Chapter 4
Functionally-Fitted Energy-Preserving Integrators for Poisson Systems

This chapter presents a class of energy-preserving integrators for Poisson systems based on the functionally-fitted strategy, and these energy-preserving integrators can have arbitrarily high order. This approach permits us to obtain the energy-preserving methods proposed in [1] by Cohen and Hairer and [2] by Brugnano et al. for Poisson systems.

4.1 Introduction

It is well known that Poisson systems arise in many applications. Moreover, it is noted that Poisson systems often have periodic or oscillatory solutions. This chapter is devoted to efficient numerical integrators for solving Poisson systems (non-canonical Hamiltonian systems)

$$y'(t) = B(y(t))\nabla H(y(t)), \quad y(0) = y_0 \in \mathbb{R}^d, \quad t \in [0, T], \tag{4.1}$$

where the prime denotes $\frac{\mathrm{d}}{\mathrm{d}t}$, $B(y)$ is a skew-symmetric matrix, $H(y)$ is a scalar function, and both are sufficiently smooth. It is assumed that the system (4.1) has a unique solution $y = y(t)$ defined for all $t \in [0, T]$. An important feature of (4.1) is that the energy $H(y)$ is preserved along the exact solution $y(t)$, since we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H(y(t)) = \nabla H(y(t))^{\intercal} y'(t) = \nabla H(y(t))^{\intercal} B(y(t))\nabla H(y(t)) = 0.$$

Numerical integrators that preserve $H(y)$ are termed energy-preserving (EP) integrators. The main aim of this chapter is to formulate and analyse some EP integrators for efficiently solving Poisson systems. Other geometric properties of the Poisson systems such as the preservation of Casimir functions and the Poisson map of the flow will not be considered in this chapter.

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If the matrix $B(y)$ is independent of y, d is an even number and

$$B = J = \begin{pmatrix} 0_{\frac{d}{2}} & I_{\frac{d}{2}} \\ -I_{\frac{d}{2}} & 0_{\frac{d}{2}} \end{pmatrix},$$

then the system (4.1) is a canonical Hamiltonian system. Much effort has been made for solving this system, and we refer the reader to [3–13] and references therein. For canonical Hamiltonian systems, EP integrators are important and efficient methods and a variety of EP methods have been derived and studied in the past few decades, such as the average vector field (AVF) method (see, e.g. [14–16]), discrete gradient methods (see, e.g. [17, 18]), Hamiltonian Boundary Value Methods (HBVMs) (see, e.g. [19, 20]), EP collocation methods (see, e.g. [21]) and exponential/trigonometric EP methods (see, e.g. [22–26]).

Among typical EP methods for solving $\dot{y} = J\nabla H(y)$ is the well-known AVF method given by Quispel and McLaren [16] as follows:

$$y_1 = y_0 + h\int_0^1 J\nabla H(y_0 + \sigma(y_1 - y_0))\mathrm{d}\sigma. \tag{4.2}$$

Quispel and McLaren in [16] pointed out that this method is a B-series method. Hairer extended this second-order method to higher-order schemes by introducing the so-called continuous-stage Runge–Kutta methods [21]. On noticing the fact that the dependence of the matrix $B(y)$ should be discretised in a different manner, an additional strategy will be required for Poisson systems. This means that it is necessary to design and analyse the EP methods specifically for Poisson systems. As is known, McLachlan et al. [18] discussed DG methods for various kinds of ODEs including Poisson systems, and Cohen et al. [1] succeeded in constructing arbitrary high-order EP schemes for Poisson systems and the following second-order EP scheme for (4.1) was derived

$$y_1 = y_0 + hB\big((1/2)(y_1 + y_0)\big)\int_0^1 \nabla H(y_0 + \sigma(y_1 - y_0))\mathrm{d}\sigma. \tag{4.3}$$

In the light of HBVMs, Brugnano et al. gave an alternative derivation of such methods and presented a new proof of their orders in [27]. EP exponentially-fitted integrators for Poisson systems were researched by Miyatake [28]. Using discrete gradients, Dahlby et al. [29] constructed useful methods that simultaneously preserve several invariants in systems of type (4.1). With regard to other multiple invariant-preserving integrators we refer the reader to [2, 20, 30–32].

We note that the functionally-fitted (FF) technique is a very useful approach to the construction of effective numerical methods for solving differential equations. In general, an FF method can be derived by requiring it to integrate members of a given finite-dimensional function space X exactly. The corresponding methods are termed trigonometrically-fitted (TF) or exponentially-fitted (EF) methods if

X is generated by trigonometrical or exponential functions. Using the FF/TF/EF technique, many efficient methods have been constructed for canonical Hamiltonian systems including the symplectic methods (see, e.g. [33–40]) and EP methods (see, e.g. [23, 41]). This technique has also been used successfully for Poisson systems in [28] and second- and fourth-order schemes were derived. In this chapter, using the functionally-fitted technology, we will design and analyse efficient EP integrators for Poisson systems. These integrators of arbitrarily high order can be derived in a routine and convenient manner, and different EP schemes can be obtained by considering different function spaces. We will show that choosing a special function space allows us to obtain the EP schemes proposed by Cohen and Hairer [1] and Brugnano et al. [27].

4.2 Functionally-Fitted EP Integrators

In order to derive the EP integrators for Poisson systems (4.1), we first define a vector function space Y=span$\{\Phi_0(t), \cdots, \Phi_{r-1}(t)\}$ on $[0, T]$ by (see [41])

$$Y = \left\{ w : w(t) = \sum_{i=0}^{r-1} \Phi_i(t) W_i,\ t \in [0, T],\ W_i \in \mathbb{R}^d \right\},$$

where the real functions $\{\Phi_i(t)\}_{i=0}^{r-1}$ are linearly independent and $\mathbb{C}^1$ $([0, T] \to \mathbb{R})$. In this chapter, we choose a stepsize $h > 0$ and consider the following two function spaces

$$Y_h = \text{span}\{\Phi_0(\tau h), \cdots, \Phi_{r-1}(\tau h)\}, \quad X_h = \text{span}\left\{1, \int_0^{\tau h} \Phi_0(s)\mathrm{d}s, \cdots, \int_0^{\tau h} \Phi_{r-1}(s)\mathrm{d}s\right\}, \tag{4.4}$$

where τ is a variable with $\tau \in [0, 1]$, the stepsize h is a positive parameter with $0 < h \leqslant h_0 \leqslant T$, and h_0 depends on the underlying problem.

We now introduce a projection (see [41]), which will be used in this chapter and we summarise its definition as follows.

Definition 4.1 (See [41]) Let $\mathscr{P}_h$ be a linear operator that maps d-vector valued real functions defined on $[0, h]$ into the finite dimensional space Y_h. The definition of $\mathscr{P}_h w(\tau h)$ is given by

$$\langle v(\tau h), \mathscr{P}_h w(\tau h)\rangle = \langle v(\tau h), w(\tau h)\rangle, \quad \text{for any } v \in Y_h, \tag{4.5}$$

where $w(\tau h)$ be a continuous $\mathbb{R}^d$-valued function for $\tau \in [0, 1]$ and $\mathscr{P}_h w(\tau h)$ is a projection of w onto Y_h. The scalar product $\langle \cdot, \cdot \rangle$ is defined as an inner product in

$C([0,1] \to \mathbb{R}^d)$ so that for

$$u = u(\tau h) = (u_1(\tau h), \cdots, u_d(\tau h))^\intercal, \quad v = v(\tau h) = (v_1(\tau h), \cdots, v_d(\tau h))^\intercal,$$

$\langle u, v\rangle$ is a d-dimensional vector with components $\int_0^1 u_i(\tau h)v_i(\tau h)\, \mathrm{d}\tau$ for $i = 1, \cdots, d$.

