Chapter 7
Global Error Bounds of One-Stage Explicit ERKN Integrators for Semilinear Wave Equations

In this chapter, we analyse global error bounds for one-stage explicit extended Runge–Kutta–Nyström integrators for semilinear wave equations with periodic boundary conditions. We show optimal second-order convergence without requiring Lipschitz continuity and higher regularity of the exact solution.

7.1 Introduction

First of all, we denote by H^s the Sobolev space $H^s(\mathbb{T})$. In this chapter we pursue the error analysis of one-stage explicit extended Runge–Kutta–Nyström (ERKN) integrators for the semilinear wave equation with some integer $p \geqslant 2$

$$u_{tt} = u_{xx} + u^p, \quad u = u(x,t), \quad t \in [t_0, T]. \tag{7.1}$$

The initial values are given by $u(\cdot, t_0) \in H^{s+1}$ and $u_t(\cdot, t_0) \in H^s$ for $s \geqslant 0$. We consider here real-valued solutions to (7.1) with 2π-periodic boundary conditions in one space dimension ($x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$). It is noted that the energy is finite in the special case $s = 0$.

Using a semidiscretisation in space, we can transform equation (7.1) into a system of second-order ordinary differential equations (ODEs) of the form

$$\ddot{y}(t) = My(t) + f(y(t)), \tag{7.2}$$

where the matrix M describes the discretised second spatial derivative and $f(y)$ denotes the polynomial nonlinearity. It is very important to note that the eigenvalues of the matrix M range from 0 to $\mathscr{O}(K)$, where $2K$ stands for the number of internal discretisation points in space (see, e.g. [1, 2]). This implies that the spatial semidiscretisation exhibits oscillations with a variety of frequencies, and the

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solution of (7.2) typically contains high-frequency oscillatory terms. Many effective integrators have been researched (see, e.g. [3–10], and the references therein) for (7.2). Gautschi-type methods have been well researched and analysed in [6, 11]. Exponential integrators have been widely developed and the reader is referred to [12–14] for instance. These methods have been applied to semilinear wave equations (see, e.g. [15–19]). As a standard form of trigonometric integrator (TI), ERKN integrators were formulated for highly oscillatory second-order differential equations in [20]. Further researches of these integrators are contained in [21–23].

As is known, the error analysis of TI for ODEs has been researched by many papers (see, e.g. [11–13, 24–27]). Unfortunately, however, this work is obviously insufficient because the nonlinearity is assumed to be Lipschitz continuous in all these publications. There is also much work about the error analysis of TI for PDEs (see, e.g. [28–31]). The author in [32] showed error bounds of TI for wave equations without requiring higher regularity of the exact solution, which was achieved by performing the error analysis in two stages. These two-stage arguments have also been used by many researchers such as in [33–37]. Recently, an error analysis has been presented for different schemes for quasilinear wave equations (see, e.g. [38–40]).

We note the fact that the error analysis of ERKN integrators has not been well researched yet in the literature for spatial semidiscretisations of (7.1) with initial values of finite energy. Thus, in this chapter, using the approach described in [32], we will analyse and present error bounds for one-stage explicit ERKN integrators when applied to a spectral semidiscretisation in space, requiring only that the exact solution is of finite energy. First, low-order error bounds will be considered in a higher-order Sobolev space, where the nonlinearity is, at least locally, Lipschitz continuous. From this low-order error bound, suitable regularity of the ERKN integrator will be obtained. Then higher-order error bounds will be shown in these spaces based on the regularity of the ERKN integrator. Optimal second-order convergence will be achieved without requiring Lipschitz continuity and higher regularity of the exact solution. Moreover, this approach to the error analysis is not restricted to spectral semidiscretisations in space.

7.2 Preliminaries

7.2.1 Spectral Semidiscretisation in Space

We consider the following trigonometric polynomial as an ansatz for the solution of the nonlinear wave equation (7.1)

$$u_{\mathscr{K}}(x,t) = \sum_{j\in\mathscr{K}} y_j(t)\mathrm{e}^{\mathrm{i}jx} \quad \text{with} \quad \mathscr{K} = \{-K, -K+1, \cdots, K-1\}, \tag{7.3}$$

where $y_j(t)$ for $j \in \mathcal{K}$ are the Fourier coefficients (see, e.g. [32, 41]). Inserting this ansatz into (7.1) and evaluating at the collocation points $x_k = \pi k/K$ with $k \in \mathcal{K}$, we obtain a system of second-order ODEs

$$\ddot{y}(t) = -\Omega^2 y(t) + f(y(t)), \tag{7.4}$$

where $y(t) = (y_j(t))_{j\in\mathcal{K}} \in \mathbb{C}^{\mathcal{K}}$ is the vector of Fourier coefficients, Ω is a nonnegative diagonal matrix $\Omega = \operatorname{diag}(\omega_j)_{j\in\mathcal{K}}$ with $\omega_j = |j|$, and the nonlinearity f is given by

$$f(y) = \underbrace{y * \cdots * y}_{p \text{ times}} \quad \text{with} \quad (y * z)_j = \sum_{k+l\equiv j \bmod 2K} y_k z_l, \quad j \in \mathcal{K}. \tag{7.5}$$

Here, '$*$' denotes the discrete convolution. The initial values $y(t_0)$ and $\dot{y}(t_0)$ for (7.4) are given respectively by

$$y_j(t_0) = \sum_{k\in\mathbb{Z}:\ k\equiv j \bmod 2K} u_k(t_0), \quad \dot{y}_j(t_0) = \sum_{k\in\mathbb{Z}:\ k\equiv j \bmod 2K} \dot{u}_k(t_0), \quad j \in \mathcal{K}, \tag{7.6}$$

where $u_k(t)$ and $\dot{u}_k(t)$ are the Fourier coefficients of $u(\cdot, t)$ and $u_t(\cdot, t)$, respectively. Once the initial values $u(\cdot, t_0)$ and $u_t(\cdot, t_0)$ are given in terms of their Fourier coefficients, we have the simpler expression:

$$y_j(t_0) = u_j(t_0), \quad \dot{y}_j(t_0) = \dot{u}_j(t_0), \quad j \in \mathcal{K}. \tag{7.7}$$

It is clear that the exact solution of the semidiscrete system (7.4) can be expressed by

$$\begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix} = R(t-t_0)\begin{pmatrix} y(t_0) \\ \dot{y}(t_0) \end{pmatrix} + \int_{t_0}^{t} \begin{pmatrix} \cos(h\Omega) & (t-\tau)\operatorname{sinc}(h\Omega) \\ -\Omega\sin(h\Omega) & \cos(h\Omega) \end{pmatrix}\begin{pmatrix} 0 \\ f(y(\tau)) \end{pmatrix} d\tau, \tag{7.8}$$

where $\operatorname{sinc} x = \sin x/x$ and

$$R(t) = \begin{pmatrix} \cos(t\Omega) & t\operatorname{sinc}(t\Omega) \\ -\Omega\sin(t\Omega) & \cos(t\Omega) \end{pmatrix}.$$

Throughout this chapter, we measure the error by the norm (see, e.g. [32, 41])

$$\|y\|_s := \left(\sum_{j\in\mathcal{K}} \langle j\rangle^{2s}|y_j|^2\right)^{1/2} \quad \text{with} \quad \langle j\rangle = \max(1, |j|) \tag{7.9}$$

for $y \in \mathbb{C}^{\mathscr{K}}$, where $s \in \mathbb{R}$. This norm is (equivalent to) the Sobolev H^s-norm of the trigonometric polynomial $\sum_{j\in\mathscr{K}} y_j \mathrm{e}^{\mathrm{i}jx}$. Clearly, using this norm, we have $\|y\|_{s_1} \leqslant \|y\|_{s_2}$ if $s_1 \leqslant s_2$. The following result presented in [32] is needed in this chapter.

