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Contact Geometry of Slant Submanifolds

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Preface

An almost Hermitian manifold is an almost complex manifold (M, J) equipped with a Riemannian metric g which satisfies $g(JX, JY) = g(X, Y)$ for vector fields X, Y tangent to M. By a submanifold of an almost Hermitian manifold (M, g_M, J), we mean the image of an isometric immersion

$$\phi : (N, g_N) \to (M, g_M, J)$$

from a Riemannian manifold (N, g_N) into (M, g_M, J).

Dual to the notion of isometric immersions, there exists the notion of Riemannian submersions introduced by B. O'Neill in [11]. By definition, a Riemannian submersion is a surjective map

$$\pi : (M, g_M) \to (B, g_B)$$

from a Riemannian manifold (M, g_M) onto another Riemannian manifold (B, g_B) which preserves the scalar products of vectors normal to fibers.

Based on the action of the almost complex structure J on the tangent bundle of a submanifold, there are three important classes of submanifolds of an almost Hermitian manifold (M, g_M, J), namely the classes of complex, totally real and slant submanifolds.

In terms of the almost complex structure J, a submanifold N of an almost complex manifold (M, g, J) is called a *complex submanifold* (respectively, *totally real submanifold*) if

$$J(T_pN) \subseteq T_pN \quad \text{(respectively, } J(T_pN) \subseteq T_p^{\perp}N) \tag{1}$$

for any point $p \in N$, where $T_p^{\perp}N$ denotes the normal space of N in M at p.

For a unit tangent vector $X \in T_pN$ of a submanifold N in an almost Hermitian manifold (M, g_M, J) at a point $p \in N$, the angle $\theta(X)$ between JX and T_pN is called the Wirtinger angle of X.

In 1990, a more general class of submanifolds than complex and totally real submanifolds was introduced in [5] as follows.

Definition 1 A submanifold N of an almost Hermitian manifold (M, g, J) is called a *slant submanifold* if the Wirtinger angle $\theta(X)$ is independent of the choice of the unit vector $X \in T_pN$ and of $p \in N$. In this case, the constant θ is called the *slant angle*. A slant submanifold with slant angle θ is said to be θ-*slant*.

It follows from the definitions that complex submanifolds and totally real submanifolds are nothing but θ-slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. From J-action points of view, slant submanifolds are the simplest and the most natural submanifolds of an almost Hermitian manifold. In [7, 10], the notion of pointwise slant submanifolds of an almost Hermitian manifold was defined as a generalization of slant submanifolds.

The first results on slant submanifolds were collected in the book [6]. Since then the study of slant submanifolds and of slant submersions has been attracting more and more researchers and a lot of interesting results have been achieved during the past 30 years.

A Riemannian $(2n + 1)$-manifold (M^{2n+1}, g) is called an *almost contact metric manifold* (cf. [1]) if there exist a (1, 1) tensor field φ, a vector field ξ (called the *structure vector field*), and a 1-form η on M^{2n+1} such that

$$\eta(\xi) = 1, \varphi^2(X) = -X + \eta(X)\xi, \varphi\xi = 0, \eta \circ \varphi = 0,$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi)$$

for any vector fields X, Y tangent to M^{2n+1}. An almost contact metric structure is called a *contact metric structure* if it satisfies

$$d\eta(X, Y) = g(X, \varphi Y).$$

A contact metric structure is called *normal* if it satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

where ∇ is the Levi-Civita connection of g. A manifold M endowed with a normal contact metric structure is called a *Sasakian manifold*. A Sasakian manifold with constant φ-sectional curvature is called a *Sasakian space form*.

The study of slant submanifolds was extended by A. Lotta [8] in 1996 to contact slant submanifolds in almost contact geometry as follows. Let N be a submanifold of an almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$. Then N is called *contact slant* if the Wirtinger angle $\theta(X)$ between ϕX and T_pN is a global constant, so that it is independent of the choice of the point $p \in N$ and the vector $X \in T_pN$ such that X and ξ_p are linearly independent. In particular, for $\theta = 0$ and $\theta = \frac{\pi}{2}$, the θ-slant submanifolds of $(M^{2n+1}, \varphi, \xi, \eta, g)$ are called invariant and anti-invariant submanifolds, respectively.

In [8], A. Lotta proved that if M^{2n+1} is a contact metric manifold, then the structure vector field ξ is tangent to every non-anti-invariant slant submanifold. After Lotta's work there are a lot of works done on contact slant submanifolds.

Dual to slant submanifolds, B. Sahin introduced in [12] the notion of slant submersions. Roughly speaking, a Riemannian submersion from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (B, g_B) is called a slant submersion if its vertical distribution is a slant distribution.

Similar to Sahin's work on slant submersions from almost Hermitian manifolds onto Riemannian manifolds, I. K. Erken and C. Murathan defined and studied in [9] slant submersions from Sasakian manifolds onto Riemannian manifolds. Cabrerizo et al. studied in [2, 3] slant, semi-slant, hemi-slant, and bi-slant submanifolds in contact slant geometry. Further, as a generalization of slant submanifolds and semi-slant submanifolds, K. S. Park [4] defined the notion of pointwise slant submanifolds and pointwise semi-slant submanifolds of an almost contact metric manifold.

Given the huge amount of work on contact slant submanifolds and submersions published since the appearance of the last monograph [6], the editors thought it is appropriate to invite a number of specialists to contribute one or more papers to illustrate the state of the art in the theory of contact slant geometry with focuses on contact slant submanifolds and contact slant submersions and many colleagues answered our call. The editors express their gratitude to all the contributors.

The editors hope that the readers will find this book both a good introduction and a useful reference of contact slant geometry to perform their research more successfully and creatively.

East Lansing, Michigan, USA — Bang-Yen Chen
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Bang-Yen Chen, a Taiwanese-American mathematician, is University Distinguished Professor Emeritus at Michigan State University, USA, since 2012. He completed his Ph.D. degree at the University of Notre Dame, USA, in 1970, under the supervision of Prof. Tadashi Nagano. He received his M.Sc. degree from National Tsing Hua University, Hsinchu, Taiwan, in 1967, and B.Sc. degree from Tamkang University, Taipei, Taiwan, in 1965. Earlier at Michigan State University, he served as University Distinguished Professor (1990–2012), Full Professor (1976), Associate Professor (1972), and Research Associate (1970–1972). He taught at Tamkang University, Taiwan, from 1966 to 1968, and at National Tsing Hua University, Taiwan, during the academic year 1967–1968.

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General Properties of Slant Submanifolds in Contact Metric Manifolds

A. Lotta

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1 Slant Submanifolds of Almost Contact Manifolds

B. Y. Chen's concept of a slant submanifold can be translated into the context of contact metric geometry in a very natural fashion. In this chapter, we shall discuss the basic facts concerning this variant of the theory.

Our standard reference for contact geometry is Blair's book [2], to which we refer the reader for the terminology, the notation and the relevant facts.

Let M be an almost contact metric manifold with structure (φ, ξ, η, g). By a *slant submanifold* of M, we shall mean an immersed submanifold N such that for any $x \in N$ and for any tangent vector $X \in T_x N$, linearly independent on ξ, the angle between φX and $T_x N$ is a constant $\theta \in [0, \frac{\pi}{2}]$, called the *slant angle* of N in M.

Like in complex geometry, when the ambient manifold is a *contact metric* manifold, for $\theta = \frac{\pi}{2}$ one recovers the notion of *anti-invariant* submanifold. For $\theta = 0$, this class coincides with that of *invariant* submanifolds, i.e. those for which each tangent space of the submanifold is invariant under φ. We remark that it is known that such a submanifold must be tangent to the Reeb vector field ξ (see [2, Sect. 8.1], p. 152). We shall see below that this property also holds in the larger class of non-anti-invariant slant submanifolds.

We shall denote by $\bar{\nabla}$ the Levi-Civita connection of the ambient manifold, by ∇ the corresponding connection relative to the metric induced on a submanifold, while the second fundamental form will be denoted by α.

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We shall also denote the kernel of η by D, which is a distribution on M of rank $\dim(M) - 1$. Using the same notation as in the complex case, for every tangent vector $X \in TN$ we write

$$\varphi X = PX + FX$$

where PX is tangent and FX is normal to the submanifold. Then $P : TN \to TN$ is a skew-symmetric (1, 1) tensor field on N with respect to the induced metric. We shall also denote by Q the symmetric operator P^2.

We begin by discussing the following basic result showing that the class of non-anti-invariant slant submanifolds of a given almost contact metric manifold splits into two sub-classes, characterized by the position of the characteristic vector field ξ with respect to the submanifold (cf. [11]).

Theorem 1.1 *Let N be an immersed slant submanifold of the almost contact metric manifold M with structure tensors (φ, ξ, η, g). Let $n = \dim(N)$. Assume that N is not anti-invariant. Then*

$$n \text{ is odd} \iff \xi \text{ is tangent to } N$$

$$n \text{ is even} \iff \xi \text{ is normal to } N.$$

Proof For every point $x \in N$, the orthogonal complement $E \subset T_x N$ of $\mathrm{Ker}(Q_x)$ is even dimensional. Observe that if $X \in \mathrm{Ker}(Q_x)$, then φX is normal to N. By definition of a slant submanifold, this forces that X be a scalar multiple of ξ_x, because we are assuming that the slant angle $\theta \neq \frac{\pi}{2}$. Thus, we have proved that

$$\mathrm{Ker}(Q_x) \subset \mathbb{R}\xi_x.$$

Now, if n is odd, then $\mathrm{Ker}(Q_x) \neq \{0\}$ for every $x \in N$, which yields that ξ is everywhere tangent to N. If n is even, we must have $\mathrm{Ker}(Q_x) = \{0\}$ for all $x \in N$. Fix x and consider an eigenspace H of Q_x relative to the eigenvalue λ. By definition of the slant angle, for every non-null $X \in H$ we have

$$\cos\theta = \frac{||PX||}{||\varphi X||} = \sqrt{-\lambda}\frac{||X||}{||\varphi X||}. \tag{1}$$

On the other hand, since $\dim(H) \geq 2$, H contains some non-null $X \in H$ belonging to D_x, for which $||\varphi X|| = ||X||$. Substituting in (1) yields $\lambda = -\cos^2\theta$. We have thus showed that $Q_x = -\cos^2\theta\, \mathrm{Id}$ and moreover, coming back again to (1) we have that $||\varphi X|| = ||X||$ for every $X \in T_x N$, which implies that $T_x N \subset D_x$. We conclude that ξ is everywhere orthogonal to N. □

We remark that the assumption that N is not anti-invariant in the above result is essential. A significant example is provided by a well-known result of Blair, who classified the contact metric manifolds of dimension at least five, whose curvature tensor annihilates ξ (i.e. $R(X, Y)\xi = 0$ for every vector fields X, Y). The simply connected, complete ones are the Riemannian products

$$M = \mathbb{R}^{n+1} \times \mathbb{S}^n(4), \quad n > 1$$

where $\mathbb{S}^n(4)$ denotes a sphere endowed with a standard metric of constant sectional curvature 4. See [2], Theorem 7.5. Then, it turns out that both the standard immersions of $\mathbb{R}^{n+1}$ and of $\mathbb{S}^n(4)$ are anti-invariant in M. Moreover, for the first one ξ is everywhere tangent, while for the second one ξ is always normal, so that letting $n > 1$ vary, we provide a series of counterexamples to the equivalences in Theorem 1.1.

For a submanifold N tangent to ξ, as a further application of formula (1) the following characterization is readily verified, involving the symmetric tensor Q and the normal bundle valued 1-form $F : TN \to TN^{\perp}$ (see [3]).

Theorem 1.2 *Let N be a submanifold of an almost contact metric manifold M. Assume that ξ is tangent to N. Then the following are equivalent:*

(a) N is slant in M with slant angle θ;
(b) $Q = -\cos^2\theta(I - \eta \otimes \xi)$;
(c) For every unit vector tangent to N and orthogonal to ξ, one has

$$||PX|| = \cos\theta;$$

d) For every unit vector tangent to N and orthogonal to ξ, one has

$$||FX|| = \sin\theta.$$

We remark that, in the general context of almost contact metric manifolds, one can provide simple examples showing that both possibilities regarding the position of ξ with respect to a slant submanifold can occur. Namely, given any almost Hermitian manifold (M, J, g_o), the product $M \times \mathbb{R}$ carries a standard almost contact metric structure (φ, ξ, η, g), where

$$\varphi(X, a\frac{d}{dt}) = (JX, 0), \ \xi = (0, \frac{d}{dt}), \ \eta = dt,$$

g being the product metric of g_o and the standard metric on the real line.

Now, given any θ-slant submanifold N of M, it is not difficult to verify that $N \times \{0\}$ and $N \times \mathbb{R}$ are both θ-slant in $M \times \mathbb{R}$ (cf. [11]). More generally, the same is true if instead of the product metric one considers a warped product metric g on $M \times I$, where $I \subset \mathbb{R}$ is an open interval, namely $g = \lambda^2\pi_1^* g_o + \pi_2^* dt \otimes dt$, where $\lambda : I \to \mathbb{R}$ is a smooth positive function and $\pi_1 : M \times I \to M$ and $\pi_2 : M \times I \to I$ are the canonical projections (see [6]).

Explicit examples of slant submanifolds (most of them in the Sasakian space form $\mathbb{R}^5$) are exhibited in [3]. Other examples and some general results concerning slant submanifolds of some particular classes of almost contact metric manifolds can be found in the recent papers [6, 7] by de Candia and Falcitelli.

We report here the following fact concerning even dimensional submanifolds (it is proved in [7] for the class of $C_5 \oplus C_{12}$-almost contact metric manifolds according to the Chinea-Gonzalez classification scheme [5]).

Theorem 1.3 *Let $(M, \varphi, \xi, \eta, g)$ an almost contact metric manifold and assume that φ is η-parallel, i.e.*

$$g((\bar{\nabla}_X \varphi)Y, Z) = 0$$

for every X, Y, Z vector fields orthogonal to ξ.

Let N be an even dimensional θ-slant submanifold of M, $\theta \neq \frac{\pi}{2}$. Then M induces on N an almost Kähler structure (J, g) where $J = \sec\theta P$.

Proof We know that ξ is normal to N. Moreover, $Q = -\cos^2\theta \mathrm{Id}$. Hence, $J = \sec\theta P$ is an almost complex structure on N, which is Hermitian with respect to the induced metric. Moreover, by the η-parallelism of φ, for every X, Y, Z vector fields tangent to N we get

$$g(\bar{\nabla}_X PY, Z) + g(\bar{\nabla}_X FY, Z) - g(\varphi\nabla_X Y, Z) - g(\varphi\alpha(X, Y), Z) = 0,$$

yielding

$$g((\nabla_X P)Y, Z) = g(A_{FY}X, Z) - g(A_{FZ}X, Y).$$

It follows that

$$\mathfrak{S}_{X,Y,Z}\, g((\nabla_X J)Y, Z) = 0,$$

where $\mathfrak{S}$ is a cyclic sum, and this ensures that the almost Hermitian structure (J, g) is almost Kähler. □

2 Slant Submanifolds of Contact Metric Manifolds

From now on, we shall consider the case when the ambient manifold is a contact metric manifold.

Theorem 2.1 *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold. Every non-anti-invariant slant submanifold N of M is tangent to ξ. Moreover, the restriction of η to N is again a contact form and N inherits canonically a contact metric structure $(\bar{\varphi}, \bar{\xi}, \bar{\eta}, \bar{g})$, where*

$$\bar{\varphi} := \sec\theta P, \;\; \bar{\xi} := \sec\theta\xi, \;\; \bar{\eta} := \cos\theta\,\eta, \;\; \bar{g} := \cos^2\theta\, g. \tag{2}$$

Proof If ξ were normal to N, from the formula

$$\bar{\nabla}_X \xi = -\varphi X - \varphi h X, \quad h := \frac{1}{2}\mathcal{L}_\xi \varphi \tag{3}$$

which is valid in the ambient manifold (cf. [2, Lemma 6.2]), for every X and Y vector fields tangent to N we would have

$$g(A_\xi X, Y) = g(\varphi X, Y) + g(\varphi h X, Y),$$

A_ξ being the Weingarten operator in the direction of ξ. But since φh and A_ξ are both symmetric operators, this would imply $g(\varphi X, Y) = 0$ identically, yielding that N is anti-invariant against the assumption. Hence according to Theorem 1.1, ξ must be tangent to N. Concerning the last statement, denoting by the same symbols the restrictions of η, ξ and g to the submanifold, setting $\bar{\varphi} := \sec\theta\, P$, it is easy to check that $(\bar{\varphi}, \xi, \eta, g)$ is an almost contact metric structure satisfying

$$d\eta = \cos\theta \bar{\Phi},$$

where $\bar{\Phi}$ is its fundamental 2-form, i.e. it is a $\cos\theta$-homothetic contact metric structure on N. This implies the last claims. □

The next proposition provides a formula linking the operator h of the ambient manifold and the analogous operator $\bar{h}$ relative to the induced contact metric structure.

Proposition 2.2 *Let N be a θ-slant, non-anti-invariant submanifold of a contact metric manifold $(M, \varphi, \xi, \eta, g)$. Then for every X, Y vectors tangent to N and orthogonal to ξ, we have*

$$g(hX, Y) = \cos^2\theta g(\bar{h}X, Y) - \sin^2\theta g(X, Y) - g(\alpha(X, \xi), FY).$$

In particular,

$$g(\alpha(X, \xi), FY) = g(\alpha(Y, \xi), FX)$$

holds with the same assumption on X, Y.

Proof We shall use (3) and the analogous formula for the induced contact metric structure (2) on N. Observing that the Levi-Civita connections of $g|_N$ and $\bar{g}$ coincide, we have

$$\begin{aligned} g(hX, Y) &= g(\varphi\bar{\nabla}_X\xi, Y) - g(X, Y) = \\ &= -g(\alpha(X, \xi), FY) - g(\nabla_X\xi, PY) - g(X, Y) = \\ &= -g(\alpha(X, \xi), FY) - \cos\theta g(\nabla_X\bar{\xi}, PY) - g(X, Y) = \\ &= -g(\alpha(X, \xi), FY) + \cos\theta g((\bar{\varphi} + \bar{\varphi}\bar{h})X, PY) - g(X, Y) = \\ &= -g(\alpha(X, \xi), FY) + g(PX, PY) + g(P\bar{h}X, PY) - g(X, Y) = \\ &= -g(\alpha(X, \xi), FY) + \cos^2\theta g(\bar{h}X, Y) - \sin^2\theta g(X, Y), \end{aligned}$$

where to deduce the last equality we used (c) in Theorem 1.2. The last claim follows since h and $\bar{h}$ are both symmetric operators. □

As another application of Theorem 1.2, we prove a result concerning contact totally umbilical submanifolds. Recall that a submanifold, tangent to ξ, of a contact metric manifold, is said to be *contact totally umbilical* provided the second fundamental form satisfies (cf. e.g. [15])

$$\alpha(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\}H + \eta(X)\alpha(Y, \xi) + \eta(Y)\alpha(X, \xi). \tag{4}$$

Here, H is the mean curvature normal vector field. If in addition $H = 0$, one speaks of a *contact totally geodesic* submanifold.

Given a proper slant submanifold, we shall consider the orthogonal splitting

$$TN^{\perp} = F(TN) \oplus E = F(\bar{D}) \oplus E$$

of the normal bundle, where $\bar{D}$ denotes the induced contact distribution on the slant submanifold. Of course, this is meaningful because $F_x : \bar{D}_x \to T_xN^{\perp}$ is injective for each point of the submanifold.

Theorem 2.3 *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold of dimension $2m + 1$, whose almost CR structure $(D, \varphi|_D)$ is integrable, i.e. M is a strongly pseudoconvex CR manifold. Let N be a contact totally umbilical proper slant submanifold of M.*

Then N is contact totally geodesic provided $\dim(N) = m + 1$ or $D_X H \in \Gamma(E)$ for every vector field tangent to N and orthogonal to ξ.

Proof First of all, we recall that for contact metric manifolds, the integrability condition for $(D, \varphi|_D)$ is equivalent to φ being η-parallel; indeed, one has the following formula for the covariant derivative of φ:

$$(\bar{\nabla}_X\varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX);$$

see [2, Theorem 6.7]. This implies that for every X, Y vector fields tangent to N and orthogonal to ξ (i.e. sections of $\bar{D}$), the vector field $(\bar{\nabla}_X\varphi)Y$ is tangent to N.

Using this, by a standard computation similar to the case of submanifolds of Kähler manifolds, one can derive the following formula for the covariant derivative of F, with the same assumption on the vector fields X, Y:

$$(\nabla_X F)Y = f\alpha(X, Y) - \alpha(X, PY). \tag{5}$$

Here, as usual, for every normal vector field ν to N, we set

$$\varphi\nu = t\nu + f\nu$$

with $t\nu$ tangent resp. $f\nu$ normal to N.

Now, assuming (4), (5) yields, for any local unit vector field X tangent to N and orthogonal to ξ:

$$(\nabla_{PX} F)X = -g(PX, PX)H = -\cos^2\theta\, H. \tag{6}$$

Observe now that the left-hand side of this equation is orthogonal to FX; indeed, by the condition (c) in Theorem 1.2 we have, for every Z, W tangent to N and orthogonal to ξ:

$$g(FZ, FW) = \sin^2\theta g(Z, W);$$

using this, we get, assuming $||X|| = 1$:

$$\begin{aligned} g((\nabla_{PX} F)X, FX) &= g(D_{PX} FX, FX) - g(F\nabla_{PX} X, FX) = \\ &= -\sin^2\theta g(\nabla_{PX} X, X) = 0. \end{aligned}$$

As a consequence, we obtain from (6) that

$$g(H, FX) = 0$$

for every X orthogonal to ξ and tangent to N, showing that $H \in \Gamma E$. If $\dim(N) = m + 1$, this suffices to prove the result, since in this case the subbundle E is trivial, because N has codimension m. Now assume that $D_X H \in \Gamma(E)$ for all sections of $\bar{D}$. Taking the inner product of both sides of (6) with H, one gets

$$-\cos^2\theta g(H, H) = g(D_{PX} FX, H) - g(F(\nabla_{PX} X), H) = -g(FX, D_{PX} H) = 0$$

due to our assumption. □

In [8], a similar result has been proved by Gupta in the case where M is a Kenmotsu manifold.

Corollary 2.4 *Let M be a Sasakian space form with φ-sectional curvature c. Then every totally contact umbilical proper slant submanifold N of M is totally contact geodesic, provided* $\dim(N) > 3$ *or* $c = 1$.

Proof In this case, we shall verify that actually $D_X H = 0$ for every X orthogonal to ξ. This can be proved by using the Codazzi equation. Indeed, observe first that for every X, Y, Z tangent to N and orthogonal to ξ:

$$\alpha(\nabla_X Y, Z) = \{g(\nabla_X Y, Z) - \eta(\nabla_X Y)\eta(Z)\}H = g(\nabla_X Y, Z)H$$

hence the Codazzi equation reads

$$g(Y, Z)D_X H - g(X, Z)D_Y H = (\bar{R}(X, Y)Z)^{\perp}$$

where $\bar{R}$ is the curvature tensor of M. Using the explicit expression of $\bar{R}$ (cf. [2, Theorem 7.19]), we thus obtain

$$g(Y, Z)D_X H - g(X, Z)D_Y H = \frac{c-1}{4}\{g(Z, PY)FX - g(Z, PX)FY + 2g(X, PY)FZ\}.$$

Choosing now $Y = Z$ of length one and orthogonal to X yields

$$D_X H = \frac{3}{4}(c-1)g(X, PY)FY.$$

Hence, the claim follows in the case $c = 1$. If $c \neq 1$, assuming $\dim(N) > 3$ we can choose Y so that Y is also orthogonal to PX, and the same formula yields $D_X H = 0$. □

3 The K-contact Case

In this section, we consider submanifolds of K-contact metric manifolds, i.e. contact metric manifolds whose Reeb vector field ξ is Killing. This is equivalent to requiring that the tensor field h in (3) vanishes. This class contains in particular the class of Sasakian manifolds.

We shall discuss the following characterization of slant submanifolds purely in terms of curvature (cf. [11]).

Theorem 3.1 *Let N be a submanifold of a K-contact metric manifold M with structure (φ, ξ, η, g). Assume that N is tangent to ξ. Fix $\theta \in [0, \frac{\pi}{2}]$; then the following conditions are equivalent:*

(a) N is slant with slant angle θ;
(b) For every $x \in N$ the sectional curvature, with respect to the induced metric, of every 2-plane containing ξ_x equals $\cos^2\theta$.

Moreover, every non-anti-invariant slant submanifold of M is itself a K-contact metric manifold with respect to the induced contact metric structure.

Proof For every $x \in N$, any 2-plane containing ξ_x is spanned by ξ_x and some unit vector X orthogonal to ξ; the corresponding sectional curvature $K(X, \xi)$ is related to the sectional curvature $\bar{K}(X, \xi)$ of the same 2-plane computed in the ambient manifold M by the Gauss equation:

$$K(X, \xi) = \bar{K}(X, \xi) + g(\alpha(X, X), \alpha(\xi_x, \xi_x)) - ||\alpha(X, \xi_x)||^2.$$

Now, M being a K-contact metric manifold, it is known that $\bar{K}(X, \xi) = 1$; moreover, we also have $\alpha(\xi, \xi) = 0$, because $\bar{\nabla}_\xi \xi = 0$. Hence, the above formula can be rewritten as

$$K(X, \xi) = 1 - ||\alpha(X, \xi_x)||^2.$$

On the other hand, remembering (3), in this case we have

$$\alpha(X, \xi_x) = -FX.$$

In conclusion:

$$K(X, \xi) = 1 - ||FX||^2.$$

Now the equivalence of (a) and (b) is clear taking into account the characterization of slant submanifolds provided by Theorem 1.2. Finally, concerning the last claim, observe that (3) also yields

$$\nabla_X \xi = -PX$$

for every vector field tangent to N, which implies that the restriction of ξ to N is again a Killing vector field, since P is skew-symmetric (alternatively, one can infer that the flow of ξ on N consists of local isometries). Hence, assuming that N is slant, the same is true for the Reeb vector field $\bar{\xi}$ of the induced contact metric structure, which is thus K-contact. □

Corollary 3.2 *Any torus admits no slant, non-anti-invariant, isometric immersions into any K-contact metric manifold.*

This is due to the fact that a torus cannot carry a K-contact metric structure [14]. Next, we consider regular K-contact manifolds. Recall that a contact manifold (M, η) is called regular provided the Reeb vector field is, i.e. it determines a regular 1-dimensional foliation on M, so that the space $B = M/\xi$ of maximal integral curves of ξ is a manifold. When M carries a K-contact metric g associated with η, then being $\mathcal{L}_\xi \varphi = \mathcal{L}_\xi g = 0$, g induces in a natural way a metric g' on M/ξ and φ also descends to an almost complex structure J.

Denoting by $\pi : M \to B$ the canonical projection, it turns out by construction that π is a Riemannian submersion with $Ker(d\pi)_x = \mathbb{R}\xi_x$ for every $x \in N$, and

$$d\pi \circ \varphi = J \circ d\pi$$

(see also [13]). Moreover, it is proved by Ogiue in [12] that (B, J, g') is an almost Kähler manifold. If M is Sasakian, then B is Kähler.

A remarkable case is when $M = M(c)$ is a simply connected, complete Sasakian space form; then B is either a flat Euclidean space $\mathbb{C}^m$ (when $c = -3$), a complex hyperbolic space $\mathbb{C}H_m$ with negative constant holomorphic sectional curvature ($c < -3$), or a complex projective space $\mathbb{C}P_m$ with positive constant holomorphic sectional curvature ($c > -3$) (see [2] or [9] for details).

The following result relates slant submanifolds of M with slant submanifolds of B. In particular, it provides a natural way to produce examples of slant submanifolds of the Sasakian space forms, by "lifting up" slant submanifolds of the corresponding complex space form. Another approach for constructing examples in this context has been developed by Cabrerizo, Carriazo, L. M. Fernandez and M. Fernandez in [4], who established a general existence and uniqueness result for slant immersions in Sasakian space forms, along the lines of the corresponding result of Chen-Vrancken for complex space forms.

Theorem 3.3 *Let M be a regular K-contact manifold canonically fibering onto the almost Kähler manifold B, with projection $\pi : M \to B$. Fix $\theta \in [0, \frac{\pi}{2})$. Then*

(a) If S is an embedded θ-slant submanifold of B, then $\pi^{-1}(B)$ is a θ-slant submanifold of M.
(b) If N is a compact θ-slant submanifold of M, then $\pi(N)$ is a θ-slant submanifold of B.

Proof (a) Since π is a surjective submersion, it is known that $N = \pi^{-1}(S)$ is an embedded submanifold of M, having the same codimension as S. Clearly, N is tangent to ξ, because at each point $x \in N$ the tangent space T_xN is $(d\pi)_x^{-1}(T_{\pi(x)}S)$. Moreover, observe that for every normal vector $\nu \in T_xN^\perp$, we have that $(d\pi)_x(\nu)$ is normal to S, because for every $X \in T_xN$ orthogonal to ξ, from $g(\nu, X) = 0$ it follows $g'((d\pi)_x\nu, (d\pi)_xX) = 0$, π being a Riemannian submersion.

Now, let X be a unit vector tangent to T_xN and orthogonal to ξ. Then from

$$\varphi X = PX + FX$$

we get

$$J(d\pi)_xX = (d\pi)_x(PX) + (d\pi)_xFX$$

where $(d\pi)_x(PX)$ is tangent and $(d\pi)_xFX$ is orthogonal to S, yielding

$$||PX|| = ||(d\pi)_x(PX)|| = \cos\theta$$

where the last equality holds because S is θ-slant.

(b) Since N is tangent to ξ, and π is a submersion satisfying $Ker(d\pi_x) = \mathbb{R}\xi_x$, for every $x \in M$, we have that the restriction of $d\pi$ to T_xN has rank $\dim(N) - 1$. Hence, $\pi : N \to B$ is a smooth map of constant rank; N being compact, it is known that its image $\pi(N)$ is a submanifold of B (cf. [1, Theorem 3.5.18]). The verification that $S = \pi(N)$ is θ-slant is based on the same argument used in the proof of (a), taking into account that at each point $\pi(x)$ of S we have $T_{\pi(x)}S = (d\pi)_x(T_x N)$. $\square$

Observe that, under the assumption of (b), one deduces that N is also a regular contact manifold. This provides a generalization of a result by Harada [10] concerning invariant submanifolds of regular Sasakian manifolds.

Corollary 3.4 *For every $m \geq 2$, the Sasakian space form $\mathbb{R}^{2m+1}$ of φ-sectional curvature -3 admits no compact proper slant submanifold.*

This holds since $\mathbb{R}^{2m+1}$ fibers onto the flat complex Euclidean space $\mathbb{C}^m$, and Chen-Tazawa's non-compactness result for slant submanifolds of $\mathbb{C}^m$ applies.

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Curvature Inequalities for Slant Submanifolds in Pointwise Kenmotsu Space Forms

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1 Introduction

In 1969, Tano [104] proved that the automorphism group of a connected almost contact Riemannian manifold M of dimension $(2n+1)$ is of maximum dimension $(n+1)^2$, and the maximum is attained only in the case when M reduces to one of the following spaces: a homogeneous Sasaki manifold (or an ε-deformation of one) with constant ϕ-holomorphic sectional curvature, a global Riemannian product of a line or a circle with a complex space form, and a warped product of the complex space with the real line. In 1972, Kenmotsu [55] investigated the properties of this warped product and characterized it by tensor equations, giving rise to one of the newest chapters of contact geometry, nowadays called Kenmotsu geometry. Although neglected for a long time, these manifolds have attracted the attention of a large number of geometers in the last three decades, proving to be a valuable chapter of the contact geometry (see [89] and the references therein, as well as the recent articles [1, 16, 26, 41, 44, 47, 82, 92, 105, 110, 111, 114]).

The aim of this work is to present a survey on the geometry of Kenmotsu submanifolds, focusing on the curvature properties of slant submanifolds in pointwise Kenmotsu space forms. The present paper is organized as follows.

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In Sect. 2, one recalls some basic facts concerning the geometry of manifolds equipped with almost contact structures and their submanifolds. Section 3 is devoted to the presentation of the definition and some basic properties of Kenmotsu spaces. The aim of Sect. 4 is to overview the main classes of submanifolds investigated in Kenmotsu geometry.

Recall now that one of the basic problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of submanifolds. In a seminal paper published in 1993, Chen [30] proved some sharp inequalities involving such invariants of a Riemannian submanifold. The notions and techniques developed in this article have turned out to be very useful, giving rise to one of the most important research topics in submanifolds geometry: theory of Chen's invariants and inequalities. Later, B. Y. Chen's inequalities have been extensively studied by many authors for different kinds of submanifolds in several ambient spaces. In Sect. 5, we present some Chen-like inequalities for slant submanifolds in a Kenmotsu space form.

Section 6 is devoted to the investigation of the δ-Casorati curvatures of slant submanifolds in Kenmotsu space forms. It is well known that the notion of Casorati curvature has been defined in the geometry of submanifolds as the normalized square of the length of the second fundamental form of the submanifold [39]. Obviously, the Casorati curvature is an extrinsic invariant. This concept, originally introduced by Casorati in 1890 for surfaces in a Euclidean 3-dimensional space [27], was preferred by the author over the classical Gaussian curvature because it seems to correspond better with the common intuition of curvature [106]. Notice that recently, Brubaker and Suceavă [21] obtained sufficient conditions for a smooth hypersurface in Euclidean ambient space to be convex, in terms of Casorati curvatures and mean curvature. On the other hand, Kowalczyk [60] gave a geometrical interpretation of the Casorati curvature of a Riemannian submanifold, as well as a characterization of normally flat submanifolds in Euclidean spaces in terms of a relation between the Casorati curvatures and the normal curvatures of the submanifold. The first basic inequalities involving the Casorati curvatures were proved for submanifolds in real space forms by Decu, Haesen and Verstraelen [38, 39]. Later, these inequalities were generalized to other classes of submanifolds and ambient spaces [2, 6, 8, 10, 15, 40, 46, 53, 54, 62, 65, 69, 70, 76, 96–98, 102, 112, 115, 118–120].

In Sect. 7, we discuss the generalized Wintgen inequality, also referred to in the literature as the DDVV inequality or the DDVV conjecture. This famous inequality has been conjectured in [43] and solved in affirmative in [45, 74]. The aim of this section is to provide the counterpart of this inequality in Kenmotsu geometry. We point out that the study of Wintgen-like inequalities was recently started in a more general setting, namely for submanifolds in statistical manifolds [12, 13, 20, 81]. Notice that Bansal, Uddin and Shahid [16] proved the DDVV inequality for statistical submanifolds of Kenmotsu statistical manifolds of constant ϕ-sectional curvature.

2 Preliminaries

Let $(\bar{M}^m, \bar{g})$ be a Riemannian manifold of dimension m and suppose (M^n, g) is a Riemannian submanifold of $\bar{M}^m$ having dimension n. In the following, we will denote by $K(\pi)$ the sectional curvature of a 2-plane section $\pi \subset T_pM,\ p \in M$. If $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_m\}$ are orthonormal bases of T_pM and $T_p^{\perp}M$ (respectively), then it is known that the scalar curvature is defined by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j)$$

and the normalized scalar curvature is given by

$$\rho = \frac{2\tau}{n(n-1)}.$$

Let us denote by $\bar{\nabla}$ the Levi-Civita connection of the metric $\bar{g}$ and suppose ∇ denotes the covariant differentiation induced on M. Then the Gauss-Weingarten formulae are

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM)$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \forall X \in \Gamma(TM), \forall N \in \Gamma(TM^{\perp})$$

where h denotes the second fundamental form of the submanifold M, $\nabla^{\perp}$ is the metric connection in the normal bundle and A_N denotes the shape operator of the submanifold M with respect to the normal vector field N.

Let us recall that a point p of the submanifold M is called totally geodesic if h vanishes at p. Moreover, the submanifold M is said to be totally geodesic if all points of M are totally geodesic points.

On the other hand, we recall Gauss' equation that relates the curvature tensor fields $\bar{R}$ and R of the connections $\bar{\nabla}$ and ∇, respectively:

$$\begin{aligned}\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) &+ \bar{g}(h(X, W), h(Y, Z)) \\ &-\bar{g}(h(X, Z), h(Y, W)),\end{aligned} \tag{1}$$

for all vector fields X, Y, Z, W on M.

Let H be the mean curvature vector of the submanifold M defined by

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i).$$

Then the squared mean curvature $\|H\|^2$ is given by

$$\|H\|^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2,$$

where

$$h_{ij}^{\alpha} = g(h(e_i, e_j), e_\alpha),\ i, j \in \{1, \dots, n\},\ \alpha \in \{n+1, \dots, m\}.$$

On the other hand, the squared norm of the second fundamental form h over the dimension of the submanifold M is an extrinsic geometric invariant known as the Casorati curvature of M. Therefore, this invariant is defined by

$$\mathcal{C} = \frac{1}{n} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{n} \left(h_{ij}^{\alpha}\right)^2.$$

One can also define the Casorati operator of M as a (1, 1)-tensor field given by [51]

$$A^C = \sum_{\alpha=n+1}^{m} A_{e_\alpha}^2,$$

where A_{e_α} denotes the shape operator of the submanifold M with respect to e_α, $\alpha = n+1, \dots, m$. It is easy to see that $\mathcal{C}$ and A^C are linked by

$$\mathcal{C} = \frac{1}{n} \mathrm{Tr} A^C.$$

If $L \subset T_pM$ is a subspace of dimension s, with $s \geq 2$, and $\{e_1, \dots, e_s\}$ is an orthonormal basis of L, then the scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{1 \leq \alpha < \beta \leq s} K(e_\alpha \wedge e_\beta).$$

3 Kenmotsu Manifolds

An almost contact metric manifold is a quintuple $(\bar{M}, \phi, \xi, \eta, \bar{g})$ consisting in a Riemannian manifold $(\bar{M}, \bar{g})$ of dimension $(2n+1)$ equipped with a (1, 1)-tensor field ϕ, a vector field ξ and a 1-form η satisfying the compatibility conditions [19]

$$\begin{aligned} &\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = \eta \circ \phi = 0 \\ &\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y) \\ &\eta(X) = \bar{g}(X, \xi) \end{aligned} \tag{2}$$

for all $X, Y \in \Gamma(T\bar{M})$. Moreover, if the Levi-Civita connection $\bar{\nabla}$ of the metric $\bar{g}$ satisfies

$$\left(\bar{\nabla}_X \phi\right)(Y) = \bar{g}(\phi X, Y)\xi - \eta(Y)\phi X, \quad \bar{\nabla}_X \xi = X - \eta(X)\xi \tag{3}$$

then $(\bar{M}, \phi, \xi, \eta, \bar{g})$ is said to be a Kenmotsu manifold [89].

As remarkable examples of Kenmotsu manifolds, we have the following.

i. The hyperbolic space $\mathbb{H}^{2n+1} = \{(x_1, \ldots, x_{2n+1}) \in \mathbb{R}^{2n+1} | x_1 > 0\}$ equipped with the almost contact structure (ϕ, ξ, η, g) constructed by Chinea and Gonzales in [34] is a Kenmotsu manifold.
ii. The product manifold of a Kähler manifold with a Kenmotsu manifold can be equipped with a Kenmotsu structure [108]. In particular, it follows that the product manifolds $P^n\mathbb{C} \times \mathbb{H}^{2n+1}$, $\mathbb{D}^n \times \mathbb{H}^{2n+1}$ and $\mathbb{C}^n \times \mathbb{H}^{2n+1}$ can be endowed with Kenmotsu structures.
iii. A special class of orientable hypersurfaces of Kähler manifolds admits natural Kenmotsu structures. These submanifolds are called natural Kenmotsu hypersurfaces. For details, see [89, Example 2.1.18].

Recall that a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$ with dimension $2n + 1 \geq 5$ is called a pointwise Kenmotsu space form, if the ϕ-sectional curvature of any ϕ-holomorphic plane $\{X, \phi X\}$, where $X \in T_p\bar{M}$, depends only on the point $p \in \bar{M}$, being independently on the ϕ-holomorphic plane at p. A connected Kenmotsu pointwise space form whose ϕ-sectional curvature does not depend on the point is said to be a *Kenmotsu space form*. It is known that a Kenmotsu manifold has constant ϕ-sectional curvature c at a point if and only if the curvature tensor $\bar{R}$ is given by [89]

$$\begin{aligned}\bar{R}(X, Y)Z = \frac{c-3}{4}\{\bar{g}(Y, Z)X - \bar{g}(X, Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + \eta(Y)\bar{g}(X, Z)\xi - \eta(X)\bar{g}(Y, Z)\xi \\ - \bar{g}(\phi X, Z)\phi Y + \bar{g}(\phi Y, Z)\phi X + 2\bar{g}(X, \phi Y)\phi Z\}.\end{aligned} \tag{4}$$

The (pointwise) Kenmotsu space forms are denoted by $\bar{M}(c)$. It is important to point out that any Kenmotsu space form $\bar{M}(c)$ has constant sectional curvature equal to $c = -1$ (see [55, Theorem 13]). As an example of Kenmotsu space form, we have the warped product $\bar{M} = \mathbb{R} \times_f \mathbb{C}^n$, where $f(t) = \exp t$. For details, see [63, Example 1].

Regarding the topology of Kenmotsu manifolds, we recall the following important result, that is, a consequence of the Green theorem and of the fact that $div\xi = 2n$.

Theorem 3.1 ([55]) *Any Kenmotsu manifold is non-compact.*

The local characterization of Kenmotsu manifolds is the following.

Theorem 3.2 ([55]) *Let $(\bar{M}, \phi, \xi, \eta, \bar{g})$ be a Kenmotsu manifold. Then any point of $\bar{M}$ has a neighborhood isometric to the warped product $I \times_f V$, where $I = (-\epsilon, \epsilon)$ is an open interval of $\mathbb{R}$, V is a Kählerian manifold and $f(t) = c \exp t$, $c > 0$.*

For other general properties concerning the geometry and topology of Kenmotsu manifolds, see [18, 42, 57, 88, 103] and [89, Chap. 2].

4 Submanifolds of Kenmotsu Manifolds

4.1 Invariant and Anti-Invariant Submanifolds

If (M, g) is a Riemannian submanifold of a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$, then we have the following decomposition for any vector field X tangent to M:

$$\phi X = PX + FX, \tag{5}$$

where PX and FX represent the tangential and the normal components of ϕX, respectively. If the dimension of the submanifold M is $(m + 1)$, then one can define

$$||P||^2 = \sum_{i,j=1}^{m+1} g(e_i, Pe_j)^2, \tag{6}$$

where $\{e_1, e_2, \dots, e_{m+1}\}$ is a local orthonormal frame of M. Notice that the squared norm of the endomorphism P of TM defined above does not depend on the chosen orthonormal frame. We also point out that F is a normal bundle-valued 1-form on the tangent bundle TM.

On the other hand, for any vector field V normal to M, we have the following decomposition:

$$\phi V = tV + fV, \tag{7}$$

where tV and fV denote the tangential and the normal component of ϕV, respectively.

A Riemannian submanifold M of a Kenmotsu manifold $\bar{M}$ is said to be an invariant submanifold [58] if $F \equiv 0$. On the other hand, if $P \equiv 0$, then the submanifold M is called anti-invariant [93]. These classes of submanifolds were investigated from two perspectives, accordingly as the structure vector field ξ is tangent or normal to the submanifold M. We recall next some results from [58].

Theorem 4.1 ([58]) *Let $(\bar{M}, \phi, \xi, \eta, \bar{g})$ be a Kenmotsu manifold and N be a submanifold of $\bar{M}$ tangent to ξ. Then*

i. N is an invariant submanifold iff t is parallel.
ii. N is an anti-invariant submanifold iff P is parallel.
iii. N is a totally geodesic submanifold iff the second fundamental form h of the submanifold is parallel.

Theorem 4.2 ([58]) *Let $\bar{M}(c)$ be a (pointwise) Kenmotsu space form, and N be a submanifold of $\bar{M}(c)$ tangent to the structure vector field of $\bar{M}(c)$. Suppose that t and f are parallel. Then N is also a (pointwise) Kenmotsu space form.*

Several curvature properties and geometric inequalities involving intrinsic and extrinsic invariants of invariant and anti-invariant submanifolds of (pointwise) Kenmotsu space forms tangent and normal to the structure vector field ξ can be found in [11, 56, 84, 107].

4.2 Contact Semi-Invariant and Normal Semi-Invariant Submanifolds

In 1983, Papaghiuc [86] investigated the concept of semi-invariant submanifold in Kenmotsu geometry, introducing two kinds of such submanifolds: contact semi-invariant submanifolds and normal semi-invariant submanifolds. A submanifold N of a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$ is called contact semi-invariant if its tangent bundle splits orthogonally into smooth distributions $\mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle \xi \rangle$ such that ϕ maps $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) into itself (resp. into the normal bundle of N). Notice that $\mathcal{D}$ is usually called the invariant distribution of the submanifold N, while $\mathcal{D}^{\perp}$ is said to be the anti-invariant distribution of the submanifold N.

In [86], the author investigated the integrability of certain subbundles of the tangent bundle of a contact semi-invariant submanifold of a Kenmotsu manifold, proving the following result.

Theorem 4.3 ([86]) *Let $(\bar{M}, \phi, \xi, \eta, \bar{g})$ be a Kenmotsu manifold and (N, g) be a contact semi-invariant submanifold of $\bar{M}$. Then*

i. *$\mathcal{D}^{\perp}$, $\mathcal{D}^{\perp} \oplus \langle \xi \rangle$ and $\mathcal{D} \oplus \mathcal{D}^{\perp}$ are integrable distributions.*
ii. *$\mathcal{D}$ and $\mathcal{D} \oplus \langle \xi \rangle$ are integrable distributions iff the second fundamental form h satisfies*

$$h(X, \phi Y) = h(\phi X, Y), \ \forall X, Y \in \Gamma(D).$$

Moreover, Sinha and Srivastava [99] derived two criteria for a submanifold of a Kenmotsu space form to be contact semi-invariant, while Papaghiuc [86] obtained some natural conditions that imply the constancy of sectional curvature for a contact semi-invariant submanifold of a pointwise Kenmotsu space form.

The concept of normal semi-invariant submanifold of a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$ was defined in [86] by considering that the structure vector field ξ is normal to the submanifold and imposing also the condition that the tangent bundle splits orthogonally as $\mathcal{D} \oplus \mathcal{D}^{\perp}$, where $\mathcal{D}$ is an invariant distribution and $\mathcal{D}^{\perp}$ is an anti-invariant distribution. In particular if $\mathcal{D} = 0$ (resp. $\mathcal{D}^{\perp} = 0$), then the submanifold is said to be normal anti-invariant (resp. normal invariant). Papaghiuc [86] investigated the integrability of both distributions involved in the definition of a normal semi-invariant submanifold, proving the following result.

Theorem 4.4 ([86]) *Let* $(\bar{M}, \phi, \xi, \eta, \bar{g})$ *be a Kenmotsu manifold and* (N, g) *be a normal semi-invariant submanifold of* $\bar{M}$*. Then*

i. $\mathcal{D}^{\perp}$ *is integrable.*
ii. $\mathcal{D}$ *is integrable iff the second fundamental form* h *satisfies*

$$h(X, \phi Y) - h(\phi X, Y) = 2g(\phi X, Y)\xi, \ \forall X, Y \in \Gamma(D).$$

In 1991, Kobayashi [59] investigated normal semi-invariant submanifolds of codimension 2 obtaining conditions under which such submanifolds are space forms or locally symmetric spaces. Several interesting results on the geometry of contact and normal semi-invariant submanifolds of Kenmotsu manifolds were obtained in [58, 78, 91, 113]. Notice that Kobayashi [58] obtained some characterizations of semi-invariant products of Kenmotsu space forms and recalled the following very interesting result concerning the Betti numbers of even order of compact contact semi-invariant submanifolds.

Theorem 4.5 ([89]) *Let* $(\bar{M}, \phi, \xi, \eta, \bar{g})$ *be a Kenmotsu manifold and* (N, g) *be an* n*-dimensional compact contact semi-invariant submanifold of* $\bar{M}$*. Suppose that* $\mathcal{D}$ *is integrable and* $\mathcal{D}^{\perp}$ *is minimal. Then* $b_{2k}(N) \geq 1$*, for* $k = 1, \ldots, n$.

We also want to point out that Matsumoto, Mihai and Shahid introduced in 1998 a new class of submanifolds of Kenmotsu manifolds, called generalized contact CR-submanifolds, as a natural generalization of contact semi-invariant submanifolds, obtaining several curvature properties [77].

4.3 Slant Submanifolds

The notion of slant submanifolds was originally introduced in 1990 in complex geometry by Chen [29] as a very natural generalization of totally real and holomorphic submanifolds. In 1996, Lotta [73] extends this notion in almost contact geometry. Recall that a submanifold N of a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \overline{g})$, tangent to the structure vector field ξ, is called *slant* if for each non-zero vector $X_p \in T_pN - \{\xi_p\}$, the angle $\theta(X)$ between the vector ϕX and the tangent space T_pN, called the slant angle, does not depend on the choice of $p \in N$ and $X_p \in T_pN - \{\xi_p\}$. Obviously, an invariant submanifold is a slant submanifold with $\theta = 0$ and an anti-invariant submanifold is a slant submanifold with $\theta = \frac{\pi}{2}$. A slant submanifold with $0 < \theta < \frac{\pi}{2}$ is said to be a proper slant submanifold. Notice that a slant submanifold with slant angle $\theta \neq \frac{\pi}{2}$ has odd dimension (see [73, Theorem 3.3]).

We remark that using (5), one can obtain the characterization of slant submanifolds by the existence of a constant $\lambda \in [0, 1]$ such that [23]

$$P^2 = -\lambda(Id - \eta \otimes \xi).$$

Therefore, we derive that a submanifold N of a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$, tangent to the structure vector field ξ, is a slant submanifold iff ϕ^2 and P^2 are collinear operators on N. Next, we denote by $\mathfrak{D}$ the orthogonal distribution to ξ in TN and we put $Q = P^2$. It is easy to check that the endomorphism Q is self-adjoint.

In 2004, Gupta, Haider and Shahid [49] give the following interesting characterization of slant submanifolds of Kenmotsu manifolds.

Theorem 4.6 ([49]) *Let $(\bar{M}, \phi, \xi, \eta, \bar{g})$ be a Kenmotsu manifold and N be a submanifold of $\bar{M}$ tangent to the structure vector field ξ. Then N is a slant submanifold iff:*

(a) The endomorphism $Q|_{\mathfrak{D}}$ has only one eigenvalue at each point of N.
(b) There exists a function $\lambda : N \to [0, 1]$ such that

$$(\nabla_X Q)Y = \lambda(g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X),$$

for all vector fields X, Y on N.

Moreover, $\lambda = \cos^2\theta$, where θ is the slant angle of N.

Later, Gupta and Pandey [50] give an intrinsic characterization of slant immersions of Kenmotsu manifolds in terms of slant angle and sectional curvature of an arbitrary plane section containing structure vector field ξ. They also provide a large class of examples of slant submanifolds in Kenmotsu ambient spaces. Another characterization of slant submanifolds of a Kenmotsu manifold with Killing structure tensor field was obtained in [83]. Recently, Uddin, Ahsan and Yaakub [109] classified totally umbilical slant submanifolds of a Kenmotsu manifold, proving that a totally umbilical slant submanifold N of a Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$ is either invariant or anti-invariant or $dimN = 1$ or the mean curvature vector H of N lies in the invariant normal subbundle. The notion of slant submanifold was later generalized to the concept of bi-slant submanifold (see, e.g., [3]). It is important to note that bi-slant submanifolds in almost contact geometry naturally englobe not only slant submanifolds, but also semi-slant submanifolds [22], hemi-slant submanifolds [25] and also semi-invariant submanifolds (also known as contact CR-submanifolds) [17]. For definitions, basic properties and examples of such submanifolds, the readers are referred to [3, 22, 117]. We only recall here the definition of bi-slant submanifolds. A submanifold N of Kenmotsu manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$ is called bi-slant if there exist two orthogonal distributions D_1 and D_2 on N, such that
(i) $TN = D_1 \oplus D_2 \oplus \xi$;
(ii) $JD_1 \perp D_2$ and $JD_2 \perp D_1$;
(iii) D_i is a slant distribution with slant angle θ_i, for $i = 1, 2$.

Moreover, a bi-slant submanifold is said to be proper if $d_1 d_2 \neq 0$ and $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

5 B. Y. Chen Inequalities for Slant Submanifolds in (Pointwise) Kenmotsu Space Forms

We start this section by recalling some basic notions originally introduced by B. Y. Chen (see [33]).

Let N be an n-dimensional Riemannian manifold. If we denote by $K(\pi)$ the sectional curvature of a plane section $\pi \subset T_pN$, $p \in N$, and by τ the scalar curvature, then Chen's first invariant is defined by

$$\delta_N(p) = \tau(p) - (\inf K)(p), \tag{8}$$

where

$$(\inf K)(p) = \inf\{K(\pi)|\pi \subset T_pN, \dim\pi = 2\}. \tag{9}$$

Suppose next that L is an r-dimensional subspace of T_pN, with $r \geq 2$, and we denote by $\tau(L)$ the scalar curvature of L. For a given integer $k \geq 0$, we will denote by $S(n,k)$ the set of all k-tuples $(n_1, \dots, n_k)$ of integers > 1 satisfying the conditions

$$n_1 < n, \ n_1 + \cdots + n_k \leq n. \tag{10}$$

For a fixed n, we will denote by $S(n)$ the set of all unordered k-tuples with $k \geq 0$. For each k-tuples $(n_1, \dots, n_k) \in S(n)$, Chen defined a new Riemannian invariant $\delta(n_1, \dots, n_k)$ by

$$\delta(n_1, \dots, n_k)(p) = \tau(p) - S(n_1, \dots, n_k)(p), \tag{11}$$

where

$$S(n_1, \dots, n_k)(p) = \inf\{\tau(L_1) + \cdots + \tau(L_k)\}, \tag{12}$$

$L_1, \dots, L_k$ running over all k mutually orthogonal subspaces of T_pN such that $\dim L_j = n_j, \ j \in \{1, \dots, k\}$.

Also, we denote by $d(n_1, \dots, n_k)$ and $b(n_1, \dots, n_k)$ the real constants given by

$$d(n_1, \dots, n_k) = \frac{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}{2(n + k - \sum_{j=1}^{k} n_j)}$$

and

$$b(n_1, \dots, n_k) = \frac{n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1))}{2}.$$

If L is a k-dimensional subspace of T_pN, $p \in N$ and $X \in L$ is a unit vector, then we can take an orthonormal basis $\{e_1, \ldots, e_k\}$ of L with $e_1 = X$ and we can define the Ricci curvature of L at X, usually denoted by $Ric_L(X)$ and also called a k-Ricci curvature, by

$$Ric_L(X) = \sum_{j=2}^{k} K(X \wedge e_j). \tag{13}$$

For a fixed integer k, $2 \leq k \leq n$, B. Y. Chen has introduced a new Riemannian invariant, denoted as Θ_k, by

$$\Theta_k(p) = \frac{1}{k-1} \inf\{Ric_L(X)|L, X\}, \;\; p \in M, \tag{14}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

We would like to point out now that in the Kenmotsu setting, one can define some modified Chen's invariants as follows. Let $(\bar{M}, \phi, \xi, \eta, \bar{g})$ be a Kenmotsu manifold and N be a submanifold of $\bar{M}$ tangent to the structure vector field ξ. Then one of Chen's invariants for the submanifold N is defined as

$$\delta'_N(p) = \tau(p) - \inf K(\pi),$$

where π ranges over all plane sections in T_pN invariant by the endomorphism P (see (5)). Likewise, for any set of integers $n_1, \ldots, n_k$ greater than 1 for which condition (10) is satisfied, one defines

$$\delta'(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\}(p),$$

where $L_1, \ldots, L_k$ run over all sets of k mutually orthogonal subspaces of T_pN invariant by P, such that $\dim L_j = n_j$, $j \in \{1, \ldots, k\}$, and $\tau(L_j)$ denotes the scalar curvature along the n_j-dimensional plane section L_j.

Next, if we denote by D the orthogonal distribution to ξ in TN, we have the orthogonal direct decomposition $TN = D \oplus \xi$, and we can consider

$$\delta_N^D(p) = \tau(p) - \inf_D K(\pi),$$

where π ranges over all plane sections in D_p.

Now, it is clear that $\delta_N^D \leq \delta_N$. In 2004, Gupta, Ahmad and Haider [48] obtained the following inequality involving the invariant δ_N^D and the squared mean curvature for a 3-dimensional slant submanifold N of a 5-dimensional (pointwise) Kenmotsu space form.

Theorem 5.1 ([48]) *Let $\bar{M}(c)$ be a 5-dimensional (pointwise) Kenmotsu space form and N be a 3-dimensional slant submanifold of $\bar{M}(c)$. Then the following inequality holds:*

$$\delta_N \leq \frac{9}{4}||H||^2 - 2.$$

Moreover, the equality case in the above inequality holds iff N is a minimal submanifold.

Later, Pandey, Gupta and Sharfuddin [85] generalized the above theorem to the case of higher dimensions for the modified invariant δ'_N.

Theorem 5.2 ([85]) *Let $\bar{M}(c)$ be a $(2m+1)$-dimensional (pointwise) Kenmotsu space form and N be an $(m+1)$-dimensional proper slant submanifold of $\bar{M}(c)$ with slant angle θ. Then the following inequality holds:*

$$\delta'_N \leq \frac{(m+1)^2(m-1)}{2m}||H||^2 + \frac{(m+1)(m-2)(c-3)}{8} + \frac{3(c+1)(m-2)}{8}\cos^2\theta - m. \tag{15}$$

Moreover, the equality holds at a point $p \in N$ iff there exist an orthonormal basis $\{e_1, \ldots, e_m, e_{m+1} = \xi\}$ of the tangent space T_pN and an orthonormal basis $\{e_{m+2}, \ldots, e_{2m+1}\}$ of the normal space $T_p^{\perp}N$ such that the shape operators $A_r \equiv A_{e_r}$, $r \in \{m+2, \ldots, 2m+1\}$, take the following forms:

$$A_{m+2} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & \ldots & 0 \\ 0 & 0 & a+b & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a+b \end{pmatrix} \tag{16}$$

and

$$A_r = \begin{pmatrix} a_r & b_r & 0 & \ldots & 0 \\ a_r & -b_r & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0, \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad r = m+3, \ldots, 2m+1, \tag{17}$$

where a, b, a_r, b_r, $r = m+3, \ldots, 2m+1$, are real functions on N.

Notice that similar inequalities were proved by Costache [35, 36]. We only recall the following result stated in [36].

Theorem 5.3 ([36]) *If N is an $(n+1)$-dimensional non-anti-invariant θ-slant submanifold of a $(2m+1)$-dimensional (pointwise) Kenmotsu space form $\bar{M}(c)$, then the following inequality is satisfied:*

$$\delta'(n_1,\ldots,n_k) \le d(n_1,\ldots,n_k)||H||^2 + b(n_1,\ldots,n_k)\frac{c-3}{4} + \frac{c+1}{8}\left[3\left(n - \sum_{j=1}^{k} n_j\right)\cos^2\theta - 2n\right],$$

for any k-tuple $(n_1,\ldots,n_k) \in S(n)$.

Moreover, the equality in the above inequality holds at $p \in N$ *if and only if there exist an orthonormal basis* $\{e_1,\ldots,e_{n+1}\}$ *of* T_pN *and an orthonormal basis* $\{e_{n+2},\ldots,e_{2m+1}\}$ *of* $T_p^{\perp}N$ *such that the shape operators* $A_r \equiv A_{e_r}$, $r \in \{n+2,\ldots,2m+1\}$, *take the following forms:*

$$A_{n+2} = \begin{pmatrix} a_1 & 0 & 0 & \ldots & 0 \\ 0 & a_2 & 0 & \ldots & 0 \\ 0 & 0 & a_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_n \end{pmatrix}$$

and

$$A_r = \begin{pmatrix} B_1^r & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & B_k^r & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}, \ r \in \{n+3,\ldots,2m+1\},$$

where $a_1,\ldots,a_n$ *satisfy*

$$a_1 + \cdots + a_{n_1} = \cdots = a_{n_1+\cdots n_{k-1}+1} + \cdots + a_{n_1+\cdots n_k} = a_{n_1+\cdots n_k+1} = \cdots = a_n$$

and each B_j^r *is a symmetric* $n_j \times n_j$ *submatrix satisfying*

$$\text{Trace } B_1^r = \cdots = \text{Trace } B_k^r = 0.$$

Notice that an improved Chen inequality for some special slant submanifolds in Kenmotsu space forms, as well as an inequality for the scalar curvature of such submanifolds, were derived in [37].

Recall now that B. Y. Chen established in [31] a relationship between the sectional curvature and the shape operator for submanifolds in a real space form. He also gave in [32] a relationship between the shape operator and the k-Ricci curvature for a submanifold of arbitrary codimension. Similar relations for slant submanifolds of a Kenmotsu space form were established in [68].

Theorem 5.4 ([68]) *Let $\bar{M}(c)$ be a $(2n+1)$-dimensional (pointwise) Kenmotsu space form and N be an $(n+1)$-dimensional θ-slant submanifold of $\bar{M}(c)$. Suppose that at a point $p \in N$ there exists a number $b > \frac{c-3}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}$ such that the sectional curvature $K \geq b$ at p. Then the shape operator A_H at the mean curvature vector satisfies*

$$A_H > \frac{n}{n+1}\left[b - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right] I_n,$$

at p, where I_n denotes the identity map identified with $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$.

Theorem 5.5 ([68]) *Let $\bar{M}(c)$ be a $(2n+1)$-dimensional (pointwise) Kenmotsu space form and N be an $(n+1)$-dimensional θ-slant submanifold of $\bar{M}(c)$. Then, for any integer k, $2 \leq k \leq n+1$, and any point $p \in N$, we have*

i. If $\Theta_k(p) \neq \frac{c-3}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}$, then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{n}{n+1}\left[\Theta_k(p) - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right] I_n,$$

at p, where I_n denotes the identity map identified with $\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$.

ii. If $\Theta_k(p) = \frac{c-3}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}$, then $A_H > 0$ at p.

iii. A unit vector $X \in T_pM$ satisfies

$$A_H X = \frac{n}{n+1}\left[\Theta_k(p) - \frac{c-3}{4} - \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}\right] X$$

if and only if

$$\Theta_k(p) = \frac{c-3}{4} + \frac{(3n\cos^2\theta - 2n)(c+1)}{4n(n+1)}$$

and $X \in \mathcal{N}_p$, where $\mathcal{N}_p$ is the relative null space of the submanifold N at the point $p \in N$ defined by $\mathcal{N}_p = \{Z \in T_pN | h(Z, Y) = 0, \forall Y \in T_pM\}$.

6 Inequalities for Casorati Curvatures of Slant Submanifolds in (Pointwise) Kenmotsu Space Forms

Suppose $(\bar{M}, \bar{g})$ is an m-dimensional Riemannian manifold and let (M, g) be an n-dimensional Riemannian submanifold of $\bar{M}$. If $L \subset T_pM$ is a subspace of dimension s, with $s \geq 2$, and $\{e_1, \ldots, e_s\}$ is an orthonormal basis of L, then one can define the

Casorati curvature $\mathcal{C}(L)$ of L by [38]

$$\mathcal{C}(L) = \frac{1}{s} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{s} \left(h_{ij}^{\alpha}\right)^2 .$$

With the help of $\mathcal{C}(L)$, one can define the notions of normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ of the n-dimensional submanifold M by [38]

$$[\delta_c(n-1)]_p = \frac{1}{2}\mathcal{C}_p + \frac{n+1}{2n}\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}$$

and

$$\left[\widehat{\delta}_c(n-1)\right]_p = 2\mathcal{C}_p - \frac{2n-1}{2n}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\}.$$

Analogously, for any positive number $r \neq n^2 - n$, one can define the concepts of generalized normalized δ-Casorati curvatures $\delta_C(r; n-1)$ and $\widehat{\delta}_C(r; n-1)$ by [39]

$$\left[\delta_C(r; n-1)\right]_p = r\mathcal{C}_p + \frac{(n-1)(n+r)(n^2-n-r)}{rn}\inf\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\},$$

if $0 < r < n^2 - n$, and

$$\left[\widehat{\delta}_C(r; n-1)\right]_p = r\mathcal{C}_p - \frac{(n-1)(n+r)(r-n^2+n)}{rn}\sup\{\mathcal{C}(L)|L \text{ a hyperplane of } T_pM\},$$

if $r > n^2 - n$.

It can be easily checked that $\delta_C(r; n-1)$ and $\widehat{\delta}_C(r; n-1)$ generalize the notions of normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ (see [67, 87]). In fact, we have that

$$[\delta_c(n-1)]_p = \frac{1}{n(n-1)}\left[\delta_C\left(\frac{n(n-1)}{2}; n-1\right)\right]_p \tag{18}$$

and

$$\left[\widehat{\delta}_c(n-1)\right]_p = \frac{1}{n(n-1)}\left[\widehat{\delta}_C\left(2n(n-1); n-1\right)\right]_p . \tag{19}$$

The first inequalities involving the Casorati curvatures were obtained for submanifolds in real space forms by Decu, Haesen and Verstraelen [38, 39] and further extended in other ambient spaces by many authors (see, e.g., [5, 7, 14, 24, 61, 64, 66, 71, 94, 95, 100, 101]). In Kenmotsu geometry, some basic inequalities involving the extrinsic Casorati curvatures were proved in [63, 72].

Theorem 6.1 ([72]) *Let M be an $(m+1)$-dimensional submanifold of a $(2n+1)$-dimensional (pointwise) Kenmotsu space form $\bar{M}(c)$ such that the structure vector field ξ is tangent to M. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_C(r;m)$ satisfies

$$\delta_C(r;m) \geq m(m+1)\left(\rho - \frac{c-3}{4}\right) + \frac{m(c+1)}{2} - \frac{3(c+1)}{4}||P||^2 \quad (20)$$

for any real number r such that $0 < r < m(m+1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_C(r;m)$ satisfies

$$\widehat{\delta}_C(r;m) \geq m(m+1)\left(\rho - \frac{c-3}{4}\right) + \frac{m(c+1)}{2} - \frac{3(c+1)}{4}||P||^2 \quad (21)$$

for any real number $r > m(m+1)$.

Moreover, the equality cases of (20) and (21) hold identically at a point $p \in M$ if and only if p is a totally geodesic point.

Due to the fact that for an $(m+1)$-dimensional slant submanifold with slant angle θ we have

$$||P||^2 = m\cos^2\theta, \quad (22)$$

the above theorem implies the following result.

Corollary 6.2 ([72]) *Let M be an $(m+1)$-dimensional θ-slant submanifold of a $(2n+1)$-dimensional (pointwise) Kenmotsu space form $\bar{M}(c)$. Then*

(i) The generalized normalized δ-Casorati curvature $\delta_C(r;m)$ satisfies

$$\delta_C(r;m) \geq m(m+1)\left(\rho - \frac{c-3}{4}\right) + \frac{m(c+1)}{2} - \frac{3m(c+1)}{4}\cos^2\theta \quad (23)$$

for any real number r such that $0 < r < m(m+1)$.

(ii) The generalized normalized δ-Casorati curvature $\widehat{\delta}_C(r;m)$ satisfies

$$\widehat{\delta}_C(r;m) \geq m(m+1)\left(\rho - \frac{c-3}{4}\right) + \frac{m(c+1)}{2} - \frac{3m(c+1)}{4}\cos^2\theta \quad (24)$$

for any real number $r > m(m+1)$.

Moreover, the equality cases of (23) and (24) hold identically at a point $p \in M$ if and only if p is a totally geodesic point.

Using (18) and (19) in the above Corollary, we derive the following.

Theorem 6.3 ([63]) *Let M be an $(m+1)$-dimensional θ-slant submanifold of a $(2n+1)$-dimensional (pointwise) Kenmotsu space form $\bar{M}(c)$. Then* *(i) The normalized δ-Casorati curvature $\delta_C(m)$ satisfies*

$$\delta_C(m) \geq \rho + \frac{c+1}{2(m+1)} - \frac{c-3}{4} - \frac{3(c+1)}{4(m+1)} \cos^2\theta. \tag{25}$$

(ii) The normalized δ-Casorati curvature $\widehat{\delta}_C(m)$ satisfies

$$\widehat{\delta}_C(m) \geq \rho + \frac{c+1}{2(m+1)} - \frac{c-3}{4} - \frac{3(c+1)}{4(m+1)} \cos^2\theta. \tag{26}$$

Moreover, the equality cases of (25) *and* (26) *hold identically at a point $p \in M$ if and only if p is a totally geodesic point.*

For examples of submanifolds satisfying the equality cases in the above inequalities, see [63, 72]. Such submanifolds are called Casorati ideal submanifolds [112]. We only recall here the following simple example. Consider the Kenmotsu space form $\bar{M} = \mathbf{R} \times_f \mathbf{C}^2$ and a 3-dimensional submanifold of $\bar{M}$ defined by the immersion $x : \mathbf{R}^3 \to \mathbf{R}^5$, given by

$$x(t, u, v) = (t, u\cos\theta, u\sin\theta, v, 0),$$

where $\theta \in (0, \pi/2)$. Then $M = \operatorname{Im} x$ is a totally geodesic θ-slant submanifold of $\bar{M}$, attaining equality in the inequalities (23), (24), (25) and (26) at all points.

Notice that the statistical counterpart of Theorem 6.1 was recently obtained in [41].

7 Generalized Wintgen Inequalities for Submanifolds in (Pointwise) Kenmotsu Space Forms

The generalized Wintgen inequality, also known as the normal scalar conjecture or the DDVV conjecture, was first formulated by De Smet, Dillen, Verstraelen and Vrancken for submanifolds in real space forms [43]. This conjecture, proven independently by Lu [74], and Ge and Tang [45], states that an isometric immersion $f : N \to \bar{M}(c)$ into a real space form $\bar{M}(c)$ satisfies

$$\rho \leq \|\mathcal{H}\|^2 - \rho^\perp + c,$$

where ρ denotes the normalized scalar curvature and $\rho^\perp$ represents the normalized normal scalar curvature. Recall that if N is a submanifold of dimension n in a Riemannian manifold $(\bar{M}, \bar{g})$ of dimension m, $\{e_1, \ldots, e_n\}$ is a tangent orthonormal frame and $\{\xi_1, \ldots, \xi_{2m-n}\}$ is a normal orthonormal frame on N, then normalized scalar curvature is given by

$$\rho = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} K(e_i \wedge e_j) \tag{27}$$

and the normalized normal scalar curvature is defined as

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1 \le i < j \le n} \sum_{1 \le r < s \le 2m-n} (R^{\perp}(e_i, e_j, \xi_r, \xi_s))^2}, \tag{28}$$

where $R^{\perp}$ denotes the normal curvature tensor on N.

In recent years, there has been an increasing interest in DDVV-type inequalities, the classical generalized Wintgen inequality being extended to many other ambient space forms [9, 16, 75, 79, 80, 90]. Recently, such inequalities were proved for different kinds of submanifolds in generalized Sasakian space forms [4]. Recall that a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ is an almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$ with Riemannian curvature tensor satisfying

$$\begin{aligned} \overline{R}(X, Y)Z = {} & f_1\{g(Y, Z)X - g(X, Z)Y\} \\ & + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ & - g(Y, Z)\eta(X)\xi\}, \end{aligned} \tag{29}$$

for all vector fields X, Y, Z on $\bar{M}$, where f_1, f_2, f_3 are differentiable functions on $\bar{M}$. Notice that Kenmotsu space forms are just generalized Sasakian space forms with $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$.

A Riemannian manifold N of an almost contact metric manifold $(\bar{M}, \phi, \xi, \eta, \bar{g})$ is said to be a C-totally real submanifold of $\bar{M}$ if ξ is a normal vector field on $\bar{M}$. Consequently, it follows easily that ϕ maps T_pN into $T_pN^{\perp}$, for all $p \in N$. We also recall that a C-totally real submanifold N of $\bar{M}$ is said to be a Legendrian submanifold if $\dim N = \frac{\dim \bar{M} - 1}{2}$.

The DDVV inequality for a Legendrian submanifold in a generalized Sasakian space form has been stated in [4] as follows.

Theorem 7.1 ([4]) *Let N be a Legendrian submanifold of a $(2n+1)$-dimensional generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$. Then*

$$\begin{aligned} (\rho^{\perp})^2 \le {} & \left(\|H\|^2 - \rho + f_1\right)^2 + \frac{2}{n(n-1)} f_2^2 \\ & + \frac{4f_2}{n(n-1)}\left(\rho - f_1\right) \end{aligned} \tag{30}$$

and the equality holds at a point $p \in N$ if and only if the shape operator A of N in $\bar{M}(f_1, f_2, f_3)$ with respect to some suitable orthonormal bases $\{e_1, \ldots, e_n\}$ of T_pN and $\{\xi_1, \ldots, \xi_{n+1}\}$ of $T_p^{\perp}N$ takes the following forms:

$$A_{\xi_1} = \begin{pmatrix} \gamma_1 & \nu & 0 & \dots & 0 \\ \nu & \gamma_1 & 0 & \dots & 0 \\ 0 & 0 & \gamma_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_1 \end{pmatrix},$$

$$A_{\xi_2} = \begin{pmatrix} \gamma_2 + \nu & 0 & 0 & \dots & 0 \\ 0 & \gamma_2 - \nu & 0 & \dots & 0 \\ 0 & 0 & \gamma_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_2 \end{pmatrix},$$

$$A_{\xi_3} = \begin{pmatrix} \gamma_3 & 0 & 0 & \dots & 0 \\ 0 & \gamma_3 & 0 & \dots & 0 \\ 0 & 0 & \gamma_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \gamma_3 \end{pmatrix}, \quad A_{\xi_4} = \cdots = A_{\xi_{n+1}} = 0,$$

where $\gamma_1, \gamma_2, \gamma_3$ and ν are real functions on N.

Recall now that those submanifolds attaining equality pointwise in a generalized Wintgen inequality are called Wintgen ideal submanifolds [52]. We point out that recently, Xie, Li, Ma and Wang classified the Wintgen ideal submanifolds into three classes: the reducible ones, the irreducible minimal ones in space forms (up to Möbius transformations) and the generic (irreducible) ones [116]. For examples of Wintgen Legendrian ideal submanifolds, see [80]. As a corollary of the above theorem, we derive the generalized Wintgen inequality for Legendrian submanifolds in (pointwise) Kenmotsu space forms.

Corollary 7.2 ([4]) *Let N be a Legendrian submanifold of a $(2n+1)$-dimensional (pointwise) Kenmotsu space form $\bar{M}(c)$. Then*

$$(\rho^{\perp})^2 \leq \left(\|H\|^2 - \rho + \frac{c-3}{4}\right)^2 + \frac{(c+1)^2}{8n(n-1)} + \frac{c+1}{n(n-1)}\left(\rho - \frac{c-3}{4}\right) \tag{31}$$

and the equality holds at a point $p \in N$ if and only if the shape operator takes the forms as in Theorem 7.1 with respect to some suitable tangent and normal orthonormal bases.

On the other hand, the generalized Wintgen-type inequality for proper bi-slant submanifolds in generalized Sasakian space form was stated as follows.

Theorem 7.3 ([4]) *Let N be a proper bi-slant submanifold of dimension n in a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ of dimension $(2m+1)$, with slant angles θ_1, θ_2 and $dim D_i = d_i$, $i = 1, 2$. Then*

$$\rho_N \leq \|H\|^2 - \rho + f_1 + \frac{3f_2}{n(n-1)}(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - \frac{2}{n}f_3. \tag{32}$$

As an immediate consequence of the above Theorem, we obtain the following result.

Corollary 7.4 ([4]) *Let N be a proper bi-slant submanifold of dimension n in a (pointwise) Kenmotsu space form $\bar{M}(c)$ of dimension $(2m+1)$, with slant angles θ_1, θ_2 and $dim D_i = d_i$, $i = 1, 2$. Then*

$$\rho_N \leq \|H\|^2 - \rho + \frac{c-3}{4} + \frac{3(c+1)}{4n(n-1)}(d_1 cos^2\theta_1 + d_2 cos^2\theta_2) - \frac{c+1}{2n}. \tag{33}$$

From the above corollary, we derive the generalized Wintgen-type inequality for proper slant submanifolds in (pointwise) Kenmotsu space forms.

Corollary 7.5 *Let N be a proper slant submanifold of dimension n in a (pointwise) Kenmotsu space form $\bar{M}(c)$ of dimension $(2m+1)$, with slant angle θ. Then*

$$\rho_N \leq \|H\|^2 - \rho + \frac{c-3}{4} + \frac{3(c+1)}{4n} cos^2\theta - \frac{c+1}{2n}. \tag{34}$$

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Some Basic Inequalities on Slant Submanifolds in Space Forms

Adela Mihai and Ion Mihai

1 Introduction

In Differential Geometry, Kähler and Sasaki manifolds and their submanifolds are probably the most studied geometric objects, because of their interesting properties. In particular, the behavior of submanifolds in complex space forms and Sasakian space forms was investigated by many geometers.

Slant submanifolds of an almost Hermitian manifold, endowed with (almost) complex structures J, were defined by B.-Y. Chen in [7] and they represent a generalization of complex and totally real submanifolds. More precisely, if, for any nonzero vector X tangent to the submanifold N at a point $p \in N$, the angle $\theta(X)$ between JX and the tangent space T_pN is constant (independent of the choice of $p \in N$ and $X \in T_pN$), then N is said to be a slant submanifold. Complex and totally real submanifolds are characterized by $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. Examples and properties of slant submanifolds were first discussed in [8], and afterward the subject was developed in different ambient spaces. There are also generalizations or other particular cases of slant submanifolds, as purely real submanifolds, bi-slant submanifolds, etc.

Another branch of the modern theory of submanifolds involves inequalities relating the intrinsic and extrinsic invariants. B.-Y. Chen has an important contribution in this respect by introducing the δ-invariants (also called Chen invariants), which are different in nature from the classical intrinsic invariants (see [15]), giving new answers to the famous Nash's embedding theorem.

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This chapter is a survey of certain papers on the geometry of slant submanifolds in complex and Sasakian space forms published after the B.-Y. Chen's book [8]. It is divided into two sections, each of them containing all necessary definitions and formulae. They put together results (most of them obtained by the authors and their coworkers) on Euler inequality, Chen-Ricci inequality, shape operator, generalized Wintgen inequality for submanifolds in complex and Sasakian space forms, respectively (see also [34]). Moreover, Sect. 2 contains a classification of quasi-minimal slant surfaces in $\mathbb{C}_1^2$. Most of the theorems are proved, for the rest we gave complete references.

2 Slant Submanifolds in Complex Space Forms

Let $\tilde{M}$ be an n-dimensional complex manifold and J its canonical almost complex structure. We denote by $\Gamma(T\tilde{M})$ the set of sections of the tangent bundle $T\tilde{M}$.

A Hermitian metric on $\tilde{M}$ is a Riemannian metric g invariant by J, that is,

$$g(JX, JY) = g(X, Y), \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

A complex manifold $\tilde{M}$ endowed with a Hermitian metric g is called a Hermitian manifold.

We recall that each complex manifold admits a Hermitian metric.

Any Hermitian metric g on the complex manifold $\tilde{M}$ determines a non-degenerate 2-form $\omega(X, Y) = g(JX, Y),\ X, Y \in \Gamma(T\tilde{M})$, called the fundamental (Kähler) 2-form.

Definition. A Hermitian manifold is said to be a *Kähler manifold* if the fundamental 2-form ω is closed.

It is known that on a Kähler manifold the almost complex structure J is parallel with respect to the Levi-Civita connection $\tilde{\nabla}$ ($\tilde{\nabla} J = 0$).

A Kähler manifold $\tilde{M}$ is said to be a *complex space form* if the holomorphic sectional curvature function is constant for all holomorphic plane sections π in $T_p\tilde{M}$ and all points $p \in \tilde{M}$.

A complex space form with constant holomorphic sectional curvature c is denoted by $\tilde{M}(c)$.

The curvature tensor $\tilde{R}$ of a complex space form $\tilde{M}(c)$ is given by

$$\tilde{R}(X, Y, Z, W) = \frac{c}{4}[g(X, Z)g(Y, W) - g(X, W)g(Y, Z)-$$

$$-g(JX, W)g(JY, Z) + g(JX, Z)g(JY, W) + 2g(X, JY)g(Z, JW)],$$

for any vector fields X, Y, Z, W on $\tilde{M}(c)$.

Each complex space form is an Einstein space.

Examples:

1. The complex Euclidean space $\mathbb{C}^n$ endowed with the Euclidean metric is a flat complex space form ($c = 0$).

2. The complex projective space $P^n(\mathbb{C})$ endowed with the Fubini-Study metric has positive constant holomorphic sectional curvature ($c = 4$).

3. The complex unit disk D^n endowed with the Bergman metric has negative constant holomorphic sectional curvature ($c = -4$).

According to the behavior of the tangent spaces of a submanifold M under the action of the complex structure of the ambient Kähler manifold $\tilde{M}$, we distinguish two basic classes of submanifolds:

(a) *Complex submanifolds* (the complex structure preserves all tangent spaces of the submanifold: $J(T_pM) = T_pM$, for any $p \in M$).

(b) *Totally real submanifolds* (the complex structure transforms all tangent spaces into the normal spaces of the submanifold: $J(T_pM) \subset T_p^{\perp}M$, for any $p \in M$). In particular, if $\dim M = \dim_{\mathbb{C}} \tilde{M}$, the submanifold M is called *Lagrangian*.

Afterward, interesting generalizations of the above classes of submanifolds were introduced: slant submanifolds, purely real submanifolds, *CR*-submanifolds, etc.

Definition. [8] A *slant submanifold* is a submanifold M of a Kähler manifold $(\tilde{M}, J, g)$ such that, for any nonzero vector X in T_pM, the angle $\theta(X)$ between JX and the tangent space T_pM is a constant (which is independent of the choice of the point $p \in M$ and the choice of the tangent vector X in the tangent plane T_pM).

It is obvious that complex submanifolds and totally real submanifolds are special classes of slant submanifolds.

A slant submanifold is called *proper* if it is neither a complex submanifold nor a totally real submanifold.

Examples of slant submanifolds are given in the above book of Chen [8] and in another chapter of this book [44], respectively.

For any vector field X tangent to a submanifold M in a Kähler manifold $(\tilde{M}, J, g)$, one decomposes $JX = PX + FX$, where PX and FX are the tangential and normal components of JX, respectively.

A proper slant submanifold is said to be *Kählerian slant* if the canonical endomorphism P is parallel ($\nabla P = 0$), where ∇ is the Levi-Civita connection on M.

A Kählerian slant submanifold is a Kähler manifold with respect to the induced metric and the almost complex structure $\tilde{J} = (\sec\theta)J$, where θ is the slant angle.

Let M be an n-dimensional Kählerian slant submanifold in an n-dimensional Kähler manifold $\tilde{M}$, $p \in M$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis of T_pM. If we put $e_i^* = \frac{1}{\sin\theta} Fe_i$, $i = 1, \ldots, n$, then $\{e_1^*, \ldots, e_n^*\}$ is an orthonormal basis of $T_p^{\perp}M$. The coefficients $h_{ij}^k = g(h(e_i, e_j), e_k^*)$ of the second fundamental form have the symmetry property: $h_{ij}^k = h_{jk}^i = h_{ki}^j$, for all $i, j, k = 1, \ldots, n$.

A submanifold M of a Kähler manifold is called a *purely real submanifold* if every eigenvalue of $Q = P^2$ lies in $(-1, 0]$, or equivalently, $FX \neq 0$, for any nonzero vector X tangent to M.

Thus, by definition, the class of purely real submanifolds contains both slant submanifolds and totally real submanifolds (in particular Lagrangian submanifolds).

2.1 Euler Inequality

For surfaces M of the Euclidean space $\mathbb{E}^3$, the (classical) Euler inequality

$$G \leq \|H\|^2$$

is fulfilled, where G is the (intrinsic) Gauss curvature of M and $\|H\|^2$ is the (extrinsic) squared mean curvature of M.

Furthermore, $G = \|H\|^2$ everywhere on M if and only if M is totally umbilical (the second fundamental form h satisfies $h(X, Y) = g(X, Y)H$, for any vector fields X and Y), or still, by a theorem of Meusnier, if and only if M is (a part of) a plane $\mathbb{E}^2$ or it is (a part of) a round sphere S^2 in $\mathbb{E}^3$.

In [10], B.-Y. Chen generalized the Euler inequality for any dimensional submanifolds M in real space forms $\tilde{M}(c)$.

Let M be an n-dimensional submanifold ($n \geq 2$) of an m-dimensional real space form $\tilde{M}(c)$. One denotes as usual by K the sectional curvature on M. Let $p \in M$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis of T_pM.

The scalar curvature at p is defined by

$$\tau(p) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j);$$

denote by $\rho = \frac{2\tau}{n(n-1)}$ the normalized scalar curvature.

The generalized Euler inequality states (see [10])

$$\rho \leq \|H\|^2 + c.$$

Moreover, the equality holds identically if and only if M is a totally umbilical submanifold.

B.-Y. Chen [17] proved the following sharp estimate of the squared mean curvature in terms of the scalar curvature for Kählerian slant submanifolds in complex space forms.

Theorem 2.1.1 *Let M be an n-dimensional ($n \geq 2$) Kählerian slant submanifold of an n-dimensional complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then*

$$\|H\|^2 \geq \frac{2(n+2)}{n^2(n-1)}\tau - \frac{n+2}{n}\left(1 + 3\frac{\cos^2\theta}{n-1}\right)c, \tag{2.1.1}$$

where θ is the slant angle of M.

In particular, for Lagrangian submanifolds, one derives:

Corollary 2.1.2 *Let M be a Lagrangian submanifold of an n-dimensional $(n \geq 2)$ complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then*

$$\|H\|^2 \geq \frac{2(n+2)}{n^2(n-1)}\tau - \frac{n+2}{n}c. \tag{2.1.2}$$

The inequality (2.1.2) was first obtained in [11].

On the other hand, it is known that any proper slant surface is Kählerian slant. Thus, the previous theorem implies the following:

Corollary 2.1.3 *Let M be a proper slant surface in a complex space form $\tilde{M}(4c)$ of complex dimension 2. Then the squared mean curvature $\|H\|^2$ and the Gaussian curvature G of M satisfy*

$$\|H\|^2 \geq 2[G - (1 + 3\cos^2\theta)c], \tag{2.1.3}$$

at each point $p \in M$, where θ is the slant angle of the slant surface.

The above inequality was obtained by B.-Y. Chen in [13] and, recently, as a corollary of a result from [18].

Theorem 2.1.4 *Let M be a purely real surface in a complex space form $\widetilde{M}(4c)$ of complex dimension 2. Then*

$$\|H\|^2 \geq 2[G - \|\nabla\alpha\|^2 - (1 + 3\cos^2\theta)c] + 4g(\nabla\alpha, Jh(e_1, e_2))\csc\alpha,$$

with respect to any orthonormal frame $\{e_1, e_2\}$ satisfying $g(\nabla\alpha, e_2) = 0$ (α is the Wirtinger angle $(\cos\alpha = g(Je_1, e_2))$, and $\nabla\alpha$ is the gradient of α).

The first author generalized Theorem 2.1.1 for purely real submanifolds with P parallel with respect to the Levi-Civita connection, $\nabla P = 0$. Such submanifolds are a generalization of Kählerian slant submanifolds.

Theorem 2.1.5 ([33]) *Let M be a purely real n-dimensional $(n \geq 2)$ submanifold with $\nabla P = 0$ of an n-dimensional complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then*

$$\|H\|^2 \geq \frac{2(n+2)}{n^2(n-1)}\tau - \frac{n+2}{n}\left[1 + 3\frac{\|P\|^2}{n(n-1)}\right]c. \tag{2.1.4}$$

Proof Let $p \in M$ and $\{e_1, e_2, \ldots, e_n\}$ an orthonormal basis of the tangent space T_pM such that all e_j's are eigenvectors of P^2. An orthonormal basis $\{e_1^*, e_2^*, \ldots, e_n^*\}$ of the normal space $T_p^{\perp}M$ is defined by $e_i^* = \frac{Fe_i}{\|Fe_i\|}$, $i = \overline{1, n}$.

For a purely real submanifold with $\nabla P = 0$ one has

$$A_{FX}Y = A_{FY}X, \quad \forall X, Y \in \Gamma(TM),$$

or equivalently,

$$h_{ij}^k = h_{ik}^j = h_{kj}^i,$$

where A means the shape operator and $h_{ij}^k = g(h(e_i, e_j), e_k^*)$, $i, j, k = 1, \ldots, n$.

From the Gauss equation, it follows that

$$2\tau = n^2 \|H\|^2 - \|h\|^2 + c[n(n-1) + 3\|P\|^2].$$

By the definition, the squared mean curvature is given by

$$n^2 \|H\|^2 = \sum_i \left[\sum_j (h_{jj}^i)^2 + 2 \sum_{j<k} h_{jj}^i h_{kk}^i \right].$$

We derive

$$\tau = \frac{n(n-1) + 3\|P\|^2}{2} c + \sum_i \sum_{j<k} h_{jj}^i h_{kk}^i - \sum_{i \neq j} (h_{jj}^i)^2 - 3 \sum_{i<j<k} (h_{ij}^k)^2.$$

If we denote $m = \frac{n+2}{n-1}$, we get

$$n^2 \|H\|^2 - m[2\tau - n(n-1)c - 3\|P\|^2 c] =$$

$$= \sum_i (h_{ii}^i)^2 + (1+2m) \sum_{i \neq j} (h_{jj}^i)^2 + 6m \sum_{i<j<k} (h_{ij}^k)^2 - 2(m-1) \sum_i \sum_{j<k} h_{jj}^i h_{kk}^i =$$

$$= \sum_i (h_{ii}^i)^2 + 6m \sum_{i<j<k} (h_{ij}^k)^2 + (m-1) \sum_i \sum_{j<k} (h_{jj}^i - h_{kk}^i)^2 +$$

$$+ [1 + 2m - (n-2)(m-1)] \sum_{i \neq j} (h_{jj}^i)^2 - 2(m-1) \sum_{i \neq j} h_{ii}^i h_{jj}^i =$$

$$= 6m \sum_{i<j<k} (h_{ij}^k)^2 + (m-1) \sum_{i \neq j,k} \sum_{j<k} (h_{jj}^i - h_{kk}^i)^2 +$$

$$+ \frac{1}{n-1} \sum_{i \neq j} [h_{ii}^i - (n-1)(m-1) h_{jj}^i]^2 \geq 0.$$

It follows that

$$n^2 \|H\|^2 - m[2\tau - n(n-1)c - 3\|P\|^2 c] \geq 0,$$

equivalent to

$$n^2 \|H\|^2 - \frac{n+2}{n-1}[2\tau - n(n-1)c - 3\|P\|^2 c] \geq 0,$$

which is, in fact, the inequality to prove.

2.2 *Chen-Ricci Inequality*

Let (M, g) be an n-dimensional submanifold of a Riemannian manifold $\tilde{M}$. For any $p \in M$ and any unit tangent vector $X \in T_pM$, we consider an orthonormal basis $\{e_1 = X, e_2, \ldots, e_n\} \subset T_pM$. The Ricci curvature of X is defined by

$$Ric(X) = \sum_{i=2}^{n} K(X \wedge e_i).$$

One denotes by $\ker h_p = \{X \in T_pM | \, h(X, Y) = 0, \forall Y \in T_pM\}$.

In [14], B.-Y. Chen obtained a sharp inequality between the Ricci curvature and the squared mean curvature for any Riemannian submanifold of dimension n in a real space form $\tilde{M}(c)$ of constant sectional curvature c:

$$Ric(X) \leq (n-1)c + \frac{n^2}{4}\|H\|^2. \tag{2.2.1}$$

This relation is known as the *Chen-Ricci inequality*. For Lagrangian submanifolds in a complex space form $\tilde{M}(4c)$ the same inequality holds (see [16]).

In [32], K. Matsumoto and the present authors extended the Chen-Ricci inequality for arbitrary submanifolds in complex space forms.

Theorem 2.2.1 ([32]) *Let M be an n-dimensional submanifold of a complex m-dimensional complex space form $\tilde{M}(4c)$. Then:*
(i) For each unit vector $X \in T_pM$ we have

$$Ric(X) \leq (n-1)c + \frac{n^2}{4}\|H\|^2 + 3c\|PX\|^2,$$

where J is the standard almost complex structure on $\tilde{M}(4c)$ and PX is the tangential component of JX.

(ii) If $H(p) = 0$*, then a unit tangent vector* X *at* p *satisfies the equality case if and only if* $X \in \ker h_p$*.*
(iii) The equality case holds identically for all unit tangent vectors at p *if and only if* p *is a totally geodesic point or* $n = 2$ *and* p *is a totally umbilical point.*

For the particular case of θ-slant submanifolds, we derive:

Corollary 2.2.2 ([32]) *Let* M *be an* n*-dimensional* θ*-slant submanifold of a complex space form* $\tilde{M}(4c)$*. Then:*
(i) For each unit vector $X \in T_pM$ *we have*

$$Ric(X) \leq (n-1)c + \frac{n^2}{4}\|H\|^2 + 3c\cos^2\theta.$$

(ii) If $H(p) = 0$*, then a unit tangent vector* X *at* p *satisfies the equality case if and only if* $X \in \ker h_p$*.*
(iii) The equality case holds identically for all unit tangent vectors at p *if and only if* p *is a totally geodesic point or* $n = 2$ *and* p *is a totally umbilical point.*

We recall that a point $p \in M$ is called a *H-umbilical point* if the second fundamental form is given by

$$h(e_1, e_1) = \lambda e_1^*, h(e_2, e_2) = \cdots = h(e_n, e_n) = \mu e_1^*, h(e_i, e_j) = 0\,(2 \leq i < j \leq n),$$

where $\{e_1, \ldots, e_n\}$ and $\{e_1^*, \ldots, e_n^*\}$ are orthonormal bases in T_pM and $T_p^{\perp}M$, $\lambda, \mu \in \mathbb{R}$.

Also, the Chen-Ricci inequality was improved later for Lagrangian submanifolds.

Theorem 2.2.3 ([25]) *Let* M *be a Lagrangian submanifold of dimension* $n \geq 2$ *in a complex space form* $\tilde{M}(4c)$ *of constant holomorphic sectional curvature* $4c$ *and* X *a unit tangent vector in* T_pM*,* $p \in M$*. Then, we have*

$$Ric(X) \leq (n-1)\left(c + \frac{n}{4}\|H\|^2\right).$$

The equality sign holds for any unit tangent vector at p *if and only if either*:
(i) p *is a totally geodesic point, or*
(ii) $n = 2$ *and* p *is an H-umbilical point with* $\lambda = 3\mu$*.*

In the same paper [25], the author determined Lagrangian submanifolds in complex space forms which satisfy identically the equality case.

The Whitney 2-sphere in $\mathbb{C}^2$ is a nontrivial example of a Lagrangian submanifold which satisfies the equality case of the improved Chen-Ricci inequality identically.

The first author and I.N. Rădulescu extended the Theorem 2.2.3 to Kählerian slant submanifolds in complex space forms, by applying the two algebraic lemmas from [25].

Lemma 1 *Let $f_1(x_1, x_2, \ldots, x_n)$ be a function on $\mathbb{R}^n$ defined by:*

$$f_1(x_1, x_2, \ldots, x_n) = x_1 \sum_{j=2}^{n} x_j - \sum_{j=2}^{n} x_j^2.$$

If $x_1 + x_2 + \cdots + x_n = 2na$, then we have

$$f_1(x_1, x_2, \ldots, x_n) \leq \frac{n-1}{4n} (x_1 + x_2 + \cdots + x_n)^2,$$

with the equality sign holding if and only if $\frac{1}{n+1} x_1 = x_2 = \cdots = x_n = a$.

Lemma 2 *Let $f_2(x_1, x_2, \ldots, x_n)$ be a function on $\mathbb{R}^n$ defined by:*

$$f_2(x_1, x_2, \ldots, x_n) = x_1 \sum_{j=2}^{n} x_j - x_1^2.$$

If $x_1 + x_2 + \cdots + x_n = 4a$, then we have

$$f_2(x_1, x_2, \ldots, x_n) \leq \frac{1}{8} (x_1 + x_2 + \cdots + x_n)^2,$$

with the equality sign holding if and only if $x_1 = a$ and $x_2 + \cdots + x_n = 3a$.

The improved Chen-Ricci inequality for Kählerian slant submanifolds is given in the following theorem.

Theorem 2.2.4 ([35]) *Let M be an n-dimensional Kählerian θ-slant submanifold in a complex n-dimensional complex space form $\tilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then for any unit tangent vector X to M, we have*

$$Ric(X) \leq (n-1)\left(c + \frac{n}{4} \|H\|^2\right) + 3c\cos^2\theta. \tag{2.2.2}$$

The equality sign of (2.2.2) holds identically if and only if either
(i) $c = 0$ and M is totally geodesic, or
(ii) $n = 2$, $c < 0$ and M is a slant H-umbilical surface with $\lambda = 3\mu$.

Proof Let $p \in M$ and X a unit vector in T_pM; one considers the orthonormal bases $\{e_1 = X, e_2, \ldots, e_n\} \subset T_pM$ and

$$\left\{e_1^* = \frac{Fe_1}{\sin\theta}, \ldots, e_n^* = \frac{Fe_n}{\sin\theta}\right\} \subset T_p^{\perp}M.$$

Now we put in the equation of Gauss $X = Z = e_1$ and $Y = W = e_j$, for $j = 2, .., n$. Then Gauss equation gives

$$\widetilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - g(h(e_1, e_1), h(e_j, e_j)) + g(h(e_1, e_j), h(e_1, e_j)),$$

or, equivalently,

$$\widetilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - \sum_{r=1}^{n} (h^r_{11} h^r_{jj} - (h^r_{1j})^2),\ \forall j \in \{2, \ldots, n\}.$$

Since the Riemannian curvature tensor of M is expressed by

$$\widetilde{R}(X, Y, Z, W) = c[g(X, Z)g(Y, W) - g(X, W)g(Y, Z)+$$

$$+g(JX, Z)g(JY, W) - g(JX, W)g(JY, Z) + 2g(JX, Y)g(JZ, W)],$$

we find

$$\widetilde{R}(e_1, e_j, e_1, e_j) = c[1 + 3g^2(Je_1, e_j)]. \tag{2.2.3}$$

By summing after $j \in \{2, \ldots, n\}$, we get

$$(n - 1 + 3\,\|PX\|^2)c = Ric(X) - \sum_{r=1}^{n} \sum_{j=2}^{n} \left[h^r_{11} h^r_{jj} - (h^r_{1j})^2\right],$$

or,

$$(n - 1 + 3\cos^2\theta)c = Ric(X) - \sum_{r=1}^{n} \sum_{j=2}^{n} \left[h^r_{11} h^r_{jj} - (h^r_{1j})^2\right].$$

It follows that

$$Ric(X) - (n - 1 + 3\cos^2\theta)c = \sum_{r=1}^{n} \sum_{j=2}^{n} \left[h^r_{11} h^r_{jj} - (h^r_{1j})^2\right] \leq \tag{2.2.4}$$

$$\leq \sum_{r=1}^{n} \sum_{j=2}^{n} h^r_{11} h^r_{jj} - \sum_{j=2}^{n} (h^1_{1j})^2 - \sum_{j=2}^{n} (h^j_{1j})^2.$$

Because M is a Kählerian slant submanifold, one has $h^1_{1j} = h^j_{11}$ and $h^j_{1j} = h^1_{jj}$, and then

$$Ric(X) - (n - 1 + 3\cos^2\theta)c \leq \sum_{r=1}^{n} \sum_{j=2}^{n} h^r_{11} h^r_{jj} - \sum_{j=2}^{n} (h^j_{11})^2 - \sum_{j=2}^{n} (h^1_{jj})^2. \tag{2.2.5}$$

Now we put

$$f_1(h^1_{11}, h^1_{22}, \dots, h^1_{nn}) = h^1_{11}\sum_{j=2}^{n} h^1_{jj} - \sum_{j=2}^{n}(h^1_{jj})^2$$

and

$$f_r(h^r_{11}, h^r_{22}, \dots, h^r_{nn}) = h^r_{11}\sum_{j=2}^{n} h^r_{jj} - (h^r_{11})^2,\ \forall r \in \{2, \dots, n\}.$$

Denote by $H^r = g(H, e^*_r)$, $r = 1, \dots, n$. Since $nH^1 = h^1_{11} + h^1_{22} + \cdots + h^1_{nn}$, we obtain by using Lemma 1 that

$$f_1(h^1_{11}, h^1_{22}, \dots, h^1_{nn}) \le \frac{n-1}{4n}(nH^1)^2 = \frac{n(n-1)}{4}(H^1)^2. \tag{2.2.6}$$

By Lemma 2 for $2 \le r \le n$, we get

$$f_r(h^r_{11}, h^r_{22}, \dots, h^r_{nn}) \le \frac{1}{8}(nH^r)^2 = \frac{n^2}{8}(H^r)^2 \le \frac{n(n-1)}{4}(H^r)^2. \tag{2.2.7}$$

From (2.2.5)–(2.2.7), we obtain

$$Ric(X) - (n-1+3\cos^2\theta)c \le \frac{n(n-1)}{4}\sum_{r=1}^{n}(H^r)^2 = \frac{n(n-1)}{4}\,\|H\|^2.$$

Thus, we have

$$Ric(X) \le (n-1+3\cos^2\theta)c + \frac{n(n-1)}{4}\,\|H\|^2,$$

which implies (2.2.2).

We will study the equality case. For $n \ge 3$, we choose Fe_1 parallel to H. Then we obtain $H^r = 0$, for $r \ge 2$. Thus, by Lemma 2, we get

$$h^1_{1j} = h^j_{11} = \frac{nH^j}{4} = 0,\ \forall j \ge 2,$$

and

$$h^1_{jk} = 0,\ \forall j, k \ge 2,\ j \ne k.$$

From Lemma 1, we have $h^1_{11} = (n+1)a$ and $h^1_{jj} = a$, $\forall j \ge 2$, with $a = \frac{H^1}{2}$.

In (2.2.4), we computed $Ric(X) = Ric(e_1)$. Similarly, by computing $Ric(e_2)$ and using the equality, we get

$$h^r_{2j} = h^2_{jr} = 0, \quad \forall r \neq 2, \quad j \neq 2, \quad r \neq j.$$

Then we obtain

$$\frac{h^2_{11}}{n+1} = h^2_{22} = \cdots = h^2_{nn} = \frac{H^2}{2} = 0.$$

The argument is also true for matrices $\left(h^r_{jk}\right)$ because the equality holds for all unit tangent vectors; so, $h^2_{2j} = h^j_{22} = \frac{H^j}{2} = 0, \ \forall j \geq 3$.

The matrix $\left(h^2_{jk}\right)$ (respectively, the matrix $\left(h^r_{jk}\right)$) has only two possible nonzero entries $h^2_{12} = h^2_{21} = h^1_{22} = \frac{H^1}{2}$ (respectively, $h^r_{1r} = h^r_{r1} = h^1_{rr} = \frac{H^1}{2}$, for all $r \geq 3$). For $X = Z = e_2$ and $Y = W = e_j$, $j = 2, \ldots, n$, in the Gauss equation, we get

$$\widetilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \left(\frac{H^1}{2}\right)^2, \forall j \geq 3.$$

If we consider $X = Z = e_2$ and $Y = W = e_1$ in the Gauss equation, we get

$$\widetilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - (n+1)\left(\frac{H^1}{2}\right)^2 + \left(\frac{H^1}{2}\right)^2.$$

From the last two relations, we derive

$$Ric(e_2) - (n - 1 + 3\cos^2\theta)c = 2(n-1)\left(\frac{H^1}{2}\right)^2.$$

On the other hand, the equality case of (2.2.2) implies that

$$Ric(e_2) - (n - 1 + 3\cos^2\theta)c = \frac{n(n-1)}{4}\|H\|^2 = n(n-1)\left(\frac{H^1}{2}\right)^2.$$

Since $n \neq 1, 2$, from the last 2 equations we get $H^1 = 0$. Thus, $\left(h^r_{jk}\right)$ are all zero; then M is a totally geodesic submanifold in $\tilde{M}(4c)$ and consequently M is a curvature-invariant submanifold of $\tilde{M}(4c)$. When $c \neq 0$, a result of Chen and Ogiue [21] implies that M is either a complex submanifold or a Lagrangian submanifold of $\tilde{M}(4c)$, which is a contradiction, because M is a non-proper θ-slant submanifold. Then, we obtain either

(1) $c = 0$ and M is totally geodesic, or

(2) $n = 2$.

If (1) holds, we find (i) of the theorem.

In the second case, for $n = 2$, by a result of Chen [13] it is known that for M a proper slant surface in a complex 2-dimensional complex space form $\tilde{M}(4c)$ which satisfies identically the equality case of (2.2.2), either M is totally geodesic or $c < 0$. If M is not totally geodesic, we get

$$h(e_1, e_1) = \lambda e_1^*, h(e_2, e_2) = \mu e_1^*, h(e_1, e_2) = \mu e_2^*,$$

with $\lambda = 3\mu = \frac{3H^1}{2}$; then M is H-umbilical, i.e., the case (ii).

Because a proper slant surface is Kählerian slant (see [8]), we rediscovered the following result.

Theorem 2.2.5 ([13]) *If M is a proper slant surface in a complex space form $\widetilde{M}(4c)$ of complex dimension 2, then the squared mean curvature and the Gaussian curvature of M satisfy:*

$$\|H\|^2 \geq 2[G - (1 + 3\cos^2\theta)c]$$

at each point $p \in M$, where θ is the slant angle of the slant surface.

A nontrivial example of a slant surface satisfying the equality case identically is given in the same paper [35].

2.3 Shape Operator A_H

B.-Y. Chen established a relationship between the sectional curvature function K and the shape operator for submanifolds in real space forms [10]. We obtained a similar inequality for a slant submanifold M into an m-dimensional complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c (see [31]).

Theorem 2.3.1 ([31]) *Let $x : M \to \tilde{M}(c)$ be an isometric immersion of an n-dimensional θ-slant submanifold into an m-dimensional complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature $c \geq 0$. If there exists a point $p \in M$ and a number $b > \frac{c}{4}(1 + \frac{3}{n-1}\cos^2\theta)$ such that $K \geq b$ at p, then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\left[b - \frac{c}{4} - 3\frac{c}{4(n-1)}\cos^2\theta\right] I_n, \tag{2.3.1}$$

where I_n is the identity map.

Proof Let $p \in M$ and a number $b > \frac{c}{4}(1 + \frac{3}{n-1}\cos^2\theta)$ such that $K \geq b$ at p. We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}\}$ at p such that e_{n+1} is parallel to the mean curvature vector H and $e_1, \ldots, e_n$ diagonalize the shape operator A_{n+1}.

Then we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix}, \tag{2.3.2}$$

$$A_r = (h^r_{ij}),\ i, j = 1, \ldots, n,\ r = n+2, \ldots, 2m,$$

$$\text{trace } A_r = \sum_{i=1}^{n} h^r_{ii} = 0. \tag{2.3.3}$$

For $i \neq j$, we denote by $u_{ij} = a_i a_j$.
From Gauss equation for $X = Z = e_i$, $Y = W = e_j$, we get

$$u_{ij} \geq b - \frac{c}{4} - 3\frac{c}{4} g^2(e_i, Je_j) - \sum_{r=n+2}^{2m} [h^r_{ii} h^r_{jj} - (h^r_{ij})^2]. \tag{2.3.4}$$

We will prove that u_{ij} have the following properties:
1. For any fixed $i \in \{1, \ldots, n\}$, one has

$$\sum_{i \neq j} u_{ij} \geq (n-1)(b - \frac{c}{4}) - 3\frac{c}{4} \cos^2 \theta.$$

2. $u_{ij} \neq 0$, for $i \neq j$.
3. For distinct $i, j, k \in \{1, \ldots, n\}$, $a_i^2 = \dfrac{u_{ij} u_{ik}}{u_{jk}}$.
4. We denote by $S_k = \{B \subset \{1, \ldots, n\};\ |B| = k\}$ and for any $B \in S_k$ we denote by $\overline{B} = \{1, \ldots, n\} \setminus B$. Then, for a fixed k, $1 \leq k \leq \left[\frac{n}{2}\right]$ and each $B \in S_k$, we have

$$\sum_{j \in B} \sum_{t \in \overline{B}} u_{jt} > 0.$$

5. For distinct $i, j \in \{1, \ldots, n\}$, $u_{ij} > 0$.

1. Indeed, (2.3.4) implies:

$$\sum_{j \neq i} u_{ij} \geq (n-1)(b - \frac{c}{4}) - 3\frac{c}{4} \|Pe_i\|^2 - \sum_{r=n+2}^{2m} [h^r_{ii}(\sum_{j \neq i} h^r_{jj}) - \sum_{j \neq i} (h^r_{ij})^2] =$$

$$= (n-1)(b - \frac{c}{4}) - 3\frac{c}{4} \cos^2 \theta - \sum_{r=n+2}^{2m} [h^r_{ii}(-h^r_{ii}) - \sum_{j \neq i} (h^r_{ij})^2] =$$

$$= (n-1)(b - \frac{c}{4}) - 3\frac{c}{4} \cos^2 \theta + \sum_{r=n+2}^{2m} \sum_{j=1}^{n} (h^r_{ij})^2 \geq$$

$$\geq (n-1)(b - \frac{c}{4}) - 3\frac{c}{4} \cos^2 \theta > 0.$$

2. If $u_{ij} = 0$, for $i \neq j$, then $a_i = 0$ or $a_j = 0$. Assume $a_i = 0$; then $u_{it} = a_i a_t = 0$, for any $t \in \{1, \dots, n\}$, $t \neq i$. It follows that

$$\sum_{j \neq i} u_{ij} = 0,$$

in contradiction with 1.

3. $\dfrac{u_{ij} u_{ik}}{u_{jk}} = \dfrac{a_i a_j a_i a_k}{a_j a_k} = a_i^2.$

4. Let $B = \{1, \dots, k\}$ and $\overline{B} = \{k+1, \dots, n\}$. Then

$$\sum_{j \in B} \sum_{t \in \overline{B}} u_{jt} \geq k(n-k)(b - \frac{c}{4}) - 3\frac{c}{4} \sum_{j=1}^{k} \sum_{t=k+1}^{n} g^2(Je_i, e_j) -$$

$$- \sum_{r=n+2}^{2m} \{\sum_{j=1}^{k} \sum_{t=k+1}^{n} [h_{jj}^r h_{tt}^r - (h_{jt}^r)^2]\}.$$

We choose $\{e_1, \dots, e_n\}$ an adapted slant basis. Then we distinguish 2 cases:

(i) If k is odd,

$$\sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} \geq k(n-k)(b - \frac{c}{4}) - 3\frac{c}{4} \cos^2\theta + \sum_{r=n+2}^{2m} [\sum_{j=1}^{k} \sum_{t=k+1}^{n} (h_{jt}^r)^2 + \sum_{j=1}^{k} (h_{jj}^r)^2] \geq$$

$$\geq k(n-k)(b - \frac{c}{4}) - 3\frac{c}{4} \cos^2\theta > 0.$$

(ii) If k is even,

$$\sum_{j \in B} \sum_{t \in \bar{B}} u_{jt} \geq k(n-k)(b - \frac{c}{4}) > 0.$$

5. Assume $u_{1n} < 0$. From 3, we get $u_{1i} u_{in} < 0$, for $1 < i < n$.

Without loss of generality, we may assume

$$\begin{cases} u_{12}, \dots, u_{1l}, u_{(l+1)n}, \dots, u_{(n-1)n} > 0, \\ u_{1(l+1)}, \dots, u_{1n}, u_{2n}, \dots, u_{\mathrm{ln}} < 0, \end{cases}$$

for some $\left[\frac{n+1}{2}\right] \leq l \leq n-1$.

If $l = n-1$, then $u_{1n} + u_{2n} + \cdots + u_{(n-1)n} < 0$, which contradicts to 1. Thus, $l < n-1$.

From 3, we get

$$a_n^2 = \frac{u_{in} u_{tn}}{u_{it}} > 0,$$

where $2 \leq i \leq l, l+1 \leq t \leq n-1$. From the last two relations we obtain $u_{it} < 0$, which implies

$$\sum_{i=1}^{l} \sum_{t=l+1}^{n} u_{it} = \sum_{i=2}^{l} \sum_{t=l+1}^{n-1} u_{it} + \sum_{i=1}^{l} u_{in} + \sum_{t=l+1}^{n} u_{1t} < 0.$$

This contradicts to 4.

Now, we return to the proof of the theorem.