The following property of $\mathscr{P}_h$ is also needed, which has been proved in [41].

Lemma 4.1 (See [41]) *The projection $\mathscr{P}_h w(\tau h)$ can be explicitly expressed as*

$$\mathscr{P}_h w(\tau h) = \langle P_{\tau,\sigma}, w(\sigma h)\rangle_\sigma,$$

where

$$P_{\tau,\sigma} = \sum_{i=0}^{r-1} \psi_i(\tau h)\psi_i(\sigma h),$$

and $\{\psi_0, \cdots, \psi_{r-1}\}$ is a standard orthonormal basis of Y_h under the inner product $\langle\cdot,\cdot\rangle$.

With these preliminaries, we first present the definition of the integrators and then show that they exactly preserve the energy of Poisson system (4.1).

Definition 4.2 Let $u = u(\tau h)$ be the unique solution of the following initial value problem

$$\frac{1}{h}\frac{\mathrm{d}u(\tau h)}{\mathrm{d}\tau} = B(u(\tau h))\mathscr{P}_h\big(\nabla H(u(\tau h))\big), \quad u(0) = y_0, \quad \tau \in [0,1]. \tag{4.6}$$

If $u \in X_h$, then the numerical solution after one step is defined by $y_1 = u(h)$. In this chapter, the integrator is termed a functionally-fitted EP (FFEP) integrator.

Remark 4.1 It is important to note that the exact solution of the Poisson system (4.1) may not belong to the function space X_h. In this definition, the function $u \in X_h$ is considered as a numerical approximation of the exact solution. This approach is similar to that given by Cohen and Hairer in [1], where they consider a polynomial function as the approximation of the exact solution. In particular, we remark that, for the Euler equation considered as a numerical experiment in Sect. 4.7, the solution of (4.6) belongs to X_h.

Theorem 4.1 *The FFEP integrator (4.6) exactly preserves the energy, i.e.,*

$$H(y_1) = H(y_0).$$

Proof It follows from $u \in X_h$ that $u' \in Y_h$. Using the definition of $\mathscr{P}_h$ yields

$$\int_0^1 u'(\tau h)_i \Big(\mathscr{P}_h\big(\nabla H(u(\tau h))\big)\Big)_i \mathrm{d}\tau = \int_0^1 u'(\tau h)_i \big(\nabla H(u(\tau h))\big)_i \mathrm{d}\tau, \quad i = 1, 2, \cdots, d,$$

where $(\cdot)_i$ denotes the i-th entry of a vector. We then obtain

$$\int_0^1 u'(\tau h)^\intercal \mathscr{P}_h\big(\nabla H(u(\tau h))\big)\mathrm{d}\tau = \int_0^1 u'(\tau h)^\intercal \nabla H(u(\tau h))\mathrm{d}\tau.$$

Hence, we have

$$\begin{aligned} H(y_1) - H(y_0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau} H(u(\tau h))\mathrm{d}\tau \\ &= h \int_0^1 u'(\tau h)^\intercal \nabla H(u(\tau h))\mathrm{d}\tau \\ &= h \int_0^1 u'(\tau h)^\intercal \mathscr{P}_h\big(\nabla H(u(\tau h))\big)\mathrm{d}\tau. \end{aligned}$$

Inserting (4.6) into this formula gives

$$H(y_1) - H(y_0) = h \int_0^1 \mathscr{P}_h\big(\nabla H(u(\tau h))\big)^\intercal B(u(\tau h))^\intercal \mathscr{P}_h\big(\nabla H(u(\tau h))\big)\mathrm{d}\tau.$$

This proves the result by considering that $B(u)$ is a skew-symmetric matrix. □

Remark 4.2 If $B(y)$ is a constant skew-symmetric matrix, (4.1) is a canonical Hamiltonian system. In this case, the FFEP integrator (4.6) is identical to the functionally-fitted EP method derived in Li and Wu [41]. Apart from this, if Y_h is generated by the shifted Legendre polynomials on [0, 1], then the FFEP integrator (4.6) reduces to the EP collocation method given by Cohen and Hairer [21] and Brugnano et al. [27].

4.3 Implementation Issues

We next pay attention to practical implementation issues of the FFEP integrator. We choose the generalised Lagrange interpolation functions $\hat{l}_i(\tau) \in Y_h$ with respect to

r distinct points $\hat{d}_i \in [0, 1]$ for $i = 1, \cdots, r$ as follows:

$$(\hat{l}_1(\tau), \cdots, \hat{l}_r(\tau)) = (\Phi_0(\tau h), \Phi_1(\tau h), \cdots, \Phi_{r-1}(\tau h)) \cdot \begin{pmatrix} \Phi_0(\hat{d}_1 h) & \Phi_1(\hat{d}_1 h) & \cdots & \Phi_{r-1}(\hat{d}_1 h) \\ \Phi_0(\hat{d}_2 h) & \Phi_1(\hat{d}_2 h) & \cdots & \Phi_{r-1}(\hat{d}_2 h) \\ \vdots & \vdots & & \vdots \\ \Phi_0(\hat{d}_r h) & \Phi_1(\hat{d}_r h) & \cdots & \Phi_{r-1}(\hat{d}_r h) \end{pmatrix}^{-1}. \tag{4.7}$$

Clearly, $\{\hat{l}_i(\tau)\}_{i=1}^r$ provides another basis of Y_h, satisfying $\hat{l}_i(\hat{d}_j) = \delta_{ij}$. As $u' \in Y_h$, u' can be expressed in

$$u'(\tau h) = \sum_{i=1}^r \hat{l}_i(\tau) u'(\hat{d}_i h).$$

It follows from Lemma 4.1 that the FFEP integrator (4.6) is identical to

$$u'(\tau h) = B(u(\tau h)) \int_0^1 P_{\tau,\sigma} \nabla H(u(\sigma h)) \mathrm{d}\sigma,$$

which leads to

$$u'(\hat{d}_i h) = B(u(\hat{d}_i h)) \int_0^1 P_{\hat{d}_i,\sigma} \nabla H(u(\sigma h)) \mathrm{d}\sigma.$$

We then obtain

$$u'(\tau h) = \sum_{i=1}^r \hat{l}_i(\tau) u'(\hat{d}_i h) = \sum_{i=1}^r \hat{l}_i(\tau) \Big(B(u(\hat{d}_i h)) \int_0^1 P_{\hat{d}_i,\sigma} \nabla H(u(\sigma h)) \mathrm{d}\sigma \Big). \tag{4.8}$$

Integrating (4.8) gives

$$\begin{aligned} u(\tau h) &= y_0 + \int_0^{\tau h} u'(x) \mathrm{d}x = y_0 + h \int_0^{\tau} u'(\alpha h) \mathrm{d}\alpha \\ &= y_0 + h \int_0^{\tau} \sum_{i=1}^r \hat{l}_i(\alpha) \mathrm{d}\alpha B(u(\hat{d}_i h)) \int_0^1 P_{\hat{d}_i,\sigma} \nabla H(u(\sigma h)) \mathrm{d}\sigma. \end{aligned}$$

Let $y_\sigma = u(\sigma h)$, and we are now in a position to present the FFEP integrator (4.6) for Poisson system (4.1).

