Iterated Function Systems—A Topological Approach. Attractors

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Abstract Various types of basins, attractors and their fiberings are defined and shortly discussed in the realm of iterated function systems on normal topological spaces.

Keywords Iterated function system · Strict attractor · Pointwise basin · Fast basin

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1 Introduction

The aim of this article is to present some topological basics on attractors of IFSs in view of recent advances in the fractal geometry. It is based on the series of articles: [2–6, 8]. We introduce the concepts of basin, pointwise basin, fast basin, strict attractor, pointwise strict attractor, point-fibred attractor, strongly fibred attractor and homoclinic attractor. Relation of these concepts with the chaos game algorithm and fractal manifolds is mentioned in passing. For a thorough discussion of the existence of attractors, invariant sets and measures in contractive and non-contractive IFSs we refer to survey [9].

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2 IFS

Throughout the paper, X will be a normal topological space. As usual, $\overline{S}$ stands for the closure and $\mathrm{Int}(S)$ for the interior of $S \subseteq X$.

We distinguish the following collections of sets:

- 2^X, all subsets of X;
- $\mathcal{C}(X)$, nonempty closed sets;
- $\mathcal{CB}(X)$, nonempty bounded closed sets (provided X is a metric space);
- $\mathcal{K}(X)$, nonempty compact sets.

The *Vietoris topology* in $\mathcal{C}(X)$ is generated by subbasic sets of two forms

$$V^+ = \{C \in \mathcal{C}(X) : C \subseteq V\},$$
$$V^- = \{C \in \mathcal{C}(X) : C \cap V \neq \emptyset\},$$

where V runs through all open subsets of X. If X is a metric space, then the Vietoris topology and the Hausdorff metric topology agree on $\mathcal{K}(X)$. If a sequence of closed sets $S_n \subseteq X$ converges to $S \subseteq X$ with respect to the Vietoris topology, then we write $S_n \to S$.

An *iterated function system* $\mathcal{F} = \{w_i : i \in I\}$, *IFS* for short, is a finite collection of maps $w_i : X \to X$. Note that we do not assume continuity of w_i.

The *Hutchinson operator* $\mathcal{F} : 2^X \to 2^X$ induced by the IFS $\mathcal{F}$ is defined as follows

$$\mathcal{F}(S) := \bigcup_{i \in I} \overline{w_i(S)} \quad \text{for} \quad S \subseteq X.$$

Note that, without ambiguity, we denote the IFS and the associated Hutchinson operator by the same symbol $\mathcal{F}$. Symbol $\mathcal{F}^n$ will stand for the n-fold composition of $\mathcal{F}$. (Conveniently $\mathcal{F}^0 = \mathrm{id}$.)

Under additional conditions, we can restrict $\mathcal{F}$ to smaller collections of sets. We shall tacitly assume the following condition

$$w_i(K) \in \mathcal{K}(X) \quad \text{for all} \quad K \in \mathcal{K}(X), i \in I,$$

whenever we write $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$. This condition is satisfied when all maps w_i are continuous.

If $\mathcal{F}$ comprises continuous maps, then the Hutchinson operator $\mathcal{F} : \mathcal{C}(X) \to \mathcal{C}(X)$ is continuous with respect to the Vietoris topology. If X is a metric space, then $\mathcal{F} : \mathcal{K}(X) \to \mathcal{K}(X)$ is continuous in both, the Vietoris topology and the Hausdorff metric topology, while $\mathcal{F} : \mathcal{CB}(X) \to \mathcal{CB}(X)$ may fail to be continuous with respect to the Hausdorff metric. See [2] for more information about the continuity of $\mathcal{F}$.