Proposition 7.1 (See [32]) *Assume that $\sigma, \sigma' \in \mathbb{R}$ with $\sigma' \geqslant |\sigma|$ and $\sigma' \geqslant 1$. If $\|y\|_{\sigma'} \leqslant M$ and $\|z\|_{\sigma'} \leqslant M$, then we have*

$$\|f(y) - f(z)\|_{\sigma} \leqslant C\,\|y - z\|_{\sigma}\,, \tag{7.10}$$

$$\|f(y)\|_{\sigma'} \leqslant C, \tag{7.11}$$

with a constant C depending only on M, $|\sigma|$, σ', and p.

7.2.2 ERKN Integrators

It is known that ERKN integrators are oscillation preserving for (7.4), as stated in Chap. 1. In this chapter, we consider one-stage explicit ERKN integrators which are formulated as follows.

Definition 7.1 (See [20]) A one-stage explicit ERKN integrator with stepsize h for solving (7.4) is defined by

$$\begin{cases} y^{(n+c_1)} = \phi_0(c_1^2 V)y^n + hc_1\phi_1(c_1^2 V)\dot{y}^n, \\ y^{(n+1)} = \phi_0(V)y^n + h\phi_1(V)\dot{y}^n + h^2\bar{b}_1(V)f(y^{(n+c_1)}), \\ \dot{y}^{(n+1)} = -h\Omega^2\phi_1(V)y^n + \phi_0(V)\dot{y}^n + hb_1(V)f(y^{(n+c_1)}), \end{cases} \tag{7.12}$$

where c_1 is real constant, $b_1(V)$ and $\bar{b}_1(V)$ are matrix-valued functions of $V \equiv h^2\Omega^2$, and $\phi_j(V) := \sum_{k=0}^{\infty} \dfrac{(-1)^k V^k}{(2k+j)!}$ for $j = 0, 1, \cdots$.

In particular, for $V = h^2\Omega^2$, we have

$$\phi_0(V) = \cos(h\Omega), \qquad \phi_1(V) = \mathrm{sinc}(h\Omega), \qquad \phi_2(V) = (h\Omega)^{-2}(I - \cos(h\Omega)).$$

In this chapter, we present five practical one-stage explicit ERKN integrators whose coefficients are displayed in Table 7.1. It can be seen from Table 7.1 that there are many different one-stage explicit ERKN integrators, and various methods with different properties can be constructed.

Table 7.1 Five one-stage explicit ERKN integrators

Methods	c_1	$\bar{b}_1(V)$	$b_1(V)$	Symmetric	Symplectic
ERKN1	1/2	$\phi_2(V)$	$\phi_0(V/4)$	Non	Non
ERKN2	1/2	$\phi_2(V)$	$\phi_1(V)$	Symmetric	Non
ERKN3	1/2	$1/2\ \phi_1(V/4)$	$\phi_0(V/4)$	Symmetric	Symplectic
ERKN4	1/2	$1/2\ \phi_1^2(V/4)$	$\phi_1(V/4)\phi_0(V/4)$	Symmetric	Non
ERKN5	1/2	$1/2\ \phi_1(V)\phi_1(V/4)$	$\phi_1(V)\phi_0(V/4)$	Symmetric	Non

7.3 Main Result

In order to present the error bounds, we need the following assumptions for the coefficients of the ERKN integrators. Similar assumptions on the filter functions of some trigonometric methods have been considered in [32].

Assumption 7.1 It is assumed that for a given $-1 \leqslant \beta \leqslant 1$, there exists a constant c such that

$$|\bar{b}_1(\xi^2)| \leqslant c\xi^{\beta}, \qquad \text{if} \qquad -1 \leqslant \beta \leqslant 0, \tag{7.13}$$

$$|1/2\text{sinc}^2(\xi/2) - \bar{b}_1(\xi^2)| \leqslant c\xi^{\beta}, \qquad \text{if} \qquad 0 < \beta \leqslant 1, \tag{7.14}$$

$$|1 - b_1(\xi^2)| \leqslant c\xi^{(1+\beta)}, \tag{7.15}$$

for all $\xi = h\omega_j$ with $j \in \mathscr{K}$ and $\omega_j \neq 0$. Furthermore, we assume that $c_1 = \dfrac{1}{2}$ for the ERKN integrators determined by (7.12).

It is easy to see that all the ERKN integrators displayed in Table 7.1 satisfy this assumption uniformly for $-1 \leqslant \beta \leqslant 1$ and $h > 0$. Under this assumption, we have the following property, which can be verified easily by the definition of the norm (7.9).

Proposition 7.2 *With the conditions of Assumption* 7.1 *it holds that*

$$\|y - b_1(V)y\|_{s-\beta} \leqslant ch^{(1+\beta)} \|y\|_{s+1}$$

for $s \in \mathbb{R}$. *Moreover, we have*

$$\left\|\bar{b}_1(V)y\right\|_{s-\beta} \leqslant ch^{\beta} \|y\|_s$$

for $-1 \leqslant \beta \leqslant 0$, *and*

$$\left\|\left(\frac{1}{2}\text{sinc}^2\left(\frac{h\Omega}{2}\right) - \bar{b}_1(V)\right)y\right\|_{s-\beta} \leqslant ch^{\beta} \|y\|_s$$

for $0 < \beta \leqslant 1$.

The following theorem presents the main result of this chapter.

Theorem 7.1 *Let $c \geqslant 1$ and $s \geqslant 0$. Assume that the exact solution $(y(t), \dot{y}(t))$ of the spatial semidiscretisation (7.4) satisfies*

$$\|y(t)\|_{s+1} + \|\dot{y}(t)\|_s \leqslant M \qquad \text{for} \qquad 0 \leqslant t - t_0 \leqslant T. \tag{7.16}$$

Under Assumption 7.1 with the constant c for $\beta = 0$ and $\beta = \alpha \in [-1, 1]$, there exists $h_0 > 0$ such that for $0 < h \leqslant h_0$, the error bound for the numerical solution $(y^n, \dot{y}^n)$ obtained from the ERKN integrator (7.12) is given by

$$\left\| y(t_n) - y^n \right\|_{s+1-\alpha} + \left\| \dot{y}(t_n) - \dot{y}^n \right\|_{s-\alpha} \leqslant Ch^{(1+\alpha)} \quad \text{for} \quad 0 \leqslant t_n - t_0 = nh \leqslant T,$$

where the constants C and h_0 depend only on M and s from (7.16), the power p, the final time T, and the constant c in Assumption 7.1.

Using the two-stage arguments described in [32, 33, 35–37], we divide the proof of Theorem 7.1 into two parts. We first show the proof of the lower-order error bounds in higher-order Sobolev spaces (i.e., $-1 \leqslant \alpha \leqslant 0$) in Sect. 7.4. We then present the proof of the higher-order error bounds in lower-order Sobolev spaces (i.e., $0 < \alpha \leqslant 1$) in Sect. 7.5.

