Semi-Slant $\xi^{\perp}$-, Hemi-Slant $\xi^{\perp}$-Riemannian Submersions and Quasi Hemi-Slant Submanifolds

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1 Introduction

A differentiable map $\pi : (M, g_M) \longrightarrow (N, g_N)$ between Riemannian manifolds (M, g_M) and (N, g_N) is called a Riemannian submersion if π_* is onto and it satisfies

$$g_N(\pi_* X_1, \pi_* X_2) = g_M(X_1, X_2) \tag{1.1}$$

for X_1, X_2 vector fields tangent to M, where π_* denotes the derivative map. The study of Riemannian submersions were studied by O'Neill [1] and Gray [2] see also [3]. Riemannian submersions have several applications in mathematical physics. Indeed, Riemannian submersions have their applications in the Yang–Mills theory [42, 43], Kaluza–Klein theory [44, 45], supergravity and superstring theories [46, 47] and more. Later, such submersions according to the conditions on the map $\pi : (M, g_M) \longrightarrow (N, g_N)$, we have the following submersions: Riemannian submersions [4], almost Hermitian submersions [5], invariant submersions [6–8], anti-invariant submersions [7–13], lagrangian submersions [14, 15], semi-invariant submersions [16, 17], slant submersions [18–22], semi-slant submersions [23–26],

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quaternionic submersions [27, 28], hemi-slant submersions [29, 30], pointwise slant submersions [31, 32], etc. In [33], Lee defined anti-invariant $\xi^{\perp}$-Riemannian submersions from almost contact metric manifolds and studied the geometry of such maps.

As a generalization of anti-invariant $\xi^{\perp}$-Riemannian submersions, Akyol et al. in [34] defined the notion of semi-invariant $\xi^{\perp}$-Riemannian submersions from almost contact metric manifolds and investigated the geometry of such maps. In 2017, Mehmet et al. [35], as a generalization of anti-invariant $\xi^{\perp}$-Riemannian submersions, semi-invariant $\xi^{\perp}$-Riemannian submersions and slant Riemannian submersions, defined and studied semi-slant $\xi^{\perp}$-Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. Very recently Ramazan Sari and Mehmet Akif Akyol [36] also introduced and studied Hemi-slant $\xi^{\perp}$-submersions and obtained interesting results. On the other hand, in 1996, using Chen's notion on slant submanifold, Lotta [37] introduced the notion of slant submanifold in almost contact metric manifold which was further generalized as semi-slant, hemi-slant and bi-slant submanifolds. Motivated from these studies, Rajendra Prasad et al. introduced and studied quasi hemi-slant submanifolds of cosymplectic manifolds.

The aim of this chapter is to discuss briefly some results of semi-slant $\xi^{\perp}$-submersions [35], hemi-slant $\xi^{\perp}$-submersions [36] and quasi hemi-slant submanifolds [38].

2 Riemannian Submersions

Let (M, g_M) and (N, g_N) be two Riemannian manifolds. A Riemannian submersion $\pi : M \longrightarrow N$ is a map of M onto N satisfying the following axioms:

(i) π has maximal rank, and
(ii) The differential π_* preserves the lenghts of horizontal vectors, that is π_* is a linear isometry.

The geometry of Riemannian submersion is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined as follows:

$$\mathcal{T}(E_1, E_2) = \mathcal{H}\overset{M}{\nabla}_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}\overset{M}{\nabla}_{\mathcal{V}E_1}\mathcal{H}E_2 \tag{2.1}$$

and

$$\mathcal{A}(E_1, E_2) = \mathcal{H}\overset{M}{\nabla}_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}\overset{M}{\nabla}_{\mathcal{H}E_1}\mathcal{H}E_2 \tag{2.2}$$

for any $E_1, E_2 \in \Gamma(M)$, where $\overset{M}{\nabla}$ is the Levi-Civita connection on g_M. Note that we denote the projection morphisms on the vertical distribution and the horizontal distribution by $\mathcal{V}$ and $\mathcal{H}$, respectively. One can easily see that $\mathcal{T}$ is vertical, $\mathcal{T}_{E_1} = \mathcal{T}_{\mathcal{V}E_1}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_{E_1} = \mathcal{A}_{\mathcal{H}E_1}$. We also note that

$$\mathcal{T}_U V = \mathcal{T}_V U \text{ and } \mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y],$$

for $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $U, V \in \Gamma(ker\pi_*)$.

On the other hand, from (2.1) and (2.2), we obtain

$$\overset{M}{\nabla}_V W = \mathcal{T}_V W + \hat{\nabla}_V W; \tag{2.3}$$

$$\overset{M}{\nabla}_V X = \mathcal{T}_V X + \mathcal{H}(\overset{M}{\nabla}_V X); \tag{2.4}$$

$$\overset{M}{\nabla}_X V = \mathcal{V}(\overset{M}{\nabla}_X V) + \mathcal{A}_X V; \tag{2.5}$$

$$\overset{M}{\nabla}_X Y = \mathcal{A}_X Y + \mathcal{H}(\overset{M}{\nabla}_X Y), \tag{2.6}$$

for any $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $V, W \in \Gamma(ker\pi_*)$. Moreover, if X is basic, then $\mathcal{H}(\overset{M}{\nabla}_V X) = \mathcal{A}_X V$. It is easy to see that for $U, V \in \Gamma(ker\pi_*)$, $\mathcal{T}_U V$ coincides with the fibres as the second fundamental form and $\mathcal{A}_X Y$ reflecting the complete integrability of the horizontal distribution.

A vector field on M is called vertical if it is always tangent to fibres. A vector field on M is called horizontal if it is always orthogonal to fibres. A vector field Z on M is called basic if Z is horizontal and π-related to a vector field $\bar{Z}$ on N, i.e., $\pi_* Z_p = \bar{Z}_{\pi_*(p)}$ for all $p \in M$.

Lemma 2.1 (see [1, 3]) *Let $\pi : M \longrightarrow N$ be a Riemannian submersion. If X and Y basic vector fields on M, then we get:*

(i) $g_M(X, Y) = g_N(\bar{X}, \bar{Y}) \circ \pi$,
(ii) $\mathcal{H}[X, Y]$ *is a basic and* $\pi_*\mathcal{H}[X, Y] = [\bar{X}, \bar{Y}] \circ \pi$;
(iii) $\mathcal{H}(\overset{M}{\nabla}_X Y)$ *is a basic, π-related to* $(\overset{N}{\nabla}_{\bar{X}} \bar{Y})$, *where* $\overset{M}{\nabla}$ *and* $\overset{N}{\nabla}$ *are the Levi-Civita connection on M and N;*
(iv) $[X, V] \in \Gamma(ker\pi_*)$ *is vertical, for any* $V \in \Gamma(ker\pi_*)$.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $\pi : M \longrightarrow N$ is a differentiable map. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(X, Y) = \overset{\pi}{\nabla}_X \pi_* Y - \pi_*(\nabla_X Y) \tag{2.7}$$

for $X, Y \in \Gamma(TM)$, where $\overset{\pi}{\nabla}$ is the pull back connection and ∇ is the Levi-Civita connections of the metrics g_M and g_N.

Finally, let (M, g_M) be a $(2m + 1)$-dimensional Riemannian manifold and TM denote the tangent bundle of M. Then M is called an almost contact metric manifold if there exists a tensor φ of type $(1, 1)$ and global vector field ξ and η is a 1-form of ξ, then we have

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1 \tag{2.8}$$

$$\varphi\xi = 0, \quad \eta o \varphi = 0 \quad \text{and} \quad g_M(\varphi X, \varphi Y) = g_M(X, Y) - \eta(X)\eta(Y), \tag{2.9}$$

where X, Y are any vector fields on M. In this case, $(\varphi, \xi, \eta, g_M)$ is called the almost contact metric structure of M. The almost contact metric manifold $(M, \varphi, \xi, \eta, g_M)$

is called a contact metric manifold if

$$\Phi(X, Y) = d\eta(X, Y)$$

for any $X, Y \in \Gamma(TM)$, where Φ is a 2-form in M defined by $\Phi(X, Y) = g_M(X, \varphi Y)$. The 2-form Φ is called the fundamental 2-form of M. A contact metric structure of M is said to be normal if

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is Nijenhuis tensor of φ. Any normal contact metric manifold is called a Sasakian manifold. Moreover, if M is Sasakian [39, 40], then we have

$$(\overset{M}{\nabla}_X \varphi)Y = g_M(X, Y)\xi - \eta(Y)X \text{ and } \overset{M}{\nabla}_X \xi = -\varphi X, \tag{2.10}$$

where $\overset{M}{\nabla}$ is the connection of Levi-Civita covariant differentiation.

3 Semi-slant $\xi^\perp$-Riemannian Submersions

In 2017, Mehmet et al. [35], as a generalization of anti-invariant $\xi^\perp$-Riemannian submersions, semi-invariant $\xi^\perp$-Riemannian submersions and slant Riemannian submersions, defined and studied semi-slant $\xi^\perp$-Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. In this Sect. 3, we will discuss some results of this paper briefly.

Definition 3.1 Let $(M, \varphi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. Suppose that there exists a Riemannian submersion $\pi : M \longrightarrow N$ such that ξ is normal to $ker\pi_*$. Then $\pi : M \longrightarrow N$ is called semi-slant $\xi^\perp$-Riemannian submersion if there is a distribution $D_1 \subseteq \ker \pi_*$ such that

$$ker\pi_* = D_1 \oplus D_2, \quad \varphi(D_1) = D_1, \tag{3.1}$$

and the angle $\theta = \theta(U)$ between φU and the space $(D_2)_p$ is constant for nonzero $U \in (D_2)_p$ and $p \in M$, where D_2 is the orthogonal complement of D_1 in $ker\pi_*$. As it is, the angle θ is called the semi-slant angle of the submersion.