From 5, it follows that $a_1, \ldots, a_n$ have the same sign. Assume $a_j > 0, \forall j \in \{1, \ldots, n\}$. Then

$$\sum_{j \neq i} u_{ij} = a_i(a_1 + \ldots a_n) - a_i^2 \geq (n-1)(b - \frac{c}{4}) - 3\frac{c}{4}\cos^2\theta.$$

From the above relation, we have

$$a_i n \|H\| \geq (n-1)(b - \frac{c}{4}) - 3\frac{c}{4}\cos^2\theta + a_i^2 > (n-1)(b - \frac{c}{4}) - 3\frac{c}{4}\cos^2\theta.$$

This equation implies that

$$a_i \|H\| > \frac{n-1}{n}(b - \frac{c}{4} - 3\frac{c}{4(n-1)}\cos^2\theta),$$

and consequently the inequality (2.3.1).

Similarly, we can prove the following

Theorem 2.3.2 ([31]) *Let $x : M \to \tilde{M}(c)$ be an isometric immersion of an n-dimensional θ-slant submanifold into an m-dimensional complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature $c < 0$. If there exists a point $p \in M$ and a number $b > \frac{c}{4}$ such that $K \geq b$ at p, then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\left[b - \frac{c}{4} - 3\frac{c}{4(n-1)}\cos^2\theta\right] I_n.$$

In particular, for totally real submanifolds, one has

Corollary 2.3.3 *Let M be an n-dimensional totally real submanifold of a complex space form $\tilde{M}(c)$ and $p \in M$. If there exists a number $b > \frac{c}{4}$ such that $K \geq b$ at p, then*

$$A_H > \frac{n-1}{n}(b - \frac{c}{4})I_n.$$

Let (M, g) be an n-dimensional Riemannian manifold, k an integer, $2 \leq k \leq n$, $p \in M$ and $L \subset T_pM$ a k-plane section. The k-Ricci curvature of a unit vector $X \in L$ is given by

$$Ric_L(X) = \sum_{i=2}^{k} K(X \wedge e_i),$$

where $\{e_1 = X, e_2, \ldots, e_k\}$ is an orthonormal basis of L.

The Riemannian invariant Θ_k is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X),\ p \in M,$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

B.-Y. Chen established a relationship between the k-Ricci curvature and the shape operator for a submanifold with arbitrary codimension (see also [10]).

We proved a corresponding inequality for a slant submanifold M of an m-dimensional complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c.

Theorem 2.3.4 ([31]) *Let $x : M \to \tilde{M}(c)$ be an isometric immersion of an n-dimensional θ-slant submanifold M into a complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have:*

(i) If $\Theta_k(p) \neq \frac{c}{4}(1 + \frac{3}{n-1}\cos^2\theta)$, then the shape operator at the mean curvature satisfies

$$A_H > \frac{n-1}{n}\left[\Theta_k(p) - \frac{c}{4} - \frac{3c}{4(n-1)}\cos^2\theta\right] I_n. \tag{2.3.5}$$

(ii) If $\Theta_k(p) = \frac{c}{4}(1 + \frac{3}{n-1}\cos^2\theta)$, then $A_H \geq 0$ at p.

(iii) A unit vector $X \in T_pM$ satisfies

$$A_H X = \frac{n-1}{n}\left[\Theta_k(p) - \frac{c}{4} - \frac{3c}{4(n-1)}\cos^2\theta\right] X$$

if and only if $\Theta_k(p) = \frac{c}{4}(1 + \frac{3}{n-1}\cos^2\theta)$ and $X \in \ker h_p$.

(iv) $A_H = \frac{n-1}{n}[\Theta_k(p) - \frac{c}{4} - \frac{3c}{4(n-1)}\cos^2\theta]I_n$ at p if and only if p is a totally geodesic point.

Proof (i) Let $\{e_1, \ldots e_n\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1 \ldots i_k}$ the k-plane section spanned by $e_{i_1}, \ldots, e_{i_k}$. It is easily seen by the definitions that

$$\tau(L_{i_1\dots i_k}) = \frac{1}{2}\sum_{i\in\{i_1,\dots,i_k\}} Ric_{L_{i_1\dots i_k}}(e_i),$$

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}}\sum_{1\le i_1<\dots<i_k\le n} \tau(L_{i_1\dots i_k}).$$

Combining the above equations, we find

$$\tau(p) \ge \frac{n(n-1)}{2}\Theta_k(p). \tag{2.3.6}$$

From the equation of Gauss for $X = Z = e_i$, $Y = W = e_j$, by summing, we obtain

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - \frac{c}{4}[n(n-1) + 3\|P\|^2].$$

We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2m}\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \dots, e_n$ diagonalize the shape operator A_{n+1}. Then the shape operators have the forms (2.3.2) and (2.3.3).

It follows that

$$n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m}\sum_{i,j=1}^{n}(h_{ij}^r)^2 - \frac{c}{4}[n(n-1) + 3\|P\|^2].$$

On the other hand, the Cauchy–Schwarz inequality implies

$$\sum_{i=1}^{n} a_i^2 \ge n\|H\|^2.$$

Consequently, we have

$$n^2\|H\|^2 \ge 2\tau + n\|H\|^2 - \frac{c}{4}[n(n-1) + 3\|P\|^2],$$

or, equivalently,

$$\|H\|^2 \ge \frac{2\tau}{n(n-1)} - \frac{c}{4} - \frac{3c\|P\|^2}{4n(n-1)}.$$

By using (2.3.6), we obtain

$$\|H\|^2(p) \ge \Theta_k(p) - \frac{c}{4} - \frac{3c\|P\|^2}{4n(n-1)} = \Theta_k(p) - \frac{c}{4} - \frac{3c}{4(n-1)}\cos^2\theta.$$

This shows that $H(p) = 0$ may occurs only when $\Theta_k(p) \le \frac{c}{4}(1 + \frac{3}{n-1}\cos^2\theta)$. Consequently, if $H(p) = 0$, statements i) and ii) hold automatically. Therefore, without loss of generality, we may assume $H(p) \neq 0$.

From the equation of Gauss, we get

$$a_i a_j = K_{ij} - \frac{c}{4}[1 + 3g^2(e_i, Je_j)] - \sum_{r=n+2}^{2m}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2],$$

which implies

$$a_1(a_{i_2} + \cdots + a_{i_k}) = Ric_{L_{1i_2\ldots i_k}}(e_1) - (k-1)\frac{c}{4} - 3\frac{c}{4}\sum_{j=2}^{k} g^2(e_1, Je_{i_j}) - \tag{2.3.7}$$

$$-\sum_{r=n+2}^{2m}\sum_{j=2}^{k}[h^r_{11}h^r_{i_ji_j} - (h^r_{1i_j})^2]$$

and consequently

$$a_1(a_2 + \cdots + a_n) = \frac{1}{C^{k-2}_{n-2}}\sum_{2\le i_2<\cdots<i_k\le n} Ric_{L_{1i_2\ldots i_k}}(e_1) - (n-1)\frac{c}{4} -$$

$$-3c\sum_{j=2}^{n} g^2(e_1, Je_{j)+} + \sum_{r=n+2}^{2m}\sum_{j=1}^{n}(h^r_{1j})^2.$$

We find

$$a_1(a_2 + \cdots + a_n) \ge (n-1)[\Theta_k(p) - \frac{c}{4} - \frac{3c}{4(n-1)}\cos^2\theta].$$

Then

$$a_1(a_1 + a_2 + \cdots + a_n) = a_1^2 + a_1(a_2 + \cdots + a_n) \ge \tag{2.3.8}$$

$$\ge a_1^2 + (n-1)[\Theta_k(p) - \frac{c}{4} - \frac{3c}{4(n-1)}\cos^2\theta] \ge$$

$$\ge (n-1)[\Theta_k(p) - \frac{c}{4} - \frac{3c}{4(n-1)}\cos^2\theta].$$

Since $n\,\|H\| = a_1 + \cdots + a_n$, (2.3.8) implies

$$A_H \ge \frac{n-1}{n}[\Theta_k(p) - \frac{c}{4} - \frac{3c}{4(n-1)}\cos^2\theta]I_n.$$

The equality does not hold because in our case $H(p) \neq 0$.

The assertion (ii) is obvious.

(iii) Let $X \in T_pM$ be a unit vector satisfying the equality case. Then one has $a_1 = 0$ and $h^r_{1j} = 0, \forall j \in \{1, \ldots, n\}, r \in \{n+2, \ldots, 2m\}$, respectively. The above conditions imply $\Theta_k(p) = \frac{c}{4}(1 + \frac{3}{n-1}\cos^2\theta)$ and $X \in \ker h_p$.

The converse is clear.

(iv) The equality holds for any $X \in T_pM$ if and only if $\ker h_p = T_pM$ (p is a totally geodesic point).

Corollary 2.3.5 ([31]) *Let $x : M \to \tilde{M}(c)$ be an isometric immersion of an n-dimensional totally real submanifold M into a complex space form $\tilde{M}(c)$ of constant holomorphic sectional curvature c. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have:*

(i) If $\Theta_k(p) \neq \frac{c}{4}$, then the shape operator at the mean curvature vector satisfies

$$A_H > \frac{n-1}{n}[\Theta_k(p) - \frac{c}{4}]I_n \text{ at } p.$$

(ii) If $\Theta_k(p) = \frac{c}{4}$, then $A_H \geq 0$ at p.

(iii) A unit vector $X \in T_pM$ satisfies

$$A_H X = \frac{n-1}{n}\left(\Theta_k(p) - \frac{c}{4}\right)X$$

if and only if $\Theta_k(p) = \frac{c}{4}$ and $X \in \ker h_p$.

(iv) $A_H = \frac{n-1}{n}\left(\Theta_k(p) - \frac{c}{4}\right)I_n$ at p if and only if p is a totally geodesic point.

2.4 Chen First Inequality

In [9], B.-Y. Chen established a basic inequality involving the sectional curvature, the scalar curvature and the mean curvature of a submanifold M in a real space form $\tilde{M}(c)$. Later, this inequality was called the *Chen first inequality*. The author studied the equality case. Recall Chen's first inequality.

Theorem 2.4.1 *Let $\tilde{M}(c)$ be a real space form of constant sectional curvature c and M an n-dimensional submanifold. Then, one has:*

$$\inf K \geq \tau - \frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2 + (n+1)c\right]. \tag{2.4.1}$$

The *Chen first invariant* is defined by $\delta_M = \tau - \inf K$.

The present authors obtained in [45] a Chen first inequality for slant submanifolds in complex space forms. We considered the Riemannian invariant

$$\delta'_M(p) = \tau - \inf\{K(\pi) | \pi \subset T_pM \text{ 2-plane section invariant by } P\}.$$

Theorem 2.4.2 ([45]) *Let $\tilde{M}(4c)$ be an m-dimensional complex space form $\tilde{M}(4c)$ and M an n-dimensional θ-slant submanifold ($n \geq 3$). Then, we have:*

$$\delta'_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + (n+1+3\cos^2\theta)c\right\}. \tag{2.4.2}$$

The equality case of the inequality holds at a point $p \in M$ if and only if there exist an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}\}$ of $T_p^\perp M$ such that the shape operators of M in $\tilde{M}(4c)$ at p have the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & \ldots & 0 \\ 0 & 0 & \mu & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mu \end{pmatrix}, \quad a + b = \mu, \tag{2.4.3}$$

$$A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \ldots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \; r \in \{n+2, \ldots, 2m\}, \tag{2.4.4}$$

where one denotes by $A_r = A_{e_r}$, $r = n+1, \ldots, 2m$, and $h^r_{ij} = g(h(e_i, e_j), e_r)$, $i, j = 1, \ldots, n$, $r = n+1, \ldots, 2m$.

Theorem 2.4.3 ([38]) *Let M be an n-dimensional Kählerian slant submanifold of an n-dimensional complex space form satisfying identically the equality in Chen first inequality. Then M is a minimal submanifold.*

Proof Let M be an n-dimensional Kählerian slant submanifold of an n-dimensional complex space form satisfying identically the equality in Chen first inequality. Then the shape operators take the forms (2.4.3) and (2.4.4). Obviously, trace $A_r = 0$, for any $r \in \{n+2, \ldots, 2n\}$.

On the other hand,

$$\mu = h^1_{33} = h^3_{13} = 0.$$

Then trace $A_{n+1} = 0$. Therefore M is a minimal submanifold.

In [33] the first author extended the above inequality to purely real submanifolds M in complex space forms $\tilde{M}(4c)$.

For a 2-plane section $\pi \subset T_pM$, $p \in M$, denote by

$$\Phi(\pi) = g^2(Je_1, e_2),$$

where $\{e_1, e_2\}$ is an orthonormal basis of π (see [6]). Then $\Phi(\pi)$ is a real number in $[0, 1]$, which is independent of the choice of the orthonormal basis $\{e_1, e_2\}$ of π.

The following optimal inequality was proved by the first author. In the proof the well-known Chen's Lemma was used.

Lemma 3 ([9]) *Let $n \geq 3$ be an integer and $a_1, \ldots, a_n, b$ real numbers satisfying*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then $2a_1a_2 \geq b$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Theorem 2.4.4 ([33]) *Let M be an n-dimensional ($n \geq 3$) purely real submanifold of an m-dimensional complex space form $\tilde{M}(4c)$, $p \in M$ and $\pi \subset T_pM$ a 2-plane section. Then*

$$\tau(p) - K(\pi) \leq \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + [(n+1)(n-2) + 3\|P\|^2 - 6\Phi(\pi)]\frac{c}{2}. \tag{2.4.5}$$

Moreover, the equality case of the inequality holds at a point $p \in M$ if and only if there exist an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}\}$ of $T_p^{\perp}M$ such that the shape operators takes the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \ldots & 0 \\ 0 & b & 0 & \ldots & 0 \\ 0 & 0 & \mu & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mu \end{pmatrix}, \quad a + b = \mu,$$

$$A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \ldots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad r \in \{n+2, \ldots, 2m\}.$$

Proof Let $p \in M$, $\pi \subset T_pM$ a 2-plane section and $\{e_1, e_2\}$ an orthonormal basis of π. We construct $\{e_1, e_2, e_3, \ldots, e_n\}$ an orthonormal basis of T_pM.

The Gauss equation implies

$$2\tau = n^2\|H\|^2 - \|h\|^2 + [n(n-1) + 3\|P\|^2]c.$$

We put

$$\varepsilon = 2\tau - \frac{n^2(n-2)}{n-1} \|H\|^2 - [n(n-1) + 3\|P\|^2]c.$$

From the above two equations, we get

$$n^2 \|H\|^2 = (n-1)(\varepsilon + \|h\|^2). \tag{2.4.6}$$

We take e_{n+1} parallel with H and construct $\{e_{n+1}, \ldots, e_{2m}\}$ an orthonormal basis of $T_p^{\perp}M$. The equation (2.4.6) becomes

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)[\sum_{r=n+1}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \varepsilon],$$

or equivalently,

$$(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1)\left[\sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \varepsilon\right].$$

By applying Chen's Lemma, we obtain

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \varepsilon.$$

The Gauss equation gives

$$K(\pi) = [1 + 3\Phi(\pi)]c + \sum_{r=n+1}^{2m} [h_{11}^r h_{22}^r - (h_{12}^r)^2] \geq$$

$$\geq [1 + 3\Phi(\pi)]c + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \frac{\varepsilon}{2} + \sum_{r=n+2}^{2m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m} (h_{12}^r)^2 =$$

$$= [1 + 3\Phi(\pi)]c + \frac{1}{2} \sum_{i\neq j>2} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{3\leq i<j\leq n} (h_{ij}^r)^2 +$$

$$+\frac{1}{2} \sum_{r=n+2}^{2m} (h_{11}^r + h_{22}^r)^2 + \sum_{r=n+1}^{2m} \sum_{j=3}^{n}[(h_{1j}^r)^2 + (h_{2j}^r)^2] + \frac{\varepsilon}{2} \geq$$

$$\geq [1 + 3\Phi(\pi)]c + \frac{\varepsilon}{2},$$

which implies the inequality (2.4.5).

We have equality at a point $p \in M$ if and only if all the above inequalities become equalities and the equality case of Chen's Lemma holds. Thus, the shape operators take the desired forms.

For n-dimensional Kählerian slant submanifolds in n-dimensional complex space form $\widetilde{M}(4c)$ an improved Chen first inequality was obtained.

Theorem 2.4.5 ([33]) *Let M be an n-dimensional $(n \geq 3)$ Kaehlerian slant submanifold in the complex space form $\tilde{M}(4c)$, $\dim_{\mathbf{C}} \tilde{M}(4c) = n$, and $p \in M$, $\pi \subset T_pM$ a 2-plane section. Then*

$$\tau(p) - K(\pi) \leq \frac{n^2(2n-3)}{2(2n+3)} \|H\|^2 + [(n+1)(n-2) + 3n\cos^2\theta - 6\Phi(\pi)]\frac{c}{2}. \tag{2.4.7}$$

Moreover, the equality case of the inequality (2.4.7) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ at p such that with respect to this basis the second fundamental form takes the following form:

$$\begin{aligned} &h(e_1, e_1) = ae_1^* + 3be_3^*, \quad h(e_1, e_3) = 3be_1^*, \quad h(e_3, e_j) = 4be_j^*,\\ &h(e_2, e_2) = -ae_1^* + 3be_3^*, \quad h(e_2, e_3) = 3be_2^*, \quad h(e_j, e_k) = 4be_3^*\delta_{jk},\\ &h(e_1, e_2) = -ae_2^*, \quad h(e_3, e_3) = 12be_3^*, \quad h(e_1, e_j) = h(e_2, e_j) = 0, \end{aligned}$$

for some numbers a, b and $j, k = 4, \ldots, n$, where $e_i^* = \frac{Fe_i}{\sin\theta}$, $i = 1, \ldots, n$.

Proof Let $p \in M$, $\pi \subset T_pM$ a 2-plane section and $\{e_1, e_2, \ldots, e_n\}$ an orthonormal basis of the tangent space T_pM such that $e_1, e_2 \in \pi$. An orthonormal basis $\{e_1^*, e_2^*, \ldots, e_n^*\}$ of the normal space $T_p^{\perp}M$ is defined by $e_i^* = \frac{Fe_i}{\sin\theta}$, $i = \overline{1, n}$. We denote by $h_{ij}^k = g(h(e_i, e_j), e_k^*)$.

The Gauss equation implies

$$\tau(p) = \sum_{r=1}^{n} \sum_{1 \leq i < j \leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + [(n(n-1) + 3n\cos^2\theta]\frac{c}{2}, \tag{2.4.8}$$

and

$$K(\pi) = \sum_{r=1}^{n} [h_{11}^r h_{22}^r - (h_{12}^r)^2] + [1 + 3\Phi(\pi)]c, \tag{2.4.9}$$

respectively. Since M is a Kählerian slant submanifold, we have $h_{ij}^k = h_{jk}^i = h_{ki}^j$.

From formulas (2.4.8) and (2.4.9), we obtain

$$\tau(p) - K(\pi) = \sum_{r=1}^{n} \left\{ \sum_{j=3}^{n} (h_{11}^r + h_{22}^r)h_{jj}^r + \sum_{3 \leq i < j \leq n} h_{ii}^r h_{jj}^r - \sum_{j=3}^{n} [(h_{1j}^r)^2 + (h_{2j}^r)^2] \right\} + \tag{2.4.10}$$

$$+[(n+1)(n-2) + 3n\cos^2\theta - 6\Phi(\pi)]\frac{c}{2}.$$

It follows that

$$\tau(p) - K(\pi) \leq$$

$$\leq \sum_{r=1}^{n}[\sum_{j=3}^{n}(h_{11}^r + h_{22}^r)h_{jj}^r + \sum_{3\leq i<j\leq n} h_{ii}^r h_{jj}^r] - \sum_{j=3}^{n}(h_{11}^j)^2 - \sum_{j=3}^{n}(h_{jj}^1)^2 - \sum_{2\leq i\neq j\leq n}(h_{jj}^i)^2 +$$

$$+[(n+1)(n-2) + 3n\cos^2\theta - 6\Phi(\pi)]\frac{c}{2}.$$

In order to achieve the proof, we will use some ideas and results from [4].

We point-out the following inequalities (see [4]):

$$\sum_{j=3}^{n}(h_{11}^r + h_{22}^r)^2 h_{jj}^r + \sum_{3\leq i<j\leq n} h_{ii}^r h_{jj}^r - \sum_{j=3}^{n}(h_{jj}^r)^2 \leq \tag{2.4.11}$$

$$\leq \frac{n-2}{2(n+1)}(h_{11}^r + \cdots + h_{nn}^r)^2 \leq \frac{2n-3}{2(2n+3)}(h_{11}^r + \cdots + h_{nn}^r)^2,$$

for $r = 1, 2$. The first inequality is equivalent to

$$\sum_{j=3}^{n}(h_{11}^r + h_{22}^r - 3h_{jj}^r)^2 + 3\sum_{3\leq i<j\leq n}(h_{ii}^r - h_{jj}^r)^2 \geq 0,$$

with equality holding if and only if $3h_{jj}^r = h_{11}^r + h_{22}^r, \forall j = 3, \ldots, n.$

The equality holds in the second inequality if and only if $h_{11}^r + h_{22}^r + \cdots + h_{nn}^r = 0$.

Also, we have

$$\sum_{j=3}^{n}(h_{11}^r + h_{22}^r)^2 h_{jj}^r + \sum_{3\leq i<j\leq n} h_{ii}^r h_{jj}^r - \sum_{j=3}^{n}(h_{jj}^r)^2 \leq \frac{2n-3}{2(2n+3)}(h_{11}^r + \cdots + h_{nn}^r)^2,$$

for $r = 3, \ldots, n,$ which is equivalent to (see [4])

$$\sum_{3\leq j\leq n, j\neq r}[2(h_{11}^r + h_{22}^r) - 3h_{jj}^r]^2 + (2n+3)(h_{11}^r - h_{22}^r)^2 + \tag{2.4.12}$$

$$+6\sum_{3\leq j\leq n, j\neq r}(h_{ii}^r - h_{jj}^r)^2 + 2\sum_{j=3}^{n}(h_{rr}^r - h_{jj}^r)^2 + 3[h_{rr}^r - 2(h_{11}^r + h_{22}^r)]^2 \geq 0.$$

The equality holds in (2.4.12) if and only if

$$\begin{cases} h^r_{11} = h^r_{22} = 3\lambda^r, \\ h^r_{jj} = 4\lambda^r, \ \forall j = 3, \dots, n, \ j \neq r, \qquad \lambda^r \in \mathbf{R}. \\ h^r_{rr} = 12\lambda^r, \end{cases}$$

By summing the inequalities (2.4.11) and (2.4.12), we obtain the inequality (2.4.7).

Combining the above equality cases, we get the desired forms of the second fundamental form.

In particular, we derive the following:

Theorem 2.4.6 ([33]) *Let M be an n-dimensional ($n \geq 3$) Kählerian slant submanifold in the complex space form $\tilde{M}(4c)$, $\dim_{\mathbf{C}} \tilde{M}(4c) = n$, $p \in M$. Then*

$$\delta'_M(p) \leq \frac{n^2(2n-3)}{2(2n+3)} \|H\|^2 + (n-2)[n+1+3\cos^2\theta]\frac{c}{2}.$$

The equality case of the inequality holds at a point $p \in M$ if and only if, with respect to a suitable orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of T_pM, the second fundamental form h takes the same form as in Theorem 2.4.4.

Proof If M is Kählerian slant and $\pi \subset T_pM$ is a 2-plane section invariant by P, one has $\Phi(\pi) = \cos^2\theta$.

In contrast with the standard Chen's first inequality, the equality case of the improved Chen first inequality does not imply the minimality of the submanifold. However, we stated the following result.

Theorem 2.4.7 ([33]) *Let M be an n-dimensional Kählerian slant submanifold in the complex space form $\tilde{M}(4c)$, $\dim_{\mathbf{C}} \tilde{M}(4c) = n$ and $n \geq 4$. If the equality case holds identically in (2.4.7), then M is a minimal submanifold.*

The *proof* follows the same steps as that of Theorem 3 from [4].

In the case $n = 3$, there is an example of non-minimal Lagrangian submanifold in $\mathbb{C}P^3$ satisfying the equality case of (2.4.7) (see [3]).

2.5 Generalized Wintgen Inequality

P. Wintgen [48] proved that the Gauss curvature G, the squared mean curvature $\|H\|^2$ and the normal curvature $G^\perp$ of any surface M in $\mathbb{E}^4$ always satisfy the inequality

$$G \leq \|H\|^2 - |G^\perp|$$

and, moreover, the equality holds if and only if the ellipse of curvature of M in $\mathbb{E}^4$ is a circle.

Example. The Whitney 2-sphere satisfies the equality case identically.

In 1999, P.J. De Smet, F. Dillen, L. Verstraelen, and L. Vrancken [24] formulated the conjecture on generalized Wintgen inequality which is also known as the *DDVV conjecture*.

Conjecture. *Let $f : M \to \tilde{M}(c)$ be an isometric immersion of an n-dimensional submanifold M in an $(n+m)$-dimensional real space form $\tilde{M}(c)$ of constant sectional curvature c. Then*

$$\rho \leq \|H\|^2 - \rho^{\perp} + c,$$

where ρ is the normalized scalar curvature (intrinsic invariant) and $\rho^{\perp}$ is the normalized normal scalar curvature (extrinsic invariant).

One denotes by K and $R^{\perp}$ the sectional curvature function and the normal curvature tensor on M, respectively. Then the normalized scalar curvature is given by

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j),$$

where τ is the scalar curvature, and the normalized normal scalar curvature by

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1 \leq i < j \leq n} \sum_{1 \leq \alpha < \beta \leq m} (R^{\perp}(e_i, e_j, \xi_\alpha, \xi_\beta))^2}.$$

This conjecture was settled for the general case by Z. Lu [30] and independently by J. Ge and Z. Tang [26].

Theorem 2.5.1 *The Wintgen inequality*

$$\rho \leq \|H\|^2 - \rho^{\perp} + c$$

holds for every submanifold M in any real space form $\tilde{M}(c)$ ($n \geq 2$, $m \geq 2$).

The equality case holds identically if and only if, with respect to suitable orthonormal frames $\{e_i | i = 1, \ldots, n\}$ and $\{\xi_\alpha | \alpha = 1, \ldots, m\}$, the shape operators of M in $\tilde{M}(c)$ take the forms

$$A_{\xi_1} = \begin{pmatrix} \lambda_1 & \mu & 0 & \cdots & 0 \\ \mu & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{pmatrix},$$

$$A_{\xi_2} = \begin{pmatrix} \lambda_2 + \mu & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 - \mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{pmatrix}, \quad A_{\xi_3} = \begin{pmatrix} \lambda_3 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_3 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3$ and μ are real functions on M, and

$$A_{\xi_4} = \cdots = A_{\xi_m} = 0.$$

The second author [39] proved the generalized Wintgen inequality for Lagrangian submanifolds and slant submanifolds, respectively, in complex space forms.

Let (M, g) be an n-dimensional submanifold of an m-dimensional complex space form $\tilde{M}(4c)$. Following [49], we put

$$K_N = \frac{1}{4} \sum_{r,s=1}^{2m-n} \text{Trace}[A_r, A_s]^2$$

and call it the *scalar normal curvature* of M. The normalized scalar normal curvature is given by $\rho_N = \frac{2}{n(n-1)}\sqrt{K_N}$.

For submanifolds in real space forms, one has $\rho^{\perp} = \rho_N$.

Obviously,

$$K_N = \frac{1}{2} \sum_{1 \le r < s \le 2m-n} \text{Trace}[A_r, A_s]^2 = \sum_{1 \le r < s \le 2m-n} \sum_{1 \le i < j \le n} (g([A_r, A_s]e_i, e_j))^2.$$

Generalized Wintgen inequality for slant submanifolds in complex space forms has the following form.

Theorem 2.5.1 ([39]) *Let M be an n-dimensional θ-slant submanifold of an m-dimensional complex space form $\tilde{M}(4c)$. Then*

$$||H||^2 \ge \rho + \rho_N - c - \frac{3c}{n-1}\cos^2\theta. \tag{2.5.1}$$

Proof Let M be a θ-slant submanifold of a complex space form $\tilde{M}(4c)$, $\{e_1, \dots, e_n\}$ an orthonormal frame on M and $\{\xi_1, \dots, \xi_{2m-n}\}$ an orthonormal frame in the normal bundle $T^{\perp}M$.

The Gauss and Ricci equations are

$$R(X, Y, Z, W) = c[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)+$$

$$+g(JX, Z)g(JY, W) - g(JX, W)g(JY, Z) + 2g(JX, Y)g(JZ, W)]+$$

$$+g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)),$$

for any vector fields $X, Y, Z, W \in \Gamma(TM)$,

$$R^{\perp}(X, Y, \xi, \eta) = c[g(JX, \xi)g(JY, \eta) - g(JX, \eta)g(JY, \xi) + 2g(JX, Y)g(J\xi, \eta)]-$$

$$-g([A_{\xi}, A_{\eta}]X, Y),$$

for any vector fields $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(T^{\perp}M)$.

Similarly, as in the proof of Lemma 2.4 from [39], we get

$$n^2||H||^2 - n^2\rho_N \geq \frac{2n}{n-1} \sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2]. \tag{2.5.2}$$

The Gauss equation implies

$$\tau = \sum_{1 \leq i < j \leq n} R(e_i, e_j, e_i, e_j) = \left[\frac{n(n-1)}{2} + \frac{3}{2}n\cos^2\theta\right]c+ \tag{2.5.3}$$

$$+\sum_{r=1}^{2m-n} \sum_{1 \leq i < j \leq n} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2].$$

Substituting (2.5.3) in (2.5.2), we obtain

$$||H||^2 - \rho_N \geq \rho - c - \frac{3c}{n-1}\cos^2\theta.$$

Corollary 2.5.2 *Let M be an n-dimensional θ-slant submanifold of $\mathbb{C}^m$. Then*

$$||H||^2 \geq \rho + \rho^{\perp}.$$

2.6 Quasi-minimal Slant Surfaces in $\mathbb{C}^2_1$

B.-Y. Chen and the second author [20] classified quasi-minimal slant surfaces in the Lorentzian complex plane $\mathbb{C}^2_1$. More precisely, they proved that there exist five large families of quasi-minimal proper slant surfaces in $\mathbb{C}^2_1$. Conversely, quasi-minimal slant surfaces in $\mathbb{C}^2_1$ are either Lagrangian or locally obtained from one of the five families. Moreover, quasi-minimal slant surfaces in a non-flat Lorentzian complex space form are Lagrangian.

A Lorentzian complex space form $(\tilde{M}_1^n(4c), \langle\, ,\,\rangle, J)$ is an indefinite Kähler manifold of constant holomorphic sectional curvature $4c$ and with complex index one. The curvature tensor $\tilde{R}$ of $\tilde{M}_1^n(4c)$ is given by

$$\tilde{R}(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ\},$$

where J is the almost complex structure on $\tilde{M}_1^n(4c)$. The simplest Lorentzian complex form is the Lorentzian complex n-plane $\mathbb{C}_1^n$ with complex coordinates $z_1, \ldots, z_n$ endowed with the flat complex Lorentzian metric:

$$g = -dz_1 d\bar{z}_1 + \sum_{j=2}^{n} dz_j d\bar{z}_j.$$

For a Lorentzian surface M in a Lorentzian complex space form $\tilde{M}_1^2(4c)$, let g be the induced metric on M and $\langle\ ,\ \rangle$ the inner product associated with $\tilde{g}$.

The notion of slant surfaces in Lorentzian Kähler surfaces can be defined as in Kähler surfaces (see [20]). Quasi-minimal Lagrangian surfaces in Lorentzian complex space forms have been classified in [19].

Recall that a submanifold M in a Lorentzian manifold $\tilde{M}$ is said to be *quasi-minimal* [47] if its mean curvature vector is a null (or light-like) vector field.

Quasi-minimal surfaces are also known as *marginally trapped* surfaces in general relativity.

Theorem 2.6.1 ([20]) *There do not exist quasi-minimal proper slant surfaces in any Lorentzian complex space form $\tilde{M}_1^2(4c)$ with $c \neq 0$.*

Therefore, we classify the quasi-minimal slant surfaces in the Lorentzian complex plane $\mathbb{C}_1^2$.

Theorem 2.6.2 *Let θ be a nonzero real number. Then we have:*

(I) *If $z(s)$ is a null curve in the light cone $\mathcal{LC}$ satisfying $\langle z', iz\rangle = 2\sinh\theta$, then*

$$L(s,t) = z(s)t^{\frac{1}{2}(1-i\,csch\theta)}$$

defines a flat quasi-minimal θ-slant surface in the Lorentzian complex plane $\mathbb{C}_1^2$.

(II) *If $z(s)$ is a null curve lying in $\mathcal{LC}$ which satisfies $\langle z, iz'\rangle = 1$, then*

$$L(s,y) = z(s)e^{(i-\sinh\theta)y}$$

defines a flat quasi-minimal θ-slant surface in $\mathbb{C}_1^2$.

(III) *For any given function* $\varphi(t)$ *defined on an open interval* $I \ni 1$,

$$L(s,t) = \frac{2\sinh\theta}{t^{\frac{1}{2}(i\,csch\theta-1)}}\left(\int_1^t t\varphi(t)dt + t^{\frac{1}{2}(i\,csch\theta-1)}\int_1^t \varphi(t)t^{\frac{1}{2}(3-i\,csch\theta)}dt - \right.$$
$$-\frac{i}{2} - \frac{s}{2}csch\theta, \int_1^t t\varphi(t)dt + t^{\frac{1}{2}(i\,csch\theta-1)}\int_1^t \varphi(t)t^{\frac{1}{2}(3-i\,csch\theta)}dt +$$
$$\left. + \frac{i}{2} - \frac{s}{2}csch\theta\right)$$

defines a flat quasi-minimal θ-slant surface in $\mathbb{C}_1^2$.

(IV) *Let* $\mu(t)$ *and* $\varphi(t)$ *be two functions defined on an open interval* $I \ni 0$. *Put* $F(t) = \int_0^t \mu(t)dt$ *and* $\Phi(t) = \varphi(t)e^{-2F(t)\sinh\theta}$. *Then*

$$L(s,t) = \left(se^{(i-\sinh\theta)F(t)} + (\sinh\theta - i)\int_0^t \Phi(t)\left(\int_0^t e^{(i+\sinh\theta)F(u)}du\right)dt + \right.$$
$$+ (1 + i\sinh\theta)\left(\int_0^t e^{(i+\sinh\theta)F(t)}dt\right)\left(\frac{1}{2} + i\int_0^t \Phi(t)dt\right),$$
$$se^{(i-\sinh\theta)F(t)} + (\sinh\theta - i)\int_0^t \Phi(t)\left(\int_0^t e^{(i+\sinh\theta)F(u)}du\right)dt +$$
$$\left. + (i - \sinh\theta)\left(\int_0^t e^{(i+\sinh\theta)F(t)}dt\right)\left(\frac{i}{2} + \int_0^t \Phi(t)dt\right)\right)$$

defines a flat quasi-minimal θ-slant surface in $\mathbb{C}_1^2$.

(V) *Let* $q(s)$ *be a function defined on on open interval* $I \ni 0$, $\phi(s,y)$ *be a solution of the second differential equation*

$$\phi_{ss} - q(s)\phi = \cosh^2\theta e^{-\frac{4}{3}y\sinh\theta}$$

and z be a null curve in $\mathbf{C}_1^2$ *satisfying* $\langle z'', z''\rangle = 0$ *and* $\langle z', iz''\rangle = \cosh^2\theta$. *If* ϕ *is not the product of two functions of single variable, then*

$$L(s,y) = \int_0^y \frac{\phi z'' - \phi_s z' + \cosh^2\theta e^{-\frac{4}{3}y\sinh\theta}z}{(\sinh\theta + i)e^{-y(i+\frac{1}{3}\sinh\theta)}}dy + z(s)e^{y(i-\sinh\theta)}$$

defines a non-flat quasi-minimal θ-slant surface in $\mathbb{C}_1^2$.

Conversely, quasi-minimal slant surfaces in $\mathbb{C}_1^2$ *are either Lagrangian or, up to dilations and rigid motions of* $\mathbb{C}_1^2$, *obtained locally from the five families of proper slant surfaces.*

Now, we provide some examples of quasi-minimal slant surfaces of type (V).

Example 1 If we choose $q(s) = 0$, then

$$\phi = u(y)s + v(y) + \frac{1}{2}s^2 e^{-\frac{4}{3}y\sinh\theta}\cosh^2\theta,$$
$$z = c_1 s + c_2 s^2 + c_3$$

for arbitrary functions $u(y)$, $v(y)$ and some vectors $c_1, c_2, c_3 \in \mathbb{C}_1^2$. It follows that the immersion of the quasi-minimal slant surface is congruent to

$$L(s, y) = \frac{1}{i + \sinh\theta}\left(\int_0^y (2c_2 v(y) - c_1 u(y))e^{y(i+\frac{1}{3}\sinh\theta)}dy\right) + \frac{c_1 s + c_2 s^2}{e^{y(\sinh\theta - i)}}.$$

So, after choosing suitable initial conditions, we obtain the following example of quasi-minimal proper slant surfaces of type (V):

$$L = \left(\frac{4s\cosh^2\theta + is^2}{4e^{y(\sinh\theta - i)}} + \frac{\int_0^y (iv(y) - 2\cosh^2\theta u(y))e^{y(i+\frac{1}{3}\sinh\theta)}dy}{2(i + \sinh\theta)},\right.$$
$$\left.\frac{4s\cosh^2\theta - is^2}{4e^{y(\sinh\theta - i)}} - \frac{\int_0^y (iv(y) + 2\cosh^2\theta u(y))e^{y(i+\frac{1}{3}\sinh\theta)}dy}{2(i + \sinh\theta)}\right).$$

A direct computation shows that the Gauss curvature of the surface is given by

$$K = \frac{6(u'v - uv')e^{\frac{4}{3}y\sinh\theta} - s\cosh^2\theta(3su' + 6v' + (4su + 8v)\sinh\theta)}{6\phi^3 e^{\frac{2}{3}y\sinh\theta}}.$$

Example 2 If we choose $q(s)$ to be a negative number $-b^2$ ($b > 0$), then we have

$$\phi = u(y)\cos bs + v(y)\sin bs + \frac{1}{b^2}e^{-\frac{4}{3}y\sinh\theta}\cosh^2\theta,$$
$$z = c_1\cos bs + c_2\sin bs + c_3$$

for arbitrary functions $u(y)$, $v(y)$ and vectors $c_1, c_2, c_3 \in \mathbb{C}_1^2$. In this case, the quasi-minimal slant surface is congruent to

$$L = \frac{b^2(i\operatorname{sech}\theta - \tanh\theta)}{\cosh\theta}\left(\int_0^y (c_1 u(y) + c_2 v(y))e^{y(i+\frac{1}{3}\sinh\theta)}dy\right) +$$
$$+ \frac{c_1\cos bs + c_2\sin bs}{e^{y(\sinh\theta - i)}}.$$

After choosing suitable initial conditions, we obtain the following example:

$$L = \frac{1}{2}\Bigg(\frac{2\cosh^2\theta + ib^3 \tan bs}{2b^3 e^{y(\sinh\theta - i)} \sec bs} - \frac{\int_0^y (2u(y)\cosh^2\theta + ib^3 v(y)) e^{y(i+\frac{1}{3}\sinh\theta)} dy}{2b(i+\sinh\theta)},$$
$$\frac{2\cosh^2\theta - ib^3 \tan bs}{2b^3 e^{y(\sinh\theta - i)} \sec bs} - \frac{\int_0^y (2u(y)\cosh^2\theta - ib^3 v(y)) e^{y(i+\frac{1}{3}\sinh\theta)} dy}{2b(i+\sinh\theta)}\Bigg).$$

A direct computation shows that the Gauss curvature of the surface is non-constant.

Example 3 If we choose $q(s)$ to be a positive number b^2 ($b > 0$), then we have

$$\phi = u(y)\cosh bs + v(y)\sinh bs - \frac{1}{b^2} e^{-\frac{4}{3} y \sinh\theta} \cosh^2\theta,$$
$$z = c_1 \cosh bs + c_2 \sinh bs + c_3$$

for arbitrary functions $u(y)$, $v(y)$ and vectors $c_1, c_2, c_3 \in \mathbb{C}_1^2$. It follows that

$$L = \frac{b^2}{i+\sinh\theta}\left(\int_0^y (c_1 u(y) - c_2 v(y)) e^{y(i+\frac{1}{3}\sinh\theta)} dy\right) -$$
$$- \frac{c_1 \cosh bs + c_2 \sinh bs}{e^{y(\sinh\theta - i)}}.$$

Hence, after choosing suitable initial conditions, we obtain the following example:

$$L = \Bigg(\frac{\int_0^y (2u(y)\cosh^2\theta - ib^3 v(y)) e^{y(i+\frac{1}{3}\sinh\theta)} dy}{2b(i+\sinh\theta)} - \frac{2\cosh^2\theta + ib^3 \tanh bs}{2b^3 e^{y(\sinh\theta - i)} \operatorname{sech} bs},$$
$$\frac{\int_0^y (2u(y)\cosh^2\theta + ib^3 v(y)) e^{y(i+\frac{1}{3}\sinh\theta)} dy}{2b(i+\sinh\theta)} - \frac{2\cosh^2\theta - ib^3 \tanh bs}{2b^3 e^{y(\sinh\theta - i)} \operatorname{sech} bs}\Bigg).$$

A direct computation shows that such a quasi-minimal slant surface has non-constant Gauss curvature.

Remark. K. Kenmotsu and D. Zhou proved in [27] that every surface in a complex space form $\tilde{M}(4c)$ of complex dimension 2 is proper slant if it has constant curvature and nonzero parallel mean curvature vector. The above examples show that this result does not hold in Lorentzian settings.

3 Slant Submanifolds in Sasakian Space Forms

Roughly speaking, a Sasakian manifold is the odd-dimensional correspondent of a Kähler manifold.

A $(2m+1)$-dimensional Riemannian manifold $(\tilde{M}, g)$ it said to be a *Sasakian manifold* if it admits an endomorphism φ of its tangent bundle $T\tilde{M}$, a vector field ξ and a 1-form η, satisfying

$$\begin{cases} \varphi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \eta \circ \varphi = 0, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi), \\ (\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \ \tilde{\nabla}_X \xi = -\varphi X, \end{cases}$$

for any vector fields X, Y on $T\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection with respect to g.

A plane section π in $T\tilde{M}$ is called a φ-section if it is spanned by X and φX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a φ-section is called a *φ-sectional curvature*. A Sasakian manifold with constant φ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\tilde{M}(c)$.

The curvature tensor $\tilde{R}$ of a Sasakian space form $\tilde{M}$ is given by

$$\tilde{R}(X, Y)Z = \frac{c+3}{4}[g(Y, Z)X - g(X, Z)Y]+$$

$$+\frac{c-1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi+$$

$$+g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z],$$

for any tangent vector fields X, Y, Z to $\tilde{M}(c)$.

As examples of Sasakian space forms we mention $\mathbb{R}^{2m+1}$ and S^{2m+1} with standard Sasakian structures (see [1, 50]).

The class of slant submanifolds of almost contact metric manifolds was introduced by A. Lotta [29] and studied by many authors [5]. In [41] the authors defined special contact slant submanifolds of Sasakian space forms and proved the minimality of such submanifolds satisfying the equality case of a Chen-Ricci inequality, identically.

A submanifold M tangent to ξ in a Sasakian manifold is called a *contact θ-slant submanifold* [5] if for any $p \in M$ and any $X \in T_pM$ linearly independent on ξ_p, the angle between φX and T_pM is a constant θ, called the *slant angle* of M.

A *proper contact θ-slant submanifold* is a contact slant submanifold which is neither invariant nor anti-invariant, that is, $\theta \neq 0$ and $\theta \neq \frac{\pi}{2}$.

A proper contact θ-slant submanifold is a *special contact θ-slant submanifold* [41] if

$$(\nabla_X T)Y = \cos^2\theta[g(X, Y)\xi - \eta(Y)\xi], \ \forall X, Y \in \Gamma TM,$$

where TX and NX are the tangential and normal components of φX, for any vector field X tangent to M.

We denote by $\|T\|^2 = \sum_{i,j=1}^{n} g^2(Te_i, e_j)$, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM, $p \in M$.

We remark that any 3-dimensional proper contact slant submanifold of a Sasakian manifold is a special contact slant submanifold [5].

Before giving some examples of special contact slant submanifolds, we remind the following result from [5].

Theorem. *Suppose that*

$$x(u,v) = (f_1(u,v),\ f_2(u,v),\ f_3(u,v),\ f_4(u,v))$$

defines a slant surface S in the complex space $\mathbb{C}^2$ *with its usual Kählerian structure, such that* $\partial/\partial u$ *and* $\partial/\partial v$ *are nonzero and perpendicular. Then,*

$$y(u,v,t) = 2(f_1(u,v),\ f_2(u,v),\ f_3(u,v),\ f_4(u,v),t)$$

defines a 3*-dimensional slant submanifold M in* $(\mathbb{R}^5, \varphi_0, \eta, g)$, *such that if we put*

$$e_1 = \frac{\partial}{\partial u} + (2f_3\frac{\partial f_1}{\partial u} + 2f_4\frac{\partial f_2}{\partial v})\frac{\partial}{\partial t}$$

and

$$e_2 = \frac{\partial}{\partial v} + (2f_3\frac{\partial f_1}{\partial v} + 2f_4\frac{\partial f_2}{\partial v})\frac{\partial}{\partial t},$$

then $\{e_1, e_2, \xi\}$ *is an orthogonal basis of the tangent bundle of the submanifold.*

Example. For any constant $k \neq 0$,

$$x(u,v,t) = 2(u, k\cos v, v, k\sin v, t)$$

defines a special contact slant submanifold M with slant angle $\theta = \cos^{-1}(1/\sqrt{1+k^2})$.

Proof Let consider on the manifold $\mathbb{R}^5$ the standard Sasakian structure, as follows:

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{2} y^i dx^i),$$

$$g = \eta \otimes \eta + \frac{1}{4}(\sum_{i=1}^{2} dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\varphi_0(\sum_{i=1}^{2}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) = \sum_{i=1}^{2}(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}) + \sum_{i=1}^{2} Y_i y^i \frac{\partial}{\partial z}.$$

So, we have

$$\frac{\partial x}{\partial u} = 2(1,0,0,0,0),\ \frac{\partial x}{\partial v} = (0, -2k\sin v, 2, 2k\cos v, 0),\ \frac{\partial x}{\partial t} = (0,0,0,0,1),$$

and

$$\frac{\partial f_1}{\partial u} = 1,\ \frac{\partial f_2}{\partial u} = 0,\ \frac{\partial f_1}{\partial v} = 0,\ \frac{\partial f_2}{\partial v} = -k\sin v.$$

It follows that

$$e_1 = (2, 0, 0, 0, 2v),\ \ e_2 = (0, -2k \sin v, 2, 2k \cos v, -2k^2 \sin^2 v).$$

By orthonormalisation, we find an orthonormal frame $\{e_1^*, e_2^*, \xi\}$.

Since the dimension is 3, x is a special contact slant immersion.

Other examples can be found in [5].

3.1 Euler Inequality

In [36], the first author et al. established for special contact slant submanifolds of Sasakian space forms a Chen inequality which involves the scalar curvature (also known as Euler inequality) and a Chen-Ricci inequality, as the contact versions of the inequalities obtained in [33, 35], respectively.

First, we state the corresponding result of Corollary 2.1.3 for proper contact slant 3-dimensional submanifolds in 5-dimensional Sasakian space forms.

Theorem 3.1.1 ([43]) *Let M be a 3-dimensional proper contact slant submanifold of a 5-dimensional Sasakian space form $\tilde{M}(c)$. Then we have:*

$$\|H\|^2 \geq \frac{8}{9}\tau - \frac{2}{9}[c + 3 + (3c + 5)\cos^2\theta].$$

Moreover, the equality sign holds at a point $p \in M$ if and only if with respect to some suitable adapted slant orthonormal basis $\{e_0 = \xi, e_1, e_2, e_3, e_4\}$ at p, the shape operators at p take the following forms:

$$A_3 = \begin{pmatrix} 3\lambda & 0 & \sin\theta \\ 0 & \lambda & 0 \\ \sin\theta & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & \sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}.$$

B.-Y. Chen and Y. Tazawa [22] proved that there do not exist minimal proper slant surfaces in a non-flat complex space form. In [43], the authors showed that there do not exist minimal 3-dimensional proper contact slant submanifolds in a 5-dimensional Sasakian space form $\tilde{M}(c)$, with $c \neq 1$.

A Sasakian space form $\tilde{M}(1)$ is locally isometric to a sphere S^{2n+1}. Minimal 3-dimensional proper contact slant submanifolds in S^5 are characterized in the following:

Proposition 3.1.2 ([43]) *A 3-dimensional proper contact slant submanifold in the 5-dimensional sphere S^5 is minimal if and only if with respect to some suitable local* *adapted slant orthonormal frame* $\{e_0 = \xi, e_1, e_2, e_3, e_4\}$ *the shape operators take the following forms:*

$$A_3 = \begin{pmatrix} -\lambda & 0 & \sin\theta \\ 0 & \lambda & 0 \\ \sin\theta & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & \sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}.$$

The next theorem generalizes Theorem 3.1.1 for special contact slant submanifolds in Sasakian space forms of arbitrary dimensions, more precisely one has the following Chen inequality for the scalar curvature.

Theorem 3.1.3 ([36]) *Let* M *be an* $(n+1)$*-dimensional special contact slant submanifold of a* $(2n+1)$*-dimensional Sasakian space form* $\widetilde{M}(c)$. *Then*

$$\|H\|^2 \geq \frac{2(n+2)}{(n-1)\,(n+1)}\tau - \frac{n(n+2)}{(n-1)(n+1)} \cdot \frac{c+3}{4} - \tag{3.1.1}$$

$$-\frac{n(n+2)}{(n-1)\,(n+1)^2}\left(3\cos^2\theta - 2\right)\frac{c-1}{4} + \frac{n}{(n+1)^2}\sin^2\theta.$$

The equality holds at any point $p \in M$ *if and only if there exists a real function* μ *on* M *such that the second fundamental form satisfies the relations*

$$h(e_1, e_1) = 3\mu e_1^*,\ h(e_2, e_2) = \cdots = h(e_n, e_n) = \mu e_1^*,$$

$$h(e_1, e_j) = \mu e_j^*,\ h(e_j, e_k) = 0\,(2 \leq j \neq k \neq n),$$

with respect to a suitable orthonormal frame $\{e_0 = \xi, e_1, \ldots, e_n\}$ *on* M, *where* $e_k^* = \frac{1}{\sin\theta} N e_k$, $k \in \{1, \ldots, n\}$.

If the second fundamental form of a submanifold M satisfies the previous relations, then M is a H*-umbilical* submanifold.

Remark. In particular, for $c = -3$ and $\theta = \frac{\pi}{2}$ (M is an anti-invariant submanifold in $\mathbb{R}^{2n+1}$), we find a result from [2].

Corollary 3.1.4 ([2]) *Let* M *be an* $(n+1)$*-dimensional anti-invariant submanifold of the Sasakian space form* $\mathbb{R}^{2n+1}$. *Then, at any point* $p \in M$, *the squared mean curvature and the scalar curvature satisfy the inequality*

$$\|H\|^2 \geq \frac{2(n+2)}{(n-1)\,(n+1)}\tau.$$

Moreover, the equality holds at any point $p \in M$ *if and only if there exists a real function* μ *on* M *such that the second fundamental form satisfies the relations*

$$h(e_1, e_1) = 3\mu\varphi e_1,\ h(e_2, e_2) = \cdots = h(e_n, e_n) = \mu\varphi e_1,$$

$$h(e_1, e_j) = \mu\varphi e_j,\ h(e_j, e_k) = 0\,(2 \leq j \neq k \neq n),$$

with respect to a suitable orthonormal frame $\{e_0 = \xi, e_1, \ldots, e_n\}$ *on* M.

As a proper example of an anti-invariant submanifold in the Sasakian space form $\mathbb{R}^{2n+1}$ satisfying identically the equality case of the previous inequality, we consider the Riemannian product of the Whitney n-sphere with the real line $\mathbb{R}$.

3.2 Chen-Ricci Inequality

B.-Y. Chen [14] proved the Chen-Ricci inequality, more precisely he estimated the mean curvature of a submanifold M in a real space form $\tilde{M}(c)$ of constant sectional curvature c, by using its Ricci curvature.

The same inequality was obtained for Lagrangian submanifolds in a complex space form $\tilde{M}(4c)$ (see [16]).

On the other hand, the second author [37] proved Chen-Ricci inequalities for submanifolds in Sasakian space forms.

Theorem 3.2.1 ([37]) *Let M be an n-dimensional C-totally real submanifold of a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then, for each unit vector $X \in T_pM$, we have*

$$Ric(X) \leq \frac{1}{4}[(n-1)(c+3) + n^2\|H\|^2].$$

Theorem 3.2.2 ([37]) *Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form and M an n-dimensional submanifold tangent to ξ. Then, for each unit vector $X \in T_pM$ orthogonal to ξ, we have*

$$Ric(X) \leq \frac{1}{4}[(n-1)(c+3) + 3(||TX||^2 - 2)(c-1) + n^2\|H\|^2].$$

In particular, for contact slant submanifolds in Sasakian space forms we derive the following.

Corollary 3.2.3 ([37]) *Let $\tilde{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form and M an n-dimensional contact θ-slant submanifold. Then, for each unit vector $X \in T_pM$ orthogonal to ξ, we have*

$$Ric(X) \leq \frac{1}{4}[(n-1)(c+3) + 3(\cos^2\theta - 2)(c-1) + n^2\|H\|^2].$$

I. Mihai and I.N. Rădulescu [42] improved the inequality from Theorem 3.2.1 for Legendrian submanifolds in Sasakian space forms.

Theorem 3.2.4 ([42]) *Let M be an n-dimensional Legendrian submanifold in a Sasakian space form $\tilde{M}(c)$ of constant φ-sectional curvature c. Then, for any unit tangent vector X to M, we have:*

$$Ric(X) \leq \frac{n-1}{4}(c+3+n||H||^2).$$

The equality sign holds identically if and only if either:
(i) M is totally geodesic, or
(ii) $n=2$ and M is a H-umbilical Legendrian surface with $\lambda = 3\mu$.

For special contact slant submanifolds in Sasakian space forms, the inequality was improved in Theorem 3.2.5.

Let M be an $(n+1)$-dimensional special contact slant submanifold of a $(2n+1)$-dimensional Sasakian space form $\tilde{M}(c)$. We will take an orthonormal basis $\{e_0 = \xi, e_1, \ldots, e_n\}$ of T_pM, respectively, $\{e_1^* = \frac{1}{\sin\theta}Ne_1, \ldots, e_n^* = \frac{1}{\sin\theta}Ne_n\}$ of $T_p^{\perp}M$. For a contact θ-slant submanifold $\sum_{j=2}^{n} g^2(Te_1, e_j) = cos^2\theta$.

By considering $X = Z = e_1$, $Y = W = e_j$, $j = 2, \ldots, n$, in the formula of the curvature tensor $\widetilde{R}$ of the Sasakian space form $\tilde{M}(c)$, we obtain

$$\tilde{R}(e_1, e_j, e_1, e_j) = \frac{c+3}{4}\left[g(e_1,e_1)g(e_j,e_j) - g(e_j,e_1)g(e_1,e_j)\right] + \tag{3.2.1}$$

$$+\frac{c-1}{4}\left[-\eta(e_1)\eta(e_1)g(e_j,e_j) + \eta(e_j)\eta(e_1)g(e_1,e_j)\right] -$$

$$-g(e_1,e_1)\eta(e_j)\eta(e_j) + g(e_j,e_1)\eta(e_1)\eta(e_j) -$$

$$-g(\varphi e_j, e_1)g(\varphi e_1, e_j) + g(\varphi e_1, e_1)g(\varphi e_j, e_j) + 2g(\varphi e_1, e_j)g(\varphi e_1, e_j) =$$

$$= \frac{c+3}{4} + \frac{3}{4}g^2(\varphi e_1, e_j)(c-1).$$

Then

$$\sum_{j=2}^{n} \tilde{R}\left(e_1, e_j, e_1, e_j\right) + \tilde{R}\left(e_1, e_0, e_1, e_0\right) =$$

$$= n\frac{c+3}{4} + \frac{3}{4}\sum_{j=2}^{n} g^2(\varphi e_1, e_j)(c-1) + 1 = n\frac{c+3}{4} + \frac{3}{4}(c-1)\cos^2\theta + 1.$$

We put $e_1 = X$. The Gauss equation gives

$$Ric(X) = n\frac{c+3}{4} + \frac{3}{4}(c-1)\cos^2\theta + 1 +$$

$$+\sum_{r=1}^{n}[h_{11}^r h_{00}^r - (h_{10}^r)^2] + \sum_{j=2}^{n}[h_{11}^r h_{jj}^r - \left(h_{1j}^r\right)^2].$$

But $e_0 = \xi$ and

$$h_{00}^r = g\left(h\left(e_0, e_0\right), e_r^*\right) = g\left(h\left(\xi, \xi\right), e_r^*\right) = g\left(\tilde{\nabla}_\xi \xi, e_r^*\right) = 0,$$

because $\tilde{\nabla}_\xi \xi = -\varphi\xi = 0$.

Also

$$\sum_{r=1}^{n}\left(h_{10}^r\right)^2 = \sum_{r=1}^{n} g^2\left(h\left(e_1, e_0\right), e_r^*\right) = \sum_{r=1}^{n} g^2\left(\tilde{\nabla}_{e_1} e_0, e_r^*\right) = \sum_{r=1}^{n} g^2\left(\phi e_1, e_r^*\right) = \sin^2\theta.$$

Then one has

$$Ric(X) = n\frac{c+3}{4} + \frac{3}{4}(c-1)\cos^2\theta + 1 - \sin^2\theta + \tag{3.2.2}$$

$$+\sum_{r=1}^{n}\sum_{j=1}^{n}\left[h_{11}^r h_{jj}^r - (h_{1j}^r)^2\right].$$

Using the same arguments as in the proof of Theorem 3.3 from [35], we obtain from (3.2.2)

$$Ric(X) - n\frac{c+3}{4} - \frac{3}{4}(c-1)\cos^2\theta - \cos^2\theta \leq \frac{(n-1)(n+1)}{4}\|H\|^2.$$

Therefore, we proved the following improved Chen-Ricci inequality.

Theorem 3.2.5 ([36]) *Let M be an $(n+1)$-dimensional special contact slant submanifold of a $(2n+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then, for any unit tangent vector X to M, we have*

$$Ric(X) \leq \frac{(n-1)(n+1)}{4}\|H\|^2 + n\frac{c+3}{4} + \frac{3}{4}(c-1)\cos^2\theta + \cos^2\theta. \tag{3.2.3}$$

The equality holds at every point $p \in M$ if and only if either:

(i) M is a totally contact geodesic submanifold, i.e.,

$$h(X, Y) = \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi),$$

for any $X, Y \in \Gamma TM$*;*
or

(ii) $n = 2$ *and* M *is a* 3*-dimensional* H*-umbilical contact slant submanifold, i.e.,*

$$h(e_1, e_1) = 3\mu e_1^*,\ h(e_2, e_2) = \mu e_1^*,\ h(e_1, e_2) = \mu e_2^*,$$

with respect to an orthonormal frame $\{e_0 = \xi, e_1, e_2\}$*.*

Remark. The inequality (3.2.3) also holds for anti-invariant submanifolds in Sasakian space forms.

3.3 Shape Operator A_H

B.-Y. Chen [10] investigated relations between the shape operator A_H in the direction of the mean curvature vector and the sectional curvature and k-Ricci curvature, respectively, for submanifolds in real space forms. Corresponding inequalities for slant submanifolds in complex space forms were obtained by K. Matsumoto and the present authors (see [31]).

Y.H. Kim, C.W. Lee, and D.W. Yoon [28] studied such inequalities for contact slant submanifolds in Sasakian space forms.

We state some of these results. The proofs follow the same ideas as in [31].

Theorem 3.3.1 ([28]) *Let* $x : M \to \tilde{M}(c)$ *be an isometric immersion of an* $(n+1)$*-dimensional contact* θ*-slant submanifold into an* $(2m+1)$*-dimensional Sasakian space form* $\tilde{M}(c)$ *of constant* φ*-sectional curvature* c*. If there exists a point* $p \in M$ *and a number* $b > \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)} \cos^2\theta$ *such that* $\inf_D K(p) = K \geq b$ *at* p*, then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\left(b - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right) I_n, \text{ at } p,$$

where $D_p = \{\xi_p\}^\perp$ *and* I_n *is the identity map of* D_p*.*

In particular, for anti-invariant and invariant submanifolds, one has.

Corollary 3.3.2 ([28]) *Let* M *be an* $(n+1)$*-dimensional anti-invariant submanifold of a Sasakian space form* $\tilde{M}(c)$ *tangent to the Reeb vector field and* $p \in M$*. If there exists a number* $b > \frac{c+3}{4}$ *such that* $\inf_D K(p) = K \geq b$ *at* p*, then*

$$A_H > \frac{n-1}{n}\left(b - \frac{c+3}{4}\right) I_n, \text{ at } p.$$

Corollary 3.3.3 ([28]) *Let M be an $(n+1)$-dimensional invariant submanifold into an $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$ tangent to the Reeb vector field. If there exists a point $p \in M$ and a number $b > \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)}$ such that $\inf_D K(p) = K \geq b$ at p, then the shape operator at the mean curvature vector satisfies*

$$A_H > \frac{n-1}{n}\left[b - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\right] I_n, \text{ at } p.$$

Moreover, the authors estimated the shape operator A_H in terms of the k-Ricci curvature for contact slant submanifolds in Sasakian space forms. They considered the Riemannian invariant Θ_k^D defined by

$$\Theta_k^D(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \ p \in M,$$

where L runs over all k-plane sections in D_p and X runs over all unit vectors in L.

Theorem 3.3.4 ([28]) *Let $x : M \to \tilde{M}(c)$ be an isometric immersion of an $(n+1)$-dimensional contact θ-slant submanifold M into a Sasakian space form $\tilde{M}(c)$ of constant φ-sectional curvature c. Then, for any integer k, $2 \leq k \leq n$, and any point $p \in M$, we have:*

(i) If $\Theta_k^D(p) \neq \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)} \cos^2\theta$, then the shape operator at the mean curvature satisfies

$$A_H > \frac{n-1}{n}\left[\Theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right] I_n, \text{ at } p.$$

(ii) If $\Theta_k^D(p) = \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)} \cos^2\theta$, then $A_H \geq 0$ at p.

(iii) A unit vector $X \in D_p$ satisfies

$$A_H X = \frac{n-1}{n}\left[\Theta_k^D(p) - \frac{c+3}{4} - \frac{3(c-1)}{4(n-1)}\cos^2\theta\right] X$$

if and only if $\Theta_k^D(p) = \frac{c+3}{4} + \frac{3(c-1)}{4(n-1)} \cos^2\theta$ and $X \in D_p \cap \ker h_p$.

In particular, similar inequalities hold for invariant and anti-invariant submanifolds tangent to ξ in Sasakian space forms.

3.4 Chen Inequalities

In [23], the first author and D. Cioroboiu established Chen inequalities for contact slant submanifolds in Sasakian space forms, by using subspaces orthogonal to the Reeb vector field ξ.

We state Chen's first inequality for contact slant submanifolds in a Sasakian space form. We give the whole proof for illustrating the techniques and the particular choice of the orthonormal basis of the tangent space. We pointed out that we considered 2-plane sections π orthogonal to ξ and invariant by T. It is known that the sectional curvature of a plane section tangent to ξ is 1.

Theorem 3.4.1 ([23]) *Let M be an $(n = 2k + 1)$-dimensional contact θ-slant submanifold in a $(2m + 1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then we have:*

$$\delta'_M \leq \frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2 + \frac{(c+3)(n+1)}{4}\right] + \frac{c-1}{8}[3(n-3)\cos^2\theta - 2(n-1)]. \tag{3.4.1}$$

The equality case of the inequality (3.4.1) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_n = \xi\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_{2m}, e_{2m+1}\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\tilde{M}(c)$ at p have the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & . & . & . & 0 \\ 0 & b & 0 & . & . & . & 0 \\ 0 & 0 & & & \mu I_{n-2} & & \end{pmatrix}, \quad a + b = \mu, \tag{3.4.2}$$

$$A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & . & . & . & 0 \\ h^r_{12} & -h^r_{11} & 0 & . & . & . & 0 \\ 0 & 0 & & & 0_{n-2} & & \end{pmatrix}, \quad r \in \{n+2, \dots, 2m+1\}. \tag{3.4.3}$$

Proof Since $\tilde{M}(c)$ is a Sasakian space form, then we have

$$\tilde{R}(X, Y, Z, W) = \frac{c+3}{4}\{-g(Y, Z)g(X, W) + g(X, Z)g(Y, W)\}+ \tag{3.4.4}$$

$$+\frac{c-1}{4}\{-\eta(X)\eta(Z)g(Y, W) + \eta(Y)\eta(Z)g(X, W) - g(X, Z)\eta(Y)g(\xi, W)+$$

$$+g(Y, Z)\eta(X)g(\xi, W) - g(\varphi Y, Z)g(\varphi X, W) + g(\varphi X, Z)g(\varphi Y, W)+$$

$$+2g(\varphi X, Y)g(\varphi Z, W)\}, \quad X, Y, Z, W \in \Gamma(TM).$$

Let $p \in M$ and $\{e_1, \dots, e_n = \xi\}$ an orthonormal basis of T_pM and $\{e_{n+1}, \dots, e_{2m}, e_{2m+1}\}$ an orthonormal basis of $T_p^{\perp}M$. For $X = Z = e_i$, $Y = W = e_j$, $i, j \in \{1, \dots, n\}$. From the relation (3.4.4), it follows that

$$\tilde{R}(e_i, e_j, e_i, e_j) = \frac{c+3}{4}(-n + n^2)+ \tag{3.4.5}$$

$$+\frac{c-1}{4}\{-2(n-1) + 3\sum_{i,j=1}^{n} g^2(\varphi e_i, e_j)\}.$$

Let $M \subset \tilde{M}(c)$ be a contact θ-slant submanifold, $\dim M = n = 2k+1$.
For $X \in \Gamma(TM)$, we put

$$\varphi X = TX + NX, \quad TX \in \Gamma(TM),\ NX \in \Gamma(T^{\perp}M).$$

Let $p \in M$ and $\{e_1, \ldots, e_n = \xi\}$ an orthonormal basis of T_pM, with

$$e_1, e_2 = \frac{1}{\cos\theta} Te_1, \ldots, e_{2k} = \frac{1}{\cos\theta} Te_{2k-1}, e_{2k+1} = \xi.$$

We have

$$g(\varphi e_1, e_2) = g(\varphi e_1, \frac{1}{\cos\theta} Te_1) = \frac{1}{\cos\theta} g(\varphi e_1, Te_1) = \frac{1}{\cos\theta} g(Te_1, Te_1) = \cos\theta$$

and, in the same way,

$$g^2(\varphi e_i, e_{i+1}) = \cos^2\theta;$$

then

$$\sum_{i,j=1}^{n} g^2(\varphi e_i, e_j) = (n-1)\cos^2\theta.$$

By using the relation (3.4.5), we get

$$\tilde{R}(e_i, e_j, e_i, e_j) = \frac{c+3}{4}(n^2 - n) + \frac{c-1}{4}[3(n-1)\cos^2\theta - 2(n-1)]. \quad (3.4.6)$$

Denoting by

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)), \quad (3.4.7)$$

the relation (3.4.6) implies that

$$\frac{c+3}{4} n(n-1) + \frac{c-1}{4}[3(n-1)\cos^2\theta - 2n + 2] = 2\tau - n^2\|H\|^2 + \|h\|^2, \quad (3.4.8)$$

or equivalently,

$$2\tau = n^2\|H\|^2 - \|h\|^2 + \frac{c+3}{4} n(n-1) + \frac{c-1}{4}[3(n-1)\cos^2\theta - 2n + 2]. \quad (3.4.9)$$

If we put

$$\varepsilon = 2\tau - \frac{n^2}{n-1}(n-2)\|H\|^2 - \frac{c+3}{4} n(n-1) - \frac{c-1}{4}[3(n-1)\cos^2\theta - 2n + 2], \quad (3.4.10)$$

we obtain

$$n^2\|H\|^2 = (n-1)(\varepsilon + \|h\|^2). \tag{3.4.11}$$

Let $p \in M$, $\pi \subset T_pM$ a 2-plane section orthogonal to ξ and invariant by T, $\pi = sp\{e_1, e_2\}$. We put $e_{n+1} = \frac{H}{\|H\|}$. The relation (3.4.11) becomes

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left[\sum_{i,j=1}^{n}\sum_{r=n+1}^{2m+1}(h_{ij}^r)^2 + \varepsilon\right],$$

or equivalently,

$$\left(\sum_{i=1}^{n} h_{ii}^{n+1}\right)^2 = (n-1)\left[\sum_{i=1}^{n}[(h_{ii}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 + \varepsilon\right].$$

By using the algebraic Chen's Lemma (see Lemma 3 from subsection 1.4), we derive

$$2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \varepsilon. \tag{3.4.12}$$

From the Gauss equation for $X = Z = e_1$, $Y = W = e_2$, we obtain

$$K(\pi) = \frac{c+3}{4} + 3\cos^2\theta \cdot \frac{c-1}{4} + \sum_{r=n+1}^{2m+1}[h_{11}^r h_{22}^r - (h_{12}^r)^2] \geq$$

$$\geq \frac{c+3}{4} + 3\cos^2\theta \cdot \frac{c-1}{4} + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{r=n+2}^{2m+1}(h_{ij}^r)^2 + \frac{\varepsilon}{2} +$$

$$+ \sum_{r=n+2}^{2m+1} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m+1}(h_{12}^r)^2 =$$

$$= \frac{c+3}{4} + 3\cos^2\theta \cdot \frac{c-1}{4} + \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j>2}(h_{ij}^r)^2 +$$

$$+ \frac{1}{2}\sum_{r=n+2}^{2m+1}(h_{11}^r + h_{22}^r)^2 + \sum_{j>2}[(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{\varepsilon}{2} \geq$$

$$\geq \frac{c+3}{4} + 3\cos^2\theta \cdot \frac{c-1}{4} + \frac{\varepsilon}{2},$$

or equivalently,

$$K(\pi) \geq \frac{c+3}{4} + 3\cos^2\theta \cdot \frac{c-1}{4} + \frac{\varepsilon}{2}.$$

Then

$$\inf K \geq \frac{c+3}{4} + 3\cos^2\theta \cdot \frac{c-1}{4} + \tau -$$

$$-\frac{c+3}{8}(n^2-n) + \frac{c-1}{8}[3(n-1)\cos^2\theta - 2n + 2]\} - \frac{n^2(n-2)}{2(n-1)}\|H\|^2.$$

The last inequality implies

$$\tau - \inf K(\pi) \leq \frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2 + \frac{(c+3)(n+1)}{4}\right] +$$
$$+\frac{(c-1)}{8}[3(n-3)\cos^2\theta - 2(n-1)].$$

This relation represents the inequality to prove.

The case of equality at a point $p \in M$ holds if and only if it achieves the equality in the previous inequality and we have the equality in the Chen's Lemma, which means

$$\begin{cases} h^{n+1}_{ij} = 0, \quad \forall i \neq j, i, j > 2, \\ h^r_{ij} = 0, \quad \forall i \neq j, i, j > 2, r = n+1, \dots, 2m+1, \\ h^r_{11} + h^r_{22} = 0, \quad \forall r = n+2, \dots, 2m+1, \\ h^{n+1}_{1j} = h^{n+1}_{2j} = 0, \quad \forall j > 2, \\ h^{n+1}_{11} + h^{n+1}_{22} = h^{n+1}_{33} = \dots = h^{n+1}_{nn}. \end{cases}$$

We may choose $\{e_1, e_2\}$ such that $h^{n+1}_{12} = 0$ and we denote by $a = h^r_{11}, b = h^r_{22}, \mu = h^{n+1}_{33} = \dots = h^{n+1}_{nn}$. It follows that the shape operators take the desired forms.

A. Carriazo [6] has established another version of the first Chen inequality for submanifolds tangent to the structure vector field ξ of a Sasakian space form. More precisely, he has proved the following theorem for proper slant submanifolds in Sasakian space forms.

Theorem 3.4.2 *Let $\varphi : M \to \tilde{M}(c)$ be an isometric immersion from a Riemannian $(n+1)$-manifold into a Sasakian space form $\widetilde{M}(c)$, of dimension $2m+1$ such that $\xi \in TM$. Then, for any point $p \in M$ and any plane sections $\pi \subset D_p$, we have*

$$\tau - K(\pi) \leq \frac{(n+1)^2(n-1)}{2n}\|H\|^2 + \frac{1}{2}(n+1)(n-2)\frac{c+3}{4} +$$

$$+n + \frac{3}{2}\|T\|^2\frac{c-1}{4} - 3\Phi(\pi)\frac{c-1}{4} - \|N\|^2,$$

where $\Phi(\pi) = g^2(\varphi e_1, e_2)$ *is independent on the orthonormal basis* $\{e_1, e_2\}$ *of* π.

Corollary 3.4.3 ([23]) *Let* M *be an* $(n = 2k + 1)$*-dimensional invariant submanifold in a* $(2m + 1)$*-dimensional Sasakian space form* $\tilde{M}(c)$*. Then we have:*

$$\delta'_M \leq \frac{(c+3)(n-2)(n+1)}{8} + \frac{(c-1)(n-7)}{8},$$

where $\delta'_M(p) = \tau(p) - \inf\{K(\pi) | \pi \subset D_p, \phi(\pi) \subset \pi\}$.

Corollary 3.4.4 ([23]) *Let* M *be an* n*-dimensional anti-invariant submanifold in a* $(2m + 1)$*-dimensional Sasakian space form* $\tilde{M}(c)$*. Then we have:*

$$\delta_M \leq \frac{n-2}{2}\left[\frac{n^2}{n-1}\|H\|^2 + \frac{(c+3)(n+1)}{4}\right] - \frac{(c-1)(n-1)}{4}.$$

The inequality from Theorem 3.4.1 was improved in [46] for special contact slant submanifolds in Sasakian space forms.

Theorem 3.4.5 *Let* M *be an* $(n + 1)$*-dimensional special contact slant submanifold into a Sasakian space form* $\tilde{M}(c)$ *and* $p \in M$, $\pi \subset T_pM$ *a* 2*-plane section orthogonal to* ξ*. Then*

$$\tau(p) - K(p) \leq \frac{n^2(2n-3)}{2(2n+3)}\|H\|^2 + \frac{(n+1)(n-2)}{8}(c+3) + \\ + 3n\cos^2\theta\frac{(c-1)}{8} - \frac{3\Phi(\pi)}{4}(c-1) + n\cos^2\theta.$$

Moreover, the equality case of the inequality holds for some plane section π *at a point* $p \in M$ *if and only if there exists an orthonormal basis* $\{e_0 = \xi, e_1, e_2, \cdots, e_n\}$ *at* p *such that* $\pi = span\{e_1, e_2\}$ *and with respect to this basis the second fundamental form takes the following form*

$$h(e_1, e_1) = aNe_1 + 3bNe_3, \quad h(e_1, e_3) = 3bNe_1, \quad h(e_3, e_j) = 4bNe_j,$$

$$h(e_2, e_2) = -aNe_1 + 3bNe_3, \quad h(e_2, e_3) = 3bNe_2, \quad h(e_j, e_k) = 4bNe_3\delta_{jk},$$

$$h(e_1, e_2) = -aNe_2, \quad h(e_3, e_3) = 12bNe_3, \quad h(e_1, e_j) = h(e_2, e_j) = 0,$$

for some numbers a, b *and* $j, k = 4, \ldots, n$.

The proof follows the ideas from the proof of Theorem 2.4.5.

Let $k \in \mathbb{N}^*$ and $n_1, \ldots, n_k$ integers ≥ 2 such that $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. The *Chen invariant* $\delta(n_1, \ldots, n_k)$ at $p \in M$ is defined by [12]

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf\{\tau(L_1) + \cdots + \tau(L_k)\},$$

where $L_1, \ldots, L_k$ are mutually orthogonal subspaces of T_pM, $\dim L_j = n_j$, for all $j = 1, \ldots, k$.

Theorem 3.4.1 was generalized for the Chen invariants $\delta(n_1, \ldots, n_k)$. We notice that we consider only subspaces orthogonal to ξ.

Theorem 3.4.6 ([23]) *Let M be an $(n = 2k + 1)$-dimensional contact θ-slant submanifold in a $(2m + 1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then we have*

$$\delta(n_1, \ldots, n_k) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k) \frac{c+3}{8} +$$

$$+ \frac{c-1}{8} \left[3(n-1)\cos^2\theta - 6 \sum_{j=1}^{k} n_j \cos^2\theta \right],$$

where

$$d(n_1, \ldots, n_k) = \frac{n^2(n + k - 1 - \sum_{j=1}^{k} n_j)}{2(n + k - \sum_{j=1}^{k})}, \quad b(n_1, \ldots, n_k) = \frac{1}{2}[n(n-1) - \sum_{j=1}^{k} n_j(n_j - 1)].$$

The proof is based on the following

Lemma 3.4.7 ([23]) *Let M be an $(n = 2k + 1)$-dimensional contact θ-slant submanifold in a $(2m + 1)$-dimensional Sasakian space form $\tilde{M}(c)$. Let $n_1, \ldots, n_k$ be integers ≥ 2 satisfying $n_1 < n$ and $n_1 + \cdots + n_k \leq n$. For $p \in M$, let $L_j \subset T_pM$ be mutually orthogonal subspaces of T_pM, $\dim L_j = n_j$, $\forall j \in \{1, \ldots, k\}$. Then we have*

$$\tau - \sum_{j=1}^{k} \tau(L_j) \leq d(n_1, \ldots, n_k) \|H\|^2 + \left[\frac{c+3}{8} n(n-1) + \frac{c-1}{8}(3\|P\|^2 - 2n + 2) \right] -$$

$$- \sum_{j=1}^{k} \left[\frac{c+3}{8} n_j(n_j - 1) + \frac{c-1}{4} 3\Phi(L_j) \right],$$

where $\Phi(L) = \sum_{1 \leq i < j \leq r} g^2(Tu_i, u_j)$ and $\{u_1, \ldots, u_r\}$ is an orthonormal basis of the r-dimensional subspace L of T_pM.

This Lemma is a contact version of a Lemma from [15].

Corollary 3.4.8 ([23]) *Let M be an $(n = 2k + 1)$-dimensional invariant submanifold in a $(2m + 1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then we have:*

$$\delta(n_1,\ldots,n_k) \le b(n_1,\ldots,n_k)\frac{c+3}{8} + \frac{c-1}{8}\left[3(n-1) - 6\sum_{j=1}^{k} n_j\right].$$

Corollary 3.4.9 ([23]) *Let M be an $(n = 2k+1)$-dimensional anti-invariant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then we have:*

$$\delta(n_1,\ldots,n_k) \le d(n_1,\ldots,n_k)\,\|H\|^2 + b(n_1,\ldots,n_k)\frac{c+3}{8}.$$

3.5 Generalized Wintgen Inequality

A Wintgen inequality for 3-dimensional contact slant submanifolds M in a 5-dimensional Sasakian space form $\tilde{M}(c)$ was obtained by the second author and Y. Tazawa.

Let $p \in M$ and $\{e_0 = \xi, e_1, e_2, e_3, e_4\}$ an adapted slant orthonormal basis of $T_p\tilde{M}(c)$ such that $e_0, e_1, e_2 \in T_pM$. We define the scalar normal curvature $\tau^\perp$ at p by $\tau^\perp = g(R^\perp(e_1, e_2)e_4, e_3)$.

Theorem 3.5.1 ([43]) *Let M be a 3-dimensional proper contact slant submanifold of a 5-dimensional Sasakian space form $\tilde{M}(c)$. Then we have*

$$\|H\|^2 \ge \frac{4}{9}(\tau + \tau^\perp) - \frac{2}{9}(c+1) - \frac{8}{9}\cos^2\theta.$$

Moreover, the equality sign holds at a point $p \in M$ if and only if with respect to some suitable adapted slant orthonormal basis $\{e_0 = \xi, e_1, e_2, e_3, e_4\}$ at p, the shape operators at p take the following forms:

$$A_3 = \begin{pmatrix} -\lambda & \mu & \sin\theta \\ \mu & \lambda & 0 \\ \sin\theta & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \mu & \lambda & 0 \\ \lambda & -\mu & \sin\theta \\ 0 & \sin\theta & 0 \end{pmatrix}.$$

Corollary 3.5.2 ([43]) *Each 3-dimensional proper contact slant submanifold M of a 5-dimensional Sasakian space form $\tilde{M}(c)$ which satisfies the equality case of the above inequality at every point $p \in M$ is a minimal submanifold.*

We state a generalized Wintgen inequality for slant submanifolds in Sasakian space forms.

Theorem 3.5.3 ([40]) *Let M be an n-dimensional contact θ-slant submanifold of a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then*

$$||H||^2 \ge \rho + \rho_N - \frac{c+3}{4} - \frac{(3\cos^2\theta - 2)(c-1)}{4n}.$$

In particular, we derive:

Corollary 3.5.4 ([40]) *Let M be an n-dimensional contact θ-slant submanifold of* S^{2m+1}. *Then*

$$||H||^2 \geq \rho + \rho^{\perp} - 1.$$

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Geometry of Warped Product Semi-Slant Submanifolds in Almost Contact Metric Manifolds

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1 Introduction

The concept of a warped product is important in general relativity theory, and it is a useful tool since general relativity theory provides us with the best mathematical model for our universe. In order to construct basic cosmological models for the cosmos, the warped product method was successfully employed in general relativity and semi-Riemannian geometry. The Robertson-Walker space–time, Friedman cosmological models, and standard static space–time, for example, are all represented as warped product manifolds [30]. Warped product manifolds are a good setting for modeling space–time near black holes or entities with large gravitational forces in more cosmological applications. The relativistic model of Schwarzschild space–time, which describes the outer space around a big star or a black hole, for example, enables a warped product construction. Bishop and O'Neill [11] proposed the concept of warped product manifolds with negative curvature manifolds. The study of warped product submanifolds in nearly Hermitian and almost contact metric manifolds has long been a research topic, particularly since Chen [18, 19] introduced

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the concept of CR-warped products in a Kaehler manifold. Furthermore, we refer to a survey of warped product submanifolds and associated geometric obstructions in various configurations [21, 22].

It is well known from literature that a $(2m+1)$-dimensional manifold $\widetilde{M}$ endowed with almost contact structure (φ, ξ, η, g) is called an almost contact metric manifold when it satisfies the following properties:

$$\varphi^2 = -I + \eta \otimes \xi, \;\; \eta(\xi) = 1, \;\; \varphi(\xi) = 0, \;\; \eta \circ \varphi = 0, \tag{1.1}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad \text{and} \quad \eta(U) = g(U, \xi), \tag{1.2}$$

for any $U, V \in \Gamma(T\widetilde{M})$, where φ, g, ξ, and η are called $(1,1)$-tensor fields, a structure vector field, and dual 1-form, respectively, and the symbol $\Gamma(T\widetilde{M})$ denotes Lie algebra of vector fields on a manifold $\widetilde{M}$.

Furthermore, an almost contact metric manifold is known to be a *Sasakian manifold* (cf. [10]) if

$$(\widetilde{\nabla}_U \varphi)V = g(U, V)\xi - \eta(V)U, \quad \widetilde{\nabla}_U \xi = -\varphi U, \tag{1.3}$$

for any $U, V \in \Gamma(T\widetilde{M})$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g. An almost contact metric structure (φ, η, ξ) is said to be *nearly trans-Sasakian manifold (cf.* [28]*)* that is, if

$$\begin{aligned}(\widetilde{\nabla}_U \varphi)V + (\widetilde{\nabla}_V \varphi)U =& \alpha\Big(2g(U, V)\xi - \eta(U)V - \eta(V)U\Big)\\ &- \beta\Big(\eta(V)\varphi U + \eta(U)\varphi V\Big),\end{aligned} \tag{1.4}$$

for any $U, V \in \Gamma(T\widetilde{M})$, where $\widetilde{\nabla}$ is the Riemannian connection metric g on $\widetilde{M}$. Here α and β are some smooth functions on $\widetilde{M}$. If we replace $U = \xi, \;\; V = \xi$ in (1.4), we find that $(\widetilde{\nabla}_\xi \varphi)\xi = 0$, which implies that $\varphi \widetilde{\nabla}_\xi \xi = 0$. Now applying φ and using (1.1), we get $\widetilde{\nabla}_\xi \xi = 0$. For more classification, see [25, 26].

Remark 1.1 (i) If $\alpha = 0$ $\beta = 0$ in (1.4), then *nearly trans-Sasakian* becomes *nearly cosymplectic* manifold, if $\alpha = 1$ *and* $\beta = 0$ in (1.4). Thus it is called *nearly Sasakian* manifold.

$$(\widetilde{\nabla}_U \varphi)V + (\widetilde{\nabla}_V \varphi)U = 2g(U, V)\xi - \eta(V)U - \eta(U)V. \tag{1.5}$$

(ii) If $\alpha = 0$ *and* $\beta = 1$ in (1.4), then *nearly trans-Sasakian* turns into *nearly Kenmotsu* manifold.

(iii) Similarly, *nearly* α*-Sasakian* manifold and *nearly* $\beta-$*Kenmotsu* manifold can be defined from *nearly trans-Sasakian* manifold by substituting $\beta = 0$ *and* $\alpha = 0$ in (1.4), respectively.

Let M be an n-dimensional submanifold of a $(2m+1)$-dimensional almost contact metric manifold $\widetilde{M}$ with the induced metric g, if ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Then Gauss and Weingarten formulas are given by

$$(i)\quad \widetilde{\nabla}_U V = \nabla_U V + h(U, V), \quad (ii)\quad \widetilde{\nabla}_U N = -\mathcal{A}_N U + \nabla^{\perp}_U N, \tag{1.6}$$

for any $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and $\mathcal{A}_N$ are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\widetilde{M}$ and they are related as

$$g(h(U, V), N) = g(\mathcal{A}_N U, V), \tag{1.7}$$

where g denotes the Riemannian metric on $\widetilde{M}$ as well as the metric induced on M. Now, for any $U \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, we put

$$(i)\quad \varphi U = PU + FU, \quad (ii)\quad \varphi N = tN + fN, \tag{1.8}$$

where PU (respectively tN) and FU (respectively fN) are tangential and normal components of φU (respectively φN). From (1.1) and (1.8) (i), it is easy to observe that for each $U, V \in \Gamma(TM)$, we have

$$(i)\quad g(PU, V) = -g(U, PV), \quad (ii)\quad ||P||^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{1.9}$$

Further, the covariant derivative of the endomorphism φ is defined as

$$(\widetilde{\nabla}_U \varphi)V = \widetilde{\nabla}_U \varphi V - \varphi \widetilde{\nabla}_U V, \tag{1.10}$$

for any $U, V \in \Gamma(T\widetilde{M})$.

For a submanifold M, the Gauss equation is defined as

$$\widetilde{R}(U, V, Z, W) = R(U, V, Z, W) + g(h(U, Z), h(V, W)) - g(h(U, W), h(V, Z)), \tag{1.11}$$

for any $U, V, Z, W \in \Gamma(TM)$, where $\widetilde{R}$ and R are the curvature tensors on $\widetilde{M}$ and M, respectively.

A Sasakian manifold is said to be Sasakian space form with constant φ-sectional curvature c if and only if the Riemannian curvature tensor $\widetilde{R}$ is given by (see [10])

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = \frac{c+3}{4}\Big\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\Big\}\\ +\frac{c-1}{4}\Big\{&\eta(X)\eta(Z)g(Y,W) + \eta(W)\eta(Y)g(X,Z)\\ &- \eta(Y)\eta(Z)g(X,W) - \eta(X)g(Y,Z)\eta(W)\\ &+ g(\varphi Y,Z)g(\varphi X,W)\\ &- g(\varphi X,Z)g(\varphi Y,W) + 2g(X,\varphi Y)g(\varphi Z,W)\Big\}. \end{aligned} \quad (1.12)$$

The mean curvature vector H for an orthonormal frame $\{e_1, e_2, \dots e_n\}$ of tangent space TM on M is defined by

$$H = \frac{1}{n}trace(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \quad \text{and} \quad ||H||^2 = \frac{1}{n^2}\Big(\sum_{i=1}^{n} h(e_i, e_i)\Big)^2, \quad (1.13)$$

where $n = \dim(M)$. Also, we set

$$h^r_{ij} = g(h(e_i, e_j), e_r), \quad \text{and} \quad ||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \quad (1.14)$$

The scalar curvature τ for a submanifold M of almost Hermitian manifolds $\widetilde{M}$ is given by

$$\tau(TM) = \sum_{1\le i\neq j\le n} K(e_i \wedge e_j), \quad (1.15)$$

where $K(e_i \wedge e_j)$ is the sectional curvature of plane section spanned by e_i and e_j. Let G_r be a r-plane section on TM and $\{e_1, e_2, \dots, e_r\}$ any orthonormal basis of G_r. Then, the scalar curvature $\tau(G_r)$ of G_r is given by

$$\tau(G_r) = \sum_{1\le i\neq j\le r} K(e_i \wedge e_j). \quad (1.16)$$

A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be *totally umbilical* and *totally geodesic* if $h(U, V) = g(U, V)H$ and $h(U, V) = 0$, respectively, for all $U, V \in \Gamma(TM)$, where H is the mean curvature vector of M. Furthermore, if $H = 0$, then M is *minimal* submanifold in $\widetilde{M}$. The covariant derivative of the endomorphism φ is defined as

$$(\widetilde{\nabla}_U\varphi)V = \widetilde{\nabla}_U\varphi V - \varphi\widetilde{\nabla}_U V, \quad U, V \in \Gamma(T\widetilde{M}). \quad (1.17)$$

In [32], A. Lotta introduced the notion of slant immersion in almost contact metric manifolds.

Definition 1.2 ([12]) Let $\widetilde{M}$ be a Sasakian manifold with an almost contact structure (φ, ξ, η) and M be a submanifold tangent to the structure vector field ξ isometrically immersed in $\widetilde{M}$. Then M is called invariant if $\varphi(T_xM) \subseteq T_xM$ and M is called anti-invariant if $\varphi(T_xM) \subset T_x^{\perp}M$ for every $x \in M$ where T_xM denotes the tangent bundle of M at the point x. Moreover, M is called slant submanifold if for all nonzero vector U tangent to M at a point x, the angle of $\theta(U)$ between φU and T_xM is constant, that is, it does not depend on the choice of $x \in M$ and $U \in T_xM - < \xi(x) >$, where $< \xi(x) >$ is a one-dimensional distribution spanned by $\xi(x)$ for each point $x \in M$.

Chen and Gray [23] derived some classifications of pointwise slant submanifolds in almost Hermitian manifolds. It has been studied in almost contact manifolds by Park in [40], Balgeshir [31], and Mihai [38]. They defined these submanifolds as follows.

Definition 1.3 An odd-dimensional submanifold M of an almost contact metric manifold $\widetilde{M}$ is called *pointwise slant* submanifold if any nonzero vector X tangent to M at $x \in M$ such that X is not proportional to ξ_x, and the Wirtinger angle $\theta(X)$ between φX and $T^*M = TM - \{0\}$ is independent of the choice of nonzero vector $X \in T^*M$. The Wirtinger angle becomes a real-valued function defined on T^*M such that $\theta : T^*M \to \mathbb{R}$, which is said to be a *Wirtinger function (slant function)*.

Lemma 1.4 ([40]) *Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$. Then M is pointwise slant if and only if there exists a constant $\delta \in [0, 1]$ such that*

$$P^2 = \delta\big(-I + \eta \otimes \xi\big). \tag{1.18}$$

Furthermore, in such a case, θ is real-valued function defined on the tangent bundle $T^*M = \bigcup_{x\in M} M_x = \bigcup_{x\in M}\{X \in T_xM : \ g(X, \xi(x)) = 0\}$, then it satisfies that $\delta = \cos^2\theta$.

Hence, for a pointwise slant submanifold M of an almost contact metric manifold $\widetilde{M}$, the following relations are consequences of Lemma 1.4.

$$g(PU, PV) = \cos^2\theta\Big(g(U, V) - \eta(U)\eta(V)\Big), \tag{1.19}$$

$$g(FU, FV) = \sin^2\theta\Big(g(U, V) - \eta(U)\eta(V)\Big), \tag{1.20}$$

for any $U, V \in \Gamma(T^*M)$.

The study of slant and semi-slant submanifolds in almost Hermitian manifolds started with the works of Chen [17] and Papaghiuc [39], where the slant submanifolds act as a natural generalization of complex (holomorphic) and totally real submanifolds. The notion of pointwise slant submanifolds in almost Hermitian manifolds was initiated by Etayo [27] under the name of quasi-slant submanifolds as a generalization of slant and semi-slant submanifolds. Further, Sahin [42] studied the pointwise semi-slant submanifold as a natural generalization of CR-submanifolds of almost Hermitian manifolds in terms of slant function, and his results were extended to the settings of contact manifolds by Park [40]. These submanifolds are defined as follows:

Definition 1.5 ([27]) A submanifold M of an almost contact metric manifold $\widetilde{M}$ is said to be a pointwise semi-slant submanifold if there exists two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\theta \oplus <\xi>$,
(ii) *$\mathcal{D}$ is invariant, that is, $\varphi(\mathcal{D}) \subseteq \mathcal{D}$,*
(iii) *$\mathcal{D}^\theta$ is pointwise slant distribution with slant function $\theta : M \to \mathbb{R}$.*

Let us denote p and q are the dimensions of the *invariant* distribution $\mathcal{D}$ and the *pointwise slant* distribution $\mathcal{D}^\theta$ of a pointwise semi-slant submanifold in an almost contact metric manifold $\widetilde{M}$. Then, we can make the following remarks.

Remark 1.6 (i) M is invariant if $p = 0$, and pointwise slant if $q = 0$.
(ii) If the slant function $\theta : M \to \mathbb{R}$ is globally constant on M and $\theta = \frac{\pi}{2}$, then M is called a *contact CR-submanifold*.
(iii) If the slant function $\theta : M \to (0, \frac{\pi}{2})$ and $p = q \neq 0$, then M is called *proper pointwise semi-slant submanifold*.
(iv) Let μ is an *invariant* subspace under φ of normal bundle $T^\perp M$, then in case of a semi-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as $T^\perp M = F\mathcal{D}^\theta \oplus \mu$.

For the involute conditions of distributions involved in the definition of the pointwise semi-slant submanifold, we refer to [35, 40]. Now, we provide an example which supports our result as follows.

Example 1 ([38]) Assume that $(\mathbb{R}^7, \varphi, \eta, \xi, g)$ be an almost contact manifold with cartesian coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, z)$ and almost contact structure $\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \varphi\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \varphi\left(\frac{\partial}{\partial z}\right) = 0, 1 \leq i, j \leq 3$, where $\xi = \frac{\partial}{\partial z}, \ \ \eta = dz$, and g is the standard Euclidean metric on $\mathbb{R}^7$. Let us consider the submanifold M^5 of $\mathbb{R}^7$, that is $\ell(u, v, w, t, z) = (u + v, -u + v, t\cos w, t\sin w, w\cos t, w\sin t, z)$ such that $t \neq w$ are nonvanishing real-valued functions on M^5. Thus the tangent space TM is spanned as follows.

$$X_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}$$
$$X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}$$
$$X_3 = -t\sin w\frac{\partial}{\partial x_2} + t\cos w\frac{\partial}{\partial y_2} + \cos t\frac{\partial}{\partial x_3} + \sin t\frac{\partial}{\partial y_3}$$

$$X_4 = \cos w \frac{\partial}{\partial x_2} + \sin w \frac{\partial}{\partial y_2} - w \sin t \frac{\partial}{\partial x_3} + w \cos t \frac{\partial}{\partial y_3}$$
$$X_5 = \xi = \frac{\partial}{\partial z}.$$

From the above vector fields, it can be easily shown that the invariant distribution $\mathcal{D}$ is spanned by $X_1, X_2,$ that is, $\mathcal{D} = \{X_1, X_2\}$. Moreover, pointwise slant distribution $\mathcal{D}_\theta$ is spanned by X_3, X_4, that is, $\mathcal{D}_\theta = \{X_3, X_4\}$ with pointwise slant function defined by

$$\theta = \cos^{-1}\left(\frac{t - w}{\sqrt{(t^2+1)(w^2+1)}}\right).$$

Therefore, M^5 is a pointwise semi-slant submanifold of $\mathbb{R}^7$ such that $\xi = \frac{\partial}{\partial z}$ is tangent to M^5.

Definition 1.7 ([13]) A submanifold M of an almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$ is said to be a *bi-slant submanifold* if there exists a pair of orthogonal distributions $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ on M such that

(i) The tangent space TM admits the orthogonal direct decomposition $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus < \xi >$;
(ii) $P\mathcal{D}^{\theta_1} \perp \mathcal{D}^{\theta_2}$ and $P\mathcal{D}^{\theta_2} \perp \mathcal{D}^{\theta_1}$;
(iii) Each distribution $\mathcal{D}_{\theta_i}$ is slant with slant angle θ_i for $i = 1, 2$.

Remark 1.8 A bi-slant submanifold M of almost contact metric manifolds is called *proper* if its bi-slant angles $\theta_i \neq 0, \frac{\pi}{2}$, for $i = 1, 2$. Otherwise,

(i) when $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then M is a *contact CR-submanifold* [46],
(ii) when $\theta_1 = 0$ and $\theta_2 \neq 0, \frac{\pi}{2}$, then M is a *semi-slant submanifold* (defined and studied in [13]),
(iii) when $\theta_1 = \frac{\pi}{2}$ and $\theta_2 \neq 0, \frac{\pi}{2}$, then M is a *pseudo-slant submanifold* (defined in [15, 16] under the name anti-slant submanifold).

For a bi-slant submanifold M of an almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$, the normal bundle $T^\perp M$ is decomposed as

$$T^\perp M = F\mathcal{D}^{\theta_1} \oplus F\mathcal{D}^{\theta_2} \oplus \mu, \tag{1.21}$$

where μ is a φ-invariant normal subbundle of M.

Now, let f be a differential function defined on M. Thus, the gradient ∇f is given as

$$(i)\ \ g(\nabla f, X) = Xf, \quad \text{and} \quad (ii)\ \ \nabla f = \sum_{i=1}^{n} e_i(f)e_i. \tag{1.22}$$

Thus, from the above equation, the Hamiltonian in a local orthonormal frame is defined by

$$\mathcal{H}(\mathrm{d}f, x) = \frac{1}{2}\sum_{j=1}^{n} df(e_j)^2 = \frac{1}{2}\sum_{j=1}^{n} e_j(f)^2 = \frac{1}{2}||\nabla f||^2. \tag{1.23}$$

Moreover, the Laplacian Δf of f is also given by

$$\Delta f = \sum_{i=1}^{n}\{(\nabla_{e_i} e_i) f - e_i(e_i(f))\} = -\sum_{i=1}^{n} g(\nabla_{e_i} gradf, e_i). \tag{1.24}$$

Similarly, the Hessian tensor of the function f is given by

$$\Delta f = -TraceH^f = -\sum_{i=1}^{n} H^f(e_i, e_i), \tag{1.25}$$

where H^f is Hessian of function f.

The compact manifold M will be considered as without boundary, that is, $\partial M = \emptyset$. Thus, for a compact-oriented Riemannian manifold M without boundary, we have the following formula:

$$\int_M \Delta f \mathrm{d}\mathcal{V} = 0, \tag{1.26}$$

such that $\mathrm{d}\mathcal{V}$ denotes the volume of M (see [45]).

2 Characterization of Warped Product Semi-Slant Submanifolds

In [11], Bishop and O'Neill introduced the notion of warped product manifolds to construct examples of Riemannian manifolds with negative curvature. They are defined as follows.

Definition 2.1 ([11]) Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and $f : M_1 \to (0, \infty)$, a positive differentiable function on M_1. Consider the product manifold $M_1 \times M_2$ with its canonical projections $\gamma_1 : M_1 \times M_2 \to M_1$, $\gamma_2 : M_1 \times M_2 \to M_2$ and the projection maps given by $\gamma_1(t, s) = t$, and $\gamma_2(t, s) = s$, for every $l = (t, s) \in M_1 \times M_2$. Thus, the warped product $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian structure such that

$$||X||^2 = ||\gamma_1 *(U)||^2 + f^2(\gamma_1(t))||\pi_2 *(X)||^2, \tag{2.1}$$

for any tangent vector $U \in \Gamma(T_l M)$, where $*$ is the symbol of tangent maps and we have the metric $g = g_1 + f^2 g_2$. Here the function f is called the warping function on M.

The following lemma can be viewed as a direct consequence of warped product manifolds.

Lemma 2.2 ([11]) *Let $M = M_1 \times_f M_2$ be a warped product manifold. If for any $X, Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, then*

(i) $\nabla_X Y \in \Gamma(TM_1)$,
(ii) $\nabla_Z X = \nabla_X Z = (X \ln f)Z$,
(iii) $\nabla_Z W = \nabla'_Z W - g(Z, W)\nabla \ln f$,

where ∇ and ∇' denote the Levi-Civita connections on M and M_2, and $\nabla \ln f$ is the gradient of $\ln f$ *which is defined as $g(\nabla \ln f, U) = U \ln f$.*

Remark 2.3 ([11])

(i) A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* or a simply Riemannian product if the warping function f is constant .
(ii) If $M = M_1 \times_f M_2$ is a warped product manifold, then M_1 is totally geodesic and M_2 is totally umbilical submanifold of M, respectively.

Assume that $\ell : M = M_1 \times_f M_2 \to \widetilde{M}$ be an isometric immersion from a warped product $M_1 \times_f M_2$ into a Riemannian manifold $\widetilde{M}$ of constant sectional curvature c. Let $p, \ q$, and n be the dimensions of $M_1, \ M_2$, and $M_1 \times_f M_2$, respectively. Then, for any unit vector fields X and Z tangent to M_1 and M_2, respectively, we get

$$\begin{aligned} K(X \wedge Z) &= g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) \\ &= \frac{1}{f}\{(\nabla_X X) f - X^2 f\}. \end{aligned} \tag{2.2}$$

If we consider the local orthonormal frame $\{e_1, e_2, \ldots e_n\}$ such that $\{e_1, e_2, \ldots e_p\}$ are tangent to M_1 and $\{e_{p+1}, \ldots e_n\}$ are tangent to M_2, we have

$$\frac{\Delta f}{f} = \sum_{i=1}^{p} K(e_i \wedge e_j)$$

or

$$\sum_{i=1}^{p}\sum_{j=1}^{q} K(e_i \wedge e_j) = \frac{q\Delta f}{f} = q\Big(\Delta(\ln f) - ||\nabla(\ln f)||^2\Big), \tag{2.3}$$

for each $j = p+1, \ldots, n$. Now, we would prove the general characterizations of two types of warped product pointwise semi-slant submanifolds as follows.

(i) $M_\theta \times_f M_T$
(ii) $M_T \times_f M_\theta$.

Moreover, the geometry of warped product semi-slant submanifolds in a Sasakian manifold does not exist (see [8]), while in the case of warped product pointwise semi-slant submanifolds exist as in the form $M = M_T \times_f M_\theta$, where M_T and M_θ are invariant and pointwise slant submanifolds, respectively. We recall the following results given by Park for the classification of above-mentioned cases.

Theorem 2.4 ([40]) *There does not exist any proper warped product pointwise semi-slant submanifolds $M = M_\theta \times_f M_T$ in a Sasakian manifold M such that M_θ is a proper pointwise slant submanifold tangent to the structure vector filed ξ and M_T is an invariant submanifold of M.*

Lemma 2.5 ([40]) *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\widetilde{M}$ such that ξ is tangent to M_T. Then*

$$\begin{aligned}(i)\ g(h(X,Z), FPW) =& -(\varphi X \ln f) g(Z, PW) - \cos^2\theta (X \ln f) g(Z, W) \\ &+ \eta(X) g(W, PZ), \\ (ii)\ g(h(Z, \varphi X), FW) =& (X \ln f) g(Z, W) - (\varphi X \ln f) g(Z, PW),\end{aligned}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Recently, Akram [4] studied characterization theorems on warped product semi-slant submanifolds in Kenmotsu manifolds. Now, here we study warped product semi-slant submanifolds and their characterization of type $M = M_T \times_f M_\theta$. For the first case, we recall the following result which was obtained by Mustafa for warped product semi-slant submanifolds of nearly trans-Sasakian manifolds.

Theorem 2.6 ([36]) *There do not exist warped product semi-slant submanifolds $M = M_\theta \times_f M_T$ in a nearly trans-Sasakian manifold $\widetilde{M}$, where M_θ and M_T are proper slant and invariant submanifolds of $\widetilde{M}$, respectively.*

3 Geometric Inequalities for Warped Product Submanifolds with a Slant Factor

Munteanu used the Coddazi equation in [34] to obtain a good inequality in terms of the Laplacian of a warping function for the second fundamental form of contact CR-warped product in a Sasakian space form, which was motivated by Chen's investigation. Several research and classifications have been published that link contact CR-warped product to various ambient manifolds (see [9, 10, 29, 33, 37]). The major goal is to use the Gauss equation instead of the Codazzi equation to derive such an inequality for a warped product pointwise semi-slant submanifold that is isometrically immersed in a Sasakian space form. We first explore certain geometric features of the mean curvature for warped product pointwise semi-slant submanifolds in this section, and then we construct a general inequality from these results.

Assume that $M = M_T \times_f M_\theta$ be an $n = (p+q)$-dimensional warped product pointwise semi-slant submanifold of $(2m+1)$-dimensional Sasakian manifold $\widetilde{M}$ with M_T of dimension $p = 2p+1$ and M_θ of dimension $q = 2q$, where M_θ and M_T are integral manifolds of $\mathcal{D}^\theta$ and $\mathcal{D} \oplus \xi$, respectively. Then we consider that $\{e_1, e_2, \ldots, e_p, e_{p+1} = \varphi e_1, \ldots, e_{2\alpha} = \varphi e_p, e_{2p+1} = \xi, \}$ and $\{e_{2p+2} = e_1^*, \ldots, e_{2p+1+q} = e_q^*, e_{2p+q+2} = e_{q+1}^* = \sec\theta P e_1^*, \ldots, e_{p+q} = e_q^* = \sec\theta P e_q^*\}$ are orthonormal frames of TM_T and TM_θ, respectively. Thus, the orthonormal frames of the normal subbundles, $F\mathcal{D}^\theta$ and μ, respectively are $\{e_{n+1} = \bar{e}_1 = \csc\theta F e_1^*, \ldots, e_{n+q} = \bar{e}_q = \csc\theta F e_1^*, e_{n+q+1} = \bar{e}_{q+1} = \csc\theta \sec\theta F P e_1^*, \ldots\ldots\ldots,$
$e_{n+2q} = \bar{e}_{2q} = \csc\theta \sec\theta F P e_q^*\}$ and $\{e_{n+2q+1}, \ldots, e_{2m+1}\}$.

Proposition 3.1 ([2]) *Let us consider a warped product pointwise semi-slant submanifold M in a Sasakian manifold $\widetilde{M}$. Then*

$$g(h(X,X), FZ) = g(h(X,X), FPZ) = 0, \tag{3.1}$$
$$g(h(\varphi X, \varphi X), FZ) = g(h(\varphi X, \varphi X), FPZ) = 0, \tag{3.2}$$
$$g(h(X,X), \rho) = -g(h(\varphi X, \varphi X), \rho), \tag{3.3}$$

for any $X \in \Gamma(TM_T)$, $Z \in \Gamma(TM_\theta)$ and $\rho \in \Gamma(\mu)$.

Proposition 3.2 ([2]) *Let $\ell : M = M_T \times_f M_\theta \to \widetilde{M}$ be an isometrically immersion of a warped product pointwise semi-slant submanifold $M_T \times_f M_\theta$ into a Sasakian manifold $\widetilde{M}$ such that M_T is invariant submanifold tangent to ξ of $\widetilde{M}$ and M_θ is a pointwise slant submanifold of $\widetilde{M}$. Then the squared norm of mean curvature of M is given by*

$$||H||^2 = \frac{1}{n^2} \sum_{r=n+1} \left(h^r_{p+1p+1} +, \cdots + h^r_{nn} \right)^2,$$

where H is the mean curvature vector and p, q, n, and $2m+1$ are dimensions of M_T, M_θ, $M_T \times_f M_\theta$, and $\widetilde{M}$, respectively. Moreover, M_T is called ℓ-minimal submanifold in $\widetilde{M}$.

Proof We skip the proof of the above proposition due to similarity to the proof of Lemma 5.2 in [5] for warped product pointwise semi-slant submanifolds in a Kaehler manifold. □

Theorem 3.3 ([2]) *Let $\ell : M = M_T \times_f M_\theta \to \widetilde{M}$ be an isometrically immersion from a warped product pointwise semi-slant submanifold $M_T \times_f M_\theta$ into a Sasakian manifold $\widetilde{M}$. Then*

(i) The squared norm of the second fundamental form of M is given by

$$||h||^2 \geq 2\left\{q||\nabla \ln f||^2 + \widetilde{\tau}(TM) - \widetilde{\tau}(TM_T) - \widetilde{\tau}(TM_\theta) - q\Delta \ln f\right\}, \quad (3.4)$$

where q is the dimension of a pointwise slant submanifold M_θ.

(ii) The equality holds in (3.4) if and only if M_T is totally geodesic and M_θ is totally umbilical submanifolds in $\widetilde{M}$. Moreover, M is minimal submanifold in $\widetilde{M}$.

Proof The proof is similar to the proof of Theorem 5.1 in [5] for warped product pointwise semi-slant submanifolds of Kaehler manifolds and for contact version in [35]. □

Remark 3.4 The Chen second inequality for the second fundamental form and its application to warping functions, which includes the slant immersions obtained in [20], is rather difficult to produce. As a result, Theorem 3.3 comes in handy when constructing Chen's type inequality in terms of slant functions.

We deduce certain obstructions to a warped product pointwise semi-slant submanifold of Sasakian space forms as a direct application of Theorem 3.3.

Theorem 3.5 ([2]) *Assume that $\ell : M = M_T \times_f M_\theta \to \widetilde{M}(c)$ be an isometric immersion from an n-dimensional warped product pointwise semi-slant submanifold $M_T \times_f M_\theta$ into a Sasakian space form $\widetilde{M}(c)$. Then*

(i) The squared norm of the second fundamental form of M is defined as

$$||h||^2 \geq 2q\left(||\nabla \ln f||^2 + \frac{c+3}{4}p - \frac{c-1}{4} - \Delta \ln f\right), \quad (3.5)$$

where p and q are the dimensions of the invariant M_T and the pointwise slant submanifold M_θ, respectively.

(ii) The equality sign holds in (3.5) if and only if M_T is totally geodesic and M_θ is totally umbilical submanifolds in $\widetilde{M}(c)$. Moreover, M is minimal submanifold in $\widetilde{M}(c)$.