3 Basins and Pointwise Basins

Definition 3.1 (Barnsley et al. [3, 4]) Let $A \in \mathcal{K}(X)$ and $\mathcal{F}$ be an IFS on X. We define the *pointwise basin* of A to be the set

$$\mathcal{B}_1(A) = \{x \in X: \mathcal{F}^n(\{x\}) \to A\},$$

and the *basin* of A to be the set

$$\mathcal{B}(A) = \bigcup \mathcal{U}(A),$$
$$\mathcal{U}(A) = \{U \subseteq X: A \subseteq U - \text{open}, \mathcal{F}^n(S) \to A \text{ for all } S \in \mathcal{K}(U)\}.$$

A nonempty compact set A is

(i) a *pointwise strict attractor* of $\mathcal{F}$, when $\mathrm{Int}(\mathcal{B}_1(A)) \supseteq A$;
(ii) a *strict attractor* of $\mathcal{F}$, when $\mathcal{B}(A) \neq \emptyset$.

Proposition 3.2 (Barnsley et al. [3] Propositions 8 and 11) *(i) If A is a pointwise strict attractor of $\mathcal{F}$, then* $\mathrm{Int}(\mathcal{B}_1(A)) = \mathcal{B}_1(A)$ *and* $\mathcal{F}(\mathcal{B}_1(A)) \subseteq \mathcal{B}_1(A)$.

(ii) If A is a strict attractor, then A is a pointwise strict attractor, and $\mathcal{B}(A) = \mathcal{B}_1(A)$.

The following criterion explains that pointwise strict attractors which are not strict attractors can exist only in highly non-contractive IFSs.

Proposition 3.3 (Barnsley et al. [3] Lemma 10) *Let $\mathcal{F} = \{w_i: i \in I\}$ be an IFS consisting of nonexpansive maps $w_i: X \to X$ acting on a metric space (X, d). If A is a pointwise strict attractor of $\mathcal{F}$, then A is a strict attractor of $\mathcal{F}$.*

We list now a couple of characteristic examples.

Example 3.4 (Strict attractor is a local concept) Let $w: X \to X$ be a continuous map with two attractive fixed points $x_1, x_2 \in X$, i.e. there exist open neighbourhoods $U_l \ni x_l$, $l = 1, 2$, such that $w^n(x) \to x_l$ for $x \in U_l$. Then, $A_l = \{x_l\}$, $l = 1, 2$, are two pointwise strict attractors of the same $\mathcal{F} = \{w\}$. (If w is locally contractive around x_1, x_2 in a complete metric space X, then we get strict attractors.)

In view of the above example and the example below, let us note that a strict attractor A of the IFS comprising global contractions is global in the sense that $\mathcal{B}(A) = X$.

Example 3.5 (Strict attractor is a topological concept) Let $\mathbb{C}$ be the complex plane. We endow $\mathbb{C}$ with two equivalent metrics: $d(z_1, z_2) = |z_1 - z_2|$ for $z_1, z_2 \in \mathbb{C}$ and $d_1 = \frac{d}{1+d}$. Fix three distinct points $a_1, a_2, a_3 \in \mathbb{C}$. Define $w_i(z) = \frac{1}{2} \cdot (z + a_i)$ for $z \in \mathbb{C}, i = 1, 2, 3$ and consider $\mathcal{F} = \{w_i: i \in \{1, 2, 3\}\}$. It is known that the Sierpiński triangle A with vertices a_1, a_2, a_3 is the *Hutchinson attractor* of $\mathcal{F}$ in $(\mathbb{C}, d)$, i.e. for all nonempty closed and bounded subsets S of $(\mathbb{C}, d)$, the set convergence $\mathcal{F}^n(S) \to$

A takes place with respect to the Hausdorff metric d_H in $\mathcal{CB}(\mathbb{C})$ induced by d. The Hausdorff metric induced by d_1 is not equivalent to d_H, because d_1 and d are not uniformly equivalent. Moreover, $\mathcal{F}^n(\mathbb{C}) = \mathbb{C} \neq A$, and the set $\mathbb{C}$ is closed and bounded in $(\mathbb{C}, d_1)$. Therefore, A is not the Hutchinson attractor of $\mathcal{F}$ in $(\mathbb{C}, d_1)$. On the other hand, A is a strict attractor of $\mathcal{F}$ regardless of the choice of equivalent metric in $\mathbb{C}$.