Now, let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, for $U \in \Gamma(\ker \pi_*)$, we put

$$U = \mathcal{P}U + \mathcal{Q}U \tag{3.2}$$

where $\mathcal{P}U \in \Gamma(D_1)$ and $\mathcal{Q}U \in \Gamma(D_2)$. For $Z \in \Gamma(TM)$, we have

$$Z = \mathcal{V}Z + \mathcal{H}Z \tag{3.3}$$

where $\mathcal{V}Z \in \Gamma(ker\pi_*)$ and $\mathcal{H}Z \in \Gamma((ker\pi_*)^{\perp})$. For $V \in \Gamma(ker\pi_*)$, we get

$$\varphi V = \phi V + \omega V \tag{3.4}$$

where ϕV and ωV are vertical and horizontal components of φV, respectively. Similarly, for any $X \in \Gamma((ker\pi_*)^{\perp})$, we have

$$\varphi X = \mathcal{B}X + CX \tag{3.5}$$

where $\mathcal{B}X$ (resp. CX) is the vertical part (resp. horizontal part) of φX. Then the horizontal distribution $(ker\pi_*)^{\perp}$ is decomposed as

$$(\ker \pi_*)^{\perp} = \omega D_2 \oplus \mu, \tag{3.6}$$

here μ is the orthogonal complementary distribution of ωD_2 and it is both invariant distribution of $(ker\pi_*)^{\perp}$ with respect to φ and contains ξ. By (2.9), (3.4) and (3.5), we have

$$g_M(\phi U_1, V_1) = -g_M(U_1, \phi V_1) \tag{3.7}$$

and

$$g_M(\omega U_1, X) = -g_M(U_1, \mathcal{B}X) \tag{3.8}$$

for $U_1, V_1 \in \Gamma(\ker \pi_*)$ and $X \in \Gamma((\ker \pi_*)^{\perp})$. From (3.4), (3.5) and (3.6), we have

Lemma 3.2 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we obtain:*

$$\begin{aligned} &(a)\ \ \phi D_1 = D_1, \quad (b)\ \ \omega D_1 = 0, \\ &(c)\ \ \phi D_2 \subset D_2, \quad (d)\ \ \mathcal{B}(ker\pi_*)^{\perp} = D_2, \\ &(e)\ \ \mathcal{T}_{U_1}\xi = \phi U_1, \quad (f)\ \ \hat{\nabla}_{U_1}\xi = -\omega U_1, \end{aligned}$$

for $U_1 \in \Gamma(\ker \pi_*)$ and $\xi \in \Gamma((ker\pi_*)^{\perp})$.

Using (3.4), (3.5) and the fact that $\varphi^2 = -I + \eta \otimes \xi$, we have

Lemma 3.3 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we get*

$$\begin{aligned} &(i)\ \ \phi^2 + \mathcal{B}\omega = -id, \quad (ii)\ \ C^2 + \omega\mathcal{B} = -id, \\ &(iii)\ \ \omega\phi + C\omega = 0, \quad (iv)\ \ \mathcal{B}C + \phi\mathcal{B} = 0, \end{aligned}$$

where I is the identity operator on the space of π.

Let $(M, \varphi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. Let $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a semi-slant $\xi^{\perp}$-Riemannian submersion. We now examine how the Sasakian structure on M effects the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of a semi-slant $\xi^{\perp}$-Riemannian submersion $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$.

Lemma 3.4 *Let $(M, \varphi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) a Riemannian manifold. Let $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a semi-slant $\xi^{\perp}$-Riemannian submersion. Then we have*

$$\mathcal{B}\mathcal{T}_U V + \phi\hat{\nabla}_U V = \hat{\nabla}_U \phi V + \mathcal{T}_U \omega V, \tag{3.9}$$

$$g_M(U, V)\xi + \mathcal{C}\mathcal{T}_U V + \omega\hat{\nabla}_U V = \mathcal{T}_U \phi V + \mathcal{H}\overset{M}{\nabla}_U \omega V, \tag{3.10}$$

$$\phi\mathcal{T}_U X + \mathcal{B}\overset{M}{\nabla}_U X - \eta(X)U = \hat{\nabla}_U \mathcal{B}X + \mathcal{T}_U \mathcal{C}X, \tag{3.11}$$

$$\omega\mathcal{T}_U X + \mathcal{C}\overset{M}{\nabla}_U X = \mathcal{T}_U \mathcal{B}X + \mathcal{H}\overset{M}{\nabla}_U \mathcal{C}X, \tag{3.12}$$

$$g_M(X, Y)\xi - \omega\mathcal{A}_X Y + \mathcal{C}\mathcal{H}\overset{M}{\nabla}_X Y = \mathcal{A}_X \mathcal{B}Y + \overset{M}{\nabla}_X \mathcal{C}Y + \eta(Y)X, \tag{3.13}$$

$$\phi\mathcal{A}_X Y + \mathcal{B}\mathcal{H}\overset{M}{\nabla}_X Y = \mathcal{V}\overset{M}{\nabla}_X \mathcal{B}Y + \mathcal{A}_X \mathcal{C}Y, \tag{3.14}$$

for all $X, Y \in \Gamma((\ker \pi_*)^{\perp})$ and $U, V \in \Gamma(\ker \pi_*)$.

Proof Given $U, V \in \Gamma(ker\pi_*)$, by virtue of (2.10) and (3.4), we have

$$g_M(U, V)\xi - \eta(V)U = \overset{M}{\nabla}_U \phi V + \overset{M}{\nabla}_U \omega V - \varphi\overset{M}{\nabla}_U V.$$

Making use of (2.3), (2.4), (3.4) and (3.5), we have

$$\begin{aligned} g_M(U, V)\xi &= \mathcal{T}_U \phi V + \hat{\nabla}_U \phi V + \mathcal{T}_U \omega V + \mathcal{H}\overset{M}{\nabla}_U \omega V \\ &\quad - \mathcal{B}\mathcal{T}_U V - \mathcal{C}\mathcal{T}_U V - \phi\hat{\nabla}_U V - \omega\hat{\nabla}_U V. \end{aligned} \tag{3.15}$$

Comparing horizontal and vertical parts, we get (3.9) and (3.10). The other assertions can be obtained in a similar method. □

Theorem 3.5 *Let $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\phi^2 W = -\cos^2\theta W, \quad W \in \Gamma(D_2), \tag{3.16}$$

where θ denotes the semi-slant angle of D_2.

Lemma 3.6 *Let $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold*

(N, g_N) *with a semi-slant angle* θ. *Then we have*

$$g_M(\phi W_1, \phi W_2) = \cos^2\theta g_M(W_1, W_2), \tag{3.17}$$
$$g_M(\omega W_1, \omega W_2) = \sin^2\theta g_M(W_1, W_2), \tag{3.18}$$

for any $W_1, W_2 \in \Gamma(D_2)$.

3.1 Integrable and Parallel Distributions

In this section, we will discuss integrability conditions of the distributions involved in the definition of a semi-slant $\xi^{\perp}$-Riemannian submersion. First, we have

Theorem 3.7 *Let* π *be a semi-slant* $\xi^{\perp}$*-Riemannian submersion from a Sasakian manifold* $(M, \varphi, \xi, \eta, g_M)$ *onto a Riemannian manifold* (N, g_N) *with a semi-slant angle* θ. *Then:*

(i) D_1 *is integrable* $\Leftrightarrow$ $(\nabla\pi_*)(U, \varphi V) - (\nabla\pi_*)(V, \varphi U) \notin \Gamma(\pi_*\mu)$
(ii) D_2 *is integrable* $\Leftrightarrow$ $g_N(\pi_*\omega W, (\nabla\pi_*)(Z, \varphi U)) + g_N(\pi_*\omega Z, (\nabla\pi_*)(W, \varphi U)) = g_M(\phi W, \hat{\nabla}_Z\varphi U) + g_M(\phi Z, \hat{\nabla}_W\varphi U)$

for $U, V \in \Gamma(D_1)$ *and* $Z, W \in \Gamma(D_2)$.

Proof For $U, V \in \Gamma(D_1)$ and $X \in \Gamma((ker\pi_*)^{\perp})$, since $[U, V] \in \Gamma(ker\pi_*)$, we have $g_M([U, V], X) = 0$. Thus, D_1 is integrable $\Leftrightarrow$ $g_M([U, V], Z) = 0$ for $Z \in \Gamma(D_2)$. Since M is a Sasakian manifold, by (2.9) and (2.10), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_U V, Z) &= g_M(\overset{M}{\nabla}_U \varphi V - g_M(U, V)\xi - \eta(V)U, \varphi Z) \\ &= g_M(\overset{M}{\nabla}_U \varphi V, \varphi Z). \end{aligned} \tag{3.19}$$

Using (3.4) in (3.19), we get

$$g_M([U, V], Z) = -g_M(\overset{M}{\nabla}_U V, \varphi\phi Z) + g_M(\mathcal{H}\overset{M}{\nabla}_U \varphi V, wZ) - g_M(\overset{M}{\nabla}_V U, \varphi\phi Z) - g_M(\mathcal{H}\overset{M}{\nabla}_V \varphi U, wZ).$$

Now, by using (2.7) and (3.16), we get

$$\begin{aligned} g_M([U, V], Z) = \cos^2\theta g_M(\overset{M}{\nabla}_U V, Z) - g_N((\nabla\pi_*)(U, \varphi V) + \nabla^{\pi}_U \pi_*\varphi V, \pi_* wZ) \\ - \cos^2\theta g_M(\overset{M}{\nabla}_V U, Z) + g_N((\nabla\pi_*)(V, \varphi U) + \nabla^{\pi}_V \pi_*\varphi U, \pi_* wZ). \end{aligned}$$

Thus, we have

$$(\sin^2\theta) g_M([U, V], Z) = -g_N((\nabla\pi_*)(U, \varphi V) - (\nabla\pi_*)(V, \varphi U), \pi_* wZ),$$

□

which completes the proof.

Now for the geometry of leaves of D_1, we have

Theorem 3.8 *Let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the distribution D_1 is parallel if and only if*

$$g_N((\nabla\pi_*)(U, \varphi V), \pi_*\omega Z) = g_M(\mathcal{T}_U\omega\phi Z, V) \tag{3.20}$$

and

$$-g_N((\nabla\pi_*)(U, \varphi V), \pi_*\mathcal{C}X) = g_M(V, \hat{\nabla}_U\phi\mathcal{B}X + \mathcal{T}_U\omega\mathcal{B}X) + g_M(V, \varphi U)\eta(X) \tag{3.21}$$

for $U, V \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $X \in \Gamma((\ker\pi_*)^\perp)$.