Remark 7.1 We remark that the authors in [31] present an error analysis of ERKN integrators when applied to wave equations. The result is given by using the norm of a matrix and is proved by following [27, 42]. It is noted that the normal result and its proof, given in this chapter, are different from those in [31]. Moreover, Lipschitz continuity and higher regularity of the exact solution are not required in the analysis of this chapter, which is also different from [31].

Remark 7.2 One-stage ERKN integrators contain some trigonometric integrators of [32], and some ERKN integrators can be considered as trigonometric integrators of [32]. However, there is no inclusive relation for these two kinds of methods, which means that the analysis of [32] cannot be directly used for one-stage ERKN integrators. The analysis presented here essentially follows from [32] with some modifications arising from the ERKN discretisation.

7.4 The Lower-Order Error Bounds in Higher-Order Sobolev Spaces

Throughout the proof in this subsection, we assume that $0 < h \leqslant 1$ and use the norm $|||(y, \dot{y})|||_\sigma = (\|y\|^2_{\sigma+1} + \|\dot{y}\|^2_\sigma)^{1/2}$ on $H^{\sigma+1} \times H^\sigma$ for $\sigma \in \mathbb{R}$.

7.4.1 Regularity Over One Time Step

We first show the preservation of regularity of (7.12) over one time step.

Lemma 7.1 *Let* $s \geqslant 0$ *and* $-1 \leqslant \alpha \leqslant 0$. *Suppose that Assumption 7.1 holds for* $\beta = \alpha$ *with a constant* c *and* $|||(y^0, \dot{y}^0)|||_s \leqslant M$, *then for the solution given by the ERKN integrator* (7.12), *we have* $|||(y^1, \dot{y}^1)|||_s \leqslant C$, *where* C *depends only on* $M, s, p,$ *and* c.

Proof On noticing $\mathrm{sinc}(0) = 1 \leqslant h^{-1}$ and the bound $|\mathrm{sinc}(\xi)| \leqslant \xi^{-1}$ for $\xi > 0$, it follows from (7.12) that

$$\left\| y^{\frac{1}{2}} \right\|_{s+1} \leqslant \left\| y^0 \right\|_{s+1} + \left\| \dot{y}^0 \right\|_s \leqslant 2M, \tag{7.17}$$

which gives

$$\left\| f(y^{\frac{1}{2}}) \right\|_{s+1} \leqslant C, \tag{7.18}$$

by considering (7.11) with $\sigma' = s + 1$. On noticing the fact that $-1 \leqslant \alpha \leqslant 0$ and the bound (7.13) of $\bar{b}_1$, we have

$$\begin{aligned} \left\| y^1 \right\|_{s+1} &\leqslant \left\| y^0 \right\|_{s+1} + \left\| \dot{y}^0 \right\|_s + h^{2+\alpha} \left\| f(y^{\frac{1}{2}}) \right\|_{s+1+\alpha} \\ &\leqslant \left\| y^0 \right\|_{s+1} + \left\| \dot{y}^0 \right\|_s + h^{2+\alpha} \left\| f(y^{\frac{1}{2}}) \right\|_{s+1}. \end{aligned}$$

It follows from (7.18) that $\left\| y^1 \right\|_{s+1} \leqslant C$. Similarly, we obtain $\left\| \dot{y}^1 \right\|_s \leqslant C$, and then the proof is complete. □

7.4.2 Local Error Bound

We now turn to the local error of the ERKN integrator (7.12).

Lemma 7.2 (Local Error in $H^{s+1-\alpha} \times H^{s-\alpha}$ **for** $-1 \leqslant \alpha \leqslant 0$**)** *With the conditions of Lemma 7.1, if* $|||(y(\tau), \dot{y}(\tau))|||_s \leqslant M$ *for* $t_0 \leqslant \tau \leqslant t_1$, *it holds that* $|||(y(t_1), \dot{y}(t_1)) - (y^1, \dot{y}^1)|||_{s-\alpha} \leqslant Ch^{2+\alpha}$, *where the constant* C *depends only on* $M, s, p,$ *and* c.

Proof Throughout the proof, C *stands for a generic constant* depending only on $M, s, p,$ and c.

(I) The local error of $y(t_1) - y^1$.
Using (7.8) and (7.12) we obtain

$$y(t_1) - y^1 = \int_{t_0}^{t_1} (t_1 - \tau)\text{sinc}((t_1 - \tau)\Omega) f(y(\tau))\text{d}\tau - h^2 \bar{b}_1(V) f(y^{\frac{1}{2}}).$$

We note the fact that for $\xi > 0$ and $-1 \leqslant \alpha \leqslant 0$, $|\text{sinc}(\xi)| \leqslant \xi^\alpha$, and $h^\alpha \geqslant 1$. By these results, (7.13) and (7.17), we have

$$\left\| y(t_1) - y^1 \right\|_{s+1-\alpha} \leqslant h^{2+\alpha} \sup_{t_0 \leqslant \tau \leqslant t_1} \| f(y(\tau)) \|_{s+1} + ch^{2+\alpha} \left\| f(y^{\frac{1}{2}}) \right\|_{s+1}.$$

It follows from (7.11) and (7.18) that $\| f(y(\tau)) \|_{s+1} \leqslant C$, which leads to

$$\left\| y(t_1) - y^1 \right\|_{s+1-\alpha} \leqslant Ch^{2+\alpha}.$$

(II) The local error of $\dot{y}(t_1) - \dot{y}^1$.
It follows from (7.8) and (7.12) that

$$\begin{aligned} \dot{y}(t_1) - \dot{y}^1 = & \int_{t_0}^{t_1} [\cos((t_1 - \tau)\Omega) - I] f(y(\tau))\text{d}\tau && (7.19) \\ & + \int_{t_0}^{t_1} f(y(\tau))\text{d}\tau - hf\left(y\left(\frac{t_0 + t_1}{2}\right)\right) && (7.20) \\ & + hf\left(y\left(\frac{t_0 + t_1}{2}\right)\right) - hf(y^{\frac{1}{2}}) && (7.21) \\ & + h(I - b_1(V)) f(y^{\frac{1}{2}}). && (7.22) \end{aligned}$$

- Bound of (7.19). For $\xi > 0$ and $-1 \leqslant \alpha \leqslant 0$, it is easy to obtain that $|\cos(\xi) - 1| \leqslant 2\xi^{1+\alpha}$. On noticing (7.11) with $\sigma' = s + 1$, one arrives at

$$\left\| \int_{t_0}^{t_1} [\cos((t_1 - \tau)\Omega) - I] f(y(\tau))\text{d}\tau \right\|_{s-\alpha} \leqslant 2h^{1+\alpha} \int_{t_0}^{t_1} C\text{d}\tau \leqslant Ch^{2+\alpha}.$$