Definition 4.3 An FFEP integrator (4.6) for Poisson system (4.1) is defined by

$$\begin{cases} y_\tau = y_0 + h \sum_{i=1}^{r} \int_0^1 \Big(P_{\hat{d}_i,\sigma} \int_0^\tau \hat{l}_i(\alpha)\mathrm{d}\alpha \Big) B(y_{\hat{d}_i}) \nabla H(y_\sigma)\mathrm{d}\sigma, \quad 0 < \tau < 1, \\ y_1 = y_0 + h \sum_{i=1}^{r} \int_0^1 \Big(P_{\hat{d}_i,\sigma} \int_0^1 \hat{l}_i(\alpha)\mathrm{d}\alpha \Big) B(y_{\hat{d}_i}) \nabla H(y_\sigma)\mathrm{d}\sigma. \end{cases} \tag{4.9}$$

Remark 4.3 It can be observed from (4.9) that this integrator has a pattern similar to the formula (2.4) given by Cohen and Hairer in [1]. We need the first formula of (4.9) only for $\tau = \hat{d}_1, \cdots, \hat{d}_r$ and this leads to a nonlinear system of equations for the unknowns $y_{\hat{d}_1}, \cdots, y_{\hat{d}_r}$ which can be solved by a fixed-point iteration method.

Remark 4.4 It is noted that the integrals $\int_0^\tau \hat{l}_i(\alpha)\mathrm{d}\alpha$ and $\int_0^1 \hat{l}_i(\alpha)\mathrm{d}\alpha$ can be calculated exactly if Y_h is generated by many kinds of functions such as polynomials, exponential and trigonometrical functions. The integral $\int_0^1 P_{\hat{d}_i,\sigma} \nabla H(y_\sigma)\mathrm{d}\sigma$ appearing in (4.9) can also be calculated exactly for many cases. If these integrals cannot be explicitly calculated, they can be approximated by quadrature to any desired degree of accuracy.

4.4 The Existence, Uniqueness and Smoothness

We note that the FFEP integrator (4.6) fails to be well defined unless its existence and uniqueness is shown. This section is devoted to this issue.

In what follows, we assume that the solution $y = y(t)$ of (4.1) is contained in the following ball for $t \in [0, h_0]$

$$\bar{B}(y_0, R) = \Big\{ y \in \mathbb{R}^d : ||y - y_0|| \leqslant R \Big\},$$

where R is a positive constant and $\|\cdot\|$ is a fixed norm in $\mathbb{R}^d$ which is the same as that stated in Assumption 4.1 below. Besides, it has been shown in [41] that $P_{\tau,\sigma}$ is a smooth function of h. In this setting, we assume that $A_n = \max_{\tau,\sigma,h\in[0,1]} \left| \frac{\partial^n P_{\tau,\sigma}}{\partial h^n} \right|$ for $n = 0, 1$. Furthermore, the nth-order derivative of ∇H at y is a multilinear map from $\underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{n-fold}$ to $\mathbb{R}^d$ defined by

$$\nabla H^{(n)}(y)(z_1, \cdots, z_n) = \sum_{1 \leqslant \alpha_1, \cdots, \alpha_n \leqslant d} \frac{\partial^n \nabla H}{\partial y_{\alpha_1} \cdots \partial y_{\alpha_n}}(y) z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

where $y = (y_1, \cdots, y_d)^\intercal$ and $z_i = (z_i^1, \cdots, z_i^d)^\intercal$ for $i = 1, \cdots, n$. The same notation is used for $B(y)$. Before presenting the result, we also need the following assumption.

Assumption 4.1 Denote $D_0 = \max_{y \in \bar{B}(y_0, R)} ||\nabla H(y)||$ and $C_0 = \max_{y \in \bar{B}(y_0, R)} ||B(y)||$. It is assumed that ∇H and $\nabla H^{(1)}$ are Lipschitz-continuous, i.e., there exist $D_1, D_2 > 0$ such that

$$||\nabla H(y_1) - \nabla H(y_2)|| \leqslant D_1 ||y_1 - y_2||, \quad ||\nabla H^{(1)}(y_1) - \nabla H^{(1)}(y_2)|| \leqslant D_2 ||y_1 - y_2||$$

for all $y_1, y_2 \in \bar{B}(y_0, R)$. The same assumption is required for $B(y)$ and $B^{(1)}(y)$, and the corresponding Lipschitz constants are denoted by C_1 and C_2, respectively.

Theorem 4.2 *Under the assumptions stated above, the FFEP integrator* (4.6) *has a unique solution* $u(\tau h)$ *provided the stepsize* h *satisfies*

$$0 \leqslant h \leqslant \delta < \min\left\{\frac{1}{A_0 C_0 D_1 + A_0 C_1 D_0}, \frac{R}{A_0 C_0 D_0}, h_0, 1\right\}. \tag{4.10}$$

Moreover, $u(\tau h)$ *is smoothly dependent on* h *for any fixed* $\tau \in (0, 1]$.

Proof **Existence and uniqueness.** It follows from Lemma 4.1 that the FFEP integrator (4.6) can be written as

$$u'(\tau h) = B(u(\tau h)) \int_0^1 P_{\tau,\sigma} \nabla H(u(\sigma h)) \mathrm{d}\sigma.$$

By integration we obtain

$$u(\tau h) = y_0 + h \int_0^\tau B(u(\alpha h)) \int_0^1 P_{\alpha,\sigma} \nabla H(u(\sigma h)) \mathrm{d}\sigma \mathrm{d}\alpha. \tag{4.11}$$

The formula (4.11) generates a function series $\{u_n(\tau h)\}_{n=0}^\infty$ by the following recursive definition

$$u_{n+1}(\tau h) = y_0 + h \int_0^1 \left(\int_0^\tau B(u_n(\alpha h)) P_{\alpha,\sigma} \mathrm{d}\alpha\right) \nabla H(u_n(\sigma h)) \mathrm{d}\sigma, \quad n = 0, 1, \cdots, \tag{4.12}$$

which will be shown to be uniformly convergent by proving the uniform convergence of the infinite series $\sum_{n=0}^\infty (u_{n+1}(\tau h) - u_n(\tau h))$. Then the integrator (4.6) has a solution $\lim\limits_{n\to\infty} u_n(\tau h)$.

We next prove the uniform convergence of $\sum_{n=0}^\infty (u_{n+1}(\tau h) - u_n(\tau h))$. First, it is clear that $||u_0(\tau h) - y_0|| = 0 \leqslant R$. We assume that $||u_n(\tau h) - y_0|| \leqslant R$ for $n = 0, \cdots, m$. It then follows from (4.10) and (4.12) that

$$||u_{m+1}(\tau h) - y_0|| \leqslant h A_0 C_0 D_0 \leqslant R,$$

which implies that $u_n(\tau h)$ are uniformly bounded by $||u_n(\tau h) - y_0|| \leqslant R$ for $n = 0, 1, \cdots$. Then using (4.12) and the Lipschitz conditions, we obtain

$$
\begin{aligned}
&\|u_{n+1}(\tau h) - u_n(\tau h)\|_c \\
&\leqslant h \int_0^1 \int_0^\tau \left\| \Big[B(u_n(\alpha h)) P_{\alpha,\sigma} \nabla H(u_n(\sigma h)) - B(u_{n-1}(\alpha h)) P_{\alpha,\sigma} \nabla H(u_{n-1}(\sigma h)) \Big] \right\|_c \mathrm{d}\alpha \mathrm{d}\sigma \\
&\leqslant h \int_0^1 \int_0^\tau \Big\| \Big[B(u_n(\alpha h)) P_{\alpha,\sigma} \nabla H(u_n(\sigma h)) - B(u_n(\alpha h)) P_{\alpha,\sigma} \nabla H(u_{n-1}(\sigma h)) \\
&\quad + B(u_n(\alpha h)) P_{\alpha,\sigma} \nabla H(u_{n-1}(\sigma h)) - B(u_{n-1}(\alpha h)) P_{\alpha,\sigma} \nabla H(u_{n-1}(\sigma h)) \Big] \Big\|_c \mathrm{d}\alpha \mathrm{d}\sigma \\
&\leqslant h(A_0 C_0 D_1 + A_0 C_1 D_0) \int_0^1 ||u_n(\sigma h) - u_{n-1}(\sigma h)|| \mathrm{d}\sigma \leqslant \beta ||u_n(\tau h) - u_{n-1}(\tau h)||_c,
\end{aligned}
$$

where $\beta = \delta(A_0 C_0 D_1 + A_0 C_1 D_0)$ and $||w||_c = \max_{\tau \in [0,1]} ||w(\tau h)||$ for a continuous $\mathbb{R}^d$-valued function w on [0, 1]. This implies that

$$
||u_{n+1} - u_n||_c \leqslant \beta ||u_n - u_{n-1}||_c
$$

and then

$$
||u_{n+1} - u_n||_c \leqslant \beta^n ||u_1 - y_0||_c \leqslant \beta^n R, \quad n = 0, 1, \cdots.
$$

Using the Weierstrass M-test and the fact that $\beta < 1$, we confirm that $\sum_{n=0}^{\infty} (u_{n+1}(\tau h) - u_n(\tau h))$ is uniformly convergent.

