \dfrac{\alpha\pi}{2}$ [44] (see Fig. 11.1).

Since it is known that systems with memory are typically more stable than their memoryless counterparts, we expect that "*fractional-order differential equations are, at least, as stable as their integer-order counterpart.*"

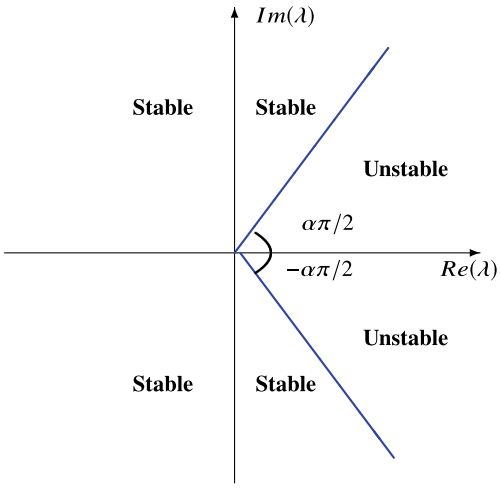

Fig. 11.1 Stability region of the fractional-order system (11.9) when $0 < \alpha \leq 1$

The equilibria of (11.8) are the points of intersections at which $D^{\alpha}x(t) = 0$ and $D^{\alpha}y(t) = 0$. Thus, we arrive at the following proposition: For the model system (11.8), there always exist trivial equilibrium $\mathcal{E}_0 = (0, 0)$ and semi-trivial equilibrium $\mathcal{E}_1 = (K, 0)$. However, if the threshold parameter

$$\mathcal{R}_0 = \frac{K[\beta - \sigma a]}{a} > 1, \tag{11.10}$$

there also exists an interior equilibrium $\mathcal{E}_+ = (x^*, y^*)$, where

$$\mathcal{E}_+ = (x^*, y^*) = \left(\frac{a}{\beta - \sigma a}, \frac{rx^{*2}}{Ka}(\mathcal{R}_0 - 1) \right). \tag{11.11}$$

11.3.1 Trivial and Semi-trivial Equilibria and Their Stabilities

The theorem matrix of the linearized system of model (11.8) is

$$J = \begin{pmatrix} r - \dfrac{2rx^*}{K} - \dfrac{\beta y^*}{(1+\sigma x^*)^2} & -\dfrac{\beta x^*}{1+\sigma x^*} e^{-\lambda\tau} \\ \dfrac{\beta y^*}{(1+\sigma x^*)^2} & \dfrac{\beta x^*}{1+\sigma x^*} e^{-\lambda\tau} - a \end{pmatrix} \tag{11.12}$$

Using (11.12), the characteristic equation[1] at the trivial equilibrium point $\mathcal{E}_0 = (0, 0)$ reduces to

$$(\lambda^{\alpha} - r)(\lambda^{\alpha} + a) = 0. \tag{11.13}$$

Clearly, Eq. (11.13) has a positive root $\lambda^{\alpha} = r$ $(0 < \alpha \leq 1)$. Then, the trivial equilibrium $\mathcal{E}_0$ of system (11.8) is always unstable (saddle point). However, at the semi-trivial equilibrium $\mathcal{E}_1 = (K, 0)$, the Jacobian matrix (11.12) reduces to

$$J_{\text{semi-trivial}} = \begin{pmatrix} -r & -\dfrac{\beta K}{1+\sigma K} e^{-\lambda\tau} \\ 0 & \dfrac{\beta K}{1+\sigma K} e^{-\lambda\tau} - a \end{pmatrix} \tag{11.14}$$

with characteristic equation

$$(\lambda^{\alpha} + r)\left(\lambda^{\alpha} + a\left[1 - \frac{\mathcal{R}_0 + \sigma K}{1+\sigma K} e^{-\lambda\tau} \right] \right) = 0. \tag{11.15}$$

[1] We may note that the characteristic equation of a system with delay has infinite roots.

It is obvious from Eq. (11.15) that the two roots are real and negative if $\mathcal{R}_0 < 1$ (when $\tau = 0$) and the equilibrium $\mathcal{E}_1$ is then asymptotically stable. In case of $\tau > 0$, we assume that the root of (11.15) $\lambda = \xi i$ must satisfy

$$\xi^{2\alpha} = a^2 \left[\frac{\mathcal{R}_0 + \sigma K}{1 + \sigma K} - 1 \right] < 0.$$

Then, when $\mathcal{R}_0 < 1$, there are no positive real roots ξ. Hence, according to Lemma 11.1, we can get the following theorem to indicate the stability of $\mathcal{E}_1$:

Theorem 11.1 *If $\mathcal{R}_0$ is defined by (11.10), then semi-trivial equilibrium $\mathcal{E}_1 = (K, 0)$ of system (11.8) is asymptotically stable when $\mathcal{R}_0 < 1$ (for all values of $\tau > 0$), unstable when $\mathcal{R}_0 > 1$, and linearly neutrally stable if $\mathcal{R}_0 = 1$.*