Proof Putting $X = W = e_i, \quad Y = Z = e_j$ in equation (1.12), we derive

$$\begin{aligned}\widetilde{R}(e_i, e_j, e_j, e_i) = \frac{c-3}{4}&\left\{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)g(e_j, e_i)\right\}\\ +\frac{c-1}{4}&\left\{\eta(e_i)\eta(e_j)g(e_i, e_j) - \eta(e_j)\eta(e_j)g(e_i, e_i)\right.\\ &+ \eta(e_i)\eta(e_j)g(e_i, e_j) - \eta(e_i)\eta(e_i)g(e_j, e_j)\\ &+ g(\varphi e_j, e_j)g(\varphi e_i, e_i) - g(\varphi e_i, e_j)g(\varphi e_j, e_i)\\ &\left.+ 2g^2(e_i, \varphi e_j)\right\}.\end{aligned}$$

Summing up over the vector fields on TM_T in the above equation, one can show that

$$2\widetilde{\tau}(TM_T) = \left(\frac{c+3}{4}\right)p(p-1) + \left(\frac{c-1}{4}\right)\left\{3||P||^2 - 2(p-1)\right\}.$$

As $\xi(x)$ is tangent to TM_T for p-dimensional invariant submanifold, we have $||P||^2 = p-1$, then we get

$$2\widetilde{\tau}(TM_T) = \left(\frac{c+3}{4}\right)p(p-1) + \left(\frac{c-1}{4}\right)(p-1). \tag{3.6}$$

Similarly, for pointwise slant submanifold TM_θ, we put $||P||^2 = q\cos^2\theta$, from Lemma 1.4 and (1.9) (ii), then

$$\widetilde{\tau}(TM_\theta) = \left(\frac{c+3}{4}\right)q(q-1) + \left(\frac{c-1}{4}\right)3q\cos^2\theta. \tag{3.7}$$

Summing up over basis vectors of TM such that $1 \le i \neq j \le n$, it is easy to obtain that

$$2\widetilde{\tau}(TM) = \left(\frac{c+3}{4}\right)n(n-1) + \left(\frac{c-1}{4}\right)\left\{3\sum_{1\le i\neq j\le n} g^2(\varphi e_i, e_j) - 2(n-1)\right\}. \tag{3.8}$$

Hence, M be a proper pointwise semi-slant submanifold of a Sasakian space form $\widetilde{M}(c)$. Thus, we set the following frame according to [24]

$e_1, e_2 = \varphi e_1, \ldots, e_{2d_1-1}, e_{2d_1} = \varphi e_{2d_1-1}, e_{2d_1+1}, e_{2d_1+2} = \sec\theta P e_{2d_1+1}, \ldots,$ $e_{2d_1-1}, e_{2d_1} = \sec\theta P e_{2d_1-1}, \ldots, e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec\theta P e_{2d_1-1}, e_{2d_1+2d_2},$ $e_{2d_1+2d_2+1} = \xi.$

Obviously, we derive

$$g^2(\varphi e_i, e_{i+1}) = \begin{cases} 1, \ \ for \ \ each, \ \ i \in \{1, \ldots, 2p-1\} \\ \cos^2\theta, \ \ for \ \ each \ \ i \in \{2p+1, \ldots, 2p+2q-1\}. \end{cases}$$

It is easy to obtain that

$$\sum_{i,j=1}^{n} g^2(Pe_i, e_j) = 2\big(p + q\cos^2\theta\big). \tag{3.9}$$

From (3.8) and (3.9), it follows that

$$2\widetilde{\tau}(TM) = \frac{c+3}{4}n(n-1) + \frac{c-1}{4}\bigg(6(p+q\cos^2\theta) - 2(n-1)\bigg). \tag{3.10}$$

Therefore, using the above relations (3.6), (3.7), and (3.10) in Theorem 3.3, we derive the required result (3.5). Moreover, the equality case holds according to the second statement of Theorem 3.3. This completes the proof of the theorem. □

Corollary 3.6 ([2]) *Let $\widetilde{M}(c)$ be Sasakian space forms with $c \leq -3$. Then there does not exist a warped product pointwise semi-slant $M_T \times_f M_\theta$ into $\widetilde{M}(c)$ such that* $\ln f$ *is a eigenfunction of Laplacian on M_T with respect to eigenvalue $\gamma > 0$.*

Corollary 3.7 ([2]) *Assume that $\widetilde{M}(c)$ be Sasakian space forms with $c \leq -3$. Then there does not exist a warped product pointwise semi-slant $M_T \times_f M_\theta$ into $\widetilde{M}(c)$ such that* $\ln f$ *is harmonic function on invariant submanifold M_T.*

Based on the minimal principle property of positive differentiable function, the results can be found in [2].

Corollary 3.8 ([2]) *Assume that $\ell : M = M_T \times_f M_\perp \to \widetilde{M}$ be an isometrically immersion from a contact CR-warped product $M_T \times_f M_\perp$ into a Sasakian space form $\widetilde{M}(c)$. Then*

(i) The squared norm of the second fundamental form of M is given by

$$||h||^2 \geq 2q\left(||\nabla \ln f||^2 + \frac{c+3}{4}p - \frac{c-1}{4} - \Delta \ln f\right), \tag{3.11}$$

where q is the dimension of anti-invariant submanifold $M_\perp$.

(ii) The equality sign holds in (3.11) if and only if M_T is totally geodesic and $M_\perp$ is totally umbilical submanifolds in $\widetilde{M}(c)$. Moreover, M is minimal in $\widetilde{M}(c)$.

Remark 3.9 Assume that $\mathbb{S}$ be a $(2m+1)$-sphere, we put $Jz = \xi$, for any point $z \in \mathbb{S}$ and J is an almost complex structure of complex $n+1$-space $\mathbb{C}$. Let us consider the orthogonal projection map $\pi : T_z\mathbb{C} \to T_z\mathbb{S}$ such that $\varphi = \pi \circ J$. It can be easily seen that (φ, η, ξ, g) is a Sasakian structure in $\mathbb{S}$, where η is a one-form dual to ξ and g is standard metric tensor field on $\mathbb{S}$. Therefore, $\mathbb{S}$ can be considered a Sasakian manifold of constant φ-sectional curvature one.

Therefore, the sphere $\mathbb{S}$ is a Sasakian manifold of constant sectional curvature one (see [34, 37]), then Theorem 3.5 can be written as follows.

Theorem 3.10 ([2]) *Assume that $\ell : M = M_T \times_f M_\theta \to \mathbb{S}$ be an isometric immersion from an n-dimensional warped product pointwise semi-slant submanifold $M_T \times_f M_\theta$ into a sphere $\mathbb{S}$. Then*

(i) The squared norm of the second fundamental form of M is defined as

$$||h||^2 \geq 2q\left(||\nabla \ln f||^2 + p - \Delta \ln f\right), \tag{3.12}$$

where p and q are the dimensions of the invariant M_T and the pointwise slant submanifold M_θ, respectively.

(ii) *The equality sign holds in (3.5) if and only if M_T is totally geodesic and M_θ is totally umbilical submanifolds in $\widetilde{M}(c)$. Moreover, M is minimal submanifold in $\widetilde{M}(c)$.*

Remark 3.11 It should be noted that the necessary and sufficient conditions for slant submanifold [12] is same as pointwise slant submanifolds [40] in (1.18) with slant function must be constant. Now, for further computations, we will use (1.18), (1.19), and (1.20) for slant submanifold with subject to constant pointwise slant function.

Many geometers have obtained the various inequalities in [3, 7, 36, 43, 44] for different warped product submanifolds, which is intriguing. We expand our work to a warped product pseudo-slant submanifold in a nearly Sasakian manifold in this chapter. We start with non-trivial warped product submanifolds of the form $M = M_\theta \times_f M_\perp$, often known as warped product pseudo-slant submanifolds, where M_θ and $M_\perp$ are slant and anti-invariant submanifolds. The following lemma is obtained by studying warped product pseudo-slant submanifolds of a nearly Sasakian manifold.

Lemma 3.12 ([6]) *Let $M = M_\theta \times_f M_\perp$ be a non-trivial warped product pseudo-slant submanifold of a nearly Sasakian manifold $\bar{M}$. Then we have*

(i) $g(h(Z, Z), FX) = g(h(Z, X), \varphi Z) + \{2\eta(X) + (PX \ln f)\}||Z||^2$,
(ii) $g(h(Z, Z), FPX) = g(h(Z, X), \varphi Z) - (X \ln f)\cos^2\theta||Z||^2$,

for any $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, where the structure vector field ξ is tangent to M_θ.

In terms of the second fundamental form and warping functions, we now have a geometric inequality for the warped product pseudo-slant submanifolds.

Theorem 3.13 ([6]) *Let $M = M_\theta \times_f M_\perp$ be an $n+1$-dimensional mixed totally geodesic warped product pseudo-slant submanifold of a $2m+1$-dimensional nearly Sasakian manifold $\widetilde{M}$ such that q is dimension of $M_\perp$ and $2p+1$ is dimension of M_θ. Then we have the following results:*

(i) *The squared norm of the second fundamental form of M satisfies*

$$||h||^2 \geq q\cot^2\theta||\nabla^\theta \ln f||^2. \tag{3.13}$$

(ii) *If the equality holds identically in (3.13), then M_θ is a totally geodesic submanifold and $M_\perp$ is a totally umbilical submanifold of $\widetilde{M}$.*

Proof First, we define an orthogonal frame. Let $M = M_\theta \times_f M_\perp$ be an $n+1$-dimensional warped product pseudo-slant submanifold of a $2m+1$-dimensional nearly Sasakian manifold $\widetilde{M}$ with M_θ of dimension $2p+1$ and $M_\perp$ of dimension

q, where M_θ and $M_\perp$ are the integral manifolds of $\mathcal{D}^\theta$ and $\mathcal{D}^\perp$, respectively, such that $n+1 = 2p+q+1$. Assuming that $\{e_1, e_2 \cdots e_q\}$ and $\{e_{q+1} = e_1^*, \ldots, e_{q+p} = e_p^*, e_{q+p+1} = e_{p+1}^* = \sec\theta P e_1^*, \ldots, e_{q+2p} = e_{2p}^* = \sec\theta P e_p^*, e_{q+2p+1} = e_{2p+1}^* = \xi\}$ be orthonormal frames of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$, respectively. Thus the orthonormal frames of the normal subbundles $\varphi\mathcal{D}^\perp$, $F\mathcal{D}^\theta$, and μ, respectively, are $\{\bar{e}_1 = \varphi e_1, \ldots, \bar{e}_q = \varphi e_q\}$, $\{\bar{e}_{q+1} = \widetilde{e}_1 = \csc\theta F e_1^*, \ldots, \bar{e}_{q+p} = \widetilde{e}_p = \csc\theta F e_p^*, \bar{e}_{q+p+1} = \widetilde{e}_{p+1} = \csc\theta \sec\theta F P e_1^*, \ldots, \bar{e}_{q+2p} = \widetilde{e}_{2p} = \csc\theta\sec\theta F P e_p^*\}$ and $\{\bar{e}_{2p+q+1}, \ldots, \bar{e}_{2m-n}\}$. From the definition of the second fundamental form, we have

$$||h||^2 = ||h(\mathcal{D}^\theta, \mathcal{D}^\theta)||^2 + ||h(\mathcal{D}^\perp, \mathcal{D}^\perp)||^2 + 2||h(\mathcal{D}^\theta, \mathcal{D}^\perp)||^2.$$

By using the property of a mixed totally geodesic submanifold, we have

$$||h||^2 = ||h(\mathcal{D}^\perp, \mathcal{D}^\perp)||^2 + ||h(\mathcal{D}^\theta, \mathcal{D}^\theta)||^2. \tag{3.14}$$

Leaving the second term and the relations (1.14) in the first term, we obtain by using the components of $\varphi\mathcal{D}^\perp$, $F\mathcal{D}^\theta$, and μ,

$$\begin{aligned} ||h||^2 \geq & \sum_{l,r,k=1}^{q} g(h(e_r, e_k), \bar{e}_l)^2 + \sum_{l=q+1}^{2p+q} \sum_{r,k=1}^{q} g(h(e_r, e_k), \bar{e}_l)^2 \\ & + \sum_{l=2p+q+1}^{2m-n} \sum_{r,k=1}^{q} g(h(e_r, e_k), \bar{e}_l)^2. \end{aligned} \tag{3.15}$$

Now, leaving all terms except the second term, then we get

$$||h||^2 \geq \sum_{l=1}^{2p} \sum_{r,k=1}^{q} g(h(e_r, e_k), \widetilde{e}_l)^2. \tag{3.16}$$

By following the adapted frame for $F\mathcal{D}^\theta$, we derive

$$\begin{aligned} ||h||^2 \geq & \csc^2\theta \sum_{j=1}^{p} \sum_{r=1}^{q} g(h(e_r, e_r), F e_j^*)^2 \\ & + \csc^2\theta \sec^2\theta \sum_{j=1}^{p} \sum_{r=1}^{q} g(h(e_r, e_r), F P e_j^*)^2. \end{aligned} \tag{3.17}$$

Using Lemma 3.12, for a mixed totally geodesic submanifold, we get

$$\begin{aligned}||h||^2 \geq \csc^2\theta \sum_{j=1}^{p}\sum_{r=1}^{q}\{2\eta(e_j^*) + Pe_j^*\ln f)\}^2 g(e_r, e_r)^2 \\ + \cot^2\theta \sum_{j=p+1}^{2p}\sum_{r=1}^{q}(e_j^*\ln f)^2 g(e_r, e_r)^2.\end{aligned}$$

The above expression can be written as

$$\begin{aligned}||h||^2 \geq \csc^2\theta \sum_{j=1}^{p}\sum_{r=1}^{q}(Pe_j^*\ln f)^2 g(e_r, e_r)^2 + 4\csc^2\theta \sum_{j=1}^{p}\sum_{r=1}^{q}\eta(e_j^*)^2 g(e_r, e_r)^2 \\ + 4\csc^2\theta \sum_{j=1}^{p}\sum_{r=1}^{q}(Pe_j^*\ln f)\eta(e_j^*) g(e_r, e_r)^2 \\ + \cot^2\theta \sum_{j=pA+1}^{2p}\sum_{r=1}^{q}(e_j^*\ln f)^2 g(e_r, e_r)^2. \qquad (3.18)\end{aligned}$$

The second and third terms in (3.18) are identically zero by the given frames. Thus, the above relations give the following inequality:

$$||h||^2 \geq \csc^2\theta \sum_{j=1}^{p}\sum_{r=1}^{q}(Pe_j^*\ln f)^2 g(e_r, e_r)^2 + \cot^2\theta \sum_{j=1}^{p}\sum_{r=1}^{q}(e_j^*\ln f)^2 g(e_r, e_r)^2.$$

Add and subtract the same terms, we derive

$$\begin{aligned}||h||^2 \geq & q\csc^2\theta \sum_{j=1}^{2p+1}(Pe_j^*\ln f)^2 - q\csc^2\theta \sum_{j=p+1}^{2p}(Pe_j^*\ln f)^2 \\ & - q\csc^2\theta(\xi\ln f)^2 + q\cot^2\theta \sum_{j=1}^{p}\sum_{r=1}^{q}(e_j^*\ln f)^2.\end{aligned}$$

Since $\xi \ln f = 0$, we obtain

$$\begin{aligned}||h||^2 \geq & q\csc^2\theta||P\nabla^\theta \ln f||^2 + q\cot^2\theta \sum_{j=1}^{p}(e_j^*\ln f)^2 \\ & - q\csc^\theta \sum_{j=1}^{p}(e_{j+p}^* P\nabla^\theta \ln f)^2,\end{aligned}$$

by some simplifications. Apply the property (1.19) in the above equation, we get

$$||h||^2 \geq q\cot^2\theta||\nabla^\theta \ln f||^2 - q\cot^2\theta(\xi \ln f)^2 + q\cot^2\theta\sum_{j=1}^{p}(e_j^* \ln f)^2$$
$$- q\csc^2\theta\sec^2\theta\sum_{j=1}^{p} g(Pe_j^*, P\nabla^\theta \ln f)^2.$$

From (1.19), it can be easily seen that

$$||h||^2 \geq q\cot^2\theta||\nabla^\theta \ln f||^2 + q\cot^2\theta\sum_{j=1}^{p}(e_j^* \ln f)^2 - q\cot^2\theta\sum_{j=1}^{p}(e_j^* \ln f)^2,$$

which implies that

$$||h||^2 \geq q\cot^2\theta||\nabla^\theta \ln f||^2.$$

This is the inequality (3.13). If the equality holds in (3.13), then leaving the terms in (3.14) and (3.15), we obtain the following condition:

$$||h(\mathcal{D}, \mathcal{D})||^2 = 0, \quad g(h(\mathcal{D}^\perp, \mathcal{D}^\perp), \varphi\mathcal{D}^\perp) = 0,$$

and

$$g(h(\mathcal{D}^\perp, \mathcal{D}^\perp), \mu) = 0,$$

where $\mathcal{D} = \mathcal{D}^\theta \oplus \xi$. It means that M_θ is totally geodesic in $\widetilde{M}$ and $h(\mathcal{D}^\perp, \mathcal{D}^\perp) \subset F\mathcal{D}^\theta$. Now from Lemma (3.12), for a mixed totally geodesic, we have

$$g(h(Z, W), FX) = (PX \ln f)g(Z, W),$$

for $Z, W \in \Gamma(TM_\perp)$ and $X \in \Gamma(TM_\theta)$. The above equations imply that $M_\perp$ is totally umbilical in $\widetilde{M}$. So the equality cases hold too. It completes the proof of the theorem. □

The notion of warped product bi-slant submanifolds of a nearly trans-Sasakian manifold is defined as

Definition 3.14 A warped product $M_{\theta_1} \times_f M_{\theta_2}$ of two slant submanifolds M_{θ_1} and M_{θ_2} with slant angles θ_1 and θ_2, respectively, of a nearly trans-Sasakian manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$ is called a *warped product bi-slant submanifold.*

For a differentiable function f on a Riemannian manifold M of dimension n, the gradient of f, ∇f, is defined by

$$g(\nabla f, X) = Xf, \tag{3.19}$$

for any $X \in \Gamma(TM)$. As a consequence, we have $||\nabla f||^2 = \sum_{i=1}^{n}(e_i(f))^2$ for a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M.

We consider the warped product bi-slant submanifold $M = M_{\theta_1} \times_f M_{\theta_2}$ of a nearly trans-Sasakian manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$ such that the structure vector field ξ is tangent to M_{θ_1}. The following lemma plays an crucial role in the next result.

Lemma 3.15 ([41]) *Let $M = M_{\theta_1} \times_f M_{\theta_2}$ be a warped product bi-slant submanifold of a nearly trans-Sasakian manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$. Then*

(i) $(\xi \ln f) = \beta$,
(ii) $g(h(X, Z), FZ) = g(h(Z, Z), FX) - \big[\alpha\eta(X) + (PX \ln f)\big]||Z||^2$,
(iii) $g(h(X, PZ), FPZ) = g(h(PZ, PZ), FX) - \big[\alpha\eta(X) + (PX \ln f)\big]\cos^2\theta_2$ $||Z||^2$,
(iv) $g(h(PX, Z), FZ) = g(h(Z, Z), FPX) + \big[(X \ln f)\cos^2\theta_1 - \beta\eta(X)\cos^2\theta_1\big]$ $||Z||^2$,
(v) $g(h(PX, PZ), FPZ) = g(h(PZ, PZ), FPX) + \big[(X \ln f)\cos^2\theta_1 - \beta\eta(X)$ $\cos^2\theta_1\big]\cos^2\theta_2||Z||^2$,
(vi) $g(h(X, PZ), FZ) = -g(h(X, Z), FPZ) = \frac{1}{3}\big[-(X \ln f) + \beta\eta(X)\big]\cos^2\theta_2$ $||Z||^2$,

for $X \in \Gamma(TM_{\theta_1})$ and $Z \in \Gamma(TM_{\theta_2})$.

Let $M = M_{\theta_1} \times_f M_{\theta_2}$ be a warped product bi-slant submanifold of a nearly trans-Sasakian manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$, where M_{θ_1} and M_{θ_2} are proper slant submanifolds with slant angles θ_1 and θ_2, respectively. Further, we assume that $\dim(\widetilde{M}) = 2m + 1$, $\dim(M_{\theta_1}) = 2p + 1$, $\dim(M_{\theta_2}) = 2q$ and $\dim(M) = n = 2p + 2q + 1$. Let $\mathcal{D}^{\theta_1}$ and $\mathcal{D}^{\theta_2}$ be the tangent bundles on M_{θ_1} and M_{θ_2}, respectively. We assume that [41]

(i) $\{e_1, \ldots, e_p, e_{p+1} = \sec\theta_1 Pe_1, \ldots, e_{2p} = \sec\theta_1 Pe_p, e_{2p+1} = \xi\}$ is a local orthonormal frame of $\mathcal{D}^{\theta_1}$.
(ii) $\{e_{2p+2} = e_1^*, \ldots, e_{2p+q+1} = e_q^*, e_{2p+q+2} = e_{q+1}^* = \sec\theta_2 Pe_1^*, \ldots, e_n = e_{2p+2q+1} = e_{2q}^* = \sec\theta_2 Pe_q^*\}$ is a local orthonormal frame of $\mathcal{D}^{\theta_2}$.
(iii) $\{e_{n+1} = \widetilde{e}_1 = \csc\theta_1 Fe_1, \ldots, e_{n+p} = \widetilde{e}_p = \csc\theta_1 Fe_p, e_{n+p+1} = \widetilde{e}_{p+1} = \csc\theta_1 \sec\theta_1 FPe_1, \ldots, e_{n+2p} = \widetilde{e}_{2p} = \csc\theta_1 \sec\theta_1 FPe_p\}$ is a local orthonormal frame of $F\mathcal{D}^{\theta_1}$.
(iv) $\{e_{n+2p+1} = \widetilde{e}_{2p+1} = \csc\theta_2 Fe_1^*, \ldots, e_{n+2p+1} = \widetilde{e}_{2p+q} = \csc\theta_2 Fe_q^*, \ldots, e_{n+2p+q+1} = \widetilde{e}_{2p+q+1} = \csc\theta_2 \sec\theta_2 FPe_1^*, \ldots, e_{2n-1} = \widetilde{e}_{2p+2q} = \csc\theta_2 \sec\theta_2 FPe_q^*\}$ is a local orthonormal frame of $F\mathcal{D}^{\theta_2}$.
(v) $\{e_{2n}, \ldots, e_{2m+1}\}$ is a local orthonormal frame of μ.

We now show that the squared norm of the second fundamental form for bi-slant submanifolds with any codimension of nearly trans-Sasakian manifolds is constrained below by the gradient of a warping function and that the equivalence holds under certain conditions.

Theorem 3.16 ([41]) *Let $M = M_{\theta_1} \times_f M_{\theta_2}$ be a warped product bi-slant submanifold of a nearly trans-Sasakian manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$ such that M_{θ_1} and M_{θ_2} are* *proper slant submanifolds with slant angles* θ_1 *and* θ_2*, respectively. If* M *is* $\mathcal{D}^{\theta_2}$*-totally geodesic, then we have the following:*

(i) The squared norm of the second fundamental form h *of* M *satisfies*

$$||h||^2 \geq 4q \csc^2 \theta_2 \big[(\cos^2 \theta_1 + \frac{1}{9} \cos^2 \theta_2)(||\nabla \ln f||^2 - \beta^2) + \alpha^2\big]. \tag{3.20}$$

Furthermore,

(a) For a nearly Sasakian manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$*,* h *of* M *satisfies*

$$||h||^2 \geq 4q \csc^2 \theta_2 \big[(\cos^2 \theta_1 + \frac{1}{9} \cos^2 \theta_2)(||\nabla \ln f||^2) + 1\big]. \tag{3.21}$$

(b) For a nearly Kenmotsu manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$*,* h *of* M *satisfies*

$$||h||^2 \geq 4q \csc^2 \theta_2 (\cos^2 \theta_1 + \frac{1}{9} \cos^2 \theta_2)(||\nabla \ln f||^2 - 1). \tag{3.22}$$

(c) For a nearly cosymplectic manifold $(\widetilde{M}, \varphi, \xi, \eta, g)$*,* h *of* M *satisfies*

$$||h||^2 \geq 4q \csc^2 \theta_2 (\cos^2 \theta_1 + \frac{1}{9} \cos^2 \theta_2)(||\nabla \ln f||^2). \tag{3.23}$$

(ii) If the equality sign holds in all four cases, then M_{θ_1} *is totally geodesic submanifold of* $\widetilde{M}$ *and* M_{θ_2} *is totally umbilical submanifold of* $\widetilde{M}$*. In other words,* M *is a minimal submanifold of* $\widetilde{M}$*.*

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Slant and Semi-slant Submanifolds of Some Almost Contact and Paracontact Metric Manifolds

Viqar Azam Khan and Meraj Ali Khan

1 Introduction

In an almost Hermitian manifold $(\bar{M}, J, g)$, the almost complex structure J turns a vector field to another vector field perpendicular to it. The impact of this property onto a submanifold M of $\bar{M}$ yields invariant (complex or holomorphic) and anti invariant (totally real) distributions on M, where a distribution D on M is *holomorphic* if $JD_x = D_x$ for each $x \in M$ and *totally real* if $JD_x \subset T_x^{\perp}M$ for each $x \in M$. A submanifold M in $\bar{M}$ is holomorphic (resp. totally real) if the tangent bundle $T(M)$ of M is holomorphic (resp. totally real). Initially, holomorphic and totally real submanifolds of an almost Hermitian manifold were explored by many differential geometers [1–3] until Bejacu [4, 5] provided a single setting to study these submanifolds by introducing the notion of CR-submanifolds of Kaehler manifolds. Infact, a submanifold M of an almost Hermitian manifold is called a *CR-submanifold* if there is a holomorphic distribution D on M such that its complementary distribution $D^{\perp}$ is totally real. With regard to the applications, the study of CR-submanifold helps to understand various phenomenon in Relativity and Mechanics [6–8].

It is easy to notice that a submanifold M of an almost Hermitian manifold $(\bar{M}, J, g)$ is holomorphic if for every non zero vector $X \in T_x(M)$ at any point $x \in M$, the angle between JX and $T_x(M)$ is equal to zero and a submanifold is totally real if the angle between JX and $T_x(M)$ is $\pi/2$. This leads to the generalization of holomorphic and totally real submanifolds [9]. In 1990 Chen [10] introduced a more general class of submanifolds namely the class of slant submanifolds, which naturally includes the class of holomorphic and totally real submanifolds.

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As far as contact geometry is concerned, several results can be found in literature on invariant and anti-invariant submanifolds of Sasakian manifold. The anti-invariant submanifolds of a Sasakian manifold turn out to be slant submanifolds with slant angle equal to $\pi/2$. Lotta [11], further showed that slant submanifolds of a Sasakian manifold with slant angle equal to zero are the invariant submanifolds, Cabrerizo et al. [12] worked out a characterization for the existence of a slant submanifold in almost contact metric manifolds.

In [13] N. Papaghiuc introduced a class of submanifolds in an almost Hermitian manifold, called semi-slant submanifolds. This class includes the class of proper CR-submanifolds and Slant submanifolds. Cabrerizo et al. [14] initiated the study of contact version of semi-slant submanifolds. Naturally, both semi-invariant and contact slant submanifolds are particular cases of the introduced notion. They obtained examples of semi-slant submanifolds in Sasakian manifolds and as a step forward they also introduced the idea of Bi-slant submanifolds which include semi-slant submanifold as a particular case. Moreover, Khan and Khan [15] studied a special case of bi-slant submanifolds namely Pseudo-slant submanifolds. They described a general method of constructing a pseudo-slant submanifold in an almost contact metric manifold. Moreover, they also worked out the integrability conditions for the distributions involved in the setting of pseudo-slant submanifold Sasakian manifolds.

The present chapter is divided in five sections, Sect. 1 is introductory which contains the brief history of theory of submanifolds. In Sect. 2 we discuss slant submanifolds of almost contact metric manifolds. In Sect. 3, we collect some results related to semi-slant submanifolds of almost contact metric manifolds. Section 4 deals with the study of pseudo-slant submanifolds of almost contact metric manifolds, more precisely we compile some results related to pseudo-slant submanifolds of Sasakian and trans-Sasakian manifolds. The last section is based on the study of Atceken [16], which covers slant and semi-slant submanifolds of almost paracontact metric manifold.

2 Slant Submanifolds of Almost Contact Metric Manifolds

Let $\bar{M}$ be a manifold of dimension $2n+1$. An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ consists of a tensor field ϕ of type $(1, 1)$, a vector field ξ (known as structure vector field), a $1-$ form, η and a metric tensor field g on $\bar{M}$, such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector field $X, Y \in T\bar{M}$. These conditions also imply that

$$\phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi).$$

A submanifold M of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ is said to be invariant submanifold if $\phi T_x M \subseteq T_x M$ for all $x \in M$. A submanifold M of an almost contact metric manifold is said to be anti-invariant submanifold if $\phi T_x M \subseteq T_x^{\perp} M$ $\forall x \in M$.

The study of differential geometry of semi-invariant or contact CR- submanifolds as a generalization of invariant and anti-invariant submanifolds of an almost contact metric manifold was initiated by Bejancu and Papaghiuc [17] and followed by several geometers [16, 18, 19] etc.).

Throughout this chapter, for a submanifold M of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$, we assume that the structure vector field ξ is tangential to submanifold M and therefore the tangent bundle TM admits the following decomposition

$$TM = D \oplus \langle \xi \rangle, \tag{1}$$

where $\langle \xi \rangle$ is the one dimensional distribution on M spanned by the structure vector field ξ.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a *semi-invariant submanifold* if there exits a pair of orthogonal distributions $(D, D^{\perp})$ satisfying the following conditions:

1. $TM = D \oplus D^{\perp} \oplus \langle \xi \rangle$,
2. D is invariant distribution with respect to ϕ i.e., $\phi D = D$,
3. The distribution $D^{\perp}$ is anti-invariant i.e., $\phi D^{\perp} \subseteq T^{\perp} M$,

where TM and $T^{\perp}M$ denote the tangent and normal bundle to M respectively. On a semi-invariant submanifold of an almost contact metric manifold, it follows that the normal bundle splits as

$$T^{\perp} M = \phi D^{\perp} \oplus \mu.$$

The notion of slant submanifolds is extended to the setting of almost contact metric manifolds by Lotta [11]. Later, Cabrerizo et al. [12] and Carriazo et al. [20] studied these submanifolds in a more specialized setting of Sasakian, K-contact and S-manifolds.

Let $\bar{M}$ be an almost contact metric manifold with structure tensors (ϕ, ξ, η, g) and M be an immersed submanifold of $\bar{M}$. For any $x \in M$ and $X \in T_x M$, if the vectors X and ξ are linearly independent, the angle $\theta(X) \in [0, 1]$ between ϕX and $T_x M$ is well defined. If $\theta(X)$ does not depend on the choice of $x \in M$ and $X \in T_x M$, then M is said to be slant submanifold in $\bar{M}$. The constant angle $\theta(X)$ is then called the slant angle of M in $\bar{M}$.

In particular anti-invariant submanifolds of an almost contact metric manifold are slant with a slant angle $\pi/2$. Lotta [11] proved that slant submanifolds of an almost contact metric manifold with slant angle zero are invariant submanifolds. This fact is not trivial, since by definition an invariant submanifold must have odd dimension and the characteristic vector field ξ of the ambient manifold is required to be tangent to the submanifold. More generally, Lotta showed that these properties are always satisfied by any non-anti-invariant slant submanifold of a contact metric manifold.

On a submanifold M of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ for any $X \in TM$, we denote the tangential and normal components of ϕX by TX and NX respectively. Whereas the tangential and normal components of ϕV, for $V \in T^{\perp}M$ are denoted by tV and nV respectively i.e.,

$$\phi X = TX + NX \tag{2}$$

and

$$\phi V = tV + fV. \tag{3}$$

The normal bundle of M can be decomposed as follows:

$$T^{\perp}M = ND \oplus \mu, \tag{4}$$

where μ is the invariant sub bundle of $T^{\perp}M$ and D is the distribution orthogonal complement of $\langle\xi\rangle$.

It is easy to observe the following:

$$NT + fN = 0. \tag{5}$$

With the help of tensorial equation of almost contact manifold and (2), we have

$$g(TX, Y) = -g(X, TY), \tag{6}$$

which implies that

$$g(T^2X, Y) = g(X, T^2Y), \tag{7}$$

for all $X, Y \in TM$ that is $T^2(= Q)$ is a self-adjoint endomorphism on TM. It is also easy to observe that the eigen values of Q belong to $[-1, 0]$ and that each non vanishing eigen value of Q has even multiplicity.

The covariant derivative of endomorphism Q is defined as

$$(\bar{\nabla} Q)Y = \nabla_X QY - Q\nabla_X Y, \tag{8}$$

for all $X, Y \in TM$.

Now, we have the following characterization

Lemma 2.1 [11] *Let M be a slant submanifold of an almost contact metric manifold $\bar{M}$. Denote by θ the slant angle of M. Then, at each point $x \in M$, $Q|_D$ has only one eigen value $\lambda_1 = -\cos^2\theta$.*

Lotta [11] also provided the following method to construct example of slant submanifolds in the setting of almost contact metric manifold

Example 1 [11] If M is a slant submanifold in an almost Hermitian manifold $\bar{M}$, then $M \times R$ is a slant submanifold in the almost contact metric manifold $\bar{M} \times R$ with usual product structure.

Further Cabrerizo et al. [12] studied slant submanifolds of an almost contact metric manifold. They obtained some new results on slant submanifolds of almost contact metric manifolds and established some characterizations for slant submanifolds of K-contact and Sasakian manifolds.

Let $\bar{M}(\phi, \xi, \eta, g)$ be an almost contact metric manifold and Φ denotes the fundamental 2-form on $\bar{M}$ given by $\Phi(X, Y) = g(X, \phi Y)$ for all $X, Y \in T\bar{M}$. If ξ is Killing vector field with respect to g, the contact metric structure is called a $K-$contact structure. It is known that a contact metric manifold is K-contact if and only if $\bar{\nabla}_X \xi = -\phi X$ for any $X \in T\bar{M}$, where $\bar{\nabla}$ denotes the Levi-Civita connection of $\bar{M}$.

The almost contact structure of $\bar{M}$ is said to be normal if $[\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ. A Sasakian manifold is a normal contact metric manifold. The tensorial equation characterizing a Sasakian manifold is given by

$$(\bar{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{9}$$

for any $X, Y \in T\bar{M}$.

Moreover, on a Sasakian manifold

$$\bar{\nabla}_X \xi = -\phi X \tag{10}$$

for any $X \in T\bar{M}$.

We will denote by $(R^{2n+1}, \phi, \xi, \eta, g)$ the manifold R^{2n+1} with its usual Sasakian structure given by,

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \quad \xi = 2\frac{\partial}{\partial z}$$

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i)$$

$$\phi(\sum_{i=1}^{n}(X_i \frac{\partial}{\partial x^i} + Y_i \frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) = \sum_{i=1}^{n}(Y_i \frac{\partial}{\partial x^i} - X_i \frac{\partial}{\partial y^i}) + \sum_{i=1}^{n} Y_i y^i \frac{\partial}{\partial z},$$

where (x^i, y^i, z), $i = 1, \ldots, n$ are the Cartesian coordinates.

Let M be a submanifold of a Sasakian manifold and λ be a function on M such that

$$(\nabla_X T)Y = \lambda(g(X, Y)\xi - \eta(Y)X), \tag{11}$$

for any $X, Y \in TM$, then

$$(\nabla_X Q)Y = \lambda(g(X, TY)\xi - \eta(Y)TX), \tag{12}$$

for any $X, Y \in TM$.

Now, we have the following characterization for slant immersion

Theorem 2.2 ([12]) *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$Q = -\lambda(I - \eta \otimes \xi). \tag{13}$$

Furthermore, in such case if θ is the slant angle of M, it satisfies that $\lambda = \cos^2\theta$.

Corollary 2.3 ([12]) *Let M be a slant submanifold of an almost contact metric manifold $\bar{M}$, with slant angle θ. Then for any $X, Y \in TM$, we have*

$$g(TX, TY) = \cos^2\theta(g(X, Y) - \eta(X)\eta(Y)), \tag{14}$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) - \eta(X)\eta(Y)). \tag{15}$$

Example 1 provides a method to construct slant submanifolds of almost contract metric manifold. Making use of this construction, Cabrerizo et al. [12] found many examples of slant submanifolds of Sasakian manifolds. Now, we have some examples of slant submanifolds in R^5 with its usual Sasakian structure.

Example 2 ([12]) For any $\theta \in [0, \pi/2]$

$$x(u, v, t) = 2(u\cos\theta, u\sin\theta, v, 0, t)$$

defines a slant submanifold of dimension 3 with slant angle θ.

Example 3 ([12]) For any constant k,

$$x(u, v, t) = 2(e^{ku}\cos u\cos v, e^{ku}\sin u\cos v, e^{ku}\cos u\sin v, e^{ku}\sin u\sin v, t)$$

defines a slant submanifold of dimension 3 with slant angle $\theta = \cos^{-1}(\frac{|k|}{\sqrt{1+k^2}})$.

Cabrerizo et al. [12] studied the impact of parallelism of Q on slant submanifolds and obtained the following characterization:

Proposition 2.4 ([12]) *Let M be a slant submanifold of a K-contact manifold $\bar{M}$. Then $\nabla Q = 0$ if and only if M is an anti-invariant submanifold.*

Proof Let θ be the slant angle of M, then for any $X, Y \in TM$ by (13) we have

$$Q\nabla_X Y = -\cos^2\theta\nabla_X Y + \cos^2\theta\eta(\nabla_X Y)\xi. \tag{16}$$

On the other hand, by taking the covariant derivative with respect to X in both sides of (13), we have

$$\nabla_X QY = -\cos^2\theta\nabla_X Y + \cos^2\theta\eta(\nabla_X Y)\xi + \cos^2\theta g(Y, \nabla_X\xi)\xi + \cos^2\theta\eta(Y)\nabla_X\xi). \tag{17}$$

Hence, $\nabla Q = 0$ if and only if the right hand sides of (16) and (17) are the same which leads to $\nabla_X\xi = 0$. Since $\bar{M}$ is a $K-$contact manifold, we have $\nabla_X\xi = -TX$ and thus the result holds.

Infact, by using (16), (17) and the formula $\nabla_X\xi = -TX$, we can see that if M is a slant submanifold of $K-$contact manifold $\bar{M}$, then

$$(\nabla_X Q)Y = \cos^2\theta(g(X, TY)\xi - \eta(Y)TX), \tag{18}$$

for any $X, Y \in TM$, where θ denotes the slant angle of M.

Now, we have another characterization for slant submanifold

Theorem 2.5 ([12]) *Let M be a submanifold of a $K-$contact manifold $\bar{M}$ such that $\xi \in TM$. Then, M is slant if and only if*

1. *The endomorphism $Q|_D$ has only one eigen value at each point of M,*
2. *There exists a function $\lambda : M \to [0, 1]$ such that*

$$(\nabla_X Q)Y = \lambda(g(X, TY)\xi - \eta(Y)TX),$$

for any $X, Y \in TM$. Moreover, in this case if θ is the slant angle of M, we have $\lambda = \cos^2\theta$.

Proof Statements 1 and 2 follow directly from Lemma 1 and formula (2.1) respectively. Conversely, let $\lambda_1(x)$ be the only eigen value of $Q|_D$ at each point $x \in M$. Let $Y \in D$ be an unit eigen vector associated with λ_1 i.e., $QY = \lambda_1 Y$. Then by virtue of statement 2, we have

$$X(\lambda_1)Y + \lambda_1\nabla_X Y = \nabla_X(QY) = Q(\nabla_X Y) + \lambda g(X, TY)\xi,$$

for any $X \in TM$, since $Y \in D$, and both $\nabla_X Y$ and $Q(\nabla_X Y)$ are perpendicular to Y we conclude that λ_1 is constant on M.

To prove that M is slant, in view of (13), it is enough to show that there is a constant μ such that $Q = -\mu I + \mu\eta \otimes \xi$. Let X be in TM, then $X = \bar{X} + \eta(X)\xi$ where $\bar{X} = X - \eta(X)\xi \in D$. Hence $QX = Q\bar{X}$. Since $Q|_D = -\lambda_1 I$, we have $Q\bar{X} = \lambda_1\bar{X}$ and so $QX = \lambda_1\bar{X} = \lambda_1 X - \lambda_1\eta(X)\xi$. By taking $\mu = -\lambda_1$, we obtain the result. Moreover, if M is slant, by virtue of (18), it must be $\lambda = -\lambda_1 = \mu = \cos^2\theta$, where θ denotes the slant angle of M.

Now, we have the following corollary, which can be verified by Theorem 2.5.

Corollary 2.6 ([12]) *Let M be a submanifold of dimension 3 of a K-contact manifold* $\bar{M}$ *such that* $\xi \in TM$*. Then, M is slant if and only if there exists a function* $\lambda : M \rightarrow [0, 1]$ *such that*

$$(\nabla_X Q)Y = \lambda(g(X, TY)\xi - \eta(Y)TX),$$

for any $X, Y \in TM$*. Moreover, in this case if* θ *is the slant angle of* M*, we have* $\lambda = \cos^2\theta$.

Gherghe [21] introduced the notion of the nearly trans-Sasakian structure, which generalizes trans-Sasakian structure in the same sense as nearly Sasakian structure generalizes Sasakian structure. An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is a nearly trans-Sasakian structure if

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \alpha[2g(X, Y)\xi - \eta(Y)X - \eta(X)Y] - \beta[\eta(Y)\phi X + \eta(X)\phi Y], \tag{19}$$

for any $X, Y \in TM$.

A trans-Sasakian structure is always a nearly trans-Sasakian structure. Moreover, a nearly trans-Sasakian structure of type (α, β) is nearly Sasakian or nearly Kenmotsu or nearly cosymplectic accordingly as $\alpha = 1, \beta = 0$ or $\alpha = 0, \beta = 1$ or $\alpha = 0$, $\beta = 0$.

Now denoting by $P_X Y$ and $Q_X Y$ the tangential and normal parts of $(\bar{\nabla}_X\phi)Y$ and making use of (2), (3) and Gauss-Weingarten formulae, on a submanifold M of $\bar{M}$ we derive

$$P_X Y = (\bar{\nabla}_X T)Y - A_{NY}X - th(X, Y), \tag{20}$$

$$Q_X Y = (\bar{\nabla}_X N)Y + h(X, TY) - fh(X, Y). \tag{21}$$

Recently, Al-Solamy and Khan [22] extended the study of slant submanifolds to the setting of nearly trans-Sasakian manifold. This study generalizes all the previously discussed results. Basically, they have obtained the following result.

Theorem 2.7 ([22]) *Let M be a slant submanifold of an almost contact metric manifold* $\bar{M}$*. Then Q is parallel if and only if at least one of the following is true*

1. *M is anti-invariant,*
2. $dim M \geq 3$,
3. *M is trivial.*

Proof Let θ be the slant angle of M in $\bar{M}$, then for any $X, Y \in TM$, by Eq. (18)

$$(\nabla_X Q)Y = \cos^2\theta(g(Y, \nabla_X\xi)\xi + \eta(Y)\nabla_X\xi). \tag{22}$$

Now, if Q is parallel, then from (22), it follows that either $\cos\theta = 0$ i.e., M is anti-invariant which accounts for case 1 or else we have

$$g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi = 0.$$

The above equation has a solution if and only if $\nabla_X \xi = 0$ and therefore either $D = \{0\}$ or we can pick at least two linearly independent vectors X and TX (belonging to unique non zero eigen value of Q) to span D. In this case, the eigen value is necessarily non zero as $\theta = \pi/2$ has already been taken care off. Hence, the $dim(M) \geq 3$.

Now, we will see the implications of the formulae (13) and (19) in order to study the parallelism of Q on a slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$.

In view of decomposition (4), we may write

$$h(X, \xi) = h_{ND}(X, \xi) + h_{\mu}(X, \xi),$$

for any $X \in TM$, where $h_{ND}(X, \xi) \in ND$ and $h_{\mu}(X, \xi) \in \mu$.

Further, by Eqs. (19), (20) and (21)

$$P_X Y + P_Y X = \alpha(2g(X, Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)TX + \eta(X)TY) \tag{23}$$

and

$$Q_X Y + Q_Y X = -\beta(\eta(Y)NX + \eta(X)NY). \tag{24}$$

In particular, for $Y = \xi$, the above equations yield

$$P_X \xi + P_\xi X = \alpha(\eta(X)\xi - X) - \beta TX, \tag{25}$$

$$Q_X \xi + Q_\xi X = -\beta NX. \tag{26}$$

Lemma 2.8 ([22]) *Let M be a proper slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$ with $\xi \in TM$. Then for any $X \in D$*

$$h_{ND}(X, \xi) = -\alpha \csc^2 \theta NX, \tag{27}$$

where θ is the slant angle of M in $\bar{M}$.

Proof By Eqs. (20) and (23), we obtain

$$(\bar{\nabla}_X T)\xi + (\bar{\nabla}_\xi T)X - A_{NX}\xi - 2th(X, \xi) = -(\alpha X + \beta TX),$$

which on further simplification yields

$$-T\nabla_X\xi + \nabla_\xi TX - T\nabla_\xi X - A_{NX}\xi - 2th(X,\xi) = -(\alpha X + \beta TX). \tag{28}$$

As M is proper slant, by Theorem 2.7, we have

$$\nabla_X\xi = 0. \tag{29}$$

Making use of this fact while taking product with X in (28), we get

$$g(h(X,\xi), NX) = -\alpha g(X,X)$$

or

$$g(h(X,\xi), NY) = -\alpha g(X,Y),$$

for each $X, Y \in D$. This relation in view of the formulae (28) and (29) proves the assertion.

Theorem 2.9 ([22]) *Let M be a slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$ with structure vector field ξ tangential to M, then Q is parallel on M if and only if M is either anti-invariant or a trivial submanifold of $\bar{M}$.*

Proof Let θ be the slant angle of M. Then from (22) it follows that either M is anti-invariant or

$$g(Y, \nabla_X\xi)\xi + \eta(Y)\nabla_X\xi = 0, \tag{30}$$

(30) holds if and only if $\nabla_X\xi = 0$.

Now taking $X \in D$ and writing $Q_X\xi + Q_\xi X$ by formula (21), we obtain

$$Q_X\xi + Q_\xi X = (\bar{\nabla}_X N)\xi + (\bar{\nabla}_\xi N)X + h(TX,\xi) - 2fh(X,\xi).$$

Substituting the value of $Q_X\xi + Q_\xi X$ from (26) into the above equation and taking product with FX, it is deduced that

$$g(N\nabla_X\xi, NX) - g(\nabla^\perp_\xi NX, NX) + g(N\nabla_\xi X, NX) - g(h(TX,\xi), NX)$$
$$+2g(fh(X,\xi), FX) = \beta g(FX, FX).$$

Making use of Eqs. (29), (15), and (5) the above equation yields,

$$g(h(TX,\xi), FX) + 2g(h(X,\xi), FTX) = \beta \sin^2\theta g(X,X).$$

In view of Lemma 2.8, we get $\theta = 0$ or $D = \{0\}$.

On the other hand, differentiating the identity $g(\phi X, \xi) = 0$ covariantly with respect to $X \in D$, we get

$$g(\bar{\nabla}_X \phi X, \xi) + g(\phi X, \bar{\nabla}_X \xi) = 0,$$

or

$$g((\bar{\nabla}_X \phi)X + \phi \bar{\nabla}_X X, \xi) + g(\phi X, \bar{\nabla}_X \xi) = 0. \quad (31)$$

If $\theta = 0$, then $\phi X = TX$. Making use of this fact and Eqs. (19) and (29), it can be deduced from (31) that

$$\alpha \|X\|^2 = 0,$$

which means if $\theta = 0$, then either $\alpha = 0$ on M or $D = \{0\}$. This rules out the possibility of slant angle being zero as long as M is a non trivial slant submanifold of $\bar{M}$ with $\nabla Q = 0$. In other words, M cannot be invariant. Hence if $\nabla_X \xi = 0$ then M is anti-invariant.

Further, Khan et al. [23] studied totally umbilical slant submanifolds of a nearly trans-Sasakian manifold and obtained the following results

Theorem 2.10 ([23]) *Let M be a totally umbilical proper slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$, then*

1. $H \in \mu$,
2. $\alpha = \frac{g(\nabla_{TX} \xi, X)}{2(\|X\|^2 - \eta^2(X))}$,

for any $X \in TM$.

Theorem 2.11 ([23]) *Let M be a totally umbilical slant submanifold of a nearly trans-Sasakian manifold $\bar{M}$ such that $\alpha = 0$ on M then one of the following statements is true*

1. $H \in \mu$,
2. *M is an anti-invariant submanifold,*
3. *If M is a proper slant submanifold then dim $M \geq 3$,*
4. *M is trivial,*
5. *ξ is Killing vector field on M.*

3 Semi-slant Submanifolds of Almost Contact Metric Manifolds

The notion of Semi-slant submanifolds of an almost Hermitian manifolds was introduced by Papaghiuc [13]. The Semi-slant submanifolds are generalized version of CR-submanifolds. Cabrerizo et al. [14] studied this class of submanifolds in almost contact metric manifolds.

Let M be a submanifold, isometrically immersed in an almost Hermitian manifold $(\bar{M}, J, g)$. A *differentiable distribution* D on M is said to be a slant distribution if for any $x \in M$, and a nonzero vector $X \in D_x$, the angle between JX and vector space D_x is constant i.e., it is independent of the choice of $x \in M$ and $X \in D_x$. This constant angle is called the wirtinger angle or slant angle of the slant distribution D.

A submanifold M is said to be a *Semi-slant submanifold* if there exist on M two differentiable distributions D and D_θ such that $TM = D \oplus D_\theta$, where D is a holomorphic distribution i.e., $JD = D$ and D_θ is a slant distribution with slant angle $\theta \neq 0$.

Remark Given a point $x \in M$, if $\xi_x \in T_xM$, then it can be observed that $\xi_x \notin (D_\theta)_x$, where D_θ is a slant distribution on M with slant angle $\theta \in (0, \pi/2]$.

The following theorem provides a useful characterization for the existence of a slant distribution on a contact metric manifold

Theorem 3.1 ([14]) *Let D be a distribution on M, orthogonal to ξ. Then D is slant if and only if there exits a constant $\lambda \in [0, 1]$ such that $(PT)^2X = -\lambda X$, for any $X \in D$, where P denotes the orthogonal projection on D. Furthermore, $\lambda = \cos^2\theta$.*

If $\bar{M}$ is an almost contact metric manifold with contact metric structure (ϕ, ξ, η, g), then a submanifold M of $\bar{M}$ is said to be semi-slant submnaifold if there exist two orthogonal distributions D and D_θ such that $TM = D \oplus D_\theta \oplus \langle\xi\rangle$ such that the distribution D is invariant under ϕ and D_θ is slant with slant angle θ.

Cabrerizo et al. [14] provide the following example of semi-slant submanifold of a Sasakian manifold

Example 4 ([14]) For any $\theta \in (0, \pi/2]$, the immersion

$$x(u, v, w, s, t) = 2(u, 0, w, 0, v, 0, s\cos\theta, s\sin\theta, z)$$

defines a 5-dimensional semi-slant submanifold M, with slant angle θ in R^9 with its usual Sasakian structure (ϕ_0, ξ, η, g) [12]. In fact, it is easy to see that

$$e_1 = 2(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}), \quad e_2 = 2(\frac{\partial}{\partial y^1}), \quad e_3 = 2(\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial z}),$$

$$e_4 = 2(\cos\theta\frac{\partial}{\partial y^3} + \sin\theta\frac{\partial}{\partial y^4}), e_5 = 2\frac{\partial}{\partial z} = \xi,$$

form a local orthonormal frame TM. Then the distribution $D = span\{e_1, e_2\}$ and $D_\theta = span\{e_3, e_4\}$ are the invariant and slant distribution.

For a semi-slant submanifold M, we denote by P_1 and P_2 the projections on to D and D_θ respectively. Then for any $X \in TM$

$$X = P_1X + P_2X + \eta(X)\xi, \tag{32}$$

where P_1X, P_2X denote the components of X in D and D_θ respectively. If we denote $T_1 = P_1 \circ T$ and $T_2 = P_2 \circ T$, then we have

$$\phi X = T_1X + T_2X + NX, \tag{33}$$

for any $X \in TM$.

Applying ϕ on both side, Eq. (32) takes the form:

$$\phi X = \phi P_1X + TP_2X + NP_2X, \tag{34}$$

for any $X \in TM$.

Now, we have the following initial result

Lemma 3.2 [14] *If M is a semi-slant submanifold of an almost contact metric manifold, then for any $X \in TM$*

1. *$\phi P_1X = TP_1X$ and $NP_1X = 0$,*
2. *$TP_2X \in D_\theta$.*

Lemma 3.3 ([14]) *Let M be a semi-slant submanifold, with slant angle θ, of a K-contact manifold $\bar{M}$. Then, for any $X, Y \in TM$*

$$g(TX, TP_2Y) = \cos^2\theta g(X, P_2Y), \quad g(NX, NP_2Y) = \sin^2\theta(X, P_2Y). \tag{35}$$

Lemma 3.4 ([14]) *Let M be a semi-slant submanifold of a Sasakian manifold $\bar{M}$. Then for any $X, Y \in TM$, we have*

$$P_1(\nabla_X\phi P_1Y) + P_1(\nabla_XTP_2Y) = \phi P_1(\nabla_XY) + P_1A_{NP_2Y}X - \eta(Y)P_1X, \tag{36}$$

$$\begin{aligned} P_2(\nabla_X\phi P_1Y) + P_2(\nabla_XTP_2Y) = TP_2(\nabla_XY) + P_2A_{NP_2Y}X \\ + th(X, Y) - \eta(Y)P_2X, \end{aligned} \tag{37}$$

$$\eta(\nabla_X\phi P_1Y) + \eta(\nabla_XTP_2Y) = \eta(A_{NP_2Y}X) + g(\phi X, \phi Y), \tag{38}$$

$$h(\phi P_1Y, X) + h(TP_2Y, X) + \nabla^\perp_X NP_2Y = NP_2(\nabla_XY) + fh(X, Y). \tag{39}$$

Proposition 3.5 ([14]) *Let M be a semi-slant submanifold, with slant angle θ, of a K-contact manifold $\bar{M}$. Then, for any $X, Y \in TM$, we have*

$$\eta(\nabla_X\phi P_1Y) = g(X, P_1Y), \tag{40}$$

$$\eta(\nabla_XTP_2Y) = \cos^2\theta g(X, P_2Y), \quad \eta(A_{NP_2Y}X) = -\sin^2\theta g(X, P_2Y). \tag{41}$$

In the following results, we study the integrability conditions of the distributions D and D_θ

Proposition 3.6 ([14]) *Let M be a semi-slant submanifold of a Sasakian manifold $\bar{M}$. Then, the invariant distribution D is not integrable.*

Proposition 3.7 ([14]) *Let M be a semi-slant submanifold of a Sasakian manifold $\bar{M}$. Then the slant distribution D_θ is integrable if and only if M is a semi-invariant submanifold.*

Proof It is easy to see that $g([X, Y], \xi) = 2g(Y, T_2 X)$, for any $X, Y \in D_\theta$. If D_θ is integrable, then $T_2 = 0$ and so, $\theta = \pi/2$. Then M is a semi-invariant submanifold.

However $D \oplus \langle\xi\rangle$ and $D_\theta \oplus \langle\xi\rangle$ are involutive under some constraints

Proposition 3.8 ([14]) *Let M be a semi-slant submanifold of a Sasakian manifold $\bar{M}$. Then, we have*

1. *The distribution $D \oplus \langle\xi\rangle$ is integrable if and only if*

$$h(X, \phi Y) = h(Y, \phi X),$$

for any $X, Y \in D$.

2. *The distribution $D_\theta \oplus \langle\xi\rangle$ is integrable if and only if*

$$P_1(\nabla_X TY - \nabla_Y TX) = P_1(A_{NY}X - A_{NX}Y),$$

for any $X, Y \in D_\theta \oplus \langle\xi\rangle$.

For a slant submanifolds of Sasakian manifolds, from Eq. (18), we have

$$(\nabla_X T)Y = \cos^2\theta(g(X, Y)\xi - \eta(Y)X), \tag{42}$$

for any $X, Y \in TM$. In fact they generalize the above equation for the semi-slant submanifolds of Sasakian manifolds. From Example 4, one can conclude

$$(\nabla_X T)Y = g(P_1 X, Y)\xi - \eta(Y)P_1 X + \cos^2\theta(g(P_2 X, Y)\xi - \eta(Y)P_2 X), \tag{43}$$

for any $X, Y \in TM$. If we put $X = P_2 X + \eta(X)\xi,\ \ Y = P_2 Y + \eta(Y)\xi \in D_\theta \oplus \langle\xi\rangle$, and formula (43) implies that

$$(\nabla_X T)Y = \cos^2\theta(g(X, Y)\xi - \eta(Y)X).$$

If ξ is tangential to a slant submanifold M of a Sasakian manifold $\bar{M}$, then $TM = D_\theta \oplus \langle\xi\rangle$ where D_θ is the slant distribution on M with slant angle θ, and on M,

$$(\nabla_X T)Y = \cos^2\theta(g(X, Y)\xi - \eta(Y)X).$$

In particular, on an invariant submanifold, the above condition reduces to

$$(\nabla_X T)Y = g(X, Y)\xi - \eta(Y)X. \tag{44}$$

Lemma 3.9 ([14]) *Let M be a proper semi-slant submanifold, with slant angle θ, of a Sasakian manifold $\bar{M}$. For any $X, Y \in TM$, we have*

$$(\nabla_X TY) = A_{NP_2Y}X + th(X, Y) + g(X, Y)\xi - \eta(Y)X. \tag{45}$$

Hence, M satisfies (43) if and only if

$$A_{NP_2Y}X = A_{NP_2X}Y - \sin^2\theta(\eta(X)P_2Y - \eta(Y)P_2X), \tag{46}$$

for any $X, Y \in TM$.

Proof Equation (45) can be obtained by using (32), (34) and (36)–(38). Now, suppose that M is a proper slant submanifold satisfying (43). Then by applying (32), (46) follows directly from (43) and (45).

Conversely, suppose that we have (46) for any $X, Y \in TM$. Then, it is easy to see that

$$g(A_{NP_2Y}Z, X) = -g(th(Y, Z), X) - \sin^2\theta g(g(P_2Y, Z)\xi - \eta(Y)P_2Z, X),$$

for any $X, Y, Z \in TM$. Now, by applying (45), we obtain

$$\begin{aligned}(\nabla_Z T)Y &= g(Z, Y)\xi - \eta(Y)Z - \sin^2\theta(g(P_2Y, Z)\xi - \eta(Y)P_2Z)\\ &= g(P_1Z, Y)\xi - \eta(Y)P_1Z + \cos^2\theta(g(P_2Z, Y)\xi - \eta(Y)P_2Z),\end{aligned}$$

for any $Y, Z \in TM$ and the proof concludes.

The following theorem shows that formula (43) provides a generalization of (18).

Theorem 3.10 ([14]) *Let M be a proper semi-slant submanifold with slant angle θ of a Sasakian manifold $\bar{M}$. Then the following statements are equivalent:*

1. *M satisfies (43),*
2. *$(\nabla_X TP_2)Y = \cos^2\theta(g(P_2X, Y)\xi - \eta(Y)P_2X)$, for any $X, Y \in TM$.*

Proof Suppose that M satisfies (43). Then, by Lemma 3.9, we have

$$th(X, Y) + A_{NP_2Y}X + \sin^2\theta(g(P_2X, Y)\xi - \eta(Y)P_2X) = 0, \tag{47}$$

for any $X, Y \in TM$. By operating P_1 in (47), it is easy to see that

$$P_1A_{NP_2Y}X = 0, \tag{48}$$

for any $X, Y \in TM$.

Writing (47) with $Y \in D \oplus \langle\xi\rangle$, we find that

$$th(X, Y) = \sin^2\theta\eta(Y)P_2X, \tag{49}$$

for any $X \in TM$ and $Y \in D \oplus \langle\xi\rangle$. On the other hand by (39)

$$h(\phi Y, X) = NP_2\nabla_X Y + fh(X, Y), \tag{50}$$

for any $X \in TM$ and $Y \in D \oplus \langle\xi\rangle$. It follows from (49) and (50) that

$$h(\phi Y, X) = NP_2\nabla_X Y + \phi h(X, Y) - \sin^2\theta\eta(Y)P_2X. \tag{51}$$

Now, for $X \in TM$ and $Y \in D \oplus \langle\xi\rangle$,on making use of the fact that $\phi Y \in D$ and $P_1 A_{NP_2\nabla_X Y}X = 0$, we find that

$$g(NP_2\nabla_X Y, h(\phi Y, X)) = g(A_{NP_2\nabla_X Y}X, \phi Y) = 0. \tag{52}$$

Moreover, for $Y \in D$ by using (48) and Lemma 3.2, it is easy to see that

$$g(NP_2\nabla_X Y, \phi h(X, Y)) = g(A_{NP_2TP_2\nabla_X Y}X, Y) = 0. \tag{53}$$

Therefore, by (51)–(53), it follows that $NP_2\nabla_X Y = 0$, for any $X \in TM$ and $Y \in D$. Since M is a proper semi-slant submanifold, it follows from this equation that $P_2\nabla_X Y$ must vanish. Hence,

$$\nabla_X Y \in D \oplus \langle\xi\rangle, \tag{54}$$

for any $X \in TM$ and $Y \in D$. In particular, this implies $\nabla_X Z \in D_\theta \oplus \langle\xi\rangle$, for any $X \in TM$ and $Z \in D_\theta$. Then by applying Lemma 3.2

$$P_1(\nabla_X TP_2Y) = 0, \tag{55}$$

for any $X, Y \in TM$. From (36), (46), (54), and (55), we have

$$(\nabla_X \phi P_1)Y = g(P_1X, Y)\xi - \eta(Y)P_1X. \tag{56}$$

We also have

$$(\nabla_X TP_2)Y = (\nabla_X T)Y - (\nabla_X \phi P_1)Y, \tag{57}$$

for any $X, Y \in TM$, so by virtue of (43) and (56), statement 2 holds.

Conversely, suppose that M satisfies 2.Then by virtue of (13) and Lemma 3.2, it is easy to see that

$$P_1(\nabla_X Z) = -\eta(Z)T_1X, \tag{58}$$

for any $X \in TM$ and $Z \in D_\theta \oplus \langle\xi\rangle$. Hence

$$\nabla_X Z \in D_\theta \oplus \langle\xi\rangle, \tag{59}$$

for any $X \in TM$ and $Z \in D_\theta$. Thus, we can deduce from (59) that $\nabla_X Y \in D \oplus \langle\xi\rangle$, for any $X \in TM$ and $Y \in D$. Therefore, by applying Lemmas 3.2, (36) and (58), we compute

$$(\nabla_X \phi P_1)Y = P_1 A_{NP_2Y} X + g(P_1 X, Y) - \eta(Y) P_1 X, \tag{60}$$

for any $X, Y \in TM$. On the other hand, condition 2, (37), (41), and (59) imply that

$$P_1 A_{NP_2Y} X = 0, \tag{61}$$

for any $X, Y \in TM$. Finally, Eq. (43) follows from Lemmas 3.2, (60), (61), and condition 2.

From the above result, one can conclude the following

Corollary 3.11 ([14]) *If M is a proper semi-slant submanifold of a Sasakian manifold $\bar{M}$ satisfying (43), then*

$$\nabla_X Y \in D \oplus \langle\xi\rangle, \quad \nabla_X Z \in D_\theta \oplus \langle\xi\rangle, \tag{62}$$

for any $X \in TM$, $Y \in D$ and $Z \in D_\theta$. In particular, distribution $D \oplus \langle\xi\rangle$ and $D_\theta \oplus \langle\xi\rangle$ are integrable.

4 Pseudo Slant Submanifolds of Almost Contact Metric Manifolds

In this section we consider pseudo-slant submanifolds of a Sasakian manifold and some integrability conditions for the distributions on the submanifolds. The study leads to characterization under which a submanifold of Sasakian manifold is pseudo-slant.

Let $\bar{M}$ be an almost contact metric manifold with an almost contact structure (ϕ, ξ, η, g) and M be a submanifold isometrically immersed into $\bar{M}$ with structure vector field ξ tangent to M. Now, we define the Pseudo-slant submanifold of $\bar{M}$.

Definition A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a pseudo-slant submanifold if there exist two distributions $D^\perp$ and D_θ on M such that

1. TM admits the orthogonal direct decomposition $TM = D^\perp \oplus D_\theta \oplus \langle\xi\rangle$,
2. The distribution $D^\perp$ is anti-invariant i.e., $\phi D^\perp \subseteq T^\perp M$,
3. The distribution D_θ is slant with slant angle $\theta \neq \pi/2$.

Example 5 ([15]) Consider R^9, a 9-dimensional Sasakian manifold with its usual contact structures and consider the following submanifold M isometrically immersed in R^9 as

$$M = 2(u, 0, w, 0, 0, v, s\cos\theta, s\sin\theta, t).$$

Then M is a pseudo-slant submanifold of R^9. In this case, the vectors

$$e_1 = 2(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z)}, \quad e_2 = 2\frac{\partial}{\partial y^2}, \quad e_3 = 2(\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial z})$$

$$e_4 = 2(\cos\theta\frac{\partial}{\partial y^3} + \sin\theta\frac{\partial}{\partial y^4}), \quad e_5 = 2\frac{\partial}{\partial z} = \xi,$$

form a local orthonormal frame on TM. The distributions $D^\perp$ and D_θ are defined as $D^\perp = span\{e_1, e_2\}$ and $D_\theta = span\{e_3, e_4\}$ are the anti-invariant and the slant distributions with slant angle θ respectively on M and $TM = D^\perp \oplus D_\theta \oplus \langle\xi\rangle$.

Suppose that M s a pseudo-slant submanifold of an almost contact metric manifold. Then for any $X \in TM$, put

$$X = P_1X + P_2X + \eta(X)\xi, \tag{63}$$

where P_1 and P_2 are projection maps on the distributions $D^\perp$ and D_θ respectively. Operating ϕ on both sides of the above equation and using Eq. (2), we find that

$$TX = TP_2X, \quad NX = NP_1X + NP_2X, \tag{64}$$

$$\phi P_1X = NP_1X, \quad TP_1X = 0, \tag{65}$$

$$TP_2X \in D_\theta, \tag{66}$$

for any $X \in D_\theta$.

Now, we have the following characterization for a submanifold M to be a pseudo-slant submanifold in an almost contact metric manifold

Theorem 4.1 ([15]) *Let M be a submanifold of an almost contact metric manifold $\bar{M}$, such that $\xi \in TM$. Then M is a pseudo-slant submanifold if and only if there exists a constant $\lambda \in (0, 1]$ such that*

1. *$D = \{X \in D | T^2X = -\lambda X\}$ is a distribution on M.*
2. *For any $X \in TM$ orthogonal to D, $TX = 0$.*

In this case $\lambda = \cos^2\theta$, where θ denotes the slant angle of the distribution D.

Proof Suppose that M is a pseudo slant submanifold of a contact metric manifold. Set $\lambda = \cos^2\theta$, then it follows from (13) that the distribution D is slant i.e., $D = D_\theta$ for some $\theta \in [0, \frac{\pi}{2}]$. Conversely, consider the orthogonal direct decomposition $TM = D \oplus D^\perp \oplus \langle\xi\rangle$. It is evident that $TD \subset D$. Hence by statement 2 it is clear that $D^\perp$ is an anti-invariant distribution. Moreover, Theorem 3.1 and the statement 1 imply that D is a slant distribution, with the slant angle θ satisfying $\lambda = \cos^2\theta$.

Now, we investigate the integrability conditions for the canonical distributions on a pseudo-slant submanifold of a Sasakian manifold.

If μ is the invariant sub bundle of the normal bundle $T^{\perp}M$ then for the pseudo-slant submanifold M, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \mu \oplus \phi D^{\perp} \oplus ND_{\theta}. \tag{67}$$

Now for any $X \in D_{\theta}$ and $Z \in D^{\perp}$ it is easy to observe that $g(\phi Z, NX) = 0$. That means the decomposition (67) is an orthogonal direct decomposition.

For a pseudo-slant submanifold of a Sasakian manifold, we have the following relations that can be obtained by using formula (2), Gauss and Weingarten formulae

$$(\bar{\nabla}_X T)Y = A_{NY}X - th(X, Y) - \eta(Y)X + g(X, Y)\xi \tag{68}$$

and

$$(\bar{\nabla}_X N)Y = fh(X, Y) - h(X, TY), \tag{69}$$

for any $X, Y \in TM$.

The following initial results play an important role in working out the integrability conditions for the distributions on a pseudo slant submanifold.

Lemma 4.2 ([15]) *Let M be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then*

$$A_{\phi Y}X = A_{\phi X}Y,$$

for all $X, Y \in D^{\perp}$.

Lemma 4.3 ([15]) *Let M be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$, then*

$$[X, \xi] \in D^{\perp},$$

for all $X \in D^{\perp}$.

Lemma 4.4 ([15]) *Let M be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then for any $X, Y \in D^{\perp} \oplus D_{\theta}$*

$$g([X, Y], \xi) = 2g(X, TY).$$

Proposition 4.5 ([15]) *Let M be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then, the anti-invariant distribution $D^{\perp}$ is always integrable*

Proof For any $X, Y \in D^{\perp}$ and $Z \in D_{\theta}$

$$g([X, Y], TP_2Z) = -g(\phi[X, Y], P_2Z),$$

which on using (9) and Weingarten formula, gives

$$g([X, Y], TP_2Z) = g(A_{\phi Y}X - A_{\phi X}Y, P_2Z).$$

Now, the assertion follows on taking account of Lemmas 4.2 and 4.4.

Corollary 4.6 ([15]) *On a pseudo-slant submanifold M of a Sasakian manifold $\bar{M}$, the distribution $D^{\perp} \oplus \langle\xi\rangle$ is also integrable.*

The corollary follows from Proposition 4.5 and Lemma 4.4.

Lemma 4.7 ([15]) *Let M be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then the slant distribution is not integrable*

The proof of the lemma is straight forward in view of Lemma 4.4 and the definition of pseudo-slant submanifold.

Proposition 4.8 ([15]) *Let M be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$ then the distribution $D_\theta \oplus \langle\xi\rangle$ is integrable if and only if*

$$h(Z, TW) - h(W, TZ) + \nabla^{\perp}_{Z}NW - \nabla^{\perp}_{W}NZ$$

lies in ND_θ for each $Z, W \in D_\theta \oplus \langle\xi\rangle$.

To study the condition (18) on a pseudo-slant submanifold, we first observe that for the pseudo-slant submanifold of R^9 given in example 5

$$(\bar{\nabla}_X T)Y = \cos^2\theta(g(P_2X, Y)\xi - \eta(Y)P_2X), \tag{70}$$

for any $X, Y \in TM$. If we take $X, Y \in D_\theta \oplus \langle\xi\rangle$, then (70) implies that

$$(\bar{\nabla}_X T)Y = \cos^2\theta(g(X, Y)\xi - \eta(Y)X). \tag{71}$$

Thus the condition (70) is satisfied on $D_\theta \oplus \langle\xi\rangle$. On the other hand if $X, Y \in D^{\perp} \oplus \langle\xi\rangle$ then it follows from (70) that

$$(\bar{\nabla}_X T)Y = 0. \tag{72}$$

This indicates that anti-invariant submanifolds satisfies Eq. (70). Moreover, the condition (70) is natural condition for pseudo-slant submanifolds of a Sasakian manifold analogous to the conditions (18) and (43) for slant and semi-slant submanifolds respectively.

Now, we have the following theorem

Theorem 4.9 ([15]) *Let M be a proper pseudo-slant submanifold with angle θ of a Sasakian manifold $\bar{M}$. Then M satisfies (70) if and only if*

$$A_{NY} = A_{NX}Y + \eta(Y)P_1X - \eta(X)P_1Y - \sin^2\theta(\eta(X)P_2Y - \eta(Y)P_2X) \quad (73)$$

for any $X, Y \in TM$, where $NX = NP_1X + NP_2X$.

Proof Suppose M is a proper pseudo-slant submanifold satisfying (70). Then by (63) and (68)

$$\cos^2\theta(g(P_2X, Y)\xi - \eta(Y)P_2X) = A_{NP_1Y}X + A_{NP_1X}Y + th(X, Y) + g(P_1X, Y)\xi$$
$$+g(P_2X, Y)\xi - \eta(Y)P_1X - \eta(Y)P_2X,$$

or

$$A_{NY}X = -th(X, Y) - g(P_1X, Y)\xi + \eta(Y)P_1X$$
$$- \sin^2\theta(g(P_2X, Y)\xi - \eta(Y)P_2X). \quad (74)$$

Similarly

$$A_{NX}Y = -th(X, Y) - g(P_1Y, X)\xi + \eta(X)P_1Y$$
$$- \sin^2\theta(g(P_2Y, X)\xi - \eta(X)P_2Y). \quad (75)$$

From the above relations, on using the fact $g(P_1X, Y) = g(X, P_1Y)$, it follows that

$$A_{NY}X = A_{NX}Y + \eta(Y)P_1X - \eta(X)P_1Y - \sin^2\theta(\eta(X)P_2Y - \eta(Y)P_2X).$$

Conversely, suppose (70) holds, then for any $Z \in TM$,

$$g(A_{NY}X, Z) = -g(th(Y, Z), X) + \eta(Y)g(P_1X, Z) - \eta(X)g(P_1Y, Z)-$$
$$- \sin^2\theta(g(P_2Y, Z)\eta(X) - \eta(Y)g(P_2X, Z)).$$

Interchanging X and Z and making use of the fact that $g(P_1X, Y) = g(X, P_1Y)$, for each $X, Y \in TM$, we get

$$g(A_{NY}X, Z) = -g(th(X, Y), Z) + \eta(Y)g(P_1X, Z) - \eta(Z)g(P_1X, Y)$$
$$- \sin^2\theta(g(P_2X, Y)\eta(Z) - \eta(Y)g(P_2X, Z)).$$

Taking account of Eq. (63) the above equation yields

$$g(A_{NY}X, Z) = -g(th(X, Y), Z) + \eta(Y)g(X, Z) - \eta(Z)g(X, Y)-$$
$$-\cos^2\theta(\eta(Z)g(P_2X, Y) - \eta(Y)g(P_2X, Z)).$$

Now, by using (68) we obtain

$$(\nabla_X T)Y = \cos^2\theta(g(P_2X, Y)\xi - \eta(Y)P_2X),$$

which proves the assertion.

Theorem 4.10 ([15]) *Let M be a proper pseudo-slant submanifold of a Sasakian manifold $\bar{M}$, with slant angle θ, then*

1. *M satisfies (70) if and only if*

$$(\nabla_X TP_2)Y = \cos^2\theta(g(P_2X, Y)\xi - \eta(Y)P_2X),$$

2. *If M satisfies (70), then the distributions $D^\perp \oplus \langle\xi\rangle$ and $D_\theta \oplus \langle\xi\rangle$ are parallel on M.*

Proof On a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$, $T = TP_2$ and therefore statement 1 follows from (70) and (64). Suppose that M satisfies (70), then by the statement 1

$$(\nabla_X TP_2)Y = \cos^2\theta(g(P_2X, Y)\xi - \eta(Y)P_2X).$$

From the above equation it is evident that

$$P_1\nabla_X TP_2Y = 0, \tag{76}$$

i.e.,

$$\nabla_X TP_2Y \in D_\theta \oplus \langle\xi\rangle, \tag{77}$$

or equivalently

$$\nabla_X W \in D_\theta \oplus \langle\xi\rangle, \tag{78}$$

for any $X \in TM$ and $W \in D_\theta$. Which implies that $\nabla_X Z \in D^\perp \oplus \langle\xi\rangle$ for $Z \in D^\perp \oplus \langle\xi\rangle$, this proves statement 2.

As a consequence, we have

Corollary 4.11 ([15]) *Let M be a pseudo-slant submanifold of a Sasakian manifold $\bar{M}$ such that M satisfies (3.14). Then $D^\perp \oplus \langle\xi\rangle$ and $D_\theta \oplus \langle\xi\rangle$ are integrable and their leaves are totally geodesic in M.*

In [24], Umit Yildirim studied the geometry of the pseudo-slant submanifold with the name of contact pseudo-slant submanifolds of a Sasakian manifold. He studied some properties of the component tensor acting on the underlying submanifold and worked out the necessary and sufficient conditions for them to be parallel. In this section we will quote some results of Umit Yildirim

Theorem 4.12 ([24]) *Let M be a contact pseudo-slant submanifold in Sasakian manifold $\bar{M}$ such that $\xi \in D_\theta$. Then we have*

$$g(A_{ND^\perp} D_\theta - T\nabla_{D_\theta} D^\perp - th(D_\theta, D^\perp), D_\theta) = 0.$$

Theorem 4.13 ([24]) *Let M be a proper contact pseudo slant submanifold of a Sasakian manifold $\bar{M}$. Then the tensor N is parallel if and only if t is parallel.*

Theorem 4.14 ([24]) *Let M be a proper contact pseudo slant submanifold of a Sasakian manifold $\bar{M}$. Then the covariant derivation of f is skew-symmetric.*

Theorem 4.15 ([24]) *Let M be a proper contact pseudo slant submanifold of a Sasakian manifold $\bar{M}$. If t is parallel, then either M is a mixed geodesic or an anti-invariant submanifold.*

Theorem 4.16 ([24]) *Let M be a proper contact pseudo slant submanifold of a Sasakian manifold $\bar{M}$. If $\bar{t}$ is parallel, then either M is a $D^\perp$- geodesic or an anti-invariant submanifold of $\bar{M}$.*

In the next theorem, we observe the impact of parallelism of the tensor N on the submanifold

Theorem 4.17 ([24]) *Let M be a proper contact pseudo slant submanifold of a Sasakian manifold $\bar{M}$. If N is parallel, then either M is a mixed geodesic or an anti-invariant submanifold.*

Theorem 4.18 ([24]) *Let M be a proper contact pseudo slant submanifold of a Sasakian manifold $\bar{M}$. Then the anti-invariant distribution $D^\perp$ defines totally geodesic foliation in M if and only if*

$$-A_{NZ}TX + A_{FTX}Z \in D_\theta$$

for any $X \in D_\theta$ and $Z \in D_\perp$.

Theorem 4.19 ([24]) *Let M be a proper contact pseudo slant submanifold of a Sasakian manifold $\bar{M}$. Then the slant distribution D_θ defines a totally geodesic foliation on M if and only if*

$$A_{\phi Z}TX + A_{FTX}Z \in D^\perp$$

for any $X \in D_\theta$ and $Z \in D_\perp$.

In 2011, De and Sarkar [25] studied pseudo-slant submanifolds of trans-Sasakian manifolds. They obtained some basic results and integrability conditions of the distributions. Moreover, they also construct the example of pseudo-slant submanifold. Now, we have the following results related to the integrability of the distributions.

Theorem 4.20 [25] *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then the distribution $D^{\perp} \oplus \langle\xi\rangle$ is integrable.*

Remark [25] In particular the above result holds for $\alpha-$ Sasakian, $\beta-$Kenmotsu and Cosymplectic manifolds. For the Sasakian case the above result is already discussed in this section.

Theorem 4.21 [25] *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then the anti-invariant distribution $D^{\perp}$ is always integrable.*

Corollary 4.22 [25] *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then the anti-invariant distribution $D^{\perp} \oplus \langle\xi\rangle$ is integrable.*

Theorem 4.23 [25] *Let M be a pseudo-slant submanifold of a trans-Sasakian manifold $\bar{M}$. Then the slant distribution D_{θ} is not integrable.*

Now, we have the following main result

Theorem 4.24 [25] *Let M be a submanifold of a trans-Sasakian manifold $\bar{M}$ with $TM = D \oplus D_{\theta} \oplus \langle\xi\rangle$. Then M is pseudo-slant submanifold if and only if*

1. *The endomorphism $Q|_{D_{\theta}}$ has only one eigenvalue at each point of M,*
2. *there exist a function $\lambda : M \to [0, 1]$ such that*

$$(\nabla_X Q)Y = \lambda\{\alpha(g(X, TY)\xi - \eta(Y)TX) + \beta(g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X)\},$$

for any $X, Y \in D_{\theta} \oplus \langle\xi\rangle$. Moreover, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

For an α-Sasakian manifold the above theorem yields the following

Corollary 4.25 [25] *Let M be a submanifold of a α-Sasakian manifold $\bar{M}$ with $TM = D \oplus D_{\theta} \oplus \langle\xi\rangle$. Then M is pseudo-slant submanifold if and only if*

1. *The endomorphism $Q|_{D_{\theta}}$ has only one eigenvalue at each point of M,*
2. *there exist a function $\lambda : M \to [0, 1]$ such that*

$$(\nabla_X Q)Y = \lambda\{\alpha(g(X, TY)\xi - \eta(Y)TX)$$

for any $X, Y \in D_{\theta} \oplus \langle\xi\rangle$. Moreover, if θ is the slant angle of M, then $\lambda = \cos^2\theta$.

For the $\beta-$Kenmotsu manifold, it is easy to conclude the following

Corollary 4.26 [25] *Let M be a submanifold of a $\beta-$Kenmotsu manifold $\bar{M}$ with $TM = D \oplus D_{\theta} \oplus \langle\xi\rangle$. Then M is pseudo-slant submanifold if and only if*

1. *The endomorphism* $Q|_{D_\theta}$ *has only one eigenvalue at each point of* M,
2. *there exist a function* $\lambda : M \to [0, 1]$ *such that*

$$(\nabla_X Q)Y = \lambda\{\beta(g(X, Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(Y)X)\},$$

for any $X, Y \in D_\theta \oplus \langle\xi\rangle$. *Moreover, if* θ *is the slant angle of* M, *then* $\lambda = \cos^2\theta$.

5 Slant and Semi-slant Submanifolds of Almost Paracontact Metric Manifolds

In the previous sections we study slant and generalized slant submanifolds of almost contact metric manifolds. In this section our aim is to give a brief introduction of slant and semi-slant submanifolds in the setting of almost paracontact metric manifolds. In [16] M. Atceken introduced the notion of slant submanifolds and semi-slant submanifolds.

First, we define the almost paracontact metric manifolds.

Let $\bar{M}$ be a $(n + 1$- dimensional differentiable manifold. If there exist on $\bar{M}$ a $(1, 1)$ type tensor field F, a vector field ξ and 1-form η satisfying

$$F^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{79}$$

then $\bar{M}$ is said to be an almost paracontact manifold. In the almost paracontact manifold, the following relations hold

$$F\xi = 0, \quad \eta \circ F = 0, \quad rank(F) = n. \tag{80}$$

An almost paracontact manifold $\bar{M}$ is said to be an almost paracontact metric manifold if there exists a Riemannian metric g on $\bar{M}$ satisfying the following:

$$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{81}$$

for all $X, Y \in TM$.

From (80) and (81), one can easily deduce the following relation

$$g(FX, Y) = g(X, FY). \tag{82}$$

Let M be a submanifold isometrically immersed in $\bar{M}$. We denote by ∇ and $\bar{\nabla}$ the Levi-civita connections on M and $\bar{M}$ respectively. We also denote the Riemannian metric g on $\bar{M}$ as well as on M.

For any vector field $X \in TM$, we put

$$FX = fX + \omega X, \tag{83}$$

where fX and ωX denote the tangential and normal components of FX, respectively. Similarly, for any vector field V normal to M, we put

$$FV = BV + CV, \tag{84}$$

where BV and CV denote the tangential and normal components of FV, respectively. The submanifold M is said to be invariant if ω is identically zero on M i.e., $FX = fX$ for any $X \in TM$. On the other hand, M is said to be an anti-invariant submanifold if f is identically zero on M i.e. $FX = \omega X$ for any $X \in TM$. Suppose that the vector field ξ is tangent to M. If we denote by $\mathcal{D}$ the orthogonal complementary distribution to $\langle\xi\rangle$ in TM, then $TM = \mathcal{D} \oplus \langle\xi\rangle$, where $\langle\xi\rangle$ is the distribution spaned by the structure vector field ξ.

For each nonzero vector field X tangent to M at any point x such that X is not proportional to ξ_x , we denote by $\theta(X)$ the angle between FX and T_xM. Since, $F\xi = 0$, so the θ is well define angle between FX and $\mathcal{D}_\theta$. The submanifold M is said to be slant submanifold if the angle $\theta(X)$ is constant, which is independent of the choice of $x \in M$ and $X \in T_xM - \langle\xi\rangle$. Invariant and anti-invariant submanifolds are the slant submanifolds with slant angles $\theta = 0$ and $\theta = \pi/2$, respectively. A slant immersion that is neither invariant nor anti-invariant is called a proper slant submanifold.

Recently, Atceken [16] constructed the following example of slant submanifold in an almost paracontact metric manifold.

Example 6 ([16]) Let R^7 be the Euclidean space endowed with the usual Euclidean metric and with coordinates $(x_1, x_2, y_1, y_2, y_3, y_4, t)$. Define an almost paracontact metric structure on R^7 by

$$F(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}, \quad F(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial y_j}, \quad j = 1, 2, 3, 4, \quad F(\frac{\partial}{\partial t}) = 0,$$

$$\xi = \frac{\partial}{\partial t}, \quad \eta = dt.$$

Then the following immersion

$$\phi(u, v) = (u, v, -k \sin u, -k \sin v, k \cos u, k \cos v)$$

defines a slant submanifold in R^7 with slant angle $\theta = \cos^{-1}(\frac{1-k^2}{1+k^2})$.

Now, we have the following characterization for slant submanifolds of an almost paracontact metric manifold.

Theorem 5.1 ([16]) *Let M be an immersed submanifold of an almost paracontact metric manifold $\bar{M}$.*

1. *Let ξ be tangent to M. Then M is slant if and only if there exist a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda(I - \eta \otimes \xi)$.*

2. *Let ξ be normal to M. Then M is slant if only if there exist a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda I$.*

Further, if θ is the slant angle of M, it satisfies $\lambda = \cos^2\theta$.

Proof (i) suppose M is slant submanifold and ξ is tangent to M. Also, we assume $\cos\theta(X) = \frac{\|fX\|}{\|FX\|}$, where $\theta(X)$ is the slant angle. From (82) and (83) we have

$$g(f^2X, X) = \cos^2\theta(X)g(FX, FX) = \cos^2\theta(X)g(X - \eta(X)\xi, X)$$

for all $X \in TM$. Since g is a Riemannian metric, we have

$$f^2X = \cos^2\theta(X - \eta(X)\xi).$$

Let $\lambda = \cos^2\theta$. Then $\lambda \in [0, 1]$ and $f^2 = \lambda(I - \eta \otimes \xi)$.

Conversely, suppose that there exists a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda(I - \eta \otimes \xi)$, then by using (81) and (82) we have

$$\cos\theta(X) = \lambda\frac{\|FX\|}{\|fX\|},$$

for any $X \in TM$. On the other hand, since $\cos\theta(X) = \frac{\|fX\|}{\|FX\|}$, we conclude that $\cos^2\theta(X) = \lambda$, that is $\theta(X)$ is constant and so M is slant.

(ii) If ξ is the normal vector field to M , then we conclude that $\eta(X) = 0$. Thus from part (i), it mean that M is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that $f^2 = \lambda I$. Moreover, if θ is the slant angle of M, it satisfies $\lambda = \cos^2\theta$.

Following are the immediate corollaries

Corollary 5.2 ([16]) *Let M be a slant submanifold of an almost paracontact metric manifold $\bar{M}$ with slant angle θ such that ξ is tangent to M . Then we have*

$$g(fX, fY) = \cos^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}$$

$$g(\omega X, \omega Y) = \sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}$$

for any $X, Y \in TM$.

Corollary 5.3 ([16]) *Let M be an immersed submanifold of an almost paracontact metric manifold $\bar{M}$.*

1. *Let ξ be tangent to M. Then M is slant submanifold of $\bar{M}$ if and only if there exists a constant $\mu \in [0, 1]$ such that $B\omega = \mu(I - \eta \otimes \xi)$.*

2. *Let ξ be normal to $\bar{M}$. Then M is slant submanifold of $\bar{M}$ if and only if there exists a constant $\mu \in [0, 1]$ such that $B\omega = \mu I$.*

Furthermore, if θ is the slant angle of M, it satisfies $\mu = \sin^2 \theta$

Let M be a submanifold isometrically immersed in an almost paracontact metric manifold $\bar{M}$ and D be a distribution on $\bar{M}$. Suppose D be an orthogonal complementary distribution to D. Then for any $X \in TM$, we can write

$$FX = P_1 fX + P_2 fX + \omega X,$$

where P_1 and P_2 are the orthogonal projections on to D and D, respectively. Thus, we have the following characterization for slant distribution.

Theorem 5.4 ([16]) *Let D be a differentiable distribution on M such that ξ is tangent to D. Then D is a slant distribution if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$(P_1 f)^2 = \lambda(I - \eta \otimes \xi).$$

Further, in such case, if θ is the slant angle of D, then $\lambda = \cos^2 \theta$.

Lemma 5.5 ([16]) *Let M be a submanifold of an almost paracontact metric manifold $\bar{M}$ and D is a distribution on M. Then M is a slant submanifold if and only if D is slant distribution with the same slant angle.*

Proof If M is slant submanifold, then it is easy to see that D is a slant distribution with the same slant angle because $\theta(X) = \theta_D(X)$ for any $X \in D$.
Conversely, for $X \in TM - \langle\xi\rangle$, we have

$$\cos\theta(X) = \frac{g(fX, FX)}{\|fX\|\|FX\|} = \frac{\|fX\|}{\sqrt{\|X\|^2 - \eta^2(X)}}.$$

On the other hand, taking into account $X - \eta(X)\xi \in D$, we derive

$$\cos\theta_D = \frac{\|P(X - \eta(X)\xi\|}{\|X - \eta(X)\|},$$

where P denotes the orthogonal projection of F on D. But in almost paracontact manifolds, we have $\sqrt{\|X\|^2 - \eta^2(X)} = \|X - \eta(X)\|$ and $fX = P(X - \eta(X)\xi)$, hence $\cos\theta(X) = \cos\theta_D$, which proves the theorem.

Suppose M is a proper semi-slant submanifold of an almost paracontact manifold $\bar{M}$ with slant distribution D_θ and invariant distribution D. Then for any $X \in TM$, we have

$$X = P_1 X + P_2 X + \eta(X)\xi \;\text{ and }\; FX = fP_1 X + fP_2 X + \omega P_2 X \tag{85}$$

and

$$g(fX, fP_2Y) = \cos^2\theta g(X, P_2Y) \text{ and } g(\omega X, \omega P_2Y) = \sin^2\theta g(X, P_2Y), \quad (86)$$

for any $X, Y \in TM$.

Now, we have the following characterizing theorem for semi-slant submanifolds in almost paracontact metric manifold.

Theorem 5.6 ([16]) *Let M be a submanifold immersed in an almost paracontact metric manifold $\bar{M}$. Then M is a semi-slant submanifold if and only if there exist a constant $\lambda \in [0, 1)$ such that*

1. *$D' = \{X : f^2X = \lambda X\}$ is a distribution on M,*
2. *For any $X \in TM$ orthogonal to D', $\omega X = 0$.*

Moreover, if θ is the slant angle of M then $\lambda = \cos^2\theta$.

Proof Let M be a semi-slant submanifold of $TM = D \oplus D_\theta \oplus \langle\xi\rangle$, where D is invariant and D_θ is the slant distribution. We put $\lambda = \cos^2\theta$. For any $X \in D'$, if $X \in D$, then

$$X = F^2X - \eta(X)\xi = F^2X = (fP_1)^2X = \lambda X.$$

It follows that $\lambda = 1$, but this is a contradiction to $\lambda \in [0, 1)$, that is $D' \subseteq D_\theta$. On the other hand, since D is a slant distribution, we have $f^2X = (fP_2)^2X = \lambda X$ it implies that $D_\theta \subseteq D'$. Thus $D_\theta = D'$.

The following theorem deals with the integrability of the distribution D and D_θ

Theorem 5.7 ([16]) *Let M be a semi-slant submanifold of almost paracontact metric manifold $\bar{M}$. Then we have*

1. *The distribution D is integrable if and only if*

$$h(X, fY) = h(fX, Y),$$

for any $X, Y \in D$.

2. *The distribution D_θ is integrable if and only if*

$$P_1(\nabla_X fY - \nabla_Y fX) = P_1(A_{\omega P_2Y}X - A_{\omega P_2X}Y),$$

for any $X, Y \in D_\theta$.

Theorem 5.8 ([16]) *Let M be a semi-slant submanifold of an almost paracontact metric manifold $\bar{M}$. If $\nabla f = 0$, then the distribution D and D_θ are integrable and their leaves are totally geodesic in $\bar{M}$.*

Theorem 5.9 ([16]) *Let M be a semi-slant submanifold of an almost paracontact metric manifold $\bar{M}$. Then M is a semi-slant product if and only its second fundamental form satisfies*

$$Bh(U, X) = 0 \text{ and } h(U, fX) = Ch(U, X)$$

for any $U \in TM$ and $X \in D$.

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The Slant Submanifolds in the Setting of Metric f-Manifolds

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1 Introduction

In this survey paper, we present a brief summary concerning the slant geometry for submanifolds in metric f-manifolds, together with some applications. The notion of f-structure was introduced by Yano [11] as a tensor field f of type (1,1) satisfying $f^3 + f = 0$, and it generalizes both almost complex and almost contact structures. It can be proved that always exists a Riemannian metric compatible with the f-structure. A manifold endowed with an f-structure and a compatible Riemannian metric is called a metric f-manifold. In this context, it is possible to study invariant (resp., anti-invariant) submanifolds of metric f-manifolds as those ones such that any tangent vector field to the submanifold is sent by f to a tangent (resp., normal) vector field and introduce slant submanifolds generalizing them.

We organize this chapter as follows. In Sect. 2 we present definitions and some basic properties concerning metric f-manifolds and their submanifolds. In Sects. 3 and 4 we devote to the main aspects of the slant geometry in metric f-manifolds and

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in S-manifolds, respectively. In the last section, we present some results on Casorati inequalities for bi-slant immersions in a T-space form.

Due to this chapter which is a survey one, we do not provide most of the proofs.

2 Preliminaries

A Riemannian manifold $\widetilde{M}$ with Riemannian metric g on $\widetilde{M}$ is said to be a metric f-manifold $(\widetilde{M}, g)$ if it is associated with an f-structure f [11] and global vector fields $\{\xi_i\}_1^s$ (referred to as structural vector fields) such that

(i)

$$\begin{cases} f\xi_\alpha = 0;\ \eta_\alpha \circ f = 0; \\ f^2 = -I + \sum_{\alpha=1}^s \eta_\alpha \otimes \xi_\alpha; \\ g(\mathcal{X}, \mathcal{Y}) = g(f\mathcal{X}, f\mathcal{Y}) + \sum_{\alpha=1}^s \eta_\alpha(\mathcal{X})\eta_\alpha(\mathcal{Y}), \quad \alpha = 1, \ldots, s, \end{cases} \tag{2.1}$$

$\forall\, \mathcal{X}, \mathcal{Y} \in \chi(\widetilde{M})$, where dual 1-forms of $\{\xi_i\}_1^s$ are $\{\eta_i\}_1^s$.

(ii) f-structure satisfies normality condition, i.e.

$$[f, f] + 2\sum_{i=1}^{s} \xi_i \otimes d\eta_i = 0$$

where $[f, f]$ is the Nijenhuis tensor of f.

(iii) $\eta_1 \wedge \cdots \wedge \eta_s \wedge (d\eta_i) \neq 0$ and for each i, $d\eta_i = 0$.

Given the aforementioned circumstances

$$g(\mathcal{X}, f\mathcal{Y}) = -g(f\mathcal{X}, \mathcal{Y}). \tag{2.2}$$

A 2-form Φ , given by $\Phi(\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, f\mathcal{Y})$, $\forall\, \mathcal{X}, \mathcal{Y} \in \chi(\widetilde{M})$, can be considered. If $\Phi = d\eta_\alpha$, $\forall\, \alpha = 1, \ldots, s$, $\widetilde{M}$ is said to be a metric f-contact manifold associated with metric f-contact structure. In a particular case, if we take $s = 1$, then metric f-contact manifolds becomes metric contact manifold.

The structure f is normal if

$$[f, f] + 2\sum_{\alpha=1}^{s} \xi_\alpha \otimes d\eta_\alpha = 0, \tag{2.3}$$

Metric f-manifold $\widetilde{M}$ is called K manifold if it is normal and $d\Phi = 0$ (see [3]). It is known that the structure vector fields $\{\xi_i\}_{i=1}^s$ in a K-manifold are killing vector fields. Furthermore, if $\Phi = d\eta_\alpha$ for every $\alpha = 1, \ldots, s$, a K-manifold is called an S-manifold. Note that a K-manifold is a Kaehlerian manifold for $s = 0$, a quasi-

Sasakian manifold for $s = 1$, and an S-manifold is a Sasakian manifold for $s = 1$. When $s \geq 2$, there are some non-trivial examples in [3, 9].

In an S-manifold,

$$\eta_1 \wedge \cdots \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0, \tag{2.4}$$

for any $\alpha = 1, \ldots, s$. S-manifolds are obviously metric f-contact manifolds.

We notice that the following relations are satisfied for the Riemannian connection $\widetilde{\nabla}$ of S-manifold $\widetilde{M}$ [5]

$$(\widetilde{\nabla}_{\mathcal{X}} f)\mathcal{Y} = \sum_{\alpha=1}^{s} (g(f\mathcal{X}, f\mathcal{Y})\xi_\alpha + \eta_\alpha(\mathcal{Y}) f^2 \mathcal{X}) \tag{2.5}$$

and

$$\widetilde{\nabla}_{\mathcal{X}} \xi_\alpha = -f\mathcal{X}, \tag{2.6}$$

$\forall \mathcal{X}, \mathcal{Y} \in \chi(\widetilde{M})$.

An S-manifolds $\widetilde{M}$ is called S-space form if its f-sectional curvature c is constant $'c'$ and is denoted by $\widetilde{M}(c)$. The curvature tensor $\widetilde{R}$ of S-space form is given by [4]

$$\begin{aligned}
\widetilde{R}(\mathcal{X}, \mathcal{Y})\mathcal{Z} = {} & \frac{c+3s}{4}\big[g(f\mathcal{X}, f\mathcal{Z}) f^2\mathcal{Y} - g(f\mathcal{Y}, f\mathcal{Z}) f^2\mathcal{X}\big] \\
& + \frac{c-s}{4}\big[g(\mathcal{X}, f\mathcal{Z}) f\mathcal{Y} - g(\mathcal{Y}, f\mathcal{Z}) f\mathcal{X} + 2g(\mathcal{X}, f\mathcal{Y}) f\mathcal{Z}\big] \\
& + \sum_{\alpha,\beta} \big[\eta^\alpha(\mathcal{X})\eta^\beta(\mathcal{Z}) f^2\mathcal{Y} - \eta^\alpha(\mathcal{Y})\eta^\beta(Z) f^2\mathcal{X} \\
& - g(f\mathcal{X}, f\mathcal{Z})\eta^\alpha(\mathcal{Y})\xi_\beta + g(f\mathcal{Y}, f\mathcal{Z})\eta^\alpha(\mathcal{X})\xi_\beta\big].
\end{aligned} \tag{2.7}$$

and the curvature tensor of T-space form is given by

$$\begin{aligned}
\widetilde{R}(\mathcal{X}, \mathcal{Y})\mathcal{Z}) = {} & \frac{c}{4}\Big\{g(\mathcal{X}, \mathcal{Z})\mathcal{Y} - g(\mathcal{Y}, \mathcal{Z})\mathcal{X} - g(\mathcal{X}, \mathcal{Z}) \sum \eta^\alpha(\mathcal{Y})\xi_\alpha \\
& - \mathcal{Y} \sum \eta^\alpha(\mathcal{Z})\eta^\alpha(\mathcal{X}) + \mathcal{X} \sum \eta^\alpha(\mathcal{Y})\eta^\alpha(\mathcal{Z}) \\
& + g(\mathcal{Y}, \mathcal{Z}) \sum \eta^\alpha(\mathcal{X})\xi_\alpha + \Big(\sum \eta^\alpha(\mathcal{Z})\eta^\alpha(\mathcal{X})\Big)\Big(\sum \eta^\alpha(\mathcal{Y})\xi_\alpha\Big) \\
& - \Big(\sum \xi_\alpha \eta^\alpha(\mathcal{X})\Big)\Big(\sum \eta^\alpha(\mathcal{Y})\eta^\alpha(\mathcal{Z})\Big) + g(\mathcal{Y}, \phi\mathcal{Z})\phi\mathcal{X} \\
& - g(\mathcal{X}, \phi\mathcal{Z})\phi\mathcal{Y} - 2g(\mathcal{X}, \phi\mathcal{Y})\phi\mathcal{Z}\Big\}
\end{aligned} \tag{2.8}$$

Let TM (resp., $T^{\perp}M$) be the tangent (resp. normal) space on $\widetilde{M}$.

Next, write

$$fX = TX + NX, \tag{2.9}$$

for every $X \in TM$, where TX (resp. NX) denotes the tangential (resp. normal) components of fX. Similarly,

$$fV = \mathcal{T}V + \mathcal{N}V, \tag{2.10}$$

for any $V \in T^{\perp}M$ where $\mathcal{T}V$ (resp. $\mathcal{N}V$) is the tangential (resp. normal) components of fV.

In [10] Lotta introduced slant submanifold in almost contact manifold.

If X is not proportional to ξ, then the angle between fX & TM is denoted by $\theta(X)$.

Definition 2.1 The submanifold M is called a slant submanifold if the angle $\theta(X)$ is constant $\forall\, X \in TM - \{\xi\}$. The submanifold M is an invariant submanifold if $\theta = 0$, anti-invariant submanifold if $\theta = \frac{\pi}{2}$.

It is noticed that $T = f$ on TM for invariant submanifolds and hence

$$T^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha, \tag{2.11}$$

for anti-invariant submanifolds, whereas $T^2 = 0$ for anti-invariant submanifolds.

A submanifold M of $\widetilde{M}$ is called a *bi-slant submanifold* if it has a couple of orthogonal distributions Λ_1 and Λ_2 of M such that

(i) $TM = \Lambda_1 \oplus \Lambda_2 \oplus \{\xi_\alpha\}$,
(ii) $f\Lambda_i \perp \Lambda_j$, for $i \neq j = 1, 2$,
(iii) Each distribution Λ_i (for $i = 1, 2$) is slant with slant angle θ_i.

3 Slant Submanifolds in Metric f-Manifolds

Fernández and Hans-Uber [8] developed some general results concerning slant submanifolds in metric f-manifolds. A submanifold M of a metric f-manifold $\widetilde{M}$ is slant with tangent structure vector fields if

$$T^2 = -\lambda I + \lambda \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha = \lambda f^2, \tag{3.1}$$

Additionally, for slant angle θ of M, it follows $\lambda = \cos^2\theta$. From f-metric manifold condition together with (2.9) and (3.1), we have

$$g(TX, TY) = \cos^2\theta[g(X, Y) - \sum_{\alpha=1}^{s}\eta_\alpha(X)\eta_\alpha(Y)], \tag{3.2}$$

$$g(NX, NY) = \sin^2\theta[g(X, Y) - \sum_{\alpha=1}^{s}\eta_\alpha(X)\eta_\alpha(Y)], \tag{3.3}$$

for $X, Y \in TM$.

Moreover, if M^{m+s} is a non-invariant slant submanifold of a metric f-manifold $\widetilde{M}^{2m+s}$ associated with slant angle θ and an orthonormal frame $\{E_1, \ldots, E_m, \xi_1, \ldots, \xi_s\}$ of TM, then it directly implies $\{(\csc\theta)NE_i\}_1^m$ is a normal orthonormal frame M.

Consequently, we have the following proposition [5].

Proposition 3.1 *Let M^{2+s} be a proper slant submanifold of a metric f-manifold $\widetilde{M}^{4+s}$ associated with slant angle θ. Let E_1 is a unit vector field tangent to M and normal to the structural vector fields and*

$$E_2 = (\sec\theta)TE_1,\ E_3 = (\csc\theta)NE_1 and E_4 = (\csc\theta)NE_2.$$

Then, $E_1 = -(\sec\theta)TE_2$ and $\{E_1, E_2, E_3, E_4, \xi_1, \ldots, \xi_s\}$ is an orthonormal frame of $\chi(\widetilde{M})$ where $E_1, E_2, \xi_1, \ldots, \xi_s$ are tangent and E_3, E_4 are normal to M. In addition,

$$\mathcal{T}E_3 = -\sin\theta E_1, \mathcal{N}E_3 = -\cos\theta E_4, \mathcal{T}E_4 = -\sin\theta E_2, \mathcal{N}E_4 = \cos\theta E_3.$$

More details can be found in the papers [5, 6].

Next, let $\bar{f} = (\sec\theta)T$. Then, from (3.1) and (3.2), we have [8]

$$\bar{f}^2X = \sec^2\theta T^2X = -X + \sum_{\alpha}\eta_\alpha(X)\xi_\alpha,$$

and

$$g(\bar{f}X, \bar{f}Y) = \sec^2\theta g(TX, TY) = g(X, Y) - \sum_{\alpha}\eta_\alpha(X)\xi_\alpha,$$

where $X, Y \in TM$, where $\bar{f}$ is an f-structure on M. M together with $\bar{f}$ and structure vector fields $\xi_1, \ldots, \xi_s$ become a metric f-manifold. Accordingly, a slant submanifold of a metric f-contact manifold with the above induced structure becomes metric f-contact manifold when it is an invariant submanifold and vice-versa.

4 Slant Submanifolds of *S*-Manifolds

4.1 Examples

Thus, in the light of Examples 2.1, 2.3, and 2.4 given in [6], we have

Example 4.1 Consider

$$x(u, v, \gamma_1, \dots, \gamma_s) = 2(u\cos\theta, u\sin\theta, v, 0, \gamma_1, \dots, \gamma_s), \tag{4.1}$$

where $\theta \in [0, \frac{\pi}{2}]$. Then, (4.1) with slant angle θ defines a minimal $2+s$-dimensional slant submanifold M .

Actually, for $\alpha = 1, \dots, s$ we assume a basis of TM built as

$$\begin{aligned} E_1 &= \frac{\partial}{\partial u} + \sum_{\alpha=1}^{s} 2v\cos\theta\frac{\partial}{\partial\gamma_\alpha} \\ &= \cos\theta\Big(2\Big(\frac{\partial}{\partial x^1} + \sum_{\alpha=1}^{s} y^1\frac{\partial}{\partial z^\alpha}\Big)\Big) + \sin\theta\Big(2\Big(\frac{\partial}{\partial x^2} + \sum_{\alpha=1}^{s} y^2\frac{\partial}{\partial z^\alpha}\Big)\Big), \\ E_2 &= \frac{\partial}{\partial v} = 2\frac{\partial}{\partial y^1}, \; E_{2+\alpha} = \frac{\partial}{\partial\gamma_\alpha} = 2\frac{\partial}{\partial z_\alpha} = \xi_\alpha. \end{aligned}$$

Further proceeding in same direction, we get the normal basis on M as

$$\begin{aligned} E_{3+s} &= -\sin\theta\Big(2\Big(\frac{\partial}{\partial x^1} + \sum_{\alpha=1}^{s} y^1\frac{\partial}{\partial z^\alpha}\Big)\Big) + \cos\theta\Big(2\Big(\frac{\partial}{\partial x^2} + \sum_{\alpha=1}^{s} y^2\frac{\partial}{\partial z^\alpha}\Big)\Big), \\ E_{4+s} &= 2\frac{\partial}{\partial y^2}. \end{aligned}$$

Both bases can be proved to be orthonormal bases. Then, it yields

$$[E_1, E_2] = -2\cos\theta\sum_{\alpha=1}^{s} E_{2+\alpha}, [E_1, E_{4+s}] = -2\sin\theta\sum_{\alpha=1}^{s} E_{2+\alpha},$$

$$[E_2, E_{3+s}] = -2\sin\theta\sum_{\alpha=1}^{s} E_{2+\alpha}, [E_{3+s}, E_{4+s}] = -2\cos\theta\sum_{\alpha=1}^{s} E_{2+\alpha},$$

whereby $\widetilde{\nabla}_{E_i}^{E_i} = 0$, for any $i = 1, \dots, 2+s$. Furthermore

$$\begin{aligned} &\sigma(E_1, E_1) = 0, \quad \sigma(E_1, E_2) = 0, \quad \sigma(E_1, \xi_\alpha) = \sin\theta E_{4+s}, \\ &\sigma(E_2, E_2) = 0, \quad \sigma(E_2, \xi_\alpha) = \sin\theta E_{3+s}, \quad \sigma(\xi_\alpha, \xi_\beta) = 0, \end{aligned}$$

and, consequently M is minimal.

Example 4.2 Consider

$$x(u, v, \gamma_1, \dots, \gamma_s) = 2(e^{ku}\cos u\cos v, e^{ku}\sin u\cos v, e^{ku}\cos u, \sin v, e^{ku}\sin u\sin v, \gamma_1, \dots, \gamma_s), \tag{4.2}$$

where k is any constant. Then, (4.2) with slant angle

$$\theta = \arccos\Big(\frac{|k|}{\sqrt{1+k^2}}\Big)$$

defines a $(2+s)$-dimensional slant submanifold M and mean curvature is given by

$$|H| = \frac{2e^{-ku}}{(2+s)\sqrt{1+k^2}}.$$

Hence, the submanifold is not minimal.

Under this consideration, the orthonormal basis $\{E_1, E_2, \dots, E_{2+s}\}$ of TM is given by

$$E_1 = \frac{e^{-ku}}{\sqrt{1+k^2}}\Big(\frac{\partial}{\partial u} + \sum_{\alpha=1}^{s} ke^{2ku}\sin(2v)\theta\frac{\partial}{\partial\gamma_\alpha}\Big),$$
$$E_2 = e^{-ku}\Big(\frac{\partial}{\partial v} - 2\sum_{\alpha=1}^{s} ke^{2ku}\sin^2 v\theta\frac{\partial}{\partial\gamma_\alpha}\Big),\ E_{2+\alpha} = \frac{\partial}{\partial\gamma_\alpha} = 2\frac{\partial}{\partial z^\alpha} = \xi_\alpha.$$

Keeping in mind, at the points of the submanifold

$$4e^{2ku} = (x^1)^2 + (x^2)^2 + (y^1)^2 + (y^2)^2,$$

the value of $|H|$ is obtained with few steps of computations.

Example 4.3 For any constant k,

$$x(u, v, \gamma_1, \dots, \gamma_s) = 2(u, k\cos v, v, k\sin v, \gamma_1, \dots, \gamma_s)$$

defines a $(2+s)$-dimensional slant submanifold M with slant angle

$$\theta = \cos^{-1}\Big(\frac{1}{\sqrt{1+k^2}}\Big)$$

and mean curvature

$$|H| = \frac{|k|}{(2+s)(1+k^2)}.$$

Further, following statements are equivalent:

(a) $k = 0$,
(b) M is invariant,
(c) M is minimal.

Here, orthonormal frame $\{E_1, E_2, \xi_1, \ldots, \xi_s\}$ of $\mathcal{X}(M)$ is given by

$$E_1 = \frac{\partial}{\partial u} + \sum_{\alpha=1}^{s} 2v \frac{\partial}{\partial \gamma_\alpha} = 2\frac{\partial}{\partial x^1} 2y^1 \sum_{\alpha=1}^{s} \frac{\partial}{\partial z^\alpha},$$

$$E_2 = \frac{1}{\sqrt{1+k^2}} \left(\frac{\partial}{\partial v} + \sum_{\alpha=1}^{s} (-2k^2 \sin^2 v) \frac{\partial}{\partial \gamma_\alpha} \right) =$$

$$= \frac{1}{\sqrt{1+k^2}} \left(-y^2 \frac{\partial}{\partial x_2} + 2\frac{\partial}{\partial y^1} + x^2 \frac{\partial}{\partial y^2} - (y^2)^2 \sum_{\alpha=1}^{s} \frac{\partial}{\partial z^\alpha} \right),$$

$$E_{2+\alpha} = \frac{\partial}{\partial \gamma_\alpha} = 2\frac{\partial}{\partial z^\alpha} = \xi_\alpha.$$

Keeping in mind, for submanifold with $x^2 = 2k \cos v$ and $y^2 = 2k \sin v$, we obtain

$$\widetilde{\nabla}_{E_1} E_1 = 0$$

and

$$\widetilde{\nabla}_{E_2} E_2 = \frac{1}{1+k^2} \left(-x^2 \frac{\partial}{\partial x^2} - y^2 \frac{\partial}{\partial y^2} - x^2 y^2 \sum_{\alpha=1}^{s} \frac{\partial}{\partial z^\alpha} \right).$$

Furthermore, it is easily seen

$$\sigma(E_1, E_1) = 0, \; \sigma(E_2, E_2) = \widetilde{\nabla}_{E_2} E_2$$

and

$$H = -\frac{1}{(2+s)(1+k^2)} \left(x^2 \frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial y^2} + x^2 y^2 \sum_{\alpha=1}^{s} \frac{\partial}{\partial z^\alpha} \right).$$

Example 4.4 Consider $\mathbb{E}^{8+s}$ equipped with the usual S structure given in Example 4.2 from [1]. For any constant k,

$$x(u, v, w, m, t_1, \ldots, t_s) = 2(e^{ku} \cos u \cos v, e^{ku} \sin u \cos v, e^{ku} \cos u \sin v, e^{ku} \sin u$$
$$\sin v, e^{ku} \cos w \cos m, e^{ku} \sin w \cos m, e^{ku} \cos w \sin m, e^{ku} \sin w \sin m, t_1, \ldots, t_s)$$

defines a bi-slant submanifold M^{4+s} with the bi-slant angles (θ_1, θ_2) where

$$\theta_1 = \cos^{-1}\left(\frac{k}{\sqrt{2k^2+1}}\right) \quad and \quad \theta_2 = \cos^{-1}\left(\frac{-k}{\sqrt{k^2+1}}\right).$$

4.2 Ricci Curvature of Slant and Bi-slant Submanifolds of S-Space Forms

Consider an orthonormal frame

$$\{E_1, \ldots, E_n, E_{n+1}, \ldots, E_{2m}, E_{2m+1} = \xi_1, \ldots, E_{2m+s} = \xi_s\}, \tag{4.3}$$

Given a unit vector field $U \in \Lambda$ and taking into account an orthonormal frame as given in (4.3) with $E_1 = U$, one has $|TU|^2 = \cos^2\theta$, we have the following:

Theorem 4.5 ([7]) *Let M^{n+s} be θ-slant submanifold of an S-space form $\widetilde{M}(c)$. Then*

(i) $4\text{Ric(U)} \leq (n+s)^2|H|^2 + (n-1)(c+3s) + \cos^2\theta(3c+s)$ *for every* $U \in \Lambda$.
(ii) The equality holds in (i) if and only if either M is a totally f-geodesic submanifold or M^{2+s} is a totally f–umbilical submanifold.