- Bound of (7.20). Since $1 \leqslant \xi^{1+\alpha} + \xi^\alpha$ for $\xi > 0$, we rewrite (7.20) as

$$\begin{aligned} & \left\| \int_{t_0}^{t_1} f(y(\tau))\text{d}\tau - hf\left(y\left(\frac{t_0 + t_1}{2}\right)\right) \right\|_{s-\alpha} \leqslant h^{1+\alpha} \\ & \times \left\| \int_{t_0}^{t_1} f(y(\tau))\text{d}\tau - hf\left(y\left(\frac{t_0 + t_1}{2}\right)\right) \right\|_{s+1} \\ & + h^\alpha \left\| \int_{t_0}^{t_1} f(y(\tau))\text{d}\tau - hf\left(y\left(\frac{t_0 + t_1}{2}\right)\right) \right\|_{s}. \end{aligned}$$

It then follows from (7.11) with $\sigma' = s+1$ that

$$\left\| \int_{t_0}^{t_1} f(y(\tau))\mathrm{d}\tau - hf\left(y\left(\frac{t_0+t_1}{2}\right)\right)\right\|_{s+1} \leqslant \int_{t_0}^{t_1} C\mathrm{d}\tau + Ch \leqslant Ch.$$

For an estimate in the norm $\|\cdot\|_s$, it is remarked that (7.20) is the quadrature error of the mid-point rule. With its first-order Peano kernel $K_1(\tau)$ and by the Peano kernel theorem, we obtain

$$\begin{aligned}&\left\| \int_{t_0}^{t_1} f(y(\tau))\mathrm{d}\tau - hf\left(y\left(\frac{t_0+t_1}{2}\right)\right)\right\|_{s}\\ &\quad = h^2 \left\| \int_{t_0}^{t_1} K_1(\tau)\frac{\mathrm{d}}{\mathrm{d}t} f(y(t_0+\tau h))\mathrm{d}\tau\right\|_s \leqslant Ch^2,\end{aligned}$$

where we have used (3.4a) in [32]. Thus, it is true that

$$\left\| \int_{t_0}^{t_1} f(y(\tau))\mathrm{d}\tau - hf\left(y\left(\frac{t_0+t_1}{2}\right)\right)\right\|_{s-\alpha} \leqslant Ch^{2+\alpha}. \tag{7.23}$$

- Bound of (7.21). Using (7.10) with $\sigma = s-\alpha$, we have

$$\left\| hf\left(y\left(\frac{t_0+t_1}{2}\right)\right) - hf(y^{\frac{1}{2}})\right\|_{s-\alpha} \leqslant Ch\left\| y\left(\frac{t_0+t_1}{2}\right) - y^{\frac{1}{2}}\right\|_{s-\alpha}.$$

It follows from (7.8) and (7.12) that

$$y\left(\frac{t_0+t_1}{2}\right) - y^{\frac{1}{2}} = \int_{t_0}^{\frac{t_0+t_1}{2}} \left(\frac{t_0+t_1}{2} - \tau\right)\mathrm{sinc}\left(\left(\frac{t_0+t_1}{2}-\tau\right)\Omega\right) f(y(\tau))\mathrm{d}\tau. \tag{7.24}$$

In a similar way to the first part of this proof, we obtain

$$\left\| y\left(\frac{t_0+t_1}{2}\right) - y^{\frac{1}{2}}\right\|_{s+1-\alpha} \leqslant h^{2+\alpha} \sup_{t_0 \leqslant \tau \leqslant \frac{t_0+t_1}{2}} \|f(y(\tau))\|_{s+1} \leqslant Ch^{2+\alpha}.$$

Then, it is true that

$$\left\| y\left(\frac{t_0+t_1}{2}\right) - y^{\frac{1}{2}}\right\|_{s-\alpha} \leqslant \left\| y\left(\frac{t_0+t_1}{2}\right) - y^{\frac{1}{2}}\right\|_{s+1-\alpha} \leqslant Ch^{2+\alpha},$$

which leads to $\left\| hf\left(y\left(\frac{t_0+t_1}{2}\right)\right) - hf(y^{\frac{1}{2}})\right\|_{s-\alpha} \leqslant Ch^{3+\alpha}.$

- Bound of (7.22). According to (7.18) and the bound (7.15), we have

$$\left\| h(I - b_1(V)) f(y^{\frac{1}{2}}) \right\|_{s-\alpha} \leqslant Ch^{2+\alpha} \left\| f(y^{\frac{1}{2}}) \right\|_{s+1} \leqslant Ch^{2+\alpha}.$$

Clearly, all these estimates imply $\| \dot{y}(t_1) - \dot{y}^1 \|_{s-\alpha} \leqslant Ch^{2+\alpha}$.

The proof is complete. □

7.4.3 Stability

In this subsection we analyse the stability of the ERKN integrator (7.12).

Lemma 7.3 (Stability in $H^{s+1-\alpha} \times H^{s-\alpha}$ for $-1 \leqslant \alpha \leqslant 0$) *Under the conditions of Lemma 7.1, if we consider the ERKN integrator (7.12) with different initial values $(y_0, \dot{y}_0)$ and $(z_0, \dot{z}_0)$ satisfying $|||(y_0, \dot{y}_0)|||_s \leqslant M$ and $|||(z_0, \dot{z}_0)|||_s \leqslant M$, then it holds that*

$$|||(y^1, \dot{y}^1) - (z^1, \dot{z}^1)|||_{s-\alpha} \leqslant (1 + Ch)|||(y^0, \dot{y}^0) - (z^0, \dot{z}^0)|||_{s-\alpha},$$

where the constant C depends only on M, s, p, and c.

Proof It follows from the result (3.8) in [32] and ERKN integrators (7.12) that

$$\begin{aligned} |||(y^1, \dot{y}^1) - (z^1, \dot{z}^1)|||_{s-\alpha} \leqslant\ & |||(y^0, \dot{y}^0) - (z^0, \dot{z}^0)|||_{s-\alpha} \\ & + h|\dot{y}_0^0 - \dot{z}_0^0| \qquad (7.25) \\ & + h^2 \left\| \bar{b}_1(V) \left(f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}}) \right) \right\|_{s+1-\alpha} \qquad (7.26) \\ & + h \left\| b_1(V) \left(f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}}) \right) \right\|_{s-\alpha}. \qquad (7.27) \end{aligned}$$

- It is trivial for (7.25), that $h|\dot{y}_0^0 - \dot{z}_0^0| \leqslant h \| \dot{y}_0^0 - \dot{z}_0^0 \|_{s-\alpha}$.
- With regard to (7.26), combining the bound (7.13) of $\bar{b}_1$ and (7.10) with $\sigma = \sigma' = s + 1$ yields

$$h^2 \left\| \bar{b}_1(V) \left(f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}}) \right) \right\|_{s+1-\alpha} \leqslant Ch^{2+\alpha} \left\| y^{\frac{1}{2}} - z^{\frac{1}{2}} \right\|_{s+1}.$$

Using the formula for ERKN integrators (7.12) again, we confirm that $\left\| y^{\frac{1}{2}} - z^{\frac{1}{2}} \right\|_{s+1} \leqslant \| y^0 - z^0 \|_{s+1} + \| \dot{y}^0 - \dot{z}^0 \|_s$. This implies

$$h^2 \left\| \bar{b}_1(V) \left(f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}}) \right) \right\|_{s+1-\alpha} \leqslant Ch^{2+\alpha} \left\| y^0 - z^0 \right\|_{s+1} + Ch^{2+\alpha} \left\| \dot{y}^0 - \dot{z}^0 \right\|_s.$$