With regard to the uniqueness, we suppose that the integrator has another solution $v(\tau h)$. We then have

$$
||u(\tau h) - v(\tau h)|| \leqslant \beta ||u(\tau h) - v(\tau h)|| \leqslant \beta ||u - v||_c,
$$

and

$$
\|u - v\|_c \leqslant \beta ||u - v||_c.
$$

Hence, we obtain $||u - v||_c = 0$ and $u(\tau h) \equiv v(\tau h)$. Therefore, the solution of the FFEP integrator (4.6) exists and is unique.

Smoothness We next prove the result that $u(\tau h)$ is smoothly dependent on h for any fixed $\tau \in (0, 1]$. This is true if the series $\left\{ \dfrac{\partial^k u_n}{\partial h^k}(\tau h) \right\}_{n=0}^{\infty}$ is uniformly convergent for $k \geqslant 1$. We note that the analysis of this part needs the bounds on $\nabla H^{(1)}(y)$ and $B^{(1)}(y)$, which are also denoted by D_1 and C_1, respectively.

Differentiating (4.12) with respect to h yields

$$\begin{aligned}\frac{\partial u_{n+1}}{\partial h}(\tau h) =& \int_0^1 \left(\int_0^\tau B(u_n(\alpha h))P_{\alpha,\sigma}\mathrm{d}\alpha\right)\nabla H(u_n(\sigma h))\mathrm{d}\sigma \\ &+h\int_0^1 \left(\int_0^\tau B^{(1)}(u_n(\alpha h))\frac{\partial u_n(\alpha h)}{\partial h}P_{\alpha,\sigma}\mathrm{d}\alpha\right)\nabla H(u_n(\sigma h))\mathrm{d}\sigma \\ &+h\int_0^1 \left(\int_0^\tau B(u_n(\alpha h))\frac{\partial P_{\alpha,\sigma}}{\partial h}\mathrm{d}\alpha\right)\nabla H(u_n(\sigma h))\mathrm{d}\sigma \\ &+h\int_0^1 \left(\int_0^\tau B(u_n(\alpha h))P_{\alpha,\sigma}\mathrm{d}\alpha\right)\nabla H^{(1)}(u_n(\sigma h))\frac{\partial u_n(\sigma h)}{\partial h}\mathrm{d}\sigma. \quad (4.13)\end{aligned}$$

We then have

$$\left\|\frac{\partial u_{n+1}}{\partial h}\right\|_c \leqslant \alpha+\beta\left\|\frac{\partial u_n}{\partial h}\right\|_c \qquad \text{with } \alpha = A_0C_0D_0+\delta A_1C_0D_0,$$

which yields that $\left\{\dfrac{\partial u_n}{\partial h}(\tau h)\right\}_{n=0}^{\infty}$ is uniformly bounded as follows:

$$\left\|\frac{\partial u_n}{\partial h}\right\|_c \leqslant \alpha(1+\beta+\cdots+\beta^{n-1}) \leqslant \frac{\alpha}{1-\beta} =: C^*, \quad n=0,1,\cdots.$$

It follows from (4.13) that

$$\begin{aligned}&\frac{\partial u_{n+1}}{\partial h}-\frac{\partial u_n}{\partial h} \\ =& \int_0^1\int_0^\tau \Big[B(u_n(\alpha h))P_{\alpha,\sigma}\nabla H(u_n(\sigma h)) - B(u_{n-1}(\alpha h))P_{\alpha,\sigma}\nabla H(u_{n-1}(\sigma h))\Big]\mathrm{d}\alpha\mathrm{d}\sigma \\ &+h\int_0^1\int_0^\tau \Big[B^{(1)}(u_n(\alpha h))\frac{\partial u_n(\alpha h)}{\partial h}P_{\alpha,\sigma}\nabla H(u_n(\sigma h)) \\ &-B^{(1)}(u_{n-1}(\alpha h))\frac{\partial u_{n-1}(\alpha h)}{\partial h}P_{\alpha,\sigma}\nabla H(u_{n-1}(\sigma h))\Big]\mathrm{d}\alpha\mathrm{d}\sigma \\ &+h\int_0^1\int_0^\tau \Big[B(u_n(\alpha h))\frac{\partial P_{\alpha,\sigma}}{\partial h}\nabla H(u_n(\sigma h)) \\ &-B(u_{n-1}(\alpha h))\frac{\partial P_{\alpha,\sigma}}{\partial h}\nabla H(u_{n-1}(\sigma h))\Big]\mathrm{d}\alpha\mathrm{d}\sigma \\ &+h\int_0^1\int_0^\tau \Big[B(u_n(\alpha h))P_{\alpha,\sigma}\nabla H^{(1)}(u_n(\sigma h))\frac{\partial u_n(\sigma h)}{\partial h} \\ &-B(u_{n-1}(\alpha h))P_{\alpha,\sigma}\nabla H^{(1)}(u_{n-1}(\sigma h))\frac{\partial u_{n-1}(\sigma h)}{\partial h}\Big]\mathrm{d}\alpha\mathrm{d}\sigma.\end{aligned}$$

Adding and removing some expressions with careful simplifications gives

$$\left\|\frac{\partial u_{n+1}}{\partial h}-\frac{\partial u_n}{\partial h}\right\|_c \leqslant \gamma\beta^{n-1}+\beta\left\|\frac{\partial u_n}{\partial h}-\frac{\partial u_{n-1}}{\partial h}\right\|_c,$$

where

$$\begin{aligned}\gamma=&(C_0A_0D_1+C_1A_0D_0+\delta C_0A_1D_1+\delta C_1A_1D_0+2\delta C_1A_0D_1C^*\\&+\delta A_0D_0C^*C_2+\delta C_0A_0C^*D_2)R.\end{aligned}$$

Hence, by induction, it is true that

$$\left\|\frac{\partial u_{n+1}}{\partial h}-\frac{\partial u_n}{\partial h}\right\|_c \leqslant n\gamma\beta^{n-1}+\beta^nC^*, \quad n=1,2,\cdots,$$

which confirms the uniform convergence of $\sum_{n=0}^{\infty}\left(\frac{\partial u_{n+1}}{\partial h}(\tau h)-\frac{\partial u_n}{\partial h}(\tau h)\right)$. Thus, $\left\{\frac{\partial u_n}{\partial h}(\tau h)\right\}_{n=0}^{\infty}$ is uniformly convergent.

Likewise, the uniform convergence of other function series $\left\{\frac{\partial^k u_n}{\partial h^k}(\tau h)\right\}_{n=0}^{\infty}$ for $k\geqslant 2$ can be shown as well. Therefore, $u(\tau h)$ is smoothly dependent on h. □

4.5 Algebraic Order

We consider the algebraic order of the FFEP integrator in this section. For this purpose, we begin with the regularity of the integrators. Following [41], if an h-dependent function $w(\tau h)$ can be expanded as

$$w(\tau h)=\sum_{n=0}^{r-1}w^{[n]}(\tau h)h^n+\mathscr{O}(h^r),$$

then $w(\tau h)$ is termed regular, where $w^{[n]}(\tau h)=\frac{1}{n!}\frac{\partial^n w(\tau h)}{\partial h^n}\Big|_{h=0}$ is a vector-valued function with polynomial entries of degrees $\leqslant n$.

