Study on Analytical Solutions of K-dV Equation, Burgers Equation, and Schamel K-dV Equation with Different Methods

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Abstract The study on analytical solutions of differential equations is quite useful in Modeling in fluid dynamics, physics, etc. In this review work we studied the analytical solutions of Korteweg–de Vries equation (K-dV), Burgers equation, Schamel equation, and Schamel–Korteweg–de Vries equations by using different analytical methods such as tanh method, sech method, sine-Gordon equation method, $\left(\frac{G'}{G}\right)$ expansion method, and tanh–coth methods. The $\left(\frac{G'}{G}\right)$ method has different types that are used to solved Schamel equation and Schamel–K-dV equation.

Keywords K-dV equation · Burgers equation · Schamel equation · Schamel–K-dV equation · tanh method · coth method · sech method · $\left(\frac{G'}{G}\right)$ methods · Sine-Gordon method · tanh-coth method

1 Introduction

Nonlinear evolution equations are used to describe the physical existence or physical models. The study of applications study on analytical solutions of the nonlinear partial differential equations is in fluid dynamics, plasma physics, nonlinear optics, engineering, mathematical physics and modeling, and so on. The important and applied nonlinear evolution equations are K-dV equation, Burgers equation, Schamel equation. Schamel–K-dV equation, and so on. To find out the analytical solutions of the nonlinear equations, many authors provide many methods and out of those

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S. R. Mishra et al. (eds.), *Recent Trends in Applied Mathematics*, Lecture Notes in Mechanical Engineering, https://doi.org/10.1007/978-981-15-9817-3_9

methods tanh method, sech method, tanh–coth method, sine–cosine method, $\left(\frac{G'}{G}\right)$ method, and sine-Gordon method are quite famous.

The tanh method is used to solve different nonlinear evolution equations in recent years. In 1996, Willy Malfliet solved K-dV Burgers, Dissipative-Dispersion, combined K-dV-MKdV, and extended MKdV–Bureges equations [1]. In 2000, Fan explores the solution of K-dV–Burgers–Kuramoto, 2-dimensional K-dV–Burgers and generalized Burgers–Fisher equations [2]. In 2004, Wazwaz solved generalized K-dV, generalized Fishers equations [3]. In 2005, Evans and Raslan studied the improved K-dV equation, equal width wave equation (EWE), Regularized long Wave and Coupled Burgers equations [4]. In 2007, Wazwaz studied the analytical solutions of the fifth-order K-dV equation, Lax equation, Sawada–Kotera (SK) equations, etc. [5]. In 2009, Sarma solved K-dV equation and MKdV equations [6, 7]. In 2010, Jawad et al. solved Burgers, K-dV–Burgers, Coupled Burgers, generalized time-delayed Burgers, Perturbed Burgers equations [8]. In 2013, Karimi solved a modified K-dV equation [9]. In 2016, Zhang and Yin solved Burgers equation [10]. Adem solved Coupled KP equation [11], Tariq and Akram solved Cahn-Allen and Phi-4 equations [12] and Ralson et al. solved time-fractional EW and MEW equations [13]. In 2019, Ali et al. solved variable coefficients of Burgers equation [14] and so on.

The tanh–coth method is used to solve analytical solutions of nonlinear evolutions equations is quite useful. In 2007 and In 2008, Wazwaz solved Fisher, Newell–Whitehead, Allen–Cahn and Fitz–Hugh–Nagumo, Burgers–Fisher, Burgers, Kodomtsev–Petviashvili, Pochhammer–Chree equations [15–17]. In 2009, Wazzan solved K-dV and K-dV Burgers equations [18]. In 2010, Parkes solved the MKdV equation, Salas and Gomezs solved K-dV equation of sixth order and MKdV equation of sixth order [19, 20]. In 2013, Jawad solved one-dimensional Burgers equation, K-dV–Burgers equation, Coupled Burgers equation, and generalized time-delayed Burgers equation [21]. In 2017 Chukkol et al. solved K-dV–Burgers equation with forcing term [22]. Asokan and Vinodh solved Sawad–Kotera equation [23]. In 2018, Asokan and Vinodh solved generalized K-dV–BBM and potential K-dV–BBM equations [24] and so on.

The sine-Gordon method is used to solved nonlinear evolution equations. In 2003, Yan solved K-dV equation, MKdV equation, and Complex NLS positive equation [25]. In 2016, Alquran and Krishnan solved generalized Phi-4 equation, generalized regularized long-wave equation and equal width equation and regularized long-wave equations [26]. In 2020, Korkmaz et al. solved conformable time-fractional RLW equation [27]. Guirao et al. solved $(3 + 1)$-dimensional B-type Kadomtsev-Petviashvili–Boussinesq equation [28].

The sine–cosine method is used to get analytical solutions of nonlinear equations. In 2004 and 2005, Wazwaz solved K-dV equation, generalized K-dV equation, Boussinesq equation, RLW equation, and Phi-4 equation, complex modified K-dV equation and complex generalized K-dV equation [29, 30]. In 2014, Bibi and Mohyudi-Din solved a modified K-dV equation [31]. In 2015, Yang and Tang solved the Schamel–K-dV equation [32]. In 2017, Raslan solved coupled general equal width wave equation [33]and so on.

The sech method is used to solve nonlinear evolution equations. In 2004, Baldwin et al. solved Hirota Satsuma System of Coupled K-dV equation [34]. In 2007, Wazwaz solved Jaulent–Miodek equation [35]. In 2008, Ganji and Abdollahzadeh solved Lax seventh-order K-dV equation [36]. In 2011, Wei and Tang solved coupled ZK equation [37]. In 2016, Jawad solved modified ZK equations, Dubrovsky equations [38], and so on.