- Concerning (7.27), it follows from (7.15) that $|b_1(\xi)| \leqslant 1 + c\xi^{1+\alpha}$, and then we have

$$
\begin{aligned}
& h \left\| b_1(V)\Big(f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}})\Big)\right\|_{s-\alpha} \\
\leqslant\ & h \left\| f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}})\right\|_{s-\alpha} + ch^{2+\alpha}\left\| f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}})\right\|_{s+1} \\
\leqslant\ & Ch \left\| y^{\frac{1}{2}} - z^{\frac{1}{2}}\right\|_{s-\alpha} + Ch^{2+\alpha}\left\| y^{\frac{1}{2}} - z^{\frac{1}{2}}\right\|_{s+1} \\
\leqslant\ & C(h + h^{2+\alpha})\left\| y^0 - z^0\right\|_{s+1} + C(h + h^{2+\alpha})\left\| \dot{y}^0 - \dot{z}^0\right\|_{s}.
\end{aligned}
$$

The above estimates of (7.25)–(7.27) with $-1 \leqslant \alpha \leqslant 0$ complete the proof. □

7.4.4 *Proof of Theorem 7.1 for* $-1 \leqslant \alpha \leqslant 0$

We are now in a position to present the proof of Theorem 7.1 for $-1 \leqslant \alpha \leqslant 0$, based on the three lemmas stated above.

Proof

(I) We begin with the proof for the case where $\alpha = 0$. Let C_1 and C_2 be the constants of Lemmas 7.2 and 7.3 with $\alpha = 0$, respectively. It is noted that Lemma 7.3 is considered with $2M$ instead of M. Let $h_0 = M/(C_1 T \mathrm{e}^{C_2 T})$ and we show by induction on n that for $h \leqslant h_0$

$$
|||(y^n, \dot{y}^n) - (y(t_n), \dot{y}(t_n))|||_s \leqslant C_1 \mathrm{e}^{C_2 nh} nh^2, \tag{7.28}
$$

as long as $t_n - t_0 = nh \leqslant T$.

We first have $|||(y^0, \dot{y}^0) - (y(t_0), \dot{y}(t_0))|||_s = 0 \leqslant C_1$. We assume that the result (7.28) is true for $n = 0, \cdots, m-1$. This implies that

$$
|||(y^{m-1}, \dot{y}^{m-1}) - (y(t_{m-1}), \dot{y}(t_{m-1}))|||_s \leqslant C_1 \mathrm{e}^{C_2(m-1)h}(m-1)h^2,
$$

which gives

$$
|||(y^{m-1}, \dot{y}^{m-1})|||_s \leqslant M + C_1 \mathrm{e}^{C_2(m-1)h}(m-1)h^2 \leqslant M + C_1 \mathrm{e}^{C_2 T} Th \leqslant 2M,
$$

as long as $t_{m-1} - t_0 = (m-1)h \leqslant T$. Denoting by $\mathscr{E}$ one time step with the ERKN integrator (7.12), we obtain

$$
\begin{aligned}
& |||(y^m, \dot{y}^m) - (y(t_m), \dot{y}(t_m))|||_s = |||\mathscr{E}(y^{m-1}, \dot{y}^{m-1}) - (y(t_m), \dot{y}(t_m))|||_s \\
\leqslant\ & |||\mathscr{E}(y^{m-1}, \dot{y}^{m-1}) - \mathscr{E}(y(t_{m-1}), \dot{y}(t_{m-1}))|||_s \qquad (7.29) \\
& + |||\mathscr{E}(y(t_{m-1}), \dot{y}(t_{m-1})) - (y(t_m), \dot{y}(t_m))|||_s. \qquad (7.30)
\end{aligned}
$$

In terms of Lemma 7.3, (7.29) admits the bound

$$
\begin{aligned}
&|||\mathscr{E}(y^{m-1}, \dot{y}^{m-1}) - \mathscr{E}(y(t_{m-1}), \dot{y}(t_{m-1}))|||_s \\
&\leqslant (1 + C_2 h)|||(y^{m-1}, \dot{y}^{m-1}) - (y(t_{m-1}), \dot{y}(t_{m-1}))|||_s \\
&\leqslant (1 + C_2 h) C_1 \mathrm{e}^{C_2 (m-1) h} (m-1) h^2.
\end{aligned}
$$

With regard to (7.30), it follows from Lemma 7.2 that $|||\mathscr{E}(y(t_{m-1}), \dot{y}(t_{m-1})) - (y(t_m), \dot{y}(t_m))|||_s \leqslant C_1 h^2$. We then obtain

$$
|||(y^m, \dot{y}^m) - (y(t_m), \dot{y}(t_m))|||_s \leqslant (1 + C_2 h) C_1 \mathrm{e}^{C_2 (m-1) h} (m-1) h^2 + C_1 h^2.
$$

Using Taylor expansions, we obtain that

$$
(1 + C_2 h) C_1 \mathrm{e}^{C_2 (m-1) h} (m-1) h^2 + C_1 h^2 \leqslant C_1 \mathrm{e}^{C_2 m h} m h^2.
$$

Consequently, (7.28) holds, and hence

$$
|||(y^n, \dot{y}^n) - (y(t_n), \dot{y}(t_n))|||_s \leqslant C_1 T \mathrm{e}^{C_2 T} h \leqslant C h,
$$

which proves the statement of Theorem 7.1 for $\alpha = 0$.

(II) We next consider the case $-1 \leqslant \alpha < 0$. Let h_0 be as above and let C_1 and C_2 be as above but for the new α instead of $\alpha = 0$. We then prove, by induction on n, that

$$
|||(y^n, \dot{y}^n) - (y(t_n), \dot{y}(t_n))|||_{s-\alpha} \leqslant C_1 \mathrm{e}^{C_2 n h} n h^{2+\alpha}, \tag{7.31}
$$

as long as $t_n - t_0 = nh \leqslant T$.

Obviously, this holds for $n = 0$. It follows from the proof stated above for the case $\alpha = 0$ that $|||(y^{n-1}, \dot{y}^{n-1})|||_s \leqslant 2M$, as long as $t_{n-1} - t_0 = (n-1)h \leqslant T$. This allows us to apply Lemmas 7.2 and 7.3 to (7.31), which gives

$$
\begin{aligned}
&|||(y^n, \dot{y}^n) - (y(t_n), \dot{y}(t_n))|||_{s-\alpha} \leqslant |||\mathscr{E}(y^{n-1}, \dot{y}^{n-1}) \\
&- \mathscr{E}(y(t_{n-1}), \dot{y}(t_{n-1}))|||_{s-\alpha} \\
&+ |||\mathscr{E}(y(t_{n-1}), \dot{y}(t_{n-1})) - (y(t_n), \dot{y}(t_n))|||_{s-\alpha} \\
&\leqslant (1 + C_2 h) C_1 \mathrm{e}^{C_2 (n-1) h} (n-1) h^{2+\alpha} + C_1 h^{2+\alpha} \leqslant C_1 \mathrm{e}^{C_2 n h} n h^{2+\alpha}.
\end{aligned}
$$

This confirms that (7.31) is true, and then we have

$$
|||(y^n, \dot{y}^n) - (y(t_n), \dot{y}(t_n))|||_{s-\alpha} \leqslant C_1 T \mathrm{e}^{C_2 T} h^{1+\alpha} \leqslant C h^{1+\alpha}.
$$

The proof is complete. □

Remark 7.3 It follows from the above proof for $\alpha = 0$ that the numerical solutions are bounded in $H^{s+1} \times H^s$

$$|||(y^n, \dot{y}^n)|||_s \leqslant 2M \qquad \text{for} \qquad 0 \leqslant t_n - t_0 = nh \leqslant T. \tag{7.32}$$

This regularity of the numerical solution is essential for the proof of Theorem 7.1 for $0 < \alpha \leqslant 1$ in the next section.