Proof Making use of (3.19), (3.4) and (2.3), for $U, V \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we have

$$g_M(\overset{M}{\nabla}_U V, Z) = -g_M(\overset{M}{\nabla}_U V, \phi^2 Z) - g_M(\overset{M}{\nabla}_U V, \omega\phi Z) + g_M(\mathcal{H}\overset{M}{\nabla}_U \varphi V, \omega Z).$$

By virtue of (2.7) and (3.16), we get

$$g_M(\overset{M}{\nabla}_U V, Z) = \cos^2\theta g_M(\overset{M}{\nabla}_U V, Z) - g_M(\mathcal{T}_U V, w\phi Z) + g_N((\nabla\pi_*)(U, \varphi V), \pi_*(wZ))$$

or

$$\sin^2\theta g_M(\overset{M}{\nabla}_U V, Z) = -g_M(\mathcal{T}_U w\phi Z, V) + g_N((\nabla\pi_*)(U, \varphi V), \pi_*(wZ)),$$

which gives (3.20). On the other hand, from (2.9) and (2.10), we have

$$g_M(\overset{M}{\nabla}_U V, X) = g_M(\overset{M}{\nabla}_U \varphi V, \varphi X) + g_M(V, \varphi U)\eta(X)$$

for $U, V \in \Gamma(D_1)$ and $X \in \Gamma((\ker\pi_*)^\perp)$. By using (3.5), we obtain

$$g_M(\overset{M}{\nabla}_U V, X) = g_M(V, \overset{M}{\nabla}_U \phi\mathcal{B}X) + g_M(V, \overset{M}{\nabla}_U \omega\mathcal{B}X) + g_M(\mathcal{C}X, \mathcal{H}\overset{M}{\nabla}_U \varphi V) + g_M(V, \varphi U)\eta(X).$$

Taking into account of (2.3), we write

$$\begin{aligned} g_M(\overset{M}{\nabla}_U V, X) &= g_M(V, \mathcal{T}_U\phi\mathcal{B}X + \hat{\nabla}_U\phi\mathcal{B}X) + g_M(V, \mathcal{T}_U\omega\mathcal{B}X + \mathcal{H}\overset{M}{\nabla}_U\omega\mathcal{B}X) \\ &\quad - g_N(\pi_*(\mathcal{C}X), \pi_*(\mathcal{H}\overset{M}{\nabla}_U\varphi V)) + g_M(V, \varphi U)\eta(X) \end{aligned}$$

hence,

$$g_M(\overset{M}{\nabla}_U V, X) = g_M(V, \hat{\nabla}_U \phi \mathcal{B}X) + g_M(V, \mathcal{T}_U \omega \mathcal{B}X) + g_N((\nabla \pi_*)(U, \varphi V), \pi_* CX) + g_M(V, \varphi U)\eta(X).$$

which gives (3.21). This completes the assertion. □

Similarly for D_2, we have:

Theorem 3.9 *Let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the distribution D_2 is parallel if and only if*

$$g_N(\pi_* \omega W, (\nabla \pi_*)(Z, \varphi U)) = g_M(\phi W, \hat{\nabla}_Z \varphi U) \tag{3.22}$$

and

$$g_N((\nabla \pi_*)(Z, \omega W), \pi_*(X)) - g_N((\nabla \pi_*)(Z, \omega \phi W), \pi_*(X)) = g_M(\mathcal{T}_Z \omega W, \mathcal{B}X) + g_M(W, \varphi Z)\eta(X) \tag{3.23}$$

for any $Z, W \in \Gamma(D_2)$, $U \in \Gamma(D_1)$ and $X \in \Gamma((ker\pi_*)^\perp)$.

Theorem 3.10 *Let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the distribution $(\ker \pi_*)^\perp$ is integrable if and only if*

$$g_N((\nabla \pi_*)(Y, \phi V), \pi_*(X)) + g_N((\nabla \pi_*)(X, \phi V), \pi_*(X)) = g_M(\phi V, \mathcal{V}(\overset{M}{\nabla}_X \mathcal{B}Y + \overset{M}{\nabla}_Y \mathcal{B}X)) \tag{3.24}$$

and

$$g_N((\nabla \pi_*)(X, CY) - (\nabla \pi_*)(Y, CX), \pi_* \omega W) = g_M(\mathcal{A}_X \mathcal{B}Y + \mathcal{A}_Y \mathcal{B}X, \omega W) + \eta(Y) g_M(X, \omega W) - \eta(X) g_M(Y, \omega W) \tag{3.25}$$

for $X, Y \in \Gamma((ker\pi_*)^\perp)$, $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$.

Proof Using (3.19), (2.9) and (2.10), we have for $X, Y \in \Gamma((\ker \pi_*)^\perp)$ and $V \in \Gamma(D_1)$.

$$g_M([X, Y], V) = g_M(\overset{M}{\nabla}_X \varphi Y, \varphi V) - g_M(\overset{M}{\nabla}_Y \varphi X, \varphi V).$$

Now, by using (3.5), we obtain

$$g_M([X, Y], V) = -g_M(\mathcal{B}Y, \overset{M}{\nabla}_X \varphi V) - g_M(CY, \overset{M}{\nabla}_X \varphi V) + g_M(\mathcal{B}X, \overset{M}{\nabla}_Y \varphi V) + g_M(CX, \overset{M}{\nabla}_Y \varphi V).$$

By using (2.5) and taking into account of the property of the map, we have

$$g_M([X,Y],V) = g_M(\varphi V, \mathcal{A}_Y\mathcal{B}X + \mathcal{V}\overset{M}{\nabla}_X\mathcal{B}Y) - g_N(\pi_*(\mathcal{C}Y), \pi_*(\overset{M}{\nabla}_X\varphi V))$$
$$- g_M(\varphi V, \mathcal{A}_X\mathcal{B}Y + \mathcal{V}\overset{M}{\nabla}_Y\mathcal{B}X) - g_N(\pi_*(\mathcal{C}X), \pi_*(\overset{M}{\nabla}_Y\varphi V)).$$

Thus, we have

$$g_M([X,Y],V) = g_M(\varphi V, \mathcal{V}(\overset{M}{\nabla}_X\mathcal{B}Y - \overset{M}{\nabla}_Y\mathcal{B}X)) + g_N(\pi_*(\mathcal{C}Y), (\nabla\pi_*)(X,\varphi V))$$
$$- g_N(\pi_*(\mathcal{C}X), (\nabla\pi_*)(Y,\varphi V)),$$

which gives (3.24). In a similar way, by virtue of (3.19), (2.9) and (2.10), we have for $X, Y \in \Gamma((\ker\pi_*)^\perp)$ and $W \in \Gamma(D_2)$,

$$g_M([X,Y],W) = g_M(\varphi\overset{M}{\nabla}_X Y, \phi W) + g_M(\varphi\overset{M}{\nabla}_X Y, \omega W) + \eta(Y)g_M(X,\omega W)$$
$$- g_M(\varphi\overset{M}{\nabla}_Y X, \phi W) - g_M(\varphi\overset{M}{\nabla}_Y X, \omega W) - \eta(X)g_M(Y,\omega W).$$

By virtue of (3.5) and (3.6), we have

$$g_M([X,Y],W) = -g_M(\overset{M}{\nabla}_X Y, \phi^2 W) - g_M(\overset{M}{\nabla}_X Y, \omega\phi W) + g_M(\overset{M}{\nabla}_X \mathcal{B}Y, \omega W) + g_M(\overset{M}{\nabla}_X \mathcal{C}Y, \omega W)$$
$$- g_M(\overset{M}{\nabla}_Y X, \phi^2 W) - g_M(\overset{M}{\nabla}_Y X, \omega\phi W) + g_M(\overset{M}{\nabla}_Y \mathcal{B}X, \omega W) + g_M(\overset{M}{\nabla}_Y \mathcal{C}X, \omega W)$$
$$+ \eta(Y)g_M(X,\omega W) - \eta(X)g_M(Y,\omega W).$$

Now, by using (3.16) and the property of the map, we get

$$g_M([X,Y],W) = \cos^2\theta g_M([X,Y],W) + g_N((\nabla\pi_*)(X,Y), \omega\phi W) + g_M(\mathcal{A}_X\mathcal{B}Y, \omega W)$$
$$- g_N((\nabla\pi_*)(X,\mathcal{C}Y), \pi_*\omega W) - g_N((\nabla\pi_*)(Y,X), \omega\phi W) + g_M(\mathcal{A}_Y\mathcal{B}X, \omega W)$$
$$+ g_N((\nabla\pi_*)(Y,\mathcal{C}X), \pi_*\omega W) + \eta(Y)g_M(X,\omega W) - \eta(X)g_M(Y,\omega W).$$

Thus, we have

$$\sin^2\theta g_M([X,Y],W) = g_N((\nabla\pi_*)(Y,\mathcal{C}X) - (\nabla\pi_*)(X,\mathcal{C}Y), \pi_*\omega W) + g_M(\mathcal{A}_X\mathcal{B}Y + \mathcal{A}_Y\mathcal{B}X, \omega W)$$
$$+ \eta(Y)g_M(X,\omega W) - \eta(X)g_M(Y,\omega W),$$

which gives (3.25). This completes the proof. □

For the geometry of leaves $(\ker\pi_*)^\perp$, we have

Theorem 3.11 *Let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold (M,φ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with a semi-slant angle θ. Then the distribution $(\ker\pi_*)^\perp$ is parallel if and only if*

$$g_M(V, \mathcal{V}\overset{M}{\nabla}_X\phi\mathcal{B}Y + \mathcal{A}_X\omega\mathcal{B}Y) = g_N(\pi_*(\mathcal{C}Y), (\nabla\pi_*)(X,\varphi V)) \quad (3.26)$$

and

$$g_M(\mathcal{A}_X\omega W, \mathcal{B}Y) + \eta(Y)g_M(X, \omega W) = g_N((\nabla\pi_*)(X, Y), \pi_*\omega\phi W) - g_N((\nabla\pi_*)(X, CY), \pi_*\omega W), \quad (3.27)$$

for $X, Y \in \Gamma((\ker\pi_*)^{\perp})$, $V \in \Gamma(D_1)$ *and* $W \in \Gamma(D_2)$.

Theorem 3.12 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the distribution $(ker\pi_*)$ is parallel if and only if*

$$g_M(\omega V, \mathcal{T}_U\mathcal{B}X) + g_M(V, \phi U)\eta(X) = g_N((\nabla\pi_*)(U, CX), \pi_*\omega V) - g_N((\nabla\pi_*)(U, X), \pi_*\omega\phi V) \quad (3.28)$$

for any $U \in \Gamma(D_1)$, $V \in \Gamma(D_2)$ and $X \in \Gamma((ker\pi_*)^{\perp})$.

By virtue of Theorems 3.8, 3.9 and 3.11, we have the following theorem;

Theorem 3.13 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the total space M is a locally product manifold of the leaves of D_1, D_2 and $(ker\pi_*)^{\perp}$, i.e., $M = M_{D_1} \times M_{D_2} \times M_{(ker\pi_*)^{\perp}}$, if and only if*

$$g_N((\nabla\pi_*)(U, \varphi V), \pi_*\omega Z) = g_M(\mathcal{T}_U\omega\phi Z, V),$$

$$-g_N((\nabla\pi_*)(U, \varphi V), \pi_*CX) = g_M(V, \hat{\nabla}_U\phi\mathcal{B}X + \mathcal{T}_U\omega\mathcal{B}X) + g_M(V, \varphi U)\eta(X),$$

$$g_N(\pi_*\omega W, (\nabla\pi_*)(Z, \varphi U)) = g_M(\phi W, \hat{\nabla}_Z\varphi U),$$

$$g_N((\nabla\pi_*)(Z, \omega W), \pi_*(X)) - g_N((\nabla\pi_*)(Z, \omega\phi W), \pi_*(X)) = g_M(\mathcal{T}_Z\omega W, \mathcal{B}X) + g_M(W, \varphi Z)\eta(X)$$

and

$$g_M(V, \mathcal{V}\overset{M}{\nabla}_X\phi\mathcal{B}Y + \mathcal{A}_X\omega\mathcal{B}Y) = g_N(\pi_*(CY), (\nabla\pi_*)(X, \varphi V)),$$

$$g_M(\mathcal{A}_X\omega W, \mathcal{B}Y) + \eta(Y)g_M(X, \omega W) = g_N((\nabla\pi_*)(X, Y), \pi_*\omega\phi W) - g_N((\nabla\pi_*)(X, CY), \pi_*\omega W)$$

for $X, Y \in \Gamma((\ker\pi_*)^{\perp})$, $U, V \in \Gamma(D_1)$ *and* $Z, W \in \Gamma(D_2)$.