11.3.2 Interior Equilibrium and Its Stability

Here, we investigate the linear stability of (11.8) at the interior equilibrium $\mathcal{E}_+ = (x^*, y^*)$ defined in (11.11). $x^* = \dfrac{a}{\beta - \sigma a} \Longrightarrow \dfrac{\beta x^*}{1 + \sigma x^*} = a$, and $y^* = \dfrac{r x^{*2}}{K a}(\mathcal{R}_0 - 1) \Longrightarrow \dfrac{\beta y^*}{1 + \sigma x^*} = r\left(1 - \dfrac{1}{\mathcal{R}_0}\right)$. We also have $\mathcal{R}_0 = \dfrac{K}{x^*}$. Therefore, the corresponding Jacobian matrix at the interior equilibrium $\mathcal{E}_+$ can be easily expressed in terms of the reproduction number $\mathcal{R}_0$, as follows:

$$J_{\text{interior}} = \begin{pmatrix} \eta_{11} & -ae^{-\lambda\tau} \\ \dfrac{r}{1 + \sigma x^*}\left(1 - \dfrac{1}{\mathcal{R}_0}\right) & ae^{-\lambda\tau} - a \end{pmatrix}, \tag{11.16}$$

where $\eta_{11} = r\left(1 - \dfrac{2}{\mathcal{R}_0}\right) - \dfrac{r}{1 + \sigma x^*}\left(1 - \dfrac{1}{\mathcal{R}_0}\right)$. The characteristic equation of (11.16) for the interior equilibrium is

$$\begin{aligned} \lambda^{2\alpha} + \lambda^{\alpha}\Bigg[& -r\left(1 - \frac{2}{\mathcal{R}_0}\right) + a(1 - e^{-\lambda\tau}) \\ & + \frac{r}{1 + \sigma x^*}\left(1 - \frac{1}{\mathcal{R}_0}\right)\Bigg] + a\Bigg[-r\left(1 - \frac{2}{\mathcal{R}_0}\right)(1 - e^{-\lambda\tau}) \\ & + \frac{r}{1 + \sigma x^*}\left(1 - \frac{1}{\mathcal{R}_0}\right)\Bigg] = 0. \end{aligned} \tag{11.17}$$

We need to find the necessary and sufficient condition for every root of the characteristic Eq. (11.17) having negative real part. Introducing

$$\varrho_1 = r\left(1 - \frac{2}{\mathcal{R}_0}\right),\ \varrho_2 = \frac{r}{1+\sigma x^*}\left(1 - \frac{1}{\mathcal{R}_0}\right),\ \varrho_3 = a. \tag{11.18}$$

Then, the characteristic Equation (11.17) can be rewritten in the form

$$\lambda^{2\alpha} + \lambda^{\alpha}(-\varrho_1 + \varrho_2 + \varrho_3) + \varrho_3(-\varrho_1 + \varrho_2) + e^{-\lambda\tau}(-\varrho_3\lambda + \varrho_1\varrho_3) = 0.$$

For simplicity, let us also assume that

$$\begin{aligned} A_1 = (-\varrho_1 + \varrho_2 + \varrho_3),\ A_2 = \varrho_3(-\varrho_1 + \varrho_2), \\ A_3 = \varrho_3,\ A_4 = \varrho_1\varrho_3. \end{aligned} \tag{11.19}$$

Then, Eq. (11.19) takes the form

$$\lambda^{2\alpha} + A_1\lambda^{\alpha} + A_2 + e^{-\lambda\tau}(-A_3\lambda + A_4) = 0. \tag{11.20}$$

We establish the existence of the parameter value τ^* for which the equilibrium solution undergoes two simultaneous Hopf bifurcations.

Theorem 11.2 *Assume that $\mathcal{R}_c = 2 + 1/(1 + 2\sigma x^*)$. Then,*

(1) the interior equilibrium $\mathcal{E}_+$ of system (11.8) is feasible and locally asymptotically stable for all $\tau \geq 0$ if $1 < \mathcal{R}_0 \leq \mathcal{R}_c$ holds;

(2) if $\mathcal{R}_0 > \mathcal{R}_c > 1$, then there exist $\tau^* > 0$, such that $\tau \in [0, \tau^*)$ the interior equilibrium $\mathcal{E}_+$ is asymptotically stable, and unstable when $\tau > \tau^*$. When $\tau = \tau^*$, the characteristic equation (11.20) has a pair of purely imaginary roots $\pm i\xi_0^{\alpha}$ with

$$\xi_0^{2\alpha} = \frac{1}{2}(2A_2 + A_3^2 - A_1^2) + \frac{1}{2}\sqrt{(2A_2 + A_3^2 - A_1^2)^2 - 4(A_2^2 - A_4^2)},$$

and

$$\tau^* = \frac{1}{\xi_0}\arccos\left(\frac{(A_4 + A_1A_3)\xi_0^2 - A_2A_4}{A_3^2\xi_0^2 + A_4^2}\right) + \frac{2j\alpha\pi}{\xi_0^{\alpha}},$$

where A_1, A_2, A_3 and A_4 are defined in (11.19).