Lemma 4.2 *The FFEP integrator (4.6) gives a regular h-dependent function $u(\tau h)$.*

Proof It has been proved in Theorem 4.2 that $u(\tau h)$ is smoothly dependent on h. We then can expand $u(\tau h)$ with respect to h at zero as follows:

$$u(\tau h)=\sum_{m=0}^{r-1}u^{[m]}(\tau h)h^m+\mathscr{O}(h^r).$$

Let $\Delta = u(\tau h) - y_0$ and it is clear that $\Delta = \mathscr{O}(h)$. Expanding $\nabla H(u(\tau h))$ at $h = 0$ and inserting the above equalities into (4.11) leads to

$$\sum_{m=0}^{r-1} u^{[m]}(\tau h)h^m$$
$$= y_0 + h\int_0^1 \int_0^\tau P_{\alpha,\sigma} B(u(\alpha h))\mathrm{d}\alpha \sum_{n=0}^{r-1} \frac{1}{n!}\nabla H^{(n)}(y_0)(\underbrace{\Delta, \cdots, \Delta}_{n-fold})\mathrm{d}\sigma + \mathscr{O}(h^r). \quad (4.14)$$

In what follows, we prove the following result by induction

$$u^{[m]}(\tau h) \in P_m^d = \underbrace{P_m([0, 1]) \times \cdots \times P_m([0, 1])}_{d-fold} \quad \text{for} \quad m = 0, 1, \cdots, r-1,$$

where $P_m([0, 1])$ consists of polynomials with degrees $\leqslant m$ on [0, 1].

First, it is clear that $u^{[0]}(\tau h) = y_0 \in P_0^d$. Assume that $u^{[n]}(\tau h) \in P_n^d$ for $n = 0, 1, \cdots, m$. Compare the coefficients of h^{m+1} on both sides of (4.14) and then we have

$$u^{[m+1]}(\tau h) = \sum_{k+n=m} \int_0^1 \int_0^\tau \big[P_{\alpha,\sigma} B(u(\alpha h))\big]^{[k]}\mathrm{d}\alpha h_n(\sigma h)\mathrm{d}\sigma, \quad h_n(\sigma h) \in P_n^d.$$

Because $P_{\alpha,\sigma}$ is regular (see [41]) and $u^{[n]}(\tau h) \in P_n^d$, it can be verified that $\big[P_{\alpha,\sigma} B(u(\alpha h))\big]^{[k]} \in P_k^{d\times d}$. Hence, with the condition $k + n = m$, we have

$$\sum_{k+n=m} \int_0^1 \int_0^\tau \big[P_{\alpha,\sigma} B(u(\alpha h))\big]^{[k]}\mathrm{d}\alpha h_n(\sigma h)\mathrm{d}\sigma \in P_{m+1}^d.$$

Thus, it is true that

$$u^{[m+1]}(\tau h) \in P_{m+1}^d.$$

□

Let us now quote a result which is needed in the analysis of algebraic order.

Lemma 4.3 (See [41]) *Given a regular function w and an h-independent sufficiently smooth function g, the composition (if it exists) is regular. Moreover, one has*

$$\mathscr{P}_h g(w(\tau h)) - g(w(\tau h)) = \mathscr{O}(h^r).$$

Before presenting the algebraic order of the integrators, we recall the following elementary theory of ordinary differential equations. Denoting by $y(\cdot, \tilde{t}, \tilde{y})$ the solution of $y'(t) = B(y(t))\nabla H(y(t))$ satisfying the initial condition $y(\tilde{t}, \tilde{t}, \tilde{y}) = \tilde{y}$ [1] for any given $\tilde{t} \in [0, h]$ and setting

$$\Phi(s, \tilde{t}, \tilde{y}) = \frac{\partial y(s, \tilde{t}, \tilde{y})}{\partial \tilde{y}},$$

one has the standard result

$$\frac{\partial y(s, \tilde{t}, \tilde{y})}{\partial \tilde{t}} = -\Phi(s, \tilde{t}, \tilde{y})B(\tilde{y})\nabla H(\tilde{y}).$$

Theorem 4.3 *The FFEP integrator* (4.6) *is of order* $2r$*, which implies*

$$u(h) - y(t_0 + h) = \mathscr{O}(h^{2r+1}).$$

Moreover, we have

$$u(\tau h) - y(t_0 + \tau h) = \mathscr{O}(h^{r+1}), \quad 0 < \tau < 1.$$

Proof On the basis of the previous preliminaries, we obtain

$$\begin{aligned}
& u(h) - y(t_0 + h) \\
&= y(t_0 + h, t_0 + h, u(h)) - y(t_0 + h, t_0, y_0) \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\alpha} y(t_0 + h, t_0 + \alpha h, u(\alpha h))\mathrm{d}\alpha \\
&= \int_0^1 \Big(h\frac{\partial y}{\partial \tilde{t}}(t_0 + h, t_0 + \alpha h, u(\alpha h)) + \frac{\partial y}{\partial \tilde{y}}(t_0 + h, t_0 + \alpha h, u(\alpha h))hu'(\alpha h)\Big) \mathrm{d}\alpha \\
&= \int_0^1 \Big(- h\frac{\partial y}{\partial \tilde{y}}(t_0 + h, t_0 + \alpha h, u(\alpha h))B(u(\alpha h))\nabla H(u(\alpha h)) \\
&\quad + \frac{\partial y}{\partial \tilde{y}}(t_0 + h, t_0 + \alpha h, u(\alpha h))hB(u(\alpha h))\mathscr{P}_h \nabla H(u(\alpha h))\Big)\mathrm{d}\alpha \\
&= -h \int_0^1 \Phi^1(\alpha)B(u(\alpha h)) \left(\nabla H(u(\alpha h)) - \mathscr{P}_h \nabla H(u(\alpha h))\right) \mathrm{d}\alpha,
\end{aligned}$$

---

[0] 1 Clearly, since the problem is autonomous, then $y(t, \tilde{t}, \tilde{y}) = y(t - \tilde{t}, 0, \tilde{y})$.

where

$$\Phi^1(\alpha) = \frac{\partial y}{\partial \tilde{y}}(t_0 + \ h, t_0 + \alpha h, u(\alpha h)).$$

It follows from Lemmas 4.2 and 4.3 that

$$\mathscr{P}_h \nabla H(u(\tau h)) - \nabla H(u(\tau h)) = \mathscr{O}(h^r).$$

Partition the matrix-valued function $\Phi^1(\alpha)$ as $\Phi^1(\alpha) = (\Phi_1^1(\alpha), \cdots, \Phi_d^1(\alpha))^{\intercal}$ and then it follows from Lemma 4.2 that

$$\Phi_i^1(\alpha) = \mathscr{P}_h \Phi_i^1(\alpha) + \mathscr{O}(h^r), \quad i = 1, 2, \cdots, d.$$

As $\mathscr{P}_h \Phi_i^1(\alpha) \in Y_h$, we obtain

$$\int_0^1 (\mathscr{P}_h \Phi_i^1(\alpha))^{\intercal} \nabla H(u(\alpha h)) \mathrm{d}\alpha = \int_0^1 (\mathscr{P}_h \Phi_i^1(\alpha))^{\intercal} \mathscr{P}_h \nabla H(u(\alpha h)) \mathrm{d}\alpha, \quad i = 1, 2, \cdots, d.$$