7.5 Higher-Order Error Bounds in Lower-Order Sobolev Spaces

The following three lemmas are needed for the proof of Theorem 7.1 in lower-order Sobolev spaces.

Lemma 7.4 *Let $s \geqslant 0$ and $0 < \alpha \leqslant 1$. Suppose that Assumption* 7.1 *holds for $\beta = \alpha$ with constant c and $|||(y^0, \dot{y}^0)|||_s \leqslant M$. We have $|||(y^1, \dot{y}^1)|||_s \leqslant C$ with a constant C depending only on $M, s, p,$ and c.*

We omit the proof of Lemma 7.4 which is quite similar to that of Lemma 7.1.

Lemma 7.5 (Local Error in $H^{s+1-\alpha} \times H^{s-\alpha}$ for $0 < \alpha \leqslant 1$) *Under the conditions of Lemma 7.4, if $|||(y(\tau), \dot{y}(\tau))|||_s \leqslant M$ for $t_0 \leqslant \tau \leqslant t_1$, then it holds that $|||(y(t_1), \dot{y}(t_1)) - (y^1, \dot{y}^1)|||_{s-\alpha} \leqslant Ch^{2+\alpha}$, where the constant C depends only on $M, s, p,$ and c.*

Proof

(I) Local error of $y(t_1) - y^1$.

It follows from (7.8), (7.12) and

$$\int_{t_0}^{t_1} (t_1 - \tau)\mathrm{sinc}((t_1 - \tau)\Omega)\mathrm{d}\tau = \frac{1}{2}h^2\mathrm{sinc}^2\left(\frac{1}{2}h\Omega\right),$$

that

$$y(t_1) - y^1 = \int_{t_0}^{t_1} (t_1 - \tau)\mathrm{sinc}((t_1 - \tau)\Omega)$$

$$\left[f(y(\tau)) - f\left(y\left(\frac{t_0 + t_1}{2}\right)\right)\right]\mathrm{d}\tau \tag{7.33}$$

$$+\frac{1}{2}h^2\mathrm{sinc}^2\left(\frac{1}{2}h\Omega\right)\left[f\left(y\left(\frac{t_0 + t_1}{2}\right)\right) - f(y^{\frac{1}{2}})\right] \tag{7.34}$$

$$+h^2\left[\frac{1}{2}\mathrm{sinc}^2\left(\frac{1}{2}h\Omega\right) - \bar{b}_1(V)\right]f(y^{\frac{1}{2}}). \tag{7.35}$$

- Bound of (7.33). For $\xi > 0$ and $0 < \alpha \leqslant 1$, it is clear that $|\mathrm{sinc}(\xi)| \leqslant \xi^{-1+\alpha}$. Using this result, the estimate (7.10) with $\sigma = s$, and the fact

$$\left\| y(\tau) - y\left(\frac{t_0+t_1}{2}\right)\right\|_s \leqslant \int_{\frac{t_0+t_1}{2}}^{\tau} \|\dot{y}(t)\|_s \,\mathrm{d}t \leqslant Ch,$$

we obtain

$$\begin{aligned}
&\left\| \int_{t_0}^{t_1} (t_1-\tau)\mathrm{sinc}((t_1-\tau)\Omega)\left[f(y(\tau)) - f\left(y\left(\frac{t_0+t_1}{2}\right)\right)\right]\mathrm{d}\tau\right\|_{s+1-\alpha} \\
&\leqslant h^{-1+\alpha}\int_{t_0}^{t_1} |t_1-\tau| \left\| f(y(\tau)) - f\left(y\left(\frac{t_0+t_1}{2}\right)\right)\right\|_s \mathrm{d}\tau \\
&\leqslant Ch^{-1+\alpha}\int_{t_0}^{t_1} |t_1-\tau| \left\| y(\tau) - y\left(\frac{t_0+t_1}{2}\right)\right\|_s \mathrm{d}\tau \leqslant Ch^{2+\alpha}.
\end{aligned}$$

- For (7.34), according to the fact that $|\mathrm{sinc}(\xi)|^2 \leqslant \dfrac{1\cdot\xi}{\xi^2} = \xi^{-1}$ for $\xi > 0$ and the estimate (7.10) with $\sigma = s-\alpha$, we have

$$\begin{aligned}
&\left\| \frac{1}{2}h^2\mathrm{sinc}^2\left(\frac{1}{2}h\Omega\right)\left[f\left(y\left(\frac{t_0+t_1}{2}\right)\right) - f(y^{\frac{1}{2}})\right]\right\|_{s+1-\alpha} \\
&\leqslant Ch\left\| f\left(y\left(\frac{t_0+t_1}{2}\right)\right) - f(y^{\frac{1}{2}})\right\|_{s-\alpha} \leqslant Ch\left\| y\left(\frac{t_0+t_1}{2}\right) - y^{\frac{1}{2}}\right\|_{s-\alpha}.
\end{aligned}$$

Furthermore, the estimate (7.11) with $\sigma' = s+1$ gives

$$\begin{aligned}
&\left\| y\left(\frac{t_0+t_1}{2}\right) - y^{\frac{1}{2}}\right\|_{s-\alpha} \leqslant \left\| y(\frac{t_0+t_1}{2}) - y^{\frac{1}{2}}\right\|_{s+2-\alpha} \\
&= \left\| \int_{t_0}^{\frac{t_0+t_1}{2}} (\frac{t_0+t_1}{2}-\tau)\mathrm{sinc}((\frac{t_0+t_1}{2}-\tau)\Omega)f(y(\tau))\mathrm{d}\tau\right\|_{s+2-\alpha} \qquad (7.36) \\
&\leqslant h^{-1+\alpha}\int_{t_0}^{\frac{t_0+t_1}{2}} \left|\frac{t_0+t_1}{2}-\tau\right| \|f(y(\tau))\|_{s+1}\,\mathrm{d}\tau \leqslant Ch^{1+\alpha}.
\end{aligned}$$

Thus, we obtain $\left\| \frac{1}{2}h^2\mathrm{sinc}^2\left(\frac{1}{2}h\Omega\right)\left[f\left(y\left(\frac{t_0+t_1}{2}\right)\right) - f(y^{\frac{1}{2}})\right]\right\|_{s+1-\alpha} \leqslant Ch^{2+\alpha}$.

- With regard to (7.35), considering (7.14) and the estimate (7.11) with $\sigma' = s+1$ yields

$$\left\| h^2 \left[\frac{1}{2}\text{sinc}^2 \left(\frac{1}{2} h\Omega \right) - \bar{b}_1(V) \right] f(y^{\frac{1}{2}}) \right\|_{s+1-\alpha} \leqslant h^{2+\alpha} \left\| f(y^{\frac{1}{2}}) \right\|_{s+1} \leqslant Ch^{2+\alpha}.$$

Finally, all the bounds of (7.33)–(7.35) imply

$$\left\| y(t_1) - y^1 \right\|_{s+1-\alpha} \leqslant Ch^{2+\alpha}.$$

(II) Local error of $\dot{y}(t_1) - \dot{y}^1$.