From Theorems 3.11 to 3.12, we have the following theorem;

Theorem 3.14 *Let* $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ *be a semi-slant* $\xi^\perp$*-Riemannian submersion from a Sasakian manifold* $(M, \varphi, \xi, \eta, g_M)$ *onto a Riemannian manifold* (N, g_N) *with a semi-slant angle* θ. *Then the total space* M *is a locally (usual) product manifold of the leaves of* $ker\pi_*$ *and* $(ker\pi_*)^\perp$, *i.e.,* $M = M_{ker\pi_*} \times M_{(ker\pi_*)^\perp}$, *if and only if*

$$g_M(V, \mathcal{V}\overset{M}{\nabla}_X \phi\mathcal{B}Y + \mathcal{A}_X\omega\mathcal{B}Y) = g_N(\pi_*(CY), (\nabla\pi_*)(X, \varphi V)),$$

$$\begin{aligned} g_M(\mathcal{A}_X\omega W, \mathcal{B}Y) + \eta(Y)g_M(X, \omega W) = {} & g_N((\nabla\pi_*)(X, Y), \pi_*\omega\phi W) \\ & - g_N((\nabla\pi_*)(X, CY), \pi_*\omega W) \end{aligned}$$

and

$$\begin{aligned} g_M(\omega V, \mathcal{T}_U\mathcal{B}X) + g_M(V, \phi U)\eta(X) = {} & g_N((\nabla\pi_*)(U, CX), \pi_*\omega V) \\ & - g_N((\nabla\pi_*)(U, X), \pi_*\omega\phi V) \end{aligned}$$

for $X, Y \in \Gamma((\ker\pi_*)^\perp)$, $U, V \in \Gamma(D_1)$ *and* $W \in \Gamma(D_2)$.

3.2 Totally Geodesic Semi-Slant $\xi^\perp$*-Submersions*

Recall that a differential map π between two Riemannian manifolds is called totally geodesic if $\nabla\pi_* = 0$ [41]. Then we have

Theorem 3.15 *Let* π *be a semi-slant* $\xi^\perp$*-Riemannian submersion from a Sasakian manifold* $(M, \varphi, \xi, \eta, g_M)$ *onto a Riemannian manifold* (N, g_N) *with a semi-slant angle* θ. *Then* π *is a totally geodesic map if*

$$\begin{aligned} -\nabla^\pi_X\pi_*Z_2 = {} & \pi_*(C(\mathcal{H}\overset{M}{\nabla}_X\omega Z_1 - \mathcal{A}_X\phi Z_1 + \mathcal{A}_X\mathcal{B}Z_2 + \mathcal{H}\overset{M}{\nabla}_X CZ_2) \\ & + \omega(\mathcal{A}_X\omega Z_1 - \mathcal{V}\overset{M}{\nabla}_X\phi Z_1 + \mathcal{V}\overset{M}{\nabla}_X\mathcal{B}Z_2 + \mathcal{A}_X CZ_2) \\ & - \eta(Z_2)CX - \eta(X)\eta(Z_2) - g_M(Y, CX)\xi) \end{aligned} \tag{3.29}$$

for any $X \in \Gamma((ker\pi_*)^\perp)$ *and* $Z = Z_1 + Z_2 \in \Gamma(TM)$, *where* $Z_1 \in \Gamma(ker\pi_*)$ *and* $Z_2 \in \Gamma((ker\pi_*)^\perp)$.

Proof Making use of (2.5), (2.9) and (2.10), we have

$$\overset{M}{\nabla}_X Z = \varphi(\overset{M}{\nabla}_X\varphi)Z - \varphi\overset{M}{\nabla}_X\varphi Z + \eta(\overset{M}{\nabla}_X Z)\xi$$

for any $Z \in \Gamma((ker\pi_*)^\perp)$ and $X \in \Gamma(TM)$. Now, from (2.7), we have

$$\begin{aligned}(\nabla\pi_*)(X,Z) &= \nabla^{\pi}_{X}\pi_*Z + \pi_*(\varphi\overset{M}{\nabla}_{X}\varphi Z - \varphi(\overset{M}{\nabla}_{X}\varphi)Z - \eta(\overset{M}{\nabla}_{X}Z)\xi)\\ &= \nabla^{\pi}_{X}\pi_*Z + \pi_*(\varphi(\overset{M}{\nabla}_{X}\varphi Z_1 + \overset{M}{\nabla}_{X}\varphi Z_2) - \eta(Z)\varphi X - \eta(\overset{M}{\nabla}_{X}Z)\xi).\end{aligned}$$

Or,

$$\begin{aligned}(\nabla\pi_*)(X,Z) = \nabla^{\pi}_{X}\pi_*Z_2 + \pi_*(&\mathcal{B}\mathcal{A}_X\phi Z_1 + \mathcal{C}\mathcal{A}_X\phi Z_1 + \phi\mathcal{V}\overset{M}{\nabla}_{X}\phi Z_1 + \omega\mathcal{V}\overset{M}{\nabla}_{X}\phi Z_1\\ &+ \phi\mathcal{A}_X\omega Z_1 + \omega\mathcal{A}_X\omega Z_1 + \mathcal{B}\mathcal{H}\overset{M}{\nabla}_{X}\omega Z_1 + \mathcal{C}\mathcal{H}\overset{M}{\nabla}_{X}\omega Z_1\\ &+ \mathcal{B}\mathcal{A}_X\mathcal{B}Z_2 + \mathcal{C}\mathcal{A}_X\mathcal{B}Z_2 + \phi\mathcal{V}\overset{M}{\nabla}_{X}\mathcal{B}Z_2 + \omega\mathcal{V}\overset{M}{\nabla}_{X}\mathcal{B}Z_2\\ &+ \phi\mathcal{A}_X\mathcal{C}Z_2 + \omega\mathcal{A}_X\mathcal{C}Z_2 + \mathcal{B}\mathcal{H}\overset{M}{\nabla}_{X}\mathcal{C}Z_2 + \mathcal{C}\mathcal{H}\overset{M}{\nabla}_{X}\mathcal{C}Z_2\\ &- \eta(Z_2)\varphi X - \eta(X)\eta(Z_2) - g_M(Z_2,\mathcal{C}X)\xi)\end{aligned}$$

for any $Z = Z_1 + Z_2 \in \Gamma(TM)$, where $Z_1 \in \Gamma(ker\pi_*)$ and $Z_2 \in \Gamma((ker\pi_*)^{\perp})$.

$$\begin{aligned}(\nabla\pi_*)(X,Z) = \nabla^{\pi}_{X}\pi_*Z_2 + \pi_*(&\mathcal{C}(\mathcal{A}_X\phi Z_1 + \mathcal{H}\overset{M}{\nabla}_{X}\omega Z_1 + \mathcal{A}_X\mathcal{B}Z_2 + \mathcal{H}\overset{M}{\nabla}_{X}\mathcal{C}Z_2)\\ &+ \omega(\mathcal{V}\overset{M}{\nabla}_{X}\phi Z_1 + \mathcal{A}_X\omega Z_1 + \mathcal{V}\overset{M}{\nabla}_{X}\mathcal{B}Z_2 + \mathcal{A}_X\mathcal{C}Z_2)\\ &- \eta(Z_2)\mathcal{C}X - \eta(X)\eta(Z_2) - g_M(Z_2,\mathcal{C}X)\xi),\end{aligned}$$

which gives (3.29). This completes the assertion. □

Theorem 3.16 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold (M,φ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with a semi-slant angle θ. Then π is a totally geodesic map if and only if*

(i) $g_M(\hat{\nabla}_{U_1}\varphi V_1,\mathcal{B}Z) = g_M(\mathcal{T}_{U_1}\mathcal{C}Z,\varphi V_1) - g_M(V_1,\phi U_1)\eta(Z),$

(ii) $(g_N(\nabla\pi_*(U_2,\omega\phi V_2)) + g_N(\nabla\pi_*(U_2,\omega V_2))),\pi_*Z = g_M(\mathcal{T}_{U_2}\omega V_2,\mathcal{B}Z) + g_M(V_2,\phi U_2)\eta(Z)$

(iii) $g_N(\nabla\pi_*(U,\mathcal{C}X),\pi_*\mathcal{C}Y) - g_N(\nabla\pi_*(U,\omega\mathcal{B}X),\pi_*Y) = g_M(\mathcal{T}_U\phi\mathcal{B}X,Y) - g_M(\mathcal{T}_U\mathcal{C}X,\mathcal{B}Y)$
$+\eta(X)g_M(\mathcal{Q}U,\varphi Y) - \eta(Y)[U\eta(X) + g_M(X,\omega U)]$

for any $U_1,V_1 \in \Gamma(D_1)$, $U_2,V_2 \in \Gamma(D_2)$, $U \in \Gamma(ker\pi_*)$ and $X,Y,Z \in \Gamma((ker\pi_*)^{\perp})$.

Theorem 3.17 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold (M,φ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with a semi-slant angle θ. Then π is a totally geodesic map if and only if*

(i) $\mathcal{C}(\mathcal{T}_U\phi V + \overset{M}{\nabla}_{U}\omega V) + \omega(\hat{\nabla}_U\phi V + \mathcal{T}_U\omega V) + g_M(\mathcal{P}V,\phi U)\xi = 0.$

(ii) $\mathcal{C}(\mathcal{A}_X\phi U + \mathcal{H}\overset{M}{\nabla}_{X}\omega U) + \omega(\mathcal{A}_X\omega U + \mathcal{V}\overset{M}{\nabla}_{X}\phi U) + g_M(\mathcal{Q}U,\mathcal{B}X)\xi = 0.$

(iii) $\mathcal{C}(\mathcal{T}_{U_1}\phi V_1 + \mathcal{H}\overset{M}{\nabla}_{U_1}\phi V_1) + \omega(\mathcal{T}_{U_1}\omega V_1 + \mathcal{V}\overset{M}{\nabla}_{U_1}\phi V_1) = 0,$

for $U_1 \in \Gamma(D_1)$, $V_1 \in \Gamma(D_2)$, $U,V \in \Gamma(ker\pi_*)$ and $X \in \Gamma((ker\pi_*)^{\perp})$.