$\left(\frac{G'}{G}\right)$ method is used to solve many nonlinear evolution equations out of those equations such equations are K-dV equation [19, 39, 40], modified-K-dV equation [41], fifth-order K-dV equation [42], seventh-order K-dV equation [43], ninth-order K-dV equation [44], K-dV–Burgers equation [45], the 2D-K-dV equation [46], Burgers equation [47, 48], equal width Burgers equation [49], K-dV-MKdV equation [50], coupled MKdV equation [51], Schrodinger–K-dV equation [52], coupled MKdV equation [53], Schamel–K-dV equation [54], and so on.

2 Solutions of K-dV Equation by *tanh* Method

Sarma in 2009 [6] evaluated the solutions of the K-dV equations of 3rd order using *tanh* method. K-dV equation of third order is of the form

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

where A and B are nonzero constants and $u = u(x, t)$. Now we are using wave transformation $X = a(x - kt)$ to convert the partial differential equation into ordinary differential equation, where a and k are nonzero constants.

$$Ba^2\frac{d^2u}{dX^2} + A\frac{u^2}{2} - ku = 0, \tag{2}$$

where $u = u(X)$. Now introducing the independent variable Z=$tanhX$ and $u(X) = \omega(Z)$. Now substituting these values in (2),

$$Ba^2(1 - Z^2)(1 - Z^2)\frac{d^2\omega}{dZ^2} - 2Ba^2Z(1 - Z^2)\frac{d\omega}{dZ} + A\frac{\omega^2}{2} - k\omega = 0 \tag{3}$$

The above equation has power series solution as it has a singular point +1 and −1.

$$\omega(Z) = \sum_{r=0}^{\infty} a_r Z^{\rho+r}$$

Then Eq. (3) becomes

$$Ba^2(1-Z^2)(1-Z^2)\sum_{r=0}^{\infty} a_r(\rho+r)(\rho+r-1)Z^{\rho+r-2}$$
$$-2Ba^2Z(1-Z^2)\sum_{r=0}^{\infty} a_r(\rho+r)Z^{\rho+r-1}$$
$$+\tfrac{A}{2}(\sum_{r=0}^{\infty} a_r Z^{\rho+r})^2 - k\sum_{r=0}^{\infty} a_r Z^{\rho+r} = 0 \quad (4)$$

Now equating the highest order derivative and highest power on nonlinear term in (4) and taking $\rho = 0$.

$$Z^{4+\rho+r-2} = Z^{(\rho+r)2}$$
$$\Rightarrow 4+r-2 = 2r$$
$$\Rightarrow r = 2.$$

For $r = 2$, we have

$$\omega(Z) = a_0 + a_1 Z + a_2 Z^2,\ a_2 \neq 0. \quad (5)$$

Now (3) becomes

$$2a_2Ba^2Z^4 + 4Ba^2a_2Z^4 + \frac{A}{2}a_2^2Z^4 + 2Ba^2a_1Z^3 + Aa_1a_2Z^3$$
$$-4a_2Ba^2Z^2 - 4Ba^2a_2Z^2 + \frac{A}{2}a_1^2Z^2 + Aa_0a_2Z^2 - ka_2Z^2$$
$$-2Ba^2a_1Z + Aa_0a_1Z - ka_1Z + \frac{A}{2}a_0^2 - ka_0 + 2a_2Ba^2 = 0 \quad (6)$$

Now equating the coefficient of Z^4, Z^3, Z^2, Z and constant terms in (6) to get the value of a_0, a_1, a_2 and k. Here $a_0 = \frac{1}{A}(k + 8Ba^2)$, $a_1 = 0$, $a_2 = -\frac{1}{A}12Ba^2$ and $k = \pm 4Ba^2$. Now substituting the values of a_0, a_1 and a_2 in (5).

Case 1 $K = 4Ba^2$ and $Z = tanhX$.

$$\Rightarrow u(X) = \frac{12}{A}Ba^2(1 - tanh^2X)$$
$$\Rightarrow u(X) = \frac{12}{A}Ba^2sech^2X$$
$$\Rightarrow u(x,t) = \frac{12}{A}Ba^2sech^2[a(x-kt)]$$
$$\Rightarrow u(x,t) = \frac{12}{A}Ba^2sech^2[a(x-4Ba^2t)] \quad (7)$$

Case 2 $K = -4Ba^2$, $Z = tanhX$

$$u(x,t) = \frac{4}{A}Ba^2(-2 - 3sech^2[a(x-kt)]) \tag{8}$$

Equations (7) and (8) are the required solutions of the K-dV equation (1).

3 Solutions of K-dV Equation by *sech* Method

Sarma in 2009 [7] evaluated the numerical solution of the K-dV equations of third order y using *tanh* method. The well-known K-dV equation in the simplest form is

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} = 0 \tag{9}$$

Now using transformation $X = a(x - kt)$.

Equation (9) is of the form

$$Ba^2\frac{d^2u}{dX^2} + A\frac{u^2}{2} - ku = 0, \tag{10}$$

where $u = u(X)$. Now introducing the independent variable $Z = sechX$ and $u(X) = \omega(Z)$.