Example 3.6 (Strict attractor in a discontinuous IFS) Let $\mathcal{F} = \{w_i : i \in I\}$ be an IFS comprising continuous maps $w_i : X \to X$. We assume that $\mathcal{F}$ admits a strict attractor, denoted A. Further, assume that A has two disjoint dense subsets $E_m \subseteq A$, $m = 1, 2$, i.e. $\overline{E_m} = A$, $E_1 \cap E_2 = \emptyset$. Let also $e_m \in E_m$ be two distinguished points. Define for $i \in I$, $m = 1, 2$

$$\widetilde{w_{i,m}}(x) = \begin{cases} w_i(x), & x \in E_m \cup (X \setminus A), \\ e_m, & x \in A \setminus E_m. \end{cases}$$

Then, the IFS $\widetilde{\mathcal{F}} = \{\widetilde{w_{i,m}} : (i, m) \in I \times \{1, 2\}\}$ is an IFS of discontinuous maps, and A is a strict attractor of $\widetilde{\mathcal{F}}$. (Indeed, the Hutchinson operators associated with $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ coincide.)

Some other notable examples of strict attractors include:

- the Alexandrov double arrow space—a nonmetrizable compact separable space ([3] Example 6);
- the Warsaw sine curve—a non-locally connected continuum ([4] Example 2).

Pointwise strict attractors, despite their generality, offer sufficiently reach theory to be worth of consideration for IFSs. For instance, the probabilistic chaos game algorithm is valid for them, cf. [3].

If A is a strict attractor of the IFS $\mathcal{F}$ comprising continuous maps, then A is an invariant set, i.e. $\mathcal{F}(A) = A$. (Indeed, $\mathcal{F}^{n+1}(A) = \mathcal{F}(\mathcal{F}^n(A)) \to \mathcal{F}(A) = A$ thanks to continuity of $\mathcal{F}$.) We will see later that attractors which are not invariant can exist in discontinuous IFSs and their existence leads to interesting questions.

4 Point-Fibred and Strongly Fibred Attractors

Let I be a finite set (with a discrete topology). The Tikhonov product I^∞ of countably many copies of I is called the *code space*. It is a Cantor space, i.e. a homeomorph of the Cantor ternary set.

Definition 4.1 (Kieninger [7] chap. 4) Let $\mathcal{F} = \{w_i : i \in I\}$ be an IFS comprising continuous maps. Let A be a strict attractor of $\mathcal{F}$. We define the *coding multifunction* $\pi : I^\infty \to \mathcal{K}(A)$ by the following formula

$$\pi(\iota) = \bigcap_{n=1}^{\infty} w_{i_1} \circ \ldots \circ w_{i_n}(A) \text{ for } \iota = (i_n)_{n=1}^{\infty} \in I^\infty.$$

The strict attractor A is said to be

- *point-fibred* if π is single-valued, i.e. $\pi(\iota)$ is a singleton for each $\iota \in I^\infty$;
- *strongly fibred* if for every open $V \subseteq X$ with $V \cap A \neq \emptyset$ there exists $\iota \in I^\infty$ such that $\pi(\iota) \subseteq V$.

Note that the coding map π provides a fibering of the attractor A into a nondisjoint union: $A = \bigcup_{\iota \in I^\infty} \pi(\iota)$.

Proposition 4.2 (Barnsley and Leśniak [1] Proposition 1) *The coding multifunction π of a strict attractor A of an IFS $\mathcal{F}$ comprising continuous maps w_i does not depend on the choice of a forward invariant compact cap $C \supseteq A$, $\mathcal{F}(C) \subseteq C$, that is for every forward invariant compact cap $C \subseteq \mathcal{B}(A)$ and every $\iota = (i_n)_{n=1}^\infty$ we have*

$$\pi(\iota) = \bigcap_{n=1}^{\infty} w_{i_1} \circ \ldots \circ w_{i_n}(C).$$

An attractor of an IFS comprising weak contractions is point-fibred. Interestingly, we can construct strongly fibred attractors from point-fibred ones.