Proof If $\lambda = \xi i$ is a root of (11.19). After substitution and separation of the real and imaginary parts, we have

$$\begin{aligned} -\xi^{2\alpha} + \varrho_3(-\varrho_1 + \varrho_2) &= \xi^{\alpha}\varrho_3\sin\xi\tau - \varrho_1\varrho_3\cos\xi\tau, \\ \xi^{\alpha}(-\varrho_1 + \varrho_2 + \varrho_3) &= \varrho_1\varrho_3\sin\xi\tau + \xi\varrho_3\cos\xi\tau, \end{aligned} \tag{11.21}$$

which are equivalent to

$$\begin{aligned} -\xi^{2\alpha} + A_2 &= \xi^{\alpha}A_3\sin\xi\tau - A_4\cos\xi\tau, \\ \xi^{\alpha}A_1 &= A_4\sin\xi\tau + \xi A_3\cos\xi\tau. \end{aligned} \tag{11.22}$$

Squaring and adding both equations yields

$$\xi^{4\alpha} + \xi^{2\alpha}(-\varrho_1 + \varrho_2)^2 + \varrho_2\varrho_3^2(-2\varrho_1 + \varrho_2) = 0, \tag{11.23}$$

which is equivalent to

$$\xi^{4\alpha} - (2A_2 + A_3^2 - A_1^2)\xi^{2\alpha} + (A_2^2 - A_4^2) = 0. \tag{11.24}$$

Equation (11.23) can also be re-written in the form

$$\xi^{4\alpha} + \xi^{2\alpha}[\varrho_1^2 + \varrho_2(-2\varrho_1 + \varrho_2)] + \varrho_2\varrho_3^2(-2\varrho_1 + \varrho_2) = 0. \tag{11.25}$$

Therefore, if $-2\varrho_1 + \varrho_2 \geq 0$ (when $\mathcal{R}_0 > 1$), then there is no positive real ξ satisfying (11.23). According to the definitions given in (11.18), the inequality $-2\varrho_1 + \varrho_2 \geq 0$, which is equivalent to $\mathcal{R}_0 \leq 2 + 1/(1 + 2\sigma x^*)$ so that all the roots ($\lambda = \xi_i$) of (11.17) are negative.

However, if $-2\varrho_1 + \varrho_2 < 0$, then (11.25) has one and only one positive root denoted by ξ_0, and the characteristic equation (11.20) has a pair of purely imaginary roots $\pm i\xi_0$. Let $\lambda(\tau) = \sigma(\tau) + i\xi(\tau)$ be the eigenvalue of (11.20), such that $\sigma(\tau^*) = 0$ and $\xi(\tau^*) = \xi_0$. From (11.22), we have

$$\tau^* = \frac{1}{\xi_0^\alpha} \arccos\left(\frac{(A_4 + A_1A_3)\xi_0^2 - A_2A_4}{A_3^2\xi_0^2 + A_4^2}\right) + \frac{2j\alpha\pi}{\xi_0^\alpha} \tag{11.26}$$

and from (11.24)

$$\xi_0^{2\alpha} = \frac{1}{2}(2A_2 + A_3^2 - A_1^2) + \frac{1}{2}\sqrt{(2A_2 + A_3^2 - A_1^2)^2 - 4(A_2^2 - A_4^2)} < 0.$$

Hence, according to Lemma 11.1, the interior equilibrium $\mathcal{E}_+$ of system (11.8) is locally asymptotically stable with $0 < \alpha \leq 1$. The proof is, thus, complete. □

11.4 Global Stability Analysis

In this section, we extend the analysis to study the global stability conditions [45, 46] for the fractional-order delay differential system. To study the global stability of the equilibrium points of (11.8), we linearize the system into the form

$$\begin{aligned} D^\alpha x(t) &= m_1 x(t) + m_2 y(t-\tau) \\ D^\alpha y(t) &= n_1 x(t) + n_2 y(t) + n_3 y(t-\tau) \end{aligned} \quad 0 < \alpha \leq 1. \tag{11.27}$$

where

$$
\begin{aligned}
m_1 &= r - \frac{2x^*}{K} - \frac{\beta y^*}{1+\sigma x^*} + \frac{\sigma\beta x^* y^*}{(1+\sigma x^*)^2},\\
m_2 &= -\frac{\beta x^*}{1+\sigma x^*}, \quad n_1 = \frac{\beta y^*}{1+\sigma x^*} - \frac{\sigma\beta x^* y^*}{(1+\sigma x^*)^2},\\
n_2 &= -a, \quad n_3 = \frac{\beta x^*}{1+\sigma x^*}.
\end{aligned}
$$

If the linear fractional differential equation has non-zero equilibrium point, we can shift equilibrium point to the origin. Put $\bar{x}(t) = x(t) - x^*$, $\bar{y}(t) = y(t) - y^*$, then the Eqs. (11.27) become

$$
\begin{aligned}
D^\alpha \bar{x}(t) &= m_1\bar{x}(t) + m_2\bar{y}(t-\tau)\\
D^\alpha \bar{y}(t) &= n_1\bar{x}(t) + n_2\bar{y}(t) + n_3\bar{y}(t-\tau)
\end{aligned}
\quad 0 < \alpha \le 1. \tag{11.28}
$$

To study the stability of system (11.8), we take a Laplace transform [47] on both sides of (11.28). Then, we have

$$
\begin{aligned}
s^\alpha X_1(s) &= m_1 X_1(s) + s^{\alpha-1}\varphi_1(0)\\
&\quad + m_2 e^{-s\tau}\left(X_2(s) + \int_{-\tau}^{0} e^{-st}\varphi_2(t)dt\right)\\
s^\alpha X_2(s) &= n_1 X_1(s) + n_2 X_2(s) + s^{\alpha-1}\varphi_2(0)\\
&\quad + n_3 e^{-s\tau}\left(X_2(s) + \int_{-\tau}^{0} e^{-st}\varphi_2(t)dt\right).
\end{aligned} \tag{11.29}
$$