3.3 Some Examples

Example 3.18 Every invariant submersion from a Sasakian manifold to a Riemannian manifold is a semi-slant $\xi^{\perp}$-Riemannian submersion with $D_2 = \{0\}$ and $\theta = 0$.

Example 3.19 Every slant Riemannian submersion from a Sasakian manifold to a Riemannian manifold is a semi-slant $\xi^{\perp}$-Riemannian submersion with $D_1 = \{0\}$.

Now, we construct some non-trivial examples of semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold. Let $(\mathbb{R}^{2n+1}, g, \varphi, \xi, \eta)$ denote the manifold $\mathbb{R}^{2n+1}$ with its usual Sasakian structure given by

$$\varphi(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) = \sum_{i=1}^{n}(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i})$$

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \ \ \xi = 2\frac{\partial}{\partial z},$$

where $(x^1, ..., x^n, y^1, ..., y^n, z)$ are the Cartesian coordinates. Throughout this section, we will use this notation.

Example 3.20 Let F be a submersion defined by

$$\begin{array}{llll} F: & \mathbb{R}^9 & \longrightarrow & \mathbb{R}^5 \\ & (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) & & (\frac{x_1+x_2}{\sqrt{2}}, \frac{y_1+y_2}{\sqrt{2}}, sin\alpha x_3 - cos\alpha x_4, y_4, z) \end{array}$$

with $\alpha \in (0, \frac{\pi}{2})$. Then it follows that

$$\begin{aligned} ker F_* = span\{Z_1 = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2},\ Z_2 = \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}, \\ Z_3 = -\cos\alpha\frac{\partial}{\partial x^3} - \sin\alpha\frac{\partial}{\partial x^4}, Z_4 = \frac{\partial}{\partial y^3}\} \end{aligned}$$

and

$$\begin{aligned} (ker F_*)^{\perp} = span\{H_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2},\ H_2 = \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2},\ H_3 = \sin\alpha\frac{\partial}{\partial x^3} - \cos\alpha\frac{\partial}{\partial x^4}, \\ H_4 = \frac{\partial}{\partial y^4},\ H_5 = \frac{\partial}{\partial z} = \xi\}. \end{aligned}$$

Hence, we have $\varphi Z_1 = -Z_2$, $\varphi Z_2 = Z_1$. Thus, it follows that $D_1 = span\{Z_1, Z_2\}$ and $D_2 = span\{Z_3, Z_4\}$ is a slant distribution with slant angle $\theta = \alpha$. Thus, F is a semi-slant submersion with semi-slant angle θ. Also, by direct computations, we obtain

$$g_N(F_*H_1, F_*H_1) = g_M(H_1, H_1),\ g_N(F_*H_2, F_*H_2) = g_M(H_2, H_2),$$

$$g_N(F_*H_3, F_*H_3) = g_M(H_3, H_3),\ g_N(F_*H_4, F_*H_4) = g_M(H_4, H_4),\ g_N(F_*\xi, F_*\xi) = g_M(\xi, \xi)$$

where g_M and g_N denote the standard metrics (inner products) of $\mathbb{R}^9$ and $\mathbb{R}^5$. Thus, F is a semi-slant $\xi^\perp$-Riemannian submersion.

Example 3.21 Let F be a submersion defined by

$$\begin{array}{llll} F: & \mathbb{R}^7 & \longrightarrow & \mathbb{R}^3 \\ & (x_1, x_2, x_3, y_1, y_2, y_3, z) & & (\frac{x_2 - y_3}{\sqrt{2}}, y_2, z). \end{array}$$

Then the submersion F is a semi-slant $\xi^\perp$-Riemannian submersion such that $D_1 = span(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1})$ and $D_2 = span(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_3}, \frac{\partial}{\partial x_3})$ with semi-slant angle $\alpha = \frac{\pi}{4}$.

Example 3.22 Let F be a submersion defined by

$$\begin{array}{llll} F: & \mathbb{R}^9 & \longrightarrow & \mathbb{R}^3 \\ & (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) & & (sin\alpha x_3 - cos\alpha x_4, y_4, z) \end{array}$$

with $\alpha \in (0, \frac{\pi}{2})$. Then the submersion F is a semi-slant $\xi^\perp$-Riemannian submersion such that $D_1 = span(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2})$ and $D_2 = span(-\cos\alpha \frac{\partial}{\partial x_3} - \sin\alpha \frac{\partial}{\partial x_4}, \frac{\partial}{\partial y_3})$ with semi-slant angle $\theta = \alpha$.

Example 3.23 Let F be a submersion defined by

$$\begin{array}{llll} F: & \mathbb{R}^{13} & \longrightarrow & \mathbb{R}^7 \\ & (x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6, z) & & (\frac{x_1 - x_2}{\sqrt{2}}, \frac{y_1 - y_2}{\sqrt{2}}, \frac{x_3 + x_4}{\sqrt{2}}, \frac{y_3 + y_4}{\sqrt{2}}, \frac{x_5 - x_6}{\sqrt{2}}, y_5, z). \end{array}$$

Then the submersion F is a semi-slant $\xi^\perp$-Riemannian submersion such that $D_1 = span(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_3} - \frac{\partial}{\partial y_4})$ and $D_2 = span(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}, \frac{\partial}{\partial y_6})$ with semi-slant angle $\alpha = \frac{\pi}{4}$.

4 Hemi-Slant $\xi^\perp$-Riemannian Submersions

Very recently Ramazan Sarıand Mehmet Akif Akyol [36] also introduced and studied hemi-slant $\xi^\perp$-submersions and obtained interesting results. In this Sect. 4, our aim is to discuss briefly some results of this paper.

Definition 4.1 Let $(M, \varphi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. Suppose that there exists a Riemannian submersion $\phi : M \longrightarrow N$ such that ξ is normal to $ker\phi_*$. Then ϕ is called a hemi-slant $\xi^\perp$-Riemannian submersion if the vertical distribution $ker\phi_*$ of ϕ admits two orthogonal complementary distributions $\mathcal{D}_\perp$ and $\mathcal{D}_\theta$ such that $\mathcal{D}_\perp$ is anti-invariant and $\mathcal{D}_\theta$ is slant, i.e., we have

$$\ker\phi_* = \mathcal{D}_\perp \oplus \mathcal{D}_\theta.$$

In this case, the angle θ is called the slant angle of the hemi-slant $\xi^\perp$-Riemannian submersion.

If $\theta \neq 0, \frac{\pi}{2}$ then we say that the submersion is proper hemi-slant $\xi^\perp$-Riemannian submersion. Now, we are going to give some proper examples in order to guarantee the existence of hemi-slant $\xi^\perp$-Riemannian submersions in Sasakian manifolds and demonstrate that the method presented in this paper is effective. Note that, $(\mathbb{R}^{2n+1}, \varphi, \eta, \xi, g_{\mathbb{R}^{2n+1}})$ will denote the manifold $\mathbb{R}^{2n+1}$ with its usual contact structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \quad \xi = 2\frac{\partial}{\partial z},$$

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\varphi(\sum_{i=1}^{n}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z) = \sum_{i=1}^{n}(Y_i\partial x^i - X_i\partial y^i)$$

where $(x_1, .., x_n, y_1, ..., y_n, z)$ denotes the Cartesian coordinates on $\mathbb{R}^{2n+1}$.

Example 4.2 Every anti-invariant $\xi^\perp$-Riemannian submersion from a Sasakian manifold onto a Riemannian manifold is a hemi-slant $\xi^\perp$-Riemannian submersion with $\mathcal{D}_\theta = \{0\}$.

Example 4.3 Every slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold onto a Riemannian manifold is a hemi-slant $\xi^\perp$-Riemannian submersion with $\mathcal{D}_\perp = \{0\}$.

Example 4.4 Let ϕ be a submersion defined by

$$\begin{array}{llll} \phi : & (\mathbb{R}^9, g_{\mathbb{R}^9}) & \to & (\mathbb{R}^5, g_{\mathbb{R}^5}) \\ & (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) & & (\frac{x_1+y_2}{\sqrt{2}}, \frac{x_2+y_1}{\sqrt{2}}, \sin\gamma x_3 - \cos\gamma x_4, y_4, z) \end{array}$$

with $\gamma \in (0, \frac{\pi}{2})$. Then it follows that

$$\ker\phi_* = Sp\{V_1 = -\partial x_1 + \partial y_2, V_2 = -\partial x_2 + \partial y_1, V_3 = -\cos\gamma\partial x_3 - \sin\gamma\partial x_4, \\ V_4 = \partial y_3\}$$

and

$$(\ker \phi_*)^\perp = Sp\{W_1 = \partial x_1 + \partial y_2, W_2 = \partial x_2 + \partial y_1, W_3 = \sin\gamma \partial x_3 - \cos\gamma \partial x_4, \\ W_4 = \partial y_4, W_5 = \partial z\}$$

hence we have $\varphi V_1 = W_2, \varphi V_2 = W_1$. Thus, it follows that $\mathcal{D}_\perp = sp\{V_1, V_2\}$ and $\mathcal{D}_\theta = sp\{V_3, V_4\}$ is a slant distribution with slant angle $\theta = \gamma$. Thus, ϕ is a slant $\xi^\perp$-submersion. Also by direct computations, we have

$$g_{\mathbb{R}^9}(W_i, W_i) = g_{\mathbb{R}^5}(\varphi W_i, \varphi W_i), \quad i = 1, ..., 5$$

which show that ϕ is a slant $\xi^\perp$-Riemannian submersion.

Example 4.5 Let F be a submersion defined by

$$\begin{array}{cccc} F: & (\mathbb{R}^9, g_{\mathbb{R}^9}) & \longrightarrow & (\mathbb{R}^5, g_{\mathbb{R}^5}) \\ & (x_1, ..., y_1, ..., z) & & (\frac{x_1+y_2}{\sqrt{2}}, \frac{x_2+y_1}{\sqrt{2}}, \frac{x_3+x_4}{\sqrt{2}}, \frac{y_3+y_4}{\sqrt{2}}, z). \end{array}$$

The submersion F is hemi-slant $\xi^\perp$-Riemannian submersion such that $\mathcal{D}_\perp = span\{\partial x_1 - \partial y_2, \partial x_2 - \partial y_1\}$ and $\mathcal{D}_\theta = span\{\partial x_3 + \partial x_4, \partial y_3 + \partial y_4\}$ with hemi-slant angle $\theta = 0$.

Example 4.6 Let π be a submersion defined by

$$\begin{array}{cccc} \pi: & (\mathbb{R}^7, g_{\mathbb{R}^7}) & \longrightarrow & (\mathbb{R}^4, g_{\mathbb{R}^4}) \\ & (x_1, ..., y_1, ..., z) & & (\frac{x_1+x_2}{\sqrt{2}}, \sin\gamma x_3 - \cos\gamma y_4, \cos\beta x_4 - \sin\beta y_3, z). \end{array}$$

The submersion π is a hemi-slant $\xi^\perp$-Riemannian submersion such that $\mathcal{D}_\perp = span\{\partial x_1 - \partial x_2\}$ and $\mathcal{D}_\theta = span\{\cos\gamma \partial x_3 - \sin\gamma \partial y_4, \sin\beta \partial x_4 - \cos\beta \partial y_3\}$ with hemi-slant angle $\theta = \alpha + \beta$.