Hence, we have

$$\begin{aligned}
&u(h) - y(t_0 + h) \\
&= -h \int_0^1 \left(\begin{pmatrix} (\mathscr{P}_h \Phi_1^1(\alpha))^{\intercal} \\ \vdots \\ (\mathscr{P}_h \Phi_d^1(\alpha))^{\intercal} \end{pmatrix} + \mathscr{O}(h^r) \right) (\nabla H(u(\alpha h)) - \mathscr{P}_h \nabla H(u(\alpha h))) \,\mathrm{d}\alpha \\
&= -h \int_0^1 \begin{pmatrix} (\mathscr{P}_h \Phi_1^1(\alpha))^{\intercal} \big(\nabla H(u(\alpha h)) - \mathscr{P}_h \nabla H(u(\alpha h))\big) \\ \vdots \\ (\mathscr{P}_h \Phi_d^1(\alpha))^{\intercal} \big(\nabla H(u(\alpha h)) - \mathscr{P}_h \nabla H(u(\alpha h))\big) \end{pmatrix} \mathrm{d}\alpha \\
&\quad - h \int_0^1 \mathscr{O}(h^r) \times \mathscr{O}(h^r) \mathrm{d}\alpha \\
&= 0 + \mathscr{O}(h^{2r+1}) = \mathscr{O}(h^{2r+1}).
\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}
u(\tau h) - y(t_0 + \tau h) &= y(t_0 + \tau h, t_0 + \tau h, u(\tau h)) - y(t_0 + \tau h, t_0, y_0) \\
&= -h \int_0^{\tau} \Phi^{\tau}(\alpha) B(u(\alpha h)) \big(\nabla H(u(\alpha h)) - \mathscr{P}_h \nabla H(u(\alpha h))\big) \mathrm{d}\alpha \\
&= -h \int_0^{\tau} \Phi^{\tau}(\alpha) B(u(\alpha h)) \mathscr{O}(h^r) \mathrm{d}\alpha = \mathscr{O}(h^{r+1}).
\end{aligned}$$

□

4.6 Practical FFEP Integrators

In what follows, we consider two illustrative examples of FFEP integrators.

Example 1 We choose

$$\Phi_k(\tau h) = (\tau h)^k, \quad k = 0, 1, \cdots, r-1,$$

for the function spaces X_h and Y_h, and then we have

$$\hat{l}_i(\tau) = \prod_{j=1, j\neq i}^{r} \frac{\tau - \hat{d}_j}{\hat{d}_i - \hat{d}_j},$$

for $i = 1, 2, \cdots, r$. The Gram–Schmidt process leads to the standard orthonormal basis of Y_h as follows:

$$\hat{p}_j(\tau h) = (-1)^j \sqrt{2j+1} \sum_{k=0}^{j} \binom{j}{k} \binom{j+k}{k} (-\tau)^k, \qquad t \in [0, 1],$$

for $j = 0, 1, \cdots, r-1$, which are the shifted Legendre polynomials on $[0, 1]$. Consequently, $P_{\tau,\sigma}$ can be determined by

$$P_{\tau,\sigma} = \sum_{i=0}^{r-1} \hat{p}_i(\tau h)\hat{p}_i(\sigma h).$$

Here it is important to note that all the above functions are independent of h. In this situation, the FFEP integrator (4.6) is identical to the EP method given by Cohen and Hairer [21] and Brugnano et al. [27].

If we choose $r = 1$ and $\hat{d}_1 = 1/2$, we obtain

$$\hat{l}_1(\tau) = 1, \quad P_{\tau,\sigma} = 1.$$

Accordingly, the integrator (4.9) yields

$$\begin{cases} y_\tau = y_0 + h\tau B(y_{\frac{1}{2}}) \int_0^1 \nabla H(y_\sigma) \mathrm{d}\sigma, \\ y_1 = y_0 + hB(y_{\frac{1}{2}}) \int_0^1 \nabla H(y_\sigma) \mathrm{d}\sigma, \end{cases} \tag{4.15}$$

which gives

$$y_\tau = y_0 + h\tau B(y_{\frac{1}{2}}) \int_0^1 \nabla H(y_\sigma) \mathrm{d}\sigma = y_0 + \tau(y_1 - y_0).$$

Let $\tau = 1/2$ for the first equality of (4.15), and then we have

$$y_{\frac{1}{2}} = y_0 + \frac{1}{2}hB(y_{\frac{1}{2}})\int_0^1 \nabla H(y_\sigma)\mathrm{d}\sigma = y_0 + \frac{1}{2}(y_1 - y_0) = \frac{1}{2}(y_0 + y_1).$$

This leads to

$$y_1 = y_0 + hB\left(\frac{1}{2}(y_0 + y_1)\right)\int_0^1 \nabla H(y_0 + \sigma(y_1 - y_0))\mathrm{d}\sigma.$$

This second-order integrator has been given by Cohen and Hairer in [1].

Example 2 Let us consider another choice for Y_h by

$$Y_h = \mathrm{span}\{\cos(\omega\tau h)\},$$

and this gives

$$\hat{l}_1(\tau) = \frac{\cos(\tau v)}{\cos(\hat{d}_1 v)}, \quad P_{\tau,\sigma} = \frac{4v\cos(\sigma v)\cos(\tau v)}{2v + \sin(2v)},$$

where $v = \omega h$. With this choice, the integrator (4.9) becomes

$$\begin{cases} y_\tau = y_0 + h\displaystyle\int_0^\tau \hat{l}_1(\alpha)\mathrm{d}\alpha B(y_{\hat{d}_1})\int_0^1 P_{\hat{d}_1,\sigma}\nabla H(y_\sigma)\mathrm{d}\sigma, \\ y_1 = y_0 + h\displaystyle\int_0^1 \hat{l}_1(\alpha)\mathrm{d}\alpha B(y_{\hat{d}_1})\int_0^1 P_{\hat{d}_1,\sigma}\nabla H(y_\sigma)\mathrm{d}\sigma. \end{cases} \tag{4.16}$$

Let $\tau = \hat{d}_1 = \dfrac{1}{2}$ in (4.16). We then obtain

$$y_{1/2} = y_0 + h\frac{\tan(v/2)}{v}B(y_{1/2})\int_0^1 P_{1/2,\sigma}\nabla H(y_\sigma)\mathrm{d}\sigma,$$
$$y_1 = y_0 + h\frac{2\sin(v/2)}{v}B(y_{1/2})\int_0^1 P_{1/2,\sigma}\nabla H(y_\sigma)\mathrm{d}\sigma.$$

It follows from these two equalities that

$$y_{1/2} = y_0 + \frac{\tan(v/2)}{v}\frac{v(y_1 - y_0)}{2\sin(v/2)} = y_0 + \frac{1}{2\cos(v/2)}(y_1 - y_0),$$
$$y_\tau = y_0 + \frac{\sin(v\tau)}{v\cos(v/2)}\frac{v(y_1 - y_0)}{2\sin(v/2)} = y_0 + \frac{\sin(v\tau)}{\sin(v)}(y_1 - y_0).$$

This then results in

$$y_1 = y_0 + h\frac{2\sin(v/2)}{v}B\Big(y_0 + \frac{y_1 - y_0}{2\cos(v/2)}\Big)\int_0^1 P_{1/2,\sigma}\nabla H\Big(y_0 + \frac{\sin(v\sigma)}{\sin(v)}(y_1 - y_0)\Big)\mathrm{d}\sigma. \tag{4.17}$$

Clearly, this integrator reduces to (4.3) when $v = 0$. We denote the second-order scheme by FFEP1.