Here, it should be mentioned that the initial values $\bar{x}(t) = \varphi_1(t)$ and $\bar{y}(t) = \varphi_2(t)$ with $t \in [-\tau, 0]$. Additionally, $X_1(s)$ and $X_2(s)$ are Laplace transforms of $\bar{x}(t)$ and $\bar{y}(t)$ with $X_1(s) = L(\bar{x}(t))$ and $X_2(s) = L(\bar{y}(t))$. The system (11.29) can be rewritten as follows:

$$
\Delta(s)\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} k_1(s) \\ k_2(s) \end{pmatrix} \tag{11.30}
$$

in which

$$
\Delta(s) = \begin{pmatrix} s^\alpha - m_1 & -m_2 e^{-s\tau} \\ -n_1 & s^\alpha - n_2 - n_3 e^{-s\tau} \end{pmatrix}
$$

and

$$
\begin{aligned}
k_1(s) &= s^{\alpha-1}\varphi_1(0) + m_2 e^{-s\tau}\textstyle\int_{-\tau}^{0} e^{-st}\varphi_2(t)dt\\
k_2(s) &= s^{\alpha-1}\varphi_2(0) + n_3 e^{-s\tau}\textstyle\int_{-\tau}^{0} e^{-st}\varphi_2(t)dt.
\end{aligned}
$$

$\Delta(s)$ is considered as characteristic matrix of system (11.8) and $\det \Delta(s)$ as its characteristic polynomial. Therefore, the distribution of the eigenvalues of the characteristic polynomial determines the stability of the system (11.8). In other words, if all roots of the characteristic equation have negative parts, then the equilibrium of the above fractional-order PP system is Lyapunov globally asymptotical stable if the equilibrium exists [45]. If we multiply both sides of (11.30) by s, we have

$$\Delta(s)\begin{pmatrix} sX_1(s) \\ sX_2(s) \end{pmatrix} = \begin{pmatrix} sk_1(s) \\ sk_2(s) \end{pmatrix} \tag{11.31}$$

Therefore, if all roots of the transcendental equation $\det \Delta(s) = 0$ lie in the open left complex plane, i.e., $\mathrm{Re}(s) < 0$, then we consider (11.31) in $\mathrm{Re}(s) \geq 0$. In this restricted area, system (11.31) has a unique solution $(sX_1(s),\ sX_2(s))$, so that

$$\lim_{s\to 0, \mathrm{Re}(s)\geq o} sX_i(s) = 0, \quad i = 1, 2.$$

From the assumption of all roots of the characteristic equation $\det \Delta(s) = 0$ and the final-value theorem of the Laplace transform [47], we get

$$\lim_{t\to+\infty} \bar{x}(t) \equiv \lim_{s\to 0, \mathrm{Re}(s)\geq 0} sX_1(s) = 0,$$
$$\text{and} \quad \lim_{t\to+\infty} \bar{y}(t) \equiv \lim_{s\to 0, \mathrm{Re}(s)\geq 0} sX_2(s) = 0.$$

It implies that the zero solution of the fractional-order PP system is Lyapunov globally asymptotically stable. Therefore, we arrive at the following result:

Theorem 11.3 *If all the roots of the characteristic equation* $\det \Delta(s) = 0$ *have negative real parts, then the positive equilibrium points* (x^*, y^*) *of system (11.8) is Lyapunov globally asymptotically stable.*

11.5 Implicit Euler's Scheme for FODDEs

Since most FODEs do not have exact analytical solutions, approximation and numerical techniques must be used. In addition, most of the resulting biological systems are stiff.[2] The *stiffness* often appears due to the differences in speed between the fastest and slowest components of the solutions and due to stability constraints. In addition, the state variables of these types of models are very sensitive to small perturbations (or changes) in the parameters that occur in the model. Therefore, efficient use of a reliable numerical method for dealing with stiff problems is necessary.

Consider the following FODDEs:

[2] One definition of stiffness is that the global accuracy of the numerical solution is determined by stability rather than local error and implicit methods are more appropriate for it.

$$\begin{aligned} D^{\alpha}y(t) &= f(t, y(t), y(t-\tau)), \quad t \in J = [0, T], \\ y(t) &= \psi(t), \quad t \in [-\tau, 0], \qquad 0 < \alpha \leq 1. \end{aligned} \tag{11.32}$$

Here $y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T$, $f : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Lipschitz condition, and there exists a positive constant $K > 0$ such that

$$\begin{aligned} &\|f(t, y(t), y(t-\tau)) - f(t, x(t), x(t-\tau))\| \\ &\leq K\{\|y(t) - x(t)\| + \|y(t-\tau) - x(t-\tau)\|\}. \end{aligned} \tag{11.33}$$

Theorem 11.4 *Problem (11.32) has a unique solution provided that the Lipschitz condition (11.33) is satisfied and* $\bar{M} = \dfrac{2K\mathrm{T}^{\alpha}}{\Gamma(\alpha+1)} < 1$.