Let ϕ be a hemi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, for $U \in \Gamma(ker\phi_*)$, we put

$$U = \mathcal{P}U + \mathcal{Q}U$$

where $\mathcal{P}U \in \Gamma(\mathcal{D}_\perp)$ and $\mathcal{Q}U \in \Gamma(\mathcal{D}_\theta)$. For $Z \in \Gamma(TM)$, we have

$$Z = \mathcal{V}Z + \mathcal{H}Z$$

where $\mathcal{V}Z \in \Gamma(ker\phi_*)$ and $\mathcal{H}Z \in \Gamma(ker\phi_*)^\perp$.

We denote the complementary distribution to $\varphi\mathcal{D}_\perp$ in $(ker\phi_*)^\perp$ by μ. Then we have

$$(ker\phi_*)^\perp = \varphi\mathcal{D}_\perp \oplus \mu,$$

where $\varphi(\mu) \subset \mu$. Hence μ contains ξ. For $V \in \Gamma(ker\phi_*)$, we write

$$\varphi V = \rho V + \omega V \tag{4.1}$$

where ρV and ωV are vertical (resp. horizontal) components of φV, respectively. Also, for $X \in \Gamma((ker\phi_*)^\perp)$, we have

$$\varphi X = \mathcal{B}X + CX, \tag{4.2}$$

where $\mathcal{B}X$ and CX are vertical (resp. horizontal) components of φX, respectively. Then the horizontal distribution $(ker\phi_*)^\perp$ is decomposed as

$$(ker\phi_*)^\perp = \varphi\mathcal{D}_\perp \oplus \mu,$$

here μ is the orthogonal complementary distribution of $\mathcal{D}_\perp$ and it is both invariant distribution of $(ker\phi_*)^\perp$ with respect to φ and contains ξ. Then by using (2.3), (2.4), (4.1) and (4.2), we get

$$(\nabla^{M}_V \rho)W = \mathcal{B}T_V W - T_V \omega W \tag{4.3}$$

$$(\nabla^{M}_V \omega)W = CT_V W - T_V \rho W \tag{4.4}$$

for $V, W \in \Gamma(ker\phi_*)$, where

$$(\nabla^{M}_V \rho)W = \hat{\nabla}_V \rho W - \rho\hat{\nabla}_V W$$

and

$$(\nabla^{M}_V \omega)W = \mathcal{H}\nabla^{M}_V \omega W - \omega\hat{\nabla}_V W.$$

Lemma 4.7 *Let $\phi : M \to N$ be a hemi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\rho^2 W = \cos^2\theta W, \ \ W \in \Gamma(\mathcal{D}_\theta), \tag{4.5}$$

where θ denotes the hemi-slant angle of $ker\phi_*$.

Lemma 4.8 *Let $\phi : M \to N$ be a hemi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$g_M(\rho U, \rho V) = \cos^2\theta g_M(U, V) \tag{4.6}$$

$$g_M(\omega U, \omega V) = \sin^2\theta g_M(U, V) \tag{4.7}$$

for all $U, V \in \Gamma(ker\phi_*)$.

4.1 Integrable and Parallel Distributions

Theorem 4.9 *Let ϕ be a hemi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then the distribution $\mathcal{D}_\perp$ is integrable if and only if we have*

$$g_M(\mathcal{T}_U\varphi V - \mathcal{T}_V\varphi U, \rho Z) = g_N((\nabla\phi_*)(V, \varphi U) - (\nabla\phi_*)(U, \varphi V), \phi_*(\omega Z))$$

for any $U, V \in \Gamma(\mathcal{D}_\perp)$ and $Z \in \Gamma(\mathcal{D}_\theta)$.

Proof For $U, V \in \Gamma(TM)$, by using (2.9) and (2.10), we have

$$g_M(\nabla^M_U V, Z) = g_M(\nabla^M_U \varphi V, \varphi Z). \tag{4.8}$$

For $U, V \in \Gamma(\mathcal{D}_\perp),\ Z \in \Gamma(\mathcal{D}_\theta)$, using (2.9) and (4.8), we have

$$g_M([U, V], Z) = g_M(\nabla^M_U \varphi V, \varphi Z) - g_M(\nabla^M_V \varphi U, \varphi Z).$$

On the other hand, we get

$$g_M([U, V], Z) = g_M(\mathcal{T}_U\varphi V - \mathcal{T}_V\varphi U, \rho Z) + g_M(\mathcal{H}(\nabla^M_U\varphi V) - \mathcal{H}(\nabla^M_V\varphi U), wZ).$$

Or,

$$\begin{aligned} g_M([U, V], Z) = {} & g_M(\mathcal{T}_U\varphi V - \mathcal{T}_V\varphi U, \rho Z) \\ & + g_N(\phi_*(\nabla^M_U\varphi V) - \phi_*(\nabla^M_V\varphi U), \phi_*(\omega Z)) \end{aligned}$$

which proves assertion. □

Theorem 4.10 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then the distribution $\mathcal{D}_\theta$ is integrable if and only if we have*

$$g_N((\nabla\phi_*)(Z, \omega W) - (\nabla\phi_*)(W, \omega Z), \varphi U) = g_M(\mathcal{T}_Z\omega\rho W - \mathcal{T}_W w\rho Z, U)$$

for any $Z, W \in \Gamma(\mathcal{D}_\theta)$ and $U \in \Gamma(\mathcal{D}_\perp)$.

Proof For $Z, W \in \Gamma(\mathcal{D}_\theta)$ and $U \in \Gamma(\mathcal{D}_\perp)$, using (2.9) and (4.8) we have

$$g_M([Z, W], U) = g_M(\nabla^M_Z\varphi W, \varphi U) - g_M(\nabla^M_W\varphi Z, \varphi U).$$

Therefore, by using (4.1), we get

$$\begin{aligned} g_M([Z,W],U) = &-g_M(\nabla^M_Z \rho^2 W, U) - g_M(\nabla^M_Z \omega\rho W, U) \\ &+g_M(\nabla^M_Z \omega W, \varphi U) + g_M(\nabla^M_W \rho^2 Z, U) \\ &+g_M(\nabla^M_W \omega\rho Z, U) - g_M(\nabla^M_W \omega Z, \varphi U). \end{aligned}$$

Now, by virtue of (3.16), we obtain

$$\begin{aligned} g_M([Z,W],U) = &\cos^2\theta g_M([Z,W],U) - g_M(\nabla^M_Z \omega\rho W, U) \\ &+g_M(\nabla^M_Z \omega W, \varphi U) + g_M(\nabla^M_W \omega\rho Z, U) \\ &-g_M(\nabla^M_W \omega Z, \varphi U). \end{aligned}$$

Then we have

$$\begin{aligned} \sin^2\theta g_M([Z,W],U) = &\, g_M(\nabla^M_W \omega\rho Z - \nabla^M_Z \omega\rho W, U) \\ &+ g_M(\nabla^M_Z \omega W - \nabla^M_W \omega Z, \varphi U). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sin^2\theta g_M([Z,W],U) &= g_M(\mathcal{T}_W \omega\rho Z - \mathcal{T}_Z \omega\rho W, U) \\ &\quad +g_M(H(\nabla^M_Z \omega W) - \mathcal{H}(\nabla^M_W \omega Z), \varphi U) \\ &= g_M(\mathcal{T}_W \omega\rho Z - \mathcal{T}_Z \omega\rho W, U) \\ &\quad +g_N(\phi_*(\nabla^M_Z \omega W) - \phi_*(\nabla^M_W \omega Z), \varphi U) \end{aligned}$$

which proves assertion. □

Theorem 4.11 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then the distribution $D_\perp$ is parallel if and only if*

$$g_M(\phi_*(\nabla_U V), \phi_*(\omega\rho Z)) = g_M(\varphi\nabla_U V, \omega Z)$$

and

$$g_M(\hat{\nabla}_U \rho V + \mathcal{T}_U \omega V, BX) = -g_M(\mathcal{T}_U \rho V + \mathcal{H}(\nabla_U \omega V), CX)$$

for any $U, V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$, $X \in \Gamma((\ker\phi_*)^\perp)$.

Proof For $U, V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$ using (2.9), we get

$$\begin{aligned} g_M(\nabla_U V, Z) &= g_M(\varphi\nabla_U V, \varphi Z) + \eta(\nabla_U V)\eta(Z) \\ &= g_M(\varphi\nabla_U V, \varphi Z). \end{aligned}$$

Or,

$$g_M(\nabla_U V, Z) = -g_M(\nabla_U V, \rho^2 Z + \omega\rho Z + \varphi\omega Z).$$

Then one obtains

$$\sin^2\theta g_M(\nabla_U V, Z) = -g_M(\mathcal{H}(\nabla_U V), \omega\rho Z) + g_M(\varphi\nabla_U V, \omega Z).$$

By property of ϕ, we get

$$\sin^2\theta g_M(\nabla_U V, Z) = -g_N(\phi_*(\nabla_U V), \phi_*(\omega\rho Z)) + g_M(\varphi\nabla_U V, \omega Z).$$

On the other hand, for $U, V \in \Gamma(D_\perp)$, $X \in \Gamma((\ker\phi_*)^\perp)$, we have

$$g_M(\nabla_U V, X) = g_M(\nabla_U\varphi V, \varphi X).$$

Now, by virtue of (2.3) and (4.1), we obtain

$$\begin{aligned} g_M(\nabla_U V, X) &= g_M(\mathcal{T}_U\rho V, CX) + g_M(\hat{\nabla}\rho V, BX) \\ &\quad + g_M(\mathcal{T}_U\omega V, BX) + g_M(\mathcal{H}(\nabla_U\omega V), CX) \end{aligned}$$

which completes the proof. □

Theorem 4.12 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then the distribution D_θ is parallel if and only if*

$$g_N(\phi_*(\omega W), (\nabla\phi_*)(Z, \varphi U)) = g_M(\rho W, \mathcal{T}_Z\varphi U)$$

and

$$\begin{aligned} &g_N((\nabla\phi_*)(\nabla_Z\omega\rho W), \phi_*(X)) - g_N((\nabla\phi_*)(\nabla_Z\omega W), \phi_*(CX)) \\ &= -g_M(\mathcal{T}_Z\omega W, BX) + g_M(\omega W, Z)\eta(X). \end{aligned}$$

for all $Z, W \in \Gamma(D_\theta)$, $U \in \Gamma(D_\perp)$, $X \in \Gamma((\ker\phi_*)^\perp)$.