For $X \in TM$, let $X = P_1X + P_2X + \sum_{\alpha=1}^{3}\eta_\alpha(X)\xi_\alpha$, where P_i are the projections. Now, we state some results obtained in [1] as follows.

Lemma 4.6 *For bi-slant submanifold of S manifold, we have*

$$g(TP_iX, TP_iY) = \cos^2\theta_i\big(g(P_iX, P_iY) - \sum_{\alpha=1}^{3}\eta_\alpha TP_iX\eta_\alpha TP_iY\big)$$

whereby $X, Y \in TM$.

Theorem 4.7 *Let $M^{2n+2m+s}$ be Λ_i -geodesic (θ_1, θ_2) bi-slant submanifold of an S manifold. Then, we have*

(i) $NX_i = 0$, *whereby* $X_i \in \Lambda_i$.
(ii) $P_iTP_j = \sin^2\theta_i I$, *for any* $i \neq j \in 1, 2$.

Theorem 4.8 *If the submanifold M is both Λ_1 and Λ_2-geodesic (θ_1, θ_2) bi-slant submanifold, then $\theta_1 + \theta_2 = \frac{\pi}{2}$.*

Theorem 4.9 *For (θ_1, θ_2) bi-slant submanifold M of an S manifold, we have*

(i) M is an invariant submanifold if it is totally umbilical submanifold.
(ii) For M to be Λ_1 and Λ_2 totally umbilical, $\theta_1 + \theta_2 = \frac{\pi}{2}$.

As a direct application of above result, it implies

Corollary 4.10 *There does not exists semi-slant and hemi-slant submanifolds of an S manifold, if either Λ_1 and Λ_2 geodesic or Λ_1 and Λ_2-totally umbilical*

Theorem 4.11 *For bi-slant submanifold $M^{(2n+2m+s)}$ of an S-space-form $\widetilde{M}(c)$, the following inequality holds:*

(i)

$$Ric(X) \leq \frac{1}{4}n^2||H||^2 + \frac{c}{4}\big\{(2n+2m-1) + 3\cos^2\theta_1 + ||P_2TX||^2\big\}, \quad (4.4)$$

for $X \in \Lambda_1$. equality holds if and only if M is Λ_1 totally geodesic.

(ii)

$$Ric(X) \leq \frac{1}{4}n^2||H||^2 + \frac{c}{4}\big\{(2n+2m-1) + 3\cos^2\theta_2 + ||P_1TX||^2\big\}, \quad (4.5)$$

for $X \in \Lambda_2$. Further, equality holds if and only if M is Λ_2 totally geodesic.

(iii) Moreover, equality condition of (4.4) and (4.5) holds, then $\theta_1 + \theta_2 = 0$.

Thus, as a consequence, we have the following result for semi-slant submanifold i.e, $\theta_1 = 0$.

Corollary 4.12 *Let $\widetilde{M}(c)$ be an S-space form and $M^{(2n+2m+s)}$ submanifolds of $\widetilde{M}(c)$, then for semi-slant submanifolds the following inequality holds:*

(i)

$$Ric(X) \leq \frac{1}{4}n^2||H||^2 + \frac{c}{4}\big\{(2n+2m+2), \quad (4.6)$$

for $X \in \Lambda_1$. Further, equality holds if and only if M is Λ_1 totally geodesic.

(ii)

$$Ric(X) \leq \frac{1}{4}n^2||H||^2 + \frac{c}{4}\big\{(2n+2m-1) + 3\cos^2\theta_2, \quad (4.7)$$

for $X \in \Lambda_2$. Further, equality holds if and only if M is Λ_2 totally geodesic.

(iii) Moreover, equality condition of both (4.6) and (4.7) does not satisfy.

Corollary 4.13 *Let $\widetilde{M}(c)$ be an S-space form and $M^{(2n+2m+s)}$ submanifolds of $\widetilde{M}(c)$, then for hemi-slant submanifolds the following inequality holds:*

(i)

$$Ric(X) \leq \frac{1}{4}n^2||H||^2 + \frac{c}{4}\big\{(2n+2m+2) + 3cos^2\theta_1, \quad (4.8)$$

for $X \in \Lambda_1$. Further, equality holds if and only if M is Λ_1 totally geodesic.

(ii)

$$Ric(X) \leq \frac{1}{4}n^2||H||^2 + \frac{c}{4}\{(2n + 2m - 1)\}, \tag{4.9}$$

for $X \in \Lambda_2$. Further, equality holds if and only if M is Λ_2 totally geodesic.

(iii) Moreover, equality condition of both (4.8) *and* (4.9) *does not satisfy.*

5 Casorati Inequalities for Bi-slant Immersions in T-Space-Form

For an immersion of a manifold M^n into a Riemannian manifold $\widetilde{M}^m$, let

$$\{E_1, \ldots, E_n, E_{n+1}, \ldots, E_m\}$$

be an orthonormal frame to $\widetilde{M}$ such that $\{E_1, \ldots, E_n\}$ and $\{E_{n+1}, \ldots, E_m\}$ are orthonormal frame of TM and $T^{\perp}M$, respectively.

Now, we first recall some basic terminologies concerning intrinsic curvatures given below. The *normalized scalar curvature* ρ of the immersion is given by

$$\rho = \frac{2\tau}{n(n-1)}, \tag{5.1}$$

for scalar curvature τ.

The *Casorati curvature* of the submanifold is stated as

$$C = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{\alpha}\right)^2. \tag{5.2}$$

The scalar curvature of the r-plane section L is given by

$$\tau(L) = \sum_{1\leq i<j\leq r} K(E_i \wedge E_j)$$

which gives formula for the Casorati curvature of the subspace L as

$$C(L) = \frac{1}{r}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(\sigma_{ij}^{\alpha}\right)^2.$$

Next, we have formula for $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ called *normalized δ-Casorati curvatures* given by

$$[\delta_c(n-1)]_x = \frac{1}{2}C(x) + \frac{n+1}{2n}\inf\{C(L) : L \text{ is a hyperplane of } T_xM\} \quad (5.3)$$

and

$$[\widehat{\delta}_c(n-1)]_x = 2C(x) + \frac{2n-1}{2n}\sup\{C(L) : L \text{ is a hyperplane of } T_xM\}. \quad (5.4)$$

Also, for $t \neq n(n-1) \in \mathbb{R}^+$ and taking

$$\mathcal{A}(t) = \frac{1}{nt}(n-1)(n+t)(n^2-n-t), \quad (5.5)$$

we have formula for the *generalized normalized* δ*-Casorati curvatures* $\delta_c(t; n-1)$ and $\widehat{\delta}_c(t; n-1)$ stated as

$$[\delta_c(t; n-1)]_x = tC(x) + \mathcal{A}(t)\inf\{C(L) : L \text{ is a hyperplane of } T_xM\}, \quad (5.6)$$

if $0 < t < n^2 - n$, and

$$[\widehat{\delta}_c(t; n-1)]_x = tC(x) + \mathcal{A}(t)\sup\{C(L) : L \text{ is a hyperplane of } T_xM\}, \quad (5.7)$$

if $t > n^2 - n$.

Now, we have [2].

Theorem 5.1 *For bi-slant submanifold* M^{n+s} *of a* T*-space-form* $\widetilde{M}^{2m+s}(c)$*, we have*

(i) $\delta_c(t; n+s-1)$ *for* $0 < t < (n+s)(n+s-1)$ *satisfies*

$$\rho \leq \frac{\delta_c(t; n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\Big\{(n(n-1) + 3(d_1\cos^2\theta_1 + d_2\cos^2\theta_2 + s(1-s))\Big\}. \quad (5.8)$$

(ii) $\widehat{\delta}_c(t; n+s-1)$ *for* $t > (n+s)(n+s-1)$ *satisfies*

$$\rho \leq +\frac{\widehat{\delta}_c(t; n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\Big\{(n(n-1) + 3(d_1\cos^2\theta_1 + d_2\cos^2\theta_2 + s(1-s))\Big\}. \quad (5.9)$$

Moreover, the equality holds in (5.8) *and* (5.9) *if and only if the shape operators* $A_\alpha = A_{E_\alpha}$, $\alpha \in \{n+s+1, \ldots, 2m+s\}$ *take the following form of matrix for tangent (resp. normal) orthonormal frame* $\{E_1, \ldots, E_{n+s}\}$ *(resp.* $\{E_{n+s+1}, \ldots, E_{2m+s}\}$*) as*

$$A_{n+s+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 0 & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{(n+s)(n+s-1)}{t}a \end{pmatrix}, A_{n+s+2} = \cdots = A_{2m+s} = 0. \quad (5.10)$$

Remark 5.2 Similar result for $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ can also be easily obtained.

Finally, we have the following result obtained in [2]:

Theorem 5.3 *Let M be a $(n+s)$-dimensional submanifold in a T-space-form $\widetilde{M}(c)$ of dimension $2m+s$. Then, inequalities for generalized normalized δ-Casorati curvatures $\delta_c(t; n+s-1)$ and $\widehat{\delta}_c(t; n+s-1)$ are given in the following table (Table 1):*

Table 1 Result in a glance for different submanifolds

M^{n+s}	Inequalities
Semi-slant	$\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{n(n-1) + 3(d_1 + d_2 cos^2\theta_2 + s(1-s))\}$ $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{n(n-1) + 3(d_1 + d_2 cos^2\theta_2 + s(1-s))\}$
Hemi-slant	$\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{n(n-1) + 3(d_1 cos^2\theta_1 + s(1-s))\}$ $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{n(n-1) + 3(d_1 cos^2\theta_1 + s(1-s))\}$
Slant	$\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1) + 3((n+s)cos^2\theta + s(1-s))\}$ $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1) + 3((n+s)cos^2\theta + s(1-s))\}$
CR	$\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1) + 3(d_1 + s(1-s))\}$ $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1) + 3(d_1 + s(1-s))\}$
Invariant	$\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-2) + 3s(2-s))\}$ $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-2) + 3s(2-s))\}$
Anti-invariant	$\rho \leq \frac{\delta_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1) + 3s(1-s))\}$ $\rho \leq \frac{\widehat{\delta}_c(t;n+s-1)}{(n+s)(n+s-1)} + \frac{c}{4(n+s)(n+s-1)}\{(n(n-1) + 3s(1-s))\}$

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Slant, Semi-slant and Pointwise Slant Submanifolds of 3-Structure Manifolds

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1 Introduction

The study of geometric properties of submanifolds in different ambient Riemannian manifolds is a very interesting area of research. In 1990 [8, 9], B.-Y. Chen introduced the notion of slant submanifolds in complex geometry as an umbrella of invariant and anti-invariant (totally real) submanifolds. Since then, the theory of slant submanifolds became one of the interesting areas of research. Later, Lotta [18] extended this notion in the framework of contact geometry. Recently, Sahin [25] defined slant and quaternion slant submanifolds of quaternion Kaehler manifolds. As a natural generalization of slant and CR-submanifolds, Papaghiuc [24] introduced the notion of semi-slant submanifolds which was further generalized by Carriazo [7] introducing bi-slant submanifolds. Further in [5], the authors investigated bi-slant and semi-slant submanifolds of Sasakian manifolds. Since then, many authors have studied these types of submanifolds in various ambient spaces endowed with other structures such as trans-Sasakian and Kenmotsu (see [6, 25, 26]). Furthermore, slant submanifolds have been extended to semi-slant, bi-slant, hemi-slant, pseudo-slant, pointwise slant, pointwise h-semi-slant submanifolds of different manifolds (like Kaehlerian, quaternionic Kaehlerian, almost contact and almost contact 3 structures) [5, 22, 28].

On the other hand, Etayo [11] extended this theory of submanifolds by defining quasi-slant submanifolds. In such submanifolds, at any given point the slant angle is independent of the choice of any non-zero vector field of submanifold. Later, Chen and Garay [10] studied and characterized these submanifolds as pointwise slant

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submanifolds. Next, pointwise h-semi slant and bi-slant submanifolds of quaternion manifolds were studied by Park in [21]. Later, B. Sahin together with Lee [17] investigated pointwise slant submersion from almost Hermitian manifolds.

In 1970, Kuo [16] and Udriste [27] were first independently defined almost contact 3-structure manifolds. 3-Sasakian and 3-cosymplectic manifolds are two classes of almost contact 3-structure manifolds which are equipped with three Killing vector fields (see [2, 3, 12]). Moreover, 3-Sasakian and 3-cosymplectic manifolds are closely related to quaternionic Kaehlerian and hyper-Kaehlerian manifolds [4, 29] and are Einstein and Ricci flat, respectively [13, 23]. In 2013, the author of the present chapter with Malek [19] introduced the notion of 3-slant submanifolds in almost contact 3-structure manifolds, focusing on the study of Sasakian slant submanifolds in which particularly established a sharp inequality involving the squared mean curvature and Ricci curvature for such submanifolds. Later, in [20] the same authors introduced 3-semi-slant and 3-bi-slant submanifolds in almost contact 3-structure manifolds. Some years later, in [15] S. Uddin and the author of the present chapter introduced pointwise hemi 3-slant submanifolds of almost contact metric 3 structures and characterized such submanifolds. They also investigated the integrability conditions for some canonical distributions.

The purpose of this chapter is to summarize the contributions of present author to the geometry of various kinds of slant submanifolds in almost contact metric 3-structure manifolds.

2 Almost Contact 3-Structure Manifolds

Let $\overline{M}$ be an odd-dimensional Riemannian manifold admitting a (1, 1)-tensor field ϕ, a vector field ξ and a 1-form η and satisfying

$$\phi^2(X) = -X + \eta(X)\xi, \;\; \eta(\xi) = 1, \tag{1}$$

for any vector field $X \in T\overline{M}$, where $T\overline{M}$ is the Lie algebra of vector fields in $\overline{M}$. Then (ϕ, ξ, η) is called an almost contact structure on $\overline{M}$.

Also, let g be a Riemannian metric on $\overline{M}$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

for all $X, Y \in T\overline{M}$. Then $(\overline{M}, \xi, \eta, \phi, g)$ is called an almost contact metric manifold. Moreover, in an almost contact metric manifold $\phi\xi = 0$ and $\eta o \phi = 0$ holds [1].

Next, we recall two important types of almost contact metric manifolds (namely, Sasakian and cosymplectic manifolds) which are defined as follows:

Let $\overline{\nabla}$ be the Levi-Civita connection of almost contact metric manifold $(\overline{M}, \xi, \eta, \phi, g)$. Then, $\overline{M}$ is called **Sasakian manifold** if

$$(\overline{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X \text{ and } \overline{\nabla}_X \xi = -\phi X, \tag{3}$$

for all $X, Y \in T\overline{M}$.

On the other hand, an almost contact metric manifold $(\overline{M}, \xi, \eta, \phi, g)$ is called **cosymplectic manifold** if it satisfies

$$(\overline{\nabla}_X \phi)Y = 0 \text{ and } \overline{\nabla}_X \xi = 0, \tag{4}$$

for all $X, Y \in T\overline{M}$.

Next, suppose $\overline{M}$ admits three almost contact metric structures $(\xi_i, \eta_i, \phi_i, g)$, for $i = 1, 2, 3$ satisfying

$$\eta_i(\xi_j) = 0, \quad \phi_i \xi_j = -\phi_j \xi_i = \xi_k, \quad \eta_i o \phi_j = -\eta_j o \phi_i = \eta_k, \tag{5}$$

$$\phi_i o \phi_j - \eta_j \otimes \xi_i = -\phi_j o \phi_i + \eta_i \otimes \xi_j = \phi_k, \tag{6}$$

$$g(\phi_i X, \phi_i Y) = g(X, Y) - \eta_i(X)\eta_i(Y), \tag{7}$$

for all $X, Y \in T\overline{M}$ and for cyclic permutation (i, j, k) to be $(1, 2, 3)$. Then, $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is called an **almost contact metric 3-structure manifold** [16] which in this work shortly called as metric 3-structure manifold.

Using relation (7), one can easily have

$$g(\phi_i X, Y) = -g(X, \phi_i Y). \tag{8}$$

Moreover, $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is called 3-Sasakian manifold if it holds

$$(\overline{\nabla}_X \phi_i)Y = g(X, Y)\xi_i - \eta_i(Y)X \text{ and } \overline{\nabla}_X \xi_i = -\phi_i X, \tag{9}$$

for all $X, Y \in T\overline{M}$.

Next, $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is called **3-cosymplectic manifold** if it satisfies

$$(\overline{\nabla}_X \phi_i)Y = 0 \text{ and } \overline{\nabla}_X \xi_i = 0, \tag{10}$$

for all $X, Y \in T\overline{M}$.

It is well known that 3-Sasakian and 3-cosymplectic manifolds are $(4n + 3)$-dimensional Einstein and Ricci flat manifolds, respectively [13, 23].

Let M be an isometrically immersed submanifold in metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. Then the Gauss and the Weingarten formulas can be written as

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y) \text{ and } \overline{\nabla}_X V = D_X V - A_V X, \tag{11}$$

for any $X, Y \in TM$ and $V \in (TM)^\perp$ where TM (resp. $T(M)^\perp$) are the set of all vector fields tangent (resp. normal) to M. Also, ∇ is the Levi-Civita connection on M, $D_X V$ is the normal connection of the immersion, $B(X, Y)$ is the second fundamental form and $A_V X$ is the shape operator. Moreover, B and A are related together by the following relation

$$g(A_V X, Y) = g(B(X, Y), V). \tag{12}$$

In addition, let $\overline{R}$ and R be the curvature tensors of $\overline{M}$ and M, respectively. Then the Gauss equation is given by

$$R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + g(B(X, W), B(Y, Z)) - g(B(X, Z), B(Y, W)), \tag{13}$$

for all $X, Y, Z, W \in TM$.

Moreover, for any $X \in TM$ and $V \in (TM)^{\perp}$ one can decompose $\phi_i X$ and $\phi_i V$ as follows.

$$\phi_i X = T_i X + N_i X \text{ and } \phi_i V = t_i V + n_i V, \tag{14}$$

where $T_i X$ (resp. $N_i X$) in the tangential (resp. normal) component of $\phi_i X$ and $t_i V$ (resp. $n_i V$) in the tangential (resp. normal) component of $\phi_i V$.

3 Slant Submanifolds of 3-Structure Manifolds

3.1 The Concept of 3-Slant Submanifolds and Examples

This section is devoted to a review some basic definitions. Later, we discuss 3-slant submanifolds of a 3-structure manifold and give some examples and characterization theorems of 3-slant submanifolds.

A submanifold M of $(\overline{M}, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$ is called an invariant submanifold if $\phi_i T_p(M) \subset T_p M$ and is called an anti-invariant submanifold if $\phi_i T_p(M) \subset T_p(M)^{\perp}$ for all $p \in M$ and $i = 1, 2, 3$.

Let M be a submanifold of an almost contact metric manifold $(\overline{M}, \xi, \eta, \phi)$. Then, M is said to be a slant submanifold if the angle between ϕX and $T_p M$ is constant at any point $p \in M$ and for any X linearly independent of ξ.

As a generalization of invariant and anti-invariant submanifolds, 3-slant submanifolds of 3-structure manifolds are defined as follows:

Definition 1 ([19]) Let M be a submanifold of a metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. Then, M is said to be a **3-slant submanifold** if for all $i \in \{1, 2, 3\}$, the angle between $\phi_i X$ and $T_p M$ is constant $\theta \in [0, \frac{\pi}{2}]$, for each $p \in M$ and each non-zero vector $X \in T_p M$ linearly independent of ξ_i.

It is noted that the slant angle does not depend on choice of $p \in M$, $X \in T_p M$ and ϕ_i, for all $i, j \in \{1, 2, 3\}$ the angle between $\phi_i X$ and $T_j X$ is θ. Here, one can easily see that a 3-slant submanifold with slant angle $\theta = 0$ (respectively $\theta = \frac{\pi}{2}$) turns out to an invariant (respectively anti-invariant) submanifold. Also for $\theta \in (0, \frac{\pi}{2})$, the submanifold is called a proper 3-slant submanifold.

Now as generalizations of examples of [19], we first give non-trivial examples of 3-slant submanifolds of 3-structure manifolds.

Example 1 Consider $\overline{M} = \mathbb{R}^{4n+3}$ with Euclidian metric $g = \sum_{i=1}^{4n+3} dx_i^2$ and ϕ_i's are given by

$$\phi_1((x_i)_{i=\overline{1,4n+3}}) = (-x_3, x_4, x_1, -x_2, \ldots, 0, -x_{4n+3}, x_{4n+2}),$$

$$\phi_2((x_i)_{i=\overline{1,4n+3}}) = (-x_4, -x_3, x_2, x_1, \ldots, x_{4n+3}, 0, -x_{4n+1}),$$

$$\phi_3((x_i)_{i=\overline{1,4n+3}}) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{4n+2}, x_{4n+1}, 0).$$

Suppose that $\xi_1 = \partial x_{4n+1}, \xi_2 = \partial x_{4n+2}, \xi_3 = \partial x_{4n+3}$ and $\eta_i(.) = g(\xi_i, .)$, $i = 1, 2, 3$, then $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is a metric 3-structure manifold.

For $\alpha \in (0, \frac{\pi}{2})$, let

$$M(u, v) = (u \sin\alpha, 0, 0, 0, u\cos\alpha, v\cos\alpha, v\cos\alpha, v\cos\alpha, 0, \ldots, 0)$$

be a submanifold of $\overline{M}$. Then

$$\{X_1 = \sin\alpha\partial x_1 + \cos\alpha\partial x_5, X_2 = \cos\alpha(\partial x_6 + \partial x_7 + \partial x_8)\}$$

is a frame for TM. By direct computations we find the slant angle θ as follows

$$\cos\theta = \frac{g(\phi_i X, T_j X)}{|\phi_i X||T_j X|} = \frac{\cos\alpha}{\sqrt{3}}, \quad i, j \in \{1, 2, 3\}.$$

So, M is a proper 3-slant submanifold of $\overline{M}$.

Example 2 Let $\overline{M} = \mathbb{R}^{15}$, $g = \sum_{i=1}^{15} dx_i^2$, $\xi_1 = \partial x_{13}, \xi_2 = \partial x_{14}, \xi_3 = \partial x_{15}$ and η_i's be the dual of ξ_i's. Also, ϕ_i's are defined as follows

$$\phi_1((x_i)_{i=\overline{1,15}}) = (-x_3, x_4, x_1, -x_2, \ldots, -x_{11}, x_{12}, x_9, -x_{10}, 0, -x_{15}, x_{14}),$$

$$\phi_2((x_i)_{i=\overline{1,15}}) = (-x_4, -x_3, x_2, x_1, \ldots, -x_{12}, -x_{11}, x_{10}, x_9, x_{15}, 0, -x_{13}),$$

$$\phi_3((x_i)_{i=\overline{1,15}}) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{10}, x_9, -x_{12}, x_{11}, -x_{14}, x_{13}, 0).$$

It is easy to show that $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is a metric 3-structure manifold.

By taking $M(t, s) = (t, s, s, s, t, 0, 0, 0, t, 0, 0, 0, t + s, t + s, t + s)$, at any point $p \in M$, T_pM is spanned by

$$X_1 = \partial x_1 + \partial x_5 + \partial x_9 + \partial x_{13} + \partial x_{14} + \partial x_{15},$$

and

$$X_2 = \partial x_2 + \partial x_3 + \partial x_4 + \partial x_{13} + \partial x_{14} + \partial x_{15}.$$

So, the slant angle θ is obtained by the following equations

$$cos\theta = \frac{g(\phi_i X_1, X_2)}{|\phi_i X_1||X_2|} = \frac{g(\phi_i X_2, X_1)}{|\phi_i X_2||X_1|} = \frac{1}{\sqrt{30}}, \quad i \in \{1, 2, 3\}.$$

Therefore M is a proper 3-slant submanifold of $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$.

3.2 Characterization Theorems for 3-Slant Submanifolds

The following theorem characterizes 3-slant submanifolds in which structure vector fields are normal to the submanifold.

Theorem 1 ([19]) *Let M be a submanifold of an almost contact metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ and all the structure vector fields are normal to M. Then, M is a 3-slant submanifold if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$T_i T_j X = \lambda X, \quad \text{for all } X \in TM \text{ and } i, j \in \{1, 2, 3\}. \tag{15}$$

Moreover, in that case $\lambda = -cos^2\theta$ where θ is the slant angle.

Proof For any $X \in TM$, by using (6), (7) and (8), we compute β and θ the angle between ϕ_i, T_j and ϕ_j, T_j, respectively, as

$$cos\beta = \frac{g(\phi_i X, T_j X)}{|\phi_i X||T_j X|} = -\frac{g(X, \phi_i T_j X)}{|X||T_j X|} = -\frac{g(X, T_i T_j X)}{|X||T_j X|}, \tag{16}$$

$$cos\theta = \frac{g(\phi_j X, T_j X)}{|\phi_j X||T_j X|} = -\frac{g(X, T_j T_j X)}{|X||T_j X|}. \tag{17}$$

So if $T_i T_j X = \lambda X$, then the angles are equal. On the other hand,

$$cos\theta = \frac{g(\phi_j X, T_j X)}{|\phi_j X||T_j X|} = \frac{g(T_j X, T_j X)}{|\phi_j X||T_j X|} = \frac{|T_j X|}{|X|}. \tag{18}$$

Hence, (17) and (18) implies that

$$cos^2\theta = -\frac{g(X, T_j T_j X)}{|X|^2}. \tag{19}$$

Now, we see that (15) yields $\lambda = -cos^2\theta$ and thus θ is constant. Conversely, let M be a 3-slant submanifold then the angles are equal and constant. So, the right side of (19) is constant and therefore $T_i T_j X = \lambda X$, where $\lambda = -cos^2\theta$. □

Next from Eq. (19) for any $X \in TM$ orthogonal to ξ_i's, we have the following corollary.

Corollary 1 ([19]) *Let M be a 3-slant submanifold of a metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. Then $T_iT_jX = -cos^2\theta X$, for $X \in TM\setminus <\xi_1, \xi_2, \xi_3>$.*

Also if the structure vector fields are tangent to M, the following theorem generalizes Theorem 2.2 of [6] for 3-structure case.

Theorem 2 ([19]) *Let M be a 3-slant submanifold of a metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$, then for any $X \in TM$ linearly independent of ξ_1, ξ_2, ξ_3,*

$$T_iT_jX = \lambda Z + \eta_i(X)\xi_j, \tag{20}$$

$$T_i^2X = \lambda Z - \eta_j(X)\xi_j - \eta_k(X)\xi_k, \tag{21}$$

where for the slant angle θ, $\lambda = -cos^2\theta$, $Z = X - \Sigma_{m=1}^3\eta_m(X)\xi_m$ and (i, j, k) is permutation of $(1, 2, 3)$.

Proof First we suppose all the structure vector fields are tangent to M. Let $TM = \mathcal{D} \oplus \xi$, where $\xi =< \xi_1, \xi_2, \xi_3 >$ and $\mathcal{D}$ is a vector space spanned by $\{e_1, \ldots, e_s\}$ such that $\{e_1, \ldots, e_s, \xi_1, \xi_2, \xi_3\}$ is a local orthonormal basis for TM. For each $X \in TM$, we put $X = Z + \sum_{m=1}^3 \eta_m(X)\xi_m$, where $Z \in \mathcal{D}$. From Corollary 1, we have $T_iT_jZ = \lambda Z$. Since $\xi_m \in TM$, by (2), $T_i\xi_j = \xi_k$. So, (8) and (14) imply $T_iT_j(\eta_i(X)\xi_i + \eta_j(X)\xi_j + \eta_k(X)\xi_k) = -\eta_i(X)T_i\xi_k = \eta_i(X)\xi_j$. Therefore, we have $T_iT_jX = \lambda Z - \eta_i(X)T_i\xi_k = \lambda Z + \eta_i(X)\xi_j$.

If at least one of the structure vector fields does not belong to TM, say $\xi_k \notin TM$, then $\eta_k(X) = 0, \quad \forall X \in TM$. So, $X = Z + \eta_i(X)\xi_i + \eta_j(X)\xi_j$ and as above, (20) will be satisfied too. Specially, when all the structure vector fields are normal to TM, (20) coincides with (15).

A computation like the one above for Eq. (21) completes the proof. □

The next corollary immediately follows from Eqs. (6), (7) and Theorem 2.

Corollary 2 ([19]) *Let M be a 3-slant submanifold of an almost contact metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$, then for all $X, Y \in TM$*

$$\begin{aligned}
g(T_iX, T_iY) &= cos^2\theta g(Z, Y) - \eta_j(X)\eta_j(Y) - \eta_k(X)\eta_k(Y)\\
&= cos^2\theta g(X, W) - \eta_j(X)\eta_j(Y) - \eta_k(X)\eta_k(Y),\\
g(T_iX, T_jY) &= cos^2\theta g(Z, Y) + \eta_i(X)\eta_j(Y)\\
&= cos^2\theta g(X, W) + \eta_j(X)\eta_i(Y),\\
g(N_iX, N_jY) &= -cos^2\theta g(X, W) - g(X, \phi_kY) + \eta_j(X)\eta_i(Y) - \eta_i(X)\eta_j(Y)\\
&= -cos^2\theta g(Z, Y) - g(\phi_kX, Y) - \eta_j(X)\eta_i(Y) + \eta_i(X)\eta_j(Y),
\end{aligned}$$

where $Z = X - \sum_{m=1}^3 \eta_m(X)\xi_m$ and $W = Y - \sum_{m=1}^3 \eta_m(Y)\xi_m$.

Proof The first and second equations follow easily from (20) and (21). For the third equation, we have $g(N_iX, N_jY) = g(\phi_iX, N_jY) = g(\phi_iX, \ \phi_jY) - g(\phi_iX, T_jY) = -g(X, \phi_i\phi_jY) + g(X, T_iT_jY)$. Applying (6) and (20) in the previous equalities imply $g(N_iX, N_jY) = -g(X, \phi_k Y) - \eta_i(X)\eta_j(Y) - cos^2(\theta)g(X, W) + \eta_j(X)\eta_i(Y)$. □

It is known that [16] if almost contact structures (ξ_1, η_1, ϕ_1) and (ξ_2, η_2, ϕ_2) satisfy in the conditions (5) and (6) on manifold $\overline{M}$, then there is an almost contact structure (ξ_3, η_3, ϕ_3) such that $(\overline{M}, \xi_i, \eta_i, \phi_i)_{i\in\{2,3\}}$ is an almost contact 3-structure manifold. The following example shows that even though almost contact structures $(\xi_i, \eta_i, \phi_i)_{i\in\{2,3\}}$ are slant structure on a submanifold but the third structure (ξ_1, η_1, ϕ_1) is not necessarily a slant structure.

Example 3 Let $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ be the metric 3-structure manifold in Example 2. We put

$$M(r, s, v, u) = \left(\frac{\sqrt{2}}{2}r, s, \frac{\sqrt{2}}{2}v, u, \frac{\sqrt{2}}{2}r, 0, \frac{\sqrt{2}}{2}v, 0, 0, 0, 0, 0, 0, 0, 0\right),$$

which is a 4-dimensional submanifold of $\overline{M}$. By direct computation we see that $(M, \xi_2, \eta_2, \phi_2, g)$ and $(M, \xi_3, \eta_3, \phi_3, g)$ are slant submanifold with slant angle $\theta = \frac{\pi}{4}$ but $(M, \xi_1, \eta_1, \phi_1, g)$ is not slant submanifold. So, M is not a 3-slant submanifold of $\overline{M}$.

4 Semi-slant and Bi-slant Submanifolds of 3-Structure Manifolds

4.1 3-Semi-slant and 3-Bi-slant Submanifolds

F. Malek and the author of this chapter have generalized the notions of invariant, anti-invariant, semi-invariant and slant submanifolds of metric 3-structure manifolds by defining the following submanifolds.

Definition 2 ([20]) Let M be a submanifold of a metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. Then M is said to be a **3-semi-slant submanifold** of $\overline{M}$, if it admits 3 orthogonal distributions $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$, where $\mathcal{D}_3 = span < \xi_1, \xi_2, \xi_3 >$ and the following conditions are satisfied:

(*a*) $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$,
(*b*) The distribution $\mathcal{D}_1$ is an invariant distribution, i.e., $\phi_i(\mathcal{D}_1) = \mathcal{D}_1, \forall i \in \{1, 2, 3\}$,
(*c*) The distribution $\mathcal{D}_2$ is a 3-slant distribution with slant angle $\theta \neq 0$, i.e., for each non-zero vector $X \in \mathcal{D}_2$ at any point $p \in M$, the angle between $\phi_i(X)$, $i = 1, 2, 3$ and $\mathcal{D}_2$ is constant and it is independent of the choice of $X \in \mathcal{D}_2$ and $p \in M$.

Definition 3 ([20]) Let M be a submanifold of a metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. M is called a **3-bi-slant submanifold** of $\overline{M}$, if there exist three orthogonal distributions $\mathcal{D}_1, \mathcal{D}_2$, and $\mathcal{D}_3$ on M, such that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$ and for $i = 1, 2$, $\mathcal{D}_i$ is a 3-slant distribution with slant angle θ_i and $\mathcal{D}_3 = span < \xi_1, \xi_2, \xi_3 >$.

Remark 1 From Definitions 2 and 3, we conclude that on a 3-bi-slant submanifold M, if $\theta_1 = 0$ then is M a 3-semi-slant submanifold. Moreover, if $dim(\mathcal{D}_1) = 0$, then both of these submanifolds become 3-slant submanifold.

Since TM can be decomposed to $\mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$, therefore, for any $X \in TM$, we put $X = P_1X + P_2X + \sum_{i=1}^{3} \eta_i(X)\xi_i$, such that $P_\alpha X$ is the projection of X on $\mathcal{D}_\alpha$, $\alpha = 1, 2$.

On the other hand, let $T_{\alpha i}X$ (respectively $t_{\alpha i}V$) be the tangential part of $\phi_i X$ (respectively $\phi_i V$) on $\mathcal{D}_\alpha$ and $N_i X$ (respectively $n_i V$) be the normal part of $\phi_i X$ (respectively $\phi_i V$), for $i \in \{1, 2, 3\}$ and $\alpha = 1, 2$. Then we put

$$\phi_i X = T_{1i}X + T_{2i}X + N_i X \quad \text{and} \quad \phi_i V = t_{1i}V + t_{2i}V + n_i V, \tag{22}$$

for all $X \in TM$ and $V \in (TM)^\perp$. By using (14), one can verify that $T_{\alpha i}X = P_\alpha \circ T_i X$ and $t_{\alpha i}V = P_\alpha \circ t_i V$.

The following theorem is a generalization of Theorem 1 for 3-bi-slant submanifolds.

Theorem 3 ([20]) *Let M be a submanifold of a metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ such that TM can be decomposed to three orthogonal distributions $\mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi_1, \xi_2, \xi_3 >$. Then M is a 3-bi-slant submanifold if and only if for $\alpha = 1, 2$, there exists a constant $\lambda_\alpha \in [-1, 0]$ such that*

$$T_i T_j X = \lambda_\alpha X, \quad \forall X \in \mathcal{D}_\alpha \text{ and } i, j \in \{1, 2, 3\}. \tag{23}$$

Moreover, in that case $\lambda_\alpha = -cos^2\theta_\alpha$, where θ_α is the slant angle of distribution $\mathcal{D}_\alpha$.

Proof Let $X \in \mathcal{D}_\alpha$ and β_α and θ_α be the angles $(\widehat{\phi_i X, T_j X})$ and $(\widehat{\phi_j X, T_j X})$, respectively. Using (6), (7) and (8) implies

$$cos\beta_\alpha = \frac{g(\phi_i X, T_j X)}{|\phi_i X||T_j X|} = -\frac{g(X, \phi_i T_j X)}{|X||T_j X|} = -\frac{g(X, T_i T_j X)}{|X||T_j X|}, \tag{24}$$

$$cos\theta_\alpha = \frac{g(\phi_j X, T_j X)}{|\phi_j X||T_j X|} = -\frac{g(X, T_j T_j X)}{|X||T_j X|}. \tag{25}$$

Therefore, if (23) is satisfied then the angles are equal. On the other hand, we have

$$cos\theta_\alpha = \frac{g(\phi_j X, T_j X)}{|\phi_j X||T_j X|} = \frac{g(T_j X, T_j X)}{|X||T_j X|} = \frac{|T_j X|}{|X|}, \tag{26}$$

and then from (25) and (26) it follows that

$$cos^2\theta_\alpha = -\frac{g(X, T_jT_jX)}{|X|^2}. \tag{27}$$

Thus, if $T_iT_jX = \lambda_\alpha X$ then $\lambda_\alpha = -cos^2\theta_\alpha$ and θ_α is constant. Conversely, if M is a 3-bi-slant submanifold then β_α and θ_α are equal and constant. Thus, (27) is satisfied and it implies $T_iT_jX = -cos^2\theta_\alpha X$. □

From Remark 1, it is obvious that 3-semi-slant submanifolds are satisfied in Theorem 3 too. As a generalization of Theorem 5.1 of [5], the next theorem is a characterization of 3-semi-slant submanifolds of 3-structure manifolds.

Theorem 4 ([20]) *Let M be a submanifold of a metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ such that all the structure vector fields are tangent to M. Then M is a 3-semi-slant submanifold if and only if $\exists\lambda \in [-1, 0)$ such that for $i, j \in \{1, 2, 3\}$:*

(a) $\mathcal{D} = \{X \in TM\backslash < \xi_1, \xi_2, \xi_3 > \mid T_iT_jX = \lambda X\}$ *is a distribution.*
(b) $\forall X \in TM$*, orthogonal to* $\mathcal{D}$*,* $N_iX = 0$.

Moreover, in that case $\lambda = -cos^2\theta$, in which θ is the slant angle of M.

Proof If M is 3-semi-slant submanifold, then by taking $\lambda = -cos^2\theta$ and using Theorem 3, we get $\mathcal{D} = \mathcal{D}_2$. On the other hand, since $\mathcal{D}_1$ is invariant, for all $X \in TM$, orthogonal to $\mathcal{D}$, $N_iX = 0$. Conversely, if we take $TM = \mathcal{D}^\perp \oplus \mathcal{D}\oplus < \xi_1, \xi_2, \xi_3 >$ then (b) implies that $\mathcal{D}^\perp$ is invariant. Using (a) and by the same way in the proof of Theorem 3, it can be proved that $\mathcal{D}$ is a 3-slant distribution with slant angle θ satisfying $\lambda = -cos^2\theta$. Thus, M is a 3-semi-slant submanifold. □

Corollary 3 ([20]) *Let M be a 3-semi-slant submanifold of metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ with slant angle θ. Then for all $X, Y \in TM$ we have*

$$g(T_iX, T_jP_2Y) = cos^2\theta g(X, P_2Y), \tag{28}$$

$$g(N_iX, N_jP_2Y) = -g(X, \phi_kP_2Y) - cos^2\theta g(X, P_2Y), \tag{29}$$

$$g(N_iX, N_iP_2Y) = sin^2\theta g(X, P_2Y). \tag{30}$$

Proof Using (8) and statement (a) of Theorem 4, implies

$$g(T_iX, T_jP_2Y) = -g(X, T_iT_jP_2Y) = cos^2\theta g(X, P_2Y),$$

since $\mathcal{D}_2$ is orthogonal to the structure vector fields. From (6), (8), (14) and (28) we have

$$\begin{aligned}-g(X, \phi_kP_2Y) = g(\phi_iX, \phi_jP_2Y) &= g(T_iX + N_iX, T_jP_2Y + N_jP_2Y)\\ &= cos^2\theta g(X, P_2Y) + g(N_iX, N_jP_2Y).\end{aligned}$$

By using (7), (14) and (28), Eq. (30) can be easily proved. □

Theorem 5 ([20]) *Let M be a submanifold of a metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ and $\xi_1, \xi_2, \xi_3 \in TM$. Then M is a 3-semi-slant submanifold if and only if $\exists \lambda \in [-1, 0)$ such that for $i, j \in \{1, 2, 3\}$*

(a) $\mathcal{D} = \{X \in TM \setminus <\xi_1, \xi_2, \xi_3> \mid t_j N_i X = -T_k X - \lambda X\}$ *is a distribution.*
(b) $\forall X \in TM$*, orthogonal to* $\mathcal{D}$*,* $N_i X = 0$.

Proof Let $X \in TM \setminus <\xi_1, \xi_2, \xi_3>$. Applying ϕ_j to (14) implies

$$-\phi_k X = T_j T_i X + t_j N_i X + N_j T_i X + n_j N_i X. \tag{31}$$

By taking tangential and normal parts of (31), we get

$$-T_k X = T_j T_i X + t_j N_i X, \quad -N_k X = N_j T_i X + n_j N_i X. \tag{32}$$

If M is a 3-semi-slant submanifold then by putting $\mathcal{D} = \mathcal{D}_2$ and using (32) and statement (a) of Theorem 4, we obtain $t_j N_i X = -T_k X + cos^2\theta X$ and also for all $X \in \mathcal{D}^\perp$, $N_i X = 0$. Conversely, by virtue of (32) and (a), we have

$$T_j T_i X = -T_k X - t_j N_i X = \lambda X.$$

Thus by Theorem 4, M is a 3-semi-slant submanifold. □

Now, we give some examples of 3-semi-slant and 3-bi-slant submanifolds of 3-structure manifolds.

Example 4 ([20]) Suppose $\overline{M} = \mathbb{R}^{15}$, $g((x_i)_{i=\overline{1,15}}, (y_i)_{i=\overline{1,15}}) = \sum_{i=1}^{15} x_i y_i$, $\xi_1 = \partial x_{13}$, $\xi_2 = \partial x_{14}$, $\xi_3 = \partial x_{15}$, η_i be the dual of ξ_i and

$$\phi_1((x_i)_{i=\overline{1,15}}) = (-x_3, x_4, x_1, -x_2, \ldots, -x_{11}, x_{12}, x_9, -x_{10}, 0, -x_{15}, x_{14}),$$

$$\phi_2((x_i)_{i=\overline{1,15}}) = (-x_4, -x_3, x_2, x_1, \ldots, -x_{12}, -x_{11}, x_{10}, x_9, x_{15}, 0, -x_{13}),$$

$$\phi_3((x_i)_{i=\overline{1,15}}) = (-x_2, x_1, -x_4, x_3, \ldots, -x_{10}, x_9, -x_{12}, x_{11}, -x_{14}, x_{13}, 0).$$

Then $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is a 3-structure manifold.

Let $M = (-u_1 - u_4, u_1 - u_4, -u_2 + u_3, -u_2 - u_3, v_1 sin\theta, v_2 sin\theta, v_2 sin\theta, v_2 sin\theta, v_1 cos\theta, 0, 0, 0, t_1, t_2, t_3)$ for $\theta \in (0, \frac{\pi}{2})$. Then M is a 9-dimensional submanifold of $\overline{M}$ and TM is spanned by

$$X_1 = -\partial x_1 + \partial x_2,\ X_2 = -\partial x_3 - \partial x_4,\ X_3 = \partial x_3 - \partial x_4,\ X_4 = -\partial x_1 - \partial x_2,$$

$$X_5 = sin\theta \partial x_5 + cos\theta \partial x_9,\ X_6 = sin\theta(\partial x_6 + \partial x_7 + \partial x_8)$$

$$X_7 = \partial x_{13},\ X_8 = \partial x_{14},\ X_9 = \partial x_{15}.$$

Set $\mathcal{D}_1 =< X_1, X_2, X_3, X_4 >$, $\mathcal{D}_2 =< X_5, X_6 >$ and $\mathcal{D}_3 =< X_7, X_8, X_9 >$. It is easy to see that $\mathcal{D}_1$ is invariant with respect to ϕ_1, ϕ_2, and ϕ_3. Moreover,

$$\phi_1(X_5) = \frac{1}{3}[X_6 + sin\theta(2\partial x_7 - \partial x_6 - \partial x_8)] + cos\theta\partial x_{11}$$

$$\Rightarrow T_{21}(X_5) = \frac{1}{3}X_6,$$

$$\phi_1(X_6) = -sin^2\theta X_5 - cos^2\theta sin\theta\partial x_5 + sin^2\theta cos\theta\partial x_9 + sin\theta(-\partial x_8 + \partial x_6)$$

$$\Rightarrow T_{21}(X_6) = -sin^2\theta X_5,$$

$$\phi_2(X_5) = \frac{1}{3}[X_6 + sin\theta(2\partial x_8 - \partial x_6 - \partial x_7)] + cos\theta\partial x_{12}$$

$$\Rightarrow T_{22}(X_5) = \frac{1}{3}X_6,$$

$$\phi_2(X_6) = -sin^2\theta X_5 - cos^2\theta sin\theta\partial x_5 + sin^2\theta cos\theta\partial x_9 + sin\theta(-\partial x_6 + \partial x_7)$$

$$\Rightarrow T_{22}(X_6) = -sin^2\theta X_5,$$

where T_{2j} is the tangent projection of ϕ_j on $\mathcal{D}_2$. Thus we have

$$cos\beta = \frac{g(\phi_i X, T_{2j}X)}{|\phi_i X||T_{2j}X|} = \frac{sin\theta}{\sqrt{3}}, \ \forall X \in \mathcal{D}_2 \text{ and } i, j \in \{1, 2, 3\}.$$

Therefore, $\mathcal{D}_2$ is a 3-slant distribution with slant angle $\beta = cos^{-1}(\frac{sin\theta}{\sqrt{3}})$. Hence, M is a 3-semi-slant submanifold of $\overline{M}$.

Example 5 ([20]) Let $\overline{M} = \mathbb{R}^{11}$ be endowed with the following almost contact metric 3-structure:

$$\phi_1((x_i)_{i=\overline{1,11}}) = (-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6, 0, -x_{11}, x_{10}),$$

$$\phi_2((x_i)_{i=\overline{1,11}}) = (-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5, x_{11}, 0, -x_9),$$

$$\phi_3((x_i)_{i=\overline{1,11}}) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7, -x_{10}, x_9, 0),$$

$g((x_i)_{i=\overline{1,11}}, (y_i)_{i=\overline{1,11}}) = \sum_{i=1}^{11} x_i y_i$, $\xi_1 = \partial x_9$, $\xi_2 = \partial x_{10}$, $\xi_3 = \partial x_{11}$ and η_i's be the dual of ξ_i's.

Let $M=(v_1cos\theta, v_1cos\theta, v_1cos\theta, v_2sin\theta+u_1cos\theta, v_2cos\theta - u_1sin\theta, u_2sin\theta, u_2sin\theta, u_2sin\theta, t_1, t_2, t_3)$. By taking

$$\mathcal{D}_1 =< cos\theta(\partial x_1 + \partial x_2 + \partial x_3), sin\theta\partial x_4 + cos\theta\partial x_5) >,$$

$$\mathcal{D}_2 =< sin\theta(\partial x_6 + \partial x_7 + \partial x_8), cos\theta\partial x_4 - sin\theta\partial x_5) >,$$

$$\mathcal{D}_3 =< \partial x_9, \partial x_{10}, \partial x_{11} >,$$

we have $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$. By direct computations it can be verified that $\mathcal{D}_1$ and $\mathcal{D}_2$ are 3-slant with slant angle $cos^{-1}(\frac{sin\theta}{\sqrt{3}})$ and $cos^{-1}(\frac{cos\theta}{\sqrt{3}})$, respectively. Therefore, M is a 3-bi-slant submanifold of $\overline{M}$.

4.2 Submanifolds of 3-Sasakian and 3-Cosymplectic Manifolds

Let $X \in TM\setminus < \xi_1, \xi_2, \xi_3 >$. Now, if $\overline{M}$ is a 3-cosymplectic manifold then using (10) follows that $g([\xi_i, \xi_j], X) = g(\overline{\nabla}_{\xi_j}\xi_i - \overline{\nabla}_{\xi_i}\xi_j, X) = 0$. Also, if $\overline{M}$ is a 3-Sasakian manifold then from (9) we get $g([\xi_i, \xi_j], X) = g(-\phi_i\xi_j + \phi_j\xi_i, X) = -2g(\xi_k, X) = 0$. Therefore, the distribution $\mathcal{D}_3 =< \xi_1, \xi_2, \xi_3 >$ is integrable in both cases.

On the other hand, if $\overline{M}$ is a 3-cosymplectic manifold, then we have $0 = \overline{\nabla}_{\xi_j}\xi_i = \nabla_{\xi_j}\xi_i + B(\xi_i, \xi_j)$. Thus $B(\xi_i, \xi_j) = 0$. Furthermore, if $\overline{M}$ is a 3-Sasakian manifold, $\nabla_{\xi_j}\xi_i + B(\xi_i, \xi_j) = \overline{\nabla}_{\xi_j}\xi_i = -\phi_i\xi_j = -\xi_k$. Thus $B(\xi_i, \xi_j) = 0$ and so the distribution $\mathcal{D}_3 =< \xi_1, \xi_2, \xi_3 >$ is totally geodesic in the both cases.

Therefore, we can state the following theorem.

Theorem 6 ([20]) *Let M be a 3-semi-slant submanifold of a 3-cosymplectic or a 3-Sasakian manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. Then the distribution $\mathcal{D}_3 =< \xi_1, \xi_2, \xi_3 >$ is integrable and totally geodesic.*

Theorem 7 ([20]) *Let M be a 3-semi-slant submanifold of a 3-cosymplectic manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$, then the distribution $\mathcal{D}_1 \oplus \mathcal{D}_2$ is integrable.*

Proof For all $X, Y \in \mathcal{D}_1 \oplus \mathcal{D}_2$ and $i \in \{1, 2, 3\}$, (10) implies

$$g([X, Y], \xi_i) = g(\overline{\nabla}_Y X - \overline{\nabla}_X Y, \xi_i) = -g(X, \overline{\nabla}_Y \xi_i) + g(Y, \overline{\nabla}_X \xi_i) = 0.$$

Thus $\mathcal{D}_1 \oplus \mathcal{D}_2$ is integrable. □

Remark 2 It should be noted that if $\overline{M}$ is a 3-Sasakian manifold, then the distribution $\mathcal{D}_1 \oplus \mathcal{D}_2$ is not integrable in general, because

$$g([X, Y], \xi_i) = -g(X, \overline{\nabla}_Y \xi_i) + g(Y, \overline{\nabla}_X \xi_i) = -2g(X, T_i Y) \tag{33}$$

which shows that $[X, Y]$ is not in $\mathcal{D}_1 \oplus \mathcal{D}_2$.

Note that if $X, Y \in \mathcal{D}_1$ or $X, Y \in \mathcal{D}_2$, then (33) is also satisfied. So if M is a 3-semi-slant submanifold of a 3-Sasakian manifold, then the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are not integrable in general. Moreover, from the same equation it follows that if $\mathcal{D}_2$ is integrable then the slant angle of this distribution is $\theta = \frac{\pi}{2}$.

Proposition 1 ([20]) *Let M be a 3-semi-slant submanifold of a 3-cosymplectic or a 3-Sasakian manifold $\overline{M}$. Then for all $X, Y \in \mathcal{D}_1$, we have*

$$P_1(\nabla_X \phi_i)Y = 0, \quad for\ all\ \ i \in \{1, 2, 3\}. \tag{34}$$

Proof First we show that if M is a 3-semi-slant submanifold, then $\phi_i(\mathcal{D}_1^\perp) \subset \mathcal{D}_1^\perp$.

Let $Z \in \mathcal{D}_1^\perp$ and $X \in \mathcal{D}_1$. Since $\mathcal{D}_1$ is invariant, using (8) implies

$$g(\phi_i Z, X) = -g(Z, \phi_i X) = 0.$$

Now let $\overline{M}$ be a 3-cosymplectic manifold. Then by Gauss formula, we obtain

$$(\overline{\nabla}_X \phi_i)Y = (\nabla_X \phi_i)Y + B(X, \phi_i Y) - \phi_i B(X, Y) = 0, \tag{35}$$

for all $X, Y \in \mathcal{D}_1$. Since we have

$$B(X, \phi_i Y) - \phi_i B(X, Y) \in \mathcal{D}_1^\perp.$$

Applying P_1 on (35), it follows that

$$P_1(\nabla_X \phi_i)Y = 0.$$

If $\overline{M}$ is a 3-Sasakian manifold, then by (9) we have

$$(\overline{\nabla}_X \phi_i)Y = (\nabla_X \phi_i)Y + B(X, \phi_i Y) - \phi_i B(X, Y) = g(X, Y)\xi_i, \tag{36}$$

for all $X, Y \in \mathcal{D}_1$, since $\eta_i(Y) = 0$. Applying P_1 to (36) completes the proof of the proposition. □

In the next theorem, we can see an important geometric property of the distribution $\mathcal{D}_1$.

Theorem 8 ([20]) *On 3-semi-slant submanifolds of 3-cosymplectic and 3-Sasakian manifolds, the distribution $\mathcal{D}_1$ is integrable if and only if $\mathcal{D}_1$ is totally geodesic.*

Proof By taking the normal parts of Eqs. (35) and (36), we find

$$N_i P_2 \nabla_X Y = -B(X, \phi_i Y) + N_i B(X, Y), \forall X, Y \in \mathcal{D}_1. \tag{37}$$

By interchanging the role of X and Y in (37) and using $B(X, Y) = B(Y, X)$, we obtain

$$N_i P_2[X, Y] = B(X, \phi_i Y) - B(\phi_i X, Y), \forall X, Y \in \mathcal{D}_1. \quad (38)$$

Equation (38) shows that $\mathcal{D}_1$ is integrable if and only if

$$B(X, \phi_i Y) = B(\phi_i X, Y). \quad (39)$$

On the other hand, from (6) and (39) we get

$$\begin{aligned} B(\phi_i X, Y) &= B(X, \phi_i Y) = B(X, \phi_j \phi_k Y) \\ &= B(\phi_j X, \phi_k Y) = B(\phi_k \phi_j X, Y) \\ &= -B(\phi_i X, Y). \end{aligned}$$

It follows that $B(X, Y) = 0$, for all $X, Y \in \mathcal{D}_1$ and thus $\mathcal{D}_1$ is totally geodesic. Conversely if $\mathcal{D}_1$ is totally geodesic, (38) implies $[X, Y] \in \mathcal{D}_1$. □

5 Pointwise Slant Submanifolds of 3-Structure Manifolds

5.1 Pointwise 3-Slant Submanifolds

In this section, we discuss the results of the author of this chapter about pointwise slant submanifolds of almost contact and almost contact 3-structure manifolds. Then we characterize them and give some examples. Later, we study some properties of pointwise slant submanifolds of Sasakian and 3-Sasakian manifolds and obtain the necessary and sufficient condition for a pointwise slant submanifold of a 3-Sasakian manifold to be a slant submanifold. Moreover, we show the non-existence of proper Sasakian pointwise 3-slant submanifolds.

Definition 4 ([14]) Let M be a submanifold of an almost contact metric manifold $\overline{M}$. Then M is said to be a **pointwise slant submanifold** with slant function $\Theta_p(X)$ if at any point $p \in M$, the Wirtinger angle between ϕX and $T_p M$ is constant for each non-zero $X \in T_p M$ linearly independent of ξ. It means that the function $\Theta_p(X)$ does not depend on choosing of X.

Definition 5 ([14]) Let M be a submanifold of a metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i \in \{1,2,3\}}$. Then M is said to be a **pointwise 3-slant submanifold** if at any point $p \in M$ and for each non-zero $X \in T_p M$ linearly independent of ξ_i, the Wirtinger angle between $\phi_i X$ and $T_p M$ is constant for all $i \in \{1, 2, 3\}$. In fact, the slant function $\Theta_p(X)$ between $\phi_i X$ and $T_j X$ only depends on the choice of p and it is independent of choosing of X and i, j.

On these submanifolds $\Theta(X)$ is considered as a function known as slant function. If for each $p \in M$, $\Theta_p = 0$ (respectively $\Theta_p = \frac{\pi}{2}$), then M is an invariant (respectively anti-invariant) submanifold. Otherwise, M is a proper pointwise 3-slant submanifold.

In special case, slant and 3-slant submanifolds are pointwise 3-slant submanifolds in which the slant angles are constant at any point.

The following theorems extend the results of Sect. 3 (mainly Theorem 1 and Corollary 1) to the pointwise slant submanifolds (cf. [14]).

Theorem 9 *Let M be a submanifold of 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$ such that ξ_i's are normal to M for $i = 1, 2, 3$. Then, M is a pointwise 3-slant submanifold if and only if there exists a real function Θ on M such that*

$$T_i T_j X = -cos^2\Theta X, \ \forall X \in TM, \ \forall i, j \in \{1, 2, 3\}. \tag{40}$$

Proof Let M be a pointwise 3-slant submanifold and Θ be the angle between $\phi_i X$ and $T_p M$. Then, (7) and (8) imply

$$cos\Theta = \frac{g(\phi_i X, T_j X)}{|\phi_i X||T_j X|} = -\frac{g(X, \phi_i T_j X)}{|X||T_j X|} = -\frac{g(X, T_i T_j X)}{|X||T_j X|}. \tag{41}$$

Also we know

$$cos\Theta = \frac{|T_j X|}{|X|}, \tag{42}$$

thus from (41) yields

$$cos^2\Theta = -\frac{g(X, T_i T_j X)}{|X|^2}, \tag{43}$$

and this implies (40). Conversely, we suppose that α and β are the angles $\phi_i \widehat{X, T_i} X$ and $\phi_i \widehat{X, T_j} X$, respectively, in the point $p \in M$. Thus, $cos\alpha = \frac{|T_i X|}{|X|}$ and $cos\beta = \frac{|T_j X|}{|X|}$. Moreover,

$$cos\alpha = \frac{g(\phi_i X, T_i X)}{|\phi_i X||T_i X|} = -\frac{g(X, T_i T_i X)}{|X||T_i X|} = -\frac{g(X, T_i T_i X)}{|X|^2 cos\alpha}, \tag{44}$$

$$cos\beta = \frac{g(\phi_i X, T_j X)}{|\phi_i X||T_j X|} = -\frac{g(X, T_i T_j X)}{|X||T_j X|} = -\frac{g(X, T_i T_j X)}{|X|^2 cos\beta}. \tag{45}$$

In account of (40), (44) and (45), we obtain that the angles are equal and does not depend on choice of X. Thus, M is a pointwise 3-slant submanifold. □

Next, we have the following result.

Theorem 10 *Let M be a pointwise 3-slant submanifold of 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$ with slant function Θ. Then, for all $X \in TM \setminus < \xi_i >$*

$$T_i T_j X = -cos^2\Theta X, \ \forall i, j \in \{1, 2, 3\}. \tag{46}$$

Proof The proof of this theorem is the same as proof of Theorem 9. □

In virtue of Eq. (8) and Theorem 10, we have the following Corollary.

Corollary 4 ([14]) *Let M be a pointwise 3-slant submanifold of 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$ with slant function Θ. Then, $\forall X, Y \in TM\setminus <\xi_i>$ and $\forall i, j \in \{1, 2, 3\}$*

$$g(T_iY, T_jX) = cos^2\Theta g(Y, X), \tag{47}$$

$$g(N_iY, N_jX) = sin^2\Theta g(Y, X). \tag{48}$$

Also when the structure of $\overline{M}$ is almost contact metric, Theorem 10 can be stated as follows.

Corollary 5 ([14]) *Let M be a pointwise slant submanifold of almost contact metric manifold $(\overline{M}, \xi, \eta, \phi, g)$ with slant function Θ. Then,*

$$T^2X = -cos^2\Theta X, \ \forall X \in TM\setminus <\xi> . \tag{49}$$

5.2 Pointwise 3-Slant Submanifolds of 3-Sasakian Manifolds

Here, we assume that the ambient manifold be a 3-Sasakian manifold and investigate its pointwise 3-slant submanifolds.

Lemma 1 ([14]) *Let M be a pointwise 3-slant submanifold of 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ with slant function Θ. Then, for any unit vector field $X \in TM\setminus <\xi_1, \xi_2, \xi_3>$ we have*

$$T_iX = cos\Theta Z, \tag{50}$$

where Z is a unit vector field in TM and orthogonal to X.

Proof For any unit vector field $X \in TM\setminus <\xi_1, \xi_2, \xi_3>$, we have $|T_iX| = cos\Theta|\phi_iX| = cos\Theta|X| = cos\Theta$. Now, let $Z = \frac{T_iX}{|T_iX|}$ be the unit vector field in the direction of T_iX. Then, $T_iX = cos\Theta Z$. Moreover since $g(\phi_iX, X) = 0$ and $g(\phi_iX, X) = g(T_iX + N_iX, X) = g(T_iX, X)$, we conclude that Z is orthogonal to X. □

The following theorem provides a necessary and sufficient condition for a pointwise 3-slant submanifold of a 3-Sasakian manifold to be a 3-slant submanifold.

Theorem 11 ([14]) *Let M be a pointwise 3-slant submanifold of a 3-Sasakian manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. Then, the slant function Θ is constant if and only if $A_{N_iX}T_iX = A_{N_iT_iX}X$.*

Proof Let $\overline{M}$ be a 3-Sasakian manifold. Then from (9) and Gauss formula, we have

$$\begin{aligned} g(X, Y)\xi_i = (\overline{\nabla}_Y\phi_i)X = \nabla_YT_iX + B(T_iX, Y) + D_YN_iX - A_{N_iX}Y - \\ T_i\nabla_YX - t_iB(X, Y) - N_i\nabla_YX - n_iB(X, Y). \end{aligned} \tag{51}$$

for $X \in TM \setminus <\xi_1, \xi_2, \xi_3>$ and $Y \in TM$.

By taking the tangential part of (51), we get

$$g(X, Y)\xi_i = \nabla_Y T_i X - A_{N_i X} Y - T_i \nabla_Y X - t_i B(X, Y). \tag{52}$$

By using (50), Eq. (52) implies

$$\begin{aligned} g(X, Y)\xi_i = Y \cos\Theta Z + \cos\Theta \nabla_Y Z - A_{N_i X} Y - T_i \nabla_Y X - t_i B(X, Y) = \\ -\sin\Theta Y(\Theta) Z + \cos\Theta \nabla_Y Z - A_{N_i X} Y - T_i \nabla_Y X - t_i B(X, Y). \end{aligned} \tag{53}$$

We apply $g(Z, .)$ on (53). Since

$$g(Z, \nabla_Y Z) = \frac{1}{2} \nabla_Y g(Z, Z) = 0,$$

$$g(Z, T_i \nabla_Y X) = -g(T_i Z, \nabla_Y X) = \cos^2\Theta \frac{1}{2} \nabla_Y g(X, X) = 0.$$

We get

$$0 = -\sin\Theta Y(\Theta) - g(Z, A_{N_i X} Y) - g(Z, t_i B(X, Y)). \tag{54}$$

So, Θ is constant if and only if

$$g(Y, A_{N_i X} Z) = g(N_i Z, B(X, Y)) = g(Y, A_{N_i Z} X).$$

Therefore, the slant function Θ is constant if and only if $A_{N_i X} Z = A_{N_i Z} X$. □

Using the same approach of the proof of the above theorem for pointwise slant submanifold of a Sasakian manifold, we have the following result.

Theorem 12 ([14]) *Let M be a pointwise slant submanifold of a Sasakian manifold $(\overline{M}, \xi, \eta, \phi, g)$. Then, the slant function Θ is constant if and only if $A_{NX} Z = A_{NZ} X$.*

Next, let M be a pointwise 3-slant submanifold of a 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i \in \{1,2,3\}}$ tangent to structure vector fields. Then, M is called a Sasakian pointwise 3-slant submanifold if

$$(\nabla_X T_i) Y = g(X, Y)\xi_i - \eta_i(Y) X, \quad \forall X, Y \in TM. \tag{55}$$

The following theorem implies that there exist no proper Sasakian pointwise 3-slant submanifolds.

Theorem 13 ([14]) *Any Sasakian pointwise 3-slant submanifolds are 3-slant submanifolds.*

Proof Let M be a Sasakian pointwise 3-slant submanifold of $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$. Then from (50) and (55), we have

$$\begin{aligned} g(X,Y)\xi_i &= (\nabla_Y T_i)X = \nabla_Y T_i X - T_i(\nabla_Y X) \\ &= \nabla_Y cos\Theta Z - T_i(\nabla_Y X) \\ &= Y(cos\Theta)Z + cos\Theta\nabla_Y Z - T_i(\nabla_Y X) \\ &= sin\Theta Y(\Theta)Z + cos\Theta\nabla_Y Z - T_i(\nabla_Y X). \end{aligned} \tag{56}$$

for any X as unit vector field in $TM\setminus<\xi_i>$ and $Y \in TM$.

Since Z is orthogonal to X and ξ_i. By applying $g(Z,.)$ on (56), we obtain

$$0 = sin\Theta Y(\Theta), \tag{57}$$

which means Θ is constant. □

Now, we give some non-trivial examples of pointwise slant submanifolds of almost contact and almost contact 3-structure manifolds (cf. [14]).

Example 6 Suppose $\overline{M} = \mathbb{R}^5$ is endowed by the following almost contact metric structure.

$$\eta = dt, \xi = \partial t, g = \sum_{i=1}^{2}(dx_i \otimes dx_i + dy_i \otimes dy_i) + dt \otimes dt,$$

$$\phi(x_1, x_2, y_1, y_2, t) = (-y_1, -y_2, x_1, x_2, 0).$$

Let $M(u, v) = (u, u, v\cos f, v\sin f, t)$, where f is a real value function on $\overline{M}$. Then, M is a pointwise slant submanifold with slant function $\Theta = \cos^{-1}(\frac{\cos f+\sin f}{\sqrt{2}})$.

Example 7 We set $\overline{M} = \mathbb{R}^{11}$ and $g = \sum_{i=1}^{11} dx_i \otimes dx_i$. We define

$$\phi_1((x_i)_{i=\overline{1,11}}) = (-x_3, x_4, x_1, -x_2, -x_7, x_8, x_5, -x_6, 0, -x_{11}, x_{10}),$$

$$\phi_2((x_i)_{i=\overline{1,11}}) = (-x_4, -x_3, x_2, x_1, -x_8, -x_7, x_6, x_5, x_{11}, 0, -x_9),$$

$$\phi_3((x_i)_{i=\overline{1,11}}) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7, -x_{10}, x_9, 0),$$

$$\xi_1 = \partial x_9, \xi_2 = \partial x_{10}, \xi_3 = \partial x_{11} \text{ and } \eta_1 = dx_9, \eta_2 = dx_{10}, \eta_3 = dx_{11}.$$

It is easy to show that $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is a metric 3-structure manifold. By taking

$$M(u, v) = (v\sin f, 0, 0, 0, ku\sin f, ku\sin f, ku\sin f, v\cos f, 0, 0, 0),$$

for $k \in \mathbb{R}^+$ and $f : \mathbb{R}^{11} \to \mathbb{R}$, M is a submanifold of $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$.

Direct computations show that M is a pointwise 3-slant submanifold of $\overline{M}$ with slant function $\Theta = \cos^{-1}(\frac{\cos f}{k\sqrt{3}})$.

5.3 Pointwise Hemi 3-Slant Submanifolds of 3-Structure Manifolds

Let M be a submanifold of a 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. Then M is called a **pointwise hemi 3-slant submanifold** (cf. [15]) if there exist three orthogonal distributions $\mathcal{D}_\theta$, $\mathcal{D}^\perp$ and Ξ on M such that

(a) $TM = \mathcal{D}_\theta \oplus \mathcal{D}^\perp \oplus \Xi$, where $\Xi = Span\{\xi_1, \xi_2, \xi_3\}$;
(b) $\mathcal{D}^\perp$ is anti-invariant with respect to ϕ_i, $\forall i = 1, 2, 3$, i.e., $\phi_i(\mathcal{D}^\perp) \subset T^\perp M$;
(c) $\mathcal{D}_\theta$ is a pointwise 3-slant distribution. That means for any $Y \in \mathcal{D}_\theta$ the angle between $\phi_i(Y)$ and $\mathcal{D}_\theta$ is independent of the choice of Y.

It is obvious that if $dim(\mathcal{D}^\perp) = 0$ (respectively $dim(\mathcal{D}_\theta \oplus \Xi) = 0$), then M is a pointwise 3-slant (respectively an anti-invariant) submanifold. In the current paper all distributions have non-zero dimension and in this case the submanifold is said to be a proper pointwise hemi 3-slant submanifold.

Here we give some examples of proper pointwise hemi 3-slant submanifolds (cf. [15]).

Example 8 On Riemannian manifold $\overline{M} = \mathbb{R}^{15}$ and $g = \Sigma_{i=1}^{15} dx^i \otimes dx^i$, we define

$$\phi_1(\partial_{4k+1}) = \partial_{4k+2}, \phi_1(\partial_{4k+2}) = -\partial_{4k+1}, \phi_1(\partial_{4k+3}) = \partial_{4k+4}, \phi_1(\partial_{4k+4}) = -\partial_{4k+3},$$

$$\phi_1(\partial_{13}) = \partial_{14}, \phi_1(\partial_{14}) = -\partial_{13}, \phi_1(\partial_{15}) = 0,$$

$$\phi_2(\partial_{4k+1}) = \partial_{4k+3}, \phi_2(\partial_{4k+2}) = -\partial_{4k+4}, \phi_2(\partial_{4k+3}) = -\partial_{4k+1}, \phi_2(\partial_{4k+4}) = \partial_{4k+2},$$

$$\phi_2(\partial_{13}) = \partial_{15}, \phi_2(\partial_{15}) = -\partial_{13}, \phi_2(\partial_{14}) = 0,$$

for $k = 0, 1, 2$. In addition, $\xi_1 = \partial_{15}$, $\xi_2 = \partial_{14}$, $\xi_3 = \partial_{13}$ and η_i's be the dual of ξ_i's for $r = 1, 2, 3$ and $\phi_3 = \phi_1 o \phi_2 - \eta_2 \otimes \xi_1$. $(\overline{M}, g, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$ is a metric 3-structure manifold.

Now, let $f, h \in C^\infty(\mathbb{R}^{15})$. Then we define a 6-dimensional submanifold M given by the immersion

$$\psi(t_1, t_2, t_3, t_4, t_5, t_6) = (t_1 f, t_2 h, t_2 h, t_2 h, t_3, 0, 0, 0, t_1 h, 0, 0, 0, t_4, t_5, t_6).$$

By taking $\mathcal{D}_\theta = \mathrm{Span}\{\mathrm{X}_1 = \mathrm{f}\partial_1 + \mathrm{h}\partial_9, \mathrm{X}_2 = \mathrm{h}(\partial_2 + \partial_3 + \partial_4)\}$, $\mathcal{D}^\perp = \mathrm{Span}\{\mathrm{X}_3 = \partial_5\}$ and $\Xi = \mathrm{Span}\{\mathrm{X}_4 = \partial_{13}, \mathrm{X}_5 = \partial_{14}, \mathrm{X}_6 = \partial_{15}\}$, it is clear that $\mathcal{D}_\theta$ is a pointwise 3-slant distribution by slant function $\Theta = cos^{-1}(\frac{h}{\sqrt{3}\sqrt{h^2+f^2}})$ and $\mathcal{D}^\perp$ is an anti-invariant distribution. Therefore, M is a pointwise hemi 3-slant submanifold of $\mathbb{R}^{15}$.

Example 9 Let M be a pointwise 3-slant submanifold of a metric 3-structure manifold $(\overline{M}, g, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$ which is given in Example 7, i.e., $(\overline{M}, g) = (\mathbb{R}^{11}, \Sigma_{i=1}^{11} dx^i \otimes dx^i)$ and

$$M = (v\sin f, 0, 0, 0, ku\sin f, ku\sin f, ku\sin f, v\cos f, 0, 0, 0).$$

Now let $N' = (y, 0, \ldots, 0)$ be a submanifold of a 4m-dimensional hyperKahler manifold $(M' = \mathbb{R}^{4m}, g', I, J, K)$ which is introduced in the example of [21]. It is obvious that $I(TN') \subset T^{\perp}N'$, $J(TN') \subset T^{\perp}N'$, $K(TN') \subset T^{\perp}N'$.

By using the above notations, we suppose that $(\tilde{M}, \tilde{g}) = (\overline{M} \times M', g \otimes g')$ and $\tilde{N} = M \times N'$. Therefore, $\tilde{M}$ is a $(4m + 11)$-dimensional almost contact 3-structure manifold. We take $\mathcal{D}_\theta \otimes \Xi = TM$ and $\mathcal{D}^{\perp} = TN'$. Thus $\tilde{N}$ is a pointwise hemi 3-slant submanifold of $\tilde{M}$. The slant function of slant distribution $\mathcal{D}_\theta$ is $\Theta = \cos^{-1}(\frac{\cos\tilde{f}}{k\sqrt{3}})$, where $k \in \mathbb{R}^{+}$ and $\tilde{f} \in C^{\infty}(\tilde{M})$ is the smooth extension of the function f.

In virtue of Theorem 9, we have the following lemma.

Lemma 2 ([15]) *Let $\mathcal{D}$ be a distribution on a submanifold of a metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ such that $\mathcal{D}$ is orthogonal to the distribution $< \xi_1, \xi_2, \xi_3 >$. Then $\mathcal{D}$ is a pointwise 3-slant distribution if and only if there exists a function $\rho \in [-1, 0)$ such that for all $Y \in \mathcal{D}$, $T_iT_jY = \rho Y$, $\forall i, j \in \{1, 2, 3\}$. Furthermore, if Θ is the slant function, then $\rho = -\cos^2\Theta$.*

The next theorem gives a characterization of pointwise hemi 3-slant submanifolds (cf. [15])

Theorem 14 *Let M be a submanifold of an almost contact metric 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)$ which ξ_i's are tangent to M for $i = 1, 2, 3$. Then M is a pointwise hemi 3-slant submanifold if and only if there exists a real-valued function $\rho \in [-1, 0)$ such that for all $i, j \in \{1, 2, 3\}$, the following conditions hold:*

(a) $\mathcal{D} = \{Y \in TM\backslash < \xi_1, \xi_2, \xi_3 > | T_iT_jY = \rho Y\}$ *is a distribution on M;*
(b) $\forall Y \in TM$ *orthogonal to distribution* $\mathcal{D}\oplus < \xi_1, \xi_2, \xi_3 >$, $T_iY = 0$.
Moreover, in that case if Θ is the slant function, then $\rho = -\cos^2\Theta$.

Proof Let M be a pointwise hemi 3-slant submanifold and $TM = \mathcal{D}_\theta \oplus \mathcal{D}^{\perp} \oplus \Xi$. From Lemma 2 we have $T_iT_jY = \rho Y$, for all $Y \in \mathcal{D}_\theta$. By taking $\mathcal{D} = \mathcal{D}_\theta$ it yields $T_iZ = 0, \forall Z \in \mathcal{D}^{\perp}$ since $\mathcal{D}^{\perp}$ is anti-invariant.

Conversely, from (a) and Lemma 2, we get $\mathcal{D}$ is a pointwise 3-slant distribution. On the other hand, (b) implies that there exists an anti-invariant distribution on M and since $\Xi \subset TM$ and does not satisfy in both of the conditions. Hence, we conclude that M is a pointwise hemi 3-slant submanifold. □

5.4 Pointwise Hemi 3-Slant Submanifolds of 3-Cosymplectic Manifolds

Here we first recall that a 3-structure manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is a 3-cosymplectic manifold if

$$(\overline{\nabla}_X \phi_i)W = 0, \quad \overline{\nabla}_W \xi_i = 0, \quad \forall\, X, W \in TM. \tag{58}$$

We can define the covariant derivative of the projection maps T_i and N_i as follows

$$(\nabla_W T_i)X = \nabla_W T_i X - T_i \nabla_W X, \tag{59}$$

$$(D_W N_i)X = D_W N_i X - N_i \nabla_W X. \tag{60}$$

Now, we study geometric properties of distributions of pointwise hemi 3-slant submanifold of a 3-cosymplectic manifold.

Theorem 15 ([15]) *Let M be a pointwise hemi 3-slant submanifold of a 3-cosymplectic manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. Then the distribution Ξ is integrable and totally geodesic.*

Proof Since M is a 3-cosymplectic manifold and the connection $\overline{\nabla}$ is symmetric, we have

$$[\xi_i, \xi_j] = \overline{\nabla}_{\xi_j}\xi_i - \overline{\nabla}_{\xi_i}\xi_j,$$

and (58) implies that $[\xi_i, \xi_j] = 0 \in \Xi$. Hence, Ξ is an integrable distribution.

Moreover, from Gauss and Weingarten formulas, we get $0 = \overline{\nabla}_{\xi_j}\xi_i = \nabla_{\xi_j}\xi_i + B(\xi_j, \xi_i)$. This means that $B(\xi_j, \xi_i) = 0$ and therefore Ξ is totally geodesic. □

Theorem 16 ([15]) *The distribution $\mathcal{D}_\theta \oplus \mathcal{D}^\perp$ of a pointwise hemi 3-slant submanifold of a 3-cosymplectic manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is integrable.*

Proof Since $\overline{\nabla}$ is symmetric and compatible with respect to g. Then for all $Y, Z \in \mathcal{D}_\theta \oplus \mathcal{D}^\perp$ and $i = 1, 2, 3$, we have

$$g(\xi_i, [Y, Z]) = g(\xi_i, \overline{\nabla}_Z Y - \overline{\nabla}_Y Z) = -g(Y, \overline{\nabla}_Z \xi_i) + g(Z, \overline{\nabla}_Y \xi_i). \tag{61}$$

Using (58) and (61), we find that $g(\xi_i, [Y, Z]) = 0$. Thus $[Y, Z] \in \mathcal{D}_\theta \oplus \mathcal{D}^\perp$, which means that the distribution $\mathcal{D}_\theta \oplus \mathcal{D}^\perp$ is integrable. □

Since $\mathcal{D}^\perp$ is an anti-invariant distribution, by some computation it can be proved that $\phi_i[Y, Z] = N_i[Y, Z]$. Consequently, we have the following theorem.

Theorem 17 ([15]) *Let M be a pointwise hemi 3-slant submanifold of a 3-cosymplectic manifold $(\overline{M}, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$. Then the anti-invariant distribution $\mathcal{D}^\perp$ is an integrable distribution.*

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Slant Submanifolds of Conformal Sasakian Space Forms

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1 Introduction

Semi-invariant [4] or contact *CR* submanifolds [25], as a generalization of invariant and anti-invariant submanifolds, of almost contact metric manifolds have been studied by a number of geometers. The concept of semi-invariant submanifold was further generalized under name of almost semi-invariant [19]. Several authors studied semi-invariant or contact *CR* submanifolds, and almost semi-invariant submanifolds of different classes of almost contact metric manifolds. Many such references are included in [4, 19, 25], and references cited therein. Since the inception of the theory of slant submanifolds in Kaehler manifolds created by Chen [9], this theory has shown an increasing development. As contact-geometric analogue, there is the concept of slant submanifolds of almost contact metric manifolds [7]. Further, generalizations of slant submanifolds of an almost contact metric manifold are given as a pointwise slant submanifold [15], a semi-slant submanifold [6], a pointwise semi-slant submanifold [15], an anti-slant submanifold [8] (or a pseudo-slant submanifold [3], or a hemi-slant submanifold [12]), a bi-slant submanifold [8], and a quasi hemi-slant submanifold [16]. However, these generalizations turn out to be particular cases of

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almost semi-invariant submanifolds in the sense of [19], which is contact-geometric analogue of generic submanifold [18] of an almost Hermitian manifold. For different kinds of submanifolds of almost Hermitian manifolds, we refer to [23]. Finally, the authors observe that in all these cases either whole or some part of submanifolds of almost contact metric manifolds is always (pointwise) slant.

The celebrated theory of J. F. Nash of isometric immersion of a Riemannian manifold into a Euclidean space of sufficiently high dimension gives very important and effective motivation to view each Riemannian manifold as a submanifold in a Euclidean space. According to B.-Y. Chen, to establish simple relationship between the main intrinsic invariants and the main extrinsic invariants of a Riemannian submanifold is one of the fundamental problems in the submanifold theory. For a Riemannian submanifold of a Riemannian manifold, the main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants: the Ricci curvature and the scalar curvature. The basic relationships discovered so far are (sharp) inequalities involving intrinsic and extrinsic invariants, and the study of this topic has attracted a lot of attention since the last decade of twentieth century. In 1999, Chen [10, Theorem 4] obtained a basic inequality involving the Ricci curvature and the squared mean curvature of submanifolds in a real space form. This inequality drew attention of several authors and they established similar inequalities for different kinds of submanifolds in ambient manifolds possessing different kinds of structures. Motivated by the result of Chen [10, Theorem 4], in [11], the authors presented a general theory for a submanifold of Riemannian manifolds by proving a basic inequality (see [11, Theorem 3.1]), called Chen-Ricci inequality [21], involving the Ricci curvature and the squared mean curvature of the submanifold. Also, in [22], an improved Chen-Ricci inequality was obtained under certain conditions.

The chapter is organized as follows. In Sect. 2, a brief introduction to Sasakian manifolds, Sasakian space forms, conformal Sasakian manifolds, and conformal Sasakian space forms are presented. In Sect. 3, the concepts of invariant, anti-invariant, semi-invariant, and almost semi-invariant submanifolds of an almost contact metric manifold are presented. It is observed that different kinds of submanifolds, like invariant, anti-invariant, semi-invariant, θ-slant, pointwise θ-slant, semi-slant, pointwise semi-slant, anti-slant, pseudo-slant, hemi-slant, bi-slant, and quasi hemi-slant submanifolds are particular cases of an almost semi-invariant submanifold of an almost contact metric manifold. In Sect. 4, Chen-Ricci inequality involving Ricci curvature and the squared mean curvature of different kinds of slant submanifolds of a conformal Sasakian space form tangent to the structure vector field are presented. Equality cases are also discussed.

2 Conformal Sasakian Space Form

An almost contact structure (φ, ξ, η) in a $(2n+1)$-dimensional smooth manifold $\widetilde{M}$ consists of a tensor field φ of type $(1, 1)$, a vector field ξ, and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{2.1}$$

The first relation and one of the remaining three relations of (2.1) imply the remaining two relations. An almost contact structure (φ, ξ, η) on $\widetilde{M}$ is said to be *normal*, if the induced almost complex structure J on $\widetilde{M} \times \mathbb{R}$ defined by $J\left(X, a\frac{d}{dt}\right) = \left(\varphi X - a\xi, \eta\,(X)\,\frac{d}{dt}\right)$ is integrable. There exists on $\widetilde{M}$ a Riemannian metric $\widetilde{g}$ compatible with the structure (φ, ξ, η) such that

$$\widetilde{g}\,(\varphi X, \varphi Y) = \widetilde{g}\,(X, Y) - \eta(X)\eta\,(Y) \tag{2.2}$$

for all vector fields X, Y on $\widetilde{M}$, and $\widetilde{M}$ is called an *almost contact metric manifold* equipped with an *almost contact metric structure* $(\varphi, \xi, \eta, \widetilde{g})$. The condition (2.2) is equivalent to

$$\widetilde{g}\,(X, \varphi Y) = -\,\widetilde{g}\,(\varphi X, Y) \;\;\text{along with}\;\; \widetilde{g}\,(X, \xi) = \eta(X) \tag{2.3}$$

for all vector fields X, Y on $\widetilde{M}$. It follows that $\widetilde{g}\,(\xi, \xi) = 1$. The *Sasaki form* is defined by $\Phi\,(X, Y) = \widetilde{g}\,(X, \varphi Y)$ for all vector fields X, Y in $\widetilde{M}$. An almost contact metric structure $(\varphi, \xi, \eta, \widetilde{g})$ is called a contact metric structure if $\Phi = d\eta$. A manifold equipped with a contact metric structure is called a *contact metric manifold*. A contact metric structure $(\varphi, \xi, \eta, \widetilde{g})$ is called *K-contact* if ξ is a Killing vector field of $\widetilde{g}$, and a manifold with such a structure is called a *K-contact manifold*. A contact metric structure is K-contact if and only if the operator h defined by $h = \frac{1}{2}\pounds_{\xi}\varphi$ vanishes. An almost contact metric structure is K-contact if and only if $\widetilde{\nabla}\xi = -\varphi$. A normal contact metric structure is called a *Sasakian structure*, and a manifold equipped with a Sasakian structure is called a *Sasakian manifold* [5]. An almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$ is a Sasakian manifold if and only if $(\widetilde{\nabla}_X\varphi)Y = \widetilde{g}\,(X, Y)\,\xi - \eta\,(Y)\,X$ for all vector fields X, Y on $\widetilde{M}$.

It is well known that the sectional curvatures of a Riemannian manifold determine the curvature. Similarly, it is also well known that the holomorphic sectional curvatures of a Kaehler manifold determine the curvature completely. Finally, it is known that the φ-sectional curvatures of a Sasakian manifold determine the curvature completely. A Sasakian manifold $\widetilde{M}$ of constant φ-sectional curvature c, denoted by $\widetilde{M}(c)$, is called a *Sasakian space form* and its Riemann curvature tensor satisfies

$$\begin{aligned}\widetilde{R}\,(X, Y)\,Z = {}& \frac{c+3}{4}\,\{\widetilde{g}\,(Y, Z)\,X - \widetilde{g}\,(X, Z)\,Y\} \\ & + \frac{c-1}{4}\,\{\eta\,(X)\,\eta\,(Z)\,Y - \eta\,(Y)\,\eta\,(Z)\,X + \widetilde{g}\,(X, Z)\,\eta\,(Y)\,\xi - \widetilde{g}\,(Y, Z)\,\eta\,(X)\,\xi \\ & + \widetilde{g}\,(\varphi Y, Z)\,\varphi X - \widetilde{g}\,(\varphi X, Z)\,\varphi Y - 2\widetilde{g}\,(\varphi X, Y)\,\varphi Z\}\,. \end{aligned} \tag{2.4}$$

For details of Sasakian geometry one may refer to [5] and the references cited therein.

Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold. Let $\overline{g}$ be a conformal change of metric given by $\overline{g} = e^{2\varkappa}\widetilde{g}$, where $\varkappa : \widetilde{M} \to \mathbb{R}$ is a smooth function. For the function $\varkappa$, the gradient $\widetilde{\nabla}\varkappa$ is given by $X(\varkappa) = d\varkappa(X) = \widetilde{g}(\widetilde{\nabla}\varkappa, X)$. Let Ψ be a symmetric (0, 2)-tensor field defined by

$$\Psi = \widetilde{\nabla} d\varkappa - d\varkappa \otimes d\varkappa + \frac{1}{2}\|\widetilde{\nabla}\varkappa\|^2 \widetilde{g} = \Psi^{\varkappa} + \frac{1}{2}\|\widetilde{\nabla}\varkappa\|^2 \widetilde{g}, \tag{2.5}$$

where

$$\Psi^{\varkappa} = \widetilde{\nabla} d\varkappa - d\varkappa \otimes d\varkappa. \tag{2.6}$$

The Levi-Civita connections $\overline{\nabla}$ of $\overline{g}$ and $\widetilde{\nabla}$ of $\widetilde{g}$ are related by

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y + d\varkappa(X)\, Y + d\varkappa(Y)\, X - \widetilde{g}(X, Y)\, \widetilde{\nabla}\varkappa. \tag{2.7}$$

Consequently, the Riemann-Christoffel curvature tensors $\overline{R}$ of $\overline{g}$ and $\widetilde{R}$ of $\widetilde{g}$ are related by

$$\begin{aligned} \widetilde{R}(X, Y, Z, W) = e^{-2\varkappa}\overline{R}(X, Y, Z, W) + \widetilde{g}(Y, Z)\,\Psi^{\varkappa}(X, W) - \widetilde{g}(X, Z)\,\Psi^{\varkappa}(Y, W) \\ + \Psi^{\varkappa}(Y, Z)\,\widetilde{g}(X, W) - \Psi^{\varkappa}(X, Z)\,\widetilde{g}(Y, W) \\ + \|\widetilde{\nabla}\varkappa\|^2 \{\widetilde{g}(Y, Z)\,\widetilde{g}(X, W) - \widetilde{g}(X, Z)\,\widetilde{g}(Y, W)\}. \end{aligned} \tag{2.8}$$

In 1980, Vaisman [24] introduced the concept of conformal changes (or deformations) of almost contact metric structures as follows. Let $\widetilde{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact metric structure $(\varphi, \xi, \eta, \widetilde{g})$. Suppose that

$$\overline{\varphi} = \varphi, \quad \overline{\xi} = e^{-\varkappa}\xi, \quad \overline{\eta} = e^{\varkappa}\eta, \quad \overline{g} = e^{2\varkappa}\widetilde{g}, \tag{2.9}$$

for some smooth function $\varkappa : \widetilde{M} \to \mathbb{R}$, then the structure $(\varphi, \xi, \eta, \widetilde{g})$ is said to be *conformally related* to the almost contact metric structure $\left(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$. Suppose $\overline{\Phi}$ is the Sasaki form of the structure $\left(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ given by $\overline{\Phi}(X, Y) = \overline{g}(X, \overline{\varphi}Y)$. Then $\overline{\Phi} = e^{2\varkappa}\Phi$.

Lemma 2.1 *Let $(\widetilde{M}, \varphi, \xi, \eta)$ be an almost contact manifold. Let $(\overline{\varphi}, \overline{\xi}, \overline{\eta})$ be an almost contact structure satisfying (2.9). Then,*

$$e^{\varkappa}\overline{h} = h + \frac{1}{2}(d\varkappa \circ \varphi) \otimes \xi, \tag{2.10}$$

where $2h = \pounds_{\xi}\varphi$ and $2\overline{h} = \pounds_{\overline{\xi}}\overline{\varphi}$.

Proof Omitted. ■

Lemma 2.2 *Let $(\widetilde{M}, \varphi, \xi, \eta)$ be an almost contact manifold. Let $(\overline{\varphi}, \overline{\xi}, \overline{\eta})$ be an almost contact structure on $\widetilde{M}$ given by (2.9). Then,*

$$e^{-\varkappa}d\overline{\eta} = d\eta + d\varkappa \wedge \eta, \tag{2.11}$$

$$[\overline{\varphi}, \overline{\varphi}] + 2d\overline{\eta} \otimes \overline{\xi} = [\varphi, \varphi] + 2\left(d\eta + d\varkappa \wedge \eta\right) \otimes \xi, \tag{2.12}$$

$$e^{-2\varkappa} d\overline{\Phi} = d\Phi + 2d\varkappa \wedge \eta. \tag{2.13}$$

Proof We have $d\overline{\eta} = d(e^{\varkappa}\eta) = e^{\varkappa}\left(d\eta + d\varkappa \wedge \eta\right)$, which gives (2.11). Next, from

$$[\overline{\varphi}, \overline{\varphi}] + 2d\overline{\eta} \otimes \overline{\xi} = [\varphi, \varphi] + 2e^{\varkappa}\left(d\eta + d\varkappa \wedge \eta\right) \otimes \left(e^{-\varkappa}\xi\right),$$

we have (2.12). Finally, we have $d\overline{\Phi} = d(e^{2\varkappa}\Phi) = e^{2\varkappa}\left(d\Phi + 2d\varkappa \wedge \eta\right)$, which gives (2.13). ■

Lemma 2.3 *Let $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$ be an almost contact metric manifold. Let $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be an almost contact metric structure on $\widetilde{M}$ obtained from (2.9). Then*

$$(\widetilde{\nabla}_X \varphi)Y = (\overline{\nabla}_X \overline{\varphi})Y - \{d\varkappa(\varphi Y)\, X - d\varkappa(Y)\,\varphi X - \widetilde{g}(X, \varphi Y)\,\widetilde{\nabla}\varkappa + \widetilde{g}(X, Y)\,\varphi(\widetilde{\nabla}\varkappa)\}, \tag{2.14}$$

$$\widetilde{\nabla}_X \xi = e^{\varkappa}\overline{\nabla}_X \overline{\xi} - \left\{d\varkappa(\xi)\, X - \eta(X)\,\widetilde{\nabla}\varkappa\right\}, \tag{2.15}$$

$$(\widetilde{\nabla}_X \eta)Y = e^{-\varkappa}(\overline{\nabla}_X \overline{\eta})Y - \{d\varkappa(\xi)\,\widetilde{g}(X, Y) - \eta(X)\, d\varkappa(Y)\} \tag{2.16}$$

for all vector fields X, Y on $\widetilde{M}$.

Proof In view of (2.7), we have

$$(\overline{\nabla}_X \overline{\varphi})Y = \widetilde{\nabla}_X \varphi Y + d\varkappa(\varphi Y)\, X - \widetilde{g}(X, \varphi Y)\,\widetilde{\nabla}\varkappa - \varphi\widetilde{\nabla}_X Y - d\varkappa(Y)\,\varphi X + \widetilde{g}(X, Y)\,\varphi(\widetilde{\nabla}\varkappa),$$

which gives (2.14). Next, using (2.7), we get

$$\overline{\nabla}_X \overline{\xi} = e^{-\varkappa}\left\{\widetilde{\nabla}_X \xi + d\varkappa(\xi)\, X - \eta(X)\,\widetilde{\nabla}\varkappa\right\}.$$

From the above relation we get (2.15). Finally, we have

$$(\overline{\nabla}_X \overline{\eta})Y = e^{\varkappa}\widetilde{g}(\widetilde{\nabla}_X \xi, Y) + e^{\varkappa} d\varkappa(\xi)\,\widetilde{g}(X, Y) - e^{\varkappa}\eta(X)\,\widetilde{g}(\widetilde{\nabla}\varkappa, Y),$$

which gives (2.16). ■

A $(2n+1)$-dimensional almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$ is called a *conformal contact metric manifold* if the structure $(\varphi, \xi, \eta, \widetilde{g})$ is conformally related to a contact metric structure $\left(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g}\right)$ by the relation (2.9). Similarly, $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$ is said to be a *conformal K-contact manifold* (resp. a *conformal Sasakian manifold*) if $(\widetilde{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a K-contact manifold (resp. a Sasakian manifold) (see [1, 24]). Moreover, $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$ is called a *conformal Sasakian space form* [1] if $(\widetilde{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a Sasakian space form.

Proposition 2.4 *Let $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$ be a conformal K-contact manifold. Then,*

$$\widetilde{\nabla}\xi = -e^{\varkappa}\varphi - \{d\varkappa(\xi)\, I - \eta \otimes \widetilde{\nabla}\varkappa\}, \tag{2.17}$$

$$\widetilde{\nabla}\eta = e^{\varkappa}\Phi - \{d\varkappa(\xi)\, \widetilde{g} - \eta \otimes d\varkappa\}\,. \tag{2.18}$$

Proof From the assumption, we see that $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a K-contact metric structure satisfying (2.9). Since $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a K-contact structure, using $\overline{\nabla}\,\overline{\xi} = -\varphi$ in (2.15), we get (2.17). Next, from (2.16) we get

$$(\widetilde{\nabla}_X \eta)Y = e^{-\varkappa}\overline{g}(\overline{\nabla}_X \overline{\xi}, Y) - \{d\varkappa(\xi)\, \widetilde{g}(X, Y) - \eta\,(X)\, d\varkappa(Y)\}\,,$$

which, in view of $\overline{\nabla}\,\overline{\xi} = -\varphi$, gives (2.18). ∎

Remark 2.5 There is another notion of a *conformal K-contact manifold* given in [14]. Accordingly, a $(2n+1)$-dimensional conformal K-contact manifold $(\widetilde{M}, \varphi, \eta, \xi, \widetilde{g})$ is a contact metric manifold in which the associated vector field ξ is a conformal Killing vector field, that is, $\pounds_\xi \widetilde{g} = \alpha\widetilde{g}$ for some smooth function $\alpha : \widetilde{M} \to \mathbb{R}$. Contracting the conformal equation gives $2\mathrm{div}\xi = (2n+1)\alpha$. But it is known that $\mathrm{div}\xi = 0$ for a contact metric manifold. So, $\pounds_\xi \widetilde{g} = 0$, that is, ξ is Killing, and hence a conformal K-contact manifold in sense of [14] reduces to a K-contact manifold. This fact was presented to the first author by R. Sharma (on 2020.07.10, Friday, 10:58 PM).

Example 2.6 ([1, Example 3.1]) Let $\mathbb{R}^{2n+1}$ be endowed with an almost contact metric structure $(\varphi, \xi, \eta, \widetilde{g})$ given by

$$\varphi\left(\sum_{i=1}^{n}\left(X^i\frac{\partial}{\partial x^i} + Y^i\frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{n}\left(Y^i\frac{\partial}{\partial x^i} - X^i\right)\frac{\partial}{\partial y^i} + \sum_{i=1}^{n} Y^i y^i \frac{\partial}{\partial z},$$

$$\xi = e^{\varkappa}\left\{2\frac{\partial}{\partial z}\right\}, \qquad \eta = e^{-\varkappa}\left\{\frac{1}{2}\left(dz - \sum_{i=1}^{n} y^i dx^i\right)\right\},$$

$$\widetilde{g} = e^{-2\varkappa}\left\{\eta\otimes\eta + \frac{1}{4}\sum_{i=1}^{n}\left\{\left(dx^i\right)^2 + \left(dy^i\right)^2\right\}\right\},$$

where

$$\varkappa = \frac{1}{2}\left\{\sum_{i=1}^{n}\left(\left(x^i\right)^2 + \left(y^i\right)^2\right) + z^2\right\}.$$

Then $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, \widetilde{g})$ is not a Sasakian manifold, but $(\mathbb{R}^{2n+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a Sasakian space form with the constant $\overline{\varphi}$-sectional curvature equal to -3, where

$$\overline{\varphi} = \varphi, \qquad \overline{\xi} = 2\frac{\partial}{\partial z}, \qquad \overline{\eta} = \frac{1}{2}\left(dz - \sum_{i=1}^{n} y^i dx^i\right),$$

$$\overline{g} = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}\left\{\left(dx^i\right)^2 + \left(dy^i\right)^2\right\}.$$

Proposition 2.7 *Let* $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$ *be a conformal Sasakian manifold. Then,*

$$\begin{aligned}(\widetilde{\nabla}_X \varphi)Y = {} & e^{\varkappa}\{\widetilde{g}(X, Y)\,\xi - \eta(Y)\,X\} \\ & -\{d\varkappa(\varphi Y)\,X - d\varkappa(Y)\,\varphi X - \widetilde{g}(X, \varphi Y)\,\widetilde{\nabla}\varkappa + \widetilde{g}(X, Y)\,\varphi(\widetilde{\nabla}\varkappa)\}\end{aligned} \tag{2.19}$$

for all vector fields X, Y *on* $\widetilde{M}$.

Proof From the assumption, we see that $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a Sasakian structure satisfying (2.9). Since $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a Sasakian structure, we have

$$(\overline{\nabla}_X \overline{\varphi})Y = \overline{g}(X, Y)\,\overline{\xi} - \overline{\eta}(Y)\,X \tag{2.20}$$

for all vector fields X, Y on $\widetilde{M}$. In view of (2.9), the above relation becomes

$$(\overline{\nabla}_X \overline{\varphi})Y = e^{\varkappa}\{\widetilde{g}(X, Y)\,\xi - \eta(Y)\,X\}.$$

Using the above relation is (2.14), we get (2.19). ■

Now suppose that $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$ is a conformal Sasakian space form, so that $(\widetilde{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a Sasakian space form of constant $\overline{\varphi}$-sectional curvature c. Then, the Riemann-Christoffel curvature tensor $\overline{R}$ of $\overline{g}$ satisfies

$$\begin{aligned}\overline{R}(X, Y, Z, W) = {} & \frac{c+3}{4}\{\overline{g}(Y, Z)\,\overline{g}(X, W) - \overline{g}(X, Z)\,\overline{g}(Y, W)\} \\ & + \frac{c-1}{4}\{\overline{\eta}(X)\,\overline{\eta}(Z)\,\overline{g}(Y, W) - \overline{\eta}(Y)\,\overline{\eta}(Z)\,\overline{g}(X, W) \\ & + \overline{g}(X, Z)\,\overline{\eta}(Y)\,\overline{\eta}(W) - \overline{g}(Y, Z)\,\overline{\eta}(X)\,\overline{\eta}(W) + \overline{g}(Y, \overline{\varphi}Z)\,\overline{g}(X, \overline{\varphi}W) \\ & - \overline{g}(X, \overline{\varphi}Z)\,\overline{g}(Y, \overline{\varphi}W) - 2\overline{g}(\overline{\varphi}X, Y)\,\overline{g}(\overline{\varphi}Z, W)\},\end{aligned}$$

which, in view of (2.9), gives

$$\begin{aligned}\overline{R}(X, Y, Z, W) = {} & \frac{c+3}{4}e^{4\varkappa}\{\widetilde{g}(Y, Z)\,\widetilde{g}(X, W) - \widetilde{g}(X, Z)\,\widetilde{g}(Y, W)\} \\ & + \frac{c-1}{4}e^{4\varkappa}\{\eta(X)\,\eta(Z)\,\widetilde{g}(Y, W) - \eta(Y)\,\eta(Z)\,\widetilde{g}(X, W) \\ & + \widetilde{g}(X, Z)\,\eta(Y)\,\eta(W) - \widetilde{g}(Y, Z)\,\eta(X)\,\eta(W) + \widetilde{g}(Y, \varphi Z)\,\widetilde{g}(X, \varphi W) \\ & - \widetilde{g}(X, \varphi Z)\,\widetilde{g}(Y, \varphi W) - 2\widetilde{g}(X, \varphi Y)\,\widetilde{g}(Z, \varphi W)\}.\end{aligned} \tag{2.21}$$

Consequently, in view of (2.21), from (2.8), the Riemann-Christoffel curvature tensor $\widetilde{R}$ of a conformal Sasakian space form $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$ satisfies

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) &= \widetilde{g}(Y,Z)\,\Psi^{\varkappa}(X,W) - \widetilde{g}(X,Z)\,\Psi^{\varkappa}(Y,W) + \Psi^{\varkappa}(Y,Z)\,\widetilde{g}(X,W) - \Psi^{\varkappa}(X,Z)\,\widetilde{g}(Y,W)\\ &+ \left(\frac{c+3}{4}e^{2\varkappa} + \|\widetilde{\nabla}\varkappa\|^2\right)(\widetilde{g}(Y,Z)\,\widetilde{g}(X,W) - \widetilde{g}(X,Z)\,\widetilde{g}(Y,W))\\ &+ \frac{c-1}{4}e^{2\varkappa}\{\eta(X)\,\eta(Z)\,\widetilde{g}(Y,W) - \eta(Y)\,\eta(Z)\,\widetilde{g}(X,W)\\ &\quad + \widetilde{g}(X,Z)\,\eta(Y)\,\eta(W) - \widetilde{g}(Y,Z)\,\eta(X)\,\eta(W) + \widetilde{g}(Y,\varphi Z)\,\widetilde{g}(X,\varphi W)\\ &\quad - \widetilde{g}(X,\varphi Z)\,\widetilde{g}(Y,\varphi W) - 2\widetilde{g}(X,\varphi Y)\,\widetilde{g}(Z,\varphi W)\}. \end{aligned} \tag{2.22}$$

Because of the presence of first four expressions in the first line on the right hand side of (2.22), a conformal Sasakian space form is not a particular case of a generalized Sasakian space form introduced in [2].

3 Slant Submanifolds of an Almost Contact Metric Manifold

Let (M, g) be a Riemannian submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ equipped with a compatible almost contact structure (φ, ξ, η). For $X \in TM$ and $N \in T^{\perp}M$ we put

$$\varphi X = PX + FX, \qquad \varphi N = tN + fN,$$

where $PX, tN \in TM$ and $FX, fN \in T^{\perp}M$. Moreover, if $\xi \in TM$ then we write $TM = \{\xi\} \oplus \{\xi\}^{\perp}$, where $\{\xi\}$ is the distribution spanned by ξ and $\{\xi\}^{\perp}$ is the complementary orthogonal distribution of $\{\xi\}$ in M. Now, we recall the definition of an almost semi-invariant submanifold given by the first author in 1996 as follows (cf. [19]).

Definition 3.1 A Riemannian submanifold (M, g) of an almost contact metric manifold $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$ with $\xi \in TM$ is said to be an *almost semi-invariant submanifold* of $\widetilde{M}$ if there are k distinct functions $\lambda_1, \ldots, \lambda_k$ defined on M with values in the open interval $(0, 1)$ such that TM is decomposed as P-invariant mutually orthogonal differentiable distributions given by

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \cdots \oplus \mathcal{D}^{\lambda_k} \oplus \{\xi\},$$

where $\mathcal{D}^1_p = \ker(F|_{\{\xi\}^{\perp}})_p$, $\mathcal{D}^0_p = \ker(P|_{\{\xi\}^{\perp}})_p$, and $\mathcal{D}^{\lambda_i}_p = \ker\left(P^2|_{\{\xi\}^{\perp}} + \lambda_i^2(p)I\right)_p$, $i \in \{1, \ldots, k\}$. If in addition, each λ_i is constant, then M is called an *almost semi-invariant* submanifold.*

An almost semi-invariant submanifold reduces to

(1) an *invariant submanifold* [4, 25] if $k = 0$ and $\mathcal{D}^0 = 0$, so that $TM = \mathcal{D}^1 \oplus \{\xi\}$;
(2) an *anti-invariant submanifold* [4, 25] if $k = 0$ and $\mathcal{D}^1 = 0$, so that $TM = \mathcal{D}^0 \oplus \{\xi\}$;
(3) a *semi-invariant submanifold* [4] (see also *contact CR-submanifold* [25]) if $k = 0$, so that $TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \{\xi\}$;
(4) a *θ-slant submanifold* [7] if $\mathcal{D}^1 = 0 = \mathcal{D}^0$, $k = 1$ and λ_1 is constant, so that $TM = \mathcal{D}^{\lambda_1} \oplus \{\xi\}$;
(5) a *pointwise slant submanifold* [15] if $\mathcal{D}^1 = 0 = \mathcal{D}^0$, $k = 1$, so that $TM = \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_1} \oplus \{\xi\}$;
(6) a *semi-slant submanifold* [6] if $\mathcal{D}^0 = 0$, $k = 1$ and λ_1 is constant, so that $TM = \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_1} \oplus \{\xi\}$;
(7) a *pointwise semi-slant submanifold* [15] if $\mathcal{D}^0 = 0$, $k = 1$, so that $TM = \mathcal{D}^1 \oplus \mathcal{D}^{\lambda_1} \oplus \{\xi\}$;
(8) an *anti-slant submanifold* [8] (or a *pseudo-slant submanifold* [3], or a *hemi-slant submanifold* [12]) if $\mathcal{D}^1 = 0$, $k = 1$ and λ_1 is constant, so that $TM = \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \{\xi\}$;
(9) a *proper bi-slant submanifold* [8] if $\mathcal{D}^1 = 0 = \mathcal{D}^0$, $k = 2$ and λ_1, λ_2 are constant, so that $TM = \mathcal{D}^{\lambda_1} \oplus \mathcal{D}^{\lambda_2} \oplus \{\xi\}$;
(10) a *quasi hemi-slant submanifold* [16] if $k = 1$ and λ_1 is constant, so that $TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\lambda_1} \oplus \{\xi\}$.

The submanifold M is invariant (resp. anti-invariant) if $\varphi X \in TM$ (resp. $\varphi X \in T^{\perp}M$) for every $X \in TM$. In case of a θ-slant submanifold, the slant angle θ is given by $\lambda_1 = \cos\theta$. A slant submanifold which is neither invariant nor anti-invariant is called a *proper θ-slant submanifold*. It is known that proper θ-slant submanifolds of almost contact metric manifolds are always odd dimensional. The definitions of pointwise bi-slant and pointwise quasi hemi-slant submanifolds are also possible as particular cases an almost semi-invariant submanifold. For an almost semi-invariant submanifold, for $X \in TM$, we may write

$$X = U^1X + U^0X + U^{\lambda_1}X + \cdots + U^{\lambda_k}X + \eta(X)\xi,$$

where $U^1, U^0, U^{\lambda_1}, \ldots, U^{\lambda_k}$ are orthogonal projection operators of TM on $\mathcal{D}^1, \mathcal{D}^0, \mathcal{D}^{\lambda_1}, \ldots, \mathcal{D}^{\lambda_k}$, respectively. Then, it follows that

$$\|X\|^2 = \|U^1X\|^2 + \|U^0X\|^2 + \|U^{\lambda_1}X\|^2 + \cdots + \|U^{\lambda_k}X\|^2 + \eta(X)^2. \tag{3.1}$$

We also have $P^2X = -U^1X - \lambda_1^2(U^{\lambda_1}X) - \cdots - \lambda_1^2(U^{\lambda_k}X)$, which implies that

$$\|PX\|^2 = \widetilde{g}(PX, PX) = -\widetilde{g}(P^2X, X) = \sum_{\lambda\in\{1,\lambda_1,\ldots,\lambda_k\}} \lambda^2\|U^{\lambda}X\|^2. \tag{3.2}$$

In particular, if M is an m-dimensional θ-slant submanifold, then $\lambda_1^2 = \cos^2\theta$ and we have

$$\|PX\|^2 = \cos^2\theta\|U^{\lambda_1}X\|^2 = \cos^2\theta\left(\|X\|^2 - \eta(X)^2\right). \tag{3.3}$$

In fact, each distribution $\mathcal{D}^{\lambda_i}$ has the slant function θ_i for $i \in \{1, \ldots, k\}$. Since for any unit vector $Z_i \in \mathcal{D}^{\lambda_i}$, $g(PZ_i, JZ_i) = g(PZ_i, PZ_i) = \theta_i^2$ it is known that $-\lambda_i^2 = -\cos^2\theta_i$, therefore the distributions $\mathcal{D}^{\lambda_i}$ can be denoted by $\mathcal{D}^{\theta_i}$ for $i \in \{1, \ldots, k\}$, and the decomposition of TM can be written as

$$TM = \mathcal{D}^1 \oplus \mathcal{D}^0 \oplus \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \cdots \oplus \mathcal{D}^{\theta_k} \oplus \{\xi\},$$

where $\mathcal{D}^1$ is invariant, $\mathcal{D}^0$ is anti-invariant, $\mathcal{D}^{\theta_i}$ is pointwise slant distribution with slant function θ_i for $i \in \{1, \ldots, k\}$.

Remark 3.2 In [17], Ronsse developed the notion of almost CR submanifolds of an almost contact metric manifold with weaker conditions (see also [4] for almost semi-invariant submanifold and [13] for almost CR submanifolds). In case, a submanifold of an almost contact metric manifold is orthogonal to the structure vector field ξ, an analogous concept of $\xi^\perp$-almost semi-invariant submanifold was introduced by the first author [20].

4 Chen-Ricci Inequality for Slant Submanifolds

Let (M, g) be an m-dimensional Riemannian manifold. Let $\{e_1, \ldots, e_m\}$ be any orthonormal basis for T_pM. The sectional curvature of a plane section spanned by orthonormal unit vectors e_i and e_j at $p \in M$, denoted K_{ij}, is given by $K_{ij} = R(e_i, e_j, e_j, e_i)$, where R is the Riemann-Christoffel curvature tensor. The Ricci contraction of Riemann-Christoffel curvature tensor is called the Ricci tensor denoted by Ric. For a fixed $i \in \{1, \ldots, m\}$, the Ricci curvature of e_i, denoted $\mathrm{Ric}(e_i)$, is given by $\mathrm{Ric}(e_i) = \sum_{j\neq i, i<j}^{m} K_{ij}$. The scalar curvature τ is given by $\tau(p) = \sum_{1\leq i<j\leq m} K_{ij} = \frac{1}{2}\sum_{i=1}^{m}\mathrm{Ric}(e_i)$. Consequently, we have

$$\mathrm{Ric}(e_1) = \tau(p) - \sum_{2\leq i<j\leq m} K_{ij} = \tau(p) - \frac{1}{2}\sum_{2\leq i\neq j\leq m} K_{ij}. \tag{4.1}$$

Let L be a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X$. The k-Ricci curvature $\mathrm{Ric}_L(X)$ is defined by [10]

$$\mathrm{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k}.$$

Thus for each fixed e_i, $i \in \{1, \ldots, k\}$ we get $\mathrm{Ric}_L(e_i) = \sum_{j\neq i, i<j}^{k} K_{ij}$.