Proof We can apply a fractional integral operator to the differential equation (11.32) and incorporate the initial conditions, thus converting the equation into the equivalent equation

$$y(t) = \psi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), y(s-\tau))ds, \tag{11.34}$$

which is also a Volterra equation of the second kind. Define the operator $\mathcal{L} : C(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n)$, such that

$$\mathcal{L}y(t) = \psi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), y(s-\tau))ds. \tag{11.35}$$

Then, we have

$$\begin{aligned} &\|\mathcal{L}y(t) - \mathcal{L}x(t)\| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \times \|f(s, y(s), y(s-\tau)) - f(s, x(s), x(s-\tau))\|ds \\ &\leq \frac{K}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \{\|y(s) - x(s)\| + \|y(s-\tau) - x(s-\tau)\|\}ds \\ &\leq \frac{K}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big\{ \sup_{s\in J} \|y(s) - x(s)\| + \sup_{s\in[-\tau,0]} \|y(s) - x(s)\| + \sup_{s\in J} \|y(s) - x(s)\| \Big\} ds \\ &\leq \frac{2K}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{s\in J} \|y(s) - x(s)\|ds \\ &\leq \frac{2K}{\Gamma(\alpha+1)} \|y - x\|\mathrm{T}^{\alpha}. \end{aligned}$$

Therefore, we obtain

$$\|\mathcal{L}y(t) - \mathcal{L}x(t)\| \leq \bar{M}\|y - x\|.$$

Using Banach contraction principle [48], we can deduce that $\mathcal{L}$ has a unique fixed point; this implies that our problem has a unique solution. □

Several numerical methods have been proposed to solve fractional-order differential equations (FODEs) [36, 49]. The predictor-corrector algorithm is an efficient and powerful technique for solving FODEs, which is a generalization of the Adams-Bashforth-Moulton method. The modification of the Adams-Bashforth-Moulton algorithm is proposed by Diethelm [50] to approximate the fractional-order derivative (See Appendix B). However, the converted Volterra integral equation (11.34) has a weakly singular kernel, such that regularization is not necessary anymore. It appears that there exists only a very small number of software packages for non-linear Volterra equations. In our case, the kernel may not be continuous and, therefore, the classical numerical algorithms for the integral part of (11.34) are unable to handle the solution of Equation (11.32). Therefore, we implement the implicit Euler's scheme to approximate the fractional-order derivative.

Given the delay fractional-order model (11.32) and mesh points $\mathcal{T} = \{t_0, t_1, \ldots, t_N\}$, such that $t_0 = 0$ and $t_N = \mathrm{T}$ with stepsize $h = \tau/m$. If $\psi(t)$ is a continuous function, then the solution $y(t)$ for $0 \le t \le \tau$ (τ is bounded) satisfies the fractional-order ordinary differential equation

$$\begin{aligned} D^{\alpha} y'(t) &= f(t, y(t), \psi(t-\tau)), \quad 0 \le t \le \tau, \\ y(0) \quad &= \psi(0) \quad 0 < \alpha \le 1. \end{aligned} \tag{11.36}$$

This equation has a unique solution, where f satisfies Lipschitz conditions and the solution of (11.36) on $[0, \tau]$ coincides with the solution of (11.32) on $[0, \tau]$. Once the solution y is known on $[0, \tau]$, we can repeat the same procedure, starting with the solution on $[0, \tau]$, to find the solution for $\tau \le t \le 2\tau$, *etc*. This procedure is called *method of steps*,[3] and yields a unique defined solution of the resulting system of FODDEs (11.32), given the initial function $\psi(t)$ on $[0, \tau]$. Therefore, FODDEs (11.32) can be numerically solved by a step-by-step fractional-order ODE integrator provided that the solution is known up to the current integration point.

Next, we will approximate the fractional derivative by a simple quadrature formula, using the Caputo fractional derivative (11.1) of order α, $0 < \alpha \le 1$, and using *implicit* Euler's approximation as follows (see [37]):

$$\begin{aligned} D_*^{\alpha} x_i(t_n) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{dx_i(s)}{ds} (t_n - s)^{-\alpha} ds \\ &\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n} \int_{(j-1)h}^{jh} \left[\frac{x_i^j - x_i^{j-1}}{h} + O(h) \right] (nh - s)^{-\alpha} ds \\ &= \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \sum_{j=1}^{n} \left\{ \left[\frac{x_i^j - x_i^{j-1}}{h} + O(h) \right] \times \left[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha} \right] \right\} h^{1-\alpha} \end{aligned}$$

[3] Method of steps is not universal, as it cannot be applied with time-varying delays, which vanish in some points.

$$= \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \frac{1}{h^{\alpha}} \sum_{j=1}^{n} \left[x_i^j - x_i^{j-1}\right] \times \left[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}\right]$$
$$+ \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \sum_{j=1}^{n} \left[x_i^j - x_i^{j-1}\right] \times \left[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}\right] O(h^{2-\alpha}).$$

Setting

$$\mathcal{G}(\alpha, h) = \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \frac{1}{h^{\alpha}}, \quad \text{and } \omega_j^{\alpha} = j^{1-\alpha} - (j-1)^{1-\alpha}, \quad (\text{where } \omega_1^{\alpha} = 1), \tag{11.37}$$

then the first-order approximation method for the computation of Caputo's fractional derivative is given by the expression

$$D_*^{\alpha} x_i(t_n) = \mathcal{G}(\alpha, h) \sum_{j=1}^{n} \omega_j^{\alpha} \left(x_i^{n-j+1} - x_i^{n-j}\right) + O(h). \tag{11.38}$$

From the analysis and numerical approximation, we also arrive at the following proposition:

Proposition 11.1 *The presence of a fractional differential order in a differential equation can lead to a notable increase in the complexity of the observed behavior, and the solution continuously depends on all previous states.*

11.5.1 Stability and Convergence of Implicit Scheme for FODDEs

In this section, we prove that the fractional-order implicit difference approximation (11.38) is unconditionally stable. It then follows that the numerical solution converges to the exact solution as $h \to 0$. To study the stability of the numerical method, let us consider a test problem of a linear scalar fractional differential equation

$$\begin{aligned} D_*^{\alpha} u(t) &= \rho_0 u(t) + \rho_1 u(t-\tau), \quad t \geq 0, \quad 0 < \alpha \leq 1 \\ u(t) &= \psi(t), \quad t \in [-\tau, 0], \; u(0) = u_0 \end{aligned} \tag{11.39}$$

such that $\rho_0 < 0$, $|\rho_1| < \rho_0$ and $\psi(t)$ is a continuous and bounded function.