Theorem 4.13 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then $D_\perp$ defines a totally geodesic foliation on M if and only if*

$$g_N((\nabla\phi_*)(U, \varphi V), \phi_*(\omega Z)) = -g_M(\mathcal{T}_U V, \omega\rho Z)$$

and

$$g_M(\mathcal{T}_U\varphi V, BX) = g_N((\nabla\phi_*)(U, \varphi V), \phi_*(CX))$$

for any $U, V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$, $X \in \Gamma((\ker\phi_*)^\perp)$.

Proof For $U, V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$, from (2.9), (2.3), (2.4), (4.1) to (4.5), we have

$$g_M(\nabla_U V, Z) = \cos^2\theta g_M(\nabla_U V, Z) - g_M(\mathcal{T}_U V, \omega\rho Z) + g_M(\mathcal{H}(\nabla_U \varphi V), wZ).$$

Or,

$$\sin^2\theta g_M(\nabla_U V, Z) = -g_M(\mathcal{T}_U V, w\rho Z) - g_N(\phi_*(\nabla_U \varphi V), \phi_*(\omega Z)).$$

On the other hand, for $X \in \Gamma((\ker\phi_*)^\perp)$, we have

$$g_M(\nabla_U V, X) = g_M(\mathcal{T}_U \varphi V, BX) + g_M(\mathcal{H}(\nabla_U \varphi V), CX).$$

Or,

$$g_M(\nabla_U V, X) = g_M(\mathcal{T}_U \varphi V, BX) - g_N(\phi_*(\nabla_U \varphi V), \phi_*(CX)).$$

This completes the proof. □

Theorem 4.14 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then D_θ defines a totally geodesic foliation on M if and only if*

$$g_N((\nabla\phi_*)(Z, \omega W), \phi_*(\varphi U)) = -g_M(\mathcal{T}_Z \omega\rho W, U)$$

and

$$g_N((\nabla\phi_*)(Z, \omega\rho W), \phi_*(X)) + g_N((\nabla\phi_*)(Z, \omega W), \phi_*(CX)) = g_M(\mathcal{T}_Z \omega W, BX)$$

for any $Z, W \in \Gamma(D_\theta)$, $U \in \Gamma(D_\perp)$, $X \in \Gamma((\ker\phi_*)^\perp)$.

4.2 Hemi-Slant $\xi^\perp$-Riemannian Submersions on Sasakian Space Forms

A plane section in the tangent space T_pM at $p \in M$ is called a φ-section if it is spanned by a vector X orthogonal to ξ and φX. The sectional curvature of φ-section is called φ-sectional curvature. A Sasakian manifold with constant φ-sectional curvature c is a Sasakian space form. The Riemannian curvature tensor of a Sasakian space form is given by

$$\begin{aligned} R^M(X,Y,Z,W) &= \frac{c+3}{4}\{g_M(Y,Z)g_M(X,W) - g_M(X,Z)g_M(Y,W)\} \\ &+ \frac{c-1}{4}\{g_M(Y,W)\eta(X)\eta(Z) - g_M(X,W)\eta(Y)\eta(Z) \\ &+ g_M(X,Z)\eta(Y)\eta(W) - g_M(Y,Z)\eta(X)\eta(W) \\ &+ g_M(\varphi Y,Z)g_M(\varphi X,W) - g_M(\varphi X,Z)g_M(\varphi Y,W) \\ &- 2g_M(\varphi X,Y)g_M(\varphi Z,W)\} \end{aligned} \tag{4.9}$$

for any $X, Y, Z, W \in \Gamma(TM)$ [39].

Theorem 4.15 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then we have*

$$\begin{aligned} \widehat{R}(U,V,W,S) &= \frac{c+3}{4}\{g_M(V,S)g_M(U,W) - g_M(U,S)g_M(V,W)\} \\ &+ g_M(\mathcal{T}_V W, \mathcal{T}_U S) - g_N(\mathcal{T}_U W, \mathcal{T}_V S) \end{aligned} \tag{4.10}$$

and

$$\widehat{K}(U,V) = \frac{c+3}{4}\{g_M(U,V)^2 - 1\} + g_M(\mathcal{T}_V U, \mathcal{T}_U V) - g_M(\mathcal{T}_U U, \mathcal{T}_V V) \tag{4.11}$$

for all $U, V, S, W \in \Gamma(\mathcal{D}^\perp)$.

Proof For any $U, V, S, W \in \Gamma(\mathcal{D}_\perp)$ by using (4.9), $\varphi U \in \Gamma((\ker\phi_*)^\perp)$ and $\eta(U) = 0$, then we have

$$R^M(U,V,S,W) = \frac{c+3}{4}\{g_M(V,S)g_M(U,W) - g_M(U,S)g_M(V,W)\}. \tag{4.12}$$

Hence, we have

$$\begin{aligned} \widehat{R}(U,V,W,S) &= \frac{c+3}{4}\{g_M(V,S)g_M(U,W) - g_M(U,S)g_M(V,W)\} \\ &+ g_M(\mathcal{T}_V W, \mathcal{T}_U S) - g_M(\mathcal{T}_U W, \mathcal{T}_V S) \end{aligned}$$

which completes the proof. □

Corollary 4.16 *Let ϕ be a hemi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M^m, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ and $m \geq 3$. If $\mathcal{D}_\perp$ is totally geodesic, then M is flat if and only if $c = -3$.*

Theorem 4.17 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. If $\mathcal{D}_\perp$ is totally geodesic, then*

$$\widehat{\tau}_{\perp} = \frac{c+3}{2} q(1-2q)$$

where $\widehat{\tau}_{\perp}$ *is the scaler curvature.*

Proof We have

$$\widehat{S}_{\perp}(U, V) = \sum_{i=1}^{2q} \widehat{R}(E_i, U, V, E_i)$$

where $\{E_1, ..., E_{2q}\}$ is ortonormal basis on $\Gamma(\mathcal{D}_{\perp})$ and $U, V \in \Gamma(\mathcal{D}_{\perp})$. Thus, one obtains

$$\widehat{S}_{\perp}(U, V) = \sum_{i=1}^{2q} \{\frac{c+3}{4}\{g_M(U, E_i) g_M(E_i, V) - g_M(E_i, E_i) g_M(U, V)\}\}.$$

Or,

$$\widehat{S}_{\perp}(U, V) = \frac{c+3}{4}(1-2q) g_M(U, V). \tag{4.13}$$

By taking $U = V = E_k$, $k = 1, ..., 2q$, we get the result. □

Corollary 4.18 *Let* ϕ *be a hemi-slant* $\xi^{\perp}$*-Riemannian submersion from a Sasakian manifold* $(M, \varphi, \eta, \xi, g_M)$ *onto Riemannian manifold* (N, g_N) *with a hemi-slant angle* θ. *If* $\mathcal{D}_{\perp}$ *is totally geodesic distribution, then* $\mathcal{D}_{\perp}$ *is Einstein.*

Theorem 4.19 *Let* ϕ *be a hemi-slant* $\xi^{\perp}$ *Riemannian submersion from a Sasakian manifold* $(M, \varphi, \eta, \xi, g_M)$ *onto Riemannian manifold* (N, g_N) *with a hemi-slant angle* θ. *Then we have*

$$\begin{aligned}
\widehat{R}(K, L, P, W) = & \frac{c+3}{4}\{g_M(L, P) g_M(K, W) - g_M(K, P) g_M(L, W)\} \\
& + \frac{c-1}{4}\{g_M(\varphi L, P) g_M(\varphi K, W) \\
& - g_M(\varphi K, P) g_M(\varphi L, W) - 2 g_M(\varphi K, L) g_M(\varphi P, W)\} \\
& + g_M(\mathcal{T}_L P, \mathcal{T}_K W) - g_M(\mathcal{T}_K P, \mathcal{T}_L W)
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
\widehat{K}(K, L) = & \frac{c+3}{4}\{g_M(L, K) g_M(K, L) - g_M(K, K) g_M(L, L)\} \\
& - 3\frac{c-1}{4} g_M(\varphi K, L) + g_M(T_L K, T_K L) - g_M(\mathcal{T}_K K, \mathcal{T}_L L)
\end{aligned} \tag{4.15}$$

for all $K, L, P, N \in \Gamma(\mathcal{D}_{\theta})$.

Theorem 4.20 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. If $\mathcal{D}_\theta$ is totally geodesic, then we have*

$$\widehat{k_\theta} = p\frac{(c+3)(2p-1) + 3(c-1)\cos^2\theta}{2}.$$

Proof For any $K, L \in \Gamma(\mathcal{D}_\theta)$, using (4.14), we derive

$$\widehat{S_\theta}(K, L) = \frac{c+3}{4}(2p-1)g_M(K, L) + 3\frac{c-1}{4}\cos^2\theta g_M(K, L) \tag{4.16}$$

where $\{E_1, ..., E_{2p}\}$ is orthonormal basis on $\Gamma(\mathcal{D}_\theta)$. From the above equation, we obtain the proof. □

Corollary 4.21 *Let ϕ be a hemi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. If $\mathcal{D}_\theta$ is totally geodesic distribution, then $\mathcal{D}_\theta$ is Einstein.*

5 Quasi Hemi-slant Submanifolds of Cosymplectic Manifolds

In this Sect. 5, we will finally discuss some results of quasi hemi-slant submanifolds introduced and studied by Rajendra Prasad et al. [38]. First, we have

Definition 5.1 A submanifold M of an almost contact metric manifold $\overline{M}$ is called a quasi hemi-slant submanifold if there exist distributions D, D^θ and $D^\perp$ such that (i) TM admits the orthogonal direct decomposition as

$$TM = D \oplus D^\theta \oplus D^\perp \oplus <\xi> .$$

(ii) The distribution D is ϕ invariant, *i.e.*, $\phi D = D$.
(iii) For any nonzero vector field $X \in (D^\theta)_p$, $p \in M$, the angle θ between JX and $(D^\theta)_p$ is constant and independent of the choice of point p and X in $(D^\theta)_p$.
(iv) The distribution $D^\perp$ is ϕ anti-invariant, i.e., $\phi D^\perp \subseteq T^\perp M$.