Example 4.3 (Strongly fibred attractor which is not point-fibred; [1] Example 2.1, [7] Example 4.3.19) Let $\mathcal{F} = \{w_i : i \in I\}$ be an IFS of at least two continuous maps $w_i : X \to X$ on a compact space X which contains at least two points. Assume that the images of these maps tessalate X: $\bigcup_{i \in I} w_i(X) = X$. (We do not demand Int $(w_i(X))$ to be disjoint.) Define an IFS on $X \times X$:

$$\mathcal{F}_\square = \mathrm{id} \times w_i, w_i \times \mathrm{id} : i \in I.$$

If X is a point-fibred strict attractor of $\mathcal{F}$, then $X \times X$ is a strongly fibred strict attractor of $F_\square$, but it is not point-fibred.

Example 4.4 (Non-strongly fibred attractor) Let $w : X \to X$ be a minimal map on a compact metric space X (i.e. $\overline{\{w^n(x) : n \geq 0\}} = X$ for each $x \in X$). Then, X is a strict attractor of $\mathcal{F} = \{\mathrm{id}, w\}$, and X is not strongly fibred.

The interesting fact about strongly fibred strict attractors, aside their mosaic inner structure (e.g. [5]), is that we can derandomize the chaos game algorithm for such attractors, cf. [1].

5 Fast Basins

So far we have considered the basin $\mathcal{B}(A)$ and the pointwise basin $\mathcal{B}_1(A)$ of a set A. These domains have the property that the iterations of the IFS $\mathcal{F} = \{w_i : i \in I\}$ starting there, as well as orbits $x_n = w_{i_n} \circ \ldots \circ w_{i_1}(x_0)$, $i_n \in I$, $n \geq 1$, $x_0 \in \mathcal{B}_1(A)$, are attracted by A. We are going to consider the fast basin $\widehat{\mathcal{B}}(A)$ of A, the domain with the property that all iterations (of orbits) fall into A after finite number of steps.

Definition 5.1 (Barnsley et al. [4, 6]) Let A be a strict attractor of an IFS $\mathcal{F}$. The *fast basin* of A is defined by

$$\widehat{\mathcal{B}}(A) = \{x \in X : \mathcal{F}^n(\{x\}) \cap A \neq \emptyset \ \text{ for some } \ n \geq 0\}.$$

We describe below the fast basin of the Sierpiński triangle.

Example 5.2 (Sierpiński wallpaper) Let A be the Sierpiński triangle in the complex plane with vertices $a_1 = 0, a_2 = 1, a_3 = \imath \in \mathbb{C}$, generated by the IFS from Example 3.5. Then, $\widehat{\mathcal{B}}(A) = \bigcup_{k,m\in\mathbb{Z}}(A + k \cdot 1 + m \cdot \imath)$.

It should be noted that in general neither $\widehat{\mathcal{B}}(A) \subseteq \mathcal{B}(A)$ nor $\mathcal{B}(A) \subseteq \widehat{\mathcal{B}}(A)$.

Example 5.3 (Fast basin reaching outside basin; [4] Example 5) Let $X = \mathbb{R} \cup \{\infty\}$. Define $w_1(x) = \frac{x}{2}$ for $x \neq \infty$, $w_1(\infty) = \infty$, $w_2(x) = \frac{x+3}{-2x+6}$ for $x \notin \{3, \infty\}$, $w_2(3) = \infty$, $w_2(\infty) = \frac{-1}{2}$. Then $A = [0, 1]$ is a strict attractor with basin $\mathcal{B}(A) = \left(-\infty, \frac{3}{2}\right)$. It turns out that

$$\{3 \cdot 2^k : k \geq 1\} \subseteq \widehat{\mathcal{B}}(A) \setminus \mathcal{B}(A).$$

Denote

- $\mathcal{F}^{-1}(S) = \bigcup_{i\in I} w_i^{-1}(S)$, the large counter-image of $S \subset X$;
- $\widehat{B}(\vartheta) = \bigcup_{k=0}^{\infty} w_{\theta_k}^{-1}(\ldots w_{\theta_1}^{-1}(A)\ldots)$, the *fractal continuation* of A along $\vartheta = (\theta_1, \theta_2, \ldots) \in I^\infty$.