Theorem 11.5 *The fully implicit numerical approximation (11.38), to test problem (11.39) for all $t \geq 0$, is consistent and unconditionally stable.*

Proof We assume that $\tau = mh$ and the approximate solution of (11.39) is of the form $u(t_n) \approx U^n \equiv \zeta_n$, and $u(t_n - \tau) \approx \zeta_{n-m}$; then, the Eq. (11.39) can be reduced to

$$\left(1-\frac{\rho_0}{\mathcal{G}_{\alpha,h}}\right)\zeta_n=\zeta_{n-1}+\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\zeta_{n-j}-\zeta_{n-j+1}\right)+\rho_1\zeta_{n-m}/\mathcal{G}_{\alpha,h},\ n\geq m \tag{11.40}$$

and

$$\begin{aligned}\left(1-\frac{\rho_0}{\mathcal{G}_{\alpha,h}}\right)\zeta_n=\zeta_{n-1}+\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\zeta_{n-j}-\zeta_{n-j+1}\right)\\ +\rho_1\psi(t_{n-m})/\mathcal{G}_{\alpha,h}\ ,\qquad n=2,\ldots m.\end{aligned} \tag{11.41}$$

Therefore,

$$\zeta_n=\frac{\zeta_{n-1}+\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\zeta_{n-j}-\zeta_{n-j+1}\right)+\rho_1\zeta_{n-m}/\mathcal{G}_{\alpha,h}}{\left(1-\frac{\rho_0}{\mathcal{G}_{\alpha,h}}\right)},\quad n\geq 2. \tag{11.42}$$

Since $\left(1-\frac{\rho_0}{\mathcal{G}_{\alpha,h}}\right)\geq 1$ for all $\mathcal{G}_{\alpha,h}$, then

$$\zeta_1\leq\zeta_0, \tag{11.43}$$

$$\zeta_n\leq\zeta_{n-1}+\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\zeta_{n-j}-\zeta_{n-j+1}\right),\qquad n\geq 2. \tag{11.44}$$

Thus, for $n=2$, the above inequality implies

$$\zeta_2\leq\zeta_1+\omega_2^{(\alpha)}\left(\zeta_0-\zeta_1\right).$$

Using the relation (11.43) and the positivity of the coefficients ω_2, we get

$$\zeta_2\leq\zeta_1.$$

Repeating the process, we have from (11.44)

$$\zeta_n\leq\zeta_{n-1}+\sum_{j=2}^{n}\omega_j^{(\alpha)}\left(\zeta_{n-j}-\zeta_{n-j+1}\right)\leq\zeta_{n-1},$$

since each term in the summation is negative. Thus, $\zeta_n\leq\zeta_{n-1}\leq\zeta_{n-2}\leq\cdots\leq\zeta_0$. With the assumption that $\zeta_n=|U^n|\leq\zeta_0=|U^0|$, which entails $\|U^n\|\leq\|\psi(t_0)\|$ and we have stability. □

Of course, this numerical technique can be used both for linear and for non-linear problems, and it may be extended to multi-term FODDEs as well.

11.5.2 Numerical Simulations

In this subsection, to verify the effectiveness of the obtained results, some numerical simulations for the fractional-order PP system (11.8) have been conducted. All the differential equations are solved using the method proposed in this chapter. In all numerical runs, the solution has been approximated using the parameter values given in the captions of the figures. Figures 11.2, 11.3 and 11.4 show the numerical simulations of model (11.8), with different values of the model parameters given in the corresponding captions. According to the obtained analysis, Fig. 11.2 shows that the numerical simulations of the model, for particular values of the parameters, admit limit cycles, whereas Fig. 11.3 shows that periodic solutions arise due to Hopf bifurcation. When the reproduction number $\mathcal{R}_0 < 1$, the semi-trivial equilibrium is stable (see Fig. 11.4); however, when $\mathcal{R}_0 > 1$, the semi-trivial equilibrium is unstable and interior equilibrium exists. The interior equilibrium is stable if $1 < \mathcal{R}_0 \leq \mathcal{R}_c$ and a sustained periodic solution is obtained when $\mathcal{R}_0 > \mathcal{R}_c$.

It has been seen that the fractional derivative damps the oscillation behavior of the model (see Figs. 11.5, 11.6, 11.7 and 11.8).

Remark 11.3 For $\alpha \in (0, 1]$ the fractional order is defined by Caputo sense (11.1) so that introducing a convolution integral with a power-law memory kernel is useful to describe memory effects in dynamical systems. The decay rate of the memory kernel (a time correlation function) depends on α. A lower value of α corresponds to more slowly decaying time-correlation functions (long memory) [51]. Therefore, as $\alpha \to 1$, the influence of memory decreases. Therefore, in the above figures, we observe that the phase portrait gets stretched as the order of the derivative is reduced.