In this case, we call θ the quasi hemi-slant angle of M. Suppose the dimension of distributions D, D^θ and $D^\perp$ are n_1, n_2 and n_3, respectively. Then we can easily see the following particular cases:
(i) If $n_1 = 0$, then M is a hemi-slant submanifold.
(ii) If $n_2 = 0$; then M is a semi-invariant submanifold.
(iii) If $n_3 = 0$, then M is a semi-slant submanifold.
We say that a quasi hemi-slant submanifold M is proper if $D \neq \{0\}$, $D^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

This means that the notion of quasi hemi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant, semi-slant submanifolds. Let M be a quasi hemi-slant submanifold of an almost contact metric manifold $\overline{M}$. We denote the projections of $X \in \Gamma(TM)$ on the distributions D, D^{θ} and $D^{\perp}$ by P, Q and R, respectively. Then we can write for any $X \in \Gamma(TM)$

$$X = PX + QX + RX + \eta(X)\xi. \tag{5.1}$$

Now we put

$$\phi X = TX + NX, \tag{5.2}$$

where TX and NX are tangential and normal components of ϕX on M. Using (5.1) and (5.2), we obtain

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX.$$

Since $\phi D = D$ and $\phi D^{\perp} \subseteq T^{\perp}M$, we have $NPX = 0$ and $TRX = 0$. Therefore, we get

$$\phi X = TPX + TQX + NQX + NRX. \tag{5.3}$$

Then for any $X \in \Gamma(TM)$, it is easy to see that

$$TX = TPX + TQX$$

and

$$NX = NQX + NRX.$$

For any $V \in \Gamma(T^{\perp}M)$, we can put

$$\phi V = tV + nV$$

where tV and nV are the tangential and normal componenets of ϕV on M, respectively.

An almost contact metric manifold is called a cosymplectic manifold if $(\widehat{\nabla}_X\phi)Y = 0$, $\widehat{\nabla}_X\xi = 0 \ \ \forall\ X, Y \in \Gamma(T\widehat{M})$, where $\widehat{\nabla}$ represents the Levi-Civita connection of $(\widehat{M}, g)$.

The covariant derivative of ϕ is defined as

$$(\widehat{\nabla}_X\phi)Y = \widehat{\nabla}_X\phi Y - \phi\widehat{\nabla}_X Y.$$

If $\widehat{M}$ is a cosymplectic manifold, then we have

$$\phi\widehat{\nabla}_X Y = \widehat{\nabla}_X\phi Y.$$

Let M be a Riemannian manifold isometrically immersed in $\widehat{M}$ and the induced Riemannian metric on M is denoted by the same symbol g throughout this paper. Let A and h denote the shape operator and second fundamental form, respectively, of submanifolds of M into $\widehat{M}$. The Gauss and Weingarten formulas are given by

$$\widehat{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$\widehat{\nabla}_X V = -A_V X + \nabla_X^{\perp} V$$

for any vector fields $X,\ Y \in \Gamma(TM)$ and V on $\Gamma(T^{\perp}M)$, where ∇ is the induced connection on M and $\nabla^{\perp}$ represents the connection on the normal bundle $T^{\perp}M$ of M and A_V is the shape operator of M with respect to normal vector $V \in \Gamma(T^{\perp}M)$. Moreover, A_V and the second fundamental form $h : TM \otimes TM \longrightarrow T^{\perp}M$ of M into $\widehat{M}$ are related by

$$g(h(X, Y), V) = g(A_V X, Y),$$

for any vector fields $X, Y \in \Gamma(TM)$ and V on $\Gamma(T^{\perp}M)$.

5.1 Integrability of Distributions

Theorem 5.2 *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\overline{M}$. Then the invariant distribution D is integrable if and only if*

$$g(\nabla_X TY - \nabla_Y TX, TQZ) = g(h(Y, TX) - h(X, TY), NQZ + NRZ)$$

for any $X,\ Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\theta} \oplus D^{\perp})$.

Proof For a cosymplectic manifold, we have

$$\overline{\nabla}_X \xi = 0 \ \forall \ X \in \Gamma(D). \tag{5.4}$$

If $Y \in \Gamma(D)$, then $g(Y, \xi) = 0$. Thus, one gets

$$g(\overline{\nabla}_X Y, \xi) + g(Y, \overline{\nabla}_X \xi) = 0. \tag{5.5}$$

Now, $g([X, Y], \xi) = g(\overline{\nabla}_X Y, \xi) - g(\overline{\nabla}_Y X, \xi) = 0$.
Also, we have

$$g([X, Y], Z) = g(\overline{\nabla}_X \phi Y, \phi Z) - g(\overline{\nabla}_Y \phi X, \phi Z) = g(\nabla_X TY \\ -\nabla_Y TX, TQZ) + g(h(X, TY) - h(Y, TX), NQZ + NRZ)$$

which completes the proof. □

Similarly, we have

Theorem 5.3 *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $(\overline{M}, g, \phi)$. Then the slant distribution D^θ is integrable if and only if*

$$g(A_{NW}Z - A_{NZ}W, TPX) = g(A_{NTW}Z - A_{NTZ}W, X) \\ +g(\nabla^\perp_Z NW - \nabla^\perp_W NZ, NRX)$$

for any $Z, \ W \in \Gamma(D^\theta)$ and $X \in \Gamma(D \oplus D^\perp)$.

Theorem 5.4 *Let M be a quasi hemi-slant submanifold of a cosymplectic manifold $\overline{M}$. Then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$g(T([Z, W]), TX) = g(\nabla^\perp_W NZ - \nabla^\perp_Z NW, NQX)$$

for any $Z, W \in \Gamma(D^\perp)$ and $X \in \Gamma(D \oplus D^\theta)$.

5.2 Totally Geodesic Foliations

Theorem 5.5 *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\overline{M}$. Then M is totally geodesic if and only if*

$$g(h(X, PY) + \cos^2\theta h(X, QY), U) = g(\nabla^\perp_X NTQY, U) \\ +g(A_{NQY}X + A_{NRY}X, tU) - g(\nabla^\perp_X NY, nU)$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(T^\perp M)$.

Proof For any $X, \ Y \in \Gamma(TM), \ \ U \in \Gamma\left(T^\perp M\right)$, we have

$$\begin{aligned} g(\overline{\nabla}_X Y, U) &= g(\overline{\nabla}_X PY, U) + g(\overline{\nabla}_X QY, U) + g(\overline{\nabla}_X RY, U) \\ &= g(\overline{\nabla}_X \phi PY, \phi U) + g(\overline{\nabla}_X TQY, \phi U) + g(\overline{\nabla}_X NQY, \phi U) \\ &\quad +g(\overline{\nabla}_X \phi RY, \phi U). \end{aligned}$$

$$\begin{aligned} g(\overline{\nabla}_X Y, U) &= g(h(X, PY) + \cos^2\theta h(X, QY), U) - g(\nabla^\perp_X NTQY, U) \\ &\quad -g(A_{NQY}X + A_{NRY}X, tU) + g(\nabla^\perp_X NY, nU) \end{aligned}$$

which completes the proof. □

Similarly, we have

Theorem 5.6 *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\overline{M}$. Then anti-invariant distribution $D^\perp$ defines totally geodesic foliation if and only if*

$$g(A_{\phi Y}X, TPZ + tQZ) = g(\nabla^{\perp}_{X}\phi Y, nQZ), \quad g(A_{\phi Y}X, tV) = g(\nabla^{\perp}_{X}\phi Y, nV)$$

for any $X, Y \in \Gamma(D^{\perp})$, $Z \in \Gamma(D \oplus D^{\theta})$ *and* $V \in \Gamma\left(T^{\perp}M\right)$.

Theorem 5.7 *Let* M *be a proper quasi hemi-slant submanifold of a cosymplectic manifold* $\overline{M}$. *Then the slant distribution* D^{θ} *defines a totally geodesic foliation on* M *if and only if*

$$g(\nabla^{\perp}_{X}NY, NRZ) = g(A_{NY}X, TPZ) - g(A_{NTY}X, Z), \text{ and}$$
$$g(A_{NY}X, tV) = g(\nabla^{\perp}_{X}NY, nV) - g(\nabla^{\perp}_{X}NTY, V)$$

for any $X, Y \in \Gamma(D^{\theta})$, $Z \in \Gamma(D \oplus D^{\perp})$ *and* $V \in \Gamma\left(T^{\perp}M\right)$.

5.3 Examples

Now we discuss few examples from [38]

Example 5.8 Let us consider a 15-dimensional differentiable manifold

$$\overline{M} = \{(x_i, y_i, z) = (x_1, x_2, ..., x_7, y_1, y_2, ..., y_7, z) \in \mathrm{R}^{15}\}.$$

And choose the vector fields

$$E_i = \frac{\partial}{\partial y_i}, \ E_{7+i} = \frac{\partial}{\partial x_i}, \ E_{15} = \xi = \frac{\partial}{\partial z}, \text{ for } i = 1, 2, ..., 7.$$

Let g be a Riemannian metric defined by

$$g = (dx_1)^2 + (dx_2)^2 + \cdots + (dx_7)^2 + (dy_1)^2 + (dy_2)^2 + \cdots + (dy_7)^2 + (dz)^2.$$

We define $(1, 1)$-tensor field ϕ as

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \ \phi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \quad \phi\left(\frac{\partial}{\partial z}\right) = 0 \ \forall \, i, j = 1, 2, ..., 7.$$

Thus, $(\overline{M}, \phi, \xi, \eta, g)$ is an almost contact metric manifold. Also, we can easily show that $(\overline{M}, \phi, \xi, \eta, g)$ is a cosymplectic manifold of dimension 15.

Let M be a submanifold of $\overline{M}$ defined by

$$f(u, v, w, r, s, t, q) = \left(u, w, 0, \frac{s}{\sqrt{2}}, 0, \frac{t}{\sqrt{2}}, 0, v, r\cos\theta, r\sin\theta, 0, \frac{s}{\sqrt{2}}, 0, \frac{t}{\sqrt{2}}, q\right),$$

where $0 < \theta < \frac{\pi}{2}$. Now the tangent bundle of M is spanned by the set $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$Z_1 = \frac{\partial}{\partial x_1},\ Z_2 = \frac{\partial}{\partial y_1},\ Z_3 = \frac{\partial}{\partial x_2},$$

$$Z_4 = \cos\theta \frac{\partial}{\partial y_2} + \sin\theta \frac{\partial}{\partial y_3},\ Z_5 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_5}\right),$$

$$Z_6 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_6} + \frac{\partial}{\partial y_7}\right),\ Z_7 = \frac{\partial}{\partial z}.$$

Thus, we have

$$\phi Z_1 = \frac{\partial}{\partial y_1},\ \phi Z_2 = -\frac{\partial}{\partial x_1},\ \phi Z_3 = \frac{\partial}{\partial y_2},$$

$$\phi Z_4 = -\left(\cos\theta \frac{\partial}{\partial x_2} + \sin\theta \frac{\partial}{\partial x_3}\right),\ \phi Z_5 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_4} - \frac{\partial}{\partial x_5}\right),$$

$$\phi Z_6 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_6} - \frac{\partial}{\partial x_7}\right),\ \phi Z_7 = 0.$$

Now, let the distributions $D = span\{Z_1, Z_2\}$, $D^\theta = span\{Z_3, Z_4\}$, $D^\perp = span\{Z_5, Z_6\}$. And D is invariant, D^θ is slant with slant angle θ and $D^\perp$ is anti-invariant.

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