Proposition 5.4 *(Alternative descriptions of the fast basin; Barnsley et al. [4] Propositions 2 and 3) If A is a strict attractor of $\mathcal{F}$ and $\widehat{B}(A)$ is the fast basin of A, then*

(i) $S = \widehat{B}(A)$ *is the smallest (with respect to* $\subseteq$*) solution of the equation*

$$\mathcal{F}^{-1}(S) \cup A = S;$$

(ii) $\widehat{B}(A) = \bigcup_{k=0}^{\infty}(\mathcal{F}^k)^{-1}(A) = \bigcup_{\vartheta\in I^\infty} \widehat{B}(\vartheta)$.

The IFS is said to be *invertible* if it consists of homeomorphisms. The characterization of the fast basin given in Proposition 5.4 is the key to the following theorem.

Theorem 5.5 *Let A be a strict attractor of the invertible IFS $\mathcal{F}$ acting on a normal space X. Let $\widehat{\mathcal{B}}(A)$ be the fast basin of A. Let (P) be any of the following properties of a set:*

(i) the Lebesgue topological dimension of the set equals $\delta \in \{0, 1, 2, \ldots\}$*;*
(ii) the Hausdorff fractal dimension of the set equals $\delta \in [0, \infty)$*;*
(iii) the set is connected;
(iv) the set is pathwise connected;
(v) the set is boundary (i.e. it has empty topological interior);

(vi) the set is σ-porous;

(vii) the set is hereditarily disconnected (in particular, it has a tree-like structure and admits ultrametrization).

If A has property (P), then $\widehat{\mathcal{B}}(A)$ has property (P) too. In (ii) and (vi), we need to assume that X is a metric space and the maps constituting $\mathcal{F}$ are b-Lipschitz. In (v), we need to assume that X is a Baire topological space. For (vii), we assume that X is a locally compact metric space.

The work [4] contains a gallery of fast basis. To unveil a true nature of the fast basin $\widehat{\mathcal{B}}(A)$, one has to introduce inductive topology in a flag of successive enlargements $w_{\theta_k}^{-1}(\ldots w_{\theta_1}^{-1}(A)\ldots)$ (or blow-ups) of A. These blow-ups fill up the fractal continuation $\widehat{B}(\vartheta)$. Properly glued continuations constitute branches (or leaves) of the resulting object called a *fractal manifold*. We refer to [6] for technical details of this construction. A simplistic visualization of this construction in the case of the Sierpiński wallpaper has been offered in [10].

6 Homoclinic Attractors Versus Fast Basins

We are going to address an intricate connection of the existence of non-invariant strict attractors, called *homoclinic attractors*, with the notion of fast basin.

Let $\mathcal{F} = \{w_i : i \in I\}$ be an IFS of continuous maps $w_i : X \to X$. Let A be a strict attractor with a nontrivial basin $\mathcal{B}(A) \neq A$. Fix $b \in \mathcal{B}(A) \setminus A$. Define $\widetilde{w}_i|A \equiv b$, $\widetilde{w}_i = w_i$ outside A, and

$$\widetilde{\mathcal{F}} = \{\widetilde{w}_i : i \in I\}.$$

Then, $\widetilde{\mathcal{F}}$ is a discontinuous modification of $\mathcal{F}$.

The following question arises: Whether/when A persists a strict attractor after the modification of $\mathcal{F}$? We would have then an attractor of $\widetilde{\mathcal{F}}$ which undergoes an expulsion of its content, i.e. $\widetilde{\mathcal{F}}(A) \nsubseteq A$. The answer is that it depends upon the fast basin $\widehat{\mathcal{B}}(A)$ of the original system $\mathcal{F}$.