Then Eq. (10) can be written as

$$Ba^2Z^2(1-Z^2)\frac{d^2\omega}{dZ^2} + Ba^2(Z - 2Z^3)\frac{d\omega}{dZ} + A\frac{\omega^2}{2} - k\omega = 0 \tag{11}$$

Let us assume the power series solution of the Eq. (11) as follows:

$$\omega(Z) = \sum_{r=0}^{\infty} a_r Z^{\rho+r}.$$

Now Eq. (11) becomes

$$\begin{aligned} &Ba^2Z^2(1-Z^2)\sum_{r=0}^{\infty} a_r(\rho+r)(\rho+r-1)Z^{\rho+r-2} \\ &+Ba^2(Z-2Z^3)\sum_{r=0}^{\infty} a_r(\rho+r)Z^{\rho+r-1} \\ &+\frac{A}{2}\left(\sum_{r=0}^{\infty} a_r Z^{\rho+r}\right)^2 - k\sum_{r=0}^{\infty} a_r Z^{\rho+r} = 0 \end{aligned} \tag{12}$$

Now equating the highest order derivative and highest power on nonlinear term in (12) and taking $\rho = 0$.

$$Z^{4+\rho+r-2} = Z^{(\rho+r)2}$$
$$\Rightarrow 4 + r - 2 = 2r$$
$$\Rightarrow r = 2.$$

For $r = 2$, then

$$\omega(Z) = a_0 + a_1 Z + a_2 Z^2,\ a_2 \neq 0 \tag{13}$$

Now Eq. (11) becomes

$$\begin{aligned}&-2Ba^2a_2Z^4 - 4Ba^2a_2Z^4 + \frac{A}{2}a_2^2Z^4 - 2Ba^2a_1Z^3 + Aa_1a_2Z^3\\&+4Ba^2a_2Z^2 + \frac{A}{2}a_1^2Z^2 + Aa_0a_2Z^2 - ka_2Z^2 + Ba^2a_1Z\\&+Aa_0a_1Z - ka_1Z + \frac{A}{2}a_0^2 - ka_0 = 0\end{aligned} \tag{14}$$

Now equating the coefficient of Z^4, Z^3, Z^2, Z and the constant terms in (14) to get the value of a_0, a_1, a_2, and k. We have

$$a_0 = -\frac{8}{A}Ba^2, \quad a_1 = 0, \quad a_2 = \frac{12}{A}Ba^2 \quad \text{and} \quad k = \pm 4Ba^2 \tag{15}$$

Now substituting these values in Eq. (13)

Case 1 $K = 4Ba^2$

$$u(x,t) = \frac{12}{A}Ba^2sech^2[a(x - 4Ba^2t)] \tag{16}$$

Case 2 $K = -4Ba^2$

$$u(x,t) = \frac{12}{A}Ba^2sech^2[a(x + 4Ba^2t)] \tag{17}$$

Equations (16) and (17) are the required solutions of the K-dV equation (9).

4 Solutions of K-dV Equation by *sine−cosine* Method

Wazwaz in 2009 [29] solved the solutions of the kdv equation by using *sine−cosine* method. The well-known K-dV equation in the simplest form is

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} = 0 \tag{18}$$

Now using transformation $X = a(x - kt)$.

Equation (18) is of the form

$$Ba^2\frac{d^2u}{dX^2} + A\frac{u^2}{2} - ku = 0, \tag{19}$$

where $u = u(X)$.

Case 1

Let us assume the solutions of the Eq. (19) as follows:

$$u(X) = \lambda\cos^{\beta}(\mu X), \textit{ where } |X| \leqslant \frac{\pi}{\mu}, \tag{20}$$

λ, μ are nonzero constants. Now Eq. (19) becomes

$$\begin{aligned} &-Ba^2\lambda\mu^2\beta\cos^{\beta}(\mu X) - Ba^2\lambda\mu^2\beta(\beta-1)\cos^{\beta}(\mu X) - ka\lambda\cos^{\beta}(\mu X) \\ &+Ba^2\lambda\mu^2\beta(\beta-1)\cos^{\beta-2}(\mu X) + \frac{A}{2}a\lambda^2\cos^{2\beta}(\mu X) = 0 \end{aligned} \tag{21}$$

Now equating the coefficient of Eq. (21), $\beta = -2$, $\lambda = \frac{3k}{2A}$ and $\mu = \sqrt{\frac{-k}{8Ba}}$, where $k < 0$.

Now substituting these values in Eq. (20), it becomes

$$\begin{aligned} u(X) &= \lambda\cos^{\beta}(\mu X) \\ &= \frac{3k}{2A}\cos^{-2}(\mu X) \\ \Rightarrow u(x,t) &= \frac{3k}{2A}\cos^{-2}\{\mu[a(x-kt)]\} \end{aligned} \tag{22}$$

Case 2

Let us assume the solutions of the Eq. (19) as follows:

$$u(X) = \lambda\sin^{\beta}(\mu X), \textit{ where } |X| \leqslant \frac{\pi}{\mu}, \tag{23}$$

λ, μ are nonzero constants. Now Eq. (19) becomes

$$\begin{aligned} &-Ba^2\lambda\mu^2\beta\sin^{\beta}(\mu X) - Ba^2\lambda\mu^2\beta(\beta-1)\sin^{\beta}(\mu X) - ka\lambda\sin^{\beta}(\mu X) \\ &+Ba^2\lambda\mu^2\beta(\beta-1)\sin^{\beta-2}(\mu X) + \frac{A}{2}a\lambda^2\sin^{2\beta}(\mu X) = 0 \end{aligned} \tag{24}$$

Now equating the coefficient of Eq. (24), $\beta = -2$, $\lambda = \frac{3k}{A}$ and $\mu = \sqrt{\frac{-k}{4Ba^2}}$, where $k < 0$.