Likewise, this error bound can be derived as the bound given in (II) of Lemma 7.2 with the first-order Peano kernel replaced by the second-order Peano kernel. □

Lemma 7.6 (Stability in $H^{s+1-\alpha} \times H^{s-\alpha}$ for $0 < \alpha \leqslant 1$) *With the conditions of Lemma 7.4, we consider different initial values $(y_0, \dot{y}_0)$ and $(z_0, \dot{z}_0)$ for the ERKN integrator (7.12). If* $\max\{|||(y_0, \dot{y}_0)|||_s, |||(z_0, \dot{z}_0)|||_s\} \leqslant M$*, then we have*

$$|||(y^1, \dot{y}^1) - (z^1, \dot{z}^1)|||_{s-\alpha} \leqslant (1 + Ch)|||(y^0, \dot{y}^0) - (z^0, \dot{z}^0)|||_{s-\alpha},$$

where the constant C depends only on M, s, p, and c.

Proof We begin with

$$|||(y^1, \dot{y}^1) - (z^1, \dot{z}^1)|||_{s-\alpha} \leqslant |||(y^0, \dot{y}^0) - (z^0, \dot{z}^0)|||_{s-\alpha} + h|\dot{y}_0^0 - \dot{z}_0^0|$$
$$+h^2 \left\| \bar{b}_1(V) \left(f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}}) \right) \right\|_{s+1-\alpha} \quad (7.37)$$
$$+h \left\| b_1(V) \left(f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}}) \right) \right\|_{s-\alpha}. \quad (7.38)$$

It is clear that $|\bar{b}_1(\xi)| \leqslant \frac{1}{2} + c\xi^\alpha$ due to (7.14). Hence, the bound of (7.37) is

$$h^2 \left\| \bar{b}_1(V) \left(f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}}) \right) \right\|_{s+1-\alpha} \leqslant \frac{1}{2} h^2 \left\| f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}}) \right\|_{s+1-\alpha}$$
$$+\frac{1}{2} h^{2+\alpha} \left\| f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}}) \right\|_{s+1} \leqslant Ch^2(1/2 + h^\alpha) \left\| y^{\frac{1}{2}} - z^{\frac{1}{2}} \right\|_{s+1}$$
$$\leqslant Ch^2(1/2 + h^\alpha) \left\| y^0 - z^0 \right\|_{s+1} + Ch^2(1/2 + h^\alpha) \left\| \dot{y}^0 - \dot{z}^0 \right\|_s.$$

We turn to (7.38). Clearly, $|b_1(\xi)| \leqslant 1 + c\xi^{1+\alpha}$ due to (7.15), and then we obtain

$$\begin{aligned} h\left\|b_1(V)\Big(f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}})\Big)\right\|_{s-\alpha} &\leqslant h\left\|f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}})\right\|_{s-\alpha} \\ &+ch^{2+\alpha}\left\|f(y^{\frac{1}{2}}) - f(z^{\frac{1}{2}})\right\|_{s+1} \\ &\leqslant C(h + h^{2+\alpha})\left\|y^0 - z^0\right\|_{s+1} + C(h + h^{2+\alpha})\left\|\dot{y}^0 - \dot{z}^0\right\|_{s}. \end{aligned}$$

The proof is complete as a consequence of the above bounds. □

Proof of Theorem 7.1 for $0 < \alpha \leqslant 1$.

Proof This proof is the same as that for $-1 \leqslant \alpha < 0$ given in Sect. 7.4. A key point used here is that the numerical solution is bounded in $H^{s+1} \times H^s$ on the basis of Remark 7.3. □

Remark 7.4 We only consider one-stage ERKN integrators in the error analysis. The extension of the analysis to higher-stage ERKN integrators is not obvious since there are some technical difficulties which need to be overcome. This issue needs to be considered in future investigations.

7.6 Numerical Experiments

This section presents a numerical experiment to illustrate the error bounds of two one-stage explicit ERKN integrators.

We solve the problem (7.1) with $p = 2$, and use the spatial semidiscretisation with $K = 2^6$ and $K = 2^8$. Following [32], we choose the initial conditions for the coefficients $y_j(t_0)$ and $\dot{y}_j(t_0)$ on the complex unit circle and then scale them by $\langle j\rangle^{-1.51}$ and $\langle j\rangle^{-0.51}$, respectively. Here, it is important to note that these complex numbers are chosen such that the corresponding trigonometric polynomial (7.3) takes real values at the collocation points. Then, the corresponding initial values satisfy the condition (7.16) of Theorem 7.1 at time $t = t_0$ uniformly in K for $s = 0$. For the time discretisation, we choose ERKN3 and ERKN4 whose coefficients are displayed in Table 7.1. For comparison, we also consider a one-stage RKN method, which is obtained from these ERKN integrators by letting $M = 0$.

The problem is solved on the interval $[0, 10]$ with the stepsizes $h = 1/2^j$ for $j = 0, 1, \cdots, 10$. We measure the errors

$$\text{erry} = \left\|y(t_n) - y^n\right\|_{1-\alpha}, \quad \text{errdy} = \left\|\dot{y}(t_n) - \dot{y}^n\right\|_{-\alpha}$$

in different Sobolev norms $\alpha = 1, \dfrac{1}{2}, 0, -\dfrac{1}{2}, -1$. For the RKN method, it has been checked that the errors are too large for some big stepsizes. Therefore we use smaller stepsizes $h = 1/2^j$ for $j = 4, \cdots, 14$. We plot the logarithm of the errors against the logarithm of stepsizes for the results displayed in Figs. 7.1 and 7.2.