Proposition 6.1 (Necessary condition for a homoclinic attractor; [8] Proposition 2) *If A is a strict attractor of $\widetilde{\mathcal{F}}$, then $b \notin \widehat{\mathcal{B}}(A)$.*

Theorem 6.2 [Sufficient condition for a homoclinic attractor; [8] Theorem 3] *If $b \notin \widehat{\mathcal{B}}(A)$ and the following nonresonance condition holds: there exists an open neighbourhood $A \subseteq U(A) \subseteq \mathcal{B}(A)$ such that*

$$\kappa(S) := \sup\{k \geq 0 : \mathcal{F}^k(S) \cap (\widehat{\mathcal{B}}(A) \setminus A) \neq \emptyset\} < \infty$$

for all nonempty compact $S \subseteq U(A)$, then A is a strict attractor of $\widetilde{\mathcal{F}}$.

What about more general modifications $\widetilde{\mathcal{F}}$ of $\mathcal{F}$? Say, $\mathcal{F}$ admits a strict attractor A with basin $\mathcal{B}(A)$ and fast basin $\widehat{\mathcal{B}}(A)$, further $\widetilde{\mathcal{F}}$ is such a modification of $\mathcal{F}$ that

$\widetilde{\mathcal{F}}(A) \subseteq \mathcal{B}(A)$ and $\widetilde{\mathcal{F}}(A) \nsubseteq A$. On this level of generality, Proposition 6.1 would sound like: if A is a strict attractor of $\widetilde{\mathcal{F}}$, then $\widetilde{\mathcal{F}}(A) \cap (\widehat{\mathcal{B}}(A) \setminus A) = \emptyset$. We have the following counterexample for such speculations.

Example 6.3 (Leśniak [8] Example 6) Let us consider the IFS $\mathcal{F} = \{w_i : i \in \{1, 2, 3\}\}$ on $\mathbb{C}$ from Example 5.2. Let $\widetilde{w_i} = w_i$ for $i = 2, 3$, and $\widetilde{w_1}(z) = w_1(z)$ for $z \neq 0$, $\widetilde{w_1}(0) = 2$. The Sierpiński triangle A is a strict attractor of $\mathcal{F}$. It turns out that A is a strict attractor of $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}(A) \nsubseteq A$, and $2 \in (\widehat{\mathcal{B}}(A) \setminus A) \cap \widetilde{\mathcal{F}}(A)$.

We do not know any good criteria for the existence of homoclinic attractors.

References

1. M.F. Barnsley, K. Leśniak, The chaos game on a general iterated function system from a topological point of view. Int. J. Bifurcation Chaos Appl. Sci. Eng. 24(1), 1450139/1–10 (2014)
2. M.F. Barnsley, K. Leśniak, On the continuity of the Hutchinson operator, Symmetry (Basel) 7 (2015), no. 4, 1831–1840
3. M.F. Barnsley, K. Leśniak, M. Rypka, Chaos game for IFSs on topological spaces, J. Math. Anal. Appl. 435 (2016), no. 2, 1458–1466
4. M.F. Barnsley, K. Leśniak, M. Rypka, Basic topological structure of fast basins. Fractals **26**(01), 1850011/1–11 (2018)
5. M.F. Barnsley, A. Vince, Developments in fractal geometry, Bull. Math. Sci. 3 (2013), no. 2, 299–348
6. M.F. Barnsley, A. Vince, Fast basins and branched fractal manifolds of attractors of iterated function systems. SIGMA Symmetry Integrability Geom. Methods Appl. **11**, 084 (2015)
7. B. Kieninger, *Iterated Function Systems on Compact Hausdorff Spaces* (Shaker-Verlag, Aachen, 2002)
8. K. Leśniak, Homoclinic attractors in discontinuo[u]s iterated function systems, Chaos Solitons Fract. 81 (2015), 146–149
9. K. Leśniak, N. Snigireva, F. Strobin, Weakly contractive iterated function systems and beyond: a manualJ. Difference Equ. Appl. **26**(8), 1114–1173 (2020)
10. *Sierpinski Fractal Manifold*. http://superfractals.com/wpfiles/sierpinski-fractal-manifold/ (alternative source https://www.geogebra.org/m/u43mWStD), 2016 last accessed 16 Nov 2018