Now substituting these values in Eq. (23), it becomes

$$\begin{aligned} u(X) &= \lambda \sin^{\beta}(\mu X) \\ &= \frac{3k}{A} \sin^{-2} \mu(X) \\ \Rightarrow u(x,t) &= \frac{3k}{A} \sin^{-2}\{\mu[a(x-kt)]\} \end{aligned} \tag{25}$$

Equations (24) and (25) are the required solutions of the K-dV equation (18).

5 Solutions of K-dV Equation by *Sine−Gordon* Method

Hepson, Korkmaz, Hosseini, Rezazadeh and Eslami together solved the K-dV equation bu using $sine-Gordon$ method in 2017. The well-known K-dV equation in the simplest form is

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} = 0 \tag{26}$$

Now using transformation $X = a(x - kt)$.

Equation (26) is of the form

$$Ba^2\frac{d^2u}{dX^2} + A\frac{u^2}{2} - ku = 0, \tag{27}$$

where $u = u(X)$.

The sine-Gordon equation is

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = m^2 sinu, \tag{28}$$

where $u = u(x,t)$ and using the transformation $X = a(x - ct)$. Equation (29) becomes

$$\frac{d^2u}{dX^2} = \frac{m^2 sinu}{a^2(1-k^2)} \tag{29}$$

Again introducing the new variable $\omega(X)$=$u(X)$, $\frac{d\omega}{dX} = sin\omega$, $\sin\omega(X) = sech(X)$, $\cos\omega(X) = tanh(X)$ and using some integral calculation, the predicated solution of (29) is

$$u(\omega) = A_0 + \sum_{j=1}^{n} \cos^{j-1}(A_j \cos\omega + B_j sin\omega) \tag{30}$$

Let us assume the solution of Eq. (27) of type (30). Now taking the derivatives of (30) and substituting the derivative values in (27). Then in order to balance the two-term equation it has j = 2.

Then Eq. (30) becomes

$$u(\omega) = A_0 + A_1 \cos\omega + A_2 \cos^2\omega + B_1 \sin\omega + B_2 \sin\omega \cos\omega \tag{31}$$

Now taking the derivatives of (31) and substituting in (27) then Eq. (27) becomes

$$\begin{aligned}
&\sin\omega(a^2 BB_2 + AA_2B_2 + AA_0B_2 + AA_1B_1 - kB_2)\\
&+\sin^2\omega(-4a^2BB_1 + 2AB_1B_2 - 3AA_1A_2 - AA_0A_1 - kA_1)\\
&+\sin^3\omega(-20a^3BB_2 - 5AA_2B_2 - 2AA_0B_2 - AA_1B_1 + 2kB_2)\\
&+\sin^4\omega(4a^2BB_1 - 3AB_1B_2 + 3AA_1A_2 + 2a^2BA_1)\\
&+\sin^5\omega(24a^2BB_2 + 4AA_2B_2)\\
&+\cos\omega(16a^2BA_2 - AB_2^2 - 2AA_0A_2 + AB_1^2 - A^2A_1 + 2KA_2)\\
&+\cos^3\omega(-40a^2BA_2 - 2AA_2^2 + 3AB_2^2 + 2AA_0A_2 + AB_1^2 - AA_1^2 + 2KA_2)\\
&+\cos^5\omega(24a^2BA_2 + 2AA_2^2 - 2AB_2^2)\\
&+\sin\omega\cos\omega(AA_0B_1 - kB_1 - 5a^2BB_1 - 2AA_2B_1 - 2AA_1B_2)\\
&+\sin\omega\cos^3\omega(6a^2BB_1 + 3AA_1B_2 + 3AA_2B_1) = 0
\end{aligned} \tag{32}$$

Now equating the coefficients of (32) to get the value of A_0, A_1, A_2, B_1, B_2 and k. Here

$$A_1 = 0,\ A_2 = \frac{-6a^2B}{A},\ B_1 = 0,\ B_2 = 0,\ k = -14a^2B + AA_0$$

Then

$$\begin{aligned}
u(\omega) &= A_0 - \frac{6a^2B}{A}\cos^2\omega\\
u(X) &= A_0 - \frac{6a^2B}{A}tanh^2(X)\\
u(x,t) &= A_0 - \frac{6a^2B}{A}tanh^2[a(x-kt)]\\
u(x,t) &= A_0 - \frac{6a^2B}{A}tanh^2[a(x-(-14a^2B + AA_0)t)].
\end{aligned} \tag{33}$$

Equation (33) is the solution of Eq. (26).

6 Solutions of K-dV Equation by $\frac{G'}{G}$ Method

Mehdipoor and Neirameh [39] in 2015 studied the analytic solution of K-dV equations by using $\frac{G'}{G}$ method. The well-known K-dV equation in the simplest form is

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} = 0 \tag{34}$$

Now using transformation $X = (x - kt)$.

Equation (34) is of the form

$$B\frac{d^2u}{dX^2} + A\frac{u^2}{2} - ku = 0, \tag{35}$$

where $u = u(X)$.