Let (M, g) be an m-dimensional Riemannian submanifold of a $(2n+1)$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$, and σ the second fundamental form. The equation of Gauss is given by

$$R(X, Y, Z, W) = \widetilde{R}(X, Y, Z, W) + \widetilde{g}(\sigma(X, W), \sigma(Y, Z)) - \widetilde{g}(\sigma(X, Z), \sigma(Y, W)) \quad (4.2)$$

for all $X, Y, Z, W \in TM$, where $\widetilde{R}$ and R are the Riemann-Christoffel curvature tensors of $\widetilde{M}$ and M, respectively. The mean curvature vector H is given by $H = \text{trace}(\sigma)/\dim(M)$. The submanifold M is totally geodesic in $\widetilde{M}$ if $\sigma = 0$, minimal if $H = 0$, and totally umbilical if $\sigma(X, Y) = g(X, Y)H$ for all vectors X, Y tangent to M. The *relative null space* of M at p [10] is defined by

$$\mathcal{N}_p = \{X \in T_pM : \sigma(X, Y) = 0 \text{ for all } Y \in T_pM\}.$$

Let $\{e_1, \ldots, e_m\}$ and $\{e_{m+1}, \ldots, e_{2n+1}\}$ be the orthonormal bases of the tangent space T_pM and the normal space $T_p^{\perp}M$, respectively. We put

$$\sigma_{ij}^r = \widetilde{g}(\sigma(e_i, e_j), e_r), \qquad ||\sigma||^2 = \sum_{i,j=1}^{m} \widetilde{g}(\sigma(e_i, e_j), \sigma(e_i, e_j)),$$

where $i, j \in \{1, \ldots, m\}$, $r \in \{m+1, \ldots, 2n+1\}$. Let K_{ij} and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\widetilde{M}$, respectively. Then, from the Gauss equation (4.2), we have [11]

$$K_{ij} = \widetilde{K}_{ij} + \sum_{r=m+1}^{2n+1} (\sigma_{ii}^r \sigma_{jj}^r - (\sigma_{ij}^r)^2). \quad (4.3)$$

From (4.3) it follows that

$$2\tau(p) = 2\widetilde{\tau}(T_pM) + m^2||H||^2 - ||\sigma||^2, \quad (4.4)$$

where $\widetilde{\tau}(T_pM) = \sum_{1 \le i < j \le m} \widetilde{K}_{ij}$ denotes the scalar curvature of the m-plane section T_pM in $\widetilde{M}$.

Theorem 4.1 ([1, Theorem 3.2, Corollary 3.3]) *Let M be a Riemannian submanifold of a conformal Sasakian space form $(\widetilde{M}, \varphi, \xi, \eta, \widetilde{g})$, denoted by $\widetilde{M}(c)$, such that $\widetilde{\nabla}\varkappa, \xi \in TM$. If $p \in M$ is a totally umbilical point, then p is a totally geodesic point and hence $\varphi(T_pM) \subseteq T_pM$. Consequently, a totally umbilical submanifold M of a conformal Sasakian space form $\widetilde{M}(c)$ such that $\widetilde{\nabla}\varkappa, \xi \in TM$, is a totally geodesic invariant submanifold.*

Theorem 4.2 ([1, Lemma 4.6]) *Let M be an m-dimensional invariant submanifold of a Conformal Sasakian manifold $\overline{M}$, tangent to the structure vector field $\overline{\xi}$. Then M is minimal if and only if $\widetilde{\nabla}\varkappa$ is tangent to M.*

Lemma 4.3 *Let M be an m-dimensional Riemannian submanifold of a conformal Sasakian space form $(\widetilde{M}(c), \varphi, \xi, \eta, \widetilde{g})$ such that $\xi \in TM$. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of the tangent space T_pM. Then*

$$\widetilde{K}_{ij} = \frac{1}{4}e^{2\varkappa}\left\{(c+3)+3(c-1)\,g\left(Pe_i, e_j\right)^2-(c-1)\{\eta(e_i)^2+\eta\left(e_j\right)^2\}\right\} \\ +\|\widetilde{\nabla}\varkappa\|^2+\Psi^{\varkappa}(e_i, e_i)+\Psi^{\varkappa}\left(e_j, e_j\right), \tag{4.5}$$

$$\widetilde{\mathrm{Ric}}_{(T_pM)}(X) = \frac{1}{4}e^{2\varkappa}\left\{(c+3)(m-1)+(c-1)\{3\|PX\|^2-(m-2)\,\eta(X)^2-1\}\right\} \\ +(m-1)\|\widetilde{\nabla}\varkappa\|^2+(m-2)\{(\widetilde{\nabla}_X d\varkappa)(X)-d\varkappa(X)^2\}+\mathrm{trace}(\Psi^{\varkappa}|_M), \tag{4.6}$$

$$\widetilde{\tau}\left(T_pM\right) = \frac{1}{8}e^{2\varkappa}\left\{m(m-1)(c+3)+(c-1)\{3\|P\|^2-2m+2\}\right\} \\ +\frac{1}{2}m(m-1)\|\widetilde{\nabla}\varkappa\|^2+(m-1)\,\mathrm{trace}(\Psi^{\varkappa}|_M). \tag{4.7}$$

Proof From (2.22) it follows (4.5). Using $\widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) = \sum_{j=1, j\neq i}^{m} \widetilde{K}_{ij}$ in (4.5), we get (4.6). Next, using $2\widetilde{\tau}\left(T_pM\right) = \sum_{i=1}^{m} \widetilde{\mathrm{Ric}}_{(T_pM)}(e_i)$ from (4.6), we obtain (4.7). ∎

Theorem 4.4 (cf. [1, Theorem 4.1]) *Let M be an m-dimensional $(m \geq 2)$ Riemannian submanifold of a conformal Sasakian space form $(\widetilde{M}(c), \varphi, \xi, \eta, \widetilde{g})$ such that $\xi \in TM$. Then,*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\left\{m^2\|H\|^2+e^{2\varkappa}\left\{(c+3)(m-1)+(c-1)\{3\|PX\|^2-(m-2)\,\eta(X)^2-1\}\right\}\right\} \\ +(m-1)\|\widetilde{\nabla}\varkappa\|^2+(m-2)(\widetilde{\nabla}_X d\varkappa)(X)+\mathrm{trace}(\Psi^{\varkappa}|_M) \tag{4.8}$$

for any unit vector $X \in T_pM$.

Proof From (4.3), we get

$$\frac{1}{4}m^2\|H\|^2 = \tau(p)-\widetilde{\tau}(T_pM)+\frac{1}{4}\sum_{r=m+1}^{2n+1}(\sigma_{11}^r-\sigma_{22}^r-\cdots-\sigma_{mm}^r)^2+\sum_{r=m+1}^{2n+1}\sum_{j=2}^{m}(\sigma_{1j}^r)^2-\sum_{2\leq i<j\leq m}(K_{ij}-\widetilde{K}_{ij}). \tag{4.9}$$

From (4.1), (4.9) yields to

$$\frac{1}{4}m^2\|H\|^2 = \mathrm{Ric}(e_1)-\widetilde{\mathrm{Ric}}(e_1)+\frac{1}{4}\sum_{r=m+1}^{2n+1}(\sigma_{11}^r-\sigma_{22}^r-\cdots-\sigma_{mm}^r)^2+\sum_{r=m+1}^{2n+1}\sum_{j=2}^{m}(\sigma_{1j}^r)^2. \tag{4.10}$$

Now by substituting (4.6) in (4.10), we obtain

$$\frac{1}{4}m^2\|H\|^2 = \mathrm{Ric}(e_1) - \frac{1}{4}e^{2\varkappa}\left\{(c+3)(m-1) + (c-1)\{3\|Pe_1\|^2 - (m-2)\eta(e_1)^2 - 1\}\right\}$$
$$- (m-1)\|\widetilde{\nabla}\varkappa\|^2 - (m-2)(\widetilde{\nabla}_{e_1}d\varkappa)(e_1) + (m-2)d\varkappa(e_1)^2 - \mathrm{trace}(\Psi^{\varkappa}|_M)$$
$$+\frac{1}{4}\sum_{r=m+1}^{2n+1}(\sigma_{11}^r - \sigma_{22}^r - \cdots - \sigma_{mm}^r)^2 + \sum_{r=m+1}^{2n+1}\sum_{j=2}^{m}(\sigma_{1j}^r)^2. \tag{4.11}$$

Choosing $e_1 = X$, the above equation implies (4.8). ■

Theorem 4.5 (cf. [1, Theorem 4.2]) *Let M be an m-dimensional ($m \geq 2$) Riemannian submanifold of a conformal Sasakian space form $(\widetilde{M}(c), \varphi, \xi, \eta, \widetilde{g})$ such that $\xi \in TM$. Then,*

- **(i)** *A unit vector $X \in T_pM$ satisfies the equality case of (4.8) if and only if either* **(a)** *$m = 2$ or* **(b)** *$\widetilde{\nabla}\varkappa$ is orthogonal to X, $2\sigma(X, X) = mH(p)$, and $\sigma(X, Y) = 0$ for all $Y \in \{X\}^{\perp}$.*
- **(ii)** *If M is minimal at p, then a unit vector $X \in T_pM$ satisfies the equality case of (4.8) if and only if X lies in the relative null space of M and either $m = 2$ or $\widetilde{\nabla}\varkappa$ be orthogonal to X.*

Proof Assuming $X = e_1$, from (4.11) the equality case of (4.8) becomes valid if and only if the following three relations are satisfied:

$$\sigma_{1j}^r = 0, \quad \forall\, j \in \{2, \ldots, m\}, \quad r \in \{m+1, \ldots, 2n+1\}, \tag{4.12}$$
$$\sigma_{11}^r = \sum_{i=2}^{m}\sigma_{ii}^r, \quad \forall\, r \in \{m+1, \ldots, 2n+1\}, \tag{4.13}$$
$$(m-2)d\varkappa(X)^2 = 0. \tag{4.14}$$

Satisfying (4.12), (4.13), and (4.14) is equivalent to the statement **(i)**. For proving the statement **(ii)** we note that minimality at p means $H(p) = 0$. So, in view of (4.12), (4.13) and (4.14), we conclude that X lies in the relative null space of M and either $m = 2$ or $\widetilde{\nabla}\varkappa$ be orthogonal to X. ■

Corollary 4.6 (cf. [1, Corollary 4.3]) *Let M be an m-dimensional ($m \geq 2$) Riemannian submanifold of a conformal Sasakian space form $(\widetilde{M}(c), \varphi, \xi, \eta, \widetilde{g})$ such that $\xi \in TM$. For a unit vector $X \in T_pM$, any three of the following four statements imply the remaining one.*

- **(i)** $d\varkappa(X) = 0$.
- **(ii)** *The mean curvature vector $H(p)$ vanishes.*
- **(iii)** *The unit vector X belongs to the relative null space $\mathcal{N}_p$.*
- **(iv)** *The unit vector X satisfies the following equality case*

$$\mathrm{Ric}(X) = \frac{1}{4}\left\{m^2\|H\|^2 + e^{2\varkappa}\left\{(c+3)(m-1) + (c-1)\{3\|PX\|^2 - (m-2)\eta(X)^2 - 1\}\right\}\right\}$$
$$+ (m-1)\|\widetilde{\nabla}\varkappa\|^2 + (m-2)(\widetilde{\nabla}_X d\varkappa)(X) + \mathrm{trace}(\Psi^{\varkappa}|_M).$$

Theorem 4.7 *Let M be an m-dimensional $(m > 2)$ Riemannian submanifold of a conformal Sasakian space form $(\widetilde{M}(c), \varphi, \xi, \eta, \widetilde{g})$ such that $\xi \in TM$. Then, the equality case of (4.8) is true for every unit vector $X \in T_pM$ if and only if p is a totally geodesic point and $\widetilde{\nabla}\varkappa \in T_p^{\perp}M$.*

Corollary 4.8 *Let M be an m-dimensional $(m \geq 2)$ semi-invariant submanifold of a conformal Sasakian space form $(\widetilde{M}(c), \varphi, \xi, \eta, \widetilde{g})$ such that $T_pM = \mathcal{D}_p^1 \oplus \mathcal{D}_p^0 \oplus \{\xi\}$. Then*

(i) *For every unit vector $X \in \mathcal{D}_p^1$,*

$$\begin{aligned}\mathrm{Ric}(X) \leq \frac{1}{4}\Big\{m^2\|H\|^2 + e^{2\varkappa}\{(c+3)(m-1) + 2(c-1)\}\Big\} \\ +(m-1)\|\widetilde{\nabla}\varkappa\|^2 + (m-2)(\widetilde{\nabla}_X d\varkappa)(X) + \mathrm{trace}(\Psi^{\varkappa}|_M). \quad (4.15)\end{aligned}$$

(ii) *For every unit vector $X \in \mathcal{D}_p^0$,*

$$\begin{aligned}\mathrm{Ric}(X) \leq \frac{1}{4}\Big\{m^2\|H\|^2 + e^{2\varkappa}\{(c+3)(m-1) - (c-1)\}\}\Big\} \\ +(m-1)\|\widetilde{\nabla}\varkappa\|^2 + (m-2)(\widetilde{\nabla}_X d\varkappa)(X) + \mathrm{trace}(\Psi^{\varkappa}|_M). \quad (4.16)\end{aligned}$$

Proof If M is a semi-invariant submanifold, then $\varphi(\mathcal{D}_p^1) \subseteq \mathcal{D}_p^1$ and $\varphi(\mathcal{D}_p^0) \subseteq T_p^{\perp}M$. If $X \in \mathcal{D}_p^1$, then $\eta(X) = 0$ and $\|PX\|^2 = 1$. Now using these values in (4.8) we get **(i)**. To prove **(ii)**, we note that in this case $P = 0$, rest of the proof is similar to **(i)**. ■

Theorem 4.9 *Let M be an m-dimensional $(m \geq 2)$ almost semi-invariant submanifold of a conformal Sasakian space form $(\widetilde{M}(c), \varphi, \xi, \eta, \widetilde{g})$. Then, for each unit vector $X \in T_pM$,*

$$\begin{aligned}\mathrm{Ric}(X) \leq \frac{1}{4}\Big\{m^2\|H\|^2 + e^{2\varkappa}\{(c+3)(m-1) + \\ (c-1)\{3\sum_{\lambda\in\{1,\lambda_1,\ldots,\lambda_k\}}\lambda^2\|U_p^{\lambda}X\|^2 - (m-2)\eta(X)^2 - 1\}\Big\}\Big\} \\ +(m-1)\|\widetilde{\nabla}\varkappa\|^2 + (m-2)(\widetilde{\nabla}_X d\varkappa)(X) + \mathrm{trace}(\Psi^{\varkappa}|_M), \quad (4.17)\end{aligned}$$

where $U_p^1, U_p^{\lambda_1}, \ldots, U_p^{\lambda_k}$ are orthogonal projection operators of T_pM on $\mathcal{D}_p^1, \mathcal{D}_p^{\lambda_1}, \ldots, \mathcal{D}_p^{\lambda_k}$, respectively.

Proof Using (3.2) in (4.8), we get (4.17). ■

Corollary 4.10 *Let M be an m-dimensional $(m \geq 2)$ θ-slant submanifold of a conformal Sasakian space form $(\widetilde{M}(c), \varphi, \xi, \eta, \widetilde{g})$. Then, for each unit vector $X \in T_pM$,*

$$
\begin{aligned}
\mathrm{Ric}(X) \leq \frac{1}{4} \big\{ & m^2 \|H\|^2 + e^{2\varkappa} \{(c+3)(m-1) + \\
& (c-1)\{3\cos^2\theta \left(1-\eta(X)^2\right) - (m-2)\,\eta(X)^2 - 1\}\}\big\} \\
& + (m-1)\,\|\widetilde{\nabla}\varkappa\|^2 + (m-2)\,(\widetilde{\nabla}_X d\varkappa)(X) + \mathrm{trace}(\Psi^{\varkappa}|_M). \quad (4.18)
\end{aligned}
$$

Proof Using (3.3) in (4.8), we get (4.18). ■

Corollary 4.11 *Let M be an m-dimensional ($m \geq 2$) anti-invariant submanifold of a conformal Sasakian space form $(\widetilde{M}(c), \varphi, \xi, \eta, \widetilde{g})$. Then, for each unit vector $X \in T_pM$,*

$$
\begin{aligned}
\mathrm{Ric}(X) \leq \frac{1}{4} & \left\{ m^2\|H\|^2 + e^{2\varkappa}\left\{(c+3)(m-1) - (c-1)\{(m-2)\,\eta(X)^2 + 1\}\right\}\right\} \\
& + (m-1)\,\|\widetilde{\nabla}\varkappa\|^2 + (m-2)\,(\widetilde{\nabla}_X d\varkappa)(X) + \mathrm{trace}(\Psi^{\varkappa}|_M). \qquad (4.19)
\end{aligned}
$$

Proof Put $\theta = \pi/2$ in (4.18) to get (4.19). ■

Corollary 4.12 *Let M be an m-dimensional ($m \geq 2$) (θ_1, θ_2) bi-slant submanifold of a conformal Sasakian space form $(\widetilde{M}(c), \varphi, \xi, \eta, \widetilde{g})$. Then, for each unit vector $X \in T_pM$,*

$$
\begin{aligned}
\mathrm{Ric}(X) \leq \frac{1}{4} \big\{ & m^2\|H\|^2 + e^{2\varkappa}\{(c+3)(m-1) + \\
& (c-1)\{3\left(\cos^2\theta_1 + \cos^2\theta_2\right)\left(1-\eta(X)^2\right) - (m-2)\,\eta(X)^2 - 1\}\}\big\} \\
& + (m-1)\,\|\widetilde{\nabla}\varkappa\|^2 + (m-2)\,(\widetilde{\nabla}_X d\varkappa)(X) + \mathrm{trace}(\Psi^{\varkappa}|_M). \qquad (4.20)
\end{aligned}
$$

Proof In this case, we get

$$
\|PX\|^2 = \left(\cos^2\theta_1 + \cos^2\theta_2\right)\left(1-\eta(X)^2\right). \qquad (4.21)
$$

Using (4.21) in (4.8), we get (4.20). ■

Remark 4.13 For many other results, one may refer to [1]. Many new results can be obtained for other different classes of submanifolds discussed in Sect. 3. As far as the authors know, inequalities for Chen's δ-invariants and different Casorati curvatures of a conformal Sasakian space form have not been obtained so far. Different kinds of inequalities for C-totally real and Legendrian submanifolds of a conformal Sasakian space form are also not obtained so far. Like statistical Sasakian space form, the concept of conformal statistical Sasakian space form can also be developed. Then Chen-Ricci, B.Y. Chen, Casorati, and Wingten inequalities can be obtained/studied for different kinds of submanifolds of conformal statistical Sasakian space forms.

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Slant Curves and Magnetic Curves

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1 Introduction

In 1990, Chen introduced the notion of slant submanifold in almost Hermitian manifolds [21] (see also [22]). To extend the notion of slant submanifold in *odd*-dimensional ambient spaces, in 1996, Lotta introduced a notion of slant submanifold in almost contact metric manifolds [74] (see also [12]).

Let $M = (M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold. An immersed submanifold N of M is said to be *slant* if for any $x \in N$ and $X \in T_x N$ linearly independent of ξ_x, the angle between φX and $T_x M$ is a constant $\theta \in [0, \pi/2]$, called the *slant angle* of N in M. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submanifold which is neither invariant nor anti-invariant is called a *proper slant submanifold*.

One can see that Lotta's definition excludes 1-dimensional submanifolds, that is slant curves. Thus, if one wishes to study curves in almost contact metric manifolds with slant property, one needs manipulation of Lotta's definition. For this purpose, we return to the original motivation of slant submanifold geometry. One of the motivations of Chen's work is to prove a new class of submanifolds in almost Hermitian manifolds which contains both holomorphic submanifolds and totally real submanifolds as extremal cases.

In the case of 1-dimensional submanifolds in almost contact metric manifold M, "invariant 1-dimensional submanifolds" are nothing but characteristic flows. More

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precisely, an arc length parametrized curve in M is a 1-dimensional invariant submanifold if and only if it is an integral curve of the characteristic vector field $\pm\xi$. It should be remarked that any regular curve in M is regarded as a 1-dimensional anti-invariant submanifold.

On the other hand, in almost contact metric geometry, *C-totally real submanifolds*, that is submanifolds orthogonal to ξ, have been studied as an analogue of totally real submanifolds. In particular, 1-dimensional C-totally real submanifolds are called *almost contact curves* (also called *almost Legendre curves*). In 3-dimensional contact geometry and contact topology, almost contact curves are traditionally called the *Legendre curves*. The Legendre curves play a central role in 3-dimensional contact geometry and topology [96].

If we think back to Chen's original approach, we need to look for a class of curves in almost contact metric manifolds which contains characteristic flows and almost contact curves as extremal cases.

To extend slant submanifold geometry to curves in almost contact metric manifolds, an alternative notion of "slant curve" was introduced by Cho, Inoguchi and Lee [29]. Since then, slant curves in almost contact metric manifolds have been paid much attention of differential geometers and investigated intensively; see [55].

The notion of slant curve introduced in [29] has another motivation derived from classical differential geometry.

A spatial curve is said to be a *curve of constant slope* (also called a *cylindrical helix*) if its tangent vector field has a constant angle θ with a fixed direction called the *axis*. The second name is derived from the fact that there exists a cylinder in Euclidean 3-space on which the curve moves in such a way that it cuts each ruling at a constant angle (see [89, pp. 72–73]).

These curves are characterized by the following Bertrand-Lancret-de Saint Venant Theorem (see [71, 89, 97]):

Theorem 1.1 *An arc length parametrized curve in Euclidean 3-space $\mathbb{E}^3$ with nonzero curvature is of constant slope if and only if the ratio of the torsion τ and the curvature κ is constant.*

For a curve of constant slope with nonzero curvature, the ratio τ/κ is sometimes called the *Lancret invariant* of the curve of constant slope. Barros [3] generalized the above characterization due to Bertrand-Lancret-de Saint Venant to curves in 3-dimensional space forms. Motivated by the Bertrand-Lancret-de Saint Venant theorem, slant curves in almost contact metric manifolds are defined as follows:

Definition 1.1 An arc length parametrized curve γ in an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be *slant* if its tangent vector field makes constant *contact angle* θ with ξ, i.e., $\cos\theta := \eta(\gamma')$ is constant along γ.

Here we would like to point out that in 1995, Blair, Dillen, Verstraelen and Vrancken gave a variational characterization of slant curves in K-contact manifolds [11]. Slant curves are a critical point of the length functional under characteristic variations (Sect. 4.2).

Moreover, in 1963, Tashiro and Tachibana have studied special kinds of slant curves called C-loxodromes in Sasakian manifold [102]. For the precise definition of C-loxodrome, see Definition 4.2. It should be remarked that the class of slant curves is larger than the class of C-loxodromes in Sasakian manifolds. The C-loxodrome equation has unexpected relations to *static magnetism in contact geometry*. Based on these observations, in this chapter, we also study magnetic curves in almost contact metric manifolds (with Killing characteristic vector fields).

On a Riemannian manifold (M, g) equipped with a closed 2-form F, the 2-form F is regarded as a static magnetic field. Now let us temporarily assume that F has a potential 1-form A, i.e., $F = dA$. Then the critical points of the *Landau-Hall functional*:

$$\mathrm{LH}(\gamma) = E(\gamma) + q \int_0^L A(\gamma'(s))\, ds \tag{1.1}$$

are called *magnetic curves*. Here $E(\gamma)$ denotes the kinetic energy

$$E(\gamma) = \int_0^L \frac{1}{2} g(\gamma'(s), \gamma'(s))\, ds \tag{1.2}$$

of γ and q is a real constant called the *charge*. It should be remarked that every contact metric manifold $(M, \varphi, \xi, \eta, g)$ has a static magnetic field (called the contact magnetic field, see Theorem 7.1). We will see later that contact magnetic curves in Sasakian manifolds are slant curves. It will be turned out that contact magnetic curves in Sasakian space forms lie in 3-dimensional subspaces. We will study the periodicity of those magnetic curves in Sect. 12.

Slant curves in 3-dimensional Sasakian space forms appear also in another variational problem. In fact, biharmonic curves in 3-dimensional Sasakian space forms are slant.

An arc length parametrized curve γ in a Riemannian manifold (M, g) is said to be *biharmonic* if it is a critical point of the *bienergy functional*:

$$E_2(\gamma) = \int_0^L \frac{1}{2} |\nabla_{\gamma'}\gamma'|^2\, ds.$$

We will see that biharmonic curves in 3-dimensional Sasakian space forms are slant [30].

Throughout this chapter, we denote by $\Gamma(E)$ the space of all smooth sections of a vector bundle E.

Part I Slant Curves in Almost Contact Metric Manifolds

2 Preliminaries

2.1 Frenet Frame

Let (M, g) be an m-dimensional Riemannian manifold with Levi-Civita connection ∇. We denote by $\mathrm{O}(M)$ the orthonormal frame bundle of M.

An arc length parametrized curve γ in M is said to be a *Frenet curve of osculating order* $r \geq 1$ if there exists an orthonormal frame field $\mathcal{E} = (T = \gamma', E_1, \ldots, E_{r-1})$ of rank r along γ such that

$$\nabla_{\gamma'} T = \kappa_1 E_1, \tag{2.1}$$

$$\nabla_{\gamma'} E_1 = -\kappa_1 T + \kappa_2 E_2, \tag{2.2}$$

$$\nabla_{\gamma'} E_j = -\kappa_j E_{j-1} + \kappa_{j+1} E_{j+1}, \quad 2 \leq j \leq r-2, \tag{2.3}$$

$$\nabla_{\gamma'} E_{r-1} = -\kappa_{r-1} E_{r-2} \tag{2.4}$$

for some *non-negative* functions $\kappa_1, \kappa_2, \ldots \kappa_{r-1}$. Each κ_j is called the j-th *curvature*.

For example, a *geodesic* in (M, g) is a Frenet curve of osculating order 1, and a *circle* is a Frenet curve of osculating order 2 with constant first curvature κ_1. A *helix* of order r is defined as a Frenet curve of osculating order r, such that all curvatures $\kappa_1, \kappa_2, \ldots \kappa_{r-1}$ are constant.

For more information on Frenet curves, we refer to [98, Chap. 7, B.].

2.2 Vector Fields along Curves

Here we recall some vector bundle calculus of curves for our later use.

Let $\gamma(s)$ be an arc length parametrized curve in a Riemannian manifold (M, g) defined on an interval I. Then the vector bundle $\gamma^* TM$ is defined by

$$\gamma^* TM = \bigcup_{s \in I} T_{\gamma(s)} M.$$

A section $X \in \Gamma(\gamma^* TM)$ is called a vector field along γ. The Levi-Civita connection ∇ induces a connection ∇^γ on $\gamma^* TM$ by

$$\nabla^\gamma_{\frac{d}{ds}} X = \nabla_{\gamma'} X.$$

The covariant derivative $\nabla_{\gamma'} X$ is often denoted as $\dfrac{D}{ds} X(s)$.

One can see that $(\gamma^*TM, \gamma^*g, \nabla^\gamma)$ is a Riemannian vector bundle over I, i.e., $\nabla^\gamma(\gamma^*g) = 0$.

The *mean curvature vector field* H of γ is a section of γ^*TM defined by $H = \nabla_{\gamma'}\gamma'$. By definition, geodesics are arc length parametrized curves with vanishing mean curvature vector field.

The *Laplace-Beltrami operator* Δ of $(\gamma^*TM, \nabla^\gamma)$ is defined by

$$\Delta = -\nabla^\gamma_{\frac{d}{ds}}\nabla^\gamma_{\frac{d}{ds}} = -\nabla_{\gamma'}\nabla_{\gamma'}.$$

Thus, for any $X \in \Gamma(\gamma^*TM)$, we have

$$\Delta X = -\nabla_{\gamma'}\nabla_{\gamma'}X.$$

A vector field X along γ is said to be *proper* if it satisfies $\Delta X = \lambda X$ for some function λ. In particular, X is said to be *harmonic* if $\Delta X = 0$.

2.3 Normal Connection

The *normal bundle* $T^\perp\gamma$ of the curve γ is given by

$$T^\perp\gamma = \bigcup_{s\in I} T_s^\perp\gamma, \quad T_s^\perp\gamma = (\mathbb{R}T(s))^\perp.$$

The *normal connection* $\nabla^\perp$ is the connection in $T^\perp\gamma$ defined by

$$\nabla^\perp_{\gamma'}X = \nabla_{\gamma'}X - g(\nabla_{\gamma'}X, T)T$$

for any section $X \in \Gamma(T^\perp\gamma)$.

The Laplace-Beltrami operator $\Delta^\perp = -\nabla^\perp_{\gamma'}\nabla^\perp_{\gamma'}$ of the vector bundle $(T^\perp\gamma, \nabla^\perp)$ is called the *normal Laplacian*.

3 Almost Contact Manifolds

3.1 Compatible Metrics

Let M be a manifold of odd dimension $m = 2n + 1$. Then M is said to be an *almost contact manifold* if its structure group $\mathrm{GL}_m\mathbb{R}$ of the linear frame bundle is reducible to $\mathrm{U}(n) \times \{1\}$. This is equivalent to existence of a tensor field φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1.$$

Here I is the identity transformation. From these conditions, one can deduce that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

Moreover, since $\mathrm{U}(n) \times \{1\} \subset \mathrm{SO}(2n+1)$, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \mathfrak{X}(M)$. Here $\mathfrak{X}(M) = \Gamma(TM)$ denotes the Lie algebra of all smooth vector fields on M. Such a metric is called an *associated metric* of the almost contact manifold $M = (M, \varphi, \xi, \eta)$. With respect to the associated metric g, η is metrically dual to ξ, that is

$$g(X, \xi) = \eta(X)$$

for all $X \in \mathfrak{X}(M)$. A structure (φ, ξ, η, g) on M is called an *almost contact metric structure*, and a manifold M equipped with an almost contact metric structure is said to be an *almost contact metric manifold*.

The *fundamental* 2*-form* Φ of $(M, \varphi, \xi, \eta, g)$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

An almost contact metric manifold M is said to be of *rank* $r = 2s > 0$ if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$, and of *rank* $r = 2s + 1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$.

A plane section Π at a point x of an almost contact metric manifold M is said to be *holomorphic* if it is invariant under φ_x. The sectional curvature function $\mathcal{H}$ of a holomorphic plane section is called the *holomorphic sectional curvature* (also called φ-sectional curvature).

3.2 Contact Metric Manifolds

On the other hand, a 1-form η on $(2n+1)$-dimensional manifold M is said to be a *contact form* if $(d\eta)^n \wedge \eta \neq 0$ on M. A manifold M together with a contact form is called a *contact manifold*. On a contact manifold (M, η), there exists a unique vector field ξ such that

$$\eta(\xi) = 1, \quad \iota_\xi d\eta = 0.$$

Here ι_ξ denotes the interior product by ξ. The vector field ξ is called the *Reeb vector field*. Moreover, there exists a Riemannian metric g and an endomorphism field φ such that (φ, ξ, η, g) is almost contact metric structure and $\Phi = d\eta$. The structure (φ, ξ, η, g) is called an almost contact metric structure associated to the contact form η.

Conversely, let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold satisfying $\Phi = d\eta$. Then one can see that η is a contact form. Such an almost contact metric manifold is called a *contact metric manifold*.

Every contact metric manifold is orientable. Here we recall the following fundamental fact ([10, Theorem 4.6]).

Proposition 3.1 *On a* $(2n+1)$*-dimensional contact metric manifold* $(M, \varphi, \xi, \eta, g)$*, the volume element* dv_g *induced from the associated metric* g *is related to the contact form* η *by*

$$dv_g = \frac{(-1)^n}{2^n n!}\eta \wedge (d\eta)^n.$$

Note that contact metric manifolds are of rank $2n+1$.

Remark 1 Let (M, g) be an *oriented* m-dimensional Riemannian manifold, then we can take a positively oriented local orthonormal frame field $\{e_1, e_2, \ldots, e_m\}$ and its dual coframe field $\{\vartheta^1, \vartheta^2, \ldots, \vartheta^m\}$. Then $dv_g = \vartheta^1 \wedge \vartheta^2 \wedge \cdots \wedge \vartheta^m$ defines a volume element compatible to the orientation. By definition, $\{e_1, e_2, \ldots, e_m\}$ satisfies

$$dv_g(e_1, e_2, \ldots, e_m) = \frac{1}{m!}.$$

For every positively oriented local coordinate system $(x_1, x_2, \ldots, x_m)$, dv_g is expressed as

$$dv_g = \sqrt{\det(g_{ij})}\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m, \quad g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j).$$

3.3 Normality Tensor

On the direct product manifold $M \times \mathbb{R}$ of an almost contact metric manifold and the real line $\mathbb{R}$, any tangent vector field can be represented as the form $(X, \lambda d/dt)$, where $X \in \mathfrak{X}(M)$ and λ is a function on $M \times \mathbb{R}$ and t is the Cartesian coordinate on the real line $\mathbb{R}$.

Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J(X, \lambda d/dt) = (\varphi X - \lambda\xi, \eta(X)d/dt).$$

If J is integrable, then M is said to be *normal*. Equivalently, M is normal if and only if

$$[\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi = 0,$$

where $[\varphi, \varphi]$ is the *Nijenhuis torsion* of φ defined by

$$[\varphi, \varphi](X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

for any $X, Y \in \mathfrak{X}(M)$.

For more details on almost contact metric manifolds, we refer to Blair's monograph [10].

3.4 Examples of Almost Contact Metric Manifolds

From the viewpoint of slant curves, we need to look for almost contact metric manifolds with *Killing* characteristic vector field. On the other hand from magnetic curve theory, we need to search almost contact metric manifolds with *closed* fundamental 2-form.

Example 3.1 (*Almost cosymplectic manifold*) An almost contact metric manifold M is said to be an *almost cosymplectic manifold* if $d\eta = 0$ and $d\Phi = 0$ [84]. An almost cosymplectic manifold M is called a *cosymplectic manifold* if it is normal.

Proposition 3.2 *An almost contact metric manifold M is cosymplectic if and only if $\nabla\varphi = 0$.*

One can see that the characteristic vector field of a cosymplectic manifold is Killing. Complete connected cosymplectic manifolds of constant φ-sectional curvature are called *cosymplectic space forms*.

Example 3.2 (*Cosymplectic space forms*) Let $\overline{M} = (\overline{M}, \bar{g}, J)$ be an almost Kähler manifold. Consider the Riemannian product $M = (\overline{M} \times \mathbb{R}, g)$ with $g = \bar{g} + dt^2$. Then we can equip an almost cosymplectic structure of M by

$$\xi = \frac{d}{dt}, \quad \eta = dt, \quad \varphi\left(X, f\frac{d}{dt}\right) = (JX, 0), \quad X \in \mathfrak{X}(\overline{M}).$$

The almost cosymplectic manifold M is cosymplectic if and only if $\overline{M}$ is Kähler. In particular, when $\overline{M}$ is a complex space form, that is, a Kähler manifold of constant holomorphic sectional curvature, then M is a cosymplectic manifold of constant holomorphic sectional curvature. Now let $\mathbb{C}P_n(c)$, $\mathbb{C}^n$ and $\mathbb{C}H_n(c)$ be complex projective n-space of constant holomorphic sectional curvature $c > 0$, complex Euclidean n-space and complex hyperbolic n-space of constant holomorphic sectional curvature $c < 0$, respectively. Then the cosymplectic manifolds

$$\mathbb{C}P_n(c) \times \mathbb{R}, \quad \mathbb{E}^{2n+1} = \mathbb{C}^n \times \mathbb{R}, \quad \mathbb{C}H_n(c) \times \mathbb{R}$$

are cosymplectic space forms.

The Riemannian curvature of a cosymplectic space form (of constant φ-sectional curvature c) has the explicit representation

$$R(X,Y)Z = \frac{c}{4}(X \wedge Y)Z + \frac{c}{4}\{(\varphi X \wedge \varphi Y)Z + 2\Phi(X,Y)\varphi Z\} + \frac{c}{4}\{\xi \wedge (X \wedge Y)\xi\}Z.$$

Here the curvaturelike tensor field $(X \wedge Y)Z$ is defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(Z,X)Y. \tag{3.1}$$

Example 3.3 (*K-contact manifolds*) Let M be a contact metric manifold, then its fundamental 2-form Φ is exact (and hence closed). In fact, η is a potential of Φ. However, ξ is not necessarily Killing. A contact metric manifold M is said to be a *K-contact manifold* if its Reeb vector field is Killing.

Proposition 3.3 *An almost contact metric manifold M is a K-contact manifold if and only if $\nabla\xi = -\varphi$.*

Example 3.4 (*Sasakian manifold*) A normal contact metric manifold is called a *Sasakian manifold*. One can see that Sasakian manifolds are K-contact.

Proposition 3.4 *An almost contact metric manifold M is a Sasakian manifold if and only if*

$$(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$$

holds.

Remark 2 If $\dim M = 3$, then K-contact manifold M is Sasakian.

Proposition 3.5 *If a Sasakian manifold M has constant φ-sectional curvature $\mathcal{H}$, then its Riemannian curvature tensor has the following form*:

$$\begin{aligned} R(X,Y)Z = & \frac{c+3}{4}(X \wedge Y)Z + \frac{c-1}{4}\{(\varphi X \wedge \varphi Y)Z + 2\Phi(X,Y)\varphi Z\} \\ & + \frac{c-1}{4}\{\xi \wedge (X \wedge Y)\xi\}Z. \end{aligned}$$

Complete and connected Sasakian manifolds of constant φ-sectional curvature are called *Sasakian space forms*. The odd-dimensional unit sphere $\mathbb{S}^{2n+1} = \mathbb{S}^{2n+1}(1)$ is a typical example of simply connected Sasakian space form.

Example 3.5 (*The unit sphere*) Let $\mathbb{C}^{n+1}$ be the $(n+1)$-dimensional complex Euclidean space with complex structure J. Identifying $\mathbb{C}^{n+1}$ with $2n+2$-dimensional Euclidean space $\mathbb{E}^{2n+2}$ with metric $\langle \cdot, \cdot \rangle$ via the isomorphism

$$(z_1, z_2, \ldots, z_{n+1}) \longmapsto (x_1, y_1, x_2, y_2, \ldots, x_{n+1}, y_{n+1}), \quad z_k = x_k + \sqrt{-1}y_k, \quad k = 1, 2, \ldots, n+1.$$

Then J corresponds to the linear transformation:

$$(x_1, y_1, x_2, y_2, \ldots, x_{n+1}, y_{n+1}) \longmapsto (-y_1, x_1, -y_2, x_2, \ldots, -y_{n+1}, x_{n+1}).$$

On the unit sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$, we take the unit normal vector field $\boldsymbol{n}$ by $\boldsymbol{n} = \boldsymbol{x}$, where $\boldsymbol{x}$ is the position vector field. Then the Levi-Civita connection D of $\mathbb{C}^{n+1}$ and ∇ of $\mathbb{S}^{2n+1}$ is related by the *Gauss formula*:

$$D_X Y = \nabla_X Y - \langle X, Y \rangle \boldsymbol{n}. \tag{3.2}$$

Define the vector field ξ on $\mathbb{S}^{2n+1}$ by $\boldsymbol{\xi} = -\boldsymbol{J}\boldsymbol{n}$ and set $\eta = g(\xi, \cdot)$, where g is the metric of $\mathbb{S}^{2n+1}$ induced from $\langle \cdot, \cdot \rangle$. Then η is a contact form on $\mathbb{S}^{2n+1}$ and ξ is the Reeb vector field of η. The associated endomorphism field φ is the restriction of J to $\mathbb{S}^{2n+1}$. It is well known that $\mathbb{S}^{2n+1}$ equipped with this structure is a Sasakian manifold of constant curvature 1. The Hopf fibering $\pi : \mathbb{S}^{2n+1} \to \mathbb{C}P_n(4)$ is a Riemannian submersion with totally geodesic fibres. The fibering coincides with the Boothby-Wang fibering.

Example 3.6 (*Elliptic Sasakian space forms*) Let (φ, ξ, η, g) be the Sasakian structure of the unit sphere $\mathbb{S}^{2n+1}$. For any positive constant a, we deform the structure in the following way:

$$\hat{\varphi} := \varphi, \quad \hat{\eta} := a\eta, \quad \hat{\xi} := \frac{1}{a}\xi, \quad \hat{g} := ag + a(a-1)\eta \otimes \eta.$$

Then the new structure is Sasakian and of constant φ-sectional curvature $c = 4/a - 3$. We denote by $\mathcal{M}^{2n+1}(c)$ the resulting Sasakian space form $(\mathbb{S}^{2n+1}, \hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ and call it the *elliptic Sasakian space form*. The deformation above is called a $\mathcal{D}$*-homothetic deformation*. Under the deformation, the Hopf fibering becomes a Riemannian submersion $\pi : \mathcal{M}^{2n+1}(c) \to \mathbb{C}P_n(c+3)$ onto the complex projective space of constant holomorphic sectional curvature $c + 3$. The Levi-Civita connection $\hat{\nabla}$ of $\mathcal{M}^{2n+1}(c)$ is related to the original Levi-Civita connection ∇ by

$$\hat{\nabla}_X Y = \nabla_X Y + (a-1)(\eta(X)\varphi Y + \eta(Y)\varphi X). \tag{3.3}$$

Example 3.7 (*Heisenberg group*) On the Cartesian space $\mathbb{R}^{2n+1}$ with natural coordinates $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z)$, we define a contact form η and a Riemannian metric g by

$$\eta = dz + \sum_{i=1}^{n}(y_i dx_i - x_i dy_i), \quad g = \sum_{i=1}^{n}(dx_i^2 + dy_i^2) + \eta \otimes \eta.$$

The Reeb vector field is $\xi = \partial_z$. Define the endomorphism field φ by $d\eta(X, Y) = g(X, \varphi Y)$. Then the resulting contact Riemannian manifold $(\mathbb{R}^{2n+1}, \varphi, \xi, \eta, g)$ is a Sasakian space form of constant φ-sectional curvature -3. This Sasakian manifold admits a nilpotent Lie group structure (*Heisenberg group* Nil_{2n+1}). The Sasakian structure is left invariant with respect to the Lie group structure.

Example 3.8 (*Hyperbolic Sasakian space forms*) Let us identify the 1-dimensional complex hyperbolic space $\mathbb{C}H_1(-c^2)$ of constant holomorphic sectional curvature

$-c^2$ ($c > 0$) with the upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the Poincaré metric $(dx^2 + dy^2)/(c^2y^2)$ of constant curvature $-c^2$. On the product manifold $\mathbb{C}H_1(-c^2) \times \mathbb{R}$, we define a 1-form η_ν and a Riemannian metric g_ν by

$$\eta_\nu = dt + \frac{2\nu\, dx}{c^2 y}, \quad g_\nu = \frac{dx^2 + dy^2}{c^2 y^2} + \eta_\nu \otimes \eta_\nu, \quad \nu \in \mathbb{R}, \quad \nu \geq 0.$$

Define an endomorphism field φ and a vector field ξ by

$$\varphi \frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad \varphi \frac{\partial}{\partial y} = -\frac{\partial}{\partial x} + \frac{1}{2y}\frac{\partial}{\partial t}, \quad \varphi \frac{\partial}{\partial t} = 0, \quad \xi = \frac{\partial}{\partial t}.$$

Then $M = (\mathbb{C}H_1(-c^2) \times \mathbb{R}, \varphi, \xi, \eta_\nu, g_\nu)$ is an almost contact metric manifold. In particular, when $\nu = 1$, M is a Sasakian space form of constant φ-sectional curvature $-c^2 - 3 < -3$. On the other hand, for $\nu = 0$, we obtain the cosymplectic space form $\mathbb{C}H_1(-c^2) \times \mathbb{R}$. Moreover, when $\nu = c^2/2$, we obtain the *Sasaki-lift metric* of the universal covering of the unit tangent sphere bundle $U\mathbb{H}^2(-c^2)$ of the hyperbolic plane $\mathbb{H}^2(-c^2)$ of curvature $-c^2$ (*cf.* Sect. 5.7).

The simply connected Sasakian space forms of constant φ-sectional curvature c are classified as follows:

- The $\mathcal{D}$-homothetic deformation $\mathcal{M}^{2n+1}(c)$ of $\mathbb{S}^{2n+1}$ $(\mathbb{S}^{2n+1}, \hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$ if $c > -3$ and $c \neq 1$.
- The unit sphere $\mathbb{S}^{2n+1}$ if $c = 1$.
- The Heisenberg group Nil_{2n+1} if $c = -3$.
- The product manifold $\mathbb{C}H_n(c + 3) \times \mathbb{R}$ equipped with a Sasakian structure if $c < -3$.

Blair introduced the notion of quasi-Sasakian manifold. The class of quasi-Sasakian manifold includes both Sasakian manifolds and cosymplectic manifolds.

Definition 3.1 An almost contact metric manifold M is said to be a *quasi-Sasakian manifold* if it is normal and $d\Phi = 0$.

The characteristic vector field of quasi-Sasakian manifolds is Killing. Sasakian manifolds are quasi-Sasakian manifolds of rank $2n + 1$. Cosymplectic manifolds are quasi-Sasakian manifolds of rank 0.

Remark 3 An almost contact metric manifold M is said to be [10]

- *nearly Sasakian* if $(\nabla_X \varphi)Y + (\nabla_Y \varphi)X = 2g(X, Y)\xi - \eta(X)Y - \eta(Y)X$ for all $X, Y \in \mathfrak{X}(M)$;
- *nearly cosymplectic* if $(\nabla_X \varphi)X = 0$ for all $X \in \mathfrak{X}(M)$.

The characteristic vector fields of nearly Sasakian manifolds and nearly cosymplectic manifolds are Killing. In case $\dim M = 3$, nearly Sasakian manifolds are automatically Sasakian. Analogously, every nearly cosymplectic 3-manifold is cosymplectic (see [67]). In [28], Chinea and González-Dávila displayed a Gray-Hervella-type classification for almost contact metric structures. Their classification is based on the decomposition of the space possible intrinsic torsions into irreducible $\mathrm{U}(n)$-modules. There are potentially 2^{12} classes. Martín Cabrera [75] showed the non-existence of 132 classes in the Chinea-González-Dávila classification in case $\dim M > 3$. On the other hand, De Nicola, Dileo and Yudin showed that nearly Sasakian manifolds of dimension greater than 5 are Sasakian [82]. In addition, they showed that non-normal nearly cosymplectic manifolds of dimension greater than 5 are locally isomorphic to a product

$$N^{2n} \times \mathbb{R}, \quad M^5 \times N^{2n-4}$$

where N^{2n} is a non-Kähler nearly Kähler manifold, N^{2n-4} is a nearly Kähler manifold, and M^5 is a non-cosymplectic nearly cosymplectic manifold. If one makes the further assumption that the manifold is complete and simply connected, then the isometry becomes global.

To end this section, we mention also Kenmotsu manifolds.

Definition 3.2 An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be an *almost Kenmotsu manifold* if it satisfies $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$.

An almost Kenmotsu manifold M is called a *Kenmotsu manifold* if it is normal. On a Kenmotsu manifold, we have

$$(\nabla_X \varphi)Y = -\Phi(X, Y)\xi - \eta(Y)\varphi X, \quad \nabla_X \xi = X - \eta(X)\xi.$$

The fundamental 2-form on a Kenmotsu manifold is non-closed and the characteristic vector field is non-Killing.

The odd-dimensional hyperbolic space $\mathbb{H}^{2n+1} = \mathbb{H}^{2n+1}(-1)$ of constant curvature -1 is a typical example of Kenmotsu manifold. Kenmotsu showed that every Kenmotsu manifold of constant φ-sectional curvature is of constant curvature -1.

4 Curves in Almost Contact Metric Manifolds

4.1 *φ-Torsions*

A Frenet curve of order $r \geq 3$ in an almost contact metric manifold M is called a *φ-curve* if the space spanned by $T, E_1, \ldots, E_{r-1}$ is φ-invariant. A Frenet curve of order 2 is called a φ-curve if $\{T, E_1, \xi\}$ is a φ-invariant space. Furthermore, a φ-helix of order r is defined as a φ-curve of osculating order r, such that all the curvatures are constant. The *φ-torsions* of γ are defined by $\tau_{ij} = g(E_i, \varphi E_j)$, $(0 \leq i < j \leq r-1)$.

4.2 Slant Curves

Let $\gamma(s)$ be an arc length parametrized curve of length L in an almost contact metric manifold M. Then we consider a variation γ_t through γ of the form:

$$\gamma_t(s) = \exp_{\gamma(s)}(tf(s)\xi_{\gamma(s)}) \tag{4.1}$$

for some function $f(s)$ satisfying $f(0) = f(L) = 0$. Such a variation is called a *characteristic variation* (or ξ-deformation [11]). The first variation of the length functional

$$\mathcal{L}(\gamma) = \int_0^L \sqrt{g(\gamma'(s), \gamma'(s))}\, ds$$

through the characteristic variations was obtained by [11]:

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(\gamma_t) = -\int_0^L f(s)\, g(\xi, \nabla_{\gamma'}\gamma')\, ds.$$

Now let us assume that the characteristic vector field ξ is a *Killing vector field*, then the first variation formula becomes

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(\gamma_t) = \int_0^L f(s) \left(\frac{d}{ds}\cos\theta(s)\right) ds.$$

Here θ is the contact angle as before. Slant curves in almost contact metric manifolds have the following variational characterization:

Theorem 4.1 ([11]) *Let M be an almost contact metric manifold with Killing characteristic vector field ξ. Then an arc length parametrized curve γ is a critical point of the length function through characteristic variations if and only if γ is a slant curve.*

Now we look for examples of slant curves.

Example 4.1 (*Geodesics*) Assume that ξ is a Killing vector field. Then any geodesic $\gamma(s)$ in M satisfies

$$\cos\theta(s) = g(\xi, \gamma'(s)) = \text{constant}$$

because of the so-called *conservation lemma* below [90, p. 252].

Lemma 4.1 *Let (M, g) be a Riemannian manifold and ξ a Killing vector field. Then for any geodesic γ, the restriction $\xi|_\gamma$ is a Jacobi field and $g(\gamma', \xi)$ is constant along γ.*

This observation motivates us to look for slant curves among curves which are generalizations of geodesics. In [102], Tashiro and Tachibana introduced the notion of C-loxodrome.

Example 4.2 An arc length parametrized curve γ in an almost contact metric manifold M with *Killing* characteristic vector field ξ is said to be a *C-loxodrome* if it satisfies

$$\nabla_{\gamma'}\gamma' = r\eta(\gamma')\varphi\gamma'. \tag{4.2}$$

Here r is a constant. One can see that every C-loxodrome is a slant curve.

It should be remarked that the notion of C-loxodrome is not identical to that of slant curve. In fact, if a C-loxodrome γ has constant contact angle $\pi/2$, namely, γ is a Legendre curve, then γ should be a Legendre geodesic. Thus, the class of slant curves is strictly larger than that of C-loxodromes on almost contact metric manifolds with Killing characteristic vector field.

Yanamoto [109] investigated C-loxodromes in the unit 3-sphere $\mathbb{S}^3$ equipped with canonical Sasakian structure.

Remark 4 A diffeomorphism f on a K-contact manifold is said to be a *CL-transformation* if it carries C-loxodromes to C-loxodromes [102]. Takamatsu and Mizusawa studied infinitesimal CL-transformations on compact Sasakian manifolds [99]. As an analogue of Weyl's conformal curvature tensor field, Koto and Nagao introduced CL-curvature tensor field for Sasakian manifolds. The CL-curvature tensor field is invariant under CL-transformations. Koto and Nagao showed that Sasakian space forms are characterized as Sasakian manifolds with vanishing CL-curvature tensor fields [70].

Next, we point out that C-loxodromes may be regarded as generalizations of geodesics. In fact, geodesics are C-loxodromes with $r = 0$. The characteristic flows are C-loxodromes with $\theta = 0$. Examples 4.1 and 4.2 suggest us to look for slant curves which are generalizations of geodesics. The C-loxodrome equation has unexpected relations to *static magnetism in contact geometry*. In the next section, we discuss static magnetism of almost contact manifolds.

Remark 5 (*Stability*) Any slant curve in a K-contact manifold is l-stable for some l in the sense of [26]. In particular, every Legendre curve is 0-stable (see [11]).

5 Magnetic Curves

5.1 Lorentz Equation

Magnetic curves represent, in physics, the trajectories of the charged particles moving on a Riemannian manifold under the action of magnetic fields. A *magnetic field* F on a Riemannian manifold (M, g) is a closed 2-form and the *Lorentz force* associated to F is an endomorphism field L defined by

$$g(LX, Y) = F(X, Y), \quad X, Y \in \Gamma(TM).$$

The magnetic trajectories of F are curves γ satisfying the *Lorentz equation*:

$$\nabla_{\gamma'}\gamma' = qL\gamma'. \tag{5.1}$$

Here q is a constant (called the *charge* or the *strength*). One can see that every magnetic trajectory has constant speed. Unit speed magnetic curves are called *normal magnetic curves*. A magnetic field F is said to be *uniform* if it is parallel. The study of magnetic curves in arbitrary Riemannian manifolds was further developed mostly in the early 1990s, even though related pioneer works were published much earlier. We can refer to Arnold's problems concerning charges in magnetic fields on Riemannian manifolds of arbitrary dimension, commented by Ginzburg in [46], and references therein. For more information on magnetic curves in Riemannian manifolds, we refer to [4–6].

5.2 *Landau-Hall Functional*

As we have mentioned in "Introduction", magnetic curves have variational characterization. Assume that a magnetic field F has a potential A, that is $F = 2dA$. Then the *Landau-Hall functional* LH is defined by

$$\mathrm{LH}(\gamma) = E(\gamma) + q\int_0^L A(\gamma'(s))\,ds,$$

where $E(\gamma)$ is the kinetic energy (1.2).

The Euler-Lagrange equation of the Landau-Hall functional is nothing but the Lorentz equation. It should be remarked that Lorentz equation (5.1) itself does not require the exactness of a magnetic field F.

Remark 6 (*Magnetic maps*) The Landau-Hall functional can be generalized to smooth maps between Riemannian manifolds. Let $f : N \to M$ be a smooth map between two Riemannian manifolds (N, h) and (M, g), ξ a divergence free vector field on N and F a magnetic field on M with potential A. Then the Landau-Hall functional LH is defined by

$$\mathrm{LH}(f) = \int_N \frac{1}{2}|df|^2 dv_h + q\int_N A(df(\xi))\,dv_h.$$

A critical point of this functional is called a *magnetic map* with charge q [57]. Magnetic maps appear in many branches of differential geometry; see [57, 58, 60].

5.3 Contact Magnetic Curves

Now let us consider magnetic curves in an almost contact metric manifold M with *closed* fundamental 2-form Φ. Then we choose $-\Phi$ as a magnetic field on M. We call $-\Phi$ the *contact magnetic field* of M. The Lorentz force L associated to $-\Phi$ is φ. Hence, the Lorentz equation becomes

$$\nabla_{\gamma'}\gamma' = q\varphi\gamma'. \tag{5.2}$$

As we have seen before, there exist several classes of almost contact manifolds with Killing characteristic vector fields and closed fundamental 2-form, e.g., quasi-Sasakian manifolds, K-contact manifolds, etc. Although the following fact is a direct consequence of the conservation lemma, it plays a fundamental role in slant curve geometry.

Proposition 5.1 *Let γ be a normal contact magnetic curve in an almost contact metric manifold M with Killing characteristic vector field. Then γ is a slant curve.*

Remark 7 (*C-loxodromes*) From Eq. (4.2), one can see that every C-loxodrome in a K-contact manifold is a contact magnetic curve with charge $q = r\cos\theta$.

In the next two subsections, we study contact magnetic curves in cosymplectic manifolds and Sasakian manifolds.

5.4 Magnetic Curves in Cosymplectic Manifolds

Let M be a cosymplectic manifold. Then its contact magnetic field $F = -\Phi$ is a uniform magnetic field. Contact magnetic curves in a cosymplectic manifold are classified as follows:

Theorem 5.1 ([38]) *Let M be a cosymplectic manifold and γ be a normal magnetic curve with charge q under the uniform magnetic field $-\Phi$. Then γ is a slant curve given by one of the following cases*:

(1) geodesics, obtained as integral curves of ξ;
(2) Legendre φ-circles of first curvature $\kappa_1 = |q|$;
(3) φ-helices of order 3 with first curvature $\kappa_1 = |q|\sin\theta$ and second curvature $\kappa_2 = |q\cos\theta|$ and such that $\mathrm{sgn}(\tau_{01}) = -\mathrm{sgn}(q)$ and $\theta \neq \pi/2$.

Proof In case (1), when the magnetic curve γ is a geodesic, from the Lorentz equation, it follows that $\varphi\gamma' = 0$, thus γ' is parallel to ξ. Using the fact that both of them are unit vector fields, it follows that $\gamma' = \pm\xi$, yielding that γ is an integral curve of ξ. In the sequel, we consider the non-geodesic magnetic curves, which are Frenet curves of order $r > 1$. Combining the first Frenet formula and the Lorentz equation, we have $\kappa_1 E_1 = q\varphi T$ and it follows that the first curvature is $\kappa_1 = |q| \sin\theta$. Differentiating $\kappa_1 E_1 = q\varphi T$ along γ, we have $\sin\theta\, \nabla_{\gamma'} E_1 = |q|(-T + \cos\theta\xi)$ because of $\nabla\varphi = 0$. Thus, $\nabla_{\gamma'} E_1$ is collinear to T if and only if $\theta = \pi/2$, case when γ is a Legendre circle of first curvature $\kappa_1 = |q|$. Hence, (2) is proved.

Next we suppose that $\theta \neq \pi/2$. Using $\sin\theta\, \nabla_{\gamma'} E_1 = |q|(-T + \cos\theta\xi)$ together with the Frenet formula, we get

$$|q| \cos\theta(\xi - \cos\theta\, T) = \kappa_2 \sin\theta\, E_2$$

and hence $\kappa_2 = |q \cos\theta|$. The characteristic vector field ξ is expressed as $\xi = \cos\theta\, T + \varepsilon \sin\theta\, E_2$, where $\varepsilon = \mathrm{sgn}(\cos\theta)$. Subsequently,

$$\varphi E_2 = -\mathrm{sgn}(q)\varepsilon \cos\theta\, E_1, \quad \eta(E_2) = \varepsilon \sin\theta.$$

From these results, we obtain

$$\varphi E_1 = \mathrm{sgn}(q)(-\sin\theta\, T + \varepsilon \cos\theta\, E_2).$$

Computing the φ-torsion $\tau_{01} = g(T, \varphi E_1) = -\mathrm{sgn}(q) \sin\theta$, we immediately get that $\mathrm{sgn}(\tau_{01}) = -\mathrm{sgn}(q)$. Finally, from equation $\varphi E_1 = \mathrm{sgn}(q)(-\sin\theta\, T + \varepsilon \cos\theta\, E_2)$, we deduce that $\nabla_{\gamma'} E_2 = -\kappa_2 E_1$. Thus $\kappa_3 = 0$. We conclude that the normal magnetic curves are Frenet curves of osculating order 3, with constant curvatures $\kappa_1 = |q| \sin\theta$ and $\kappa_2 = |q \cos\theta|$. Hence item (3) is shown. □

Now let us investigate contact magnetic curves in cosymplectic space forms. Let $\overline{M}_n(c)$ be a complex space form of constant holomorphic sectional curvature c. Then the Riemannian product $M = \overline{M}_n(c) \times \mathbb{R}$ admits a cosymplectic structure.

Theorem 5.2 (Codimension reduction theorem [38]) *Let γ be a normal contact magnetic curve on the cosymplectic space form $M = \overline{M}_n(c) \times \mathbb{R}$. Then γ is a normal contact magnetic curve on a subspace $\overline{M}_1(c) \times \mathbb{R} \subset M$, where $\overline{M}_1(c)$ is a complex 1-dimensional complex space form of constant holomorphic sectional curvature c which is a totally geodesic complex submanifold of $\overline{M}_n(c)$.*

We will study contact magnetic curves in $\overline{M}_1(c) \times \mathbb{R}$ in Sects. 10.3 and 10.4.

5.5 *Magnetic Curves in Sasakian Manifolds*

Now let M be a Sasakian manifold. Then its contact magnetic field $F = -\Phi$ is an *exact* magnetic field with magnetic potential $-\eta/2$. Contact magnetic curves in a Sasakian manifold are classified as follows:

Theorem 5.3 ([37]) *Let M be a Sasakian manifold and γ be a normal magnetic curve with charge q under the contact magnetic field $F = -d\eta$. Then γ is a slant curve given by one of the following cases*:

(a) *geodesics obtained as integral curves of* ξ*;*
(b) *non-geodesic* φ*-circles of curvature* $\kappa_1 = \sqrt{q^2 - 1}$ *for* $|q| > 1$ *and of constant contact angle* $\theta = \arccos \frac{1}{q}$*;*
(c) *Legendre* φ*-curves in* M *with curvatures* $\kappa_1 = |q|$ *and* $\kappa_2 = 1$*;*
(d) φ*-helices of order* 3 *with axis* ξ*, having curvatures* $\kappa_1 = |q| \sin\theta$ *and* $\kappa_2 = |q\cos\theta - 1|$*, where* $\theta \neq \frac{\pi}{2}$ *is the constant contact angle.*

Remark 8 For an arbitrary φ-helix of order 3 in a Sasakian manifold, not all φ-torsions are constant, hence a φ-helix is not necessary a magnetic curve. Yet, if the contact angle is constant, or equivalently $\tau_{02} = 0$, then the three φ-torsions are constant. Consequently,

$$\kappa_1\tau_{12} - \kappa_2\tau_{01} + g(\nu_2, \xi) = 0 \text{ and } \tau_{01}^2 + \tau_{12}^2 = 1.$$

It follows that the φ-helix is a magnetic curve with the strength $q = -\frac{\kappa_1}{\tau_{01}}$ and the contact angle is given by $\cos\theta = \frac{\tau_{01}\tau_{12}}{\kappa_2\tau_{01} - \kappa_1\tau_{12}}$. In particular, if τ_{12} vanishes, then the magnetic curve becomes the Legendre φ-curve stated at item (c) of Theorem 5.3.

From the proof of Theorem 5.3, given in [37], we may infer an interesting result.

Proposition 5.2 *Let* γ *be a non-geodesic Legendre* φ*-curve of order* 3 *in a Sasakian manifold. Then* $\kappa_2 = 1$ *and* $E_2 = \pm\xi$*.*

This statement generalizes [10, Proposition 8.2, p. 133].

Let us start our investigation of contact magnetic curves in Sasakian space forms of constant φ-sectional curvature $c > -3$. We show now that the study of trajectories associated to contact magnetic fields on Sasakian space forms with $c > -3$ reduces to their study on $\mathbb{S}^{2n+1}$.

Let $\mathcal{M}^{2n+1}(c)$ be an elliptic Sasakian space form given in Example 3.6, then the Levi-Civita connection $\hat{\nabla}$ of $\mathcal{M}^{2n+1}(c)$ is given by (3.3). The corresponding fundamental 2-form $\hat{\Phi}$ is given by $\hat{\Phi} = a\Phi$.

Let $\gamma(s)$ be a magnetic trajectory parametrized by arc length in $\mathbb{S}^{2n+1}$:

$$\nabla_{\gamma'}\gamma' = q\varphi\gamma'.$$

We study the $\mathcal{D}$-homothetic image of γ. We know that γ has constant contact angle θ. To distinguish arc length parameter of γ and that of $\mathcal{D}$-homothetic image, we denote the derivative with respect to s by dot. Since

$$\hat{g}(\dot{\gamma}, \dot{\gamma}) = a + a(a-1)\eta(\dot{\gamma})^2 = a(\sin^2\theta + a\cos^2\theta),$$

the arc length parameter $\hat{s}$ of γ with respect to $\hat{g}$ is

$$\hat{s} = ms, \text{ when } m = \sqrt{a(\sin^2\theta + a\cos^2\theta)},$$

and hence $\hat{\nabla}_{\gamma'}\gamma' = \frac{1}{m^2}\hat{\nabla}_{\dot\gamma}\dot\gamma$, where γ' denotes the derivative of γ with respect to $\hat{s}$. It follows that

$$\hat{\nabla}_{\gamma'}\gamma' = \hat{q}\varphi\gamma', \text{ where } \hat{q} = \frac{1}{m}\big(q + 2(a-1)\cos\theta\big).$$

This formula shows that $\gamma(\hat{s})$ is a magnetic trajectory for the contact magnetic field $-\hat{\Phi}$ with charge $\hat{q}$ in $\mathcal{M}^{2n+1}(c) = (\mathbb{S}^{2n+1}, \varphi, \hat{\xi}, \hat{\eta}, \hat{g})$. The contact angle $\hat{\theta}$ of $\gamma(\hat{s})$ in $\mathcal{M}^{2n+1}(c)$ is given by $\cos\hat{\theta} = \frac{a}{m}\cos\theta$.

Conversely, let $\gamma(\hat{s})$ be a normal magnetic trajectory in $\mathcal{M}^{2n+1}(c)$ with respect to the contact magnetic field $\hat{\Phi}$ with charge $\hat{q}$, namely, $\gamma(\hat{s})$ satisfies

$$\hat{\nabla}_{\gamma'}\gamma' = \hat{q}\,\varphi\gamma'.$$

Then we have $\nabla_{\dot\gamma}\dot\gamma = q\,\varphi\dot\gamma$, with arc length parameter $s = \hat{m}\hat{s}$ and strength q, where

$$\hat{m} = \frac{1}{a}\sqrt{a\sin^2\hat{\theta} + \cos^2\hat{\theta}}, \quad q = \frac{1}{\hat{m}}\left(\hat{q} - \frac{2(a-1)}{a}\cos\hat{\theta}\right).$$

As like in the case of cosymplectic space forms, the following codimension reduction theorem holds for $\mathbb{S}^{2n+1}$.

Theorem 5.4 (Codimension reduction theorem [37, 80]) *Let γ be a normal magnetic curve on the Sasakian sphere $\mathbb{S}^{2n+1}$ with respect to the contact magnetic field $-\Phi$. Then γ is a normal magnetic curve on a 3-dimensional unit sphere $\mathbb{S}^3$, embedded as a Sasakian totally geodesic submanifold in $\mathbb{S}^{2n+1}$.*

Theorem 5.5 (Codimension reduction theorem [37]) *Let γ be a normal magnetic curve on the Heisenberg group Nil_{2n+1} with respect to the contact magnetic field $-\Phi$. Then γ is a normal magnetic curve on a 3-dimensional Heisenberg group Nil_3, embedded as a Sasakian totally geodesic submanifold in Nil_{2n+1}.*

Theorem 5.6 (Codimension reduction theorem [37]) *Let γ be a normal magnetic curve on the $\mathbb{C}H_n(c+3)\times\mathbb{R}$ with respect to the contact magnetic field $-\Phi$. Then γ is a normal magnetic curve on a 3-dimensional Sasakian space form $\mathbb{C}H_1(c+3)\times\mathbb{R}$, embedded as a Sasakian totally geodesic submanifold in $\mathbb{C}H_n(c+3)\times\mathbb{R}$.*

Note that $\mathbb{C}H_1(c+3)\times\mathbb{R}$ is isomorphic to the universal covering group $\widetilde{\mathrm{SL}_2\mathbb{R}}$ of the special linear group $\mathrm{SL}_2\mathbb{R}$ equipped with a left invariant Sasakian structure (see Sect. 12.2).

5.6 Magnetic Curves in Quasi-Sasakian Manifolds

In [66, 79] Jleli, Munteanu and Nistor investigated contact magnetic curves in certain quasi-Sasakian manifolds.

On the Cartesian $2(n+p)+1$-space $\mathbb{R}^{2(n+p)+1}$ with global coordinates (x_i, y_i, z, u_j, v_j), $(1 \leq i \leq n,\ 1 \leq j \leq p)$, we equip a Riemannian metric

$$g = \sum_{i=1}^{n}(dx_i^2 + dy_i^2) + \sum_{j=1}^{p}(du_j^2 + dv_j^2) + \eta \otimes \eta, \quad \eta = dz - 2\sum_{i=1}^{n} y_i dx_i$$

and an endomorphism field φ by

$$\varphi\frac{\partial}{\partial x_i} = -\frac{\partial}{\partial y_i},\ \varphi\frac{\partial}{\partial y_i} = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial z},\ \varphi\frac{\partial}{\partial z} = 0,\ \varphi\frac{\partial}{\partial u_j} = \frac{\partial}{\partial v_j},\ \varphi\frac{\partial}{\partial v_j} = -\frac{\partial}{\partial u_j}.$$

Then the structure (φ, ξ, η, g) is a quasi-Sasakian manifold with closed fundamental 2-form

$$\Phi = 2\left(\sum_{i=1}^{n} dx_i \wedge dy_i + \sum_{j=1}^{p} du_j \wedge dv_j\right).$$

In particular, $M = (\mathbb{R}^{2(n+p)+1}, \varphi, \xi, \eta, g)$ is Sasakian when $p = 0$ and cosymplectic when $n = 0$.

Theorem 5.7 *Every normal magnetic curve is a slant helix of maximum order* 5.

Example 5.1 The curve parametrized by

$$\gamma(s) = \left(\sin\frac{s}{2}, \cos\frac{s}{2}, \frac{1}{2}\sin s, \sqrt{2}\sin\frac{s}{2}, \sqrt{2}\cos\frac{s}{2}\right)$$

is a magnetic helix of order 3 in the quasi-Sasakian $\mathbb{R}^5$ with $\theta = \pi/3$ and charge $q = 1/2$.

5.7 *Magnetic Curves in the Unit Tangent Sphere Bundles*

Let (M, g) be a Riemannian manifold, then, as is well known, its unit tangent sphere bundle UM inherits an almost contact Riemannian structure from the almost Kähler structure of the tangent bundle TM. Denote by $\bar{g}$ the Riemannian metric on UM induced from the Sasaki-lift metric of TM. Then the 1-form η dual to the geodesic spray ξ of UM is a contact form. These structure tensor fields together with the restriction φ of the almost complex structure of TM to UM define an almost contact metric structure $(\varphi, \xi, \eta, \bar{g})$ on UM. The fundamental 2-form Φ satisfies $2d\eta = \Phi$. Thus, $-\Phi$ gives a magnetic field on UM. It should be remarked that ξ is Killing if and only if M is of constant curvature 1 [10, p. 136].

Represent an arc length parametrized curve $\gamma(s)$ in UM as $\gamma(s) = (\underline{\gamma}(s); V(s))$. Here $\underline{\gamma}(s)$ is a curve in M and $V(s)$ is a unit vector field along $\underline{\gamma}(s)$. For simplicity, we restrict our attention to the case M is of constant curvature c. Then the Lorentz equation for γ has the following form:

$$\nabla_{\underline{\gamma'}}\underline{\gamma'} + c\, g(\underline{\gamma'}, \nabla_{\underline{\gamma'}} V) + (q - c\, \cos\theta)\nabla_{\underline{\gamma'}} V = 0,$$
$$\nabla_{\underline{\gamma'}}\nabla_{\underline{\gamma'}} V + (|\nabla_{\underline{\gamma'}} V|^2 + q\cos\theta)V = q\underline{\gamma'}.$$

The contact angle $\theta(s)$ is computed as

$$\cos\theta(s) = \eta(\gamma'(s)) = g_{\underline{\gamma}(s)}(\underline{\gamma'}(s), V(s)).$$

The derivative of the contact angle is given by

$$\frac{d}{ds}\cos\theta = (1-c)g(\underline{\gamma'}, \nabla_{\underline{\gamma'}} V).$$

Thus, we obtain the following:

Proposition 5.3 ([62]) *Every arc length parametrized contact magnetic curve in* $U\mathbb{S}^m$ $(m \geq 2)$ *is a slant curve.*

For more information on magnetic curves in unit tangent sphere bundles, we refer to [62].

Part II Slant Curves in 3-Dimensional Spaces

6 Curve Theory in 3-Dimensional Oriented Riemannian Manifolds

6.1 Vector Product

Let us concentrate on 3-dimensional oriented Riemannian manifold (M, g, dv_g). The volume element dv_g defines the *vector product operation* (also called the *cross product*) $\times$ on each tangent space T_xM by the rule

$$g(X \times Y, Z) = 3!\, dv_g(X, Y, Z), \quad X, Y, Z \in T_xM.$$

For any positively oriented local orthonormal frame field $\{e_1, e_2, e_3\}$, we have

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2.$$

By elementary linear algebra, we have

$$(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y = Z \times (X \times Y).$$

6.2 Curves in 3-Manifolds

Now let $\gamma(s)$ be a unit curve in the oriented Riemannian 3-manifold (M^3, g, dv) with non-vanishing acceleration $\nabla_{\gamma'}\gamma'$. Then we put $\kappa := |\nabla_{\gamma'}\gamma'|$. We can take a unit normal vector field N by the formula $\nabla_{\gamma'}\gamma' = \kappa N$. Next define a unit vector field B by $B = T \times N$. Here $T = \gamma'$. In this way, we obtain an orthonormal frame field $\mathcal{F} = (T, N, B)$ along γ which is *positively oriented*, that is, $dv_g(T, N, B) > 0$. The orthonormal frame field $\mathcal{F}$ is called the *Frenet frame field* and satisfies

$$\nabla_{\gamma'}\mathcal{F} = \mathcal{F}\begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \tag{6.1}$$

for some function τ. The functions κ and τ are called the *curvature* and *torsion* of γ, respectively. By definition $\mathcal{F}$ is a section of $\gamma^*\mathrm{SO}(M)$. Here $\mathrm{SO}(M)$ is a positive orthonormal frame bundle of M and $\gamma^*\mathrm{SO}(M)$ is the principal SO(3) bundle over γ obtained by pulling back $\mathrm{SO}(M)$ by γ.

The ordinary differential equation (6.1) is called the *Frenet-Serret formula* of γ. The unit vector fields T, N and B are called the *unit tangent vector field*, *principal normal vector field* and *binormal vector field* of γ, respectively.

Put $E_1 = N$ and $\kappa_1 = \kappa$. Then we obtain an orthonormal frame field $\mathcal{E} = (T, E_1, E_2)$ as in Sect. 2.1. The Frenet frame field $\mathcal{F}$ is related to $\mathcal{E}$ by $E_2 = \epsilon B$ and $\kappa_2 = \epsilon\tau$ with $\epsilon = \pm 1$. Here ϵ is determined by the formula $\epsilon = 3!dv_g(T, E_1, E_2)$. It represents the sign of τ.

Fundamental theorems of curve theory in (M^3, g, dv_g) are formulated as follows.

Theorem 6.1 (uniqueness theorem) *Let $\gamma_1, \gamma_2 : I \to M$ be arc length parametrized curves in an oriented Riemannian 3-manifold (M^3, g, dv_g) with curvatures and torsions (κ_1, τ_1), (κ_2, τ_2), respectively. Then γ_1 is congruent to γ_2 under orientation preserving isometries if and only if $(\kappa_1, \tau_1) = (\kappa_2, \tau_2)$.*

Theorem 6.2 (existence theorem) *Let $\kappa(s) > 0$ and $\tau(s)$ be smooth functions defined on an interval I. Then there exists an arc length parametrized curve $\gamma : I \to M$ in an oriented Riemannian 3-manifold (M, g, dv_g) with curvature κ and torsion τ.*

Based on these fundamental theorems, it is natural to take positive orthonormal frame fields along arc length parametrized curves in *oriented* Riemannian 3-manifolds.

Throughout this chapter, we take positive orthonormal frame field for arc length parametrized curves in oriented Riemannian 3-manifolds.

Lemma 6.1 *Let (M, g) be an oriented Riemannian 3-manifold and γ a non-geodesic arc length parametrized curve. Then we have*

$$\nabla_{\gamma'}H = -\kappa^2 T + \kappa' N + \kappa\tau B,$$

$$\nabla_{\gamma'}\nabla_{\gamma'}H = -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B.$$

Thus, Lemma 6.1 implies the following fundamental result.

Proposition 6.1 *Let γ be an arc length parametrized curve in an oriented Riemannian 3-manifold (M, g). Then γ has proper mean curvature vector field $(\Delta H = \lambda H)$ if and only if γ is a geodesic $(\lambda = 0)$ or a helix satisfying $\kappa^2 + \tau^2 = \lambda$.*

With respect to the normal Laplacian, we have the following fact.

Proposition 6.2 ([43, 50]) *Let γ be a non-geodesic arc length parametrized curve in an oriented Riemannian 3-manifold (M, g). Then γ satisfies $\Delta^{\perp} H = \lambda H$ if and only if γ is a geodesic $(\lambda = 0)$ or*

$$\kappa'' + \kappa(\lambda - \tau^2) = 0, \quad 2\tau\kappa' + \tau'\kappa = 0.$$

7 Magnetic Curves in 3-Manifolds

The relation between geometry and magnetic fields has a long history. As is well known, the notion of linking number can be traced back to Gauss' work on terrestrial magnetism (see [92]). The linking number connects topology and Ampère's law in magnetism. De Turck and Gluck studied magnetic curves and linking numbers in the 3-sphere $\mathbb{S}^3$ and hyperbolic 3-space $\mathbb{H}^3$ [34, 35]. In this section, we concentrate on magnetic curves in 3-dimensional Riemannian manifolds, especially on almost contact Riemannian manifolds with Killing characteristic vector field.

7.1 Magnetic Fields in Dimension 3

The dimension 3 is rather special in magnetic curve geometry, since it allows us to identify 2-forms with vector fields via the Hodge star operator and the volume element dv_g of the 3-dimensional oriented Riemannian manifold. More precisely, let us denote by $\sharp : T^*M \to TM$ and $\flat : TM \to T^*M$ the musical isomorphisms with respect to the metric g. In addition, let $*$ be the Hodge star operator of M with respect to g and dv_g. Then any vector field V on M is identified with a 2-form $F = F_V$ given by

$$F_V = 2\iota_V dv_g = 2 * (\flat V).$$

Here ι_V is the interior product by V. Conversely, any 2-form F is identified with a vector field $V = V_F$ given by $V_F = \sharp(*F)/2$. In particular, closed 2-forms are identified with divergence free vector fields.

Now let V be a divergence free vector field on a 3-dimensional oriented Riemannian manifold (M, g, dv_g). Then $F = F_V = 2\iota_V dv_g$ is a magnetic field on M. The Lorentz force L corresponding to F_V is computed as

$$g(LX, Y) = F_V(X, Y) = 3!dv_g(V, X, Y) = g(V \times X, Y).$$

Hence, the Lorentz equation becomes

$$\nabla_{\gamma'}\gamma' = qV \times \gamma'. \tag{7.1}$$

Magnetic fields corresponding to Killing vector fields are usually known as *Killing magnetic fields*. Their trajectories, called *Killing magnetic curves*, are of great importance since they are related to the Kirchhoff elastic rods.

7.2 An Equivalence

Now, we point out a close relationship between magnetic fields and almost contact structures. Let F be a magnetic field on a 3-dimensional oriented Riemannian manifold (M, g). Then the divergence free vector field V corresponding to F and the Lorentz force L associated to F satisfy

$$L^2 = -g(V, V)I + (\flat V) \otimes V, \quad LV = 0,$$

$$g(LX, LY) = g(V, V)g(X, Y) - g(V, X)g(V, Y).$$

Thus, if V is a unit vector field, then $(L, V, \flat V, g)$ is an almost contact metric structure with closed fundamental 2-form $-F$.

On the other hand, we know the following fundamental existence theorem:

Theorem 7.1 ([13]) *Let (M, g, dv_g) be a 3-dimensional oriented Riemannian manifold. Then there exists a unit vector field ξ and an endomorphism field φ such that $(\varphi, \xi, \eta = \flat\xi, g)$ is an almost contact metric structure.*

These facts suggest us to study contact magnetic curves in almost contact manifolds with *closed* fundamental 2-form. In addition, as we have seen before, codimension reduction theorems hold for contact magnetic curves in cosymplectic space forms and Sasakian space forms. The study of contact magnetic curves in 3-dimensional Sasakian manifolds are initiated by [14] (see also [33]).

Remark 9 (*Magnetic aesthetic curve*) In Euclidean 3-space, (static) magnetic field is regarded as a divergence free vector field $\boldsymbol{B}$. When the charge q is constant, $\boldsymbol{B}$ is uniform and no other forces are involved, the particle describes a helical trajectory with constant step whose axis is parallel to $\boldsymbol{B}$ (see [77]). However, in case of a variable charge or field, the particle moves along a curve with variable curvature. This strategy was proposed by Xu and Mould in [108] for plotting aesthetic planar curves using simulations of charged particles in a magnetic field. Interestingly, in the design and the production of cartoons, as well as in the description of decay processes, the solutions of the problem describing the movement of the charged particle in a magnetic field are used, even for the un-physical time dependence of the charge. Further applications of magnetic curves in CAD systems are described in [107].

8 3-Dimensional Almost Contact Metric Manifolds

8.1 The Vector Product of Almost Contact Riemannian Structure

Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional contact metric manifold. Then as we have seen before, the volume element dv_g derived from the associated metric g is related to the contact form η by

$$dv_g = -\frac{1}{2}\eta \wedge \Phi. \tag{8.1}$$

Even if M is non-contact, M is orientable by the 3-form $-\eta \wedge \Phi/2$ and the volume element dv_g coincides with this 3-form. Thus, hereafter, we orient 3-dimensional almost contact metric manifolds by $dv_g = -\eta \wedge \Phi/2$ given in (8.1). With respect to this orientation, the vector product $\times$ is computed as

$$X \times Y = -\Phi(X, Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X. \tag{8.2}$$

Note that for a unit vector field X orthogonal to ξ, the local frame field $\{X, \varphi X, \xi\}$ is positively oriented and

$$\xi \times X = \varphi X.$$

Camcı [19] called the vector product operation $\times$ given in (8.2) the new *extended cross product*. However, the operation $\times$ in nothing but the vector product induced by dv_g and hence *not* a new operation.

8.2 Normal Almost Contact Metric Manifolds

For an arbitrary almost contact metric 3-manifold M, we have the following *Olszak formula* [86]:

$$(\nabla_X \varphi)Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi, \tag{8.3}$$

where ∇ is the Levi-Civita connection on M. Moreover, we have

$$d\eta = \eta \wedge \nabla_\xi \eta + \alpha \Phi, \quad d\Phi = 2\beta\eta \wedge \Phi,$$

where α and β are the functions defined by

$$\alpha = \frac{1}{2}\text{tr}\,(\varphi\nabla\xi), \quad \beta = \frac{1}{2}\text{tr}\,(\nabla\xi) = \frac{1}{2}\text{div}\,\xi. \tag{8.4}$$

Remark 10 (*Contact metric manifolds*) When M is a 3-dimensional contact metric manifold, then we have

$$\nabla_X\xi = -\varphi(I+h)X, \quad X \in \mathfrak{X}(M),$$

where $h = \pounds_\xi\varphi/2$. Here $\pounds_\xi$ is the Lie differentiation by ξ. Hence, the covariant derivative φ is given by

$$(\nabla_X\varphi)Y = g((I+h)X, Y)\xi - \eta(Y)(I+h)X.$$

Olszak [86] showed that a 3-dimensional almost contact metric manifold M is normal if and only if $\nabla\xi\circ\varphi = \varphi\circ\nabla\xi$ or, equivalently,

$$\nabla_X\xi = -\alpha\varphi X + \beta(X - \eta(X)\xi), \quad X \in \mathfrak{X}(M). \tag{8.5}$$

We call the pair (α, β) the *type* of a normal almost contact metric 3-manifold M.

Using (8.3) and (8.5), we note that the covariant derivative $\nabla\varphi$ of a 3-dimensional normal almost contact metric manifold is given by

$$(\nabla_X\varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X). \tag{8.6}$$

Moreover, M satisfies

$$2\alpha\beta + \xi(\alpha) = 0.$$

Thus, if α is a nonzero constant, then $\beta = 0$. In particular, a 3-dimensional normal almost contact metric manifold is

- cosymplectic if $\alpha = \beta = 0$;
- quasi-Sasakian if $\beta = 0$ and $\xi(\alpha) = 0$;
- Kenmotsu if $\alpha = 1$ and $\beta = 1$.

Remark 11 (*f-Kenmotsu manifolds*) Olszak and Roşca [88] showed that an almost contact metric manifold M of dimension $2n+1 > 3$ satisfying (8.6) with $\alpha = 0$ automatically satisfies the equation $d\beta \wedge \eta = 0$. But this does not hold in general when $\dim M = 3$. Clearly, when β is a constant, this condition holds. Based on these observations, in a 3-dimensional case, Olszak and Roşca introduced the notion of f-Kenmotsu manifold as follows:

Definition 8.1 ([88]) An almost contact metric manifold M is said to be an f-*Kenmotsu manifold* if it satisfies

$$(\nabla_X\varphi)Y = f(g(\varphi X, Y)\xi - \eta(Y)\varphi X),$$

where f is a function satisfying $df \wedge \eta = 0$.

8.3 Bianchi-Cartan-Vranceanu Spaces

Here we give explicit models of Sasakian space forms. Let μ be a real number and set

$$\mathcal{D} = \left\{(x, y, z) \in \mathbb{R}^3(x, y, z) \mid 1 + \mu(x^2 + y^2) > 0\right\}.$$

Note that $\mathcal{D}$ is the whole $\mathbb{R}^3(x, y, z)$ for $\mu \geq 0$. On the region $\mathcal{D}$, we define the following Riemannian metric:

$$g_{\lambda,\mu} = \frac{dx^2 + dy^2}{\{1 + \mu(x^2 + y^2)\}^2} + \left(dz + \frac{\lambda}{2}\frac{ydx - xdy}{1 + \mu(x^2 + y^2)}\right)^2, \tag{8.7}$$

where $\lambda \in \mathbb{R}$.

The 2-parameter family $\{(\mathcal{D}, g_{\lambda,\mu}) \mid \lambda, \mu \in \mathbb{R}\}$ of 3-dimensional Riemannian manifolds is classically known by Bianchi [9], Cartan [20] and Vranceanu [105] (See also Kobayashi [69]). The Riemannian manifolds $(\mathcal{D}, g_{\lambda,\mu})$ are called the *Bianchi-Cartan-Vranceanu models* [8]. This 2-parameter family includes all the Riemannian metric with 4 or 6-dimensional isometry group other than constant negative curvature metrics. More precisely, $(\mathcal{D}, g_{\lambda,\mu})$ is (locally) isometric to one of the following spaces:

- $\mu = \lambda = 0$: Euclidean 3-space $\mathbb{R}^3$;
- $\mu = 0, \lambda \neq 0$: The Heisenberg group Nil_3 (see Example 3.7);
- $\mu > 0, \lambda \neq 0$: The special unitary group $\mathrm{SU}(2)$;
- $\mu < 0, \lambda \neq 0$: The universal covering $\widetilde{\mathrm{SL}}_2\mathbb{R}$ of $\mathrm{SL}_2\mathbb{R}$;
- $\mu > 0, \lambda = 0$: Product space $\mathbb{S}^2(4\mu) \times \mathbb{R}$;
- $\mu < 0, \lambda = 0$: Product space $\mathbb{H}^2(4\mu) \times \mathbb{R}$;
- $4\mu = \lambda^2$: The 3-sphere $\mathbb{S}^3(\mu)$ of curvature μ.

Now let us introduce almost contact structure (φ, ξ, η) on $\mathcal{D}$ compatible to the metric $g_{\lambda,\mu}$. Take the following orthonormal frame field on $(\mathcal{D}, g_{\lambda,\mu})$:

$$e_1 = \{1 + \mu(x^2 + y^2)\}\frac{\partial}{\partial x} - \frac{\lambda y}{2}\frac{\partial}{\partial z}, \ \ e_2 = \{1 + \mu(x^2 + y^2)\}\frac{\partial}{\partial y} + \frac{\lambda x}{2}\frac{\partial}{\partial z}, \ \ e_3 = \frac{\partial}{\partial z} := \xi.$$

Define the endomorphism field φ by

$$\varphi e_1 = e_2, \ \ \varphi e_2 = -e_1, \ \ \varphi e_3 = 0.$$

The dual 1-form η of the vector field ξ satisfies

$$d\eta(X, Y) = \frac{\lambda}{2} g_{\lambda,\mu}(X, \varphi Y), \quad X, Y \in \mathfrak{X}(\mathcal{D}).$$

Moreover, the structure $(\varphi, \xi, \eta, g_{\lambda,\mu})$ is a normal almost contact metric structure of type $(\lambda/2, 0)$. We denote by $\mathcal{M}^3(\lambda, \mu)$ the resulting normal almost contact metric manifold. In this way, we obtain a 2-parameter family $\{\mathcal{M}^3(\lambda, \mu)\,|\,\lambda, \mu \in \mathbb{R}\}$ of 3-dimensional normal almost contact metric manifolds. One can see that $\mathcal{M}^3(\lambda, \mu)$ is of constant φ-sectional curvature $\mathcal{H} = 4\mu - 3\lambda^2/4$. (*cf.* [8, 100]). In particular, if we choose $\lambda = 2$, then $\mathcal{M}^3(2, \mu)$ is a Sasakian manifold of constant φ-sectional curvature $\mathcal{H} = 4\mu - 3$. In addition, if $\lambda = 0$, then $\mathcal{M}^3(0, \mu)$ is a cosymplectic manifold of constant φ-sectional curvature $\mathcal{H} = 4\mu$.

Remark 12 (*Thurston geometry*) According to Thurston [104], there are eight simply connected model spaces in 3-dimensional geometries:

- The Euclidean 3-space $\mathbb{E}^3$, the 3-sphere $\mathbb{S}^3$, the hyperbolic 3-space $\mathbb{H}^3$;
- the product spaces $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$;
- the Heisenberg group Nil_3, the universal covering $\widetilde{\mathrm{SL}}_2\mathbb{R}$ of $\mathrm{SL}_2\mathbb{R}$ (see Sect. 12.2);
- the space Sol_3.

These eight model spaces admit *invariant* almost contact structure compatible to the metric. The resulting almost contact metric manifolds are homogeneous almost contact metric manifolds. Moreover, they are *normal* except Sol_3. In particular, $\mathbb{S}^3$, Nil_3 and $\widetilde{\mathrm{SL}}_2\mathbb{R}$ are (homothetic to) Sasakian space forms. Euclidean 3-space, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ are cosymplectic. The hyperbolic 3-space is the only Kenmotsu manifold in Thurston's list.

8.4 3-Dimensional Non-Sasakian Quasi-Sasakian Manifolds

Here we give an example of 3-dimensional *non-Sasakian* quasi-Sasakian manifold. To this end, we recall the following fact:

Proposition 8.1 ([85, 101]) *Let $M = (M, \varphi, \xi, \eta, g)$ be a 3-dimensional quasi-Sasakian manifold and σ a positive smooth function on M satisfying $d\sigma(\xi) = 0$. Then the structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ defined by*

$$\tilde{\varphi} := \varphi,\ \tilde{\xi} := \varepsilon\,\xi,\ \tilde{\eta} := \varepsilon\eta,\ \tilde{g} = \sigma g + (1-\sigma)\eta\otimes\eta,\ \varepsilon = \pm 1,\ \ \textit{or} \tag{8.8}$$

$$\tilde{\varphi} := \varepsilon\,\varphi,\ \tilde{\xi} := \xi,\ \tilde{\eta} := \eta,\ \tilde{g} = \sigma g + (1-\sigma)\eta\otimes\eta,\ \varepsilon = \pm 1 \tag{8.9}$$

is a quasi-Sasakian structure. The resulting structure is called the pseudo-conformal deformation of the original structure.

Let M be a 3-dimensional quasi-Sasakian manifold satisfying $\nabla\xi = -\alpha\varphi$. Then the Levi-Civita connection $\widetilde{\nabla}$ of the pseudo-conformally deformed metric $\tilde{g}$ is related to the Levi-Civita connection ∇ of the original metric g by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2\sigma}\{d\sigma(X)(Y - \eta(Y)\xi) + d\sigma(Y)(X - \eta(X)\xi)\}\} \\ - \frac{1}{2\sigma}\{g(X, Y) - \eta(X)\eta(Y)\}\operatorname{grad}\sigma - \frac{\alpha(1-\sigma)}{\sigma}\{\eta(X)\varphi Y + \eta(Y)\varphi X\}. \tag{8.10}$$

This formula implies that the new quasi-Sasakian structure satisfies $\widetilde{\nabla}\tilde{\xi} = -(\varepsilon\alpha/\sigma)\tilde{\varphi}$. In particular, if M is Sasakian, then the new structure is quasi-Sasakian and of rank 3.

Corollary 8.1 *Let M be a 3-dimensional quasi-Sasakian manifold satisfying $\nabla\xi = -\alpha\varphi$ for some function. Assume that M is of rank 3 and α has constant sign $\varepsilon = \pm 1$. Then the pseudo-conformal deformation of M with respect to $\sigma = \varepsilon\alpha > 0$ is Sasakian.*

Example 8.1 (*non-Sasakian example* [101]) Let $\mathrm{Nil}_3 = \mathbb{R}^3(x_1, y_1, z)$ be the 3-dimensional Heisenberg group equipped with Sasakian structure described in Example 3.7. Via the coordinate change $x := -x_1$, $y := y_1$, $t := z + x_1 y_1$, we obtain Sasakian manifold $\mathbb{R}^3(x, y, t)$ with metric $g = dx^2 + dy^2 + (dt + 2x dy)^2$ and contact form $\eta = dt + 2x dy$.

Take the half-space $M = \{(x, y, t) \in \mathbb{R}^3 \mid x > 0\}$ of $\mathbb{R}^3(x, y, t)$ and consider a pseudo-conformal deformation (8.9) of the Sasakian structure on M with respect to σ, then we obtain a quasi-Sasakian manifold (see Tanno [101]). The fundamental 2-form of the resulting quasi-Sasakian manifold is $2\varepsilon\sigma\, dx \wedge dy$.

For example, Wełyczko chose $\sigma = x^2$ and $\varepsilon = -1$ in [106]. Under this choice, the deformed structure satisfies $\tilde{\alpha} = -1/x^2 < 0$. The deformed metric is

$$\tilde{g} = x^2(dx^2 + dy^2) + \eta \otimes \eta, \quad \tilde{\eta} = \eta = dt + 2x dy.$$

One can see that $\tilde{g}$ is scalar flat. The endomorphism field $\tilde{\varphi} = -\varphi$ is described as

$$\tilde{\varphi}e_1 = e_2, \ \tilde{\varphi}e_2 = -e_1, \ \tilde{\varphi}e_3 = 0,$$

where

$$e_1 = \frac{1}{x}\frac{\partial}{\partial x}, \ e_2 = \frac{1}{x}\frac{\partial}{\partial y} - 2\frac{\partial}{\partial z}, \ e_3 = \frac{\partial}{\partial t} = \tilde{\xi}.$$

For more examples of non-Sasakian quasi-Sasakian 3-manifolds, see [87]. Olszak constructed explicit examples of conformally flat quasi-Sasakian 3-manifolds in [87].

9 Curve Theory in Almost Contact Metric 3-Manifolds

9.1 *Slant Curves in 3-Dimensional Quasi-Sasakian Manifolds*

Let γ be an arc length parametrized curve in a 3-dimensional almost contact metric manifold M. Then the Frenet frame field (T, N, B) satisfies

$$\xi \times T = \varphi T, \;\; \xi \times N = \varphi N, \;\; \xi \times B = \varphi B$$

and $T \times N = B$. Since ξ is expressed as

$$\xi = \eta(T)T + \eta(N)N + \eta(B)B$$

along γ, we have

$$\varphi T = -\eta(N)B + \eta(B)N, \;\; \varphi N = -\eta(B)T + \eta(T)B, \;\; \varphi B = -\eta(T)N + \eta(N)T.$$

Now let us compute the derivatives of $\eta(T)$, $\eta(N)$ and $\eta(B)$ along γ. We consider two particular cases.

Now let us assume that M is a quasi-Sasakian manifold with $\nabla\xi = -\alpha\varphi$. Then we have

$$\eta(T)' = \kappa\eta(N), \tag{9.1}$$
$$\eta(N)' = -\kappa\eta(T) + (\tau - \alpha)\eta(B), \tag{9.2}$$
$$\eta(B)' = -(\tau - \alpha)\eta(N). \tag{9.3}$$

Proposition 9.1 *A non-geodesic arc length parametrized curve γ in a 3-dimensional quasi-Sasakian 3-manifold satisfying $\nabla\xi = -\alpha\varphi$ is a slant curve if and only if $\eta(N) = 0$.*

Moreover, Eq. (9.2) shows that on a non-geodesic slant curve, $\kappa\eta(T) = (\tau - \alpha)\eta(B)$. Equation (9.3) implies that $\eta(B)$ is constant along γ. Hence we obtain the following:

Proposition 9.2 *In a 3-dimensional quasi-Sasakian manifold, the ratio of κ and $\tau - \alpha$ is constant along a non-geodesic slant curve.*

Conversely, we have the following result.

Proposition 9.3 ([31, 73]) *Let $\gamma(s)$ be a non-geodesic arc length parametrized curve in a 3-dimensional quasi-Sasakian manifold with $\nabla\xi = -\alpha\varphi$. If $\eta(N)$ and the ratio of κ and $\tau - \alpha$ are constant along γ. Then γ is a slant curve.*

Proof By the assumption, there exists a constant c such that $\tau - \alpha = c\kappa$. Then the equation (9.2) implies that $\eta(T) = c\eta(B)$. Then by using (9.1) and (9.3), we get $(1 + c^2)\kappa\eta(N) = 0$. Hence γ is a slant curve. □

We arrive at the following Bertrand-Lancret-de Saint Venant-type theorem for slant curves:

Theorem 9.1 *Let M be a 3-dimensional quasi-Sasakian manifold satisfying $\nabla\xi = -\alpha\varphi$. Then a non-geodesic arc length parametrized curve γ in M is a slant curve if and only if $\eta(N)$ and the ratio of κ and $\tau - \alpha$ are constant along γ.*

Remark 13 In case M is a contact metric manifold, we have

$$\begin{aligned}\eta(T)' &= \kappa\eta(N) - g(T, \varphi hT),\\ \eta(N)' &= -\kappa\eta(T) + (\tau - 1)\eta(B) - g(N, \varphi hT),\\ \eta(B)' &= -(\tau - 1)\eta(N) - g(B, \varphi hT).\end{aligned}$$

Example 9.1 We perform a coordinate change $\bar{x} = x$, $\bar{y} = y$, $\bar{z} = z + xy/2$ to the Bianchi-Cartan-Vranceanu model $\mathcal{M}^3(0, -1)$ exhibited in Sect. 8.3. Then the Riemannian metric $g_{0,-1}$ and the 1-form η are rewritten as $g_{0,-1} = d\bar{x}^2 + d\bar{y}^2 + \eta \otimes \eta$ and $\eta = d\bar{z} - \bar{y}d\bar{x}$, respectively. Next we perform a homothetical change $\bar{g} := g_{0,-1}/4$ and $\bar{\eta} := \eta/2$. Then $\mathbb{R}^3(\bar{x}, \bar{y}, \bar{z})$ together with a Riemannian metric $\bar{g}$ and a contact form $\bar{\eta}$ becomes a contact metric manifold, especially a Sasakian space form of constant φ-sectional curvature -3. We denote this Sasakian space form by $\mathbb{R}^3(-3)$. In [19], Camcı exhibited the following example. Define a function $\sigma(s)$ by $\sigma(s) = (1 - \cos(2\sqrt{2}s))/2$ and $\psi(s)$ be a solution to the ODE

$$\psi'(s) = -2\sigma(s) + \frac{2}{1 + \sigma(s)},$$

and define a curve $\gamma(s)$ in the Sasakian space form $\mathbb{R}^3(-3)$ with metric $(dx^2 + dy^2)/4 + \eta \otimes \eta$ and contact form $\eta = (dz - ydx)/2$ by

$$\bar{x}'(s) = -2\sqrt{1 - \sigma(s)^2}\sin\psi(s), \quad \bar{y}'(s) = 2\sqrt{1 - \sigma(s)^2}\cos\psi(s), \quad \bar{z}'(s) = 2\sigma(s) + \bar{y}(s)\bar{x}'(s).$$

Then the contact angle $\theta(s)$ is a non-constant function

$$\cos\theta(s) = \sigma(s) = \frac{1}{2}(1 - \cos(2\sqrt{2}s)).$$

This curve has curvature 2 and torsion -1 (Note that the function ψ is denoted by θ in [19]). Hence, the ratio of $\tau - 1$ and κ is constant. Thus, the curve is a *non-slant* helix in $\mathbb{R}^3(-3)$. However, Camcı's example is *not* a counterexample to Theorem 9.1. In fact,

$$\eta(N)' = \frac{1}{2}\sigma'(s) = -\sqrt{2}\sin(2\sqrt{2}s).$$

Thus, γ does not satisfy $\eta(N)' = 0$. We conclude that Camcı's example is not a counterexample to [31].

Example 9.2 (*Almost contact curves*) Let γ be a non-geodesic almost contact curve in a 3-dimensional quasi-Sasakian manifold, that is, γ' is orthogonal to ξ. Then we have $\eta(N) = 0$. Since N is orthogonal to both T and ξ, N is expressed as $N = \epsilon\varphi T$ with $\epsilon = \pm 1$. Then

$$\begin{aligned}\nabla_{\gamma'}N &= \epsilon\{(\nabla_{\gamma'}\varphi)T + \varphi(\nabla_{\gamma'}T)\}\\ &= \epsilon\{\alpha g(\gamma', \gamma')\xi + \varphi(\kappa N)\} = \epsilon(\xi + \kappa\varphi N)\\ &= -\kappa T + \epsilon\alpha\xi.\end{aligned}$$

Since (T, N, B) is positively oriented, we have $B = \epsilon T \times \varphi T$. On the other hand, B is computed as

$$\begin{aligned}B = T \times N &= -\Phi(T, N)\xi + \eta(T)\varphi N - \eta(N)\varphi T\\ &= -g(T, \varphi N)\xi = -g(T, \varphi(\epsilon\varphi T))\xi = \epsilon\xi.\end{aligned}$$

Hence, we obtain

$$\nabla_{\gamma'}N = -\kappa T + \alpha B.$$

This formula should coincide with

$$\nabla_{\gamma'}N = -\kappa T + \tau B.$$

Hence, we have $\tau = \alpha$.

Proposition 9.4 ([106]) *Every almost contact curve in a* 3*-dimensional quasi-Sasakian manifold has torsion* $\alpha(\gamma(s))$.

In case when M is Sasakian, we retrieve the following result due to Bikoussis and Blair (compare with [19]):

Corollary 9.1 ([7]) *Every Legendre curve in a Sasakian* 3*-manifold has constant torsion* 1.

From Proposition 6.2, we obtain the following result.

Corollary 9.2 *An arc length parametrized Legendre curve* γ *in a* 3*-dimensional Sasakian manifold satisfies* $\Delta^{\perp}H = \lambda H$ *for some* $\lambda \in \mathbb{R}$ *if and only if* γ *is a Legendre geodesic* $(\lambda = 0)$ *or Legendre helix with* $\lambda = 1$.

Let us compute the curvature and torsion of a slant curve γ in a 3-dimensional quasi-Sasakian manifold M with $\nabla\xi = -\alpha\varphi$.

We suppose that if γ is non-geodesic, then γ cannot be an integral curve of ξ. We find an orthonormal frame field $\{E_1, E_2, E_3\}$ along γ:

$$E_1 = T = \gamma', \quad E_2 = \frac{\varphi\gamma'}{\sin\theta}, \quad E_3 = \frac{\xi - \cos\theta\gamma'}{\sin\theta}. \tag{9.4}$$

Hence, the characteristic vector field ξ decomposes as $\xi = \cos\theta\, E_1 + \sin\theta\, E_3$.

Then for a non-geodesic slant curve γ, we have

$$\begin{cases} \nabla_{\gamma'} E_1 = \delta \sin\theta\ E_2, \\ \nabla_{\gamma'} E_2 = -\ \delta \sin\theta\ E_1 + (\alpha + \delta\cos\theta)\ E_3, \\ \nabla_{\gamma'} E_3 = -\ (\alpha + \delta\cos\theta)\ E_2, \end{cases} \tag{9.5}$$

where $\delta = g(\nabla_{\gamma'}\gamma', \varphi\gamma')/\sin^2\theta$. Moreover, we also deduce that

$$\nabla_{\gamma'}\xi = -\alpha\sin\theta\ E_2, \quad \kappa = |\delta|\sin\theta, \quad \tau = \alpha + \delta\cos\theta.$$

9.2 *Slant Curves: Fundamental Examples*

In this subsection, we exhibit some fundamental examples of slant curves in 3-dimensional space forms.

Example 9.3 (*Euclidean helices*) Let $\mathbb{E}^3(x, y, z) = \mathbb{E}^2 \times \mathbb{R}$ be the Euclidean 3-space with metric $\langle\cdot,\cdot\rangle = dx^2 + dy^2 + dz^2$. Then the standard cosymplectic structure associated to g is defined by

$$\eta = dz,\ \xi = \frac{\partial}{\partial z},\ \varphi\frac{\partial}{\partial x} = \frac{\partial}{\partial y},\ \varphi\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}.$$

Now let $\gamma(s)$ be a slant helix with constant contact angle θ. Then γ is congruent to the following *model helix*:

$$\gamma(s) = (a\cos(s/c), a\sin(s/c), bs/c),$$

where $a > 0$, $b \neq 0$ and $c = \sqrt{a^2 + b^2} > 0$ are constants. The Frenet frame of γ is given by

$$\mathcal{F} = (T, N, B) = \begin{pmatrix} -(a/c)\sin(s/c) & -\cos(s/c) & (b/c)\sin(s/c) \\ (a/c)\cos(s/c) & -\sin(s/c) & -(b/c)\cos(s/c) \\ b/c & 0 & a/c \end{pmatrix}.$$

One can see that $\det\mathcal{F} = 1$. The curvature and torsion of γ are $\kappa = a/c^2 > 0$ and $\tau = b/c^2 \neq 0$. One can see that γ has constant contact angle θ with $\cos\theta =$

b/c. In particular, every almost Legendre helix is congruent to the circle $\gamma(s) = (c\cos(s/c), c\sin(s/c), 0)$ of curvature $1/c > 0$ and torsion 0.

Example 9.4 (*Slant helices in* $\mathbb{S}^3$) Let $\mathbb{S}^3 \subset \mathbb{C}^2$ be a 3-dimensional unit sphere equipped with a Sasakian structure of constant curvature 1 (see Example 3.5). The vector product operation (8.2) derived from the Sasakian structure of $\mathbb{S}^3$ is related to the determinant form det of $\mathbb{C}^2 = \mathbb{E}^4$ by

$$\det(X, Y, Z, x) = g(X \times Y, Z), \quad X, Y, Z \in T_x\mathbb{S}^3.$$

Now let a, b and ϕ be constants such that

$$a^2\cos^2\phi + b^2\sin^2\phi = 1. \tag{9.6}$$

Then

$$\gamma(s) = (\cos\phi\cos(as), \cos\phi\sin(as), \sin\phi\cos(bs), \sin\phi\sin(bs)) \tag{9.7}$$

is an arc length parametrized curve in $\mathbb{S}^3$(see [47]). One can see that γ lies in the flat torus $\mathbb{T}^2$ of constant mean curvature $\cot(2\phi)$ given by the equations $x_1^2 + y_1^2 = \cos^2\phi$ and $x_2^2 + y_2^2 = \sin^2\phi$. The tangent vector field T of γ is

$$T = (-a\cos\phi\sin(as), a\cos\phi\cos(as), -b\sin\phi\sin(bs), b\sin\phi\cos(bs)).$$

From this equation and the Gauss formula (3.2), we get

$$\nabla_{\gamma'}\gamma' = \gamma'' + \gamma = \begin{pmatrix} (1-a^2)\cos\phi\cos(as) \\ (1-a^2)\cos\phi\sin(as) \\ (1-b^2)\sin\phi\cos(bs) \\ (1-b^2)\sin\phi\sin(bs) \end{pmatrix}.$$

Thus, the curvature κ is computed as

$$\kappa^2 = |\nabla_{\gamma'}\gamma'|^2 = a^2 + b^2 - a^2b^2 - 1 = (a^2-1)(1-b^2).$$

Hereafter, we assume that $\kappa \neq 0$. Then the principal normal N is given by

$$N = \frac{1}{\sqrt{(a^2-1)(1-b^2)}} \begin{pmatrix} (1-a^2)\cos\phi\cos(as) \\ (1-a^2)\cos\phi\sin(as) \\ (1-b^2)\sin\phi\cos(bs) \\ (1-b^2)\sin\phi\sin(bs) \end{pmatrix}.$$

On the other hand, the Reeb vector field along γ is given by

$$\xi_\gamma = -J\gamma = \begin{pmatrix} \cos\phi \sin(as) \\ -\cos\phi\cos(as) \\ \sin\phi\sin(bs) \\ -\sin\phi\cos(bs) \end{pmatrix}.$$

Hence, the contact angle θ is computed as

$$\cos\theta = \eta(T) = -(a\cos^2\phi + b\sin^2\phi). \tag{9.8}$$

Hence, γ is a slant curve. In particular, γ is a Legendre curve if and only if $a\cos^2\phi + b\sin^2\phi = 0$. For later use, we compute φT:

$$\varphi T = JT - \langle JT, \gamma\rangle\gamma = JT - \cos\theta\gamma = -\begin{pmatrix} (\cos\theta + a)\cos\phi\cos(as) \\ (\cos\theta + a)\cos\phi\sin(as) \\ (\cos\theta + b)\sin\phi\cos(bs) \\ (\cos\theta + b)\sin\phi\sin(bs) \end{pmatrix}. \tag{9.9}$$

Note that

$$\cos\theta + a = (a-b)\sin^2\phi, \quad \cos + b = (b-a)\cos^2\phi.$$

Next we compute the torsion τ of γ. The square τ^2 of the torsion is given by $\tau^2 = |\nabla_{\gamma'}N + \kappa T|^2$. Since κ is constant, we have

$$\nabla_{\gamma'}N = \frac{1}{\kappa}(\gamma'' + \gamma)' = \frac{1}{\sqrt{(a^2-1)(1-b^2)}} \begin{pmatrix} -a(1-a^2)\cos\phi\sin(as) \\ a(1-a^2)\cos\phi\cos(as) \\ -b(1-b^2)\sin\phi\sin(bs) \\ b(1-b^2)\sin\phi\cos(bs) \end{pmatrix}.$$

It follows that

$$\tau^2 = \left(\frac{(1-a^2)a}{\kappa} + a\kappa\right)^2 \cos^2\phi + \left(\frac{(1-b^2)b}{\kappa} + b\kappa\right)^2 \sin^2\phi = (ab)^2.$$

Thus, B has the form

$$B = \frac{\varepsilon}{ab\sqrt{(a^2-1)(1-b^2)}} \begin{pmatrix} -a(1-a^2)b^2\cos\phi\sin(as) \\ a(1-a^2)b^2\cos\phi\cos(as) \\ -b(1-b^2)a^2\sin\phi\sin(bs) \\ b(1-b^2)a^2\sin\phi\cos(bs) \end{pmatrix}, \quad \varepsilon = \pm 1.$$

Next we determine the sign ε. By definition, $g(T \times N, B) = 1$. On the other hand, we notice that $g(T \times N, B) = \det(T, N, B, \gamma) = -\varepsilon$. Hence, we have $\varepsilon = -1$. Thus, we get

$$\nabla_{\gamma'} B = \frac{-1}{ab\sqrt{(a^2-1)(1-b^2)}} \begin{pmatrix} -a^2(1-a^2)b^2\cos\phi\cos(as) \\ -a^2(1-a^2)b^2\cos\phi\sin(as) \\ -b^2(1-b^2)a^2\sin\phi\cos(bs) \\ -b^2(1-b^2)a^2\sin\phi\sin(bs) \end{pmatrix} = abN.$$

From the Frenet-Serret formula, we have $\tau = -ab$.

Now we concentrate on the Legendre helices. Assume that γ is Legendre, then from Eqs. (9.8) and (9.9), we get

$$N = \begin{pmatrix} a\cos\phi\cos(as) \\ a\cos\phi\sin(as) \\ b\sin\phi\cos(bs) \\ b\sin\phi\sin(bs) \end{pmatrix} = -\varphi T.$$

In this case, the binormal vector field is given by $B = -\xi_\gamma$.

From Eqs. (9.6), (9.8) and $\sin^2\theta + \cos^2\theta = 1$, we have $ab = -1$. Hence, $\mathcal{F} = (T, -\varphi T, -\xi_\gamma)$ is a positive orthonormal frame field along γ. The torsion τ is computed by $\tau = -ab = 1$. This fact is confirmed also by the formula

$$-\tau N = \nabla_{\gamma'} B = -\nabla_{\gamma'}\xi = \varphi T = -N.$$

Note that the Legendre curves cannot have $\tau = -1$. In fact, under the hypothesis $\tau = -ab = -1$, the Legendre condition $a\cos^2\phi + b\sin^2\phi = 0$ contradicts to (9.6) (compare with [19, Remark 4.1]).