Example 3 We now consider

$$Y_h = \operatorname{span}\{\cos(\omega\tau h),\ \sin(\omega\tau h)\}.$$

This choice of Y_h leads to

$$\hat{l}_1(\tau) = \frac{\sin((\tau - \hat{d}_2)v)}{\sin((\hat{d}_1 - \hat{d}_2)v)},\quad \hat{l}_2(\tau) = \frac{\sin((\tau - \hat{d}_1)v)}{\sin((\hat{d}_2 - \hat{d}_1)v)}$$

and

$$P_{\tau,\sigma} = \frac{2v(2v\cos((\sigma - \tau)v) + \sin((-2 + \sigma + \tau)v) - \sin((\sigma + \tau)v))}{-1 + 2v^2 + \cos(2v)}.$$

We here choose $\tau = \hat{d}_1$ and $\hat{d}_2$ for the integrator (4.9). We then obtain

$$\begin{cases} y_{\hat{d}_1} = y_0 + h\displaystyle\int_0^1 \big(\bar{A}_{11}(\sigma)B(y_{\hat{d}_1}) + \bar{A}_{12}(\sigma)B(y_{\hat{d}_2})\big)\nabla H(y_\sigma)\mathrm{d}\sigma,\\ y_{\hat{d}_2} = y_0 + h\displaystyle\int_0^1 \big(\bar{A}_{21}(\sigma)B(y_{\hat{d}_1}) + \bar{A}_{22}(\sigma)B(y_{\hat{d}_2})\big)\nabla H(y_\sigma)\mathrm{d}\sigma,\\ y_1 = y_0 + h\displaystyle\int_0^1 \big(\bar{b}_1(\sigma)B(y_{\hat{d}_1}) + \bar{b}_2(\sigma)B(y_{\hat{d}_2})\big)\nabla H(y_\sigma)\mathrm{d}\sigma, \end{cases} \tag{4.18}$$

where

$$\bar{A}_{ij}(\sigma) = P_{\hat{d}_j,\sigma}\int_0^{\hat{d}_i} \hat{l}_j(\alpha)\mathrm{d}\alpha,\quad \bar{b}_j(\sigma) = P_{\hat{d}_j,\sigma}\int_0^1 \hat{l}_j(\alpha)\mathrm{d}\alpha \quad i, j = 1, 2.$$

We denote this fourth-order integrator (4.18) by FFEP2. It is worth noting that when $v = 0$ and $\hat{d}_{1,2} = 1/2 \mp \sqrt{3}/6$, this scheme becomes

$$\begin{cases} y_{\hat{d}_1} = y_0 + h \int_0^1 \Big(\frac{1}{2} l_1(\sigma) B(y_{\hat{d}_1}) + \left(\frac{1}{2} - \frac{\sqrt{3}}{3}\right) l_2(\sigma) B(y_{\hat{d}_2})\Big) \nabla H(y_\sigma) \mathrm{d}\sigma, \\ y_{\hat{d}_2} = y_0 + h \int_0^1 \left(\left(\frac{1}{2} + \frac{\sqrt{3}}{3}\right) l_1(\sigma) B(y_{\hat{d}_1}) + \frac{1}{2} l_2(\sigma) B(y_{\hat{d}_2}) \right) \nabla H(y_\sigma) \mathrm{d}\sigma, \\ y_1 = y_0 + h \int_0^1 \big(l_1(\sigma) B(y_{\hat{d}_1}) + l_2(\sigma) B(y_{\hat{d}_2})\big) \nabla H(y_\sigma) \mathrm{d}\sigma, \end{cases}$$

where

$$l_1(\sigma) = \frac{\sigma - \hat{d}_2}{\hat{d}_1 - \hat{d}_2}$$

and

$$l_2(\sigma) = \frac{\sigma - \hat{d}_1}{\hat{d}_2 - \hat{d}_1}.$$

This fourth-order integrator has been proposed by Cohen and Hairer in [1].

Remark 4.5 We remark that different choices of Y_h and X_h will derive different practical integrators. We do not pursue this point further for brevity.

4.7 Numerical Experiments

To illustrate the efficiency and robustness of the integrators derived in this chapter, we apply our integrators FFEP1 and FFEP2 to the Euler equation. For comparison, we consider the second-order and fourth-order EP collocation methods given in [1] and denote them by EPCM1 and EPCM2, respectively. We also choose the following second-order trigonometrically-fitted EP method (see [28])

$$y_1 = y_0 + h \frac{2 \sinh(v/2)}{v \cosh(v/2)} B((1/2)(y_0 + y_1)) \int_0^1 \nabla H(y_0 + \sigma(y_1 - y_0)) \mathrm{d}\sigma, \tag{4.19}$$

which is denoted by TFEP1. Since these five methods are all implicit, we use fixed-point iteration. We set 10^{-16} as the error tolerance and 10 as the maximum number of each iteration.

We will use as a test problem the following Euler equations (see [28, 33]) given by

$$\dot{y} = \big((\alpha-\beta)y_2y_3, (1-\alpha)y_3y_1, (\beta-1)y_1y_2\big)^{\intercal}, \quad t \in [0, T],$$

which describes the motion of a rigid body under no forces. This system can be written as a Poisson system

$$\dot{y} = \begin{pmatrix} 0 & \alpha y_3 & -\beta y_2 \\ -\alpha y_3 & 0 & y_1 \\ \beta y_2 & -y_1 & 0 \end{pmatrix} \nabla H(y)$$

with

$$H(y) = \frac{y_1^2 + y_2^2 + y_3^2}{2}.$$

Following [28, 33], the initial value is chosen as $y(0) = (0, 1, 1)$, and the parameters are given by

$$\alpha = 1 + \frac{1}{\sqrt{1.51}}, \ \beta = 1 - \frac{0.51}{\sqrt{1.51}}.$$

The exact solution is given by

$$y(t) = (\sqrt{1.51}sn(t, 0.51), cn(t, 0.51), dn(t, 0.51))^{\intercal},$$

where sn, cn, dn are the Jacobi elliptic functions. This solution is periodic with the period

$$T_p = 7.450563209330954,$$

and thence we consider choosing $\omega = 2\pi/T_p$ for the methods FFEP1 and TFEP1. We integrate this problem with the stepsizes $h = 0.5$ and $h = 0.2$ on the interval $[0, 10000]$. The energy conservation for different methods is shown in Fig. 4.1. We then solve the problem on the interval $[0, T]$ with different stepsizes $h = 1/2^i$ for $i = 4, 5, 6, 7$. The global errors are presented in Fig. 4.2 for $T = 10, 100$.

We also consider another case. As mentioned in [28], when $\beta \approx 1$, it is expected that $\dot{y}_3 \approx 0$ and thus $y_3(t) \approx 1$. Therefore, the variables y_1 and y_2 seem to behave like the harmonic oscillator with the period $T_p = 2\pi/(\alpha - 1)$. We choose $\alpha = 2$ and $\beta = 1.01$. We integrate this problem with $h = 0.5$ and $h = 0.2$ on the interval $[0, 10000]$. The energy conservation for different methods is shown in Fig. 4.3.

Then the problem is solved on the interval $[0, T]$ with $h = 1/2^i$ for $i = 4, 5, 6, 7$, and see Fig. 4.4 for the global errors of $T = 10, 100$.