Now introducing the independent variable $Z = \frac{G'(X)}{G(X)}$ and $u(x) = \omega(X)$, where $G(X)$ satisfies the second-order differential equation

$$G'' + \lambda G' + \mu G = 0, \tag{36}$$

where λ and μ are constants. Then Eq. (35) is converted into

$$\begin{aligned} &B(-\mu - \lambda Z - Z^2)^2\frac{d^2\omega}{dZ^2} + B(-\lambda - 2Z)(-\lambda Z - \mu - Z^2)\frac{d\omega}{dZ} \\ &+\frac{A}{2}\omega^2 - k\omega + C_1 = 0 \end{aligned} \tag{37}$$

Let us assume the power series solution of the Eq. (37) as follows:

$$u(Z) = \alpha_{m1}\left(\frac{G'}{G}\right)^{m1} + \cdots, \tag{38}$$

where α_i's are constant and m1 is the positive integer, which can be determined by considering the highest order derivatives and nonlinear terms. Now Eq. (37) becomes

$$\begin{aligned} &B(-\mu - \lambda Z - Z^2)^2[\alpha_{m1}m_1(m_1 - 1)Z^{m_1-2} + \cdots] \\ &+B(-\lambda - 2Z)(-\lambda Z - \mu - Z^2)[\alpha_{m1}m1Z^{m1-1} + \cdots] \\ &+\frac{A}{2}\omega^2 - k\omega + C_1 = 0. \end{aligned} \tag{39}$$

Considering the homogeneous balance between u^2 and $\frac{d^2u}{dx^2}$ in Eq. (39), $m1 = 2$ then Eq. (38) becomes and substitute $\frac{G'}{G} = Z$.

$$u(Z) = \alpha_0 + \alpha_1 Z + \alpha_2 Z^2, \quad \alpha_2 \neq 0. \tag{40}$$

Now by using (40) in (37), it becomes

$$\begin{aligned}
&6B\alpha_2 Z^4 + \frac{A}{2}\alpha_2 Z^4 + 2B\alpha_1 Z^3 + 10B\lambda\alpha_2 Z^3 + A\alpha_1\alpha_2 Z^3 \\
&+3B\alpha_1\lambda Z^2 + 4B\alpha_2\lambda^2 Z^2 + 8B\alpha_2\mu Z^2 + \frac{A}{2}\alpha_1^2 Z^2 \\
&+A\alpha_2\alpha_0 Z^2 - k\alpha_2 Z^2 + B\alpha_1\lambda^2 Z + 2B\mu\alpha_1 Z + 6B\lambda\mu\alpha_2 Z + B\lambda\alpha_1\mu \\
&+2B\alpha_2\mu^2 + \frac{A}{2}\alpha_0^2 - k\alpha_0 + C_1 = 0
\end{aligned} \tag{41}$$

Now equating the coefficient of Z^4, Z^3, Z^2, Z and the constant terms in (41) to get the value of $\alpha_2 = \frac{-12B}{A}$, $\alpha_1 = \frac{-12\lambda B}{A}$, $k = A\alpha_0 + B\lambda^2 + 8\mu B$.

Then Eq. (40) becomes

$$u(Z) = \alpha_0 - \frac{12\lambda B}{A}Z - \frac{12B}{A}Z^2, \quad \alpha_2 \neq 0, \tag{42}$$

where $X = x - (B\lambda^2 + A\alpha_0 + 8B\mu)$.

Now considering the general solution of Eq. (36).

Case 1

Hyperbolic function traveling wave solutions when $\lambda^2 - 4\mu > 0$.

$$G(X) = C_2 e^{(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2})X} + C_3 e^{(\frac{-\lambda}{2} - \frac{\sqrt{\lambda^2-4\mu}}{2})X} \tag{43}$$

Now by using Eq. (43) in Eq. (42)

$$u(X) = -\frac{3B}{A}(\lambda^2 - 4\mu)\frac{d_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}X + d_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}X}{d_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}X + d_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}X} + \frac{3B\lambda^2}{A} + \alpha_0 = 0. \tag{44}$$

In particular, if $d_1 \neq 0$ and $d_2 = 0$ then Eq. (44) becomes

$$u(X) = \frac{3B}{A}(\lambda^2 - 4\mu)sech^2\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}X\right) + \frac{B}{12A}\mu + \alpha_0 \tag{45}$$

Case 2

Trigonometric function traveling wave solutions when $\lambda^2 - 4\mu < 0$.

$$G(X) = e^{(\frac{-\lambda}{2})X}\{C_2\cos\frac{\sqrt{\lambda^2-4\mu}}{2}X + C_3\sin\frac{\sqrt{\lambda^2-4\mu}}{2}X\} \tag{46}$$

Now by using Eq. (46) in Eq. (42)

$$u(X) = -\frac{3B}{A}(4\mu - \lambda^2)\left\{\frac{-C_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}X + C_3\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}X}{C_3\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}X + C_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}X}\right\}^2 + \frac{3B\lambda^2}{A} + \alpha_0. \tag{47}$$

In particular, if $C_2 \neq 0$ and $C_3 = 0$ in Eq. (47), it is

$$u(X) = -\frac{3B}{A}(4\mu - \lambda^2)tanh^2\frac{1}{2}\sqrt{\lambda^2 - 4\mu}X + \frac{3B\lambda^2}{A} + \alpha_0 \tag{48}$$

Case 3
Trigonometric function traveling wave solutions when $\lambda^2 - 4\mu = 0$.

$$G(X) = (C_2 + C_3X)e^{\frac{-\lambda}{2}X} \tag{49}$$

Now by using Eq. (49) in Eq. (42)

$$u(X) = -\frac{12B}{A}(\frac{C_3}{C_2 + C_3X})^2 + \frac{3B}{A}\lambda^2 + \alpha_0. \tag{50}$$

Here (44), (47), (50) are the types of solutions of the K-dV equations by using $\frac{G'}{G}$ method.