Example 9.5 (*Hyperbolic* 3-*space*) Consider the warped product model of hyperbolic 3-space $\mathbb{H}^3$ of constant curvature -1:

$$\mathbb{H}^3 = (\mathbb{R}^3(x, y, t), e^{2t}(dx^2 + dy^2) + dt^2).$$

We can equip $\mathbb{H}^3$ with a Kenmotsu structure by $\eta = dt$, $\xi = \partial_t$, $\varphi\partial_x = \partial_y$, $\varphi\partial_y = -\partial_x$, $\varphi\partial_t = 0$. Slant curves in $\mathbb{H}^3$ are parametrized as follows [18]:

$$\gamma(s) = \left(\sin\theta \int_0^s e^{-u\cos\theta}\cos\psi(u)\,du,\ \sin\theta\int_0^s e^{-u\cos\theta}\sin\psi(u)\,du,\ s\cos\theta\right).$$

The curvature and torsion are given by

$$\kappa(s) = \sin\theta\sqrt{1+\psi'(s)^2},\quad \tau = \pm\left(\cos\theta\,\psi'(s) + \frac{\psi''(s)}{\psi'(s)^2}\right),$$

where ψ is a smooth function.

Slant curves with proper mean curvature vector field in 3-dimensional f-Kenmotsu manifolds are investigated in [18].

10 Magnetic Curves in 3-Dimensional Almost Contact Manifolds

10.1 Magnetic Curves and Pseudo-Conformal Deformations

The observations in Sects. 7.1 and 7.2 and codimension reduction theorems in Sects. 5.4 and 5.5 show that contact magnetic curves in 3-dimensional quasi-Sasakian manifolds are of particular interest [64].

Let γ be a normal magnetic trajectory in a 3-dimensional quasi-Sasakian M satisfying $\nabla\xi = -\alpha\varphi$ with respect to the Lorentz force φ with charge q. Namely, γ satisfies

$$\nabla_{\gamma'}\gamma' = q\,\varphi\gamma'. \tag{10.1}$$

Since the characteristic vector field ξ is Killing, the conservation lemma (4.1) and the Lorentz equation (10.1) imply the following fact.

Proposition 10.1 *Every contact magnetic curve on a 3-dimensional quasi-Sasakian manifold is a slant curve.*

Take the Frenet frame field (T, N, B) along γ. By definition, $T = \gamma'$. Hence, the magnetic equation is written as

$$\nabla_{\gamma'}\gamma' = q\xi \times \gamma' = \kappa N. \tag{10.2}$$

Hence, we get

$$\kappa^2 = q^2 g(\xi \times \gamma', \xi \times \gamma') = q^2\big[g(\xi,\xi)g(\gamma',\gamma') - g(\gamma',\xi)^2\big] = q^2\sin^2\theta.$$

Thus, γ has constant curvature $\kappa = |q|\sin\theta$. Assume that γ is a non-geodesic normal magnetic curve, then from (10.2), we have

$$N = \frac{q}{\kappa}\varphi\gamma'. \tag{10.3}$$

Next, the binormal vector field B is defined by $B = T \times N$:

$$B = \gamma' \times N = \gamma' \times \left\{\frac{q}{\kappa}(\xi \times \gamma')\right\} = \frac{q}{\kappa}(\xi - \cos\theta\gamma'). \tag{10.4}$$

Hence, we obtain

$$\nabla_{\gamma'}B = \frac{q}{\kappa}\nabla_{\gamma'}(\xi - \cos\theta\gamma') = -\frac{q}{\kappa}(\alpha + q\cos\theta)\varphi\gamma'.$$

Comparing this with

$$\nabla_{\gamma'}B = -\tau N = -\frac{\tau q}{\kappa}\varphi\gamma',$$

we obtain the torsion

$$\tau = \alpha + q\cos\theta.$$

Next, we consider pseudo-conformal deformations of contact magnetic curves.

Take an arc length parametrized curve $\gamma(s)$ in a 3-dimensional quasi-Sasakian manifold M satisfying $\nabla\xi = -\alpha\varphi$. We perform a pseudo-conformal deformation (8.8) of M by a smooth function σ. Then the velocity vector field $\gamma'(s)$ satisfies

$$\tilde{g}(\gamma'(s), \gamma'(s)) = \cos^2\theta(s) + \sigma(\gamma(s))\sin^2\theta(s).$$

Clearly, the property "arc length parametrized" is *not* preserved under pseudo-conformal deformations.

On the other hand, the property "Legendre" is preserved under the pseudo-conformal deformations since the distribution $\mathcal{D}$ is invariant under pseudo-conformal deformations. Now we study behaviour of magnetic curves under pseudo-conformal deformation.

Now let us assume that $\gamma(s)$ is a contact magnetic curve in M satisfying $\nabla_{\gamma'}\gamma' = q\varphi\gamma'$. Then we have

$$\widetilde{\nabla}_{\gamma'}\gamma' = q\varphi\gamma' + \frac{\sigma'}{\sigma}(\gamma' - \cos\theta\xi) - \frac{\sin^2\theta}{2\sigma}\,\mathrm{grad}\,\sigma - \frac{2\alpha(1-\sigma)}{\sigma}\cos\theta\,\varphi\gamma'. \quad (10.5)$$

Here σ' denotes the derivative $\{\sigma(\gamma(s))\}'$. Thus, the property "contact magnetic" is not preserved. Even if every quasi-Sasakian 3-manifold of rank 3 is locally pseudo-conformal to Sasakian 3-manifolds, contact magnetic curves are not invariant under the deformation. Because the study of contact magnetic curves in 3-dimensional quasi-Sasakian manifolds does *not* reduce to that of 3-dimensional Sasakian manifolds, it is interesting in its own right.

Assume that γ is *non-geodesic*, i.e., $\kappa \neq 0$ and $q \neq 0$, then from (10.3) and (10.4), the unit normal N and binormal B are related to $\varphi\gamma'$ and ξ by

$$\varphi\gamma' = \frac{\kappa}{q}N, \quad \xi = \frac{\kappa}{q}B + \cos\theta T.$$

Then the formula (10.5) is rewritten as

$$\widetilde{\nabla}_{\gamma'}\gamma' = -\frac{\cos^2\theta\sigma'}{\sigma}T + \left\{q - \frac{2\alpha(1-\sigma)}{\sigma}\right\}\varphi\gamma' - \frac{\cos\theta\kappa\sigma'}{q\sigma}B - \frac{\sin^2\theta}{2\sigma}\,\mathrm{grad}\,\sigma.$$

The following relations hold true

$$g(\mathrm{grad}\,\sigma, T) = \sigma',$$
$$\kappa g(\mathrm{grad}\,\sigma, B) = qg(\mathrm{grad}\,\sigma, \xi - \cos\theta\gamma') = -q\cos\theta\sigma'.$$

Therefore, if both the σ and α are *constant along* γ, then γ is magnetic with respect to the new metric $\tilde{g}$.

For a Legendre magnetic curve $\gamma(s)$, we have

$$\widetilde{\nabla}_{\gamma'}\gamma' = q\varphi\gamma' + \frac{\sigma'}{\sigma}\gamma' - \frac{1}{2\sigma}\operatorname{grad}\sigma.$$

Thus, under the assumption "$\sigma' = \alpha' = 0$", γ is also a Legendre magnetic curve with respect to $\tilde{g}$.

For example, let us consider the pseudo-conformal deformation of Wełyczko's example given in Example 8.1 with $\sigma = 1/x^2$ and $\varepsilon = -1$. The resulting Sasakian manifold is the Sasakian space form $\mathbb{R}^3(-3)$ with metric $dx^2 + dy^2 + (dt + 2xdy)^2$. Thus, the Legendre magnetic helix in Wełyczko's example corresponds to the Legendre magnetic helix in the Sasakian space form (Heisenberg group) Nil_3 under this pseudo-conformal deformation. For the Legendre magnetic curves in the Heisenberg group Nil_3, we refer to [37].

10.2 Non-helical Magnetic Curves

Let us observe that a magnetic curve on a *non-Sasakian* quasi-Sasakian 3-manifold is *not*, in general, a helix. Let us classify magnetic curves in Wełyczko's space. The magnetic equation (10.1) of Wełyczko's space is a system of three second-order differential equations, that is [64]:

$$\begin{cases} \dfrac{(x')^2}{x} - \dfrac{5(y')^2}{x} - \dfrac{2y'z'}{x^2} + x'' = & -qy' \\ \dfrac{6x'y'}{x} + \dfrac{2x'z'}{x^2} + y'' = & qx' \\ -10x'y' - \dfrac{4x'z'}{x} + x'' = & -2qxx'. \end{cases} \tag{10.6}$$

From Proposition 10.1, we know that $\eta(\dot{\gamma}) = \cos\theta$ is constant along γ. Hence, we have

$$z' + 2xy' = \cos\theta. \tag{10.7}$$

As the curve γ is parametrized by arc length, we also have $x^2((x')^2 + (y')^2) = \sin^2\theta$. Therefore, there exists a (smooth) function u (depending on s) such that

$$xx' = \sin\theta\cos u(s), \quad xy' = \sin\theta\sin u(s).$$

Hence, when u is known, the x-coordinate may be found from the equation

$$x(s)^2 = c_0 + 2\sin\theta\int_0^s \cos u(t)dt, \tag{10.8}$$

where c_0 is a positive constant. Then, using (10.7), we get

$$z(s) = z_0 + s\cos\theta - 2\sin\theta \int_0^s \sin u(t)dt, \quad z_0 \in \mathbb{R}. \tag{10.9}$$

Finally, we compute y:

$$y(s) = y_0 + \sin\theta \int_0^s \frac{\sin u(t)}{x(t)}dt, \quad y_0 \in \mathbb{R}. \tag{10.10}$$

The key point is how to obtain u.

From (10.6), when $\sin\theta \neq 0$, we have

$$\begin{aligned} &\sin u(s)\big[2\cos\theta + \sin\theta\sin u(s) + x^2(s)\big(-q + u'(s)\big)\big] = 0, \\ &\cos u(s)\big[2\cos\theta + \sin\theta\sin u(s) + x^2(s)\big(-q + u'(s)\big)\big] = 0. \end{aligned}$$

Combining with (10.8), we deduce that u is a solution of the following integro-differential equation:

$$2\cos\theta + \sin\theta\sin u(s) + \big(-q + u'(s)\big)\Big[c_0 + 2\sin\theta \int_0^s \cos u(t)dt\Big] = 0.$$

Thus, in general, normal magnetic curves in Wełyczko's example of non-Sasakian quasi-Sasakian manifold are *not* helices. In fact, the torsion $\tau = -1/x(s)^2 + q\cos\theta$ is non-constant.

10.3 Magnetic Curves in $\mathbb{S}^2 \times \mathbb{R}$

Let us realize the cosymplectic manifold $\mathbb{S}^2 \times \mathbb{R}$ as a hypersurface $\{(x, y, z, t) \in \mathbb{E}^4 \mid x^2 + y^2 + z^2 = 1\}$ of 4-dimensional Euclidean space $\mathbb{E}^4$. Contact magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$ are classified as follows:

Theorem 10.1 ([78]) *Let $\gamma(s) = (x(s), y(s), z(s), t(s))$ be a contact magnetic curve in $\mathbb{S}^2 \times \mathbb{R}$ with charge* 1*, defined by the vector field $\xi = \frac{\partial}{\partial t}$ and satisfying the initial condition*:

$$\gamma(0) = (x_0, y_0, z_0, t_0), \quad \gamma'(0) = (u_0, v_0, w_0, \zeta_0).$$

Then γ is one of the following slant curves:

- *The vertical geodesic $\gamma(s) = (x_0, y_0, z_0, t_0 \pm s)$.*
- *The circle $\mathbb{S}^1(2) \times \{t_0\}$ of radius $1/\sqrt{2}$ parametrized as*

$$\gamma(s) = \begin{pmatrix} \frac{u_0}{\sqrt{2}}\sin(\sqrt{2}s) - \frac{a_0}{2}\cos(\sqrt{2}s) + \frac{a_0}{2} + x_0 \\ \frac{v_0}{\sqrt{2}}\sin(\sqrt{2}s) - \frac{b_0}{2}\cos(\sqrt{2}s) + \frac{b_0}{2} + y_0 \\ \frac{w_0}{\sqrt{2}}\sin(\sqrt{2}s) - \frac{c_0}{2}\cos(\sqrt{2}s) + \frac{c_0}{2} + z_0 \\ t_0 \end{pmatrix}.$$

- *A non-geodesic cylindrical helix on* $\mathbb{S}^1(r) \times \mathbb{R}$ *with radius* $r = \sqrt{\mu_0^2 - 1}/\mu_0$ *parametrized as*

$$\gamma(s) = \begin{pmatrix} \frac{u_0}{\mu_0}\sin(\mu_0 s) - \frac{a_0}{\mu_0^2}\cos(\mu_0 s) + \frac{a_0}{\mu_0^2} + x_0 \\ \frac{v_0}{\mu_0}\sin(\mu_0 s) - \frac{b_0}{\mu_0^2}\cos(\mu_0 s) + \frac{b_0}{\mu_0^2} + y_0 \\ \frac{w_0}{\mu_0}\sin(\mu_0 s) - \frac{c_0}{\mu_0^2}\cos(\mu_0 s) + \frac{c_0}{\mu_0^2} + z_0 \\ t_0 + \zeta_0 s \end{pmatrix},$$

where $\mu_0 = \sqrt{2 - \zeta_0^2}$*, and* $\mu_0 \geq 1$ *is a constant. In each case,* a_0*,* b_0 *and* c_0 *take certain values.*

The Lie algebra of Killing vector fields on the Riemannian symmetric space $\mathbb{S}^2 \times \mathbb{R}$ is generated by $\{-y\partial_x + x\partial_y, z\partial_x - x\partial_z, z\partial_y - y\partial_z, \partial_t\}$. These Killing vector fields are called *basic Killing vector fields*. Magnetic curves derived from basic Killing vector fields on $\mathbb{S}^2 \times \mathbb{R}$ other than $\xi = \partial_t$ are also classified in [78].

10.4 Magnetic Curves in $\mathbb{H}^2 \times \mathbb{R}$

Let us realize the cosymplectic manifold $\mathbb{H}^2 \times \mathbb{R}$ as a region $\{(x, y, t) \in \mathbb{R}^3 \mid y > 0\}$ of 3-dimensional Cartesian space $\mathbb{R}^3$ equipped with the metric

$$\frac{dx^2 + dy^2}{y^2} + dt^2.$$

Contact magnetic curves in $\mathbb{H}^2 \times \mathbb{R}$ are classified by Nistor as follows:

Theorem 10.2 ([83]) *Let* $\gamma(s) = (x(s), y(s), z(s), t(s))$ *be a contact magnetic curve in* $\mathbb{S}^2 \times \mathbb{R}$ *with charge* 1*, defined by the vector field* $\xi = \frac{\partial}{\partial t}$ *and satisfying the initial condition*:

$$\gamma(0) = (x_0, y_0, z_0, t_0), \quad \gamma'(0) = (u_0, v_0, w_0, \zeta_0).$$

Then γ *is one of the following slant curves*:

- *The vertical geodesic* $\gamma(s) = (x_0, y_0, t_0 \pm s)$.
- *The Riemannian circle* $(x_0 \pm s\, y_0, y_0, t_0)$.
- *A helix*

$$\left(x_0 + \frac{y_0(1 + \sin\theta)\tan\theta\sin(s\cos\theta)}{1 + \sin\theta\cos(s\cos\theta)}, \frac{y_0(1 + \sin\theta)}{1 + \sin\theta\cos(s\cos\theta)}, s\cos\theta + t_0\right).$$

- *A Riemannian circle*

$$\left(x_0 - \frac{s y_0}{s^2 + 1}, \frac{y_0}{s^2 + 1}, t_0\right).$$

The Lie algebra of Killing vector fields on the Riemannian symmetric space $\mathbb{H}^2 \times \mathbb{R}$ is generated by

$$\left\{\partial_x, \frac{x^2-y^2+1}{2}\partial_x + xy\partial_y, x\partial_x + y\partial_y, \partial_t\right\}.$$

These Killing vector fields are called basic Killing vector fields. Magnetic curves derived from basic Killing vector fields on $\mathbb{H}^2 \times \mathbb{R}$ other than $\xi = \partial_t$ are also classified in [83].

11 Magnetic Curves in Contact Metric Manifolds

In this section, we study contact magnetic curves in 3-dimensional contact metric manifolds.

11.1 Magnetic Curves in 3-Dimensional Tori

We consider a flat torus $\mathbb{T}^3 = \mathbb{E}^3/\Gamma$ with $\Gamma = \pi\mathbb{Z}^3$. Then the contact form $\tilde{\eta} = \cos(2z)dx + \sin(2z)dy$ and Euclidean metric $\tilde{g} = dx^2 + dy^2 + dz^2$ on $\mathbb{E}^3(x, y, z)$ induce a contact Riemannian structure on M. We denote by η and g the induced contact form and induced metric, respectively. It should be remarked that the Reeb vector field is divergence free, but non-Killing. Hence, it is *non-Sasakian*.

Munteanu and Nistor classified closed trajectories of contact magnetic field on $\mathbb{T}^3$.

Theorem 11.1 ([81]) *On the flat torus $\mathbb{T}^3 = \mathbb{E}^3/\pi\mathbb{Z}^3$, the closed contact magnetic curves with charge $q \neq 0$ are described as the projection images of the following curves in $\mathbb{E}^3$:*

- *the horizontal line*

$$(x_0 \pm s\cos(2z_0), y_0 \pm s\sin(2z_0), z_0), \quad x_0, y_0 \in \mathbb{R}$$

  if $\tan(2z_0) \in \mathbb{Q}$, *which is a Reeb flow;*
- *any parallel to the y-axis of the form*:

$$\left(x_0, y_0 \pm s, \frac{(2l+1)\pi}{4}\right), \quad l \in \mathbb{Z}, \quad x_0, y_0 \in \mathbb{R},$$

  which is a Reeb flow;
- *the slant helix*

$$\left(a \pm \frac{q}{4\sqrt{\zeta}}\sin 2(z_0 \pm s\sqrt{\zeta}), b \mp \frac{q}{4\sqrt{\zeta}}\cos 2(z_0 \pm s\sqrt{\zeta}), z_0 \pm s\sqrt{\zeta}\right),$$

  where a, b, $z_0 \in \mathbb{R}$, $\zeta = 1 - \frac{q^2}{4}$ and $-2 < q < 2$ and $\cos\theta = q/2$;
- *the special curve*

$$\left(\left(\lambda + \frac{q}{2} - \frac{q}{B^2}\right)s + E(As + c, B), -\frac{q}{AB^2}\mathrm{dn}(As + c, B), \mathrm{am}(As + c, B)\right),$$

where A and B are certain positive constants, $c \in \mathbb{R}$. $E(u)$ is a particular function involving elliptic integrals and Jacobi elliptic functions. Moreover, dn *and* am *are Jacobi's dn-function and amplitude function, respectively. Those constants satisfy the closing condition*:

$$\frac{\mathsf{K}}{\pi A}\left(\lambda + \frac{q}{2} - \frac{q}{B^2}\right) \in \mathbb{Q}.$$

Here $2\mathsf{K}$ is the real fundamental period of the dn*-function.*

Remark 14 The contact Riemannian structure on flat tori can be induced from those of the universal covering group $\widetilde{\mathrm{SE}}(2)$ of the Euclidean plane $\mathbb{E}^2$; see [61].

11.2 Magnetic Curves in Sol$_3$

The model space Sol$_3$ of the solve geometry is $\mathbb{R}^3(x, y, z)$ with homogenous Riemannian metric

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

The 1-form $\eta = (e^z dx + e^{-z}dy)/\sqrt{2}$ is a contact form on Sol$_3$. The Lorentz force φ corresponding to the magnetic field $F = 2d\eta$ is described as

$$\varphi\partial_x = \frac{e^z}{\sqrt{2}}\partial_z, \quad \varphi\partial_y = -\frac{e^{-z}}{\sqrt{2}}\partial_z, \quad \varphi\partial_z = \frac{1}{\sqrt{2}}(e^{-z}\partial_x - e^z\partial_y).$$

Then (φ, ξ, η, g) with $\xi = (e^{-z}\partial_x + e^z\partial_y)/\sqrt{2}$ is a homogenous almost contact metric structure on Sol$_3$ satisfying $2d\eta = -\Phi$. Note that ξ is *non-Killing*. The magnetic curve equation is the system

$$\begin{aligned}
&e^z x'' - e^{-z}y'' + 2z'(e^z x' + e^{-z}y') = \sqrt{2}qz',\\
&e^z x'' + e^{-z}y'' + 2z'(e^z x' - e^{-z}y') = 0,\\
&z'' + e^{-2z}(y')^2 - e^{2z}(x')^2 = -\frac{q}{\sqrt{2}}(e^z x' - e^{-z}y').
\end{aligned}$$

The contact angle computed as $\cos\theta = (e^z x' + e^{-z}y')/\sqrt{2}$ is non-constant, in general. Some particular solutions are obtained in [40]. Magnetic curves in Sol$_3$ derived from basic Killing vector fields are investigated in [41]. In addition, $(\eta = dz, \xi = \partial_z, g)$ defines an almost cosymplectic structure on Sol$_3$. Contact magnetic curves in Sol$_3$ equipped with this almost cosymplectic structure are investigated in [42].

11.3 Magnetic Curves in Unit Tangent Sphere Bundles

As we have seen in Sect. 5.7, every contact magnetic curve in $U\mathbb{S}^m$ is slant. Here we give an application of magnetic curve theory to curve theory in $\mathbb{S}^2$. Note that $U\mathbb{S}^2$ is homothetic to a 3-dimensional real projective space $\mathbb{R}P^3$ equipped with Sasakian structure.

Theorem 11.2 ([62]) *Let $\underline{\gamma}(t)$ be an arc length parametrized curve in $\mathbb{S}^2$ with unit normal vector field $n(t)$. Then its Gauss map $\gamma(t) = (\underline{\gamma}(t); n(t))$ is a contact magnetic curve in $U\mathbb{S}^2$ with charge $q \neq 0$ if and only if $\underline{\gamma}(t)$ satisfies the natural equation*

$$\kappa(t) = -\frac{q(t-t_0)}{\sqrt{1-q^2(t-t_0)^2}}$$

for certain constant t_0.

Note that the Gauss map is automatically a Legendre curve.

Next we study contact magnetic curves in $U\mathbb{E}^2$ under the *slant assumption*.

Theorem 11.3 ([61]) *Let $\gamma(s) = (x(s), y(s); u(s), v(s))$ be an arc length parametrized contact magnetic curve in $\mathrm{U}\mathbb{E}^2 = \mathbb{E}^2 \times \mathbb{S}^1 \subset \mathbb{E}^2 \times \mathbb{E}^2$ with charge q and initial condition $(0, q/\sqrt{1-q^2}; 1, 0)$. Assume that γ is slant and non-geodesic, then γ is parametrized as*

$$\left(-\frac{q\sin(s\sqrt{1-q^2})}{\sqrt{1-q^2}}, \frac{q\cos(s\sqrt{1-q^2})}{\sqrt{1-q^2}}; \cos(s\sqrt{1-q^2}), \sin(s\sqrt{1-q^2})\right),$$

where $0 < |q| < 1$.

12 Periodicity of Contact Magnetic Curves

In 2007, Taubes [103] proved the generalized Weinstein conjecture in dimension 3, namely, on a compact, orientable, 3-dimensional contact manifold, the Reeb vector field ξ has at least one closed integral curve. Linked to this problem, it is important to investigate the existence of periodic magnetic trajectories of the contact magnetic field defined by ξ in 3-dimensional Sasakian manifolds, in particular in Sasakian space forms. In 2009, Cabrerizo et al. [14] have been looked for periodic orbits of the contact magnetic field on the unit sphere $\mathbb{S}^3$. See also [49]. In this section, we investigate the periodicity of contact magnetic curves in elliptic Sasakian space form $\mathcal{M}^3(c)$ and $\mathrm{SL}_2\mathbb{R}$.

12.1 Berger Spheres

Let $\mathcal{M}^3(c)$ be a 3-dimensional elliptic Sasakian space form of constant φ-sectional curvature $c > -3$. Recall that $\mathcal{M}^3(c)$ is obtained from $\mathbb{S}^3$ by $\mathcal{D}$-homothetic deformation.

The Riemannian metric $\hat{g} = ag + a(a-1)\eta \otimes \eta$ is a homothetical change of the *Berger sphere metric* $g + (a-1)\eta \otimes \eta$.

Then the Hopf fibering (Boothby-Wang fibering) $\pi : \mathcal{M}^3(c) \to \mathbb{S}^2(c+3)$ onto a 2-dimensional sphere of curvature $c+3$ is a Riemannian submersion. Take a curve β in $\mathbb{S}^2(c+3)$, then its inverse image $\Sigma_\beta := \pi^{-1}\{\beta\}$ is a flat surface in $\mathcal{M}^3(c)$ called the *Hopf tube* over β.

Proposition 12.1 *If β is a curve of length L on $\mathbb{S}^2(c+3)$ of length L, then the corresponding Hopf tube Σ_β is isometric to $\mathbb{S}^1(a) \times [0, L]$, where $\mathbb{S}^1(a)$ is the unit circle endowed with the metric $a^2 dt^2$ with $a = 4/(c+3)$. Moreover, its mean curvature H in $\mathcal{M}^3(c)$ is $H = (\kappa_\beta \circ \pi)/2$, where κ_β is the signed geodesic curvature of β in $\mathbb{S}^2(c+3)$.*

If β is a closed curve, i.e., $\beta(u+L) = \beta(u)$ for all $u \in \mathbb{R}$, the Hopf tube Σ_β is an immersed flat torus (called a *Hopf torus*). One can easily see that, if β is a great circle in $\mathbb{S}^2(c+3)$, then the Hopf torus Σ_β is minimal in $\mathcal{M}^3(c)$.

Proposition 12.2 *Let β be a closed curve on $\mathbb{S}^2(c+3)$ of length L enclosing an oriented area A. Then, the corresponding Hopf torus Σ_β is isometric to $\mathbb{R}^2/\Gamma$, where the lattice Γ is generated by the vectors $(2\pi a, 0)$ and $(A(c+3)/2, L)$.*

The contact magnetic curves in $\mathcal{M}^3(c)$ with $c > -3$ are geometrically characterized as follows:

Proposition 12.3 ([59]) *Let $\gamma(s)$ be a normal magnetic curve in the elliptic Sasakian space form $\mathcal{M}^3(c)$. Then γ is a geodesic in the Hopf tube Σ_β over a circle $\beta = \pi(\gamma)$. Moreover, the geodesic curvature κ_β of the projected curve β is given by $\kappa_\beta = (\kappa^2 + \tau^2 - 1)/\kappa$, where κ and τ are the curvature and the torsion of γ, respectively.*

If β is the projection of a periodic contact magnetic curve γ in $\mathcal{M}^3(c)$, then it is a circle on $\mathbb{S}^2(c+3)$. Denote by R its radius, $R \le r := \sqrt{a}/2 = 1/\sqrt{c+3}$. We have

$$\kappa_\beta = \frac{\sqrt{r^2 - R^2}}{rR}, \quad L = 2\pi R, \quad A = 2\pi r(r - \sqrt{r^2 - R^2}).$$

Since γ is a periodic (closed) geodesic on the Hopf torus Σ_β, it corresponds to a segment in $\mathbb{R}^2$ (with identified ends). This segment is in fact the diagonal of a parallelogram constructed by taking m vectors in the fibre, hence m times $(2\pi a, 0)$ and n vectors in the horizontal direction, i.e., n times $(\frac{A}{2r^2}, L)$, $n \in \mathbb{N}$. Thus, the direction of the magnetic trajectory γ is given by

$$\left(2\pi ma + n\pi\left(1 - \sqrt{1 - \frac{R^2}{r^2}}\right), 2\pi nR\right).$$

If we put $\sigma = \cot\theta$ (here θ is the contact angle of the curve γ) and call this quantity the *slope* of γ, we have

$$\sigma = \frac{2ma + n\left(1 - \sqrt{1 - \frac{R^2}{r^2}}\right)}{2nR}.$$

Hence, we get

$$R\sigma + \frac{1}{2}\sqrt{1 - \frac{R^2}{r^2}} = \frac{m}{n}a + \frac{1}{2}.$$

We can state the following result.

Theorem 12.1 ([59]) *The set of all periodic magnetic curves of arbitrary charge on the elliptic Sasakian space form $\mathcal{M}^3(c)$ can be quantized in the set of rational numbers.*

Finally, we apply our results to the unit sphere $\mathcal{M}^3(1) = \mathbb{S}^3$. In $\mathbb{S}^3$, every normal contact magnetic curve is a slant helix. One can see that the model helix (9.7) is periodic if and only if

$$a = 1/\sqrt{p^2 \sin^2\phi + \cos^2\phi}, \quad b = pa,$$

where p is a rational number.

In the case of $\mathbb{S}^3$, we obtain the following periodicity criterion [14, 49]:

Theorem 12.2 *Let γ be a normal magnetic curve on the unit sphere $\mathbb{S}^3$. Then γ is periodic if and only if*

$$\frac{q}{\sqrt{q^2 - 4q\cos\theta + 4}} \in \mathbb{Q}.$$

In the following, we take $\theta = \arccos\frac{29}{37}$. Consider the stereographic projection of the sphere from its North pole. Then the image of γ is drawn in Fig. 1.

We know that γ lies on a Hopf tube in $\mathbb{S}^3$. In Fig. 2, we plot the image of this tube under the stereographic projection we have mentioned before.

Remark 15 During the study of area minimization problem among Lagrangian surfaces in Kähler surfaces, Schoen and Wolfson completely classified admissible singularities of area minimizing Lagrangian surfaces. These singularities are locally modelled by Hamilton-minimal Lagrangian cones in complex Euclidean plane $\mathbb{C}^2$ [95]. In particular, Schoen and Wolfson classified Hamilton-minimal Lagrangian cones in $\mathbb{C}^2$. Those Lagrangian cones are realized as cones over L-minimal Legendre curves in $\mathbb{S}^3$. As a result, closed L-minimal Legendre curves in $\mathbb{S}^3$ are classified (see Theorem 12.3 below). Here a Legendre curve in a 3-dimensional Sasakian manifold is said to be L-minimal if it is a critical point of the length functional under Legendre variations (compare with ξ-variation in Sect. 4.2). One can see that every closed L-minimal Legendre curve in $\mathbb{S}^3$ is a contact magnetic curve. Conversely, every closed contact magnetic curve in $\mathbb{S}^3$ which is Legendre with respect to the canonical Sasakian structure is L-minimal and hence induces a Hamiltonian-minimal Lagrangian cone in $\mathbb{C}^2$.

Theorem 12.3 ([65, 95]) *All of closed L-minimal Legendre curves in $\mathbb{S}^3$ are parametrized as*

$$\gamma(s) = \left(\sqrt{n}\exp\left(\sqrt{-1}\sqrt{m/n}\, s\right), \sqrt{-1}\sqrt{m}\exp\left(\sqrt{-1}\sqrt{n/m}\, s\right)\right), \quad 0 \le s \le 2\pi\sqrt{mn},$$

where (m, n) is a pair of relatively prime positive integers. These are so-called torus knots of type (m, n).

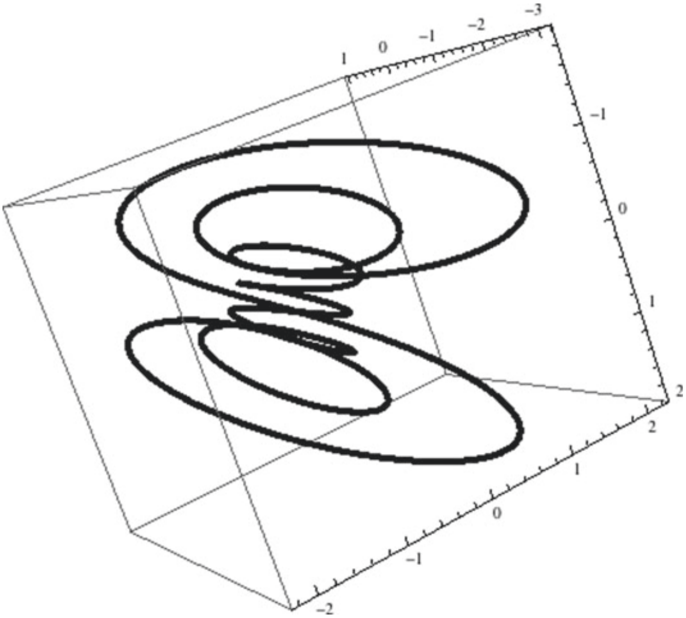

Fig. 1 $\cos\theta = 29/37$

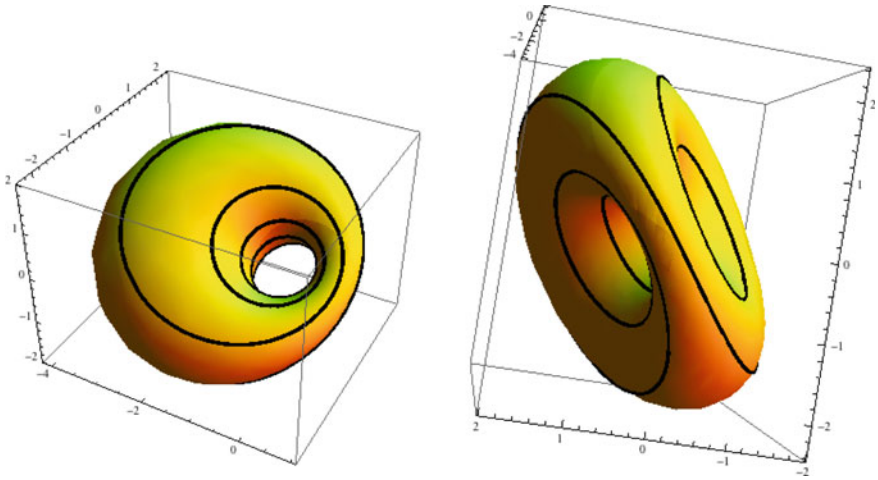

Fig. 2 The curve and the tube after stereographic projection; two different viewpoints

One can see that these Legendre knots are contact magnetic curves satisfying $|q| = |n-m|/\sqrt{mn}$. These Legendre knots are L-unstable [68].

12.2 The Special Linear Group

In this section, we consider the periodicity of contact magnetic curves in a 3-dimensional Sasakian space form of constant φ-sectional curvature $c < -3$. Without loss of generality, we may assume that $c = -7$. Moreover, we use $\mathrm{SL}_2\mathbb{R}$-model. Let $\mathrm{SL}_2\mathbb{R}$ be the real special linear group of degree 2:

$$\mathrm{SL}_2\mathbb{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R}, \; ad - bc = 1 \right\}.$$

By using the Iwasawa decomposition $\mathrm{SL}_2\mathbb{R} = NAK$ of $\mathrm{SL}_2\mathbb{R}$,

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}, \text{(Nilpotent part)}$$
$$A = \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \middle| \, y > 0 \right\}, \text{(Abelian part)}$$
$$K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \middle| \, 0 \leq t < 2\pi \right\} = \mathrm{SO}(2), \text{(Maximal torus)}$$

we can introduce the following global coordinate system (x, y, t) of $\mathrm{SL}_2\mathbb{R}$:

$$(x, y, t) \longmapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \tag{12.1}$$

The mapping

$$\mathbb{H}^2(-4) \times \mathbb{S}^1 \to \mathrm{SL}_2\mathbb{R}; \quad \psi(x, y, t) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

is a diffeomorphism onto $\mathrm{SL}_2\mathbb{R}$. We refer (x, y, t) as a global coordinate system of $\mathrm{SL}_2\mathbb{R}$. Hence $\mathrm{SL}_2\mathbb{R}$ is diffeomorphic to $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^1$ and hence diffeomorphic to $\mathbb{R}^3 \setminus \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}^+$ is diffeomorphic to open unit disc $\mathbb{D}$, $\mathrm{SL}_2\mathbb{R}$ is diffeomorphic to open solid torus $\mathbb{D} \times \mathbb{S}^1$.

The Sasakian space form $\mathrm{SL}_2\mathbb{R}$ is a principal circle bundle over the hyperbolic plan $\mathbb{H}^2(-4)$ equipped with the Poincaré metric $(dx^2 + dy^2)/(4y^2)$ (see Example 3.8). The projection is given by $\pi(x, y, t) = (x, y)$ and called the *hyperbolic Hopf fibering*.

Consider an arc length parametrized curve $\beta : \mathbb{R} \longrightarrow \mathbb{H}^2(-4)$ then its inverse image $\Sigma_\beta = \pi^{-1}\{\beta\}$ is a flat surface in $\mathrm{SL}_2\mathbb{R}$ tangent to the Reeb vector field ξ. It is called the *Hopf tube* over β.

Proposition 12.4 ([63]) *If β is a curve on $\mathbb{H}^2(-4)$ of length L, then the corresponding Hopf tube Σ_β is isometric to $\mathbb{S}^1(1) \times [0, L]$, where $\mathbb{S}^1(1)$ is the unit circle endowed with the metric dt^2. Moreover, its mean curvature in* $\mathrm{SL}_2\mathbb{R}$ *is $(\kappa_\beta \circ \pi)/2$, where κ_β is the signed geodesic curvature of β in $\mathbb{H}^2(-4)$.*

Analogous to Berger spheres, we have the following geometric characterization of contact magnetic curves:

Proposition 12.5 ([63]) *The contact magnetic curve γ in* $\mathrm{SL}_2\mathbb{R}$ *is a geodesic of the Hopf tube Σ_β over $\beta = \pi \circ \gamma$.*

For the periodicity arguments on contact magnetic curves, the following result is useful.

Proposition 12.6 *The projection image $\beta(u) = \pi(\gamma(u))$ of a contact magnetic curve is a Riemannian circle in $\mathbb{H}^2(-4)$. Hence, $\gamma(u)$ is a geodesic in a Hopf tube Σ_β over a Riemannian circle β.*

Now let us take a contact magnetic curve $\gamma(s) = (x(s), y(s), t(s))$ in $\mathrm{SL}_2\mathbb{R}$. The Lorentz equation of magnetic trajectory becomes

$$\begin{aligned}
&\frac{x''y - x'y'}{2y^2} - \frac{x'y'}{2y^2} - (\cos\theta)\frac{y'}{y} = -\frac{qy'}{2y},\\
&\frac{y''y - (y')^2}{2y^2} + \frac{(x')^2}{2y^2} + (\cos\theta)\frac{x'}{y} = \frac{qx'}{2y},\\
&t' + \frac{x'}{2y} = \cos\theta,
\end{aligned}$$

where θ is a constant contact angle.

Example 12.1 (*Reeb flows*) According to item (a) of Theorem 5.3, characteristic flows are magnetic curves. Choose $t = 0$ or π in the magnetic equations, we have $x(s) =$ constant and $y(s) =$ constant. The coordinate t is determined by $t' = \pm 1$. Hence, t is an affine function of s.

Next, we observe Legendre magnetic curves. According to item (c) of Theorem 5.3, Legendre φ-curves with $\kappa_1 = |q|$ and $\kappa_2 = 1$ are magnetic curves. The magnetic curve $\gamma(s)$ is a horizontal lift of a Riemannian circle $\beta(s) = (x(s), y(s))$ with $|\kappa_\beta| = |q|$. The third coordinate $t(s)$ is determined by the horizontal lift condition (Legendre condition):

$$t'(s) = -\frac{x'(s)}{2y(s)}$$

under the prescribed initial condition.

To look for periodic trajectories, we restrict our attention to horizontal lifts of closed Riemannian circles.

For $|\kappa_\beta| > 2$, $\beta(s)$ is a closed circle and parametrized as (see [63, Appendix]):

$$(x(s), y(s)) = \left(r \sin\mu(s) + x_0, r\left(\frac{|q|}{2} - \cos\mu(s)\right)\right),$$

where r is a positive constant and $\mu(s)$ is a solution to

$$\mu'(s) = |q| - 2\cos\mu(s).$$

Under the initial condition $\mu(0) = 0$, $\mu(s)$ is given explicitly by

$$\tan\frac{\mu(s)}{2} = \sqrt{\frac{|q|-2}{|q|+2}}\tan\left(\frac{\sqrt{q^2-4}}{2}s\right).$$

From this formula, one can deduce that $\beta(s)$ has period $2\pi/\sqrt{q^2-4}$. The t-coordinate is given by

$$t(s) = \frac{1}{2}\mu(s) - \frac{|q|}{2}s$$

under the initial condition $t(0) = 0$.

The horizontal lift is closed if and only if there exists a positive integer m such that

$$t\left(s + \frac{2m\pi}{\sqrt{q^2-4}}\right) \equiv t(s) \mod 2\pi.$$

Hence, the periodicity condition is equivalent to

$$|q| = \frac{2}{\sqrt{1-(m/k)^2}}$$

for some relatively prime positive integers m and k satisfying $m/k < 1$ (see also Kajigaya [68]). Thus, there exist countably many closed Legendre magnetic curves in $\mathrm{SL}_2\mathbb{R}$.

From the previous computations, we have

$$\mu(s) = \arctan\left(\sqrt{\frac{|q|-2}{|q|+2}}\tan\frac{\sqrt{q^2-4}}{2}\,s\right) + 2h\pi,$$

if $s \in (-\frac{\mathsf{T}}{2}, \frac{\mathsf{T}}{2})$, where $h \in \mathbb{Z}$. Fix the integers m and k as in the periodicity condition. We are looking now for a positive integer h such that $t(\frac{\mathsf{T}}{2} + h\mathsf{T}) \equiv t(-\frac{\mathsf{T}}{2}) \mod 2\pi$. This means that γ has $(h+1)$ "branches" to be periodic. The condition is equivalent to $(h+1)(1-\frac{k}{m})$ which is an even number.

In the following, we give some examples and draw the corresponding pictures on $\mathrm{SL}_2\mathbb{R}$ thought as a solid torus $\mathbb{S}^1 \times \mathbb{D}$. The pictures are drawn up to a homothetic deformation of the circle $\mathbb{S}^1$. Here, $\mathbb{D}$ is obtained from the Poincaré half plane $\mathbb{H}^2$ via the Cayley transformation

$$f : \mathbb{H}^2 \to \mathbb{D}^2,\ f(z) = \frac{z-\sqrt{-1}}{z+\sqrt{-1}},\ \text{where } z \in \mathbb{C},\ \Im m\, z > 0.$$

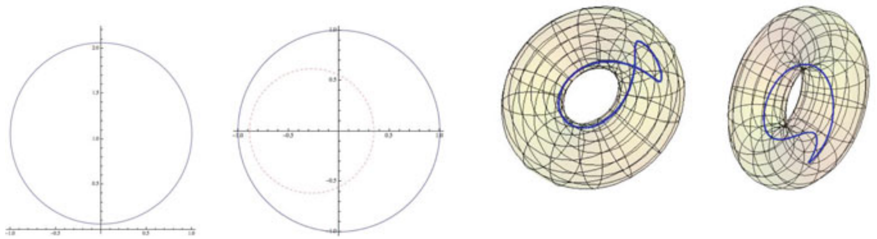

Fig. 3 Legendre magnetic curves $m = 1$, $k = 3$, $h = 0$

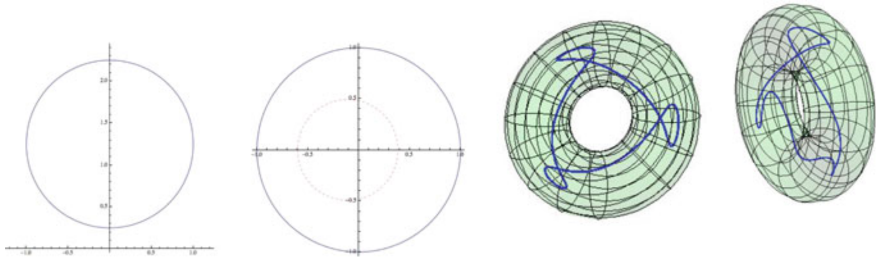

Fig. 4 Legendre magnetic curves $m = 3$, $k = 5$, $h = 2$

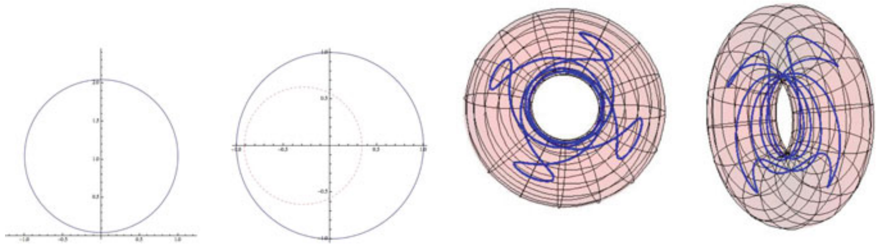

Fig. 5 Legendre magnetic curves $m = 2$, $k = 7$, $h = 3$

Every figure in the next three examples (Figs. 3, 4 and 5) is composed of four images:

- the first one represents the curve β represented in the upper half-plane;
- the second one represents the same curve β in the unit disc $\mathbb{D}$;
- the last two pictures represent the same curve γ on the solid torus $\mathbb{S}^1 \times \mathbb{D}$ from different viewpoints.

The case $\gamma(s)$ that is neither Reeb nor Legendre is much involved. Here, we give a sketch of a classification of periodic trajectories (For detailed discussions, we refer to [63]). Let $\gamma(s) = (x(s), y(s), t(s))$ be a periodic contact magnetic curve which is neither Reeb nor Legendre. Put $X = x'/(2y)$ and $Y = y'/(2y)$. Then we have $X^2 + Y^2 + \cos^2\theta = 1$, which implies that $X^2 + Y^2 = \sin^2\theta$. Moreover, we represent X and Y as

$$X(s) = \sin\theta\ \cos U(s), \quad Y(s) = \sin\theta\ \sin U(s),$$

for a certain function $U(s)$. To look for closed trajectories, we need to demand that $|q - 2\cos\sigma| > 2$. Let us denote by T the fundamental period of γ. Under the initial condition $U(0) = 0$, the periodicity $x(s+\mathsf{T}) = x(s)$ and $y(s+\mathsf{T}) = y(s)$ implies that $U(s+\mathsf{T}) \equiv U(s) \mod 2\pi$ for integer k. From this formula, one can deduce that

$$\frac{\mathsf{T}\sqrt{(q-2\cos\theta)^2 - 4\sin^2\theta}}{2} = m\pi$$

for some integer m. Next, the periodicity of $t(s)$ implies

$$t(s+\mathsf{T}) - t(s) = \left(\cos\theta - \frac{q-2\cos\theta}{2}\right)\mathsf{T} + \frac{U(s+\mathsf{T}) - U(s)}{2} \equiv 0 \mod 2\pi. \tag{12.2}$$

Finally, we obtain the

$$q = \frac{2a\cos\theta \pm \sqrt{2(1-a\cos(2\theta))}}{\frac{1+a}{2}}, \quad a = 1 - 2\left(\frac{m}{k}\right)^2. \tag{12.3}$$

We now state the following result.

Theorem 12.4 *The set of all periodic magnetic curves on* $\mathrm{SL}_2\mathbb{R}$ *equipped with Sasakian structure can be quantized in the set of rational numbers.*

In the following, we draw a picture of periodic non-Reeb and non-Legendre magnetic curves in $\mathrm{SL}_2\mathbb{R}$.

For all four images in Fig. 6 we keep the same convention as before.

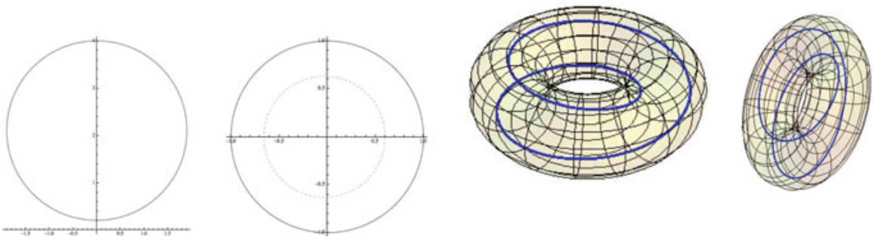

Fig. 6 $m = 1, k = 3, \theta = \frac{2\pi}{5}$

13 Biharmonic Curves

13.1 Bienergy

Let (M, g) be a Riemannian manifold. An arc length parametrized curve $\gamma(s)$ in M is said to be *biharmonic* if it is biharmonic with respect to the Levi-Civita connection ∇. The biharmonicity of γ is characterized as a geometric variational problem as follows:

Definition 13.1 Let us denote by $\Omega(M; L)$ the space of all smooth maps from a closed interval $[0, L]$ to a Riemannian manifold M. The *bienergy functional* E_2 on $\Omega(M; L)$ is defined by

$$E_2(\gamma) = \int_0^L \frac{1}{2}|\nabla_{\gamma'}\gamma'|^2\, ds.$$

An arc length parametrized curve $\gamma \in \Omega(M; L)$ is said to be a *biharmonic curve* if it is a critical point of E_2.

The Euler-Lagrange equation of this variational problem is given as follows:

Theorem 13.1 *Let $\gamma(s)$ be an arc length parametrized curve in a Riemannian manifold M defined on a closed interval* $[0, L]$. *Then γ is biharmonic if and only if it satisfies the biharmonic equation*:

$$\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\gamma' + R(\nabla_{\gamma'}\gamma', \gamma')\gamma' = 0.$$

Obviously, harmonic curves, i.e., geodesics are biharmonic. Non-geodesic biharmonic curves are often called *proper biharmonic curves*. Differential geometry of biharmonic submanifolds has two origins. One is derived from differential geometry of submanifolds with harmonic mean curvature vector field [24, 36]. Another one is derived from the theory of harmonic maps [39]. For more information on biharmonic submanifolds, we refer to Chen's book [24] and a survey [76] due to Montaldo and Oniciuc.

13.2 Biharmonic Curves in Dimension 3

Hereafter, we restrict our attention to biharmonic curves in oriented 3-dimensional Riemannian manifolds. By using the Frenet frame field (T, N, B), the biharmonic equation is given explicitly as follows:

$$-3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\tau\kappa' + \kappa\tau')B + R(\kappa N, T)T = 0. \tag{13.1}$$

When M is of constant curvature c, then we have

$$R(X, Y)Z = c(X \wedge Y)Z.$$

Thus, the biharmonic equation (13.1) reduces to

$$\kappa' = \tau' = 0, \quad \kappa^2 + \tau^2 = c.$$

This implies that there are no proper biharmonic curves in Euclidean 3-space $\mathbb{E}^3$ (*cf.* [27, 36]) or in hyperbolic 3-space $\mathbb{H}^3$ (*cf.* [15]).

Caddeo, Montaldo and Oniciuc classified proper biharmonic curves in the unit 3-sphere $\mathbb{S}^3$:

Theorem 13.2 ([15]) *Let γ be an arc length parametrized proper biharmonic curve in $\mathbb{S}^3$. Then $\kappa \leq 1$ and we have two cases*:

(1) $\kappa = 1$ and γ is a circle of radius $1/\sqrt{2}$.
(2) $0 < \kappa < 1$ and γ is a helix, which is a geodesic in the Clifford minimal torus.

In case (1), γ is congruent to

$$\frac{1}{\sqrt{2}}\left(\cos(\sqrt{2}s), \sin(\sqrt{2}s), c_1, c_2\right), \quad c_1^2 + c_2^2 = 1.$$

In case (2), γ is congruent to

$$\frac{1}{\sqrt{2}}\left(\cos(as), \sin(as), \cos(bs), \sin(bs)\right).$$

The proper biharmonic curves in $\mathbb{S}^3$ are helices satisfying $\kappa^2 + \tau^2 = 1$, hence those are curves with proper mean curvature vector field with eigenvalue $\lambda = 1$ (see Proposition 6.1).

Since $\mathbb{S}^3$ is a typical example of Sasakian manifold, these classifications motivate us to classify proper biharmonic curves in 3-dimensional Sasakian space forms.

We rephrase the above classification in terms of Sasakian structure of $\mathbb{S}^3$:

Corollary 13.1 *Let γ be an arc length parametrized proper biharmonic curve in $\mathbb{S}^3$. Then $\kappa \leq 1$ and we have two cases*:

(1) γ is a small circle with $\kappa = 1$, $\tau = 0$ and contact angle $\pi/4$. This curve is congruent to a model helix (9.7) with $a + b = 0$ and $ab = 0$.
(2) γ is congruent to a model helix (9.7) satisfying $0 < \kappa = \sqrt{(a^2 - 1)(1 - b^2)} < 1$ and $\tau = -ab \neq 0$ and $\cos\theta \neq 0$.

In both cases, γ is a geodesic in the Clifford minimal torus. In particular, there are no proper biharmonic Legendre curves in $\mathbb{S}^3$.

This corollary motivates us to study biharmonic curves in 3-dimensional Sasakian space forms as well as cosymplectic space forms. For this purpose, we use the Bianchi-Cartan-Vranceanu model given in Sect. 8.3.

The biharmonicity equation (13.1) for arc length parametrized curve γ in the Bianchi-Cartan-Vranceanu model $\mathcal{M}^3(\lambda, \mu)$ is obtained as follows(*cf.* [16, 17, 30]):

$$\kappa' = 0, \quad \tau' = (\lambda^2 - 4\mu)\eta(N)\eta(B), \quad \eta(B)\eta(N) = 0, \quad \kappa^2 + \tau^2 = \frac{\lambda^2}{4} - (\lambda^2 - 4\mu)\eta(B)^2.$$

Under the assumption γ is non-geodesic, we have the following:

Theorem 13.3 ([16, 17, 30]) *Let γ be an arc length parametrized curve in a Bianchi-Cartan-Vranceanu model $\mathcal{M}^3(\lambda, \mu)$. Then γ is proper biharmonic if and only if γ satisfies*

$$\kappa = \text{constant} \neq 0, \quad \tau = \text{constant}, \quad \eta(N) = 0, \quad \kappa^2 + \tau^2 = \frac{\lambda^2}{4} - (\lambda^2 - 4\mu)\eta(B)^2.$$

Since $\eta(N) = 0$, γ is a slant curve. In addition, by using the constant contact angle θ, $\eta(B)$ is expressed as $\eta(B) = \sin\theta$.

Now let us apply this theorem to Sasakian space forms:

Corollary 13.2 *An arc length parametrized curve $\gamma(s)$ in a 3-dimensional Sasakian space form of φ-sectional curvature c is proper biharmonic if it a slant helix satisfying $\kappa^2 + \tau^2 = 1 + (c-1)\sin^2\theta$. These helices have proper mean curvature vector field with eigenvalue $1 + (c-1)\sin^2\theta$.*

If we choose $c = 1$, then we retrieve Theorem 13.2. It should be emphasized that Sasakian space forms with constant φ-sectional curvature $c > 1$ admit proper biharmonic *Legendre curves*. Proper biharmonic Legendre curves are helices with $\kappa = \sqrt{c-1}$ and torsion $\tau = 1$. This existence was discovered in [50]. The existence of proper biharmonic Legendre curves in elliptic Sasakian space form of φ-sectional curvature greater than 1 implies that the study of biharmonic submanifolds in elliptic Sasakian space forms does not reduce to that of the unit sphere. Although the Legendre property for submanifolds in Sasakian manifolds is invariant under $\mathcal{D}$-homothetic deformations, biharmonicity is not.

On the other hand, in cosymplectic space forms, we obtain the following:

Proposition 13.1 *A non-geodesic arc length parametrized curve $\gamma(s)$ in Riemannian products $\mathbb{S}^2(4\mu) \times \mathbb{R}$ and $\mathbb{H}^2(4\mu) \times \mathbb{R}$ is biharmonic if it is a slant helix satisfying $\kappa^2 + \tau^2 = 4\mu \sin^2\theta$. These helices have a proper mean curvature vector field with eigenvalue $4\mu \sin^2\theta$. In particular, the only biharmonic curves in $\mathbb{H}^2(4\mu) \times \mathbb{R}$ are geodesics.*

One can obtain explicit parametrizations of biharmonic curves in $\mathcal{M}^3(\lambda, \mu)$. See [2, 16, 17, 30].

The existence of proper biharmonic Legendre curves (as well as Hopf tubes) in 3-dimensional elliptic Sasakian space forms [50] and that of proper biharmonic Legendre surfaces in $\mathbb{S}^5$ [93] opened up a research area "biharmonic submanifolds in Sasakian manifolds"; see [1, 94]. For biharmonic submanifolds in higher dimensional Sasakian manifolds, we refer to [44, 45].

Remark 16 The notion of biharmonic curve can be generalized to curves in manifolds equipped with linear connection. Let (M, D) be a manifold with a linear connection D. For a curve $\gamma : I \to M$, we define the *affine mean curvature* $H(\gamma; D)$ of γ with respect to D (also called the *D-mean curvature*) by $H(\gamma; D) = D_{\gamma'}\gamma'$. A curve γ is said to be *affine harmonic* with respect to D (or *D-harmonic* in short) if $H(\gamma; D) = 0$. Obviously, D-harmonic curves are geodesics with respect to D. In case when D is the Levi-Civita connection ∇ of (M, g), the ∇-mean curvature $H(\gamma; \nabla)$ coincides with the mean curvature vector field H as we introduced before.

In [31], D-biharmonicity of curves was introduced. Denote by T^D and R^D the torsion and curvature tensor field of D. Then a curve $\gamma : I \to (M, D)$ is said to be *affine biharmonic* with respect to D (or D-*biharmonic* in short) if it satisfies

$$D_{\gamma'} D_{\gamma'} H(\gamma; D) - T^D(\gamma', D_{\gamma'} H(\gamma; D)) - (D_{\gamma'} T^D)(\gamma', H(\gamma; D)) + R^D(H(\gamma; D), \gamma')\gamma' = 0. \quad (13.2)$$

Short calculation shows that (13.2) is rewritten as

$$D_{\gamma'} D_{\gamma'} H(\gamma; D) + D_{\gamma'} T^D(H(\gamma; D), \gamma') + R^D(H(\gamma; D), \gamma')\gamma' = 0.$$

Note that when γ is D-harmonic, then its Jacobi field X satisfies

$$D_{\gamma'} D_{\gamma'} X + D_{\gamma'} T^D(X, \gamma') + R^D(X, \gamma')\gamma' = 0.$$

Affine biharmonic curves and slant curves in 3-dimensional almost contact Riemannian manifolds (equipped with a generalized Tanaka-Webster-Okumura connection) are studied in [31, 32, 48, 51–56, 72, 91].

14 Concluding Remarks

Recall that a slant curve γ in a 3-dimensional Sasakian manifold satisfies $\eta(N) = 0$. Let h be the second fundamental form of γ in M. Then the second fundamental form $h(\gamma', \gamma')$ of γ in M coincides with $\nabla_{\gamma'}\gamma'$. When γ in non-geodesic curve, we notice that

$$\kappa\eta(N) = \eta(h(\gamma', \gamma')) = g(h(\gamma', \gamma'), \xi).$$

This formula implies that a non-geodesic curve is slant if and only if $g(h(\gamma', \gamma'), \xi) = 0$. This observation motivates us to introduce the following notion:

Definition 14.1 A submanifold N of a Sasakian manifold M is said to be a *Lancret submanifold* if its second fundamental form h satisfies $g(h(X, Y), \xi) = 0$ for all vector fields X and Y tangent to N.

This notion is closely related to the notion of rectifying submanifold introduced by Chen [23, 25].

Definition 14.2 Let M be a Riemannian manifold and V a non-vanishing vector field. A submanifold N of M is said to be a *rectifying submanifold* with respect to V if the normal component of V is nowhere zero and

$$g(V_x, \mathrm{Im}\, h_x) = 0$$

holds for all point $x \in N$. Here, $\mathrm{Im}\, h_x$ is the image of the second fundamental form h at x.

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Contact Slant Geometry of Submersions and Pointwise Slant and Semi-slant-Warped Product Submanifolds

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1 Introduction

Neill [5] and Gray [1] investigated the Riemannian submersion between Riemannian manifolds. These submersions were later extensively studied in differential geometry. Watson [8] first studied the Riemannian submersions between Riemannian manifolds admitting an almost complex structure. He defined a submersion between almost Hermitian manifolds and named these submersions as almost Hermitian submersion. Basically, Watson proved that the base manifold and fibers have a similar structure as the total space. Due to applications of Riemannian submersions in Mathematical Physics and Kaluza–Klein theory [23], it becomes a favorite topic for research.

Motivated by Watson [8], Sahin [7] introduced anti-invariant submersion in the setting of almost Hermitian manifold. More precisely, Sahin provided a generalized

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version of almost Hermitian submersion and anti-invariant submersion by exploring slant submersions from almost Hermitian manifolds onto Riemannian manifolds [7]. In case of almost contact metric manifolds, Kupeli Erken and Murathan [11] introduced the notion of slant submersion from Sasakian manifolds onto Riemannian manifolds and obtained some basic results. Further, Sushil Kumar et al. [26] extended the study of Kupeli Erken and C. Murathan and defined pointwise slant submersions from Kenmotsu manifolds onto Riemannian manifolds. Riemannian submersions have been widely studied for various almost Hermitian and almost contact structures (see [6, 10, 14, 15, 18]). The Riemannian submersions can be defined in the following steps.

Let (M, g_M) and (N, g_N) be two Riemannian manifolds of dimensions m and n, respectively $(m > n)$, where g_M and g_N are the their Riemannian metrics respectively. Let $f : M \to N$ be a smooth map. The Kernel space of f_* is denoted by $kerf_*$ and suppose $H = (kerf_*)^\perp$ be orthogonal complementary space of $kerf_*$. Then the tangent bundle of M has the following decomposition

$$TM = (kerf_*) \oplus (kerf_*)^\perp. \tag{1.1}$$

The range of f_* is denoted by $rangef_*$ and let $(rangef_*)^\perp$ be the orthogonal complementary space of $rangef_*$ in the tangent bundle TN of N. Thus, TN can be decomposed as follows

$$TN = (rangef_*) \oplus (rangef_*)^\perp. \tag{1.2}$$

A Riemannian submersion f is a C^∞-map from Riemannian manifold (M, g_M) onto (N, g_N) satisfying the following conditions

1. f has maximal rank,
2. The differential f_* preserves the lengths of horizontal vectors.

For each $x \in N$, $f^{-1}(x)$ is fiber that is $(m-n)$-dimensional submanifold of M. If a vector field on M is always tangent (respectively orthogonal) to fibers, then it is called vertical (respectively horizontal). A vector field X on M is said to be basic if it is horizontal and f-related to a vector field X_* on N, that is, $f_* X_p = X_{f(p)}$ for all $p \in M$. The projection morphisms on the distributions $kerf_*$ and $(kerf_*)^\perp$ are denoted by ν and $\mathcal{H}$ respectively.

A smooth map $f : M \to N$ is said to be a Riemannian submersion if and only if

$$g_M(U, V) = g_N(f_*U, f_*V),$$

for any $U, V \in (kerf_*)^\perp$.

The O. Neill's tensor $\mathcal{T}$ and $\mathcal{A}$ is defined by

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\nu E}\nu F + \nu\nabla_{\nu E}\mathcal{H}F \tag{1.3}$$

$$\mathcal{A}_E F = \nu\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\nu F, \tag{1.4}$$

for any vector field E and F on M, where ∇ is the Riemannian connection on M [5].

Lemma 1.1 ([5]) *Let f be a Riemannian submersion between Riemannian manifolds (M, g_M) and (N, g_N). If X and Y are basic vector fields on M, then*

1. *$g_M(X, Y) = g_N(f_*X, f_*Y)$,*
2. *The horizontal part $[X, Y]^{\mathcal{H}}$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$ that is, $f_*([X, Y]^{\mathcal{H}}) = [X_*, Y_*]$,*
3. *$[V, X]$ is vertical for any vector field V of $ker f_*$,*
4. *$(\nabla^M_X Y)^{\mathcal{H}}$ is vertical for any vector field corresponding to $(\nabla^N_{X_*} Y_*)$ where ∇^M and ∇^N are the Riemannian connections on M and N, respectively.*

Now, from Eqs. (1.3) and (1.4), we get

$$\nabla_X Y = \mathcal{T}_X Y + \hat{\nabla}_X Y, \tag{1.5}$$

$$\nabla_X V = \mathcal{H}\nabla_X V + \mathcal{T}_X V, \tag{1.6}$$

$$\nabla_V X = \mathcal{A}_V X + \nu\nabla_V X, \tag{1.7}$$

$$\nabla_V W = \mathcal{H}\nabla_V W + \mathcal{A}_V W, \tag{1.8}$$

for any $X, Y \in ker f_*$ and $V, W \in (ker f_*)^{\perp}$.

On the other hand, for any $E \in TM$, it is seen that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\nu E}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$.

The tensor fields $\mathcal{T}$ and $\mathcal{A}$ satisfy the equations

$$\mathcal{T}_X Y = \mathcal{T}_Y X \tag{1.9}$$

$$\mathcal{A}_V W = -\mathcal{A}_W V = \frac{1}{2}\nu[V, W], \tag{1.10}$$

for any $X, Y \in \Gamma(ker f_*)$ and $V, W \in \Gamma(ker f_*)^{\perp}$.

Moreover, if the horizontal distribution $\mathcal{H}$ is integrable if and only if $\mathcal{A} = 0$. Then it is straightforward to observe the following

$$g(\mathcal{T}_D E, G) + g(\mathcal{T}_D G, E) = 0, \tag{1.11}$$

$$g(\mathcal{A}_D E, G) + g(\mathcal{A}_D G, E) = 0, \tag{1.12}$$

for any $D, E, G \in TM$.

Another aspect of the present chapter is warped product submanifolds of almost contact metric manifolds, so we provide a brief history of warped product submanifolds in the following steps.

Bishop and Neill [21] explored the geometry of Riemannian manifolds of negative curvature and introduced the notion of warped product for these manifolds (see the definition in Sect. 5). The warped product manifolds are the natural generalization of Riemannian product manifolds. Some natural properties of warped products were investigated in [21].

In the early twentieth century, B.-Y. Chen first used the idea of warped products for CR-submanifolds of Kaehler manifolds [3]. Infact, he proved the existence of CR-warped product submanifolds of the type $N_T \times_f N_\perp$ in the setting of Kaehler manifold, where N_T and $N_\perp$ are the holomorphic and totally real submanifolds. Since then, many authors have studied warped product submanifolds in the different settings of Riemannian manifolds, and numerous existence results have been explored (see the survey article [4]).

The study of Bishop and Neill [21] has enlightened a few intrinsic properties of the warped product manifolds. Initial extrinsic studies of warped product manifold in the almost complex setting were performed by Chen [3] while obtaining some results of existence for CR-submanifold as CR-warped product submanifold in Kaehler manifolds. Furthermore, contact CR-warped product submanifolds were studied by Hasegawa et al. [13] in the almost contact settings. Warped product manifolds are also investigated in the contact setting by many other geometers and which have attained various existence results [13, 17, 28].

Warped product pointwise semi-slant submanifolds are another generalized class of warped product semi-slant submanifolds and contact CR-warped product submanifolds. In [16], Park studied such warped product submanifolds. After that, Mihai et al. [12] extended this study in Sasakian manifolds and acquired some optimal inequalities related to warping function and second fundamental form. Warped product pointwise semi-slant submanifolds for almost contact and almost complex manifolds were explored (see [19, 22]).

2 Slant Submersions from Sasakian Manifolds

In this section, we study the slant submersion from Sasakian manifolds onto Riemannian manifolds. This section consists of some important results from the study of Erken and Murathan [11].

Let M be an almost contact metric manifold. So there exist on M a (1, 1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \circ \xi = 0, \quad \eta \circ \phi = 0 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\phi X, Y) = -g(X, \phi Y), \tag{2.2}$$

for any X, Y on M.

A normal contact metric structure is called a Sasakian structure, which satisfies

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.3}$$

where ∇ denotes the Levi-civita connection of g. For a Sasakian manifold M, it is known that

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X \tag{2.4}$$

$$\nabla_X \xi = -\phi X. \tag{2.5}$$

Let $M(\phi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $f : M \to N$ is said to be slant if for any nonzero vector $X \in kerf_* - \langle\xi\rangle$, the angle $\theta(X)$ between ϕX and the space $ker(f_*)$ is constant (which is independent of the choice of $p \in M$ and $X \in kerf_* - \langle\xi\rangle$). The angle θ is called the slant angle of the slant submersion. Invariant and anti-invariant submersions are slant submersions with slant angles $\theta = 0$ and $\theta = \pi/2$, respectively. A slant submersion that is neither invariant nor anti-invariant is called proper slant submersion.

Now we have the following nontrivial example of slant submersion.

Example ([11]) Let R^5 be a Sasakian manifold and $f : R^5 \to R^2$ be a map defined by $f(x_1, x_2, y_1, y_2, z) = (x_1 - 2\sqrt{2}x_2 + y_1, 2x_1 - 2\sqrt{2}x_2 + y_1)$. Then, it is easy to see that

$$kerf_* = span\left\{V_1 = 2E_1 + \frac{1}{\sqrt{2}}E_4,\ V_2 = E_2,\ V_3 = \xi = E_5\right\}$$

and

$$(kerf_*)^\perp = span\left\{H_1 = 2E_1 - \frac{1}{\sqrt{2}}E_4,\ H_2 = E_3\right\}.$$

Then it is easy to see that f is a Riemannian submersion. In addition, $\phi V_1 = 2E_3 - \frac{1}{\sqrt{2}}E_2$ and $\phi V_2 = E_4$ imply that $g(\phi V_1, V_2) = \frac{1}{\sqrt{2}}$. So f is a slant submersion with slant angle $\theta = \pi/4$.

In above example, it is evident that the characteristic vector field ξ is vertical vector field. If ξ is orthogonal to $kerf_*$, then we have the following characterization.

Theorem 2.1 ([11]) *Let f be a slant submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). If ξ is orthogonal to $kerf_*$, then f is anti-invariant submersion.*

Proof By (1.9), (1.6), (1.11) and (2.5), we have

$$g(\phi X, Y) = -g(\nabla_X \xi, Y) = -g(\mathcal{T}_X \xi, Y) = g(\mathcal{T}_X Y, \xi)$$
$$= g(\mathcal{T}_Y X, \xi) = g(X, \phi Y)$$

for any $X, Y \in ker f_*$. Using the skew symmetry property of ϕ, we get the required result.

Remark ([11]) The above result is a submersion version of Lotta's result [2] for slant submanifold.

For a slant submersion f from an almost contact metric manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then for any $X \in ker f_*$, we put

$$\phi X = \psi X + \omega X, \tag{2.6}$$

where ψX and ωX are vertical and horizontal components of ϕX, respectively. Similarly, for any $Z \in (ker f_*)^\perp$, we have

$$\phi Z = BZ + CZ, \tag{2.7}$$

where BZ (respectively CZ) is the vertical part (respectively horizontal part) of ϕZ.
From (2.1), (2.6) and (2.7), we obtain

$$g_M(\psi X, Y) = -g_M(X, \psi Y) \tag{2.8}$$

and

$$g_M(\omega X, Z) = -g_M(X, BZ) \tag{2.9}$$

for any $X, Y \in ker f_*$ and $Z \in (ker f_*)^\perp$.
Using (1.5), (2.5) and (2.6), we obtain

$$\mathcal{T}_X \xi = -\omega X, \quad \hat{\nabla}_X \xi = -\psi X,$$

for any $X \in ker f_*$.

For two-dimensional fibers, we have the following result.

Proposition 2.2 ([11]) *Let f be a Riemannian submersion from an almost contact metric manifold onto a Riemannian manifold. If dim $(ker f_*) = 2$ and ξ is a vertical vector field, then the fibres are anti-invariant.*

Proposition 2.3 ([11]) *Let f be a Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) such that $\xi \in ker f_*$. Then f is an anti-invariant submersion if and only if D is integrable, where $D = ker f_* - \langle \xi \rangle$.*

Theorem 2.4 ([11]) *Let $M(\phi, \xi, \eta, g_M)$ be a Sasakian manifold of dimension $2m + 1$ and (N, g_N) is a Riemannian manifold of dimension n. Let $f : M \to N$ be a slant Riemannian submersion. Then the fibers are not totally umbilical.*

Proof Using (1.5) and (2.5), we obtain

$$\mathcal{T}_X \xi = -\omega X, \tag{2.10}$$

for any $X \in ker f_*$. If the fibers are totally umbilical, then we have $\mathcal{T}_X Y = g_M(X, Y)H$ for any vertical vector fields X, Y where H is the mean curvature vector field of any fiber. Since $\mathcal{T}_\xi \xi = 0$, we have $H = 0$, which shows that fibers are minimal. Hence, the fibers are totally geodesic, which is a contradiction to the fact $\mathcal{T}_X \xi = -\omega X \neq 0$.

By (1.5), (1.6), (2.6) and (2.7), we have

$$(\nabla_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \psi Y, \tag{2.11}$$

$$(\nabla_X \psi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y + R(\xi, X)Y, \tag{2.12}$$

where

$$(\nabla_X \omega)Y = \mathcal{H}\nabla_X \omega Y - \omega \hat{\nabla}_X Y$$

and

$$(\nabla_X \psi)Y = \hat{\nabla}_X \psi Y - \psi \hat{\nabla}_X Y,$$

for any $X, Y \in ker f_*$.

Now, we have the following characterization theorem for slant submersion.

Theorem 2.5 ([11]) *Let f be a Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) such that $\xi \in ker f_*$. Then f is a slant submersion if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$\psi^2 = -\lambda(I - \eta \otimes \xi). \tag{2.13}$$

Furthermore, in such a case, if θ is the slant angle of f, it satisfies $\lambda = \cos^2\theta$.

Now, we have the following lemma, which can be verified by above theorem.

Lemma 2.6 ([11]) *Let f be a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with slant angle θ. Then the following relations are valid for any $X, Y \in ker f_*$*

$$g_M(\psi X, \psi Y) = \cos^2\theta(g_M(X, Y) - \eta(X)\eta(Y)) \tag{2.14}$$

$$g_M(\omega X, \omega Y) = \sin^2\theta(g_M(X, Y) - \eta(X)\eta(Y). \tag{2.15}$$

The orthogonal complementary distribution to $\omega(kerf_*)$ in $(kerf_*)^\perp$ is denoted by μ. Then we have

$$(kerf_*)^\perp = \omega(kerf_*) \oplus \mu. \tag{2.16}$$

Lemma 2.7 ([11]) *Let f be a proper slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then μ is an invariant distribution of $(kerf_*)^\perp$ under the endomorphism ϕ.*

From the formula (2.14), we have the following consequence.

Corollary 2.8 ([11]) *Let f be a proper slant Riemannian submersion from a Sasakian manifold $M^{2m+1}(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N^n, g_N). Let $\{e_1, e_2, \ldots, e_{2m-n}, \xi\}$ be a local orthonormal frame of $(kerf_*)$, then $\{\csc\theta\omega e_1, \csc\theta\omega e_2, \ldots, \csc\theta\omega e_{2m-n}\}$ is a local orthonormal frame of $\omega(kerf_*)$.*

By using (2.15) and above corollary, one can easily prove the following proposition.

Proposition 2.9 ([11]) *Let f be a proper slant Riemannian submersion from a Sasakian manifold $M^{2m+1}(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N^n, g_N). Then dim $(\mu) = 2(n-m)$. If $\mu = \{0\}$, then $n = m$.*

By (2.8) and (2.13), we have.

Lemma 2.10 ([11]) *Let f be a proper slant Riemannian submersion from a Sasakian manifold $M^{2m+1}(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N^n, g_N). If $e_1, e_2, \ldots e_k$, ξ are orthogonal unit vector fields in $(kerf_*)$, then $\{e_1, \sec\theta\psi e_1, e_2, \sec\theta\psi e_2, \ldots e_k$, $\sec\theta\psi e_k, \xi\}$ is a local orthonormal frame of $(kerf_*)$. Moreover dim $(kerf_*) = 2m - n + 1 = 2k + 1$ and dim $N = n = 2(m-k)$.*

Lemma 2.11 ([11]) *Let f be a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). If ω is parallel, then we have*

$$\mathcal{T}_{\psi X}\psi X = -\cos^2\theta(\mathcal{T}_X X + \eta(X)\omega X) \tag{2.17}$$

for any $X \in kerf_*$.

Proof if ω is parallel, from (2.11), we obtain $C\mathcal{T}_X Y = \mathcal{T}_X \psi Y$ for $X, Y \in kerf_*$. Using antisymmetry with respect to X, Y and using (1.9), we get

$$\mathcal{T}_X \psi Y = \mathcal{T}_Y \psi X.$$

Substituting Y by ψX in the above equation and using Theorem 2.5, we get the required result.

In the following theorem, we will see the extension of harmonicity of slant submersions for almost Hermitain manifolds to harmonicity of slant submersion for setting of almost contact metric manifolds.

Theorem 2.12 ([11]) *Let f be a slant Riemannian submersion from a Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). If ω is parallel, then f is harmonic map.*

Proof We know that, f is harmonic if and only if $\sum_{i=1}^{n_1} \mathcal{T}_{e_i}^{e_i} = 0$. Hence, using the adapted frame for slant Riemannian submersion by Lemma 2.10, we can write

$$\tau = -\sum_{i=1}^{m-\frac{n}{2}} f_*(\mathcal{T}_{e_i} e_i + \mathcal{T}_{\sec\theta\psi e_i} \sec\theta\psi e_i - f_*(\mathcal{T}_\xi \xi).$$

Since $\mathcal{T}_\xi \xi = 0$, we have

$$\tau = -\sum_{i=1}^{m-\frac{n}{2}} f_*(\mathcal{T}_{e_i} e_i + \sec^2\theta \mathcal{T}_{\psi e_i} \psi e_i).$$

Using (2.17) in the above equation, we obtain

$$\tau = -\sum_{i=1}^{m-\frac{n}{2}} f_*(\mathcal{T}_{e_i} e_i + \sec^2\theta(-\cos^2\theta(\mathcal{T}_{e_i} e_i + \eta(e_i)\omega e_i))) = 0.$$

Thus f is a harmonic function.

3 Slant Submersions from Kenmotsu Manifolds

The purpose of the present section is to study pointwise slant submersion from Kenmotsu manifolds onto Riemannian manifolds with vertical and horizontal structure vector fields. The results of this section are taken from the study of Sushil Kumar et al. [26].

An almost contact metric manifold M is called a Kenmotsu manifold if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{3.1}$$

for any vector fields X, Y on M, where ∇ is the Riemannian connection of the Riemannian metric g. Moreover, on a Kenmotsu manifold, the following equation holds

$$\nabla_X \xi = X - \eta(X)\xi. \tag{3.2}$$

Let f be a Riemmanian submersion from a Kenmotsu manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). If for each $x \in M$, the angle $\theta(X)$ between ϕx and the space $ker f_*$ is independent of the choice of the nonzero vector field

$X \in ker f_* - \langle \xi \rangle$, then f is called a pointwise slant submersion and the angle θ is said to be slant function of the pointwise slant submersion.

A pointwise slant submersion is called slant if its slant function θ is independent of the choice of the point on $M(\phi, \xi, \eta, g_M)$. Then the constant θ is called the slant angle of the slant submersion [2].

Let f be a Riemannian submersion from a Kenmotsu manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Using Eqs. (3.2), (1.5), (1.7), (2.6) and (2.7), we get

$$\nabla_X \xi = X - \eta(X)\xi, \;\; \mathcal{T}_X \xi = 0, \tag{3.3}$$

We say that ω is parallel if

$$(\nabla_X \omega)Y = 0. \tag{3.4}$$

We have following lemma that can be verified easily.

Lemma 3.1 ([26]) *Let $M(\phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : M \to N$ is a pointwise slant submersion, then*

$$(\nabla_X \psi)Y = B\mathcal{T}_X Y - \mathcal{T}_X \omega Y - g(\psi X, Y)\xi + \eta(Y)\psi X \tag{3.5}$$

and

$$(\nabla_X \omega)Y = C\mathcal{T}_X Y - \mathcal{T}_X \psi Y + \eta(Y)\omega X, \tag{3.6}$$

for any $X, Y \in ker f_*$.

Remark Theorem 2.5 and Lemma 2.6 are also true for pointwise slant immersion from Kenmotsu manifold with slant function θ.

Now, we state the following theorem.

Theorem 3.2 ([26]) *If $M(\phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : M \to N$ is a pointwise slant submersion. If ω is parallel, then we have*

$$\mathcal{T}_{\psi X} \psi X = \cos^2\theta(\mathcal{T}_X X - \eta(X)\omega\psi X),$$

for any $X \in ker f_*$.

Theorem 3.3 ([26]) *If $M(\phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : M \to N$ is a pointwise slant submersion with nonzero slant function θ, then the fibers are totally geodesic submanifolds of M if and only if*

$$g_N(\nabla^N_{V'} f_*(\omega X), f_*(\omega Y)) = -g_M([X, V], Y)\sin^2\theta + V(\theta)g_M(\phi X, \phi Y)\sin 2\theta$$

$$+g_M(\mathcal{A}_V \omega\psi X, Y) - g_M(\mathcal{A}_V \omega X, \psi Y) - \eta(Y)g_M(BV, \psi X) - \eta(\nabla_V X)\eta(Y)\sin^2\theta,$$

for any $X, Y \in ker f_*$ and $V \in (ker f_*)^\perp$, where V and V' are $f-$related vector fields and ∇^N is the Riemannian connection on N.

Proof For any $X, Y \in kerf_*$ and $V \in (kerf_*)^{\perp}$, using Eqs. (2.1), (2.2), (1.5) and (2.6), we get

$$g_M(\mathcal{T}_X Y, V) = -g_M([X, Y], Y) + g_M(\nabla_V \psi^2 X, Y) + g_M(\nabla_V \omega\psi X, Y)$$
$$- g_M(\nabla_V \omega X, \phi Y) - \eta(\nabla_V X)\eta(Y).$$

From Theorem 3.2 and using Eqs. (1.5), (1.8) and (2.7), we get

$$g_M(\mathcal{T}_X Y, V)\sin^2\theta = -g_M([X, V], Y)\sin^2\theta + V(\theta) g_M(\phi X, \phi Y)\sin 2\theta +$$
$$+ g_M(\mathcal{A}_V \omega\psi X, Y) - g_N(\nabla^N_{V'} f_*(\omega X), f_*(\omega Y)) - g_M(\mathcal{A}_V \omega X, \psi Y)$$
$$-\eta(\nabla_V X)\eta(Y)\sin^2\theta - \eta(Y) g_M(BV, \psi X).$$

By considering the fibers as totally geodesic, we derive the formula in the above theorem. Conversely, it can directly verified.

Theorem 3.4 ([26]) *Let $M(\phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : M \to N$ be a pointwise slant submersion with nonzero slant function θ, then f is harmonic if and only if*

$$trace^* f_*((\nabla f_*)((.)\omega\psi(.))) - trace\mathcal{T}_{(.)}\omega(.) + traceC^* f_*(\nabla f_*)((.)\omega(.)) = 0.$$

Proof For any $X \in kerf_*$ and $V \in (kerf_*)^{\perp}$, using Eqs. (2.1), (2.2), (1.5), (2.6) and (2.7), we get

$$g_M(\mathcal{T}_X X, V) = g_M(\nabla_X X, V)\cos^2\theta - g_M(\nabla_X \omega\psi X, V) + g_M(\nabla_X \omega X, \phi V).$$

From Theorem 3.2 and using Eqs. (2.2), (3.1) and (2.6), we get

$$g_M(\mathcal{T}_X X, V) = g_M(\nabla_X X, V)\cos^2\theta - g_M(\nabla_X \omega\psi X, V) + g_M(\nabla_X \omega X, \phi V).$$

Using Eqs. (1.6), (2.7) and by definition of adjoint map $*f_*$, we have

$$g_M(\mathcal{T}_X X, V)\sin^2\theta = g_N(f_*(\nabla f_*)(X, \omega\psi X), V) - g_M(\omega\mathcal{T}_X \omega X, V)$$
$$-g_N(C^* f_*(\nabla f_*)(X, \omega X), V).$$

Conversely, a direct computation gives the proof.

Now, we study pointwise slant submersion from Kenmotsu manifolds onto Riemannian manifolds for $\xi \in (kerf_*)^{\perp}$.

When $\xi \in (kerf_*)^{\perp}$, then from Eqs. (2.1) and (2.2), we get

$$\phi^2 X = -X \tag{3.7}$$

and

$$g(\phi X, \phi Y) = g(X, Y), \tag{3.8}$$

for any $X, Y \in ker f_*$. Moreover, from Eqs. (1.6), (1.8), (3.2), (2.6) and (2.7), we get

$$\mathcal{T}_X \xi = X, \tag{3.9}$$

$$\mathcal{A}_V \xi = 0, \tag{3.10}$$

and

$$\eta(\nabla_X Y) = -g_M(X, Y), \tag{3.11}$$

for any $X, Y \in ker f_*$ and $V \in (ker f_*)^\perp$.

Corollary 3.5 ([26]) *If $M(\phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : M \to N$ be a pointwise slant submersion, then*

$$g_M(\psi X, \psi Y) = \cos^2\theta g_M(X, Y),$$

$$g_M(\omega X, \omega Y) = \sin^2\theta g_M(X, Y),$$

for any $X, Y \in ker f_*$.

Theorem 3.6 ([26]) *Let $M(\phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. Assume that $f : M \to N$ is a pointwise slant submersion with slant function θ. If ω is parallel, then*

$$\mathcal{T}_{\psi X}\psi X = \cos^2\theta \mathcal{T}_X X,$$

for any $X \in ker f_*$.

Theorem 3.7 ([26]) *Let $M(\phi, \xi, \eta, g_M)$ be a Kenmotsu manifold and (N, g_N) be a Riemannian manifold. If $f : M \to N$ is a pointwise slant submersion with nonzero slant function θ, then the fibers are totally geodesic submanifolds of M if and only if*

$$g_N((\nabla'^N_V f_*(\omega X), f_*(\omega Y)) = -g_M([X, V], Y)\sin^2\theta + V(\theta)g_M(X, Y)\sin 2\theta$$

$$+g_M(\mathcal{A}_V \omega\psi X, Y) - g_M(\mathcal{A}_V \omega X, \psi Y),$$

for any $X, Y \in ker f_*$ and $V \in (ker f_*)^\perp$, where V and V' are f-related vector fields and ∇^N is the Riemannian connection on N.

Proof For any $X, Y \in kerf_*$ and $V \in (kerf_*)^\perp$, using Eqs. (2.1), (3.1), (1.5), (1.8), (2.6), (2.7) and Theorem 3.2, we get

$$g_M(\mathcal{T}_X Y, V)\sin^2\theta = -g_M([X, V], Y)\sin^2\theta + V(\theta)g_M(X, Y)\sin 2\theta + g_M(\mathcal{A}_V\omega\psi X, Y)$$

$$-g_N(\nabla^N_{V'} f_*(\omega X), f_*(\omega Y)) - g_M(\mathcal{A}_V\omega X, \psi Y).$$

By considering the fibers as totally geodesic, we derive the formula. Conversely, it can be directly verified.

4 Slant Submersions from Almost Paracontact Metric Manifolds

Recently, Yilmaz Gündüzalp [29] investigated slant submersions whose total space is an almost paracontact metric manifold. In this section, we quote some results from the study of Yilmaz Gündüzalp.

Let M be a $(n+1)$-dimensional differentiable manifold. If there exist on M a $(1,1)$ type tensor field ϕ, a vector field ξ and 1-form η satisfying

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{4.1}$$

then $\bar{M}$ is said to be an almost paracontact manifold. In the almost paracontact manifold, the following relations hold

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad rank(\phi) = n. \tag{4.2}$$

An almost paracontact manifold $\bar{M}$ is said to be an almost paracontact metric manifold [25], if there exists a pseudo-Riemannian metric g_M on M satisfying the following

$$g_M(\phi X, \phi Y) = g_M(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{4.3}$$

for all $X, Y \in TM$.

Further, we can determine an anti-symmetric two-form Φ by $\phi(X, Y) = g_M(X, \phi Y)$, which is called the fundamental 2-form corresponding to the structure.

An almost paracontact metric structure (ϕ, ξ, η, g_M) is said to be paracosymplectic, if $\nabla\eta = 0$ and $\nabla\Phi = 0$ are closed [25] and the tensorial equation of a paracosymplectic manifold is given by

$$(\nabla_X\phi)Y = 0, \quad X, Y \in TM, \tag{4.4}$$

where ∇ denotes the Riemannian connection of the metric g_M on M. Moreover, for a paracosymplectic manifold, we know that

$$\nabla_X \xi = 0. \tag{4.5}$$

Let f be a semi-Riemannian submersion from an almost paracontact metric manifold M with the structure (ϕ, ξ, η, g_M) onto a semi-Riemannian manifold (N, g_N). Then for $X \in kerf_*$ and $Z \in (kerf_*)^{\perp}$ we consider the Eqs. (2.6) and (2.7), respectively.

If for any spacelike or timelike vertical vector field $X \in kerf_* - \langle \xi \rangle$, the quotient $\frac{g_M(\psi X, \psi X)}{g_M(\phi X, \phi Y)}$ is constant that is, it is independent of the choice of the point $p \in M$ and choice of the spacelike or timelike vertical vector field $X \in kerf_* - \langle \xi \rangle$, at that time we call that f is a slant submersion. In this case, the angle θ is called the slant angle of the slant submersion. We note that the vector field ξ is a spacelike vertical vector field.

Let $\{e_1, e_2, \xi\}$ be a local orthonormal frame of vertical vector fields with $g_M(e_1, e_2) = 1$ such that e_1 is spacelike (if both e_1 and e_2 are timelike, the situation would be similar). From (4.3) and (2.6), we have

$$-1 = g_M(\phi e_1, \phi e_1) = g_M(\psi e_1, \psi e_1) + g_M(\omega e_1, \omega e_1).$$

On the other hand, $\psi e_1 = \rho e_2$. Let $\rho \neq 0, \pm 1$, these conditions would correspond to invariant and anti-invariant submersion [29]. Clearly, ψe_1 and e_2 have the same character. Depending on it and the value of ρ, we can separate the following three conditions:

1. If ψe_1 is a timelike and $\|\rho\| > 1$, at this moment $g_M(\omega e_1, \omega e_1) = -1 + \rho^2$ and ωe_1 is spacelike.
2. If ψe_1 is a timelike and $\|\rho\| < 1$, at this moment $g_M(\omega e_1, \omega e_1) = -1 + \rho^2$ and ωe_1 is timelike.
3. If ψe_1 is a spacelike and $g_M(\omega e_1, \omega e_1) = -1 - \rho^2$ and ωe_1 is a timelike vector field.

These three conditions classify the proper slant submersion into three types submersions, which are defined as

Let f be a proper slant submersion from an almost paracontact manifold $M(\phi, \xi, \eta, g_M)$ onto a semi-Riemannian manifold (N, g_N). we say that f is of

1. type 1 if for any spacelike(timelike) vertical vector field $X \in kerf_*$, ψX is timelike (spacelike) and $\frac{\|\psi X\|}{\|\phi X\|} > 1$,
2. type 2 if for any spacelike(timelike) vertical vector field $X \in kerf_*$, ψX is timelike (spacelike) and $\frac{\|\psi X\|}{\|\phi X\|} < 1$,
3. type 3 if for any spacelike (timelike) vertical vector field $X \in kerf_*$, ψX is timelike (spacelike).

It is known that the distribution $(kerf_*)$ is integrable for a semi-Riemannian submersion between semi-Riemannian manifolds. Infact, its leaves are $f^{-}(b),\ b \in N$, that is, fibers. Thus it follows from the above definition that the fibers of a slant submersion are slant submanifold of M.

Theorem 4.1 ([29]) *Let f be a proper slant submersion from an almost paracontact manifold M with the structure (ϕ, ξ, η, g_M) onto a semi-Riemannian manifold (N, g_N). Then,*

1. *f is slant submersion of type 1 if and only if for any spacelike (timelike) vector field $X \in kerf_*$, ψX is timelike (spacelike), and there exists a constant $\mu \in (1, \infty)$ such that*
$$\psi^2 X = \mu(X - \eta(X)\xi). \tag{4.6}$$
If f is a proper slant submersion of type 1, then $\mu = \cos h^2\theta$, with $\theta > 0$.
2. *f is proper slant submersion of type 2 if and only if for any spacelike (timelike) vector field $X \in kerf_*$, ψX is timelike (spacelike), and there exists a constant $\mu \in (0, 1)$ such that*
$$\psi^2 X = \mu(X - \eta(X)\xi). \tag{4.7}$$
If f is a proper slant submersion of type 2, then $\mu = \cos h^2\theta$, with $0 < \theta < 2\pi$.
3. *f is slant submersion of type 3 if and only if for any spacelike (timelike) vector field $X \in kerf_*$, ψX is timelike (spacelike), and there exists a constant $\mu \in (-\infty, 0)$ such that*
$$\psi^2 X = \mu(X - \eta(X)\xi). \tag{4.8}$$
If f is a proper slant submersion of type 3, then $\mu = -\sin h^2\theta$, with $\theta > 0$.

In every case, the angle θ is called the slant angle of slant submersion.

Proof Part 1, If f is slant submersion of type 1, for any spacelike vertical vector field $X \in kerf_*$, ψX is timelike and by virtue of (4.3), ϕX is timelike. Furthermore, they satisfy $\frac{\|\psi X\|}{\|\phi X\|} > 1$. So, there exists $\theta > 0$ such that

$$\cos h\theta = \frac{\|\psi X\|}{\|\phi X\|} = \frac{\sqrt{-g_M(\psi X, \psi X)}}{\sqrt{-g_M(\phi X, \phi X)}}. \tag{4.9}$$

By using (4.1), (4.2), (2.6) and (4.9), we obtain

$$g_M(\psi^2 X, X) = \cos^2 h\theta g_M(X - \eta(X)\xi, X), \tag{4.10}$$

for all $X \in kerf_*$. Since g_M is a semi-Riemannian metric, from (4.10) we get

$$\psi^2 X = \cos h^2\theta(X - \eta(X)\xi), \quad X \in kerf_*. \tag{4.11}$$

Let $\mu = \cos h^2\theta$, then it is easy to see that $\mu \in (1, \infty)$ and $\psi^2 = \mu(I - \eta \otimes \xi)$. Now for timelike vector field $Y \in kerf_*$, but ψY and ϕY are spacelike and hence (4.9) can be written as

$$\cos h\theta = \frac{\|\psi Y\|}{\|\phi Y\|} = \frac{\sqrt{g_M(\psi Y, \psi Y)}}{\sqrt{g_M(\phi Y, \phi Y)}}. \tag{4.12}$$

Since $\psi^2 Y = \mu(Y - \eta(Y)\xi)$, for any spacelike or timelike Y we have that $\psi^2 = \mu(I - \eta \otimes \xi)$. The converse is straightforward. Part 2 can be proved by using similar steps.

Part 3, If f is proper slant submersion of type 3, for any spacelike vector field $X \in kerf_*$, ψX is spacelike, hence there exists $\theta > 0$ such that

$$\sin h\theta = \frac{\|\psi X\|}{\|\phi X\|} = \frac{\sqrt{g_M(\psi X, \psi X)}}{\sqrt{-g_M(\phi X, \phi X)}}. \tag{4.13}$$

Now, it is evident that $g_M(\psi^2 X, X) = -\sin h^2\theta g_M(X - \eta(X)\xi, X)$. Let $\mu = -\sin h^2\theta$, at this moment $\mu \in (-\infty, 0)$ and $\psi^2 = \mu(I - \eta \otimes \xi)$. The converse can be proved by using some easy computations.

Theorem 4.2 ([29]) *Let f be a proper slant submersion from an almost paracontact manifold M with the structure (ϕ, ξ, η, g_M) onto a semi Riemannian manifold (N, g_N). Then,*

1. *f is slant submersion of type 1 if and only if $\psi^2 X = \cos h^2\theta(X - \eta(X)\xi)$ for every spacelike vector field $X \in kerf_*$.*
2. *f is slant submersion of type 2 if and only if $\psi^2 X = \cos^2\theta(X - \eta(X)\xi)$ for every spacelike vector field $X \in kerf_*$.*

Proof (1) For every timelike vector field $Y \in kerf_*$, there exists a spacelike vector field $X \in kerf_*$ such as $\psi X = Y$. Then

$$\psi^2 Y = \psi^2\psi X = \psi^3 X = \cos h^2\theta(\psi X - \eta(\psi X)\xi) = \cos h^2\theta(Y - \eta(Y)\xi). \tag{4.14}$$

The same proof is valid for part 2, but $\psi^2 X = \cos^2\theta(X - \eta(X)\xi)$.

Theorem 4.3 ([29]) *Let f be a proper slant submersion from an almost paracontact manifold M with the structure (ϕ, ξ, η, g_M) onto a semi-Riemannian manifold (N, g_N). Then we have following classifications*

1. *f is slant submersion of type 1 if and only if $B\omega X = -\sin h^2\theta(X - \eta(X)\xi)$ for every spacelike (timelike) vertical vector field $X \in kerf_*$.*
2. *f is slant submersion of type 2 if and only if $B\omega X = -\sin^2\theta(X - \eta(X)\xi)$ for every spacelike (timelike) vertical vector field $X \in kerf_*$.*
3. *f is slant submersion of type 3 if and only if $B\omega X = \cos h^2\theta(X - \eta(X)\xi)$ for every spacelike (timelike) vertical vector field $X \in kerf_*$.*

Theorem 4.4 ([29]) *Let f be a semi-Riemannian submersion from an almost paracontact metric manifold $M^{4n+1}_{2n}(\phi, \eta, \xi, g_M)$ onto a semi-Riemannian manifold (N^{2n}_n, g_N). Then, we have*

1. *f is slant submersion of type 1 if and only if $C^2 Y = \cos h^2\theta Y$ for every spacelike (timelike) horizontal vector field $Y \in (ker f_*)^\perp$.*
2. *f is slant submersion of type 2 if and only if $C^2 Y = \cos^2\theta Y$ for every spacelike (timelike) horizontal vector field $Y \in (ker f_*)^\perp$.*

Now, we have following examples of proper slant submersions.

Example ([29]) Determine a map $f : R^5_2 \to R^2_1$ by

$$f(x_1, x_2, x_3, x_4, z) = \left(\frac{x_1 - x_3}{\sqrt{2}}, x_2\right).$$

By direct calculations, we obtain

$$ker f_* = span\left\{U_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3},\ \ U_2 = \frac{\partial}{\partial x_4},\ \ U_3 = \xi = \frac{\partial}{\partial z}\right\}$$

and

$$(ker f_*)^\perp = span\left\{X_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3},\ \ X_2 = \frac{\partial}{\partial x_2}\right\}.$$

Thus, the map f is a slant submersion of type 2 with the slant angle $\theta = \pi/4$.

Example ([29]) Define a map $f : R^5_2 \to R^2_1$ by

$$f(x_1, x_2, x_3, x_4, z) = (x_2 \sin hx + x_3 \cos hx, x_1 \sin hy + x_4 \cos hy),$$

for any $x, y \in R$. Then, by direct calculation we get

$$ker f_* = span\left\{U_1 = \cos hx\frac{\partial}{\partial x_2} - \sin hx\frac{\partial}{\partial x_3},\ U_2 = \cos hy\frac{\partial}{\partial x_1} - \sin hy\frac{\partial}{\partial x_4},\right.$$

$$\left. U_3 = \xi = \frac{\partial}{\partial z}\right\}$$

and

$$(ker f_*)^\perp = span\left\{X_1 = -\sin hx\frac{\partial}{\partial x_2} + \cos hx\frac{\partial}{\partial x_3},\ X_2 = \sin hy\frac{\partial}{\partial x_1} - \cos hy\frac{\partial}{\partial x_4}\right\}.$$

Thus, the map f is a slant submersion of type 1 with the slant angle $\cos\theta = \cos h(x - y)$.

Example ([29]) Define a map $f : R_2^5 \to R_1^2$ by

$$f(x_1, x_2, x_3, x_4, z) = (x_2 \cos hx + x_3 \sin hx, x_4),$$

for any $x \in R^+$. The map f is a slant submersion of type 3 with the slant angle $\theta = -\sin h^2 x$.

5 Warped Product Pointwise Slant and Semi-Slant Submanifolds of Almost Contact Metric Manifolds

In the previous sections, we already discussed Sasakian and Kenmotsu manifold, now we define the cosymplectic structure on an almost contact metric manifold.

The following tensorial equation characterizing a cosymplectic manifold

$$(\bar{\nabla}_X \phi)Y = 0, \tag{5.1}$$

for any $X, Y \in T\bar{M}$.

Moreover, on a cosymplectic manifold $\bar{M}$

$$\bar{\nabla}_X \xi = 0. \tag{5.2}$$

On the similar line of pointwise slant submanifold of almost Hermitian manifolds introduced by Etayo [9], K.-S.Park defined and studied pointwise slant submanifolds of almost contact metric manifolds. He defined as follows.

Definition ([16]) Let $\bar{M}(\phi, \xi, \eta, g)$ be $2n + 1$-dimensional almost contact metric manifold and M be a submanifold of $\bar{M}$. The submanifold M is called a pointwise slant submanifold if at each point $p \in M$ the angle $\theta(X)$ between ϕX and the space M_p is constant for nonzero $X \in M_p$, where $M_p = \{X \in T_pM : g(X, \xi(p)) = 0\}$. The angle θ is called slant function as a function on M.

Notice that the above definition does not depend on ξ.

Throughout this section, we consider Eqs. (2.6) and (2.7) for any $X \in TM$ and $Z \in T^{\perp}M$, respectively. Where, ψX and ωX denote the tangential and normal components of ϕX, respectively. Whereas the tangential and normal components of ϕZ are denoted by BZ and CZ, respectively.

Let $T^1M = \cup_{p \in M} M_p = \cup_{p \in M}\{X \in T_pM : g(X, \xi(p)) = 0\}$. Now, we have some initial results.

Lemma 5.1 ([16]) *Let M be a submanifold of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$. Then M is a pointwise slant submanifold of $\bar{M}$ if and only if $\psi^2 = -\cos^2\theta I$, for some function $\theta : M \to R$.*

Remark ([16])Let M be a pointwise slant submanifold of an almost contact metric manifold $\bar{M}$ with the slant function θ. By using Lemma 5.1, we easily get

$$g(\psi X, \psi Y) = \cos^2\theta g(X, Y), \tag{5.3}$$

$$g(\omega X, \omega Y) = \sin^2\theta g(X, Y), \tag{5.4}$$

for any $X, Y \in T^1M$. At each given point $p \in M$ with $0 \leq \theta(p) < \pi/2$. by using (5.3) we can choose an orthonormal basis $\{X_1, \sec\theta\psi X_1, \ldots, X_k, \sec\theta\psi X_k\}$ of M_p.

Using Lemma 5.1, we obtain.

Corollary 5.2 ([16]) *Let M be a pointwise slant submanifold of an almost contact metric manifold $\bar{M}$ with the nonconstant slant function $\theta : M \to R$. Then M is even dimensional.*

Proposition 5.3 ([16]) *Let M be a two-dimensional submanifold of an almost contact metric manifold $\bar{M}$. Then M is a pointwise slant submanifold of $\bar{M}$.*

Now, we have following characterization.

Theorem 5.4 ([16]) *Let M be a pointwise slant connected totally geodesic submanifold of a cosymplectic manifold $\bar{M}(\phi, \xi, \eta, g)$. Then M is a slant submanifold of $\bar{M}$.*

Proof Given any two points $p, q \in M$, we choose a C^∞-curve $c : [0, 1] \to M$ such that $c(0) = p$ and $c(1)q$. For nonzero $X \in M_p$, we take a parallel transport $Z(t)$ along the curve c in M such that $Z(0) = X$ and $Z(1) = Y$. Then since M is totally geodesic,

$$0 = \nabla_{c'} Z(t) = \bar{\nabla}_{c'} Z(t), \tag{5.5}$$

where ∇ and $\bar{\nabla}$ are the Levi-Civita connection on M and $\bar{M}$, respectively. By the uniqueness of parallel transports, $Z(t)$ is also a parallel transport in $\bar{M}$. Since ξ is parallel, we have

$$\frac{d}{dt} g(Z(t), \xi) = g(\bar{\nabla}_{c'} Z(t), \xi) + g(Z(t), \bar{\nabla}_{c'}\xi) = 0, \quad g(Z(0), \xi) = 0 \tag{5.6}$$

so that

$$0 = g(Z(1), \xi) = g(Y, \xi),$$

which implies $Y \in M_q$. But by (5.1),

$$\bar{\nabla}_{c'}\phi Z(t) = (\bar{\nabla}_{c'}\phi)Z(t) + \phi\bar{\nabla}_{c'} Z(t) = 0$$

so that $\phi Z(t)$ becomes a parallel transport along c in $\bar{M}$ such that $\phi Z(0) = \phi X$ and $\phi Z(1) = \phi Y$.

Define a map $\tau : T_p\bar{M} \to T_q\bar{M}$ by $\tau(U) = V$ for $U \in T_pN$ and $V \in T_qN$, where $W(t)$ is the parallel transport along c in $\bar{M}$ such that $W(0) = U$ and $W(1) = V$. Then τ is surely isometry. It is easy to check that $\tau(T_pM) = T_pM$ and $\tau(T_p^{\perp}M) = T_p^{\perp}M$ so that $\tau(\phi X) = \phi Y$ means $\tau(\psi X) = \psi Y$. Hence, $\cos\theta(p) = \frac{\|\psi X\|}{\|X\|} = \frac{\|\psi Y\|}{\|Y\|} = \cos\theta(q)$, where θ is the slant function on M. Therefore, the result follows.

Corollary 5.5 ([16]) *Let M be a two-dimensional connected totally geodesic submanifold of a cosymplectic manifold $\bar{M}$. Then M is a slant submanifold $\bar{M}$.*

Proposition 5.6 ([16]) *Let M be a submanifold of an almost contact metric manifold $\bar{M}$. Then M is a pointwise slant submanifold $\bar{M}$ if and only if*

$$g(\psi X, \psi Y) = 0 \ \ where \ \ g(X, Y) = 0 \ \ for \ \ X, Y \in M_p, \ \ p \in M. \tag{5.7}$$

Now, we have a general result, which is true for Sasakian, Kenmotsu as well as cosymplectic manifold.

Theorem 5.7 ([16]) *Let M be slant submanifold of an almost contact metric manifold $\bar{M}$ with the slant angle θ. Assume that $\bar{M}$ is one of the three manifolds cosymplectic, Sasakian and Kenmotsu. Then we have.*

$$A_{\omega X}\psi X = A_{\omega\psi X}X, \ \ for X \in T^1M. \tag{5.8}$$

Now, we have examples of pointwise slant submanifolds.

Example ([16]) Define a map $f : R^3 \to R^5$ by $f(x_1, x_2, x_3) = (y_1, y_2, y_3, y_4, t) = (x_1, \sin x_2, 0, \cos x_2, x_3)$. Let $M = \{(x_1, x_2, x_3) \in R^3 : 0 < x_2, \pi/2\}$. We define (ϕ, ξ, η, g) on R^5 as $\phi(a_1\frac{\partial}{\partial y_1} + \cdots + a_4\frac{\partial}{\partial y_4} + a_5\frac{\partial}{\partial t}) = -a_2\frac{\partial}{\partial y_1} + a_1\frac{\partial}{\partial y_2} - a_4\frac{\partial}{\partial y_3} + a_3\frac{\partial}{\partial y_4}$, $\xi = \frac{\partial}{\partial t}$, $\eta = dt$, g is the Euclidean metric on R^5, then (ϕ, ξ, η, g) is an almost contact metric structure. Then M is a pointwise slant submanifold of an almost contact metric manifold $R^5(\phi, \xi, \eta, g)$ with the slant function x_2 such that ξ is tangent to M.

Example ([16]) Define a map $f : R^2 \to R^5$ by $f(x_1, x_2) = (y_1, y_2, y_3, y_4, t) = (0, \cos x_1, x_2, \sin x_1, 0)$. Let $M = \{(x_1, x_2) \in R^2 : 0 < x_1 < \pi/2\}$. We also know that $R^5(\phi, \xi, \eta, g)$ is a cosymplectic manifold. Then M is a pointwise slant submanifold of a cosymplectic manifold of $R^5(\phi, \xi, \eta, g)$ with the slant function x_1 such that ξ is normal to M.

Now, we define the pointwise semi-slant submanifolds.

Definition ([16]) Let $\bar{M}(\phi, \xi, \eta, g)$ be an almost contact metric manifold and M be a submanifold of $\bar{M}$. The submanifold M is called a pointwise semi-slant submanifold if there is a distribution $D \subset TM$ on M such that

$$TM = D \oplus D_\theta, \ \ \phi D \subset D$$

and at each given point $p \in M$ the angle $\theta = \theta(X)$ between ϕX and the space $(D_\theta)_p$ is constant for nonzero $X \in (D_\theta)_p$, where D_θ is the orthogonal complement of D in TM. The angle θ is called a semi-slant function as a function on M.

Note that the normal bundle $T^\perp M$ of a pointwise semi-slant submanifold M is decomposed as

$$T^\perp M = \omega D_\theta \oplus \mu, \ \ \omega D_\theta \perp \mu,$$

where μ is an invariant normal subbundle of $T^\perp M$ under ϕ.

We have following characterizations for semi-slant submanifolds.

Proposition 5.8 ([16]) *Let M be a pointwise semi-slant submanifold of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$. Assume that either $D_\theta \subset \ker \eta$ or $\mu \subset \ker \eta$. Then μ is ϕ-invariant.*

Lemma 5.9 ([16]) *Let M be a pointwise semi-slant submanifold of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ with the semi-slant function θ. Then*

$$g(\psi^2 + \cos^2\theta(I - \eta \otimes \xi))(X), Y) = 0, \quad for\ X, Y \in D_\theta. \tag{5.9}$$

Proof We will prove this at each point of M. Let $p \in M$ be a point such that $p \in D_\theta$ at p is vanishing, then done. Given a nonzero $X \in D_\theta$ at p, we obtain $\cos\theta(p) = \frac{\|\psi X\|}{\|\phi X\|}$, so that $\cos^2\theta(p) g(\phi X, \phi X) = g(\psi X, \psi X) = -g(\psi^2 X, X)$. Substituting X by $X + Y, \forall\ Y \in D_\theta$ at the above equation, we have

$$g((\psi^2 + \cos^2\theta(I - \eta \otimes \xi))(X), Y) = 0.$$

Lemma 5.10 ([16]) *Let M be a pointwise semi-slant submanifold of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$. We have*

1. *Suppose, ξ is tangent to M and $\bar{M}$ is either cosymplectic or Sasakian or Kenmotsu, then the distribution D is integrable if and only if*

$$g(h(X, \phi Y) - h(Y, \phi X), \omega Z) = 0,$$

2. *Suppose, ξ is normal to M and $\bar{M}$ is either cosymplectic or Kenmotsu, then the distribution D is integrable if and only if*

$$g(h(X, \phi Y) - h(Y, \phi X), \omega Z) = 0,$$

for any $X, Y \in D$ and $Z \in D_\theta$.

Lemma 5.11 ([16]) *Let M be a pointwise semi-slant submanifold of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$. We have*

1. *Suppose, ξ is tangent to M and $\bar{M}$ is either cosymplectic or Kenmotsu, then the distribution D_θ is integrable if and only if*

$$g(A_{\omega\psi W}Z - A_{\omega\psi Z}W, X) = g(A_{\omega W}Z - A_{\omega Z}W, \phi X),$$

2. *Suppose, ξ is normal to M and $\bar{M}$ is either cosymplectic or Kenmotsu, then the distribution D_θ is integrable if and only if*

$$g(A_{\omega\psi W}Z - A_{\omega\psi Z}W, X) = g(A_{\omega W}Z - A_{\omega Z}W, \phi X),$$

for any $X \in D$ and $Z, W \in D_\theta$.

For the totally geodesicness of the leaves, we have.

Theorem 5.12 ([16]) *Let M be a pointwise semi-slant submanifold of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$. We have*

1. *Suppose, ξ is tangent to M and $\bar{M}$ is either cosymplectic or Sasakian or Kenmotsu, then the distribution D defines a totally geodesic foliation if and only if*

$$g(A_{\omega Z}\phi X - A_{\omega\psi Z}X, Y) = 0,$$

2. *Suppose, ξ is normal to M and $\bar{M}$ is either cosymplectic or Kenmotsu, then the distribution D defines a totally geodesic foliation if and only if*

$$g(A_{\omega Z}\phi X - A_{\omega\psi Z}X, Y) = 0$$

for any $X, Y \in D$ and $Z \in D_\theta$.

Theorem 5.13 ([16]) *Let M be a pointwise semi-slant submanifold of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$. Suppose, ξ is normal to M and $\bar{M}$ is either cosymplectic or Kenmotsu, then the distribution D_θ defines a totally geodesic foliation if and only if*

$$g(A_{\omega Z}\phi X - A_{\omega\psi Z}X, W) = 0$$

for any $X \in D$ and $Z, W \in D_\theta$.

Theorem 5.14 ([16]) *Let M be a pointwise semi-slant submanifold of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$. Assume that ξ is tangent to M*

1. *If $\bar{M}$ is either cosymplectic or Sasakian, then the distribution D_θ defines a totally geodesic foliation if and only if*

$$g(A_{\omega Z}\phi X - A_{\omega\psi Z}X, W) = 0,$$

2. *If $\bar{M}$ is Kenmotsu, then the distribution D_θ defines a totally geodesic foliation if and only if*

$$g(A_{\omega Z}\phi X - A_{\omega\psi Z}X, W) + \sin^2\theta\eta(X)g(W, Z) = 0,$$

for any $X \in D$ and $Z, W \in D_\theta$.