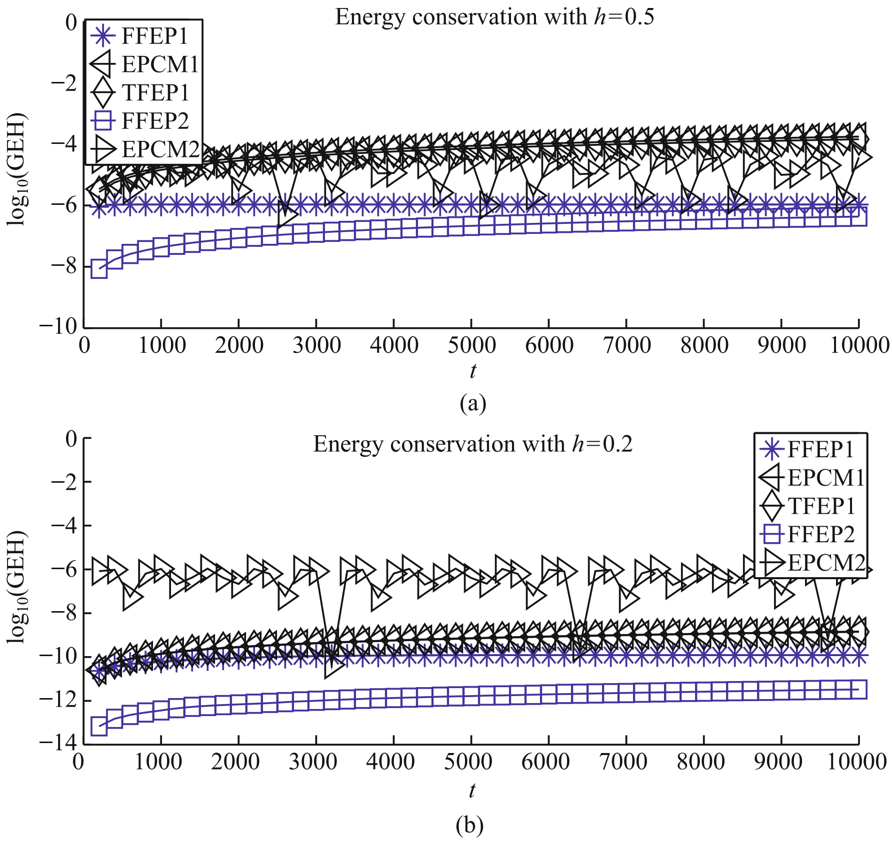

Fig. 4.1 The logarithm of the error of Hamiltonian against t

It is very clear from the numerical results that our FFEP methods when applied to the underlying Euler equations show remarkable numerical behaviour compared with the existing EP methods in the literature.

4.8 Conclusions

The Poisson system is an important model in applications. It is well known that the energy of Poisson system is preserved along its exact solution. This chapter paid attention to the analysis of preserving the energy exactly in the numerical treatment, so that we can obtain $H(y_1) = H(y_0)$ after one step of the method starting from y_0 with a time stepsize h. In this chapter, we presented functionally-fitted energy-preserving integrators for Poisson systems by using a functionally-fitted strategy. It has been shown that these integrators preserve exactly the energy of

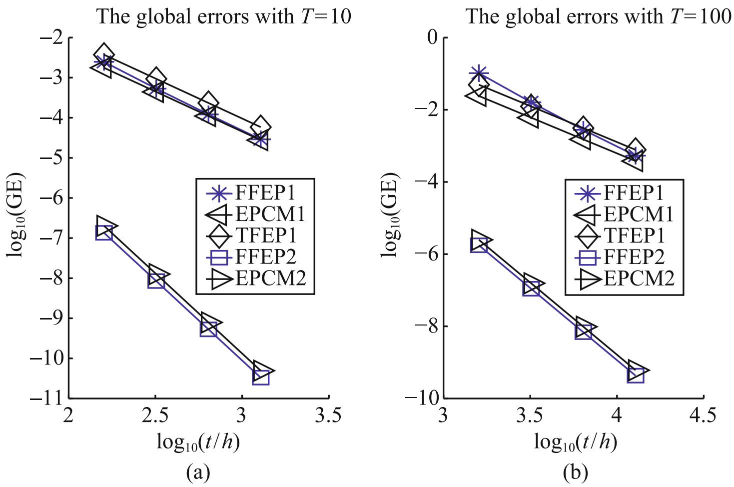

Fig. 4.2 The logarithm of the global error against the logarithm of t/h

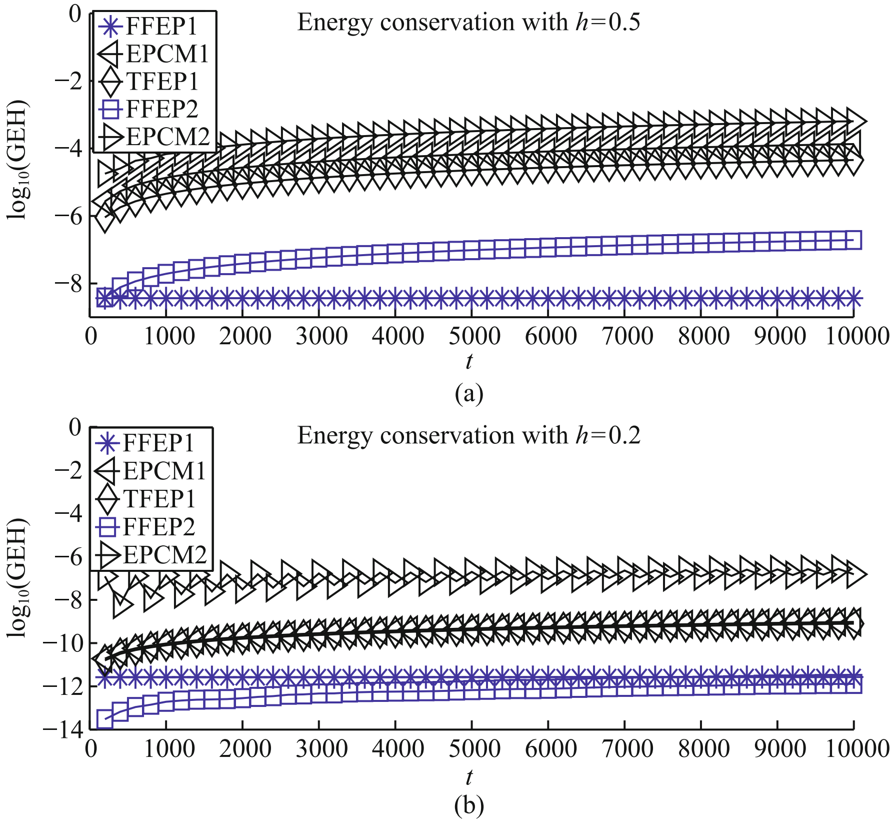

Fig. 4.3 The logarithm of the error of Hamiltonian against t

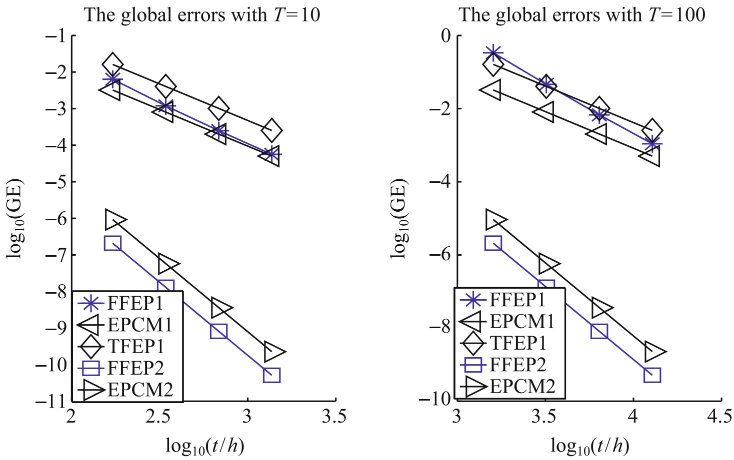

Fig. 4.4 The logarithm of the global error against the logarithm of t/h

Poisson systems and can be of arbitrarily high order by choosing a sufficiently large integer r for the function spaces Y_h and X_h. These integrators contain the energy-preserving schemes given by Cohen and Hairer [1] and Brugnano et al. [27]. The remarkable efficiency and robustness of the integrators were demonstrated through the numerical experiments for the Euler equations. In a similar way, it is possible to develop functionally-fitted energy-diminishing integrators for gradient systems.

The materials in this chapter are based on the work by Wang and Wu [42].

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