7 Solutions of K-dV Equation by *tanh−coth* Method

K-dV equation is

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} = 0 \tag{51}$$

Now using transformation $X = (x - kt)$.

Equation (51) is of the form

$$B\frac{d^2u}{dX^2} + A\frac{u^2}{2} - ku = 0, \tag{52}$$

where $u = u(X)$.

Let the power series solution of Eq. (52) of the form

$$u(X) = a_0 + \sum_{j=1}^{n}[a_jY^j(X) + b_jY^{-j}(X)], \tag{53}$$

$Y(X)$ is the solution of the Riccati equation

$$Y^{'} = A_1 + B_1Y + C_1Y^2, \tag{54}$$

where A_1, B_1, C_1 are constants.

Now using Eq. (53) and substituting the values in Eq. (52).

$$
\begin{aligned}
&Ba^2\left[\sum_{j=1}^{n}\left[a_j j(j-1)Y^{j-2}(X)+b_j(j)(j+1)Y^{-j-2}(X)\right]\right] \\
&+\frac{A}{2}\left[\sum_{j=1}^{n}\left[a_j Y^{j}(X)+b_j Y^{-j}(X)\right]\right]^2 \\
&-k\left[a_0+\sum_{j=1}^{n}\left[a_j Y^{j}(X)+b_j Y^{-j}(X)\right]\right]=0
\end{aligned} \tag{55}
$$

The parameter n is the positive constant that can be determined by balancing the linear highest term of highest order with the nonlinear term, here $n = 2$. Then Eq. (53) becomes

$$u(X) = a_0 + a_1Y + a_2Y^2 + b_1Y^{-1} + b_2Y^{-2}. \tag{56}$$

Now using Eqs. (54) and (58) then Eq. (52) is

$$
\begin{aligned}
&Y^4\left(6Ba^2a_2C_1^2+\frac{A}{2}a_2^2\right) \\
&+Y^3\left(2Ba^2a_1C_1^2+10Ba^2a_2B_1C_1+Aa_1a_0\right) \\
&+Y^2\left(3Ba^2a_1B_1C_1+4Ba^2a_2B_1^2+8Ba^2a_2B_1C_1-ka_2+\frac{A}{2}a_1^2+Aa_0a_2\right) \\
&+Y\left(Ba^2a_1B_1^2+2Ba^2a_1A_1C_1+6Ba^2a_2A_1B_1-ka_1+Aa_0a_1+Aa_0a_2\right) \\
&+Y^{-1}\left(Ba^2b_1B_1^2+2Ba^2b_1A_1C_1+6Ba^2b_2B_1C_1-2Ba^2b_2B_1^2-kb_1+Aa_0b_1+Aa_1b_2\right) \\
&+Y^{-2}\left(3Ba^2b_1A_1B_1+6Ba^2b_2B_1^2+8Ba^2b_2A_1C_1-kb_2+\frac{A}{2}b_1^2Aa-0b_2\right) \\
&+Y^{-3}\left(2Ba^2b_1+10Ba^2b_2A_1B_1+Ab_1b_2\right) \\
&+Y^{-4}\left(6Ba^2b_2A_1^2+\frac{A}{2}b_2^2\right) \\
&+Ba^2a_1A_1B_1+2Ba^2a_2A_1^2+Ba^2b_1B_1C_1+2Ba^2b_2C_1^2-ka_0+\frac{A}{2}a_0^2+Aa_1b_1+Aa_2b_2=0
\end{aligned} \tag{57}
$$

Now equating the coefficients of $Z^4, Z^3, Z^2, Z, Z^{-1}, Z^{-2}, Z^{-3}, Z^{-4}$ and the constant terms in (57) to get the values of a_0, a_1, a_2, b_1, b_2.

Then $a_1 = 60\frac{B}{A}a^2B_1C_1, a_2 = -12\frac{B}{A}a^2C_1^2, b_1 = \frac{60B^2a^4A_1^3B_1}{ABa^2-6ABa^2A_1^2}, b_2 = -12\frac{B}{A}a^2A_1^2$. Now substituting the values of the coefficients in

$$u(X) = a_0 + a_1Y + a_2Y^2 + b_1Y^{-1} + b_2Y^{-2}, \tag{58}$$

where Y is the solution of the Riccati equation.

8 Solutions of Burgers Equation by *tanh* Method

Consider the Burgers equation of the form.

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} - B\frac{\partial^2 u}{\partial x^2} = 0 \tag{59}$$

where A, B are constants and $u = u(x,t)$. Now using wave transformation $X = a(x - kt)$ to convert the partial differential equation into ordinary differential equation, where a and k are nonzero constants.