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and f be a positive differentiable function on N_1. If $a : N_1 \times N_2 \to N_1$ and $b : N_1 \times N_2 \to N_2$ are the projection maps given by $a(x, y) = x$ and $b(x, y) = y$ for every $(x, y) \in N_1 \times N_2$, then the *warped product manifold* is the product manifold $M = N_1 \times N_2$ endowed with the Riemannian structure such that

$$g(X, Y) = g_1(x_*X, x_*Y) + (f \circ x)^2 g_2(y_*X, y_*Y),$$

for all $X, Y \in TM$. The function f is called the *warping function* of the warped product manifold. If the warping function is constant, then the warped product is trivial that is, simply Riemannian product. Further, if $X \in TN_1$ and $Z \in TN_2$, then from Lemma 7.3 of [21], we have the following well known result

$$\nabla_X Z = \nabla_Z X = \left(\frac{Xf}{f}\right) Z, \tag{5.10}$$

where ∇ is the Levi-civita connection on M. For $M = N_1 \times_f N_2$, it can be seen that

$$\nabla_X Z = \nabla_Z X = X ln f Z. \tag{5.11}$$

The gradient of the function f is denoted by ∇f and is defined as

$$g(\nabla f, X) = Xf \tag{5.12}$$

for all $X \in TM$.

Let M be a Riemannian manifold M of dimension n with $\{e_1, \ldots, e_n\}$ as an orthogonal basis of TM and g a Riemannian metric of M. Then as a result of (5.12), we set

$$\|\nabla f\|^2 = \sum_{i=1}^{n} (e_i(f))^2. \tag{5.13}$$

Let N_T and N_θ are the invariant and pointwise slant submanifolds of an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$. In [16], Park studied the warped product submanifolds of the types $N_\theta \times_f N_T$ and $N_T \times_f N_\theta$ in the setting of almost contact metric manifolds and obtained various existence and nonexistence conditions. Basically, he proved the following results.

Theorem 5.15 ([16]) *Let $\bar{M}(\phi, \xi, \eta, g)$ be an almost contact metric manifold either cosymplectic or Sasakian or Kenmotsu. Then there does not exist warped product* *submanifolds of the type $N_\theta \times_f N_T$ with ξ tangential or normal to M, where N_θ and N_T are the pointwise slant and invariant submanifold of $\bar{M}$, respectively.*

Further, K. -S. Park considered the warped product of type $N_T \times_f N_\theta$ and proved the following lemma.

Lemma 5.16 ([16]) *Let $\bar{M}(\phi, \xi, \eta, g)$ be an almost contact metric manifold either cosymplectic or Sasakian or Kenmotsu. Then there does not exist warped product submanifolds of the type $N_T \times_f N_\theta$ with ξ tangential or normal to M, where N_θ and N_T are the pointwise slant and invariant submanifold of $\bar{M}$ respectively. Then*

$$g(A_{\omega Z}W, X) = g(A_{\omega W}Z, X), \tag{5.14}$$

for any $X \in TN_T$ and $Z, W \in TN_\theta$.

Lemma 5.17 ([16]) *Let $N_T \times_f N_\theta$ be a nontrivial warped product proper pointwise semi-slant submanifold of an an almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$*

1. *If $\bar{M}$ is cosymplectic, then*

$$g(A_{\omega\psi Z}W, X) = -\phi X lnf g(W, \psi W) - \cos^2\theta X lnf g(W, Z) \ \ and$$

$$g(A_{\omega Z}W, \phi X) = (X - \eta(X)\xi)(lnf)g(W, Z) - \phi X(lnf)g(\psi W, Z).$$

2. *If $\bar{M}$ is Sasakian, then*

$$g(A_{\omega\psi Z}W, X) = -\eta(X)g(W, \psi W) - \phi X lnf g(W, \psi W) - \cos^2\theta X lnf g(W, Z)$$

$$and \ \ g(A_{\omega Z}W, \phi X) = (X - \eta(X)\xi)(lnf)g(W, Z) - \phi X(lnf)g(\psi W, Z).$$

3. *If $\bar{M}$ is Kenmotsu, then*

$$g(A_{\omega\psi Z}W, X) = \cos^2\theta\eta(X)(g(Z, W) - \eta(Z)\eta(W)) - \phi X lnf g(W, \psi Z)$$

$$- \cos^2\theta X lnf g(W, Z)$$

$$and \ \ g(A_{\omega Z}W, \phi X) = (X - \eta(X)\xi)(lnf)g(W, Z) - \phi X(lnf)g(\psi W, Z)$$

for $X \in TN_T$ and $Z, W \in TN_\theta$.

Proof We only give the proof when $\bar{M}$ is a Kenmotsu manifold. For any $X \in TN_T$ and $Z, W \in TN_\theta$, On using Lemmas 5.16, 5.9 and some basic computations, we have

$$\begin{aligned} g(A_{\omega\psi Z}W, X) &= -g(g(\phi\psi W, W)\xi - \eta(W)\phi\psi W + \phi\bar{\nabla}_{\psi W}W, X) + g(\bar{\nabla}_{\psi Z}\psi W, X) \\ &= +\cos^2\theta\eta(X)(g(Z, W) - \eta(Z)\eta(W)) - \phi X lnf g(W, \psi Z) \\ &- \cos^2\theta X lnf g(W, Z). \end{aligned}$$

Replacing ψZ and X by Z and ϕX, respectively, we obtain

$$g(A_{\omega Z}W, \phi X) = (X - \eta(X)\xi)lnf\, g(Z, W) - \phi X lnf\, g(\psi W, Z).$$

Now, we have some inequalities related to the squared norm of the second fundamental form and warping function.

Theorem 5.18 ([16]) *Let $N_T \times_f N_\theta$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a (2n+1)-dimensional almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ with semi slant function θ such that ξ is tangential to M.*

1. *If $\bar{M}$ is Sasakian manifold, such that $m = m_1 + 2m_2$. Then we have*

$$\|h\|^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)\|\phi\nabla lnf\|^2 + 4m_2\sin^2\theta, \tag{5.15}$$

the equality holds if and only if $g(h(Z, W), V) = 0$, for $Z, W \in TN_\theta$ and $V \in T^\perp M$.

2. *If $\bar{M}$ is cosymplectic manifold, such that $m = m_1 + 2m_2$. Then we have*

$$\|h\|^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)\|\phi\nabla lnf\|^2, \tag{5.16}$$

the equality holds if and only if $g(h(Z, W), V) = 0$, for $Z, W \in TN_\theta$ and $V \in T^\perp M$.

3. *If $\bar{M}$ is Kenmotsu manifold, such that $m = m_1 + 2m_2$. Then we have*

$$\|h\|^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)\|\phi\nabla lnf\|^2, \tag{5.17}$$

the equality holds if and only if $g(h(Z, W), V) = 0$, for $Z, W \in TN_\theta$ and $V \in T^\perp M$.

In above theorem, the inequalities were computed when the vector field ξ is taken tangent to M. Now, in next theorem, we will see the variations in the inequalities when, ξ is taken normal to M.

Theorem 5.19 ([16]) *Let $N_T \times_f N_\theta$ be a m-dimensional nontrivial warped product proper pointwise semi-slant submanifold of a (2n+1)-dimensional almost contact metric manifold $\bar{M}(\phi, \xi, \eta, g)$ with semi slant function θ such that ξ is normal to M and $\xi \in \mu$.*

1. *If $\bar{M}$ is cosymplectic manifold, such that $m = m_1 + 2m_2$. Then we have*

$$\|h\|^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)\|\phi\nabla lnf\|^2, \tag{5.18}$$

the equality holds if and only if $g(h(Z, W), V) = 0$, for $Z, W \in TN_\theta$ and $V \in T^\perp M$.

2. *If $\bar{M}$ is Kenmotsu manifold, such that $m = m_1 + 2m_2$. Then we have*

$$\|h\|^2 \geq 4m_2(\csc^2\theta + \cot^2\theta)\|\phi\nabla lnf\|^2 + 2m_1, \tag{5.19}$$

the equality holds if and only if $g(h(Z, W), V) = 0$, for $Z, W \in TN_\theta$ and $V \in T^\perp M$.

Now, we have a nontrivial example of pointwise warped product semi-slant submanifolds of a cosymplectic manifold.

Example ([16]) Define (ϕ, ξ, η, g) on R^{11} as follows

$$\phi\left(a_1\frac{\partial}{\partial y_1} + \cdots + a_{10}\frac{\partial}{\partial y_{11}} + a_{11}\frac{\partial}{\partial t}\right) = \sum_{i=1}^{5}\left(-a_{2i}\frac{\partial}{\partial y_{2i-1}} + a_{2i-1}\frac{\partial}{\partial y_{2i}}\right)$$

$$\xi = \frac{\partial}{\partial t}, \quad \eta = dt, \quad a_i \in R, \quad 1 \leq i \leq 11,$$

g is the Euclidean metric on R^{11}. Then we know that $R^{11}(\phi, \xi, \eta, g)$ is a cosymplectic manifold. Let

$$M = \{(x_1, x_2, u, v) : 0 < x_i < 1, \ i = 1, 2, \ 0 < u\ , v < \pi/2\}.$$

Taking two points P_1 and P_2 in the unit sphere S^1 such that $P_i = (a_{1i}, a_{2i}), \ i = 1, 2$ and $a_{11}a_{12} + a_{21}a_{22} = 0, \ a_{11}a_{22} + a_{21}a_{12} = 0$. We define a map $i : M \subset R^4 \to R^{11}$ by

$$i(x_1, x_2, u, v) = (x_1\cos u, x_2\cos u, x_1\cos v, x_2\cos v, x_1\sin u, x_2\sin u, x_1\sin v,$$

$$x_2\sin v, a_{11}u + a_{12}v, a_{21}u + a_{22}v, 2020).$$

Then the tangent bundle TM is spanned by X_1, X_2, Y_1 and Y_2, where

$$X_1 = \cos u\frac{\partial}{\partial y_1} + \cos v\frac{\partial}{\partial y_3} + \sin u\frac{\partial}{\partial y_5} + \sin v\frac{\partial}{\partial y_7}$$

$$X_2 = \cos u\frac{\partial}{\partial y_2} + \cos v\frac{\partial}{\partial y_4} + \sin u\frac{\partial}{\partial y_6} + \sin v\frac{\partial}{\partial y_8},$$

$$Y_1 = -x_1\sin u\frac{\partial}{\partial y_1} - x_2\sin u\frac{\partial}{\partial y_2} + x_1 cosu\frac{\partial}{\partial y_5} + x_2\cos u\frac{\partial}{\partial y_6} + a_{11}\frac{\partial}{\partial y_9} + a_{21}\frac{\partial}{\partial y_{10}}$$

$$Y_2 = -x_1\sin v\frac{\partial}{\partial y_3} - x_2\sin v\frac{\partial}{\partial y_4} + x_1\cos v\frac{\partial}{\partial y_7} + x_2\cos v\frac{\partial}{\partial y_8} + a_{12}\frac{\partial}{\partial y_9} + a_{22}\frac{\partial}{\partial y_{10}}.$$

We can easily check that M is a proper pointwise semi-slant submanifold of a 11-dimensional cosymplectic manifold $R^{11}(\phi, \xi, \eta, g)$ such that $D = \langle, X, X_2\rangle$, $D_\theta = \langle, X, X_2\rangle$ and the semi-slant function θ is given by

$$\cos\theta = \frac{|a_{11}a_{22} + a_{21}a_{12}|}{1 + x_1^2 + x_2^2},$$

ξ is normal to M with $\xi \in \mu$. We see that the distributions D and D_θ are integrable. Denote by N_T and N_θ the integral manifolds of D and D_θ, respectively. Then we see that (M, g) is a warped product pointwise semi-slant submanifold of $R^{11}(\phi, \xi, \eta, g)$ such that

$$g = 2(dx_1^2 + dx_2^2) + (1 + x_1^2 + x_2^2)(du^2 + dv^2)$$

the warping function is $f = \sqrt{(1 + x_1^2 + x_2^2)}$. By Eq. (5.18), we obtain

$$\|h\|^2 \geq 4\frac{(1 + x_1^2 + x_2^2)^2 + (-a_{11}a_{22} + a_{21}a_{12})^2}{(1 + x_1^2 + x_2^2)^2 - (-a_{11}a_{22} + a_{21}a_{12})^2}\|\nabla(\frac{1}{2}ln(1 + x_1^2 + x_2^2))\|^2.$$

Further, Uddin and Khaldi [22] redefined the concept of pointwise slant submanifolds, basically, they consider the structure vector field ξ tangent to submanifold M and proved the following characterization.

Theorem 5.20 ([22]) *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in TM$. Then M is pointwise slant submanifold if and only if*

$$\psi^2 = \cos^2\theta(-I + \eta \otimes \xi),$$

for some real valued function θ defined on the tangent bundle TM of M.

The following corollary is an immediate consequence of the above theorem.

Corollary 5.21 ([22]) *Let M be pointwise slant submanifold of an almost contact metric manifold $\bar{M}$. Then, we have*

$$g(\psi X, \psi Y) = \cos^2\theta[g(X, Y) - \eta(X)\eta(Y)]$$

$$g(\omega X, \omega Y) = \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)]$$

for any $X, Y \in TM$.

Analogues to the definition of pseudo-slant submanifold [27], we have the following definition of point wise pseudo-slant submanifold.

Definition ([22]) A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be pointwise pseudo-slant submanifold if there exists a pair of orthogonal distributions $D^\perp$ and D_θ on M such that

1. The tangent bundle TM admits the orthogonal direct decomposition $TM = D^{\perp} \oplus D_{\theta} \oplus \langle \xi \rangle$.
2. The distribution $D^{\perp}$ is anti-invariant that is, $\phi D^{\perp} \subset T^{\perp}M$.
3. The distribution D_{θ} is pointwise slant with slant function θ.

The pointwise pseudo slant submanifold is said to be proper if $\theta \neq 0, \pi/2$ and $D^{\perp} \neq \{0\}$.

In the following lemma, we will see the integrability conditions of the distributions.

Lemma 5.22 ([22]) *Let M be a pointwise pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then the anti-invariant distribution $D^{\perp}$ is always integrable.*

Lemma 5.23 ([22]) *Let M be a pointwise pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then we have*

$$g(\nabla_Z W, \psi X) = g(h(X, Z), \phi W) - g(h(Z, W), \omega X), \tag{5.20}$$

$$\cos^2\theta g(\nabla_X Y, Z) = g(h(X, \psi Y), \phi Z) - g(h(X, Z), \omega\psi Y) - \eta(Z)g(X, \psi Y), \tag{5.21}$$

for any $X, Y \in D_{\theta}$ and $Z, W \in D^{\perp} \oplus \langle \xi \rangle$.

Uddin and Khaldi [22] studied warped product submanifolds of Sasakian manifolds by considering that one of the factors, a pointwise slant submanifold. Basically, they studied warped product submanifolds of the type $N_{\perp} \times_f N_{\theta}$ in a Sasakian manifold, where $N_{\perp}$ and N_{θ} are the anti-invariant and pointwise slant submanifolds of a Sasakian manifold, respectively. These warped products are called warped product pointwise pseudo-slant submanifold. Now, we have some basic results.

Lemma 5.24 ([22]) *Let $M = N_{\perp} \times_f N_{\theta}$ be a warped product pointwise pseudo-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_{\perp}$, where $N_{\perp}$ is an anti-invariant submanifold and N_{θ} is a proper pointwise slant submanifolds of $\bar{M}$. Then we have*

$$g(h(Y, Z), \omega\psi X) - g(h(\psi X, Z), \omega Y) = (\sin 2\theta)Z(\theta)g(X, Y), \tag{5.22}$$

$$g(h(X, Y), \phi Z) - g(h(X, Z), \omega Y) = Zlnf g(X, \psi Y) + \eta(Z)g(X, Y), \tag{5.23}$$

for any $X, Y \in TN_{\theta}$ and $Z \in TN_{\perp}$.

Lemma 5.25 ([22]) *Let $M = N_{\perp} \times_f N_{\theta}$ be a warped product pointwise pseudo-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_{\perp}$, where $N_{\perp}$ is an anti-invariant submanifold and N_{θ} is a proper pointwise slant submanifolds of $\bar{M}$. Then we have*

$$g(h(Y, Z), \omega\psi X) - g(h(\psi X, Z), \omega Y) = (2\cos^2\theta)Z(lnf)g(X, Y), \tag{5.24}$$

for any $X, Y \in TN_{\theta}$ and $Z \in TN_{\perp}$.

The following corollary is an immediate consequence of the above lemma.

Corollary 5.26 ([22]) *There does not exist any proper warped product mixed totally geodesic submanifold of the form $M = N_\perp \times_f N_\theta$ of a Sasakian manifold $\bar{M}$ such that $N_\perp$ is an anti-invariant submanifold and N_θ is a proper pointwise slant submanifold of $\bar{M}$.*

Proof From (5.24) and the mixed totally geodesic assumption, we have

$$(2\cos^2\theta)Z(lnf)g(X, Y) = 0.$$

Since g is Riemannian metric and M be a proper, then $\cos^2\theta \neq 0$. Thus, the proof follows from the above relation.

Now, from Lemmas 5.24 and 5.25, we have following result.

Theorem 5.27 ([22]) *Let $M = N_\perp \times_f N_\theta$ be a warped product pointwise pseudo-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_\perp$, where $N_\perp$ is an anti-invariant submanifold and N_θ is a proper pointwise slant submanifolds of $\bar{M}$. Then, one of the following statements holds*

1. *Either M is warped product of anti-invariant submanifolds, that is, $\theta = \pi/2$,*
2. *or if $\theta \neq \pi/2$, then $Zlnf = \tan\theta Z(\theta)$, for any $Z \in TN_\perp$.*

Proof From (5.22) and (5.24), we have

$$\cos^2\theta\{Zlnf - \tan\theta Z(\theta)\}g(X, Y) = 0. \tag{5.25}$$

Since g is Riemannian metric, therefore from above equation, we conclude that either $\cos^2\theta = 0$ or $Zlnf - \tan\theta Z(\theta) = 0$, which concludes the result.

Theorem 5.28 ([22]) *Let M be a pointwise pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then M is locally a warped product submanifold of the form $N_\perp \times_f N_\theta$ if and only if*

$$A_{\phi Z}\psi X - A_{\omega\psi X}Z = \eta(Z)\psi X - \cos^2\theta Z(\mu)X, \;\; \forall Z \in D^\perp, \;\; X \in D_\theta, \tag{5.26}$$

for some smooth function μ on M satisfying $Y(\mu) = 0$, for any $Y \in D_\theta$.

Proof Let $M = N_\perp \times_f N_\theta$ be a warped product pointwise pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then for any $X \in TN_\theta$ and $Z, W \in TN_\perp$, we have

$$g(A_{\phi Z}X, W) = Wlnf g(X, \phi Z) = 0,$$

which means that $A_{\phi Z}X$ has no component in $TN_\perp$. Similarly, we can find that $g(A_{\omega X}Z, W) = 0$. Therefore, we conclude that $A_{\phi Z}X - A_{\omega X}Z$ lies in TN_θ. Then from Lemma 5.24, we get (5.26).

Conversely, if M is a pointwise pseudo-slant submanifold such that (5.26) holds, then from (5.20), we have

$$g(A_{\phi Z}X, W) = Wlnf g(X, \phi Z) = 0.$$

Then from the condition (5.26), we get

$$g(\nabla_Z W, \psi X) = g(A_{\phi W}X - A_{\omega X}W, Z) = (Z\mu)g(X, Z) - \sec^2\theta\eta(Z)g(\psi X, Z) = 0,$$

which means that the leaves of the distribution $D_\perp \oplus \langle\xi\rangle$ are totally geodesic in M. On the other hand, from (5.21) we also have

$$\cos^2\theta g(\nabla_X Y, Z) = g(A_{\phi Z}TY - A_{\omega\psi Y}Z, X) - \eta(Z)g(X, \psi Y)$$

and

$$\cos^2\theta g(\nabla_Y X, Z) = g(A_{\phi Z}\psi X - A_{\omega\psi X}Z, Y) - \eta(Z)g(Y, \psi X).$$

From above two equations, we get $\cos^2\theta g([X, Y], Z) = 0$. Since D_θ is a proper pointwise slant distribution then one can conclude D_θ is integrable. If N_θ be a leaf of D_θ and h_θ is second fundamental form of N_θ in M, then we have

$$g(h_\theta(X, Y), Z) = -Z(\mu)g(X, Y).$$

From last equation, we find $h_\theta(X, Y) = -\tilde{\nabla}\mu g(X, Y)$. Hence, N_θ is a totally umbilical submanifold of M with mean curvature vector $-\tilde{\nabla}\mu$, where $\tilde{\nabla}\mu$ is the gradient vector of the function μ. Since, $Y(\mu) = 0$, for any $Y \in D_\theta$, then we show that $H_\theta = -\tilde{\nabla}\mu$ is parallel with respect to normal connection of N_θ. Thus N_θ is a totally umbilical submanifold of M with a nonvanishing mean curvature vector $H_\theta = -\tilde{\nabla}\mu$ that mean N_θ is an extrinsic sphere in M. Then from a result of Heipko [24], M is a warped product manifold of the form $N_\perp \times_\mu N_\theta$.

Ion Mihai et al. [12] redefined the pointwise semi-slant submanifolds of contact metric manifold by taking the structure vector field ξ tangential to the submanifold. More precisely they defined these submanifolds as follows:

Definition ([12]) A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be a pointwise semi-slant submanifold if there exists a pair of orthogonal distributions D and D_θ on M such that

1. the tangent bundle TM admits the orthogonal direct decomposition $TM = D \oplus D_\theta \oplus \langle\xi\rangle$,
2. the distribution D is invariant under ϕ that is, $\phi D = D$,
3. the distribution D_θ is pointwise slant with slant function θ.

We have the following example of semi-slant submanifolds in an almost contact metric manifold.

Example ([12]) Let $R^7(\phi, \xi, \eta, g)$ be an almost contact metric manifold with cartesian coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, z)$ and the almost contact structure

$$\phi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \ \ \phi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \ \ \phi\left(\frac{\partial}{\partial z}\right) = 0, \ \ 1 \leq i, j \leq 3$$

where $\xi = \frac{\partial}{\partial z}$, $\eta = dz$ and g is the standard Euclidean metric on R^7. Then (ϕ, ξ, η, g) is an almost contact metric structure on R^7. Consider the submanifold M of R^7 defined by $\psi(u, v, w, t, z) = (u + v, -u + v, t\cos w, t\sin w, w\cos t, w\sin t, z)$, such that $w, t\,(w \neq t)$ are nonzero real numbers. Then the tangent space TM is spanned by the following vector fields

$$X_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, \ \ X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}$$

$$X_3 = -t\sin w\frac{\partial}{\partial x_2} + t\cos w\frac{\partial}{\partial y_2} + \cos t\frac{\partial}{\partial x_3} + \sin t\frac{\partial}{\partial y_3},$$

$$X_4 = \cos w\frac{\partial}{\partial x_2} + \sin w\frac{\partial}{\partial y_2} - w\sin t\frac{\partial}{\partial x_3} + w\cos t\frac{\partial}{\partial y_3}, \ \ X_5 = \frac{\partial}{\partial z}.$$

Thus, $D = span\{X_1, X_2\}$ is an invariant distribution and $D_\theta = span\{X_3, X_4\}$ is a pointwise slant distribution with pointwise slant function $\theta = \cos^{-1}\frac{t-w}{\sqrt{(t^2+1)(w^2+1)}}$. Hence, M is a pointwise slant submanifold of R^7.

Lemma 5.29 ([12]) *Let M be a pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$. Then, we have*

1. $\sin^2\theta g(\nabla_X Y, Z) = g(h(X, \phi Y), \omega Z) - g(h(X, Y), \omega\psi Z)$,
2. $\sin^2\theta g(\nabla_Z W, X) = g(h(X, Z), \omega\psi Z) - g(h(\phi X, Y), \omega W)$,

for any $X, Y \in D \oplus \langle\xi\rangle$ and $Z, W \in D_\theta$.

Further, Mihai et al. [12] studied warped product pointwise semi-slant submanifolds of the form $M = N_T \times N_\theta$ such that ξ is tangential to N_T in the setting of Sasakian manifolds. Now we have some basic results.

Lemma 5.30 ([12]) *Let $M = N_T \times_f N_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_T$, where N_T is an invariant submanifold and N_θ is proper pointwise slant submanifold of $\bar{M}$. Then, we have*

$$g(h(X, W), \omega\psi Z) - g(h(X, \psi Z), \omega Z) = \sin 2\theta X(\theta) g(Z, W), \tag{5.27}$$

for any $X \in TN_T$ and $Z, W \in TN_\theta$.

Lemma 5.31 ([12]) *Let $M = N_T \times_f N_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_T$, where N_T is an invariant submanifold and N_θ is proper pointwise slant submanifold of $\bar{M}$, respectively. Then*

1. $g(\psi Z, W) = -\xi lnf g(Z, W)$,
2. $g(h(X, Y), \omega Z) = 0$,
3. $g(h(X, Z), \omega W) = Xlnf g(\psi Z, W) - \phi Xlnf g(Z, W) - \eta(X)g(Z, W)$,

for any $X, Y \in TN_T$ and $Z, W \in TN_\theta$.

Lemma 5.32 ([12]) *Let $M = N_T \times_f N_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_T$, where N_T is an invariant submanifold and N_θ is proper pointwise slant submanifold of $\bar{M}$. Then*

$$g(h(\phi X, Z), \omega W) = Xlnf g(Z, W) - \eta(X)g(Z, \psi W) - \phi Xlnf g(Z, \psi W), \tag{5.28}$$

for any $X \in TN_T$ and $Z, W \in TN_\theta$.

Proof Interchanging X by ϕX, for any $X \in TN_T$ in part 3 of Lemma 5.31 and using the first part of Lemma 5.31, we get the required result.

Lemma 5.33 ([12]) *Let $M = N_T \times_f N_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_T$, where N_T is an invariant submanifold and N_θ is proper pointwise slant submanifold of $\bar{M}$, respectively. Then, we have*

$$g(h(X, \psi Z), \omega W) = \phi Xlnf g(Z, \psi W) - \eta(X)g(\psi Z, W) - \cos^2\theta Xlnf g(Z, W), \tag{5.29}$$

for any $X \in TN_T$ and $Z, W \in TN_\theta$.

Proof Interchanging Z by ψZ, for any $Z \in TN_\theta$ in part 3 of Lemma 5.3 and after using Corollary 5.21, we get the required result.

Similarly, if we interchange W by ψW, for any $W \in TN_\theta$ in part 3 of Lemma 5.31, then we can obtain the following lemma.

Lemma 5.34 ([12]) *Let $M = N_T \times_f N_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_T$, where N_T is an invariant submanifold and N_θ is proper pointwise slant submanifold of $\bar{M}$. Then, we have*

$$g(h(X, Z), \omega\psi W) = \cos^2\theta Xlnf g(Z, W) - \phi Xlnf g(Z, \psi W) - \eta(X)g(Z, \psi W), \tag{5.30}$$

for any $X \in TN_T$ and $Z, W \in TN_\theta$.

Lemma 5.35 ([12]) *Let $M = N_T \times_f N_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_T$, where N_T is an invariant submanifold and N_θ is proper pointwise slant submanifold of $\bar{M}$. Then, we have*

$$g(A_{\omega W}\phi X, Z) - g(A_{\omega\psi W}X, Z) = \sin^2\theta Xlnf g(Z, W), \tag{5.31}$$

for any $X \in TN_T$ and $Z, W \in TN_\theta$.

Proof Subtracting (5.30) from (5.28), we get (5.31).

A warped product submanifold $M = N_1 \times_f N_2$ of a Sasakain manifold $\bar{M}$ is said to be mixed totally geodesic if $h(X, Z) = 0$, for any $X \in TN_1$ and $Z \in TN_2$.

From Lemma 5.35, we obtain the following result.

Theorem 5.36 ([12]) *Let $M = N_T \times_f N_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_T$, where N_T is an invariant submanifold and N_θ is proper pointwise slant submanifold of $\bar{M}$. If M is mixed totally geodesic, then M is the warped product of invariant submanifolds or warping function is constant on M.*

Proof From (5.31) and the assumption that M is mixed totally geodesic we have

$$\sin^2\theta X \ln f g(Z, W) = 0.$$

The result follows from the above equation.

Lemma 5.37 ([12]) *Let $M = N_T \times_f N_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_T$, where N_T is an invariant submanifold and N_θ is proper pointwise slant submanifold of $\bar{M}$ respectively. Then, we have*

$$g(A_{\omega\psi Z}W, X) - g(A_{\omega W}\psi Z, X) = 2\cos^2\theta X \ln f g(Z, W), \tag{5.32}$$

for any $X \in TN_T$ and $Z, W \in TN_\theta$.

Theorem 5.38 ([12]) *Let $M = N_T \times_f N_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$ such that $\xi \in TN_T$, where N_T is an invariant submanifold and N_θ is proper pointwise slant submanifold of $\bar{M}$. If M is mixed totally geodesic, then M is contact CR-warped product submanifold $N_T \times_f N_\perp$ or warping function is constant on M.*

Proof From (5.32) and the assumption that M is mixed totally geodesic we have

$$\cos^2\theta X \ln f g(Z, W) = 0.$$

The result follows from the above equation.

From Theorems 5.36 and 5.38, one can conclude.

Corollary 5.39 ([12]) *There does not exist any mixed totally geodesic proper warped product pointwise semi-slant submanifold $M = N_T \times_f N_\theta$ of a Sasakian manifold.*

Now, we have the following characterization for pointwise semi-slant warped product submanifolds of a Sasakain manifold.

Theorem 5.40 ([12]) *Let M be a pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$. Then M is locally a nontrivial warped product submanifold of the form $N_T \times_f N_\theta$, where N_T is an invariant submanifold and N_θ is a proper pointwise slant submanifold of $\bar{M}$ if and only if*

$$A_{\omega W}\phi X - A_{\omega\psi W}X = \sin^2\theta X(\mu)W, \quad \forall X \in D \oplus \langle\xi\rangle, \; W \in D_\theta, \tag{5.33}$$

for some smooth function μ on M satisfying $Z(\mu) = 0$ for any $Z \in D_\theta$.

Example ([12]) Let $R^7(\phi, \xi, \eta, g)$ be an almost contact metric manifold with cartesian coordinates $(x_1, x_2, x_3, y_1, y_2, y_3, z)$ and the almost contact structure

$$\phi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \;\; \phi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \;\; \phi\left(\frac{\partial}{\partial z}\right) = 0, \;\; 1 \le i, j \le 3.$$

Let M be a submanifold of R^7 defined by the immersions

$$x(u_1, u_2, u_3, u_4, z) = \left(u_1, u_3 cos u_4, \frac{u_3}{2}, u_2, u_3 \sin u_4, u_4, z\right)$$

for any nonzero function u_3 on M. Then the tangent space TM of M is spanned by X_1, X_2, X_3, X_4 and X_5,, where

$$X_1 = \frac{\partial}{\partial x_1}, \;\; X_2 = \frac{\partial}{\partial y_1}, \;\; X_3 = \cos u_4 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} + \sin u_4 \frac{\partial}{\partial y_2},$$

$$X_4 = -u_3 sin u_4 \frac{\partial}{\partial x_2} + u_3 \sin u_4 \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}, \;\; X_5 = \frac{\partial}{\partial z}.$$

Then M is a pointwise semi-slant submanifold with invariant distribution $D = span\{X_1, X_2\}$ and pointwise slant distribution $D_\theta = span\{X_3, X_4\}$ with the slant function $\theta = cos^{-1}(2u_3/\sqrt{1 + u_3^2})$.

In this continuation, Nadia Al-luhaibi and Meraj Ali Khan [20] studied warped product pointwise semi-slant submanifolds of Sasakian space form and obtained some existence conditions for these submanifolds. Further, they also estimate the squared norm of the second fundamental form in terms of slant function and warping function.

A Sasakian manifold $\bar{M}$ is said to be a *Sasakian space form* [9] if it has constant ϕ-holomorphic sectional curvature c and is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of Sasakian space form $\bar{M}(c)$ is given by

$$\bar{R}(X,Y)U = \frac{c-3}{4}\{g(Y,U)X - g(X,U)Y\} + \frac{c-1}{4}\{g(X,\phi U)\phi Y$$

$$-g(Y,\phi U)\phi X + 2g(X,\phi Y)\phi U + \eta(X)\eta(U)Y$$

$$-\eta(Y)\eta(U)X + g(X,U)\eta(Y)\xi - g(Y,U)\eta(X)\xi\},$$

for any vector fields X, Y, U on $\bar{M}$.

Now, we have some initial results.

Lemma 5.41 ([20]) *Suppose that $N_T \times_f N_\theta$ is a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$. Then, we have*

(i) $\xi lnf = 0$,
(ii) $g(h(\phi X, Z), \omega Z) = X lnf \|Z\|^2$,
(iii) $g(h(\phi X, Z), \phi h(X, Z)) = \|h_\mu(X, Z)\|^2 + \cos^2\theta (X lnf)^2 \|Z\|^2$,

$\forall\ X \in TN_T$ and $Z \in TN_\theta$, where h_μ is the μ component of h.

Proof From (2.12), Gauss equation, and (5.10), it is easy to see that $\xi lnf = 0$. Moreover, replacing X by ϕX in part 2 of Lemma 5.17, using (2.1) and part (i), we get the part (ii). To prove part (iii), on making use of Gauss equation and (2.10), we get

$$h(\phi X, Z) = -\eta(X)Z + \phi h(X, Z) + \phi\nabla_Z X - \nabla_Z \phi X.$$

By utilizing (5.10), the form of above equation can be changed to as follow

$$h(\phi X, Z) = -\eta(X)Z + \phi h(X, Z) + X lnf \phi Z - \phi X lnf Z.$$

By comparing the normal parts, we get

$$h(\phi X, Z) = \phi h_\mu(X, Z) + X lnf \omega Z,$$

taking inner product with $\phi h(X, Z)$, we obtain

$$g(h(\phi X, Z), \phi h(X, Z)) = \|h_\mu(X, Z)\|^2 + X lnf g(\phi h(X, Z), \omega Z). \tag{5.34}$$

Calculating the last term of above equation by using Gauss equation, (2.12), and Corollary 5.21 as follows

$$g(\phi h(X, Z), \omega Z) = g(h(\phi X, Z), \omega Z) - \sin^2\theta X lnf \|Z\|^2.$$

Utilizing part (ii), we get

$$g(\phi h(X, Z), \omega Z) = \cos^2\theta X lnf \|Z\|^2,$$

using in (5.34), we obtain the required result.

Lemma 5.42 ([20]) *Suppose that $N_T \times_f N_\theta$ is a warped product pointwise semi-slant submanifold of a Sasakian manifold $\bar{M}$. Then*

$$g(h(X, \psi W), \omega Z) = -g(h(X, Z), \omega\psi Z) = -\cos^2\theta X lnf \|Z\|^2,$$

for all $X \in TN_T$, $Z \in TN_\theta$.

Proof From the part (ii) of Lemma 5.41, the following is attained

$$g(h(\psi X, Z), \omega Z) + g(h(\psi X, Z), \omega W) = 2X lnf g(W, Z)$$

$\forall X \in TN_T$ and $W, Z \in TN_\theta$. Replacing Z by $\psi W \in D_\theta$ and using the fact that W and ψW are perpendicular, the following is obtained

$$g(h(X, \psi W), \omega W) = -g(h(X, W), \omega\psi W). \tag{5.35}$$

Further by some routine computations, we get

$$\psi X lnf W - X lnf \psi W = th(X, W) - \eta(X)W.$$

Now taking inner product with $Z \in TN_\theta$ in the above equation, we have

$$\psi X lnf g(Z, W) - X lnf g(\psi W, Z) = -g(h(X, W), \omega Z) - \eta(X)g(Z, W). \tag{5.36}$$

Interchanging W and Z and subtracting the resultant from Eq. (5.36) leads to

$$-g(h(X, W), \omega Z) + g(h(X, Z), \omega W) = 2X lnf g(W, \psi Z).$$

In particular, replacing Z by $\psi W \in D_\theta$, we get

$$g(h(X, W), \omega\psi W) - g(h(X, \psi W), \omega W) = -2\cos^2\theta X lnf \|W\|^2. \tag{5.37}$$

Using (1.4) yields

$$g(h(X, \psi W), \omega W) = -g(h(X, W), \omega\psi W) = -\cos^2\theta X lnf \|W\|^2. \tag{5.38}$$

Lemma 5.43 ([20]) *On a warped product pointwise semi-slant submanifold $M = N_T \times_f N_\theta$ of a Sasakian manifold $\bar{M}$, we obtain*

$$\sum_{i=1}^{q} \Big[\sum_{j,k=1}^{2p} g(h(\phi e_i, e^k), \omega e^j) g(h(e_i, \psi e^k), \omega e^j) - g(h(e_i, e^k), \omega e^j) g(h(\phi e_i, \psi e^k), \omega e^j) \Big] = -4p\cos^2\theta \|\nabla lnf\|^2,$$

where $\{e_0=\xi, e_1, e_2, \ldots, e_q, \phi e_1, \phi e_2, \ldots, \phi e_q\}$ and $\{e^1, e^2, \ldots, e^p, \sec\theta\psi e^1, \ldots, \sec\theta\psi e^p\}$ are the frames of the orthonormal vector fields on TN_T and TN_θ respectively.

Proof First, we expand the left-hand term in the following way

$$\begin{aligned}
\sum_{i=1}^{q}\Big[\sum_{j,k=1}^{2p} g(h(\phi e_i, e^k),\omega e^j)g(h(e_i, \psi e^k), \omega e^j)\Big]
&= \sum_{i=1}^{q}\Big[\sum_{j=1}^{2p} g(h(\phi e_i, e^j), \omega e^j)g(h(e_i, \psi e^j), \omega e^j)\\
&\quad + \sum_{j\neq k=1}^{2p} g(h(\phi e_i, e^k), \omega e^j)g(h(e_i, \psi e^k), \omega e^j)\Big]\\
&= \sum_{i=1}^{q}\Big[\sum_{j=1}^{2p} g(h(\phi e_i, e^j), \omega e^j)g(h(e_i, \psi e^j), \omega e^j)\\
&\quad + \sum_{j=1}^{p} g(h(\phi e_i, e^j), \omega e^{j+p})g(h(e_i, \psi e^j), \omega e^{j+p})\\
&\quad + \sum_{j=1}^{p} g(h(\phi e_i, e^{j+p}), \omega e^j)g(h(e_i, \psi e^{j+p}), \omega e^j)\Big]\\
&= \sum_{i=1}^{q}\Big[\sum_{j=1}^{2p} g(h(\phi e_i, e^j), \omega e^j)g(h(e_i, \psi e^j), \omega e^j)\\
&\quad + \sec^2\theta\sum_{j=1}^{p} g(h(\phi e_i, e^j), \omega\psi e^j)g(\sigma(e_i, \psi e^j), \omega\psi e^j)\\
&\quad - \sum_{j=1}^{p} g(h(\phi e_i, \psi e^j), \omega e^j)g(h(e_i, e^j), \omega e^j)\Big].
\end{aligned}$$

Using part (ii) of Lemmas 5.41, 5.42 and utilizing (5.13), we get

$$\begin{aligned}
\sum_{i=1}^{q}\Big[\sum_{j,k=1}^{2p} g(h(\phi e_i, e^k), \omega e^j)g(\sigma(e_i, \psi e^k), \omega e^j)\Big]
&= \sum_{i=1}^{q}\Big[-2p\cos^2\theta(e_i lnf)^2 - 2p\cos^2\theta(\phi e_i lnf)^2 - 2p\phi e_i lnf\,\eta(e_i)\Big]\\
&= -2p\cos^2\theta\|\nabla lnf\|^2.
\end{aligned}$$

Replacing e_i by ϕe_i in above equation, we get

$$\sum_{i=1}^{q}\left[\sum_{j,k=1}^{2p} g(h(e_i, e^k), \omega e^j j) g(h(\phi e_i, \psi e^k), \omega e^j j)\right] = 2p\cos^2\theta\|\nabla lnf\|^2.$$

By subtracting the above two findings, the required result gets attained.

Theorem 5.44 ([20]) *Suppose that $M = N_T \times_f N_\theta$ is a warped product pointwise slant submanifolds of a Sasakian space form $\bar{M}(c)$ such that N_T is a compact submanifold. Then M is a Riemannian product submanifold if the following inequalities hold*

$$\sum_{i=1}^{2q}\left[\sum_{j=1}^{2p} \|h_\mu(e_i, e^j)\|^2 \leq qp(c-1)\sin^2\theta - 2p(\cos^2\theta + 2\cot^2\theta)\|\nabla lnf\|^2\right.$$

and

$$\sum_{i=1}^{q}\sum_{j=1}^{2p} g(h_\mu(\phi e_i, e^j), h_\mu(e_i, \psi e^j)) \geq 0,$$

where h_μ stands for the component of h in μ, and $(2q+1)$ and $2p$ are the dimensions of N_T, and N_θ, respectively.

The squared norm of h with reference to the warping function and slant function is provided in the next theorem.

Theorem 5.45 ([20]) *Let $\bar{M}(c)$ be a $(2m+1)$-dimensional Sasakian space form and $M = N_T \times_f N_\theta$ be an $n-$dimensional warped product pointwise slant submanifolds such that N_T is a $2q+1-$dimensional invariant submanifold and N_θ be a $2p-$dimensional proper pointwise slant submanifold of $\bar{M}(c)$. If*

$$\sum_{i=1}^{q}\sum_{j=1}^{2p} g(h(\phi e_i, e^j), h(e_i, \psi e^j)) \geq 0,$$

then

$$(i)\ \ \|h\|^2 \geq\ qp(c-1)\sin^2\theta + 2p\csc^2\theta + 2p\sin^2\theta\|\nabla lnf\|^2 - 2p\Delta(lnf). \tag{5.39}$$

(ii) *The necessary and sufficient conditions for the equality sign of (5.39) to be held identically are*
(a) N_T *is totally geodesic invariant in* $\bar{M}(c)$*. Furthermore, it is a Sasakian space form.*
(b) N_θ *is totally umbilical in* $\bar{M}(c)$*.*
(c) $\sum_{i=1}^{q}\sum_{j=1}^{2p} g(h(\phi e_i, e^j), h(e_i, \psi e^j)) = 0.$

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Semi-Slant $\xi^{\perp}$-, Hemi-Slant $\xi^{\perp}$-Riemannian Submersions and Quasi Hemi-Slant Submanifolds

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1 Introduction

A differentiable map $\pi : (M, g_M) \longrightarrow (N, g_N)$ between Riemannian manifolds (M, g_M) and (N, g_N) is called a Riemannian submersion if π_* is onto and it satisfies

$$g_N(\pi_* X_1, \pi_* X_2) = g_M(X_1, X_2) \tag{1.1}$$

for X_1, X_2 vector fields tangent to M, where π_* denotes the derivative map. The study of Riemannian submersions were studied by O'Neill [1] and Gray [2] see also [3]. Riemannian submersions have several applications in mathematical physics. Indeed, Riemannian submersions have their applications in the Yang–Mills theory [42, 43], Kaluza–Klein theory [44, 45], supergravity and superstring theories [46, 47] and more. Later, such submersions according to the conditions on the map $\pi : (M, g_M) \longrightarrow (N, g_N)$, we have the following submersions: Riemannian submersions [4], almost Hermitian submersions [5], invariant submersions [6–8], anti-invariant submersions [7–13], lagrangian submersions [14, 15], semi-invariant submersions [16, 17], slant submersions [18–22], semi-slant submersions [23–26],

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quaternionic submersions [27, 28], hemi-slant submersions [29, 30], pointwise slant submersions [31, 32], etc. In [33], Lee defined anti-invariant $\xi^{\perp}$-Riemannian submersions from almost contact metric manifolds and studied the geometry of such maps.

As a generalization of anti-invariant $\xi^{\perp}$-Riemannian submersions, Akyol et al. in [34] defined the notion of semi-invariant $\xi^{\perp}$-Riemannian submersions from almost contact metric manifolds and investigated the geometry of such maps. In 2017, Mehmet et al. [35], as a generalization of anti-invariant $\xi^{\perp}$-Riemannian submersions, semi-invariant $\xi^{\perp}$-Riemannian submersions and slant Riemannian submersions, defined and studied semi-slant $\xi^{\perp}$-Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. Very recently Ramazan Sari and Mehmet Akif Akyol [36] also introduced and studied Hemi-slant $\xi^{\perp}$-submersions and obtained interesting results. On the other hand, in 1996, using Chen's notion on slant submanifold, Lotta [37] introduced the notion of slant submanifold in almost contact metric manifold which was further generalized as semi-slant, hemi-slant and bi-slant submanifolds. Motivated from these studies, Rajendra Prasad et al. introduced and studied quasi hemi-slant submanifolds of cosymplectic manifolds.

The aim of this chapter is to discuss briefly some results of semi-slant $\xi^{\perp}$-submersions [35], hemi-slant $\xi^{\perp}$-submersions [36] and quasi hemi-slant submanifolds [38].

2 Riemannian Submersions

Let (M, g_M) and (N, g_N) be two Riemannian manifolds. A Riemannian submersion $\pi : M \longrightarrow N$ is a map of M onto N satisfying the following axioms:

(i) π has maximal rank, and
(ii) The differential π_* preserves the lenghts of horizontal vectors, that is π_* is a linear isometry.

The geometry of Riemannian submersion is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined as follows:

$$\mathcal{T}(E_1, E_2) = \mathcal{H}\overset{M}{\nabla}_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}\overset{M}{\nabla}_{\mathcal{V}E_1}\mathcal{H}E_2 \tag{2.1}$$

and

$$\mathcal{A}(E_1, E_2) = \mathcal{H}\overset{M}{\nabla}_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}\overset{M}{\nabla}_{\mathcal{H}E_1}\mathcal{H}E_2 \tag{2.2}$$

for any $E_1, E_2 \in \Gamma(M)$, where $\overset{M}{\nabla}$ is the Levi-Civita connection on g_M. Note that we denote the projection morphisms on the vertical distribution and the horizontal distribution by $\mathcal{V}$ and $\mathcal{H}$, respectively. One can easily see that $\mathcal{T}$ is vertical, $\mathcal{T}_{E_1} = \mathcal{T}_{\mathcal{V}E_1}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_{E_1} = \mathcal{A}_{\mathcal{H}E_1}$. We also note that

$$\mathcal{T}_U V = \mathcal{T}_V U \text{ and } \mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y],$$

for $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $U, V \in \Gamma(ker\pi_*)$.

On the other hand, from (2.1) and (2.2), we obtain

$$\overset{M}{\nabla}_V W = \mathcal{T}_V W + \hat{\nabla}_V W; \tag{2.3}$$

$$\overset{M}{\nabla}_V X = \mathcal{T}_V X + \mathcal{H}(\overset{M}{\nabla}_V X); \tag{2.4}$$

$$\overset{M}{\nabla}_X V = \mathcal{V}(\overset{M}{\nabla}_X V) + \mathcal{A}_X V; \tag{2.5}$$

$$\overset{M}{\nabla}_X Y = \mathcal{A}_X Y + \mathcal{H}(\overset{M}{\nabla}_X Y), \tag{2.6}$$

for any $X, Y \in \Gamma((ker\pi_*)^{\perp})$ and $V, W \in \Gamma(ker\pi_*)$. Moreover, if X is basic, then $\mathcal{H}(\overset{M}{\nabla}_V X) = \mathcal{A}_X V$. It is easy to see that for $U, V \in \Gamma(ker\pi_*)$, $\mathcal{T}_U V$ coincides with the fibres as the second fundamental form and $\mathcal{A}_X Y$ reflecting the complete integrability of the horizontal distribution.

A vector field on M is called vertical if it is always tangent to fibres. A vector field on M is called horizontal if it is always orthogonal to fibres. A vector field Z on M is called basic if Z is horizontal and π-related to a vector field $\bar{Z}$ on N, i.e., $\pi_* Z_p = \bar{Z}_{\pi_*(p)}$ for all $p \in M$.

Lemma 2.1 (see [1, 3]) *Let $\pi : M \longrightarrow N$ be a Riemannian submersion. If X and Y basic vector fields on M, then we get:*

(i) $g_M(X, Y) = g_N(\bar{X}, \bar{Y}) \circ \pi$,
(ii) $\mathcal{H}[X, Y]$ *is a basic and* $\pi_*\mathcal{H}[X, Y] = [\bar{X}, \bar{Y}] \circ \pi$;
(iii) $\mathcal{H}(\overset{M}{\nabla}_X Y)$ *is a basic,* π*-related to* $(\overset{N}{\nabla}_{\bar{X}} \bar{Y})$, *where* $\overset{M}{\nabla}$ *and* $\overset{N}{\nabla}$ *are the Levi-Civita connection on* M *and* N;
(iv) $[X, V] \in \Gamma(ker\pi_*)$ *is vertical, for any* $V \in \Gamma(ker\pi_*)$.

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $\pi : M \longrightarrow N$ is a differentiable map. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(X, Y) = \overset{\pi}{\nabla}_X \pi_* Y - \pi_*(\nabla_X Y) \tag{2.7}$$

for $X, Y \in \Gamma(TM)$, where $\overset{\pi}{\nabla}$ is the pull back connection and ∇ is the Levi-Civita connections of the metrics g_M and g_N.

Finally, let (M, g_M) be a $(2m + 1)$-dimensional Riemannian manifold and TM denote the tangent bundle of M. Then M is called an almost contact metric manifold if there exists a tensor φ of type $(1, 1)$ and global vector field ξ and η is a 1-form of ξ, then we have

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1 \tag{2.8}$$

$$\varphi\xi = 0, \quad \eta o \varphi = 0 \quad \text{and} \quad g_M(\varphi X, \varphi Y) = g_M(X, Y) - \eta(X)\eta(Y), \tag{2.9}$$

where X, Y are any vector fields on M. In this case, $(\varphi, \xi, \eta, g_M)$ is called the almost contact metric structure of M. The almost contact metric manifold $(M, \varphi, \xi, \eta, g_M)$

is called a contact metric manifold if

$$\Phi(X, Y) = d\eta(X, Y)$$

for any $X, Y \in \Gamma(TM)$, where Φ is a 2-form in M defined by $\Phi(X, Y) = g_M(X, \varphi Y)$. The 2-form Φ is called the fundamental 2-form of M. A contact metric structure of M is said to be normal if

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is Nijenhuis tensor of φ. Any normal contact metric manifold is called a Sasakian manifold. Moreover, if M is Sasakian [39, 40], then we have

$$(\overset{M}{\nabla}_X \varphi)Y = g_M(X, Y)\xi - \eta(Y)X \quad \text{and} \quad \overset{M}{\nabla}_X \xi = -\varphi X, \tag{2.10}$$

where $\overset{M}{\nabla}$ is the connection of Levi-Civita covariant differentiation.

3 Semi-slant $\xi^\perp$-Riemannian Submersions

In 2017, Mehmet et al. [35], as a generalization of anti-invariant $\xi^\perp$-Riemannian submersions, semi-invariant $\xi^\perp$-Riemannian submersions and slant Riemannian submersions, defined and studied semi-slant $\xi^\perp$-Riemannian submersions from Sasakian manifolds onto Riemannian manifolds. In this Sect. 3, we will discuss some results of this paper briefly.

Definition 3.1 Let $(M, \varphi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. Suppose that there exists a Riemannian submersion $\pi : M \longrightarrow N$ such that ξ is normal to $ker\pi_*$. Then $\pi : M \longrightarrow N$ is called semi-slant $\xi^\perp$-Riemannian submersion if there is a distribution $D_1 \subseteq \ker \pi_*$ such that

$$ker\pi_* = D_1 \oplus D_2, \quad \varphi(D_1) = D_1, \tag{3.1}$$

and the angle $\theta = \theta(U)$ between φU and the space $(D_2)_p$ is constant for nonzero $U \in (D_2)_p$ and $p \in M$, where D_2 is the orthogonal complement of D_1 in $ker\pi_*$. As it is, the angle θ is called the semi-slant angle of the submersion.

Now, let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, for $U \in \Gamma(\ker \pi_*)$, we put

$$U = \mathcal{P}U + \mathcal{Q}U \tag{3.2}$$

where $\mathcal{P}U \in \Gamma(D_1)$ and $\mathcal{Q}U \in \Gamma(D_2)$. For $Z \in \Gamma(TM)$, we have

$$Z = \mathcal{V}Z + \mathcal{H}Z \tag{3.3}$$

where $\mathcal{V}Z \in \Gamma(ker\pi_*)$ and $\mathcal{H}Z \in \Gamma((ker\pi_*)^{\perp})$. For $V \in \Gamma(ker\pi_*)$, we get

$$\varphi V = \phi V + \omega V \tag{3.4}$$

where ϕV and ωV are vertical and horizontal components of φV, respectively. Similarly, for any $X \in \Gamma((ker\pi_*)^{\perp})$, we have

$$\varphi X = \mathcal{B}X + CX \tag{3.5}$$

where $\mathcal{B}X$ (resp. CX) is the vertical part (resp. horizontal part) of φX. Then the horizontal distribution $(ker\pi_*)^{\perp}$ is decomposed as

$$(\ker \pi_*)^{\perp} = \omega D_2 \oplus \mu, \tag{3.6}$$

here μ is the orthogonal complementary distribution of ωD_2 and it is both invariant distribution of $(ker\pi_*)^{\perp}$ with respect to φ and contains ξ. By (2.9), (3.4) and (3.5), we have

$$g_M(\phi U_1, V_1) = -g_M(U_1, \phi V_1) \tag{3.7}$$

and

$$g_M(\omega U_1, X) = -g_M(U_1, \mathcal{B}X) \tag{3.8}$$

for $U_1, V_1 \in \Gamma(\ker \pi_*)$ and $X \in \Gamma((\ker \pi_*)^{\perp})$. From (3.4), (3.5) and (3.6), we have

Lemma 3.2 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we obtain:*

$$\begin{aligned}
&(a)\ \ \phi D_1 = D_1, \quad (b)\ \ \omega D_1 = 0,\\
&(c)\ \ \phi D_2 \subset D_2, \quad (d)\ \ \mathcal{B}(ker\pi_*)^{\perp} = D_2,\\
&(e)\ \ \mathcal{T}_{U_1}\xi = \phi U_1, \quad (f)\ \ \hat{\nabla}_{U_1}\xi = -\omega U_1,
\end{aligned}$$

for $U_1 \in \Gamma(\ker \pi_*)$ and $\xi \in \Gamma((ker\pi_*)^{\perp})$.

Using (3.4), (3.5) and the fact that $\varphi^2 = -I + \eta \otimes \xi$, we have

Lemma 3.3 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we get*

$$\begin{aligned}
&(i)\ \ \phi^2 + \mathcal{B}\omega = -id, \quad (ii)\ \ C^2 + \omega\mathcal{B} = -id,\\
&(iii)\ \ \omega\phi + C\omega = 0, \quad (iv)\ \ \mathcal{B}C + \phi\mathcal{B} = 0,
\end{aligned}$$

where I is the identity operator on the space of π.

Let $(M, \varphi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. Let $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a semi-slant $\xi^\perp$-Riemannian submersion. We now examine how the Sasakian structure on M effects the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of a semi-slant $\xi^\perp$-Riemannian submersion $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$.

Lemma 3.4 *Let $(M, \varphi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) a Riemannian manifold. Let $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a semi-slant $\xi^\perp$-Riemannian submersion. Then we have*

$$\mathcal{B}\mathcal{T}_U V + \phi\hat{\nabla}_U V = \hat{\nabla}_U \phi V + \mathcal{T}_U \omega V, \tag{3.9}$$

$$g_M(U, V)\xi + \mathcal{C}\mathcal{T}_U V + \omega\hat{\nabla}_U V = \mathcal{T}_U \phi V + \mathcal{H}\overset{M}{\nabla}_U \omega V, \tag{3.10}$$

$$\phi\mathcal{T}_U X + \mathcal{B}\overset{M}{\nabla}_U X - \eta(X)U = \hat{\nabla}_U \mathcal{B}X + \mathcal{T}_U \mathcal{C}X, \tag{3.11}$$

$$\omega\mathcal{T}_U X + \mathcal{C}\overset{M}{\nabla}_U X = \mathcal{T}_U \mathcal{B}X + \mathcal{H}\overset{M}{\nabla}_U \mathcal{C}X, \tag{3.12}$$

$$g_M(X, Y)\xi - \omega\mathcal{A}_X Y + \mathcal{C}\mathcal{H}\overset{M}{\nabla}_X Y = \mathcal{A}_X \mathcal{B}Y + \overset{M}{\nabla}_X \mathcal{C}Y + \eta(Y)X, \tag{3.13}$$

$$\phi\mathcal{A}_X Y + \mathcal{B}\mathcal{H}\overset{M}{\nabla}_X Y = \mathcal{V}\overset{M}{\nabla}_X \mathcal{B}Y + \mathcal{A}_X \mathcal{C}Y, \tag{3.14}$$

for all $X, Y \in \Gamma((\ker \pi_*)^\perp)$ and $U, V \in \Gamma(\ker \pi_*)$.

Proof Given $U, V \in \Gamma(ker\pi_*)$, by virtue of (2.10) and (3.4), we have

$$g_M(U, V)\xi - \eta(V)U = \overset{M}{\nabla}_U \phi V + \overset{M}{\nabla}_U \omega V - \varphi\overset{M}{\nabla}_U V.$$

Making use of (2.3), (2.4), (3.4) and (3.5), we have

$$\begin{aligned} g_M(U, V)\xi &= \mathcal{T}_U \phi V + \hat{\nabla}_U \phi V + \mathcal{T}_U \omega V + \mathcal{H}\overset{M}{\nabla}_U \omega V \\ &\quad - \mathcal{B}\mathcal{T}_U V - \mathcal{C}\mathcal{T}_U V - \phi\hat{\nabla}_U V - \omega\hat{\nabla}_U V. \end{aligned} \tag{3.15}$$

Comparing horizontal and vertical parts, we get (3.9) and (3.10). The other assertions can be obtained in a similar method. □

Theorem 3.5 *Let $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\phi^2 W = -\cos^2\theta W, \quad W \in \Gamma(D_2), \tag{3.16}$$

where θ denotes the semi-slant angle of D_2.

Lemma 3.6 *Let $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold*

(N, g_N) with a semi-slant angle θ. Then we have

$$g_M(\phi W_1, \phi W_2) = \cos^2\theta g_M(W_1, W_2), \tag{3.17}$$
$$g_M(\omega W_1, \omega W_2) = \sin^2\theta g_M(W_1, W_2), \tag{3.18}$$

for any $W_1, W_2 \in \Gamma(D_2)$.

3.1 Integrable and Parallel Distributions

In this section, we will discuss integrability conditions of the distributions involved in the definition of a semi-slant $\xi^\perp$-Riemannian submersion. First, we have

Theorem 3.7 *Let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then:*

(i) D_1 is integrable $\Leftrightarrow$ $(\nabla\pi_*)(U, \varphi V) - (\nabla\pi_*)(V, \varphi U) \notin \Gamma(\pi_*\mu)$
(ii) D_2 is integrable $\Leftrightarrow$ $g_N(\pi_*\omega W, (\nabla\pi_*)(Z, \varphi U)) + g_N(\pi_*\omega Z, (\nabla\pi_*)(W, \varphi U)) = g_M(\phi W, \hat{\nabla}_Z\varphi U) + g_M(\phi Z, \hat{\nabla}_W\varphi U)$

for $U, V \in \Gamma(D_1)$ and $Z, W \in \Gamma(D_2)$.

Proof For $U, V \in \Gamma(D_1)$ and $X \in \Gamma((ker\pi_*)^\perp)$, since $[U, V] \in \Gamma(ker\pi_*)$, we have $g_M([U, V], X) = 0$. Thus, D_1 is integrable $\Leftrightarrow$ $g_M([U, V], Z) = 0$ for $Z \in \Gamma(D_2)$. Since M is a Sasakian manifold, by (2.9) and (2.10), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_U V, Z) &= g_M(\overset{M}{\nabla}_U \varphi V - g_M(U, V)\xi - \eta(V)U, \varphi Z) \\ &= g_M(\overset{M}{\nabla}_U \varphi V, \varphi Z). \end{aligned} \tag{3.19}$$

Using (3.4) in (3.19), we get

$$g_M([U, V], Z) = -g_M(\overset{M}{\nabla}_U V, \varphi\phi Z) + g_M(\mathcal{H}\overset{M}{\nabla}_U \varphi V, wZ) - g_M(\overset{M}{\nabla}_V U, \varphi\phi Z) - g_M(\mathcal{H}\overset{M}{\nabla}_V \varphi U, wZ).$$

Now, by using (2.7) and (3.16), we get

$$\begin{aligned} g_M([U, V], Z) = \cos^2\theta g_M(\overset{M}{\nabla}_U V, Z) - g_N((\nabla\pi_*)(U, \varphi V) + \nabla^\pi_U \pi_*\varphi V, \pi_* wZ) \\ - \cos^2\theta g_M(\overset{M}{\nabla}_V U, Z) + g_N((\nabla\pi_*)(V, \varphi U) + \nabla^\pi_V \pi_*\varphi U, \pi_* wZ). \end{aligned}$$

Thus, we have

$$(\sin^2\theta) g_M([U, V], Z) = -g_N((\nabla\pi_*)(U, \varphi V) - (\nabla\pi_*)(V, \varphi U), \pi_* wZ),$$

□

which completes the proof.

Now for the geometry of leaves of D_1, we have

Theorem 3.8 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the distribution D_1 is parallel if and only if*

$$g_N((\nabla\pi_*)(U, \varphi V), \pi_*\omega Z) = g_M(\mathcal{T}_U\omega\phi Z, V) \tag{3.20}$$

and

$$-g_N((\nabla\pi_*)(U, \varphi V), \pi_*\mathcal{C}X) = g_M(V, \hat{\nabla}_U\phi\mathcal{B}X + \mathcal{T}_U\omega\mathcal{B}X) + g_M(V, \varphi U)\eta(X) \tag{3.21}$$

for $U, V \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $X \in \Gamma((\ker\pi_*)^{\perp})$.

Proof Making use of (3.19), (3.4) and (2.3), for $U, V \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we have

$$g_M(\overset{M}{\nabla}_U V, Z) = -g_M(\overset{M}{\nabla}_U V, \phi^2 Z) - g_M(\overset{M}{\nabla}_U V, \omega\phi Z) + g_M(\mathcal{H}\overset{M}{\nabla}_U \varphi V, \omega Z).$$

By virtue of (2.7) and (3.16), we get

$$g_M(\overset{M}{\nabla}_U V, Z) = \cos^2\theta g_M(\overset{M}{\nabla}_U V, Z) - g_M(\mathcal{T}_U V, w\phi Z) + g_N((\nabla\pi_*)(U, \varphi V), \pi_*(wZ))$$

or

$$\sin^2\theta g_M(\overset{M}{\nabla}_U V, Z) = -g_M(\mathcal{T}_U w\phi Z, V) + g_N((\nabla\pi_*)(U, \varphi V), \pi_*(wZ)),$$

which gives (3.20). On the other hand, from (2.9) and (2.10), we have

$$g_M(\overset{M}{\nabla}_U V, X) = g_M(\overset{M}{\nabla}_U \varphi V, \varphi X) + g_M(V, \varphi U)\eta(X)$$

for $U, V \in \Gamma(D_1)$ and $X \in \Gamma((\ker\pi_*)^{\perp})$. By using (3.5), we obtain

$$g_M(\overset{M}{\nabla}_U V, X) = g_M(V, \overset{M}{\nabla}_U \phi\mathcal{B}X) + g_M(V, \overset{M}{\nabla}_U \omega\mathcal{B}X) + g_M(\mathcal{C}X, \mathcal{H}\overset{M}{\nabla}_U \varphi V) + g_M(V, \varphi U)\eta(X).$$

Taking into account of (2.3), we write

$$\begin{aligned} g_M(\overset{M}{\nabla}_U V, X) &= g_M(V, \mathcal{T}_U\phi\mathcal{B}X + \hat{\nabla}_U\phi\mathcal{B}X) + g_M(V, \mathcal{T}_U\omega\mathcal{B}X + \mathcal{H}\overset{M}{\nabla}_U\omega\mathcal{B}X) \\ &\quad - g_N(\pi_*(\mathcal{C}X), \pi_*(\mathcal{H}\overset{M}{\nabla}_U\varphi V)) + g_M(V, \varphi U)\eta(X) \end{aligned}$$

hence,

$$g_M(\overset{M}{\nabla}_U V, X) = g_M(V, \hat{\nabla}_U \phi \mathcal{B}X) + g_M(V, \mathcal{T}_U \omega \mathcal{B}X) + g_N((\nabla\pi_*)(U, \varphi V), \pi_* CX) + g_M(V, \varphi U)\eta(X).$$

which gives (3.21). This completes the assertion. □

Similarly for D_2, we have:

Theorem 3.9 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the distribution D_2 is parallel if and only if*

$$g_N(\pi_*\omega W, (\nabla\pi_*)(Z, \varphi U)) = g_M(\phi W, \hat{\nabla}_Z \varphi U) \tag{3.22}$$

and

$$g_N((\nabla\pi_*)(Z, \omega W), \pi_*(X)) - g_N((\nabla\pi_*)(Z, \omega\phi W), \pi_*(X)) = g_M(\mathcal{T}_Z \omega W, \mathcal{B}X) + g_M(W, \varphi Z)\eta(X) \tag{3.23}$$

for any $Z, W \in \Gamma(D_2)$, $U \in \Gamma(D_1)$ and $X \in \Gamma((ker\pi_*)^{\perp})$.

Theorem 3.10 *Let π be a semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the distribution $(\ker\pi_*)^{\perp}$ is integrable if and only if*

$$g_N((\nabla\pi_*)(Y, \phi V), \pi_*(X)) + g_N((\nabla\pi_*)(X, \phi V), \pi_*(X)) = g_M(\phi V, \mathcal{V}(\overset{M}{\nabla}_X \mathcal{B}Y + \overset{M}{\nabla}_Y \mathcal{B}X)) \tag{3.24}$$

and

$$g_N((\nabla\pi_*)(X, CY) - (\nabla\pi_*)(Y, CX), \pi_*\omega W) = g_M(\mathcal{A}_X \mathcal{B}Y + \mathcal{A}_Y \mathcal{B}X, \omega W) + \eta(Y) g_M(X, \omega W) - \eta(X) g_M(Y, \omega W) \tag{3.25}$$

for $X, Y \in \Gamma((ker\pi_*)^{\perp})$, $V \in \Gamma(D_1)$ and $W \in \Gamma(D_2)$.

Proof Using (3.19), (2.9) and (2.10), we have for $X, Y \in \Gamma((\ker\pi_*)^{\perp})$ and $V \in \Gamma(D_1)$.

$$g_M([X, Y], V) = g_M(\overset{M}{\nabla}_X \varphi Y, \varphi V) - g_M(\overset{M}{\nabla}_Y \varphi X, \varphi V).$$

Now, by using (3.5), we obtain

$$g_M([X, Y], V) = -g_M(\mathcal{B}Y, \overset{M}{\nabla}_X \varphi V) - g_M(CY, \overset{M}{\nabla}_X \varphi V) + g_M(\mathcal{B}X, \overset{M}{\nabla}_Y \varphi V) + g_M(CX, \overset{M}{\nabla}_Y \varphi V).$$

By using (2.5) and taking into account of the property of the map, we have

$$g_M([X,Y],V) = g_M(\varphi V, \mathcal{A}_Y\mathcal{B}X + \mathcal{V}\overset{M}{\nabla}_X\mathcal{B}Y) - g_N(\pi_*(\mathcal{C}Y), \pi_*(\overset{M}{\nabla}_X\varphi V))$$
$$- g_M(\varphi V, \mathcal{A}_X\mathcal{B}Y + \mathcal{V}\overset{M}{\nabla}_Y\mathcal{B}X) - g_N(\pi_*(\mathcal{C}X), \pi_*(\overset{M}{\nabla}_Y\varphi V)).$$

Thus, we have

$$g_M([X,Y],V) = g_M(\varphi V, \mathcal{V}(\overset{M}{\nabla}_X\mathcal{B}Y - \overset{M}{\nabla}_Y\mathcal{B}X)) + g_N(\pi_*(\mathcal{C}Y), (\nabla\pi_*)(X,\varphi V))$$
$$- g_N(\pi_*(\mathcal{C}X), (\nabla\pi_*)(Y,\varphi V)),$$

which gives (3.24). In a similar way, by virtue of (3.19), (2.9) and (2.10), we have for $X, Y \in \Gamma((\ker\pi_*)^\perp)$ and $W \in \Gamma(D_2)$,

$$g_M([X,Y],W) = g_M(\varphi\overset{M}{\nabla}_X Y, \phi W) + g_M(\varphi\overset{M}{\nabla}_X Y, \omega W) + \eta(Y)g_M(X,\omega W)$$
$$- g_M(\varphi\overset{M}{\nabla}_Y X, \phi W) - g_M(\varphi\overset{M}{\nabla}_Y X, \omega W) - \eta(X)g_M(Y,\omega W).$$

By virtue of (3.5) and (3.6), we have

$$g_M([X,Y],W) = -g_M(\overset{M}{\nabla}_X Y, \phi^2 W) - g_M(\overset{M}{\nabla}_X Y, \omega\phi W) + g_M(\overset{M}{\nabla}_X \mathcal{B}Y, \omega W) + g_M(\overset{M}{\nabla}_X \mathcal{C}Y, \omega W)$$
$$- g_M(\overset{M}{\nabla}_Y X, \phi^2 W) - g_M(\overset{M}{\nabla}_Y X, \omega\phi W) + g_M(\overset{M}{\nabla}_Y \mathcal{B}X, \omega W) + g_M(\overset{M}{\nabla}_Y \mathcal{C}X, \omega W)$$
$$+ \eta(Y)g_M(X,\omega W) - \eta(X)g_M(Y,\omega W).$$

Now, by using (3.16) and the property of the map, we get

$$g_M([X,Y],W) = \cos^2\theta g_M([X,Y],W) + g_N((\nabla\pi_*)(X,Y), \omega\phi W) + g_M(\mathcal{A}_X\mathcal{B}Y, \omega W)$$
$$- g_N((\nabla\pi_*)(X,\mathcal{C}Y), \pi_*\omega W) - g_N((\nabla\pi_*)(Y,X), \omega\phi W) + g_M(\mathcal{A}_Y\mathcal{B}X, \omega W)$$
$$+ g_N((\nabla\pi_*)(Y,\mathcal{C}X), \pi_*\omega W) + \eta(Y)g_M(X,\omega W) - \eta(X)g_M(Y,\omega W).$$

Thus, we have

$$\sin^2\theta g_M([X,Y],W) = g_N((\nabla\pi_*)(Y,\mathcal{C}X) - (\nabla\pi_*)(X,\mathcal{C}Y), \pi_*\omega W) + g_M(\mathcal{A}_X\mathcal{B}Y + \mathcal{A}_Y\mathcal{B}X, \omega W)$$
$$+ \eta(Y)g_M(X,\omega W) - \eta(X)g_M(Y,\omega W),$$

which gives (3.25). This completes the proof. □

For the geometry of leaves $(\ker\pi_*)^\perp$, we have

Theorem 3.11 *Let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the distribution $(\ker\pi_*)^\perp$ is parallel if and only if*

$$g_M(V, \mathcal{V}\overset{M}{\nabla}_X\phi\mathcal{B}Y + \mathcal{A}_X\omega\mathcal{B}Y) = g_N(\pi_*(\mathcal{C}Y), (\nabla\pi_*)(X,\varphi V)) \tag{3.26}$$

and

$$g_M(\mathcal{A}_X\omega W, \mathcal{B}Y) + \eta(Y)g_M(X, \omega W) = g_N((\nabla\pi_*)(X, Y), \pi_*\omega\phi W) - g_N((\nabla\pi_*)(X, CY), \pi_*\omega W), \tag{3.27}$$

for $X, Y \in \Gamma((\ker\pi_*)^\perp)$, $V \in \Gamma(D_1)$ *and* $W \in \Gamma(D_2)$.

Theorem 3.12 *Let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the distribution $(ker\pi_*)$ is parallel if and only if*

$$g_M(\omega V, \mathcal{T}_U\mathcal{B}X) + g_M(V, \phi U)\eta(X) = g_N((\nabla\pi_*)(U, CX), \pi_*\omega V) - g_N((\nabla\pi_*)(U, X), \pi_*\omega\phi V) \tag{3.28}$$

for any $U \in \Gamma(D_1)$, $V \in \Gamma(D_2)$ and $X \in \Gamma((ker\pi_*)^\perp)$.

By virtue of Theorems 3.8, 3.9 and 3.11, we have the following theorem;

Theorem 3.13 *Let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N) with a semi-slant angle θ. Then the total space M is a locally product manifold of the leaves of D_1, D_2 and $(ker\pi_*)^\perp$, i.e., $M = M_{D_1} \times M_{D_2} \times M_{(ker\pi_*)^\perp}$, if and only if*

$$g_N((\nabla\pi_*)(U, \varphi V), \pi_*\omega Z) = g_M(\mathcal{T}_U\omega\phi Z, V),$$

$$-g_N((\nabla\pi_*)(U, \varphi V), \pi_*CX) = g_M(V, \hat{\nabla}_U\phi\mathcal{B}X + \mathcal{T}_U\omega\mathcal{B}X) + g_M(V, \varphi U)\eta(X),$$

$$g_N(\pi_*\omega W, (\nabla\pi_*)(Z, \varphi U)) = g_M(\phi W, \hat{\nabla}_Z\varphi U),$$

$$g_N((\nabla\pi_*)(Z, \omega W), \pi_*(X)) - g_N((\nabla\pi_*)(Z, \omega\phi W), \pi_*(X)) = g_M(\mathcal{T}_Z\omega W, \mathcal{B}X) + g_M(W, \varphi Z)\eta(X)$$

and

$$g_M(V, \mathcal{V}\overset{M}{\nabla}_X\phi\mathcal{B}Y + \mathcal{A}_X\omega\mathcal{B}Y) = g_N(\pi_*(CY), (\nabla\pi_*)(X, \varphi V)),$$

$$g_M(\mathcal{A}_X\omega W, \mathcal{B}Y) + \eta(Y)g_M(X, \omega W) = g_N((\nabla\pi_*)(X, Y), \pi_*\omega\phi W) - g_N((\nabla\pi_*)(X, CY), \pi_*\omega W)$$

for $X, Y \in \Gamma((\ker\pi_*)^\perp)$, $U, V \in \Gamma(D_1)$ *and* $Z, W \in \Gamma(D_2)$.

From Theorems 3.11 to 3.12, we have the following theorem;

Theorem 3.14 *Let* $\pi : (M, \varphi, \xi, \eta, g_M) \longrightarrow (N, g_N)$ *be a semi-slant* $\xi^{\perp}$*-Riemannian submersion from a Sasakian manifold* $(M, \varphi, \xi, \eta, g_M)$ *onto a Riemannian manifold* (N, g_N) *with a semi-slant angle* θ*. Then the total space* M *is a locally (usual) product manifold of the leaves of* $ker\pi_*$ *and* $(ker\pi_*)^{\perp}$*, i.e.,* $M = M_{ker\pi_*} \times M_{(ker\pi_*)^{\perp}}$*, if and only if*

$$g_M(V, \mathcal{V}\overset{M}{\nabla}_X \phi\mathcal{B}Y + \mathcal{A}_X\omega\mathcal{B}Y) = g_N(\pi_*(CY), (\nabla\pi_*)(X, \varphi V)),$$

$$\begin{aligned} g_M(\mathcal{A}_X\omega W, \mathcal{B}Y) + \eta(Y)g_M(X, \omega W) &= g_N((\nabla\pi_*)(X, Y), \pi_*\omega\phi W) \\ &\quad - g_N((\nabla\pi_*)(X, CY), \pi_*\omega W) \end{aligned}$$

and

$$\begin{aligned} g_M(\omega V, \mathcal{T}_U\mathcal{B}X) + g_M(V, \phi U)\eta(X) &= g_N((\nabla\pi_*)(U, CX), \pi_*\omega V) \\ &\quad - g_N((\nabla\pi_*)(U, X), \pi_*\omega\phi V) \end{aligned}$$

for $X, Y \in \Gamma((\ker\pi_*)^{\perp})$*,* $U, V \in \Gamma(D_1)$ *and* $W \in \Gamma(D_2)$*.*

3.2 Totally Geodesic Semi-Slant $\xi^{\perp}$*-Submersions*

Recall that a differential map π between two Riemannian manifolds is called totally geodesic if $\nabla\pi_* = 0$ [41]. Then we have

Theorem 3.15 *Let* π *be a semi-slant* $\xi^{\perp}$*-Riemannian submersion from a Sasakian manifold* $(M, \varphi, \xi, \eta, g_M)$ *onto a Riemannian manifold* (N, g_N) *with a semi-slant angle* θ*. Then* π *is a totally geodesic map if*

$$\begin{aligned} -\nabla^{\pi}_X\pi_*Z_2 = \pi_*(C(&\mathcal{H}\overset{M}{\nabla}_X\omega Z_1 - \mathcal{A}_X\phi Z_1 + \mathcal{A}_X\mathcal{B}Z_2 + \mathcal{H}\overset{M}{\nabla}_X CZ_2) \\ &+ \omega(\mathcal{A}_X\omega Z_1 - \mathcal{V}\overset{M}{\nabla}_X\phi Z_1 + \mathcal{V}\overset{M}{\nabla}_X\mathcal{B}Z_2 + \mathcal{A}_X CZ_2) \\ &- \eta(Z_2)CX - \eta(X)\eta(Z_2) - g_M(Y, CX)\xi) \end{aligned} \tag{3.29}$$

for any $X \in \Gamma((ker\pi_*)^{\perp})$ *and* $Z = Z_1 + Z_2 \in \Gamma(TM)$*, where* $Z_1 \in \Gamma(ker\pi_*)$ *and* $Z_2 \in \Gamma((ker\pi_*)^{\perp})$*.*

Proof Making use of (2.5), (2.9) and (2.10), we have

$$\overset{M}{\nabla}_X Z = \varphi(\overset{M}{\nabla}_X\varphi)Z - \varphi\overset{M}{\nabla}_X\varphi Z + \eta(\overset{M}{\nabla}_X Z)\xi$$

for any $Z \in \Gamma((ker\pi_*)^{\perp})$ and $X \in \Gamma(TM)$. Now, from (2.7), we have

$$(\nabla\pi_*)(X,Z)=\nabla^\pi_X\pi_*Z+\pi_*(\varphi\overset{M}{\nabla}_X\varphi Z-\varphi(\overset{M}{\nabla}_X\varphi)Z-\eta(\overset{M}{\nabla}_X Z)\xi)$$
$$=\nabla^\pi_X\pi_*Z+\pi_*(\varphi(\overset{M}{\nabla}_X\varphi Z_1+\overset{M}{\nabla}_X\varphi Z_2)-\eta(Z)\varphi X-\eta(\overset{M}{\nabla}_X Z)\xi).$$

Or,

$$\begin{aligned}(\nabla\pi_*)(X,Z)=\nabla^\pi_X\pi_*Z_2+\pi_*(&\mathcal{B}\mathcal{A}_X\phi Z_1+\mathcal{C}\mathcal{A}_X\phi Z_1+\phi\mathcal{V}\overset{M}{\nabla}_X\phi Z_1+\omega\mathcal{V}\overset{M}{\nabla}_X\phi Z_1\\&+\phi\mathcal{A}_X\omega Z_1+\omega\mathcal{A}_X\omega Z_1+\mathcal{B}\mathcal{H}\overset{M}{\nabla}_X\omega Z_1+\mathcal{C}\mathcal{H}\overset{M}{\nabla}_X\omega Z_1\\&+\mathcal{B}\mathcal{A}_X\mathcal{B}Z_2+\mathcal{C}\mathcal{A}_X\mathcal{B}Z_2+\phi\mathcal{V}\overset{M}{\nabla}_X\mathcal{B}Z_2+\omega\mathcal{V}\overset{M}{\nabla}_X\mathcal{B}Z_2\\&+\phi\mathcal{A}_X\mathcal{C}Z_2+\omega\mathcal{A}_X\mathcal{C}Z_2+\mathcal{B}\mathcal{H}\overset{M}{\nabla}_X\mathcal{C}Z_2+\mathcal{C}\mathcal{H}\overset{M}{\nabla}_X\mathcal{C}Z_2\\&-\eta(Z_2)\varphi X-\eta(X)\eta(Z_2)-g_M(Z_2,\mathcal{C}X)\xi)\end{aligned}$$

for any $Z=Z_1+Z_2\in\Gamma(TM)$, where $Z_1\in\Gamma(ker\pi_*)$ and $Z_2\in\Gamma((ker\pi_*)^\perp)$.

$$\begin{aligned}(\nabla\pi_*)(X,Z)=\nabla^\pi_X\pi_*Z_2+\pi_*(&\mathcal{C}(\mathcal{A}_X\phi Z_1+\mathcal{H}\overset{M}{\nabla}_X\omega Z_1+\mathcal{A}_X\mathcal{B}Z_2+\mathcal{H}\overset{M}{\nabla}_X\mathcal{C}Z_2)\\&+\omega(\mathcal{V}\overset{M}{\nabla}_X\phi Z_1+\mathcal{A}_X\omega Z_1+\mathcal{V}\overset{M}{\nabla}_X\mathcal{B}Z_2+\mathcal{A}_X\mathcal{C}Z_2)\\&-\eta(Z_2)\mathcal{C}X-\eta(X)\eta(Z_2)-g_M(Z_2,\mathcal{C}X)\xi),\end{aligned}$$

which gives (3.29). This completes the assertion. □

Theorem 3.16 *Let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold (M,φ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with a semi-slant angle θ. Then π is a totally geodesic map if and only if*

(i) $g_M(\hat{\nabla}_{U_1}\varphi V_1,\mathcal{B}Z)=g_M(\mathcal{T}_{U_1}\mathcal{C}Z,\varphi V_1)-g_M(V_1,\phi U_1)\eta(Z),$

(ii) $(g_N(\nabla\pi_*(U_2,\omega\phi V_2))+g_N(\nabla\pi_*(U_2,\omega V_2))),\pi_*Z=g_M(\mathcal{T}_{U_2}\omega V_2,\mathcal{B}Z)+g_M(V_2,\phi U_2)\eta(Z)$

(iii) $g_N(\nabla\pi_*(U,\mathcal{C}X),\pi_*\mathcal{C}Y)-g_N(\nabla\pi_*(U,\omega\mathcal{B}X),\pi_*Y)=g_M(\mathcal{T}_U\phi\mathcal{B}X,Y)-g_M(\mathcal{T}_U\mathcal{C}X,\mathcal{B}Y)$
$+\eta(X)g_M(\mathcal{Q}U,\varphi Y)-\eta(Y)[U\eta(X)+g_M(X,\omega U)]$

for any $U_1,V_1\in\Gamma(D_1)$, $U_2,V_2\in\Gamma(D_2)$, $U\in\Gamma(ker\pi_*)$ and $X,Y,Z\in\Gamma((ker\pi_*)^\perp)$.

Theorem 3.17 *Let π be a semi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold (M,φ,ξ,η,g_M) onto a Riemannian manifold (N,g_N) with a semi-slant angle θ. Then π is a totally geodesic map if and only if*

(i) $\mathcal{C}(\mathcal{T}_U\phi V+\overset{M}{\nabla}_U\omega V)+\omega(\hat{\nabla}_U\phi V+\mathcal{T}_U\omega V)+g_M(\mathcal{P}V,\phi U)\xi=0.$

(ii) $\mathcal{C}(\mathcal{A}_X\phi U+\mathcal{H}\overset{M}{\nabla}_X\omega U)+\omega(\mathcal{A}_X\omega U+\mathcal{V}\overset{M}{\nabla}_X\phi U)+g_M(\mathcal{Q}U,\mathcal{B}X)\xi=0.$

(iii) $\mathcal{C}(\mathcal{T}_{U_1}\phi V_1+\mathcal{H}\overset{M}{\nabla}_{U_1}\phi V_1)+\omega(\mathcal{T}_{U_1}\omega V_1+\mathcal{V}\overset{M}{\nabla}_{U_1}\phi V_1)=0,$

for $U_1\in\Gamma(D_1)$, $V_1\in\Gamma(D_2)$, $U,V\in\Gamma(ker\pi_*)$ and $X\in\Gamma((ker\pi_*)^\perp)$.

3.3 Some Examples

Example 3.18 Every invariant submersion from a Sasakian manifold to a Riemannian manifold is a semi-slant $\xi^{\perp}$-Riemannian submersion with $D_2 = \{0\}$ and $\theta = 0$.

Example 3.19 Every slant Riemannian submersion from a Sasakian manifold to a Riemannian manifold is a semi-slant $\xi^{\perp}$-Riemannian submersion with $D_1 = \{0\}$.

Now, we construct some non-trivial examples of semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold. Let $(\mathbb{R}^{2n+1}, g, \varphi, \xi, \eta)$ denote the manifold $\mathbb{R}^{2n+1}$ with its usual Sasakian structure given by

$$\varphi(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) = \sum_{i=1}^{n}(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i})$$

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \quad \xi = 2\frac{\partial}{\partial z},$$

where $(x^1, ..., x^n, y^1, ..., y^n, z)$ are the Cartesian coordinates. Throughout this section, we will use this notation.

Example 3.20 Let F be a submersion defined by

$$F: \quad \begin{array}{ccc} \mathbb{R}^9 & \longrightarrow & \mathbb{R}^5 \\ (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) & & (\frac{x_1+x_2}{\sqrt{2}}, \frac{y_1+y_2}{\sqrt{2}}, sin\alpha x_3 - cos\alpha x_4, y_4, z) \end{array}$$

with $\alpha \in (0, \frac{\pi}{2})$. Then it follows that

$$\begin{aligned} ker F_* = span\{Z_1 = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}, \ Z_2 = \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2}, \\ Z_3 = -\cos\alpha\frac{\partial}{\partial x^3} - \sin\alpha\frac{\partial}{\partial x^4}, Z_4 = \frac{\partial}{\partial y^3}\} \end{aligned}$$

and

$$\begin{aligned} (ker F_*)^{\perp} = span\{H_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \ H_2 = \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}, \ H_3 = \sin\alpha\frac{\partial}{\partial x^3} - \cos\alpha\frac{\partial}{\partial x^4}, \\ H_4 = \frac{\partial}{\partial y^4}, \ H_5 = \frac{\partial}{\partial z} = \xi\}. \end{aligned}$$

Hence, we have $\varphi Z_1 = -Z_2$, $\varphi Z_2 = Z_1$. Thus, it follows that $D_1 = span\{Z_1, Z_2\}$ and $D_2 = span\{Z_3, Z_4\}$ is a slant distribution with slant angle $\theta = \alpha$. Thus, F is a semi-slant submersion with semi-slant angle θ. Also, by direct computations, we obtain

$$g_N(F_*H_1, F_*H_1) = g_M(H_1, H_1),\ g_N(F_*H_2, F_*H_2) = g_M(H_2, H_2),$$

$$g_N(F_*H_3, F_*H_3) = g_M(H_3, H_3),\ g_N(F_*H_4, F_*H_4) = g_M(H_4, H_4),\ g_N(F_*\xi, F_*\xi) = g_M(\xi, \xi)$$

where g_M and g_N denote the standard metrics (inner products) of $\mathbb{R}^9$ and $\mathbb{R}^5$. Thus, F is a semi-slant $\xi^{\perp}$-Riemannian submersion.

Example 3.21 Let F be a submersion defined by

$$\begin{array}{llll} F: & \mathbb{R}^7 & \longrightarrow & \mathbb{R}^3 \\ & (x_1, x_2, x_3, y_1, y_2, y_3, z) & & (\frac{x_2-y_3}{\sqrt{2}}, y_2, z). \end{array}$$

Then the submersion F is a semi-slant $\xi^{\perp}$-Riemannian submersion such that $D_1 = span(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1})$ and $D_2 = span(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_3}, \frac{\partial}{\partial x_3})$ with semi-slant angle $\alpha = \frac{\pi}{4}$.

Example 3.22 Let F be a submersion defined by

$$\begin{array}{llll} F: & \mathbb{R}^9 & \longrightarrow & \mathbb{R}^3 \\ & (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) & & (sin\alpha x_3 - cos\alpha x_4, y_4, z) \end{array}$$

with $\alpha \in (0, \frac{\pi}{2})$. Then the submersion F is a semi-slant $\xi^{\perp}$-Riemannian submersion such that $D_1 = span(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2})$ and $D_2 = span(-\cos\alpha \frac{\partial}{\partial x_3} - \sin\alpha \frac{\partial}{\partial x_4}, \frac{\partial}{\partial y_3})$ with semi-slant angle $\theta = \alpha$.

Example 3.23 Let F be a submersion defined by

$$\begin{array}{llll} F: & \mathbb{R}^{13} & \longrightarrow & \mathbb{R}^7 \\ & (x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6, z) & & (\frac{x_1-x_2}{\sqrt{2}}, \frac{y_1-y_2}{\sqrt{2}}, \frac{x_3+x_4}{\sqrt{2}}, \frac{y_3+y_4}{\sqrt{2}}, \frac{x_5-x_6}{\sqrt{2}}, y_5, z). \end{array}$$

Then the submersion F is a semi-slant $\xi^{\perp}$-Riemannian submersion such that $D_1 = span(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_3} - \frac{\partial}{\partial y_4})$ and $D_2 = span(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}, \frac{\partial}{\partial y_6})$ with semi-slant angle $\alpha = \frac{\pi}{4}$.

4 Hemi-Slant $\xi^{\perp}$-Riemannian Submersions

Very recently Ramazan Sarıand Mehmet Akif Akyol [36] also introduced and studied hemi-slant $\xi^{\perp}$-submersions and obtained interesting results. In this Sect. 4, our aim is to discuss briefly some results of this paper.

Definition 4.1 Let $(M, \varphi, \xi, \eta, g_M)$ be a Sasakian manifold and (N, g_N) be a Riemannian manifold. Suppose that there exists a Riemannian submersion $\phi : M \longrightarrow N$ such that ξ is normal to $ker\phi_*$. Then ϕ is called a hemi-slant $\xi^\perp$-Riemannian submersion if the vertical distribution $ker\phi_*$ of ϕ admits two orthogonal complementary distributions $\mathcal{D}_\perp$ and $\mathcal{D}_\theta$ such that $\mathcal{D}_\perp$ is anti-invariant and $\mathcal{D}_\theta$ is slant, i.e., we have

$$\ker\phi_* = \mathcal{D}_\perp \oplus \mathcal{D}_\theta.$$

In this case, the angle θ is called the slant angle of the hemi-slant $\xi^\perp$-Riemannian submersion.

If $\theta \neq 0, \frac{\pi}{2}$ then we say that the submersion is proper hemi-slant $\xi^\perp$-Riemannian submersion. Now, we are going to give some proper examples in order to guarantee the existence of hemi-slant $\xi^\perp$-Riemannian submersions in Sasakian manifolds and demonstrate that the method presented in this paper is effective. Note that, $(\mathbb{R}^{2n+1}, \varphi, \eta, \xi, g_{\mathbb{R}^{2n+1}})$ will denote the manifold $\mathbb{R}^{2n+1}$ with its usual contact structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \quad \xi = 2\frac{\partial}{\partial z},$$

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\varphi(\sum_{i=1}^{n}(X_i \partial x^i + Y_i \partial y^i) + Z\partial z) = \sum_{i=1}^{n}(Y_i \partial x^i - X_i \partial y^i)$$

where $(x_1, .., x_n, y_1, ..., y_n, z)$ denotes the Cartesian coordinates on $\mathbb{R}^{2n+1}$.

Example 4.2 Every anti-invariant $\xi^\perp$-Riemannian submersion from a Sasakian manifold onto a Riemannian manifold is a hemi-slant $\xi^\perp$-Riemannian submersion with $\mathcal{D}_\theta = \{0\}$.

Example 4.3 Every slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold onto a Riemannian manifold is a hemi-slant $\xi^\perp$-Riemannian submersion with $\mathcal{D}_\perp = \{0\}$.

Example 4.4 Let ϕ be a submersion defined by

$$\begin{array}{llll} \phi : & (\mathbb{R}^9, g_{\mathbb{R}^9}) & \rightarrow & (\mathbb{R}^5, g_{\mathbb{R}^5}) \\ & (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) & & (\frac{x_1+y_2}{\sqrt{2}}, \frac{x_2+y_1}{\sqrt{2}}, \sin\gamma x_3 - \cos\gamma x_4, y_4, z) \end{array}$$

with $\gamma \in (0, \frac{\pi}{2})$. Then it follows that

$$\ker\phi_* = Sp\{V_1 = -\partial x_1 + \partial y_2, V_2 = -\partial x_2 + \partial y_1, V_3 = -\cos\gamma\partial x_3 - \sin\gamma\partial x_4, \\ V_4 = \partial y_3\}$$

and

$$(\ker \phi_*)^{\perp} = Sp\{W_1 = \partial x_1 + \partial y_2, W_2 = \partial x_2 + \partial y_1, W_3 = \sin\gamma \partial x_3 - \cos\gamma \partial x_4, \\ W_4 = \partial y_4, W_5 = \partial z\}$$

hence we have $\varphi V_1 = W_2, \varphi V_2 = W_1$. Thus, it follows that $\mathcal{D}_{\perp} = sp\{V_1, V_2\}$ and $\mathcal{D}_{\theta} = sp\{V_3, V_4\}$ is a slant distribution with slant angle $\theta = \gamma$. Thus, ϕ is a slant $\xi^{\perp}$-submersion. Also by direct computations, we have

$$g_{\mathbb{R}^9}(W_i, W_i) = g_{\mathbb{R}^5}(\varphi W_i, \varphi W_i), \quad i = 1, ..., 5$$

which show that ϕ is a slant $\xi^{\perp}$-Riemannian submersion.

Example 4.5 Let F be a submersion defined by

$$\begin{array}{rccc} F: & (\mathbb{R}^9, g_{\mathbb{R}^9}) & \longrightarrow & (\mathbb{R}^5, g_{\mathbb{R}^5}) \\ & (x_1, ..., y_1, ..., z) & & (\frac{x_1+y_2}{\sqrt{2}}, \frac{x_2+y_1}{\sqrt{2}}, \frac{x_3+x_4}{\sqrt{2}}, \frac{y_3+y_4}{\sqrt{2}}, z). \end{array}$$

The submersion F is hemi-slant $\xi^{\perp}$-Riemannian submersion such that $\mathcal{D}_{\perp} = span\{\partial x_1 - \partial y_2, \partial x_2 - \partial y_1\}$ and $\mathcal{D}_{\theta} = span\{\partial x_3 + \partial x_4, \partial y_3 + \partial y_4\}$ with hemi-slant angle $\theta = 0$.

Example 4.6 Let π be a submersion defined by

$$\begin{array}{rccc} \pi: & (\mathbb{R}^7, g_{\mathbb{R}^7}) & \longrightarrow & (\mathbb{R}^4, g_{\mathbb{R}^4}) \\ & (x_1, ..., y_1, ..., z) & & (\frac{x_1+x_2}{\sqrt{2}}, \sin\gamma x_3 - \cos\gamma y_4, \cos\beta x_4 - \sin\beta y_3, z). \end{array}$$

The submersion π is a hemi-slant $\xi^{\perp}$-Riemannian submersion such that $\mathcal{D}_{\perp} = span\{\partial x_1 - \partial x_2\}$ and $\mathcal{D}_{\theta} = span\{\cos\gamma \partial x_3 - \sin\gamma \partial y_4, \sin\beta \partial x_4 - \cos\beta \partial y_3\}$ with hemi-slant angle $\theta = \alpha + \beta$.

Let ϕ be a hemi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then, for $U \in \Gamma(ker\phi_*)$, we put

$$U = \mathcal{P}U + \mathcal{Q}U$$

where $\mathcal{P}U \in \Gamma(\mathcal{D}_{\perp})$ and $\mathcal{Q}U \in \Gamma(\mathcal{D}_{\theta})$. For $Z \in \Gamma(TM)$, we have

$$Z = \mathcal{V}Z + \mathcal{H}Z$$

where $\mathcal{V}Z \in \Gamma(ker\phi_*)$ and $\mathcal{H}Z \in \Gamma(ker\phi_*)^{\perp}$.

We denote the complementary distribution to $\varphi\mathcal{D}_{\perp}$ in $(ker\phi_*)^{\perp}$ by μ. Then we have

$$(ker\phi_*)^{\perp} = \varphi\mathcal{D}_{\perp} \oplus \mu,$$

where $\varphi(\mu) \subset \mu$. Hence μ contains ξ. For $V \in \Gamma(ker\phi_*)$, we write

$$\varphi V = \rho V + \omega V \tag{4.1}$$

where ρV and ωV are vertical (resp. horizontal) components of φV, respectively. Also, for $X \in \Gamma((ker\phi_*)^\perp)$, we have

$$\varphi X = \mathcal{B}X + CX, \tag{4.2}$$

where $\mathcal{B}X$ and CX are vertical (resp. horizontal) components of φX, respectively. Then the horizontal distribution $(ker\phi_*)^\perp$ is decomposed as

$$(ker\phi_*)^\perp = \varphi\mathcal{D}_\perp \oplus \mu,$$

here μ is the orthogonal complementary distribution of $\mathcal{D}_\perp$ and it is both invariant distribution of $(ker\phi_*)^\perp$ with respect to φ and contains ξ. Then by using (2.3), (2.4), (4.1) and (4.2), we get

$$(\overset{M}{\nabla}_V \rho)W = \mathcal{B}T_V W - T_V \omega W \tag{4.3}$$

$$(\overset{M}{\nabla}_V \omega)W = CT_V W - T_V \rho W \tag{4.4}$$

for $V, W \in \Gamma(ker\phi_*)$, where

$$(\overset{M}{\nabla}_V \rho)W = \hat{\nabla}_V \rho W - \rho\hat{\nabla}_V W$$

and

$$(\overset{M}{\nabla}_V \omega)W = \mathcal{H}\overset{M}{\nabla}_V \omega W - \omega\hat{\nabla}_V W.$$

Lemma 4.7 *Let $\phi : M \to N$ be a hemi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\rho^2 W = \cos^2\theta W, \ \ W \in \Gamma(\mathcal{D}_\theta), \tag{4.5}$$

where θ denotes the hemi-slant angle of $ker\phi_*$.

Lemma 4.8 *Let $\phi : M \to N$ be a hemi-slant $\xi^\perp$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$g_M(\rho U, \rho V) = \cos^2\theta g_M(U, V) \tag{4.6}$$

$$g_M(\omega U, \omega V) = \sin^2\theta g_M(U, V) \tag{4.7}$$

for all $U, V \in \Gamma(ker\phi_*)$.

4.1 Integrable and Parallel Distributions

Theorem 4.9 *Let ϕ be a hemi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then the distribution $\mathcal{D}_{\perp}$ is integrable if and only if we have*

$$g_M(\mathcal{T}_U\varphi V - \mathcal{T}_V\varphi U, \rho Z) = g_N((\nabla\phi_*)(V, \varphi U) - (\nabla\phi_*)(U, \varphi V), \phi_*(\omega Z))$$

for any $U, V \in \Gamma(\mathcal{D}_{\perp})$ and $Z \in \Gamma(\mathcal{D}_{\theta})$.

Proof For $U, V \in \Gamma(TM)$, by using (2.9) and (2.10), we have

$$g_M(\nabla^M_U V, Z) = g_M(\nabla^M_U \varphi V, \varphi Z). \tag{4.8}$$

For $U, V \in \Gamma(\mathcal{D}_{\perp}),\ Z \in \Gamma(\mathcal{D}_{\theta})$, using (2.9) and (4.8), we have

$$g_M([U, V], Z) = g_M(\nabla^M_U \varphi V, \varphi Z) - g_M(\nabla^M_V \varphi U, \varphi Z).$$

On the other hand, we get

$$g_M([U, V], Z) = g_M(\mathcal{T}_U\varphi V - \mathcal{T}_V\varphi U, \rho Z) + g_M(\mathcal{H}(\nabla^M_U \varphi V) - \mathcal{H}(\nabla^M_V \varphi U), wZ).$$

Or,

$$\begin{aligned} g_M([U, V], Z) = {} & g_M(\mathcal{T}_U\varphi V - \mathcal{T}_V\varphi U, \rho Z) \\ & + g_N(\phi_*(\nabla^M_U \varphi V) - \phi_*(\nabla^M_V \varphi U), \phi_*(\omega Z)) \end{aligned}$$

which proves assertion. □

Theorem 4.10 *Let ϕ be a hemi-slant $\xi^{\perp}$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then the distribution $\mathcal{D}_{\theta}$ is integrable if and only if we have*

$$g_N((\nabla\phi_*)(Z, \omega W) - (\nabla\phi_*)(W, \omega Z), \varphi U) = g_M(\mathcal{T}_Z\omega\rho W - \mathcal{T}_W w\rho Z, U)$$

for any $Z, W \in \Gamma(\mathcal{D}_{\theta})$ and $U \in \Gamma(\mathcal{D}_{\perp})$.

Proof For $Z, W \in \Gamma(\mathcal{D}_{\theta})$ and $U \in \Gamma(\mathcal{D}_{\perp})$, using (2.9) and (4.8) we have

$$g_M([Z, W], U) = g_M(\nabla^M_Z \varphi W, \varphi U) - g_M(\nabla^M_W \varphi Z, \varphi U).$$

Therefore, by using (4.1), we get

$$\begin{aligned} g_M([Z,W],U) &= -g_M(\nabla^M_Z \rho^2 W, U) - g_M(\nabla^M_Z \omega\rho W, U) \\ &\quad + g_M(\nabla^M_Z \omega W, \varphi U) + g_M(\nabla^M_W \rho^2 Z, U) \\ &\quad + g_M(\nabla^M_W \omega\rho Z, U) - g_M(\nabla^M_W \omega Z, \varphi U). \end{aligned}$$

Now, by virtue of (3.16), we obtain

$$\begin{aligned} g_M([Z,W],U) &= \cos^2\theta g_M([Z,W],U) - g_M(\nabla^M_Z \omega\rho W, U) \\ &\quad + g_M(\nabla^M_Z \omega W, \varphi U) + g_M(\nabla^M_W \omega\rho Z, U) \\ &\quad - g_M(\nabla^M_W \omega Z, \varphi U). \end{aligned}$$

Then we have

$$\begin{aligned} \sin^2\theta g_M([Z,W],U) &= g_M(\nabla^M_W \omega\rho Z - \nabla^M_Z \omega\rho W, U) \\ &\quad + g_M(\nabla^M_Z \omega W - \nabla^M_W \omega Z, \varphi U). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sin^2\theta g_M([Z,W],U) &= g_M(\mathcal{T}_W \omega\rho Z - \mathcal{T}_Z \omega\rho W, U) \\ &\quad + g_M(H(\nabla^M_Z \omega W) - \mathcal{H}(\nabla^M_W \omega Z), \varphi U) \\ &= g_M(\mathcal{T}_W \omega\rho Z - \mathcal{T}_Z \omega\rho W, U) \\ &\quad + g_N(\phi_*(\nabla^M_Z \omega W) - \phi_*(\nabla^M_W \omega Z), \varphi U) \end{aligned}$$

which proves assertion. □

Theorem 4.11 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then the distribution $D_\perp$ is parallel if and only if*

$$g_M(\phi_*(\nabla_U V), \phi_*(\omega\rho Z)) = g_M(\varphi\nabla_U V, \omega Z)$$

and

$$g_M(\hat{\nabla}_U \rho V + \mathcal{T}_U \omega V, BX) = -g_M(\mathcal{T}_U \rho V + \mathcal{H}(\nabla_U \omega V), CX)$$

for any $U, V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$, $X \in \Gamma((\ker\phi_*)^\perp)$.

Proof For $U, V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$ using (2.9), we get

$$\begin{aligned} g_M(\nabla_U V, Z) &= g_M(\varphi\nabla_U V, \varphi Z) + \eta(\nabla_U V)\eta(Z) \\ &= g_M(\varphi\nabla_U V, \varphi Z). \end{aligned}$$

Or,

$$g_M(\nabla_U V, Z) = -g_M(\nabla_U V, \rho^2 Z + \omega\rho Z + \varphi\omega Z).$$

Then one obtains

$$\sin^2\theta g_M(\nabla_U V, Z) = -g_M(\mathcal{H}(\nabla_U V), \omega\rho Z) + g_M(\varphi\nabla_U V, \omega Z).$$

By property of ϕ, we get

$$\sin^2\theta g_M(\nabla_U V, Z) = -g_N(\phi_*(\nabla_U V), \phi_*(\omega\rho Z)) + g_M(\varphi\nabla_U V, \omega Z).$$

On the other hand, for $U, V \in \Gamma(D_\perp)$, $X \in \Gamma((\ker\phi_*)^\perp)$, we have

$$g_M(\nabla_U V, X) = g_M(\nabla_U\varphi V, \varphi X).$$

Now, by virtue of (2.3) and (4.1), we obtain

$$\begin{aligned} g_M(\nabla_U V, X) = {} & g_M(\mathcal{T}_U\rho V, CX) + g_M(\hat{\nabla}\rho V, BX) \\ & + g_M(\mathcal{T}_U\omega V, BX) + g_M(\mathcal{H}(\nabla_U\omega V), CX) \end{aligned}$$

which completes the proof. □

Theorem 4.12 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then the distribution D_θ is parallel if and only if*

$$g_N(\phi_*(\omega W), (\nabla\phi_*)(Z, \varphi U)) = g_M(\rho W, \mathcal{T}_Z\varphi U)$$

and

$$\begin{aligned} & g_N((\nabla\phi_*)(\nabla_Z\omega\rho W), \phi_*(X)) - g_N((\nabla\phi_*)(\nabla_Z\omega W), \phi_*(CX)) \\ & = -g_M(\mathcal{T}_Z\omega W, BX) + g_M(\omega W, Z)\eta(X). \end{aligned}$$

for all $Z, W \in \Gamma(D_\theta)$, $U \in \Gamma(D_\perp)$, $X \in \Gamma((\ker\phi_*)^\perp)$.

Theorem 4.13 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then $D_\perp$ defines a totally geodesic foliation on M if and only if*

$$g_N((\nabla\phi_*)(U, \varphi V), \phi_*(\omega Z)) = -g_M(\mathcal{T}_U V, \omega\rho Z)$$

and

$$g_M(\mathcal{T}_U\varphi V, BX) = g_N((\nabla\phi_*)(U, \varphi V), \phi_*(CX))$$

for any $U, V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$, $X \in \Gamma((\ker\phi_*)^\perp)$.

Proof For $U, V \in \Gamma(D_\perp)$, $Z \in \Gamma(D_\theta)$, from (2.9), (2.3), (2.4), (4.1) to (4.5), we have

$$g_M(\nabla_U V, Z) = \cos^2\theta g_M(\nabla_U V, Z) - g_M(\mathcal{T}_U V, \omega\rho Z) + g_M(\mathcal{H}(\nabla_U \varphi V), wZ).$$

Or,

$$\sin^2\theta g_M(\nabla_U V, Z) = -g_M(\mathcal{T}_U V, w\rho Z) - g_N(\phi_*(\nabla_U \varphi V), \phi_*(\omega Z)).$$

On the other hand, for $X \in \Gamma((\ker\phi_*)^\perp)$, we have

$$g_M(\nabla_U V, X) = g_M(\mathcal{T}_U \varphi V, BX) + g_M(\mathcal{H}(\nabla_U \varphi V), CX).$$

Or,

$$g_M(\nabla_U V, X) = g_M(\mathcal{T}_U \varphi V, BX) - g_N(\phi_*(\nabla_U \varphi V), \phi_*(CX)).$$

This completes the proof. □

Theorem 4.14 *Let ϕ be a hemi-slant $\xi^\perp$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then D_θ defines a totally geodesic foliation on M if and only if*

$$g_N((\nabla\phi_*)(Z, \omega W), \phi_*(\varphi U)) = -g_M(\mathcal{T}_Z \omega\rho W, U)$$

and

$$g_N((\nabla\phi_*)(Z, \omega\rho W), \phi_*(X)) + g_N((\nabla\phi_*)(Z, \omega W), \phi_*(CX)) = g_M(\mathcal{T}_Z \omega W, BX)$$

for any $Z, W \in \Gamma(D_\theta)$, $U \in \Gamma(D_\perp)$, $X \in \Gamma((\ker\phi_*)^\perp)$.

4.2 Hemi-Slant $\xi^\perp$-Riemannian Submersions on Sasakian Space Forms

A plane section in the tangent space T_pM at $p \in M$ is called a φ-section if it is spanned by a vector X orthogonal to ξ and φX. The sectional curvature of φ-section is called φ-sectional curvature. A Sasakian manifold with constant φ-sectional curvature c is a Sasakian space form. The Riemannian curvature tensor of a Sasakian space form is given by

$$\begin{aligned} R^M(X,Y,Z,W) &= \frac{c+3}{4}\{g_M(Y,Z)g_M(X,W) - g_M(X,Z)g_M(Y,W)\} \\ &+ \frac{c-1}{4}\{g_M(Y,W)\eta(X)\eta(Z) - g_M(X,W)\eta(Y)\eta(Z) \\ &+ g_M(X,Z)\eta(Y)\eta(W) - g_M(Y,Z)\eta(X)\eta(W) \\ &+ g_M(\varphi Y,Z)g_M(\varphi X,W) - g_M(\varphi X,Z)g_M(\varphi Y,W) \\ &- 2g_M(\varphi X,Y)g_M(\varphi Z,W)\} \end{aligned} \tag{4.9}$$

for any $X, Y, Z, W \in \Gamma(TM)$ [39].

Theorem 4.15 *Let ϕ be a hemi-slant $\xi^{\perp}$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. Then we have*

$$\begin{aligned} \widehat{R}(U,V,W,S) &= \frac{c+3}{4}\{g_M(V,S)g_M(U,W) - g_M(U,S)g_M(V,W)\} \\ &+ g_M(\mathcal{T}_V W, \mathcal{T}_U S) - g_N(\mathcal{T}_U W, \mathcal{T}_V S) \end{aligned} \tag{4.10}$$

and

$$\widehat{K}(U,V) = \frac{c+3}{4}\{g_M(U,V)^2 - 1\} + g_M(\mathcal{T}_V U, \mathcal{T}_U V) - g_M(\mathcal{T}_U U, \mathcal{T}_V V) \tag{4.11}$$

for all $U, V, S, W \in \Gamma(\mathcal{D}^{\perp})$.

Proof For any $U, V, S, W \in \Gamma(\mathcal{D}_{\perp})$ by using (4.9), $\varphi U \in \Gamma((\ker \phi_*)^{\perp})$ and $\eta(U) = 0$, then we have

$$R^M(U,V,S,W) = \frac{c+3}{4}\{g_M(V,S)g_M(U,W) - g_M(U,S)g_M(V,W)\}. \tag{4.12}$$

Hence, we have

$$\begin{aligned} \widehat{R}(U,V,W,S) &= \frac{c+3}{4}\{g_M(V,S)g_M(U,W) - g_M(U,S)g_M(V,W)\} \\ &+ g_M(\mathcal{T}_V W, \mathcal{T}_U S) - g_M(\mathcal{T}_U W, \mathcal{T}_V S) \end{aligned}$$

which completes the proof. □

Corollary 4.16 *Let ϕ be a hemi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M^m, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ and $m \geq 3$. If $\mathcal{D}_{\perp}$ is totally geodesic, then M is flat if and only if $c = -3$.*

Theorem 4.17 *Let ϕ be a hemi-slant $\xi^{\perp}$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. If $\mathcal{D}_{\perp}$ is totally geodesic, then*

$$\widehat{\tau}_{\perp} = \frac{c+3}{2} q(1-2q)$$

where $\widehat{\tau}_{\perp}$ *is the scaler curvature.*

Proof We have

$$\widehat{S}_{\perp}(U, V) = \sum_{i=1}^{2q} \widehat{R}(E_i, U, V, E_i)$$

where $\{E_1, ..., E_{2q}\}$ is ortonormal basis on $\Gamma(\mathcal{D}_{\perp})$ and $U, V \in \Gamma(\mathcal{D}_{\perp})$. Thus, one obtains

$$\widehat{S}_{\perp}(U, V) = \sum_{i=1}^{2q} \{\frac{c+3}{4}\{g_M(U, E_i)g_M(E_i, V) - g_M(E_i, E_i)g_M(U, V)\}\}.$$

Or,

$$\widehat{S}_{\perp}(U, V) = \frac{c+3}{4}(1-2q)g_M(U, V). \tag{4.13}$$

By taking $U = V = E_k, k = 1, ..., 2q$, we get the result. □

Corollary 4.18 *Let* ϕ *be a hemi-slant* $\xi^{\perp}$*-Riemannian submersion from a Sasakian manifold* $(M, \varphi, \eta, \xi, g_M)$ *onto Riemannian manifold* (N, g_N) *with a hemi-slant angle* θ. *If* $\mathcal{D}_{\perp}$ *is totally geodesic distribution, then* $\mathcal{D}_{\perp}$ *is Einstein.*

Theorem 4.19 *Let* ϕ *be a hemi-slant* $\xi^{\perp}$ *Riemannian submersion from a Sasakian manifold* $(M, \varphi, \eta, \xi, g_M)$ *onto Riemannian manifold* (N, g_N) *with a hemi-slant angle* θ. *Then we have*

$$\begin{aligned}\widehat{R}(K, L, P, W) = & \frac{c+3}{4}\{g_M(L, P)g_M(K, W) - g_M(K, P)g_M(L, W)\} \\ & + \frac{c-1}{4}\{g_M(\varphi L, P)g_M(\varphi K, W) \\ & - g_M(\varphi K, P)g_M(\varphi L, W) - 2g_M(\varphi K, L)g_M(\varphi P, W)\} \\ & + g_M(\mathcal{T}_L P, \mathcal{T}_K W) - g_M(\mathcal{T}_K P, \mathcal{T}_L W)\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\widehat{K}(K, L) = & \frac{c+3}{4}\{g_M(L, K)g_M(K, L) - g_M(K, K)g_M(L, L)\} \\ & - 3\frac{c-1}{4} g_M(\varphi K, L) + g_M(T_L K, T_K L) - g_M(\mathcal{T}_K K, \mathcal{T}_L L)\end{aligned} \tag{4.15}$$

for all $K, L, P, N \in \Gamma(\mathcal{D}_{\theta})$.