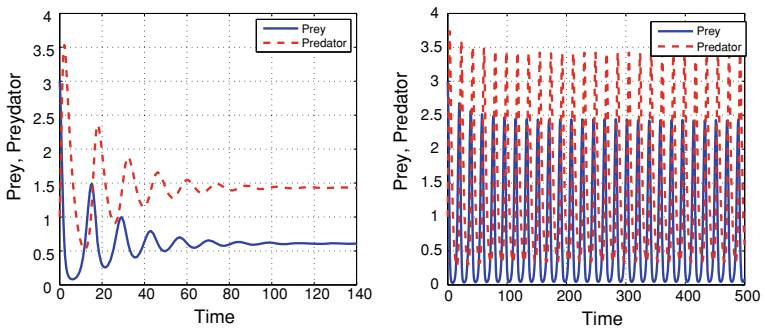

Fig. 11.2 Solution of PP model (11.8) when $r = 0.8, k = 5, \sigma = 0.01, \beta = 0.5; a = 0.3$, and $\mathcal{R}_0 > \mathcal{R}_c > 1$ with time-lag $\tau = 0.1 < \tau^*$ (top) and $\tau = \tau^* = 0.86$ (bottom), which display periodic outbreak of the disease due to a Hopf bifurcation when $\tau = \tau^*$

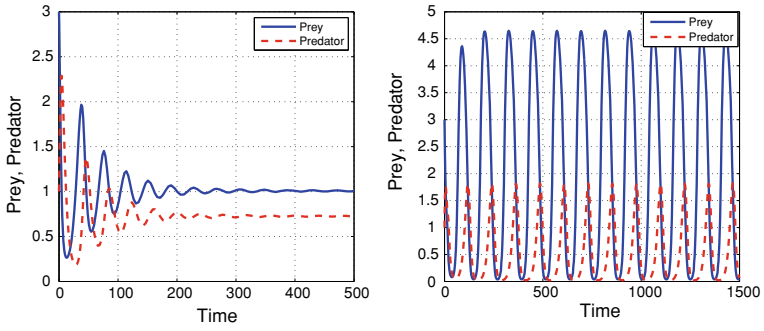

Fig. 11.3 Solution of PP model (11.8), when $r = 0.2$, $k = 5$, $\sigma = 0.01$, $\beta = 0.2$; $a = 0.2$, and $\mathcal{R}_0 = 4.7 > \mathcal{R}_c = 3$ with time-lag $\tau = 0.01 < \tau^*$ (top) and $\tau = 12$ (bottom), which display periodic outbreak of the disease due to a Hopf bifurcation when $\tau = \tau^*$

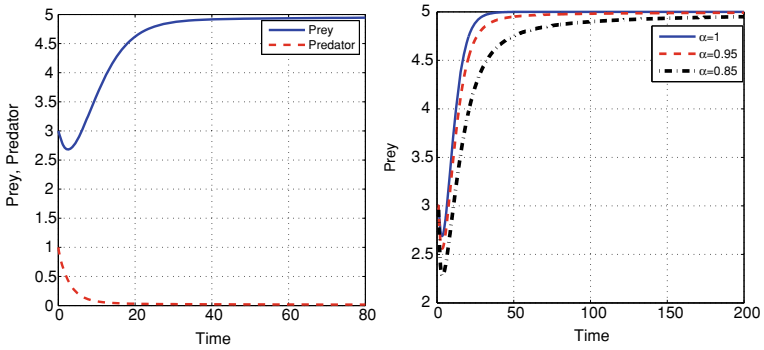

Fig. 11.4 Solution of delayed PP model (11.8). We have asymptotically stable semi-trivial equilibrium $\mathcal{E}_1 = (K, 0)$ when $\mathcal{R}_0 < 1$, with $r = 0.2$, $K = 5$, $\sigma = 0.01$, $\beta = 0.2$, $a = 1$; and $\tau = 1$

11.6 Concluding Remarks

In this chapter, we have introduced a fractional-order PP model with time-delay in the response function. We have also studied local stability and global stability behaviors of all the feasible equilibrium states of the system. It has been found that Hopf bifurcation occurs when the delay passes through a sequence of critical values τ^*, with fractional order $0 < \alpha \leq 1$. We derived the conditions in terms of the threshold parameter $\mathcal{R}_0$, which guarantees the asymptotic stability of the semi-trivial and interior equilibria. When $\mathcal{R}_0 < 1$, the semi-trivial equilibrium $\mathcal{E}_1$ is asymptotically stable for all values of $\tau > 0$ and unstable when $\mathcal{R}_0 > 1$. If all roots of the characteristic equation have negative parts, then the zero solution of the fractional-order delay PP system is Lyapunov globally asymptotical stable. If $\mathcal{R}_0 > \mathcal{R}_c > 1$, then there exists $\tau^* > 0$ such that $\tau \in [0, \tau^*)$ the interior equilibrium $\mathcal{E}_+$ is asymptotically stable, and unstable when $\tau > \tau^*$.