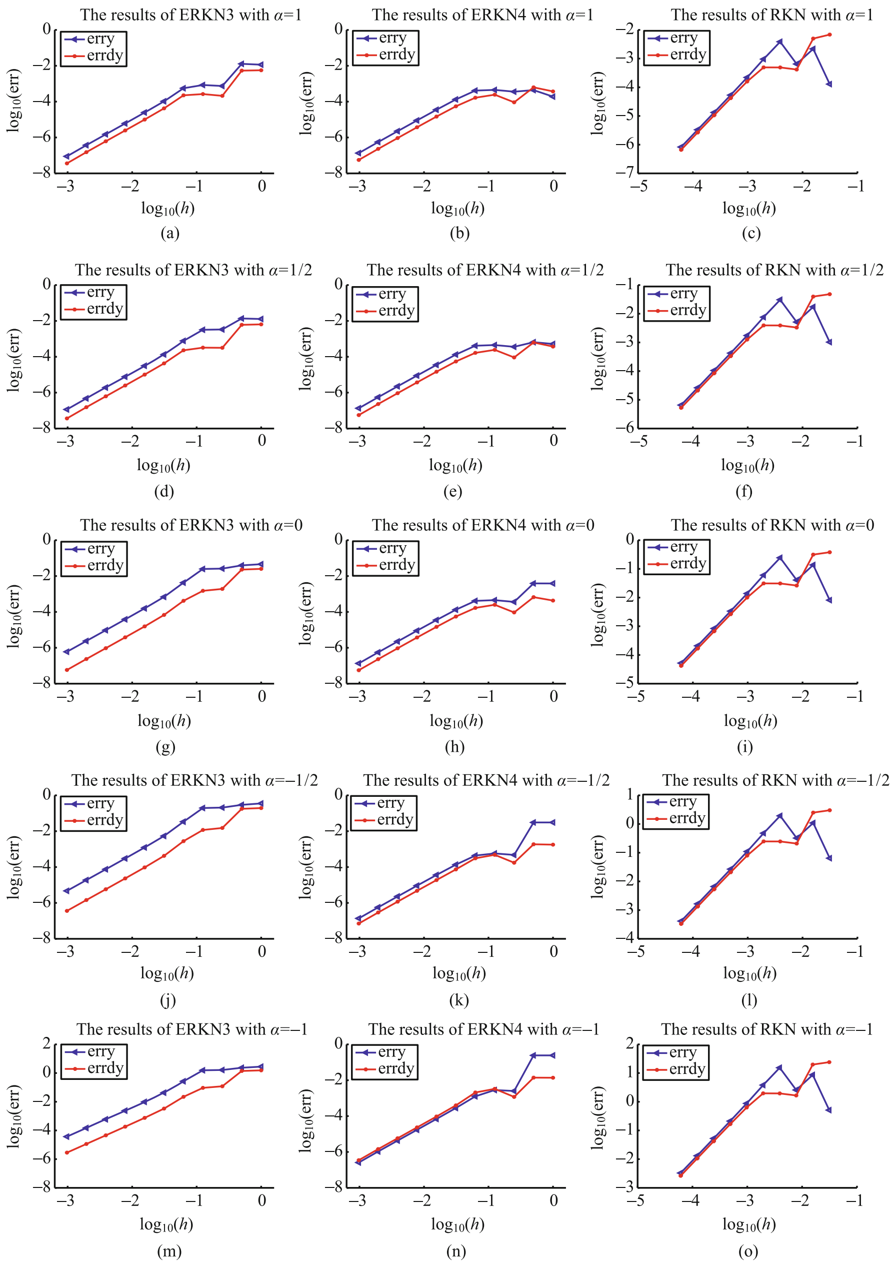

Fig. 7.1 The logarithm of the errors against the logarithm of stepsizes for $K = 2^6$

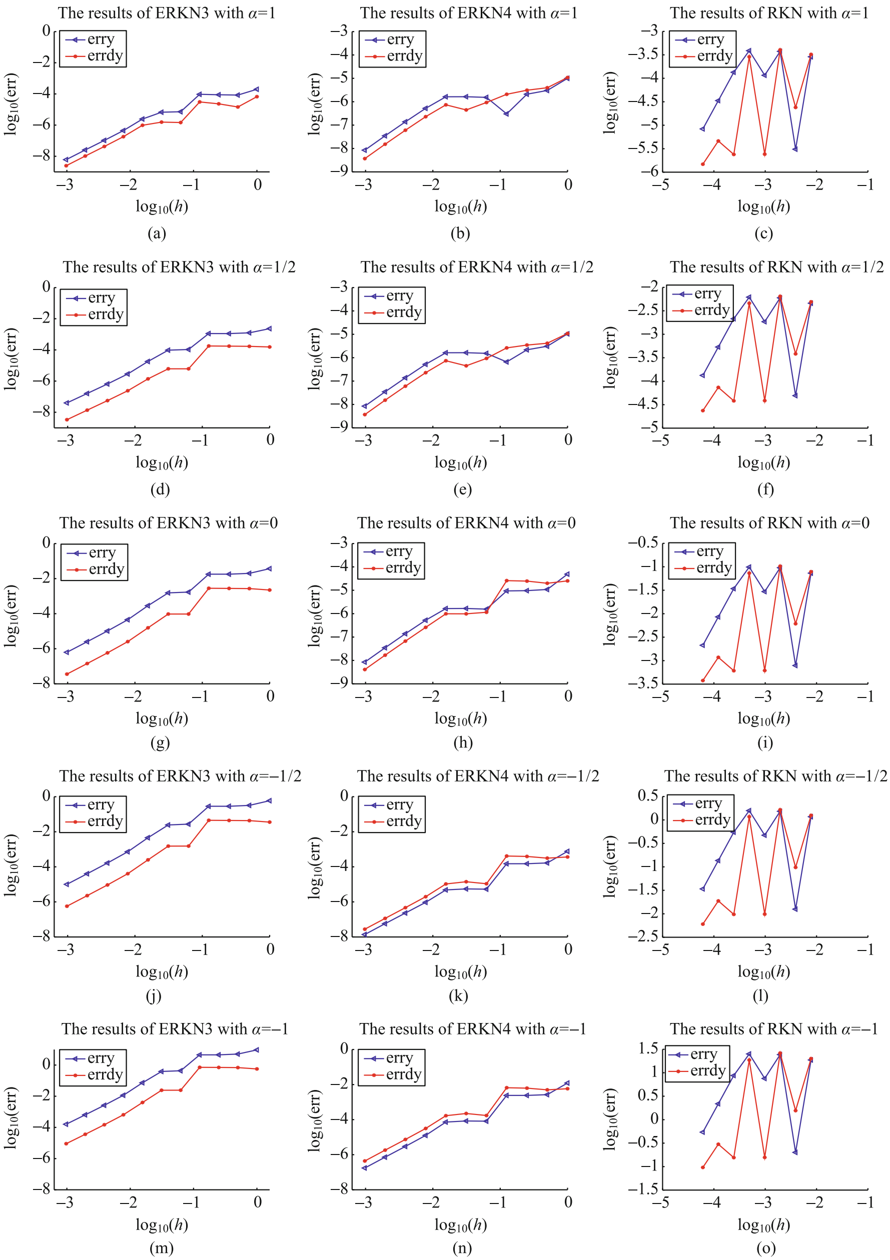

Fig. 7.2 The logarithm of the errors against the logarithm of stepsizes for $K = 2^8$

It follows from these results that the convergence order is not uniform for α, and when α goes from 1 to -1, the errors of ERKN integrators become large. This supports the result given in Theorem 7.1. Moreover, it can be observed from the computed results that ERKN integrators work much better for larger stepsizes, and they are more accurate for smaller stepsizes than the corresponding RKN method.

At the end of this section, we remark that the results for ERKN4 with a small stepsize are considered as the "exact" solutions of the underlying system for both values of K. We also note that a few errors for ERKN integrators for $K = 2^8$ are smaller than those for $K = 2^6$. This phenomenon may be caused by the choices of "exact" solutions for different K.

7.7 Concluding Remarks

In this chapter, we have analysed the error bounds of ERKN integrators when applied to spatial semidiscretisations of semilinear wave equations. Optimal second-order convergence has been obtained without requiring Lipschitz continuity and higher regularity of the exact solution. Moreover, the analysis is uniform in the spatial discretisation parameter. On the basis of this work, we are hopeful of obtaining an extension to two-stage ERKN integrators for semidiscrete semilinear wave equations. Another issue for future exploration is the error analysis of ERKN integrators in the case of quasi-linear wave equations.

The material in this chapter is based on the work by Wang and Wu [43].

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