Theorem 4.20 *Let ϕ be a hemi-slant $\xi^{\perp}$ Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. If $\mathcal{D}_\theta$ is totally geodesic, then we have*

$$\widehat{k_\theta} = p\frac{(c+3)(2p-1)+3(c-1)\cos^2\theta}{2}.$$

Proof For any $K, L \in \Gamma(\mathcal{D}_\theta)$, using (4.14), we derive

$$\widehat{S_\theta}(K, L) = \frac{c+3}{4}(2p-1)g_M(K, L) + 3\frac{c-1}{4}\cos^2\theta g_M(K, L) \tag{4.16}$$

where $\{E_1, ..., E_{2p}\}$ is orthonormal basis on $\Gamma(\mathcal{D}_\theta)$. From the above equation, we obtain the proof. □

Corollary 4.21 *Let ϕ be a hemi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold $(M, \varphi, \eta, \xi, g_M)$ onto Riemannian manifold (N, g_N) with a hemi-slant angle θ. If $\mathcal{D}_\theta$ is totally geodesic distribution, then $\mathcal{D}_\theta$ is Einstein.*

5 Quasi Hemi-slant Submanifolds of Cosymplectic Manifolds

In this Sect. 5, we will finally discuss some results of quasi hemi-slant submanifolds introduced and studied by Rajendra Prasad et al. [38]. First, we have

Definition 5.1 A submanifold M of an almost contact metric manifold $\overline{M}$ is called a quasi hemi-slant submanifold if there exist distributions D, D^θ and $D^\perp$ such that (i) TM admits the orthogonal direct decomposition as

$$TM = D \oplus D^\theta \oplus D^\perp \oplus <\xi> .$$

(ii) The distribution D is ϕ invariant, $i.e.$, $\phi D = D$.
(iii) For any nonzero vector field $X \in (D^\theta)_p$, $p \in M$, the angle θ between JX and $(D^\theta)_p$ is constant and independent of the choice of point p and X in $(D^\theta)_p$.
(iv) The distribution $D^\perp$ is ϕ anti-invariant, i.e., $\phi D^\perp \subseteq T^\perp M$.

In this case, we call θ the quasi hemi-slant angle of M. Suppose the dimension of distributions D, D^θ and $D^\perp$ are n_1, n_2 and n_3, respectively. Then we can easily see the following particular cases:
(i) If $n_1 = 0$, then M is a hemi-slant submanifold.
(ii) If $n_2 = 0$; then M is a semi-invariant submanifold.
(iii) If $n_3 = 0$, then M is a semi-slant submanifold.
We say that a quasi hemi-slant submanifold M is proper if $D \neq \{0\}$, $D^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

This means that the notion of quasi hemi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant, semi-slant submanifolds. Let M be a quasi hemi-slant submanifold of an almost contact metric manifold $\overline{M}$. We denote the projections of $X \in \Gamma(TM)$ on the distributions D, D^{θ} and $D^{\perp}$ by P, Q and R, respectively. Then we can write for any $X \in \Gamma(TM)$

$$X = PX + QX + RX + \eta(X)\xi. \tag{5.1}$$

Now we put

$$\phi X = TX + NX, \tag{5.2}$$

where TX and NX are tangential and normal components of ϕX on M. Using (5.1) and (5.2), we obtain

$$\phi X = TPX + NPX + TQX + NQX + TRX + NRX.$$

Since $\phi D = D$ and $\phi D^{\perp} \subseteq T^{\perp}M$, we have $NPX = 0$ and $TRX = 0$. Therefore, we get

$$\phi X = TPX + TQX + NQX + NRX. \tag{5.3}$$

Then for any $X \in \Gamma(TM)$, it is easy to see that

$$TX = TPX + TQX$$

and

$$NX = NQX + NRX.$$

For any $V \in \Gamma(T^{\perp}M)$, we can put

$$\phi V = tV + nV$$

where tV and nV are the tangential and normal componenets of ϕV on M, respectively.

An almost contact metric manifold is called a cosymplectic manifold if $(\widehat{\nabla}_X\phi)Y = 0$, $\widehat{\nabla}_X\xi = 0 \ \ \forall\ X, Y \in \Gamma(T\widehat{M})$, where $\widehat{\nabla}$ represents the Levi-Civita connection of $(\widehat{M}, g)$.
The covariant derivative of ϕ is defined as

$$(\widehat{\nabla}_X\phi)Y = \widehat{\nabla}_X\phi Y - \phi\widehat{\nabla}_X Y.$$

If $\widehat{M}$ is a cosymplectic manifold, then we have

$$\phi\widehat{\nabla}_X Y = \widehat{\nabla}_X\phi Y.$$

Let M be a Riemannian manifold isometrically immersed in $\widehat{M}$ and the induced Riemannian metric on M is denoted by the same symbol g throughout this paper. Let A and h denote the shape operator and second fundamental form, respectively, of submanifolds of M into $\widehat{M}$. The Gauss and Weingarten formulas are given by

$$\widehat{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$\widehat{\nabla}_X V = -A_V X + \nabla_X^{\perp} V$$

for any vector fields $X, Y \in \Gamma(TM)$ and V on $\Gamma(T^{\perp}M)$, where ∇ is the induced connection on M and $\nabla^{\perp}$ represents the connection on the normal bundle $T^{\perp}M$ of M and A_V is the shape operator of M with respect to normal vector $V \in \Gamma(T^{\perp}M)$. Moreover, A_V and the second fundamental form $h : TM \otimes TM \longrightarrow T^{\perp}M$ of M into $\widehat{M}$ are related by

$$g(h(X, Y), V) = g(A_V X, Y),$$

for any vector fields $X, Y \in \Gamma(TM)$ and V on $\Gamma(T^{\perp}M)$.

5.1 Integrability of Distributions

Theorem 5.2 *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\overline{M}$. Then the invariant distribution D is integrable if and only if*

$$g(\nabla_X TY - \nabla_Y TX, TQZ) = g(h(Y, TX) - h(X, TY), NQZ + NRZ)$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\theta} \oplus D^{\perp})$.

Proof For a cosymplectic manifold, we have

$$\overline{\nabla}_X \xi = 0 \;\; \forall \;\; X \in \Gamma(D). \tag{5.4}$$

If $Y \in \Gamma(D)$, then $g(Y, \xi) = 0$. Thus, one gets

$$g(\overline{\nabla}_X Y, \xi) + g(Y, \overline{\nabla}_X \xi) = 0. \tag{5.5}$$

Now, $g([X, Y], \xi) = g(\overline{\nabla}_X Y, \xi) - g(\overline{\nabla}_Y X, \xi) = 0$.
Also, we have

$$g([X, Y], Z) = g(\overline{\nabla}_X \phi Y, \phi Z) - g(\overline{\nabla}_Y \phi X, \phi Z) = g(\nabla_X TY - \nabla_Y TX, TQZ) + g(h(X, TY) - h(Y, TX), NQZ + NRZ)$$

which completes the proof. □

Similarly, we have

Theorem 5.3 *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $(\overline{M}, g, \phi)$. Then the slant distribution D^θ is integrable if and only if*

$$g(A_{NW}Z - A_{NZ}W, TPX) = g(A_{NTW}Z - A_{NTZ}W, X)$$
$$+g(\nabla^\perp_Z NW - \nabla^\perp_W NZ, NRX)$$

for any $Z,\ W \in \Gamma(D^\theta)$ and $X \in \Gamma(D \oplus D^\perp)$.

Theorem 5.4 *Let M be a quasi hemi-slant submanifold of a cosymplectic manifold $\overline{M}$. Then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$g(T([Z, W]), TX) = g(\nabla^\perp_W NZ - \nabla^\perp_Z NW, NQX)$$

for any $Z, W \in \Gamma(D^\perp)$ and $X \in \Gamma(D \oplus D^\theta)$.

5.2 Totally Geodesic Foliations

Theorem 5.5 *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\overline{M}$. Then M is totally geodesic if and only if*

$$g(h(X, PY) + \cos^2\theta h(X, QY), U) = g(\nabla^\perp_X NTQY, U)$$
$$+g(A_{NQY}X + A_{NRY}X, tU) - g(\nabla^\perp_X NY, nU)$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(T^\perp M)$.

Proof For any $X,\ Y \in \Gamma(TM),\ \ U \in \Gamma\left(T^\perp M\right)$, we have

$$\begin{aligned} g(\overline{\nabla}_X Y, U) &= g(\overline{\nabla}_X PY, U) + g(\overline{\nabla}_X QY, U) + g(\overline{\nabla}_X RY, U) \\ &= g(\overline{\nabla}_X \phi PY, \phi U) + g(\overline{\nabla}_X TQY, \phi U) + g(\overline{\nabla}_X NQY, \phi U) \\ &\quad +g(\overline{\nabla}_X \phi RY, \phi U). \end{aligned}$$

$$\begin{aligned} g(\overline{\nabla}_X Y, U) &= g(h(X, PY) + \cos^2\theta h(X, QY), U) - g(\nabla^\perp_X NTQY, U) \\ &\quad -g(A_{NQY}X + A_{NRY}X, tU) + g(\nabla^\perp_X NY, nU) \end{aligned}$$

which completes the proof. □

Similarly, we have

Theorem 5.6 *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\overline{M}$. Then anti-invariant distribution $D^\perp$ defines totally geodesic foliation if and only if*

$$g(A_{\phi Y}X, TPZ + tQZ) = g(\nabla^{\perp}_{X}\phi Y, nQZ), \quad g(A_{\phi Y}X, tV) = g(\nabla^{\perp}_{X}\phi Y, nV)$$

for any $X, Y \in \Gamma(D^{\perp})$, $Z \in \Gamma(D \oplus D^{\theta})$ *and* $V \in \Gamma\left(T^{\perp}M\right)$.

Theorem 5.7 *Let M be a proper quasi hemi-slant submanifold of a cosymplectic manifold $\overline{M}$. Then the slant distribution D^{θ} defines a totally geodesic foliation on M if and only if*

$$g(\nabla^{\perp}_{X}NY, NRZ) = g(A_{NY}X, TPZ) - g(A_{NTY}X, Z), \text{ and}$$
$$g(A_{NY}X, tV) = g(\nabla^{\perp}_{X}NY, nV) - g(\nabla^{\perp}_{X}NTY, V)$$

for any $X, Y \in \Gamma(D^{\theta})$, $Z \in \Gamma(D \oplus D^{\perp})$ *and* $V \in \Gamma\left(T^{\perp}M\right)$.

5.3 Examples

Now we discuss few examples from [38]

Example 5.8 Let us consider a 15-dimensional differentiable manifold

$$\overline{M} = \{(x_i, y_i, z) = (x_1, x_2, ..., x_7, y_1, y_2, ..., y_7, z) \in \mathrm{R}^{15}\}.$$

And choose the vector fields

$$E_i = \frac{\partial}{\partial y_i}, \; E_{7+i} = \frac{\partial}{\partial x_i}, \; E_{15} = \xi = \frac{\partial}{\partial z}, \text{ for } i = 1, 2, ..., 7.$$

Let g be a Riemannian metric defined by

$$g = (dx_1)^2 + (dx_2)^2 + \cdots + (dx_7)^2 + (dy_1)^2 + (dy_2)^2 + \cdots + (dy_7)^2 + (dz)^2.$$

We define $(1, 1)$-tensor field ϕ as

$$\phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \; \phi\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}, \quad \phi\left(\frac{\partial}{\partial z}\right) = 0 \;\; \forall\, i, j = 1, 2, ..., 7.$$

Thus, $(\overline{M}, \phi, \xi, \eta, g)$ is an almost contact metric manifold. Also, we can easily show that $(\overline{M}, \phi, \xi, \eta, g)$ is a cosymplectic manifold of dimension 15.

Let M be a submanifold of $\overline{M}$ defined by

$$f(u, v, w, r, s, t, q) = \left(u, w, 0, \frac{s}{\sqrt{2}}, 0, \frac{t}{\sqrt{2}}, 0, v, r\cos\theta, r\sin\theta, 0, \frac{s}{\sqrt{2}}, 0, \frac{t}{\sqrt{2}}, q\right),$$

where $0 < \theta < \frac{\pi}{2}$. Now the tangent bundle of M is spanned by the set $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$Z_1 = \frac{\partial}{\partial x_1},\ Z_2 = \frac{\partial}{\partial y_1},\ Z_3 = \frac{\partial}{\partial x_2},$$

$$Z_4 = \cos\theta\frac{\partial}{\partial y_2} + \sin\theta\frac{\partial}{\partial y_3},\ Z_5 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_5}\right),$$

$$Z_6 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_6} + \frac{\partial}{\partial y_7}\right),\ Z_7 = \frac{\partial}{\partial z}.$$

Thus, we have

$$\phi Z_1 = \frac{\partial}{\partial y_1},\ \phi Z_2 = -\frac{\partial}{\partial x_1},\ \phi Z_3 = \frac{\partial}{\partial y_2},$$

$$\phi Z_4 = -\left(\cos\theta\frac{\partial}{\partial x_2} + \sin\theta\frac{\partial}{\partial x_3}\right),\ \phi Z_5 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_4} - \frac{\partial}{\partial x_5}\right),$$

$$\phi Z_6 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y_6} - \frac{\partial}{\partial x_7}\right),\ \phi Z_7 = 0.$$

Now, let the distributions $D = span\{Z_1, Z_2\}$, $D^\theta = span\{Z_3, Z_4\}$, $D^\perp = span\{Z_5, Z_6\}$. And D is invariant, D^θ is slant with slant angle θ and $D^\perp$ is anti-invariant.

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Slant Lightlike Submanifolds of Indefinite Contact Manifolds

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1 Introduction

The geometry of submanifolds with degenerate (lightlike) metric is difficult and strikingly different from the geometry of submanifolds with non-degenerate metric because of the fact that their (of degenerate submanifolds) normal vector bundle intersects with the tangent bundle. This means that we cannot use the classical theory of submanifolds to define induced objects on a lightlike submanifold. Since the geometry of lightlike submanifolds is needed to fill a gap in the general theory of submanifolds and have significant applications in general theory of relativity, particularly in black hole theory, therefore, Duggal and Bejancu [9] introduced the geometry of lightlike submanifolds of semi-Riemannian manifolds and further established by many other geometers.

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Chen [7, 8] introduced the notion of slant submanifolds as a generalizing of holomorphic and totally real submanifolds for complex geometry. Later, slant submanifolds for contact geometry were introduced by Lotta [21]. Cabrerizo et al. [5, 6] studied slant, semi-slant, and bi-slant submanifolds in contact geometry, and then many interesting results on slant submanifolds of contact manifolds were explored by many other geometers. Most of them studied the geometry of slant submanifolds with positive definite metric; therefore this geometry may not be applicable to the other branches of mathematics and physics, where the metric is not necessarily definite. Thus, the notion of slant lightlike submanifolds of indefinite Hermitian manifolds was introduced by Sahin [30].

It is well known that there are significant uses of contact geometry in differential equations, optics, and phase spaces of a dynamical system (cf. [1, 22, 23]). Hence, the notion of screen slant lightlike submanifolds of indefinite Kenmotsu manifolds and Cosymplectic manifolds was given by Gupta et al. in [12, 13], respectively. Later, the geometry of slant lightlike submanifolds of indefinite Kenmotsu manifolds [14], of indefinite Cosymplectic manifolds [15], and of indefinite Sasakian manifolds [20, 33] was introduced and obtained necessary and sufficient conditions for their existence. Haider et al. studied screen slant lightlike submanifolds of indefinite Sasakian manifolds and hemi-slant lightlike submanifolds of indefinite Kenmotsu manifolds in [16] and [17], respectively. Shukla and Yadav studied radical transversal screen semi-slant lightlike submanifolds, screen semi-slant lightlike submanifolds, and semi-slant lightlike submanifolds of indefinite Sasakian manifolds in [34], [37], and [36], respectively. Sachdeva et al. studied the geometry of totally contact umbilical slant lightlike submanifolds of indefinite Cosymplectic manifolds, Sasakian manifolds, and Kenmotsu manifolds in [24], [25], and [27], respectively. Sachdeva et al. also studied warped product slant lightlike submanifolds and totally contact umbilical hemi-slant lightlike submanifolds of indefinite Sasakian manifolds in [26] and [28], respectively.

The major purpose of this chapter is to present a comprehensive geometry of slant lightlike submanifolds of indefinite Sasakian manifolds.

2 Lightlike Submanifolds

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$, see [9]. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as the radical (null) subspace. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$ defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $Rad(TM)$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is, $TM = Rad(TM) \perp S(TM)$, and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Since for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $(S(TM))^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM) = ltr(TM) \perp S(TM^{\perp})$, then $tr(TM)$ is a complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$ and we have $T\bar{M}\ |_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp})$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$, then for $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{1}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here, ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$, called the second fundamental form, and A_U is a linear operator on M, known as the shape operator.

Let $\mathcal{L}$ and $\mathcal{S}$ be the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then (1) becomes

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2}$$

where we put $h^l(X, Y) = \mathcal{L}(h(X, Y))$, $h^s(X, Y) = \mathcal{S}(h(X, Y))$, and $D^l_X U = \mathcal{L}(\nabla^{\perp}_X U)$, $D^s_X U = \mathcal{S}(\nabla^{\perp}_X U)$. As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued, they are called the lightlike second fundamental form and the screen second fundamental form on M, respectively. In particular,

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N) \tag{3}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{4}$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$, and $W \in \Gamma(S(TM^{\perp}))$. Further, from (2), (3), and (4), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y). \tag{5}$$

Let P be a projection morphism of TM on $S(TM)$, then we can write

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \quad \nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{6}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X PY, A^*_{\xi} X\}$ and $\{h^*(X, PY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. Here, ∇^* and ∇^{*t}_X

are linear connections on $S(TM)$ and $Rad(TM)$, respectively. Using (3), (4), and (6), we obtain

$$\bar{g}(h^l(X, PY), \xi) = g(A^*_\xi X, PY). \tag{7}$$

3 Totally Contact Umbilical Slant Lightlike Submanifolds

A semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an ϵ-almost contact metric manifold if there exists a (1, 1) tensor field ϕ, a vector field V, called a characteristic vector field, and a 1-form η, satisfying

$$\phi^2 X = -X + \eta(X)V, \quad \eta(V) = \epsilon, \quad \eta \circ \phi = 0, \quad \phi V = 0, \tag{8}$$

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \epsilon\eta(X)\eta(Y), \tag{9}$$

for all $X, Y \in \Gamma(T\bar{M})$, where $\epsilon = 1$ or -1. It follows that

$$\bar{g}(V, V) = \epsilon, \quad \bar{g}(X, V) = \eta(X), \quad \bar{g}(X, \phi Y) = -\bar{g}(\phi X, Y). \tag{10}$$

Then $(\phi, V, \eta, \bar{g})$ is called an ϵ-almost contact metric structure on $\bar{M}$ [19]. An ϵ-almost contact metric structure $(\phi, V, \eta, \bar{g})$ is called an indefinite Sasakian structure if and only if

$$(\bar{\nabla}_X \phi)Y = \bar{g}(X, Y)V - \epsilon\eta(Y)X, \tag{11}$$

where $\bar{\nabla}$ is Levi-Civita connection with respect to $\bar{g}$. A semi-Riemannian manifold endowed with an indefinite Sasakian structure is called an indefinite Sasakian manifold. From (11), we have $\bar{\nabla}_X V = -\phi X$. An indefinite almost contact metric manifold $\bar{M}$ is called an indefinite Kenmotsu manifold [18] if $(\bar{\nabla}_X \phi)Y = -\bar{g}(\phi X, Y)V + \eta(Y)\phi X$ and is called an indefinite Cosymplectic manifold [4] if $(\bar{\nabla}_X \phi)Y = 0$.

To define the notion of slant submanifolds, one needs to consider the angle between two vector fields. A lightlike submanifold has two distributions, namely, the radical distribution and the screen distribution. The radical distribution is totally lightlike, and therefore it is not possible to define an angle between two vector fields of the radical distribution. Therefore, Sahin and Yildirim [33] used two vector fields of screen distribution (as screen distribution is non-degenerate) to introduce the notion of slant lightlike submanifolds of an indefinite Sasakian manifold. Toward this direction, they proved the following important lemmas:

Lemma 3.1 *Let $(M, g, S(TM), S(TM^\perp))$ be an r-lightlike submanifold in an indefinite Sasakian manifold of constant index $2q$. Suppose $\phi(Rad(TM)) \subset S(TM)$ be a distribution on M and $V|_M \in \Gamma(S(TM))$. Then $\phi ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi ltr(TM) \cap \phi(Rad(TM)) = \{0\}$.*

Lemma 3.2 ([33]) *Under the hypothesis of Lemma 3.1 and the spacelike characteristic vector field, if* $r = q$*, then any complementary distribution to* $\phi ltr(TM) \oplus \phi(Rad(TM))$ *in screen distribution* $S(TM)$ *is Riemannian.*

Using Lemmas 3.1 and 3.2, Sahin and Yildirim [33] defined slant lightlike submanifolds of indefinite Sasakian manifolds as below:

Definition 3.3 An q-lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ of constant index $2q$ is said to be a slant lightlike submanifold if it satisfies the following conditions:

(A) $\phi(Rad(TM)) \subset S(TM)$ be a distribution on M and $V|_M \in \Gamma(S(TM))$ such that $V|_M \notin (\phi ltr(TM) \oplus \phi(Rad(TM))$.

(B) For each non-zero vector field X tangent to $\bar{D} = D \perp \{V\}$ at $x \in U \subset M$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space $\bar{D}_x$ is constant, that is, it is independent of the choice of $X \in \bar{D}_x$ and $x \in U$, where $\bar{D}$ is complementary distribution to $\phi ltr(TM) \oplus \phi Rad(TM)$ in screen distribution $S(TM)$.

The constant angle $\theta(X)$ is called the slant angle of the distribution $\bar{D}$. The slant lightlike submanifold is called proper if $\bar{D} \neq \{0\}$ and $\theta \neq 0, \pi/2$.

It should be noted that the notion of slant lightlike submanifolds is not intrinsic and depends a priori on the selected screen data $(S(TM), S(TM^{\perp}))$; therefore, we consider here slant screen structure by integrating the structure $(S(TM), S(TM^{\perp}))$ on lightlike submanifolds.

It is well known that a submanifold M is invariant or anti-invariant if $\phi T_xM \subset T_xM$ or $\phi T_xM \subset T_xM^{\perp}$, respectively, for any $x \in M$. Thus, from the definition of slant submanifolds, M is invariant or anti-invariant, accordingly, if $\theta(X) = 0$ or $\theta(X) = \frac{\pi}{2}$, respectively.

From the above definition, it is clear that for a slant lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$, the tangent bundle TM is decomposed into $TM = Rad(TM) \perp (\phi Rad(TM) \oplus \phi ltr(TM)) \perp \bar{D}$, where $\bar{D} = D \perp \{V\}$. Therefore, for any $X \in \Gamma(TM)$, we can write

$$\phi X = TX + FX, \tag{12}$$

where TX is the tangential component of ϕX and FX is the transversal component of ϕX. Similarly, for any $U \in \Gamma(tr(TM))$, we can write

$$\phi U = BU + CU, \tag{13}$$

where BU is the tangential component of ϕU and CU is the transversal component of ϕU. We denote P_1, P_2, Q_1, Q_2, and $\bar{Q}_2$, the projections on the distributions $Rad(TM)$, $\phi Rad(TM)$, $\phi ltr(TM)$, D, and $\bar{D} = D \perp V$, respectively. Then, for any $X \in \Gamma(TM)$, we can write $X = P_1X + P_2X + Q_1X + \bar{Q}_2X$, where $\bar{Q}_2X = Q_2X + \eta(X)V$. On applying ϕ, we obtain

$$\phi X = \phi P_1 X + \phi P_2 X + F Q_1 X + T Q_2 X + F Q_2 X. \tag{14}$$

Then, it is easy to prove the following observation:

Lemma 3.4 *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$, then $FQ_2X \in \Gamma(S(TM^{\perp}))$, for any $X \in \Gamma(TM)$.*

Thus, from Lemma 3.4, it follows that $F(D_p)$ is a subspace of $S(TM^{\perp})$. Hence, there exists an *invariant* subspace μ_p of $T_p\bar{M}$ such that $S(T_pM^{\perp}) = F(D_p)\perp\mu_p$, then $T_p\bar{M} = S(T_pM)\perp\{Rad(T_pM) \oplus ltr(T_pM)\}\perp\{F(D_p)\perp\mu_p\}$. Now, differentiating (14) and using (2)–(4), (12), and (13), we obtain

$$\begin{aligned} D^s(X, FQ_1Y) + D^l(X, FQ_2Y) = F\nabla_X Y - h(X, TY) + Ch^s(X, Y) \\ -\nabla^s_X FQ_2Y - \nabla^l_X FQ_1Y. \end{aligned} \tag{15}$$

Now, we recall important theorems for the existence of slant lightlike submanifolds of indefinite Sasakian manifolds from [33].

Theorem 3.5 *Let M be a lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, M is a slant lightlike submanifold, if and only if*

(i) $\phi(Rad(TM)) \subset S(TM)$ *be a distribution on M and $V|_M \in \Gamma(S(TM))$.*
(ii) $\bar{D} = \{X \in \Gamma(\bar{D}) : T^2X = -\lambda(X - \eta(X)V)\}$ *is a distribution such that it is complementary to $\phi ltr(TM) \oplus \phi Rad(TM)$, where $\lambda = -cos^2\theta$.*

Theorem 3.6 *Let M be a lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, M is a slant lightlike submanifold, if and only if*

(i) $\phi(Rad(TM)) \subset S(TM)$ *be a distribution on M and $V|_M \in \Gamma(S(TM))$.*
(ii) *For any vector field X tangent to $\bar{D}$, there exists a constant $\mu \in [-1, 0]$ such that $BFX = \mu(X - \eta(X)V)$, where $\bar{D}$ is a complementary distribution to $\phi ltr(TM) \oplus \phi Rad(TM)$ in TM and $\mu = -sin^2\theta$.*

For necessary and sufficient conditions that a lightlike submanifold of an indefinite Kenmotsu and of an indefinite Cosymplectic manifold to be a slant lightlike submanifold, see [14] and [15], respectively.

From Theorem 3.5, we have the following observations directly.

Corollary 3.7 *Let M be a slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then*

$$g(T\bar{Q}_2X, T\bar{Q}_2Y) = cos^2\theta[g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)], \tag{16}$$

$$g(F\bar{Q}_2X, F\bar{Q}_2Y) = sin^2\theta[g(\bar{Q}_2X, \bar{Q}_2Y) - \eta(\bar{Q}_2X)\eta(\bar{Q}_2Y)], \tag{17}$$

for any $X, Y \in \Gamma(TM)$.

Definition 3.8 ([39]) If the second fundamental form h of a submanifold, tangent to characteristic vector field V, of a Sasakian manifold is of the form

$$h(X,Y)=\{g(X,Y)-\eta(X)\eta(Y)\}\alpha+\eta(X)h(Y,V)+\eta(Y)h(X,V), \tag{18}$$

where α is a vector field transversal to M, then M is called a totally contact umbilical and totally contact geodesic if $\alpha=0$. This definition also holds for a lightlike submanifold M. For a totally contact umbilical lightlike submanifold M, we have

$$h^l(X,Y)=\{g(X,Y)-\eta(X)\eta(Y)\}\alpha_L+\eta(X)h^l(Y,V)+\eta(Y)h^l(X,V), \tag{19}$$

$$h^s(X,Y)=\{g(X,Y)-\eta(X)\eta(Y)\}\alpha_S+\eta(X)h^s(Y,V)+\eta(Y)h^s(X,V), \tag{20}$$

where $\alpha_L \in \Gamma(ltr(TM))$ and $\alpha_S \in \Gamma(S(TM^{\perp}))$.

Using the above definition, it is easy to prove the following lemma.

Lemma 3.9 ([25]) *Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ then $g(\nabla_X X, \phi\xi)=0$, for any $X \in \Gamma(D)$ and $\xi \in \Gamma(Rad(TM))$.*

An important classification property of slant lightlike submanifold is the following.

Theorem 3.10 ([25]) *Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, at least one of the following statements is true:*

(i) M is an anti-invariant submanifold.
(ii) $D=\{0\}$.
(iii) If M is a proper slant lightlike submanifold, then $\alpha_S \in \Gamma(\mu)$.

Proof Let M be a totally contact umbilical slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$, then for any $X=Q_2X \in \Gamma(D)$ with (18), we get $h(TQ_2X, TQ_2X)=g(TQ_2X, TQ_2X)\alpha$; therefore, from (1) and (16), we get $\bar{\nabla}_{TQ_2X}TQ_2X-\nabla_{TQ_2X}TQ_2X=cos^2\theta[g(Q_2X,Q_2X)]\alpha$. Using (12) and the fact that $\bar{M}$ is a Sasakian manifold, we obtain

$$\begin{aligned}&\phi\bar{\nabla}_{TQ_2X}Q_2X-g(TQ_2X,TQ_2X)V-\bar{\nabla}_{TQ_2X}FQ_2X-\nabla_{TQ_2X}TQ_2X\\&=cos^2\theta[g(Q_2X,Q_2X)]\alpha.\end{aligned}$$

Then, using (2)–(4) and (16), we get

$$\begin{aligned}&\phi\nabla_{TQ_2X}Q_2X+\phi h^l(TQ_2X,X)+\phi h^s(TQ_2X,X)+A_{FQ_2X}TQ_2X\\&-\nabla^s_{TQ_2X}FQ_2X-D^l(TQ_2X,FQ_2X)-\nabla_{TQ_2X}TQ_2X\\&=cos^2\theta[g(Q_2X,Q_2X)](\alpha+V).\end{aligned}$$

Thus, using (12), (13), (19), and (20), we have

$$T\nabla_{TQ_2X}Q_2X + F\nabla_{TQ_2X}Q_2X + g(TQ_2X, X)\phi\alpha^l + g(TQ_2X, X)B\alpha^s$$
$$+g(TQ_2X), X)C\alpha^s + A_{FQ_2X}TQ_2X - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X)$$
$$-\nabla_{TQ_2X}TQ_2X = cos^2\theta[g(Q_2X, Q_2X)](\alpha + V),$$

equating the transversal components, we get

$$F\nabla_{TQ_2X}Q_2X + g(TQ_2X, X)C\alpha^s - \nabla^s_{TQ_2X}FQ_2X$$
$$-D^l(TQ_2X, FQ_2X) = cos^2\theta[g(Q_2X, Q_2X)]\alpha. \quad (21)$$

On the other hand, (17) holds for any $X = Y \in \Gamma(D)$ and taking the covariant derivative with respect to TQ_2X, we obtain

$$g(\nabla^s_{TQ_2X}FQ_2X, FQ_2X) = sin^2\theta g(\nabla_{TQ_2X}Q_2X, Q_2X). \quad (22)$$

Take inner product of (21) with FQ_2X, we get $g(F\nabla_{TQ_2X}Q_2X, FQ_2X) - g(\nabla^s_{TQ_2X} FQ_2X, FQ_2X) = cos^2\theta[g(Q_2X, Q_2X)]g(\alpha_S, FQ_2X)$. Then, using (17) and (22), we get $cos^2\theta[g(Q_2X, Q_2X)]g(\alpha_S, FQ_2X) = 0$; thus, it follows that either $\theta = \frac{\pi}{2}$ or $Q_2X = 0$ or $\alpha_S \in \Gamma(\mu)$. This completes the proof. □

In [32], Sahin proved that every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic and the following theorem is the lightlike version of this result for indefinite Sasakian manifolds.

Theorem 3.11 ([25]) *Every totally contact umbilical proper slant lightlike submanifold of an indefinite Sasakian manifold is totally contact geodesic.*

Proof Since M is a totally contact umbilical slant lightlike submanifold, therefore for any $X = Q_2X \in \Gamma(D)$, using (18), we have $h(TQ_2X, TQ_2X) = g(TQ_2X, TQ_2X)\alpha$, then using (16), we get

$$h(TQ_2X, TQ_2X) = cos^2\theta[g(Q_2X, Q_2X) - \eta(Q_2X)\eta(Q_2X)]\alpha$$
$$= cos^2\theta[g(Q_2X, Q_2X)]\alpha. \quad (23)$$

Using (8) and (15) for any $X \in \Gamma(D)$, we obtain

$$h(TQ_2X, TQ_2X) = F\nabla_{TQ_2X}X + Ch(TQ_2X, X) - \nabla^s_{TQ_2X}FQ_2X$$
$$-D^l(TQ_2X, FQ_2X). \quad (24)$$

Since M is a totally contact umbilical slant lightlike submanifold, therefore $Ch(TQ_2X, X) = g(TQ_2X, X)C\alpha = 0$. Hence using (23) and (24), we get $cos^2\theta [g(Q_2X, Q_2X)]\alpha = F\nabla_{TQ_2X}X - \nabla^s_{TQ_2X}FQ_2X - D^l(TQ_2X, FQ_2X)$. Further, on taking the scalar product of both sides with respect to FQ_2X, we obtain that $cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) = \bar{g}(F\nabla_{TQ_2X}X, FQ_2X) - \bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X)$, then using (17), we get

$$cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) = sin^2\theta[g(\nabla_{TQ_2X}X, Q_2X)] \\ -\bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X). \quad (25)$$

Now, for any $X = Q_2X \in \Gamma(D)$, (17) implies that $g(FQ_2X, FQ_2X) = sin^2\theta$ $[g(Q_2X, Q_2X)]$. Taking covariant derivative of this equation with respect to $\bar{\nabla}_{TQ_2X}$, we get

$$\bar{g}(\nabla^s_{TQ_2X}FQ_2X, FQ_2X) = sin^2\theta[g(\nabla_{TQ_2X}Q_2X, Q_2X)]. \quad (26)$$

Using (26) in (25), we obtain

$$cos^2\theta[g(Q_2X, Q_2X)]\bar{g}(\alpha_S, FQ_2X) = 0. \quad (27)$$

Since M is a proper slant lightlike submanifold and g is a Riemannian metric on D, therefore we have $\bar{g}(\alpha_s, FQ_2X) = 0$. Thus, using the Lemma 3.4, we obtain $\alpha_S \in \Gamma(\mu)$. Now, using the Sasakian property of $\bar{M}$, we have $\bar{\nabla}_X\phi Y = \phi\bar{\nabla}_X Y - g(X, Y)V$, for any $X, Y \in \Gamma(D)$, then using (18), we obtain $\nabla_X TQ_2Y + g(X, TQ_2Y)\alpha - A_{FQ_2Y}X + \nabla^s_X FQ_2Y + D^l(X, FQ_2Y) = T\nabla_X Y + F\nabla_X Y + g(X, Y)\phi\alpha - g(X, Y)V$. Taking the scalar product of both sides with respect to $\phi\alpha_S$ and using the fact that μ is an invariant subbundle of $T\bar{M}$, we obtain

$$\bar{g}(\nabla^s_X FQ_2X, \phi\alpha_S) = g(Q_2X, Q_2Y)g(\alpha_S, \alpha_S). \quad (28)$$

Again, using the Sasakian character of $\bar{M}$, we have $\bar{\nabla}_X\phi\alpha_S = \phi\bar{\nabla}_X\alpha_S$. This further implies that $-A_{\phi\alpha_S}X + \nabla^s_X\phi\alpha_S + D^l(X, \phi\alpha_S) = -TA_{\alpha_S}X - FA_{\alpha_S}X + B\nabla^s_X\alpha_S + C\nabla^s_X\alpha_S + \phi D^l(X, \alpha_S)$. Taking the scalar product of both sides of above equation with respect to FQ_2Y and using invariant character of μ, that is, $C\nabla^s_X\alpha_S \in \Gamma(\mu)$ with (8) and (17), we get

$$\bar{g}(\nabla^s_X\phi\alpha_S, FQ_2Y) = -g(FA_{\alpha_S}X, FQ_2Y) = -sin^2\theta[g(A_{\alpha_S}X, Q_2Y)]. \quad (29)$$

Since $\bar{\nabla}$ is a metric connection then $(\bar{\nabla}_X g)(FQ_2Y, \phi\alpha_S) = 0$. This further implies that $\bar{g}(\nabla^s_X FQ_2Y, \phi\alpha_S) = \bar{g}(\nabla^s_X\phi\alpha_S, FQ_2Y)$; therefore using (29), we obtain

$$\bar{g}(\nabla^s_X FQ_2Y, \phi\alpha_S) = -sin^2\theta[g(A_{\alpha_S}X, Q_2Y)]. \quad (30)$$

From (28) and (30), we get $g(Q_2X, Q_2Y)g(\alpha_S, \alpha_S) = -sin^2\theta g[(A_{\alpha_S}X, Q_2Y)]$; using (5), we get $g(Q_2X, Q_2Y)g(\alpha_S, \alpha_S) = -sin^2\theta[\bar{g}(h^s(Q_2X, Q_2Y), \alpha_S)] = -sin^2\theta[g(Q_2X, Q_2Y)]g(\alpha_S, \alpha_S)$,, which implies $(1 + sin^2\theta)[g(Q_2X, Q_2Y)]g(\alpha_S, \alpha_S) = 0$. Since M is a proper slant lightlike submanifold and g is a Riemannian metric on D, therefore we obtain

$$\alpha_S = 0. \quad (31)$$

Next, for any $X \in \Gamma(D)$, using the Sasakian character of $\bar{M}$, we have $\bar{\nabla}_X\phi X = \phi\bar{\nabla}_X X$. This implies that $\nabla_X TQ_2X + h(X, TQ_2X) - A_{FQ_2X}X + \nabla^s_X FQ_2X + D^l$

$(X, FQ_2X) = T\nabla_X X + F\nabla_X X + Bh(X, X) + Ch(X, X)$. Since M is a totally contact umbilical slant lightlike submanifold, therefore using $h(X, TQ_2X) = 0$ and then comparing the tangential components, we obtain $\nabla_X TQ_2X - A_{FQ_2X}X = T\nabla_X X + Bh(X, X)$. Taking scalar product of both sides with respect to $\phi\xi \in \Gamma(\phi Rad(TM))$ and then using Lemma 3.9, we get

$$g(A_{FQ_2X}X, \phi\xi) + \bar{g}(h^l(Q_2X, Q_2X), \xi) = 0. \tag{32}$$

Using (3), $\bar{g}(h^s(X, \phi\xi), FQ_2X) + \bar{g}(\phi\xi, D^l(X, FQ_2X)) = g(A_{FQ_2X}X, \phi\xi)$. Since M is a totally contact umbilical slant lightlike submanifold therefore using (20) and (31), we obtain $g(A_{FQ_2X}X, \phi\xi) = 0$. Using this fact in (32), we obtain that $\bar{g}(h^l(Q_2X, Q_2X), \xi) = 0$, then further using (19) this implies that $g(Q_2X, Q_2X)\bar{g}(\alpha_L, \xi) = 0$. Since g is a Riemannian metric on D, therefore $\bar{g}(\alpha_L, \xi) = 0$, we obtain

$$\alpha_L = 0. \tag{33}$$

Thus, from (31) and (33), the proof is complete. □

Analogously, we have the following results:

Theorem 3.12 ([24]) *Every totally contact umbilical proper slant lightlike submanifold of an indefinite Cosymplectic manifold is totally contact geodesic.*

Theorem 3.13 ([27]) *Every totally contact umbilical proper slant lightlike submanifold of an indefinite Kenmotsu manifold is totally contact geodesic.*

Denote by $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇, respectively, then using (2)–(4), we have

$$\begin{aligned}\bar{R}(X, Y)Z = {} & R(X, Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ & - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y, Z) - (\nabla_Y h^l)(X, Z) \\ & + D^l(X, h^s(Y, Z)) - D^l(Y, h^s(X, Z)) \\ & + (\nabla_X h^s)(Y, Z) - (\nabla_Y h^s)(X, Z) \\ & + D^s(X, h^l(Y, Z)) - D^s(Y, h^l(X, Z)),\end{aligned} \tag{34}$$

where

$$(\nabla_X h^s)(Y, Z) = \nabla^s_X h^s(Y, Z) - h^s(\nabla_X Y, Z) - h^s(Y, \nabla_X Z), \tag{35}$$

$$(\nabla_X h^l)(Y, Z) = \nabla^l_X h^l(Y, Z) - h^l(\nabla_X Y, Z) - h^l(Y, \nabla_X Z). \tag{36}$$

An indefinite Sasakian space form is a connected indefinite Sasakian manifold of constant holomorphic sectional curvature c and denoted by $\bar{M}(c)$. Then the curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by (see [19])

$$\bar{R}(X,Y)Z = \frac{c+3\epsilon}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \frac{c-\epsilon}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \bar{g}(X,Z)\eta(Y)V - \bar{g}(Y,Z)\eta(X)V + \bar{g}(\phi Y,Z)\phi X - \bar{g}(\phi X,Z)\phi Y - 2\bar{g}(\phi X,Y)\phi Z\}, \quad (37)$$

for X, Y, Z vector fields on $\bar{M}$.

Theorem 3.14 ([25]) *There do not exist totally contact umbilical proper slant light-like submanifolds of an indefinite Sasakian space form $\bar{M}(c)$ such that $c \neq \epsilon$.*

Proof Suppose M be a totally contact umbilical proper lightlike submanifold of $\bar{M}(c)$ such that $c \neq \epsilon$. Then, using (37), for any $X \in \Gamma(D)$, $Z \in \Gamma(\phi ltr(TM))$ and $\xi \in \Gamma(Rad(TM))$, we obtain $\bar{g}(\bar{R}(X,\phi X)Z,\xi) = -\frac{c-\epsilon}{2}g(\phi X,\phi X)g(\phi Z,\xi)$. Using (9), we get

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = -\frac{c-\epsilon}{2}g(Q_2X,Q_2X)g(\phi Z,\xi). \quad (38)$$

On the other hand, using (18) and (34), we get

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = \bar{g}((\nabla_X h^l)(\phi X,Z),\xi) - \bar{g}((\nabla_\phi h^l)(X,Z),\xi). \quad (39)$$

Using (19) and (36), we have

$$(\nabla_X h^l)(\phi X,Z) = -g(\nabla_X \phi X,Z)\alpha_L - g(TQ_2X,\nabla_X Z)\alpha_L. \quad (40)$$

Similarly,

$$(\nabla_{\phi X} h^l)(X,Z) = -g(\nabla_{\phi X}X,Z)\alpha_L - g(X,\nabla_{\phi X}Z)\alpha_L. \quad (41)$$

Using (40) and (41) in (39), we obtain

$$\bar{g}(\bar{R}(X,\phi X)Z,\xi) = -g(\nabla_X\phi X,Z)\bar{g}(\alpha_L,\xi) - g(\phi X,\nabla_X Z)\bar{g}(\alpha_L,\xi) + g(\nabla_{\phi X}X,Z)\bar{g}(\alpha_L,\xi) + g(X,\nabla_{\phi X}Z)\bar{g}(\alpha_L,\xi). \quad (42)$$

Now using (11), we have

$$g(\phi X,\nabla_X Z) = -\bar{g}(\bar{\nabla}_X\phi X,Z) = -g(\nabla_X\phi X,Z) \quad (43)$$

and

$$g(X,\nabla_{\phi X}Z) = -\bar{g}(\bar{\nabla}_{\phi X}X,Z) = -g(\nabla_{\phi X}X,Z). \quad (44)$$

Using (43) and (44) in (42), we obtain $\bar{g}(\bar{R}(X,\phi X)Z,\xi) = 0$, and using this fact in (38), we have $\frac{c-\epsilon}{2}g(Q_2X,Q_2X)g(\phi Z,\xi) = 0$. Since g is a Riemannian metric on D and $g(\phi Z,\xi) \neq 0$, therefore $c = \epsilon$. This contradiction completes the proof. □

The following two theorems provide analogous observation for indefinite Cosymplectic and Kenmotsu space forms.

Theorem 3.15 ([24]) *There does not exist a totally contact umbilical proper slant lightlike submanifold of an indefinite Cosymplectic space form $\bar{M}(c)$ such that $c \neq 0$.*

Theorem 3.16 ([27]) *There does not exist a totally contact umbilical proper slant lightlike submanifold of an indefinite Kenmotsu space form $\bar{M}(c)$ such that $c \neq -1$.*

In [9], a minimal lightlike submanifold M was defined when M is a hypersurface of a four-dimensional Minkowski space. Then in [2], a general notion of minimal lightlike submanifold of a semi-Riemannian manifold $\bar{M}$ was introduced as follows:

Definition 3.17 A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if $h^s = 0$ on $Rad(TM)$ and $trace\ h = 0$, where trace is written with respect to g restricted to $S(TM)$.

Next, a couple of theorems provides characterizations for a slant lightlike submanifold of indefinite Sasakian manifolds to be minimal.

Theorem 3.18 ([20]) *Let M be proper slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with characteristic vector field tangent to M. Then M is minimal if and only if $trace\ A_{W_k}|_{S(TM)} = 0$, $trace\ A^*_{\xi_i}|_{S(TM)} = 0$ and $\bar{g}(D^l(X, W), Y) = 0$, for $X, Y \in \Gamma(Rad(TM))$, where $\{W_k\}_{k=1}^{l}$ is a basis of $S(TM^\perp)$ and $\{\xi_i\}_{i=1}^{r}$ is a basis of $\Gamma(Rad(TM))$.*

Theorem 3.19 ([20]) *Let M be proper slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with characteristic vector field tangent to M such that $dim(D) = dim(S(TM^\perp))$. Then M is minimal if and only if $trace\ A_{Fe_j}|_{S(TM)} = 0$, $trace\ A^*_{\xi_i}|_{S(TM)} = 0$ and $\bar{g}(D^l(X, Fe_j), Y) = 0$, for $X, Y \in \Gamma(Rad(TM))$, where $\{e_j\}$ is a basis of D and $\{\xi_i\}_{i=1}^{r}$ is a basis of $\Gamma(Rad(TM))$.*

Definition 3.20 ([10]) A lightlike submanifold is called irrotational if and only if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for all $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$.

Rashmi et al. [25] derived conditions for an irrotational slant lightlike submanifold to be a minimal submanifold as below.

Theorem 3.21 *Let M be an irrotational slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is minimal if and only if $trace\ A_{W_k}|_{S(TM)} = 0$ and $trace\ A^*_{\xi_i}|_{S(TM)} = 0$, where $\{W_k\}_{k=1}^{l}$ is a basis of $S(TM^\perp)$ and $\{\xi_i\}_{i=1}^{r}$ is a basis of $Rad(TM)$.*

Gupta et al. [14, 15] also derived some necessary and sufficient conditions for a slant lightlike submanifold of an indefinite Kenmotsu and of an indefinite Cosymplectic manifold to be a minimal lightlike submanifold.

4 Warped Product Slant Lightlike Submanifolds

Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_{M_1} and g_{M_2}, respectively, and $f > 0$ a differentiable function on M_1. Assume the product manifold $M_1 \times M_2$ with its projection $\pi : M_1 \times M_2 \to M_1$ and $\psi : M_1 \times M_2 \to M_2$. The warped product $M = M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ equipped with the Riemannian metric g, where $g = g_{M_1} + f^2 g_{M_2}$. If X is tangent to $M = M_1 \times_f M_2$ at (p, q) then we have $\|X\|^2 = \|\pi_* X\|^2 + f^2(\pi(X))\|\psi_* X\|^2$. The function f is called the warping function of the warped product. For differentiable function f on M, the gradient ∇f is defined by $g(\nabla f, X) = Xf$, for all $X \in T(M)$.

Lemma 4.1 ([3]) *Let $M = M_1 \times_f M_2$ be a warped product manifold. If $X, Y \in T(M_1)$ and $U, Z \in T(M_2)$ then*

$$\nabla_X U = \nabla_U X = \frac{Xf}{f} U = X(ln\ f)U. \tag{45}$$

Corollary 4.2 *On a warped product manifold $M = M_1 \times_f M_2$ M_1 and M_2 are totally geodesic and totally umbilical in M, respectively.*

Definition 4.3 ([11]) Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold, tangent to characteristic vector field V, of an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. Then M is said to be a contact Screen Cauchy Riemann (SCR) lightlike submanifold of $\bar{M}$ if there exist real non-null distributions $D \subset S(TM)$ and $D^\perp$ such that $S(TM) = D \oplus D^\perp \perp \{V\}, \phi D^\perp \subset (S(TM^\perp)), D \cap D^\perp = \{0\}$, where $D^\perp$ is orthogonal complementary to $D \perp \{V\}$ in $S(TM)$, and the distributions D and $Rad(TM)$ are invariant with respect to ϕ.

Theorem 4.4 *A contact SCR-lightlike submanifold M, of an indefinite Sasakian manifold $\bar{M}$, is a holomorphic or complex (resp. screen real) lightlike submanifold, if and only if $D^\perp = \{0\}$ (resp. $D = \{0\}$).*

Definition 4.5 ([29]) Let M be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is said to be a transversal lightlike submanifold if $\bar{J}(Rad(TM)) = ltr(TM)$ and $\bar{J}(S(TM)) \subseteq S(TM^\perp)$.

Next are some characterization theorems for the non-existence of warped product slant lightlike submanifolds of indefinite Sasakian manifolds.

Theorem 4.6 ([26]) *Let $\bar{M}$ be an indefinite Sasakian manifold. Then there does not exist warped product submanifold $M = M_\theta \times_f M_T$ of $\bar{M}$ such that M_θ is a proper slant lightlike submanifold of $\bar{M}$ and M_T is a holomorphic Screen Cauchy-Riemann (SCR) lightlike submanifold of $\bar{M}$.*

Proof Let X, linearly independent of V, be tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Then, using (45) $g(\nabla_{\phi X} Z, X) = Z(ln\ f)g(\phi X, X) = 0$. Therefore, using (2), (8)–(11), and (12), we get $0 = \bar{g}(\bar{\nabla}_{\phi X} Z, X) = -\bar{g}(\phi Z, \bar{\nabla}_{\phi X} \phi X) = \bar{g}(\nabla_{\phi X} TZ, \phi X) -$

$\bar{g}(FZ, h^s(\phi X, \phi X))$. Further by virtue of (45), we obtain $TZ(ln\ f)g(X, X) = \bar{g}(h^s(\phi X, \phi X), FZ)$. Thus, using polarization identity, we get

$$TZ(ln\ f)g(X, Y) = \bar{g}(h^s(\phi X, \phi Y), FZ), \tag{46}$$

for any X, Y, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. On the other hand, using (4) and (45), we have

$$g(A_{FZ}\phi X, \phi Y) = -Z(ln\ f)g(\phi X, Y) + TZ(ln\ f)g(X, Y).$$

Now, using (5), we have $\bar{g}(h^s(\phi X, \phi Y), FZ) = g(A_{FZ}\phi X, \phi Y)$; therefore we obtain

$$\bar{g}(h^s(\phi X, \phi Y), FZ) = -Z(ln\ f)g(\phi X, Y) + TZ(ln\ f)g(X, Y). \tag{47}$$

Thus, (46) and (47) imply that $Z(ln\ f)g(\phi X, Y) = 0$ for any X, Y, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Since $M_T \neq \{0\}$ is a Riemannian and invariant, therefore we obtain $Zln\ f = 0$. This shows that f is constant. Hence, the proof is complete. □

Theorem 4.7 ([26]) *Let $\bar{M}$ be an indefinite Sasakian manifold. Then there does not exist warped product submanifold $M = M_T \times_f M_\theta$ in $\bar{M}$ such that M_T is a holomorphic SCR-lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$.*

Proof Let X, linearly independent of V, be tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ. Then, using (45) $g(\nabla_{TZ}X, Z) = X(ln\ f)g(TZ, Z) = 0$. This, further using with (4), (5), and (16) implies that $\phi X(ln\ f).cos^2\theta g(Z, Z) + \bar{g}(h^s(\phi X, TZ), FZ) = 0$. Replace X by ϕX, we get

$$X(ln\ f).cos^2\theta g(Z, Z) + \bar{g}(h^s(X, TZ), FZ) = 0. \tag{48}$$

After replacing Z by TZ and then using Theorem 3.5 and (16), we obtain

$$\bar{g}(h^s(X, Z), FTZ) = X(ln\ f).cos^2\theta g(Z, Z). \tag{49}$$

Next, on the other hand, using (2), (12), (16), (45), and Theorem 3.5, for any X, linearly independent of V, tangent to $D \subset S(TM)$ of a holomorphic SCR-lightlike submanifold M_T and $Y, Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ, we have $\bar{g}(h^s(TZ, X), FY) = -cos^2\theta X(ln\ f)g(Z, Y) + \bar{g}(FTZ, h^s(X, Y)) + X(ln\ f)g(TZ, TY) = \bar{g}(FTZ, h^s(X, Y))$. Put $Y = Z$, we get

$$\bar{g}(h^s(TZ, X), FZ) = \bar{g}(FTZ, h^s(X, Z)). \tag{50}$$

Thus, from (48) to (50), we have $X(ln\ f)cos^2\theta g(Z, Z) = 0$. Since D^θ is a proper slant and Z is non-null, we obtain $X(ln\ f) = 0$. This proves our assertion. □

Theorem 4.8 ([26]) *Let $\bar{M}$ be an indefinite Sasakian manifold. Then there does not exist warped product submanifold $M = M_\perp \times_f M_\theta$ of $\bar{M}$ such that $M_\perp$ is a transversal lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$.*

Proof Let $Z \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ and X independent of V and tangent to $S(TM)$ of a transversal lightlike submanifold $M_\perp$, then using (4), (8)–(11), (12), (16), and (45), we have $g(A_{\phi X}TZ, Z) = X(ln\ f)cos^2\theta g(Z, Z) + \bar{g}(h^s(TZ, X), FZ)$, on using (5) in the left hand side, we obtain

$$\bar{g}(h^s(TZ, Z), \phi X) = X(ln\ f)cos^2\theta g(Z, Z) + \bar{g}(h^s(TZ, X), FZ). \tag{51}$$

Replace Z by TZ in (51) and then using Theorem 3.5 and (16), we get

$$\bar{g}(h^s(Z, TZ), \phi X) = -X(ln\ f)cos^2\theta g(Z, Z) + \bar{g}(h^s(Z, X), FTZ). \tag{52}$$

Also, using (4), (8)–(11), (12), (17), and (45), we obtain that $g(A_{FZ}X, TZ) = -cos^2\theta X(lnf)g(Z, Z) + \bar{g}(h^s(X, Z), FTZ) + X(lnf)cos^2\theta g(Z, Z)$. This implies $g(A_{FZ}X, TZ) = \bar{g}(h^s(X, Z), FTZ)$. Hence using (5), we obtain

$$\bar{g}(h^s(TZ, X), FZ) = \bar{g}(h^s(X, Z), FTZ). \tag{53}$$

Thus, using (51)–(53), we get $2X(ln\ f)cos^2\theta g(Z, Z) = 0$. Since M_θ is proper slant lightlike submanifold and D^θ is Riemannian, therefore we obtain $X(ln\ f) = 0$. Hence, f is constant, which proves our assertion. □

Thus using Theorems 4.6, 4.7, and 4.8, now onwards, we call $M = M_\theta \times_f M_\perp$ as a warped product slant lightlike submanifold, where M_θ is a proper slant lightlike submanifold and $M_\perp$ is a transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$.

Theorem 4.9 ([26]) *Let $M = M_\theta \times_f M_\perp$ be a warped product slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ such that $M_\perp$ is a transversal lightlike submanifold and M_θ is a proper slant lightlike submanifold of $\bar{M}$. Then $g(h^s(X, Y), JZ) = -TX(ln\ f)g(Y, Z)$, for any $X \in \Gamma(D^\theta)$ of a slant lightlike submanifold M_θ and Y, Z, independent of V and tangent to $S(TM)$ of transversal lightlike submanifold $M_\perp$.*

5 Hemi-slant Lightlike Submanifolds

Lemma 5.1 ([28]) *Let M be an r-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index q such that the characteristic vector field V is tangent to M. Assume that $\phi(Rad(TM)) \subset S(TM)$ be a distribution on M such that*

$\phi(Rad(TM)) = ltr(TM)$. If $r = q$ then the screen distribution $S(TM)$ is Riemannian.

Definition 5.2 Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index q with characteristic vector field V tangent to M. Then, M is said to be a hemi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $\phi(Rad(TM)) \subset S(TM)$ be a distribution on M such that $\phi(Rad(TM)) = ltr(TM)$.
(ii) For all $x \in \mathcal{U} \subset M$ and for each non-zero vector field X tangent to $S(TM) = D^{\theta} \perp V$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space S(TM) is constant, where D^{θ} is complementary distribution to V in screen distribution $S(TM)$.

A hemi-slant lightlike submanifold is said to be proper if $D^{\theta} \neq 0$ and $\theta \neq 0, \pi/2$. Hence, using the definition of hemi-slant lightlike submanifolds, the tangent bundle TM of M is decomposed as $TM = S(TM) \perp Rad(TM) = D^{\theta} \perp \{V\} \perp Rad(TM)$.

Example 1 ([28]) Let M be a lightlike submanifold of a semi-Euclidean space $(\mathbb{R}^9_2, \bar{g})$ and defined by $x_1 = s, x_2 = t, x_3 = u\, sinv, x_4 = sinu, y_1 = t, y_2 = s, y_3 = u\, cosv, y_4 = cosu$, where $u, v \in (0, \pi/2)$ and $\mathbb{R}^9_2$ is of signature $(-, +, +, +, -, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Then, the local frame of fields $\{\xi_1, \xi_2, Z_1, Z_2, V\}$ of TM is given by $\xi_1 = \partial x_1 + \partial y_2, \xi_2 = \partial x_2 + \partial y_1, Z_1 = sinv\partial x_3 + cosu\partial x_4 + cosv\partial y_3 - sinu\partial y_4, Z_2 = u\ cosv\partial x_3 - u\ sinv\partial y_3,\ V = \partial z$. Hence M is a 2-lightlike submanifold with $Rad(TM) = span\{\xi_1, \xi_2\}$ and $S(TM) = span\{Z_1, Z_2\} \perp V$, which is Riemannian. It can be easily seen that $S(TM)$ is a slant distribution with slant angle $\theta = \pi/4$. Further, the screen transversal bundle $S(TM^{\perp})$ is spanned by $W_1 = sinu\partial x_4 + cosu\partial y_4$, $W_2 = sinv\partial x_3 - cosu\partial x_4 + cosv\partial y_3 + sinu\partial y_4$. The transversal lightlike bundle $ltr(TM)$ is spanned by $N_1 = -\frac{1}{2}(-\partial x_1 - \partial y_2), N_2 = \frac{1}{2}(\partial x_2 - \partial y_1)$. Clearly $\phi\xi_1 = 2N_2, \phi\xi_2 = -2N_1$. Hence, M is a hemi-slant lightlike submanifold of $\mathbb{R}^9_2$.

Denote the projection morphisms from TM on D^{θ} and $Rad(TM)$ by P and Q, respectively, then any X tangent to M can be written as $X = PX + \eta(X)V + QX$. On applying ϕ to both sides and then using the definition of hemi-slant lightlike submanifolds with $\phi V = 0$, we can write

$$\phi X = TPX + FPX + FQX, \tag{54}$$

where $TPX \in \Gamma(D^{\theta})$, $FPX \in \Gamma(tr(TM))$, and $FQX \in \Gamma(ltr(TM))$. Similarly, for any $U \in \Gamma(tr(TM))$, we can write $\phi U = BU + CU$, where BU and CU are tangential and transversal components of ϕU, respectively.

Theorem 5.3 ([28]) *Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index q. Then M is a hemi-slant lightlike submanifold if and only if $\phi(ltr(TM))$ is a distribution on M and for any vector field X tangent to M, there exists a constant $\lambda \in [-1, 0]$ such that $(TP)^2 X = \lambda PX$, where $\lambda = -\cos^2\theta$.*

Analogous to Theorem 3.11, the following theorem is for hemi-slant lightlike submanifolds of indefinite Sasakian manifolds from [28].

Theorem 5.4 *Every totally contact umbilical proper hemi-slant lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ is totally contact geodesic.*

Contrary to the classical theory of submanifolds, the induced connection ∇ on a lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is not a metric connection. So as a consequence of the above theorem, we have the following important result.

Corollary 5.5 *Let M be a totally contact umbilical proper hemi-slant lightlike submanifold of $\bar{M}$. Then, the induced connection ∇ is a metric connection on M.*

Theorem 5.6 ([28]) *There do not exist totally contact umbilical proper hemi-slant lightlike submanifolds of an indefinite contact space form $\bar{M}(c)$ such that $c \neq 1$.*

Proof Let M be a totally contact umbilical hemi-slant lightlike submanifold of $\bar{M}(c)$ such that $c \neq 1$. Then using (37), for $X \in \Gamma(D^\theta)$ and $\xi, \xi' \in \Gamma(Rad(TM))$, we get

$$\bar{g}(\bar{R}(X, \phi X)\xi', \xi) = -\frac{c-1}{2} g(X, X) g(\phi\xi', \xi). \tag{55}$$

On the other hand, using (34), we get

$$\bar{g}(\bar{R}(X, \phi X)\xi', \xi) = \bar{g}((\nabla_X h^l)(\phi X, \xi'), \xi) - \bar{g}((\nabla_{\phi X} h^l)(X, \xi'), \xi). \tag{56}$$

On using (19), we get $(\nabla_X h^l)(\phi X, \xi') = -g(\nabla_X \phi X, \xi')H^l - g(\phi X, \nabla_X \xi')H^l = \bar{g}(h^l(X, TX), \xi')H^l = g(X, \phi X)\bar{g}(H^l, \xi') = 0$ and similarly $(\nabla_{\phi X} h^l)(X, \xi') = 0$. Thus, from (55) and (56), we obtain $\frac{c-1}{2} g(X, X) g(\phi\xi', \xi) = 0$. Since g is a Riemannian metric on D^θ, therefore $g(\phi\xi', \xi) \neq 0$, and hence $c = 1$. This contradiction completes the proof. □

Haider et al. [17] presented the following result for hemi-slant lightlike submanifolds of indefinite Kenmotsu manifolds.

Theorem 5.7 *There does not exist any curvature-invariant proper hemi-slant lightlike submanifold of an indefinite Kenmotsu space form $\bar{M}(c)$ with $c \neq -1$.*

Theorem 5.8 ([28]) *Let M be a totally contact umbilical proper hemi-slant lightlike submanifold of $\bar{M}$. Then M is minimal.*

Proof From Theorem 5.4, we know $H^l = 0 = H^s$ and $\eta(\xi) = 0$, for any $\xi \in \Gamma(Rad(TM))$ then using (20), we have $h^s(\xi, \xi) = 0$, that is, $h^s = 0$ on $Rad(TM)$. From (11), we have $\bar{\nabla}_V V = 0$, implies that $h(V, V) = 0$. Let $\{e_1, \ldots\ldots, e_k\}$ be an orthonormal basis of D^θ, then using the fact that $\eta(e_i) = 0$, $i \in \{1, 2, \ldots, k\}$ with (19) and (20), we have $h(e_i, e_i) = 0$; hence $trace\ h|_{S(TM)} = 0$, and this completes the proof. □

Theorem 5.9 ([28]) *Let M be an irrotational hemi-slant lightlike submanifold of $\bar{M}$. Then M is minimal, if and only if, $trace\ A_{W_q}|_{S(TM)} = 0$, $trace\ A^*_{\xi_j}|_{S(TM)} = 0$, where $\{W_q\}_{q=1}^{l}$ is a basis of $S(TM^{\perp})$ and $\{\xi_j\}_{j=1}^{r}$ is a basis of $Rad(TM)$.*

Proof Let M be an irrotational lightlike submanifold, then $h^s(X, \xi) = 0$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, which implies that h^s vanishes on $Rad(TM)$ and $\bar{\nabla}_V V = 0$, and that $h(V, V) = 0$. Hence M is minimal if and only if $trace\ h = 0$ on D^{θ}, that is, M is minimal if and only if $\sum_{i=1}^{k} h(e_i, e_i) = 0$, where $\{e_i\}_{i=1}^{k}$ be an orthonormal basis of D^{θ}. Using (5) and (7), we obtain $\sum_{i=1}^{k} h(e_i, e_i) = \sum_{i=1}^{k}\{\frac{1}{r}\sum_{j=1}^{r} g(A^*_{\xi_j} e_i, e_i)N_j + \frac{1}{l}\sum_{q=1}^{l} g(A_{W_q} e_i, e_i)W_q\}$, and the assertion follows. □

The following assertions can be proved directly.

Theorem 5.10 ([28]) *Let M be a proper hemi-slant lightlike submanifold of $\bar{M}$. Then M is minimal if and only if $trace\ A_{W_q}|_{S(TM)} = 0$, $trace\ A^*_{\xi_j}|_{S(TM)} = 0$, and $\bar{g}(D^l(X, W), Y) = 0$, for any $X, Y \in \Gamma(Rad(TM))$, where $\{W_q\}_{q=1}^{l}$ is a basis of $S(TM^{\perp})$ and $\{\xi_j\}_{j=1}^{r}$ is a basis of $Rad(TM)$.*

Lemma 5.11 ([28]) *Let M be a proper hemi-slant lightlike submanifold of $\bar{M}$ such that $dim(D^{\theta}) = dim(S(TM^{\perp}))$. If $\{e_i\}_{i=1}^{k}$ is a local orthonormal basis of $\Gamma(D^{\theta})$ then $\{csc\theta F e_i\}_{i=1}^{k}$ is a orthonormal basis of $S(TM^{\perp})$.*

Theorem 5.12 ([28]) *Let M be a proper hemi-slant lightlike submanifold of $\bar{M}$ such that $dim(D^{\theta}) = dim(S(TM^{\perp}))$. Then M is minimal if and only if $trace A_{csc\theta F e_i}|_{S(TM)} = 0$, $trace A^*_{\xi_j}|_{S(TM)} = 0$, and $\bar{g}(D^l(X, Fe_i), Y) = 0$, for any $X, Y \in \Gamma(Rad(TM))$, where $\{e_i\}_{i=1}^{k}$ is a basis of D^{θ}.*

6 Screen Slant Lightlike Submanifolds

Definition 6.1 Let M be a $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ of index $2q$ with structure vector field tangent to M such that $2q < dim(M)$. Then M is a screen slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $Rad(TM)$ is invariant with respect to ϕ, i.e., $Rad(TM) = \phi Rad(TM)$
(ii) For all $x \in U \subset M$ and for each non-zero vector field X tangent to $S(TM) = D\perp\{V\}$, if X and V are linearly independent, then the angle $\theta(X)$ between ϕX and the vector space $S(TM)$ is constant, where D is complementary distribution to V in screen distribution $S(TM)$.

Example 2 ([12]) Let $\bar{M} = (\mathbb{R}^9_2, \bar{g})$ be a semi-Euclidean space of signature $(-, -, +, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$. Consider a submanifold M of $\mathbb{R}^9_2$, defined by

$$X(u, v, \theta_1, \theta_2, t) = (u, v, \sin\theta_1, \cos\theta_1, -\theta_1 \sin\theta_2, -\theta_1 \cos\theta_2, u, v, t)$$

Then a local frame of TM is given by

$$\begin{cases} Z_1 = e^{-z}(\partial x_1 + \partial y_3), \\ Z_2 = e^{-z}(\partial x_2 + \partial y_4), \\ Z_3 = e^{-z}(\cos\theta_1 \partial x_3 - \sin\theta_1 \partial x_4 - \sin\theta_2 \partial y_1 - \cos\theta_2 \partial y_2), \\ Z_4 = e^{-z}(-\theta_1 \cos\theta_2 \partial y_1 + \theta_1 \sin\theta_2 \partial y_2), \\ Z_5 = V = \partial z \end{cases}$$

Hence, $Rad(TM) = span\{Z_1, Z_2\}$, which is invariant with respect to ϕ. Next, $S(TM) = D\perp\{V\} = \{Z_3, Z_4\}\perp\{V\}$ is slant distribution with slant angle $\frac{\pi}{4}$. By direct calculations, we get

$$S(TM^{\perp}) = \text{span} \begin{cases} W_1 = e^{-z}(\cos\theta_1 \partial x_3 - \sin\theta_1 \partial x_4 + \sin\theta_2 \partial y_1 + \cos\theta_2 \partial y_2), \\ W_2 = e^{-z}(\sin\theta_1 \partial x_3 + \cos\theta_1 \partial x_4) \end{cases}$$

and $ltr(TM) = span\{N_1 = \frac{e^{-z}}{2}(-\partial x_1 + \partial y_3), N_2 = \frac{e^{-z}}{2}(-\partial x_2 + \partial y_4)\}$. It is easy to see that conditions (i) and (ii) of Definition 6.1 hold. Hence, M is a proper screen slant lightlike submanifold of $\mathbb{R}^9_2$.

Now, we know that for any $X \in \Gamma(S(TM))$, we can write

$$\phi X = TX + \omega X, \tag{57}$$

where $TX \in \Gamma(TM)$ and $\omega X \in \Gamma(tr(TM))$ are the tangential and transversal components of ϕX, respectively. Moreover, for a screen slant lightlike submanifold, we denote by Q, P, and $\bar{P}$ the projections on the distributions $Rad(TM)$, D, and $S(TM) = D\perp\{V\}$, respectively. Then for any $X \in \Gamma(TM)$, we can write $X = QX + \bar{P}X$, where $\bar{P}X = PX + \eta(X)V$. Using (57) in the above equation, we obtain $\phi X = TQX + \phi PX = TQX + TPX + \omega PX$, for any $X \in \Gamma(TM)$. Thus, we conclude that $\phi QX = TQX$, $\omega QX = 0$, and $TPX \in \Gamma(S(TM))$. On the other hand, the screen transversal vector bundle $S(TM^{\perp})$ has the following decomposition $S(TM^{\perp}) = \omega P(S(TM))\perp\delta$, then for $W \in \Gamma(S(TM^{\perp}))$, we have $\phi W = BW + CW$, where $BW \in \Gamma(S(TM))$ and $CW \in \Gamma(\delta)$. Thus, if $X \in \Gamma(S(TM))$ then $\omega X \in \Gamma(S(TM^{\perp}))$ and if $X \in \Gamma(Rad(TM))$, then $\omega X = 0$.

Theorem 6.2 ([12]) *Let M be a $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ of index $2q$ with structure vector field tangent to M such that $2q < dim(M)$. Then M is screen slant lightlike submanifold if and only if the following conditions are satisfied:*

(a) $\phi ltr(TM) = ltr(TM)$, that is, $ltr(TM)$ is invariant
(b) There exists a constant $\lambda \in [-1, 0]$ such that

$$T^2\bar{P}X = \lambda(\bar{P}X - \eta(\bar{P}X)V) \tag{58}$$

for all $X \in \Gamma(S(TM))$ linearly independent of structure vector field V. Moreover, in such a case, $\lambda = -\cos^2\theta\,|_{S(TM)}$, where θ is the slant angle of M.

Proof Let M be $2q$-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ of index $2q$. Then $S(TM)$ is a Riemannian vector bundle. If M is a screen slant lightlike submanifold of $\bar{M}$, then $Rad(TM)$ is invariant distribution with respect to ϕ. We have that $\omega PX \in \Gamma(S(TM^{\perp}))$, for $X \in S(TM)$. Thus, using (57), we get $\bar{g}(\phi N, X) = -\bar{g}(N, TPX) - \bar{g}(N, \omega PX) = 0$, for $X \in S(TM)$ and $N \in ltr(TM)$. Hence, we conclude that ϕN does not belong to $S(TM)$. On the other hand, we find $\bar{g}(\phi N, W) = -\bar{g}(N, BW) - \bar{g}(N, CW) = 0$, for $W \in \Gamma(S(TM^{\perp}))$ and $N \in \Gamma(ltr(TM))$. Thus, ϕN does not belong to $S(TM^{\perp})$. Next, suppose that $\phi N \in \Gamma(Rad(TM))$. Then $\phi\phi N = -N + \eta(N)V = -N \in \Gamma(ltr(TM))$, as $Rad(TM)$ is invariant, and we get a contradiction. Thus (a) is proved. For $X \in \Gamma(S(TM))$, $PX \in S(TM) - \{V\}$, we have

$$\cos\theta(PX) = \frac{\bar{g}(\phi PX, TPX)}{|\phi PX||TPX|} = -\frac{\bar{g}(PX, \phi TPX)}{|\phi PX||TPX|} = -\frac{\bar{g}(PX, T^2PX)}{|PX||TPX|}. \quad (59)$$

On the other hand, we get $\cos\theta(PX) = \frac{|TPX|}{|\phi PX|}$. Thus, using this fact in (59), we find $\cos^2\theta(PX) = -\frac{\bar{g}(PX, T^2PX)}{|PX|^2}$. Since $\theta(PX)$ is constant on $S(TM)$, we conclude that

$$T^2PX = \lambda PX = \lambda(\bar{P}X - \eta(\bar{P}X)V), \lambda \in (-1, 0). \quad (60)$$

Moreover, in this case, $\lambda = -\cos^2\theta$. It is clear that Eq. (60) is valid for $\theta = 0$ and $\theta = \frac{\pi}{2}$. Hence, for $\bar{P}X \in S(TM)$, we find $T^2(\bar{P}X) = \lambda(\bar{P}X - \eta(\bar{P}X)V)$, $\lambda \in [-1, 0]$. The converse can be obtained in a similar way. □

Analogous to the last Theorem 6.2, existence theorems for a screen slant lightlike of indefinite Sasakian manifolds and of indefinite Cosymplectic manifolds are derived by Haider et al. [16] and by Gupta [13], respectively.

Corollary 6.3 *Let M be a screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ with structure vector tangent to M. Then we have*

$$g(T\bar{P}X, T\bar{P}Y) = \cos^2\theta|_{S(TM)}[g(\bar{P}X, \bar{P}Y) - \eta(\bar{P}X)\eta(\bar{P}Y)] \quad (61)$$

$$g(F\bar{P}X, F\bar{P}Y) = \sin^2\theta|_{S(TM)}[g(\bar{P}X, \bar{P}Y) - \eta(\bar{P}X)\eta(\bar{P}Y)] \quad (62)$$

for $X, Y \in \Gamma(TM)$.

From the above corollary, the following observation follows immediately.

Lemma 6.4 ([12]) *Let M be a proper screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ such that $dim(D) = dim(S(TM^{\perp}))$. If $\{e_1, \ldots, e_m\}$ is a local orthonormal basis of $\Gamma(D)$, then $\{\csc\theta Fe_1, \ldots, \csc\theta Fe_m\}$ is an orthonormal basis of $S(TM^{\perp})$.*

Then it is easy to derive the following important results.

Theorem 6.5 ([12]) *Let M be a proper screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ with structure vector field tangent to M. Then M is minimal if and only if*

$$traceA_{W_j|S(TM)} = 0, \quad traceA^*_{\xi_k|S(TM)} = 0, \quad \bar{g}(D^l(X, W), Y) = 0,$$

for $X, Y \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where $\{\xi_k\}_{k=1}^r$ is a basis of $Rad(TM)$ and $\{W_j\}_{j=1}^m$ is a basis of $S(TM^{\perp})$.

Theorem 6.6 ([12]) *Let M be a proper screen slant lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ with structure vector field tangent to M such that $dim(D) = dim(S(TM^{\perp}))$. Then M is minimal if and only if*

$$traceA_{Fe_j|S(TM)} = 0, \quad traceA^*_{\xi_k|S(TM)} = 0, \quad \bar{g}(D^l(X, Fe_j), Y) = 0,$$

for $X, Y \in \Gamma(Rad(TM))$, where $\{\xi_k\}_{k=1}^r$ is a basis of $Rad(TM)$ and $\{e_j\}_{j=1}^m$ is a basis of $\Gamma(D)$.

7 Screen Semi-slant Lightlike Submanifolds

Definition 7.1 ([37]) Let M be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$ such that $2q < dim(M)$ with structure vector field tangent to M. Then M is called a screen semi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $Rad(TM)$ is invariant with respect to ϕ, that is, $\phi(Rad(TM)) = Rad(TM)$,
(ii) there exist non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$,
(iii) the distribution D_1 is an invariant distribution, that is, $\phi D_1 = D_1$,
(iv) the distribution D_2 is slant with angle $\theta(\neq 0)$, that is, for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ between ϕX and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in \Gamma(D_2)_x$.

This constant angle θ is called the slant angle of the distribution D_2. A screen semi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta \neq \frac{\pi}{2}$. From the above definition, we have $TM = Rad(TM) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$.

Example 3 ([37]) Let $(\mathbb{R}_2^{13}, \bar{g})$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, +, -, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}_2^{13}$ given by $x^1 = u_1$, $y^1 = u_2$, $x^2 = u_1 \cos\alpha - u_2 \sin\alpha$, $y^2 = u_1 \sin\alpha + u_2 \cos\alpha$, $x^3 = -y^4 = u_3$, $x^4 = y^3 = u_4$, $x^5 = u_5 \sin u_6$, $y^5 = u_5 \cos u_6$,

$x^6 = \sin u_5$, $y^6 = \cos u_5$, $z = u_7$. Then, the local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$\begin{aligned}
Z_1 &= 2(\partial x_1 + \cos\alpha\partial x_2 + \sin\alpha\partial y_2 + y^1\partial z + \cos\alpha y^2\partial z),\\
Z_2 &= 2(\partial y_1 - \sin\alpha\partial x_2 + \cos\alpha\partial y_2 - \sin\alpha y^2\partial z),\\
Z_3 &= 2(\partial x_3 - \partial y_4 + y^3\partial z),\ Z_4 = 2(\partial x_4 + \partial y_3 + y^4\partial z),\\
Z_5 &= 2(\sin u_6\partial x_5 + \cos u_6\partial y_5 + \cos u_5\partial x_6 - \sin u_5\partial y_6\\
&\quad + \sin u_6 y^5\partial z + \cos u_5 y^6\partial z),\\
Z_6 &= 2(u_5\cos u_6\partial x_5 - u_5\sin u_6\partial y_5 + u_5\cos u_6 y^5\partial z),\\
Z_7 &= V = 2\partial z.
\end{aligned}$$

Hence, $Rad(TM) = span\{Z_1, Z_2\}$ and $S(TM) = span\{Z_3, Z_4, Z_5, Z_6, V\}$. Now, $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \cos\alpha\partial x_2 + \sin\alpha\partial y_2 - y^1\partial z + \cos\alpha y^2\partial z$, $N_2 = -\partial y_1 - \sin\alpha\partial x_2 + \cos\alpha\partial y_2 - \sin\alpha y^2\partial z$, and $S(TM^\perp)$ is spanned by

$$\begin{aligned}
W_1 &= 2(\partial x_3 + \partial y_4 + y^3\partial z),\ W_2 = 2(\partial x_4 - \partial y_3 + y^4\partial z),\\
W_3 &= 2(\sin u_6\partial x_5 + \cos u_6\partial y_5 - \cos u_5\partial x_6 + \sin u_5\partial y_6\\
&\quad + \sin u_6 y^5\partial z + \cos u_5 y^6\partial z),\\
W_4 &= 2(u_5\sin u_5\partial x_6 + u_5\cos u_5\partial y_6 + u_5\sin u_5 y^6\partial z).
\end{aligned}$$

It follows that $\phi Z_1 = -Z_2$ and $\phi Z_2 = Z_1$, which implies that $Rad(TM)$ is invariant, i.e., $\phi Rad(TM) = Rad(TM)$. On the other hand, we can see that $D_1 = span\{Z_3, Z_4\}$ such that $\phi Z_3 = -Z_4$ and $\phi Z_4 = Z_3$, which implies that D_1 is invariant with respect to ϕ and $D_2 = span\{Z_5, Z_6\}$ is a slant distribution with slant angle $\pi/4$. Hence, M is a screen semi-slant 2-lightlike submanifold of $\mathbb{R}^{13}_2$.

Now, for any vector field X tangent to M, we put $\phi X = PX + FX$, where PX and FX are tangential and transversal parts of ϕX, respectively. We denote the projections on $Rad(TM)$, D_1 and D_2 in TM by P_1, P_2, and P_3, respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ by Q_1 and Q_2, respectively. Then, for $X \in \Gamma(TM)$, we can write $X = P_1X + P_2X + P_3X + \eta(X)V$. On applying ϕ, it follows that $\phi X = \phi P_1X + \phi P_2X + \phi P_3X$, implies

$$\phi X = \phi P_1X + \phi P_2X + fP_3X + FP_3X, \tag{63}$$

where fP_3X (resp. FP_3X) denotes the tangential (resp. transversal) component of ϕP_3X. Thus, we get $\phi P_1X \in \Gamma(Rad(TM))$, $\phi P_2X \in \Gamma(D_1)$, $fP_3X \in \Gamma(D_2)$ and $FP_3X \in \Gamma(S(TM^\perp))$. Also, for any $W \in \Gamma(tr(TM))$, we can write $W = Q_1W + Q_2W$ and on applying ϕ, it follows that $\phi W = \phi Q_1W + \phi Q_2W$, which implies

$$\phi W = \phi Q_1W + BQ_2W + CQ_2W, \tag{64}$$

where BQ_2W (resp. CQ_2W) denotes the tangential (resp. transversal) component of ϕQ_2W. Thus, we get $\phi Q_1W \in \Gamma(ltr(TM))$, $BQ_2W \in \Gamma(D_2)$, and $CQ_2W \in \Gamma(S(TM^{\perp}))$.

Now, using (2)–(4), (11), (63), (64) and on identifying the components on $Rad(TM)$, D_1, D_2, and $S(TM^{\perp})$, we derive

$$P_1(\nabla_X\phi P_1Y) + P_1(\nabla_X\phi P_2Y) + P_1(\nabla_X fP_3Y) = P_1(A_{FP_3Y}X) + \phi P_1\nabla_XY - \eta(Y)P_1X, \tag{65}$$

$$P_2(\nabla_X\phi P_1Y) + P_2(\nabla_X\phi P_2Y) + P_2(\nabla_X fP_3Y) = P_2(A_{FP_3Y}X) + \phi P_2\nabla_XY - \eta(Y)P_2X, \tag{66}$$

$$P_3(\nabla_X\phi P_1Y) + P_3(\nabla_X\phi P_2Y) + P_3(\nabla_X fP_3Y) = P_3(A_{FP_3Y}X) + fP_3\nabla_XY + Bh^s(X,Y) - \eta(Y)P_3X, \tag{67}$$

$$h^s(X,\phi P_1Y) + h^s(X,\phi P_2Y) = Ch^s(X,Y) - h^s(X, fP_3Y) - \nabla^s_X FP_3Y + FP_3\nabla_XY, \tag{68}$$

Theorem 7.2 ([37]) *Let M be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then, M is a screen semi-slant lightlike submanifold of $\bar{M}$ if and only if*

(i) $ltr(TM)$ and D_1 are invariant with respect to ϕ,
(ii) there exists a constant $\lambda \in [0, 1)$ such that $P^2X = -\lambda X$.

Moreover, there also exists a constant $\mu \in (0, 1]$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM) = D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$ and $\lambda = \cos^2\theta$, θ is slant angle of D_2.

Proof Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, distributions D_1 and $Rad(TM)$ are invariant with respect to ϕ. Now, for any $N \in \Gamma(ltr(TM))$ and $X \in \Gamma(S(TM) - \{V\})$, using (10) and (63), we obtain $\bar{g}(\phi N, X) = -\bar{g}(N, \phi X) = -\bar{g}(N, \phi P_2X + fP_3X + FP_3X) = 0$. Thus, ϕN does not belong to $\Gamma(S(TM) - \{V\})$. For any $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, from (10) and (64), we have $\bar{g}(\phi N, W) = -\bar{g}(N, \phi W) = -\bar{g}(N, BW + CW) = 0$. Hence, we conclude that ϕN does not belong to $\Gamma(S(TM^{\perp}))$. Now, suppose that $\phi N \in \Gamma(Rad(TM))$. Then $\phi(\phi N) = \phi^2N = -N + \eta(N)V = -N \in \Gamma(ltrTM)$, which contradicts that $Rad(TM)$ is invariant. Thus, $ltr(TM)$ is invariant with respect to ϕ. Now, for any $X \in \Gamma(D_2)$, we have $|PX| = |\phi X|\cos\theta$, which implies

$$\cos\theta = \frac{|PX|}{|\phi X|}. \tag{69}$$

In view of (69), we get $\cos^2\theta = \frac{|PX|^2}{|\phi X|^2} = \frac{g(PX,PX)}{g(\phi X,\phi X)} = \frac{g(X,P^2X)}{g(X,\phi^2 X)}$, this gives $g(X, P^2X)$ $= \cos^2\theta\, g(X, \phi^2 X)$. Since M is a screen semi-slant lightlike submanifold, $\cos^2\theta = \lambda(constant) \in [0, 1)$, therefore, we get $g(X, P^2X) = \lambda g(X, \phi^2 X) = g(X, \lambda\phi^2 X)$, this implies $g(X, (P^2 - \lambda\phi^2)X) = 0$. Since $(P^2 - \lambda\phi^2)X \in \Gamma(D_2)$ and the induced metric $g = g|_{D_2\times D_2}$ is non-degenerate (positive definite); hence $(P^2 - \lambda\phi^2)X = 0$, which implies

$$P^2X = \lambda\phi^2 X = -\lambda X. \tag{70}$$

For any vector field $X \in \Gamma(D_2)$, we have $\phi X = PX + FX$. Applying ϕ and on taking the tangential component, we get

$$-X = P^2X + BFX. \tag{71}$$

Hence, from (70) and (71), we obtain

$$BFX = -\mu X, \tag{72}$$

where $1 - \lambda = \mu(constant) \in (0, 1]$. This proves (ii).

Conversely, suppose that conditions (i) and (ii) are satisfied. We can show that $Rad(TM)$ is invariant in similar way that $ltr(TM)$ is invariant. From (71), for any $X \in \Gamma(D_2)$, we have $-X = P^2X - \mu X$. This further implies $P^2X = -\lambda X$, where $1 - \mu = \lambda(constant) \in [0, 1)$. Now

$$\cos\theta = \frac{g(\phi X, PX)}{|\phi X||PX|} = -\frac{g(X, P^2X)}{|\phi X||PX|} = -\lambda\frac{g(X, \phi^2 X)}{|\phi X||PX|} = \lambda\frac{g(\phi X, \phi X)}{|\phi X||PX|},$$

which further implies

$$\cos\theta = \lambda\frac{|\phi X|}{|PX|}. \tag{73}$$

Therefore, (69) and (73) imply $\cos^2\theta = \lambda(constant)$. Hence, M is a screen semi-slant lightlike submanifold. □

Theorem 7.3 ([37]) *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then, $Rad(TM)$ is integrable if and only if $h^s(Y, \phi P_1 X) = h^s(X, \phi P_1 Y)$, $P_2(\nabla_X \phi P_1 Y) = P_2(\nabla_Y \phi P_1 X)$ and $P_3(\nabla_X \phi P_1 Y) = P_3(\nabla_Y \phi P_1 X)$, for all $X, Y \in \Gamma(Rad(TM))$.*

Proof Let $X, Y \in \Gamma(Rad(TM))$. Then from (68), we have $h^s(X, \phi P_1 Y) = Ch^s(X, Y) + FP_3\nabla_X Y$. This gives $h^s(X, \phi P_1 Y) - h^s(Y, \phi P_1 X) = FP_3[X, Y]$. From (66), we get $P_2(\nabla_X \phi P_1 Y) = \phi P_2 \nabla_X Y$. This implies $P_2(\nabla_X \phi P_1 Y) - P_2(\nabla_Y \phi P_1 X) = \phi P_2[X, Y]$. Also from (67), we derive $P_3(\nabla_X \phi P_1 Y) = f P_3 \nabla_X Y + Bh^s(X, Y)$, which gives $P_3(\nabla_X \phi P_1 Y) - P_3(\nabla_Y \phi P_1 X) = f P_3[X, Y]$. This completes the proof. □

Theorem 7.4 ([37]) *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then $D_1 \oplus \{V\}$ is integrable if and only if $h^s(Y, \phi P_2 X) = h^s(X, \phi P_2 Y)$, $P_1(\nabla_X \phi P_2 Y) = P_1(\nabla_Y \phi P_2 X)$ and $P_3(\nabla_X \phi P_2 Y) = P_3(\nabla_Y \phi P_2 X)$, for all $X, Y \in \Gamma(D_1 \oplus \{V\})$.*

Proof Let $X, Y \in \Gamma(D_1 \oplus \{V\})$. Then from (68), we have $h^s(X, \phi P_2 Y) = Ch^s(X, Y) + FP_3\nabla_X Y$, which gives $h^s(X, \phi P_2 Y) - h^s(Y, \phi P_2 X) = FP_3[X, Y]$. In view of (65), we get $P_1(\nabla_X \phi P_2 Y) = \phi P_1 \nabla_X Y$, implies $P_1(\nabla_X \phi P_2 Y) - P_1(\nabla_Y \phi P_2 X) = \phi P_1[X, Y]$. Also from (67), we obtain $P_3(\nabla_X \phi P_2 Y) = f P_3 \nabla_X Y + Bh^s(X, Y)$; this gives $P_3(\nabla_X \phi P_2 Y) - P_3(\nabla_Y \phi P_2 X) = f P_3[X, Y]$. This concludes the theorem. □

Theorem 7.5 ([37]) *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then $Rad(TM)$ defines a totally geodesic foliation if and only if $\bar{g}(h^l(X, PZ), \phi Y) = -\bar{g}(D^l(X, FZ), \phi Y)$, for all $X, Y \in \Gamma(Rad(TM))$ and $Z \in \Gamma(S(TM))$.*

Proof It is clear that $Rad(TM)$ defines a totally geodesic foliation if and only if $\nabla_X Y \in \Gamma(Rad(TM))$, for all $X, Y \in \Gamma(Rad(TM))$. Since $\bar{\nabla}$ is a metric connection, using (2) and (9), for any $X, Y \in \Gamma(Rad(TM))$ and $Z \in \Gamma(S(TM))$, we get $\bar{g}(\nabla_X Y, Z) = -\bar{g}(\bar{\nabla}_X PZ + \bar{\nabla}_X FZ, \phi Y)$, which implies $\bar{g}(\nabla_X Y, Z) = -\bar{g}(h^l(X, PZ) + D^l(X, FZ), \phi Y)$ and hence the proof is complete. □

Theorem 7.6 ([37]) *Let M be a screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then $D_1 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\bar{g}(\nabla_X f Z, \phi Y) = \bar{g}(A_{FZ} X, \phi Y)$ and $A_{\phi N} X$ has no component in $D_1 \oplus \{V\}$, for all $X, Y \in \Gamma(D_1 \oplus \{V\})$, $Z \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$.*

Proof To prove the distribution $D_1 \oplus \{V\}$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in \Gamma(D_1 \oplus \{V\})$, for all $X, Y \in \Gamma(D_1 \oplus \{V\})$. Since $\bar{\nabla}$ is a metric connection, from (2), (9), and (11), for any $X, Y \in \Gamma(D_1 \oplus \{V\})$ and $Z \in \Gamma(D_2)$, we obtain $\bar{g}(\nabla_X Y, Z) = -\bar{g}(\bar{\nabla}_X \phi Z, \phi Y)$, which gives $\bar{g}(\nabla_X Y, Z) = \bar{g}(A_{FZ} X - \nabla_X f Z, \phi Y)$. In view of (2), (9), and (11), for any $X, Y \in \Gamma(D_1 \oplus \{V\})$ and $N \in \Gamma(ltr(TM))$, we have $\bar{g}(\nabla_X Y, N) = -\bar{g}(\phi Y, \bar{\nabla}_X \phi N)$, which implies $\bar{g}(\nabla_X Y, N) = \bar{g}(\phi Y, A_{\phi N} X)$. This completes the proof. □

8 Radical Transversal Screen Semi-slant Lightlike Submanifolds

Definition 8.1 ([34]) Let M be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$ such that $2q < dim(M)$ with structure vector field tangent to M. Then M is a radical transversal screen semi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $\phi(Rad(TM)) = ltr(TM)$,

(ii) there exist non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$,
(iii) the distribution D_1 is invariant, that is, $\phi D_1 = D_1$,
(iv) the distribution D_2 is slant with angle $\theta(\neq 0)$.

A radical transversal screen semi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, and $\theta \neq \pi/2$. From the above definition, we have $TM = Rad(TM) \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$ and in particular

(i) if $D_2 = 0$, then M is a radical transversal lightlike submanifold;
(ii) if $D_1 = 0$ and $\theta = \pi/2$, then M is a transversal lightlike submanifold;
(iii) if $D_1 \neq 0$ and $\theta = \pi/2$, then M is a generalized transversal lightlike submanifold.

Thus, the class of radical transversal screen semi-slant lightlike submanifold of an indefinite Sasakian manifold includes radical transversal, transversal, generalized transversal lightlike submanifolds as its sub-cases and has been studied in [38, 40].

Example 4 ([34]) Let $(\mathbb{R}^{13}_2, \bar{g})$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, +, -, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}^{13}_2$ given by $x^1 = u_1$, $y^1 = -u_2$, $x^2 = u_1 \cos\alpha - u_2 \sin\alpha$, $y^2 = u_1 \sin\alpha + u_2 \cos\alpha$, $x^3 = y^4 = u_3$, $x^4 = -y^3 = u_4$, $x^5 = u_5 \cos\theta$, $y^5 = u_6 \cos\theta$, $x^6 = u_6 \sin\theta$, $y^6 = u_5 \sin\theta$, $z = u_7$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$Z_1 = 2(\partial x_1 + \cos\alpha \partial x_2 + \sin\alpha \partial y_2 + y^1 \partial z + \cos\alpha y^2 \partial z),$$
$$Z_2 = 2(-\partial y_1 - \sin\alpha \partial x_2 + \cos\alpha \partial y_2 - \sin\alpha y^2 \partial z),$$
$$Z_3 = 2(\partial x_3 + \partial y_4 + y^3 \partial z),\ Z_4 = 2(\partial x_4 - \partial y_3 + y^4 \partial z),$$
$$Z_5 = 2(\cos\theta \partial x_5 + \sin\theta \partial y_6 + y^5 \cos\theta \partial z),$$
$$Z_6 = 2(\sin\theta \partial x_6 + \cos\theta \partial y_5 + y^6 \sin\theta \partial z),$$
$$Z_7 = V = 2\partial z.$$

Hence, $Rad(TM) = span\{Z_1, Z_2\}$ and $S(TM) = span\{Z_3, Z_4, Z_5, Z_6, V\}$. Now, $ltr(TM)$ is spanned by $N_1 = -\partial x_1 + \cos\alpha \partial x_2 + \sin\alpha \partial y_2 - y^1 \partial z + \cos\alpha y^2 \partial z$, $N_2 = \partial y_1 - \sin\alpha \partial x_2 + \cos\alpha \partial y_2 - \sin\alpha y^2 \partial z$ and $S(TM^\perp)$ is spanned by

$$W_1 = 2(\partial x_3 - \partial y_4 + y^3 \partial z),\ W_2 = 2(\partial x_4 + \partial y_3 + y^4 \partial z),$$
$$W_3 = 2(\sin\theta \partial x_5 - \cos\theta \partial y_6 + y^5 \sin\theta \partial z),$$
$$W_4 = 2(\cos\theta \partial x_6 - \sin\theta \partial y_5 + y^6 \cos\theta \partial z).$$

It follows that $\phi Z_1 = -2N_2, \phi Z_2 = 2N_1$, which implies that $\phi Rad(TM) = ltr(TM)$. On the other hand, we can see that $D_1 = span\{Z_3, Z_4\}$ such that $\phi Z_3 = Z_4, \phi Z_4 = -Z_3$, which implies that D_1 is invariant with respect to ϕ and $D_2 = span\{Z_5, Z_6\}$

is a slant distribution with slant angle 2θ. Hence, M is a radical transversal screen semi-slant 2-lightlike submanifold of $\mathbb{R}^{13}_2$.

From (63) and (64), it is clear that for a radical transversal screen semi-slant lightlike submanifold, we have $\phi P_1 X \in \Gamma(ltr(TM))$ and $\phi Q_1 W \in \Gamma(Rad(TM))$. Now, by using (2)–(4), (11), (63)–(64) and on identifying the components on D_1, D_2, $ltr(TM)$, and $S(TM^{\perp})$, we obtain

$$P_2(\nabla_X \phi P_2 Y) + P_2(\nabla_X f P_3 Y) = P_2(A_{FP_3Y} X) + P_2(A_{\phi P_1 Y} X) + \phi P_2 \nabla_X Y - \eta(Y) P_2 X, \tag{74}$$

$$P_3(\nabla_X \phi P_2 Y) = P_3(A_{FP_3Y} X) + P_3(A_{\phi P_1 Y} X) - P_3(\nabla_X f P_3 Y) + f P_3 \nabla_X Y + Bh^s(X, Y) - \eta(Y) P_3 X, \tag{75}$$

$$\phi P_1 \nabla_X Y = \nabla^l_X \phi P_1 Y + h^l(X, \phi P_2 Y) + h^l(X, f P_3 Y) + D^l(X, FP_3 Y), \tag{76}$$

$$FP_3 \nabla_X Y - \nabla^s_X FP_3 Y = D^s(X, \phi P_1 Y) + h^s(X, \phi P_2 Y) + h^s(X, f P_3 Y) - Ch^s(X, Y). \tag{77}$$

Theorem 8.2 ([34]) *Let M be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then, M is a radical transversal screen semi-slant lightlike submanifold if and only if*

(i) $\phi ltr(TM)$ is a distribution on M such that $\phi ltr(TM) = Rad(TM)$,
(ii) distribution D_1 is invariant with respect to ϕ, that is, $\phi D_1 = D_1$,
(iii) there exists a constant $\lambda \in [0, 1)$ such that $P^2 X = -\lambda X$.

Moreover, there also exists a constant $\mu \in (0, 1]$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM) = D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$ and $\lambda = \cos^2\theta$, θ is slant angle of D_2.

Proof Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, distribution D_1 is invariant with respect to ϕ and $\phi Rad(TM) = ltr(TM)$. Thus, $\phi X \in \Gamma(ltr(TM))$, for all $X \in \Gamma(Rad(TM))$. Hence $\phi(\phi X) \in \Gamma(\phi(ltr(TM)))$, which implies $-X \in \Gamma(\phi(ltr(TM)))$, for all $X \in \Gamma(Rad(TM))$, which proves (i) and (ii). Now, for any $X \in \Gamma(D_2)$, similar to proof of Theorem 7.2, we have (iii).

Conversely, suppose that conditions (i), (ii), and (iii) are satisfied. In view of (i), we have $\phi N \in \Gamma(Rad(TM))$, for all $N \in \Gamma(ltr(TM))$. Hence, $\phi(\phi N) \in \Gamma(\phi(Rad(TM)))$, which implies $-N \in \Gamma(\phi(Rad(TM)))$, for all $N \in \Gamma(ltr(TM))$. Thus, $\phi Rad(TM) = ltr(TM)$. As in Theorem 7.2 for any $X \in \Gamma(D_2)$, we have our result. Hence, M is a radical transversal screen semi-slant lightlike submanifold. □

The conditions for integrability and totally geodesic foliations for the distributions of a radical transversal screen semi-slant lightlike submanifold of an indefinite Sasakian manifold are given below.

Theorem 8.3 ([34]) *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then, $Rad(TM) \oplus \{V\}$ is integrable if and only if $P_2(A_{\phi P_1 Y} X) = P_2(A_{\phi P_1 X} Y)$, $P_3(A_{\phi P_1 Y} X) = P_3(A_{\phi P_1 X} Y)$, and $D^s(Y, \phi P_1 X) = D^s(X, \phi P_1 Y)$, for all $X, Y \in \Gamma(Rad(TM) \oplus \{V\})$.*

Theorem 8.4 ([34]) *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then, $D_1 \oplus \{V\}$ is integrable if and only if $h^l(Y, \phi P_2 X) = h^l(X, \phi P_2 Y)$, $h^s(Y, \phi P_2 X) = h^s(X, \phi P_2 Y)$, and $P_3(\nabla_X \phi P_2 Y) = P_3(\nabla_Y \phi P_2 X)$, for all $X, Y \in \Gamma(D_1 \oplus \{V\})$.*

Theorem 8.5 ([34]) *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then $Rad(TM) \oplus \{V\}$ defines a totally geodesic foliation if and only if $\bar{g}(\nabla_X \phi P_2 Z + \nabla_X f P_3 Z, \phi Y) = \bar{g}(A_{FP_3 Z} X, \phi Y)$, for all $X, Y \in \Gamma(Rad(TM) \oplus \{V\})$ and $Z \in \Gamma(D_1 \oplus D_2)$.*

Theorem 8.6 ([34]) *Let M be a radical transversal screen semi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then, $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if $\bar{g}(fY, \nabla_X \phi Z) = -\bar{g}(FY, h^s(X, \phi Z))$ and $\bar{g}(fY, \nabla_X \phi N) = -\bar{g}(FY, h^s(X, \phi N))$, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1)$ and $N \in \Gamma(ltr(TM))$.*

9 Screen Pseudo-slant Lightlike Submanifolds

Definition 9.1 ([35]) Let M be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$ such that $2q < dim(M)$ with structure vector field tangent to M. Then M is a screen pseudo-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $Rad(TM)$ is invariant with respect to ϕ, that is, $\phi(Rad(TM)) = Rad(TM)$,
(ii) there exists non-degenerate orthogonal distributions D_1 and D_2 on M such that $S(TM) = D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$,
(iii) the distribution D_1 is anti-invariant, that is, $\phi D_1 \subset S(TM^{\perp})$,
(iv) the distribution D_2 is slant with angle $\theta (\neq \pi/2)$.

A screen pseudo-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$, and $\theta \neq 0$. Moreover, we have

(i) if $D_1 = 0$, then M is a screen slant lightlike submanifold,

(ii) if $D_2 = 0$, then M is a screen real lightlike submanifold,
(iii) if $D_1 = 0$ and $\theta = 0$, then M is an invariant lightlike submanifold,
(iv) if $D_1 \neq 0$ and $\theta = 0$, then M is a contact SCR-lightlike submanifold.

Thus, this new class of screen pseudo-slant lightlike submanifolds of an indefinite Sasakian manifold includes invariant, screen slant, screen real, contact screen CR-lightlike submanifolds as its sub-cases which have been studied in [11, 16, 31, 33].

Example 5 ([35]) Let $(\mathbb{R}^{13}_2, \bar{g})$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, +, -, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$. Suppose M is a submanifold of $\mathbb{R}^{13}_2$ given by $x^1 = u_1$, $y^1 = -u_2$, $x^2 = -u_1 \cos\alpha - u_2 \sin\alpha$, $y^2 = -u_1 \sin\alpha + u_2 \cos\alpha$, $x^3 = u_3 \cos\beta$, $y^3 = u_3 \sin\beta$, $x^4 = u_4 \sin\beta$, $y^4 = u_4 \cos\beta$, $x^5 = u_5$, $y^5 = u_6$, $x^6 = k\cos u_6$, $y^6 = k \sin u_6$, $z = u_7$, where k is any constant. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$, where

$$\begin{aligned}
Z_1 &= 2(\partial x_1 - \cos\alpha \partial x_2 - \sin\alpha \partial y_2 + y^1 \partial z - \cos\alpha y^2 \partial z),\\
Z_2 &= 2(-\partial y_1 - \sin\alpha \partial x_2 + \cos\alpha \partial y_2 - \sin\alpha y^2 \partial z),\\
Z_3 &= 2(\cos\beta \partial x_3 + \sin\beta \partial y_3 + y^3 \cos\beta \partial z),\\
Z_4 &= 2(\sin\beta \partial x_4 + \cos\beta \partial y_4 + y^4 \sin\beta \partial z),\\
Z_5 &= 2(\partial x_5 + y^5 \partial z),\\
Z_6 &= 2(\partial y_5 - k \sin u_6 \partial x_6 + k \cos u_6 \partial y_6 - k \sin u_6 y^6 \partial z),\\
Z_7 &= V = 2\partial z.
\end{aligned}$$

Hence, $Rad(TM) = span\{Z_1, Z_2\}$ and $S(TM) = span\{Z_3, Z_4, Z_5, Z_6, V\}$. Now, $ltr(TM)$ is spanned by $N_1 = \partial x_1 + \cos\alpha \partial x_2 + \sin\alpha \partial y_2 + y^1 \partial z + \cos\alpha y^2 \partial z$, $N_2 = -\partial y_1 + \sin\alpha \partial x_2 - \cos\alpha \partial y_2 + \sin\alpha y^2 \partial z$ and $S(TM^\perp)$ is spanned by

$$\begin{aligned}
W_1 &= 2(\sin\beta \partial x_3 - \cos\beta \partial y_3 + y^3 \sin\beta \partial z),\\
W_2 &= 2(\cos\beta \partial x_4 - \sin\beta \partial y_4 + y^4 \cos\beta \partial z),\\
W_3 &= 2(k \cos u_6 \partial x_6 + k \sin u_6 \partial y_6 + k \cos u_6 y^6 \partial z),\\
W_4 &= 2(k^2 \partial y_5 + k \sin u_6 \partial x_6 - k \cos u_6 \partial y_6 + k \sin u_6 y^6 \partial z).
\end{aligned}$$

It follows that $\phi Z_1 = Z_2$ and $\phi Z_2 = -Z_1$, which implies that $Rad(TM)$ is invariant, i.e., $\phi Rad(TM) = Rad(TM)$. On the other hand, we can see that $D_1 = span\{Z_3, Z_4\}$ such that $\phi Z_3 = W_1$ and $\phi Z_4 = W_2$, which implies that D_1 is anti-invariant with respect to ϕ and $D_2 = span\,\{Z_5, Z_6\}$ is a slant distribution with slant angle $\theta = \arccos(1/\sqrt{1+k^2})$. Hence, M is a screen pseudo-slant 2-lightlike submanifold of $\mathbb{R}^{13}_2$.

We denote the projections of $tr(TM)$ on $ltr(TM)$, $\phi(D_1)$ and D' by Q_1, Q_2, and Q_3, respectively, where D' is non-degenerate orthogonal complementary subbundle of $\phi(D_1)$ in $S(TM^\perp)$. Therefore, from (63), we get $\phi P_1 X \in \Gamma(Rad(TM))$, $\phi P_2 X \in$

$\Gamma(\phi D_1) \subset \Gamma(S(TM^{\perp}))$, $fP_3X \in \Gamma(D_2)$, and $FP_3X \in \Gamma(D')$. Also, for any $W \in \Gamma(tr(TM))$, we have $W = Q_1W + Q_2W + Q_3W$. On applying ϕ, we obtain $\phi W = \phi Q_1W + \phi Q_2W + \phi Q_3W$, this gives

$$\phi W = \phi Q_1 W + \phi Q_2 W + BQ_3 W + CQ_3 W, \tag{78}$$

where BQ_3W (resp. CQ_3W) denotes the tangential (resp. transversal) component of ϕQ_3W. Thus, we get $\phi Q_1W \in \Gamma(ltr(TM))$, $\phi Q_2W \in \Gamma(D_1)$, $BQ_3W \in \Gamma(D_2)$, and $CQ_3W \in \Gamma(D')$. Now, using (2)–(4), (11), (63), (78) and on identifying the components on $Rad(TM)$, D_2, $\phi(D_1)$ and D', we obtain

$$\begin{aligned}\nabla^{*t}_X \phi P_1 Y + P_1(\nabla_X fP_3Y) = P_1(A_{FP_3Y}X) + P_1(A_{\phi P_2 Y}X) \\ +\phi P_1 \nabla_X Y - \eta(Y)P_1X,\end{aligned} \tag{79}$$

$$\begin{aligned}P_3(A^*_{\phi P_1 Y}X) + P_3(A_{\phi P_2 Y}X) + P_3(A_{FP_3Y}X) = P_3(\nabla_X fP_3Y) \\ - fP_3(\nabla_X Y) - BQ_3h^s(X,Y) + \eta(Y)P_3X,\end{aligned} \tag{80}$$

$$\begin{aligned}Q_2\nabla^s_X \phi P_2 Y + Q_2 \nabla^s_X FP_3Y = \phi P_2 \nabla_X Y - Q_2h^s(X, \phi P_1 Y) \\ - Q_2 h^s(X, fP_3Y),\end{aligned} \tag{81}$$

$$\begin{aligned}Q_3\nabla^s_X \phi P_2 Y + Q_3\nabla^s_X FP_3Y - FP_3\nabla_X Y = CQ_3h^s(X,Y) \\ - Q_3h^s(X, fP_3Y) - Q_3h^s(X, \phi P_1 Y).\end{aligned} \tag{82}$$

Theorem 9.2 ([35]) *Let M be a $2q$-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then, M is a screen pseudo-slant lightlike submanifold of $\bar{M}$ if and only if*

(i) $ltr(TM)$ is invariant and D_1 is anti-invariant with respect to ϕ,
(ii) there exists a constant $\lambda \in (0, 1]$ such that $P^2X = -\lambda X$.

Moreover, there also exists a constant $\mu \in [0, 1)$ such that $BFX = -\mu X$, for all $X \in \Gamma(D_2)$, where D_1 and D_2 are non-degenerate orthogonal distributions on M such that $S(TM) = D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$ and $\lambda = \cos^2\theta$, θ is slant angle of D_2.

Proof Let M be a screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, D_1 is anti-invariant and $Rad(TM)$ is invariant with respect to ϕ. For any $N \in \Gamma(ltr(TM))$ and $X \in \Gamma(S(TM) - \{V\})$, from (10) and (63), we obtain $\bar{g}(\phi N, X) = -\bar{g}(N, \phi X) = -\bar{g}(N, \phi P_2X + fP_3X + FP_3X) = 0$. Thus, ϕN does not belong to $\Gamma(S(TM) - \{V\})$. Now, for any $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, from (10) and (78), we get $\bar{g}(\phi N, W) = -\bar{g}(N, \phi W) = -\bar{g}(N, \phi Q_2W$

$+ BQ_3W + CQ_3W) = 0$. Hence, we conclude that ϕN does not belong to $\Gamma(S(TM^{\perp}))$.

Now suppose that $\phi N \in \Gamma(Rad(TM))$. Then, further the proof of this theorem is analogous to Theorem 7.2. □

Theorem 9.3 ([35]) *Let M be a screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then, $Rad(TM)$ is integrable if and only if $Q_2h^s(Y, \phi P_1X) = Q_2h^s(X, \phi P_1Y)$, $Q_3h^s(Y, \phi P_1X) = Q_3h^s(X, \phi P_1Y)$ and $P_3A^*_{\phi P_1X}Y = P_3A^*_{\phi P_1Y}X$, for all $X, Y \in \Gamma(Rad(TM))$.*

Theorem 9.4 ([35]) *Let M be a screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then $D_2 \oplus \{V\}$ is integrable if and only if $P_1(\nabla_X fP_3Y - \nabla_Y fP_3X) = P_1(A_{FP_3Y}X - A_{FP_3X}Y)$ and $Q_2(\nabla^s_X FP_3Y - \nabla^s_Y FP_3X) = Q_2(h^s(Y, fP_3X) - h^s(X, fP_3Y))$, for all $X, Y \in \Gamma(D_2 \oplus \{V\})$.*

Theorem 9.5 ([35]) *Let M be a screen pseudo-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ with structure vector field tangent to M. Then D_1 defines a totally geodesic foliation if and only if $\bar{g}(h^s(X, fZ), \phi Y) = -\bar{g}(\nabla^s_X FZ, \phi Y)$ and $D^s(X, \phi N)$ has no component in $\phi(D_1)$, for all $X, Y \in \Gamma(D_1)$, $Z \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$.*

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