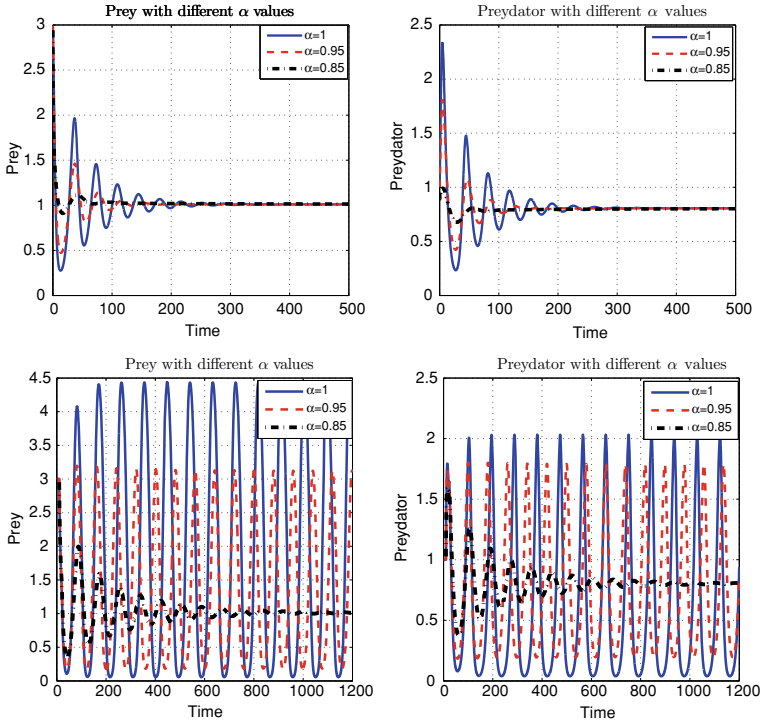

Fig. 11.5 Solution of PP model (11.8) when $r = 0.2$, $k = 5$, $\sigma = 0.01$, $\beta = 0.2$; $a = 0.2$, and $\mathcal{R}_0 = 4.7 > \mathcal{R}_c = 3$ with time-lag $\tau = 0.01 < \tau^*$ (top) and $\tau = 12$ (bottom), which display periodic outbreak of the disease due to a Hopf bifurcation when $\tau = \tau^*$

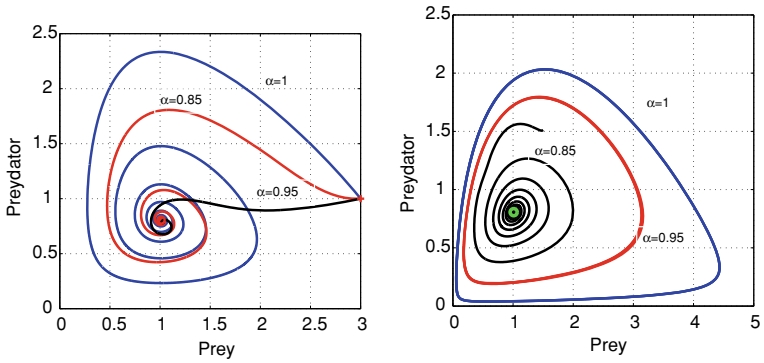

Fig. 11.6 The behavior of the PP model (11.8) with different fractional-order $0 < \alpha \leq 1$, with the same parameter values of Fig. 11.5. The fractional derivative damps the oscillation behavior

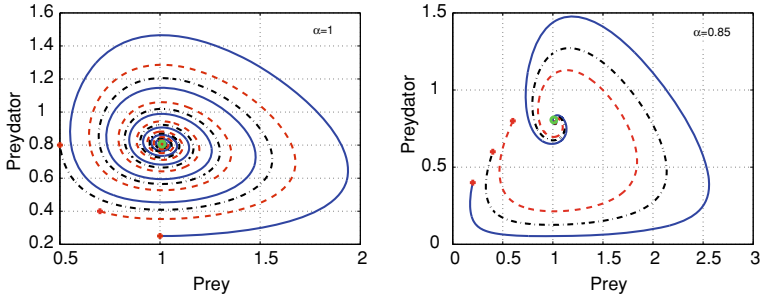

Fig. 11.7 The behavior of the PP model (11.8) with different initial conditions $\tau = 0.2 < \tau^*$. The fractional derivative damps the oscillation behavior

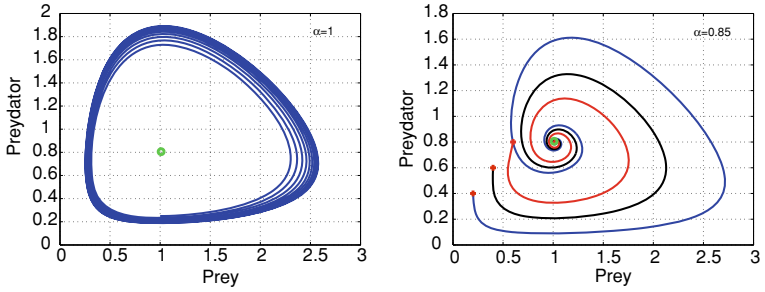

Fig. 11.8 The behavior of the PP model (11.8) with different initial conditions and $\tau = 1 > \tau^*$. The fractional derivative damps the oscillation behavior

We also introduced a suitable numerical method based on an implicit scheme for FODDEs. The numerical simulations demonstrate the accuracy and efficiency of the numerical scheme. Fractional-order models with time-delay are consistent with the dynamics of real PP interactions. We have seen from the numerical simulations that the fractional derivative improves the stability of the solutions and sometimes dampens the oscillation behavior of the solutions.

In the next chapter, we extend the analysis to investigate the dynamics of HCV infection using FODDEs.

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