Here,

$$X = a(x - kt) \Rightarrow \frac{dX}{dx} = a \quad and \quad \frac{dX}{dt} = -ka$$

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{d}{dX}\frac{dX}{dt} = \frac{d}{dX}(-ka) = -ka\frac{d}{dX} \\
\frac{\partial}{\partial x} &= \frac{d}{dX}\frac{dX}{dx} = \frac{d}{dX}(a) = a\frac{d}{dX} \\
\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\left(a\frac{d}{dX}\right) = a\frac{d}{dX}\frac{dX}{dx}\left(\frac{d}{dX}\right) = a^2\frac{d}{dX}\left(\frac{d}{dX}\right) = a^2\frac{d^2}{dX^2}
\end{aligned}$$

Now substituting these values in (59), we have

$$\begin{aligned}
&-ka\frac{du}{dX} + Aua\frac{du}{dX} - Ba^2\frac{d^2u}{dX^2} = 0 \\
&\Rightarrow -k\frac{du}{dX} + Au\frac{du}{dX} - Ba\frac{d^2u}{dX^2} = 0 \\
&\Rightarrow -ku + A\frac{u^2}{2} - Ba\frac{du}{dX} = 0,
\end{aligned} \tag{60}$$

where $u = u(X)$. Now introducing the independent variable $Z = tanhX$ and $u(X) = \omega(Z)$

$$\Rightarrow \frac{dZ}{dX} = sech^2 X \quad and \quad \frac{d\omega}{dX} = \frac{d\omega}{dZ}\frac{dZ}{dX} = \frac{d\omega}{dZ}sech^2 X$$

Now substituting these values in (60), then

$$-k\omega + A\frac{\omega^2}{2} - Ba(1 - Z^2)\frac{d\omega}{dZ} = 0 \tag{61}$$

Let us assume the power series solution of the Eq. (61) as follows:

$$\omega(Z) = \sum_{r=0}^{\infty} a_r Z^{\rho+r}$$
$$\Rightarrow \frac{d\omega}{dZ} = \sum_{r=0}^{\infty} a_r(\rho + r)Z^{\rho+r-1}$$

Now substituting these values in Eq. (61)

$$-k\sum_{r=0}^{\infty} a_r Z^{\rho+r} + \frac{A}{2}\left(\sum_{r=0}^{\infty} a_r Z^{\rho+r}\right)^2 - Ba(1 - Z^2)\sum_{r=0}^{\infty} a_r(\rho + r)Z^{\rho+r-1} = 0 \tag{62}$$

Now equating the highest order derivative and highest power on nonlinear term in (62) and taking $\rho = 0$.

$$2\rho + 2r = \rho + r + 1$$
$$\Rightarrow r = 1.$$

For $r = 1$, we have

$$\omega(Z) = a_0 + a_1 Z, \ a_1 \neq 0$$
$$\Rightarrow \frac{d\omega}{dZ} = a_1 \tag{63}$$

Now (61) becomes

$$\frac{A}{2}a_0^2 - ka_0 - Baa_1 + Aa_0a_1Z - ka_1Z + \frac{A}{2}a_1^2Z^2 + Baa_1Z^2 = 0 \tag{64}$$

Now equating the coefficient of Z^2, Z and the constant terms in (64) to get the value of a_0, a_1, and k.

We have $a_0 = \frac{k}{A}$, $a_1 = -\frac{2Ba}{A}$, $k = 2Ba$ and $k = -2Ba$.

Case 1 $K = 2Ba$

$$u(x,t) = \frac{2Ba}{A}(1 - \sqrt{1 - sech^2[a(x - 2Bat)]}) \tag{65}$$

Case 2 $K = -2Ba$

$$u(x,t) = \frac{-2Ba}{A}(1 + \sqrt{1 - sech^2[a(x + 2Bat)]}) \tag{66}$$

Equations (65) and (66) are the required solutions of the Burgers equation (59).

9 Solutions of Burgers Equation by *sech* Method

Consider the Burgers equation of the form.

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} - B\frac{\partial^2 u}{\partial x^2} = 0 \tag{67}$$

where A, B are constants and $u = u(x, t)$. Now using wave transformation $X = a(x - kt)$ to convert the partial differential equation in to ordinary differential equation, where a and k are nonzero constants.

Then Eq. (67) is of the form

$$-ku + A\frac{u^2}{2} - Ba\frac{du}{dX} = 0, \tag{68}$$

where $u = u(X)$. Now introducing the independent variable $Z = sechX$ and $u(X) = \omega(Z)$

$$\Rightarrow \frac{dZ}{dX} = -sechXtanhX \quad and \quad \frac{d\omega}{dX} = \frac{d\omega}{dZ}\frac{dZ}{dX} = -ZtanhX\frac{d\omega}{dZ}$$

Now substituting these values in (68), then

$$-k\omega + A\frac{\omega^2}{2} + Baz\sqrt{1 - Z^2}\frac{d\omega}{dZ} = 0 \tag{69}$$

Let us assume the power series solution of the Eq. (69) as follows:

$$\omega(Z) = \sum_{r=0}^{\infty} a_r Z^{\rho+r}$$
$$\Rightarrow \frac{d\omega}{dZ} = \sum_{r=0}^{\infty} a_r(\rho + r)Z^{\rho+r-1}$$

Now substituting these values in Eq. (69)

$$-k\sum_{r=0}^{\infty} a_r Z^{\rho+r} + \frac{A}{2}\left(\sum_{r=0}^{\infty} a_r Z^{\rho+r}\right)^2 - Ba(1 - Z^2)\sum_{r=0}^{\infty} a_r(\rho + r)Z^{\rho+r-1} = 0 \tag{70}$$

Now equating the highest order derivative and highest power on nonlinear term in (62) and taking $\rho = 0$.

$$2\rho + 2r = \rho + r - 1 + 2$$
$$\Rightarrow r = 1.$$

For $r = 1$, we have

$$\omega(Z) = a_0 + a_1 Z,\ \ a_1 \neq 0$$
$$\Rightarrow \frac{d\omega}{dZ} = a_1 \tag{71}$$

Now (69) becomes

$$-ka_0 + \frac{A}{2}a_0^2 - ka_1 Z + Aa_0 a_1 Z + Baa_1 Z + \frac{A}{2}a_1^2 Z^2 - \frac{1}{2}Baa_1 Z^3 = 0 \tag{72}$$

Now equating the coefficient of Z^2, Z and the constant terms in (72) to get the value of a_0, a_1, and k.

We have $a_0 = \frac{2k}{A}$, $a_1 = -\frac{Ba}{A}$, $k = 2Ba$ and $k = -Ba$.

$$u(x,t) = \frac{-Ba}{A}(1 - secha(x + Bat)) \tag{73}$$

Equation (73) is the required solutions of the Burgers equation (67).

10 Solutions of Burgers Equation by $\left(\frac{G'}{G}\right)$ Method

The well-known Burgers equation in the simplest form is

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} - B\frac{\partial^2 u}{\partial x^2} = 0 \tag{74}$$

Now using transformation $X = a(x - kt)$. Equation (74) is of the form

$$-ku + A\frac{u^2}{2} - Ba\frac{du}{dX} = 0, \tag{75}$$

where $u = u(X)$.

Now introducing the independent variable $Z = \frac{G'(X)}{G(X)}$ and $u(x) = \omega(X)$, where $G(X)$ satisfies the second-order differential equation

$$G'' + \lambda G' + \mu G = 0, \tag{76}$$

where λ and μ are constants. Then Eq. (75) is converted into

$$-k\omega + \frac{A}{2}\omega^2 - Ba(-\mu - \lambda Z - Z^2)\frac{d\omega}{dZ} = 0 \tag{77}$$

Let us assume the power series solution of the Eq. (77) as follows:

$$\omega(Z) = \alpha_{m1}(\frac{G'}{G})^{m1} + \cdots, \tag{78}$$

where α_i's are constant and m1 is the positive integer, which can be determined by considering the highest order derivatives and nonlinear terms. Now Eq. (77) becomes

$$-k(\alpha_{m1}Z^{m_1} + \cdots) + \frac{A}{2}(\alpha_{m1}Z^{m_1} + \cdots)^2 - Ba(-\mu - \lambda Z^2)(\alpha_{m1}m_1 Z^{m1-1} + \cdots) = 0 \tag{79}$$

Considering the homogeneous balance between u^2 and $\frac{du}{dx}$ in Eq. (79), $m1 = 1$ then Eq. (78) becomes and substitute $\frac{G'}{G} = Z$.

$$\omega(Z) = \alpha_0 + \alpha_1 Z, \quad \alpha_1 \neq 0.$$
$$\Rightarrow \frac{d\omega}{dZ} = \alpha_1 \tag{80}$$

Now by using (80) in (77), it becomes

$$\frac{A}{2}\alpha_1^2 Z^2 + Ba\alpha_1 Z^2 - k\alpha_1 Z + A\alpha_0\alpha_1 Z + Ba\alpha_1\lambda Z - k\alpha_0 + \frac{A}{2}\alpha_0^2 + Ba\mu\alpha_1 = 0. \tag{81}$$

Now equating the coefficient of Z^2, Z and constant terms in (81) to get the value of $\alpha_1 = \frac{-2Ba}{A}$, $\alpha_0 = \frac{K}{A} - \frac{Ba}{A}\lambda$.

Then Eq. (80) becomes

$$\omega(Z) = \frac{K}{A} - \frac{aB}{A}\lambda - \frac{2B}{A}aZ, \tag{82}$$

Now considering the general solution of Eq. (76).

Case 1

Hyperbolic function traveling wave solutions when $\lambda^2 - 4\mu > 0$.

$$G(X) = C_2 e^{\left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\mu}}{2}\right)X} + C_3 e^{\left(\frac{-\lambda}{2} - \frac{\sqrt{\lambda^2-4\mu}}{2}\right)X} \tag{83}$$

Now by using Eq. (83) in Eq. (82)

$$u(X) = \frac{K}{A} - \frac{B}{A}a\lambda - \frac{B}{A}a\sqrt{\lambda^2 - 4\mu}\frac{d_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}X + d_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}X}{d_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}X + d_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}X} = 0. \tag{84}$$

Case 2

Trigonometric function traveling wave solutions when $\lambda^2 - 4\mu < 0$.

$$G(X) = e^{(\frac{-\lambda}{2})X}\left\{C_2\cos\frac{\sqrt{\lambda^2-4\mu}}{2}X + C_3\sin\frac{\sqrt{\lambda^2-4\mu}}{2}X\right\} \tag{85}$$

Now by using Eq. (85) in Eq. (82)

$$u(X) = \frac{k}{A} - \frac{Ba}{A}\sqrt{4\mu - \lambda^2}\frac{-C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}X + C_3 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}X}{C_3 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}X + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}X}. \tag{86}$$

Case 3
Trigonometric function traveling wave solutions when $\lambda^2 - 4\mu = 0$.

$$G(X) = (C_2 + C_3 X)e^{\frac{-\lambda}{2}X} \tag{87}$$

Now by using Eq. (87) in Eq. (82)

$$u(X) = \frac{k}{A} - \frac{2Ba}{A}\left(\frac{C_3}{C_2 + C_3 X}\right). \tag{88}$$

Here (84), (86), (88) are the types of solutions of the Burgers equations by using $\frac{G'}{G}$ method.

11 Solutions of Burgers Equation by $Sine{-}Gordon$ Method

$$\frac{\partial u}{\partial t} + Au\frac{\partial u}{\partial x} - B\frac{\partial^2 u}{\partial x^2} = 0 \tag{89}$$

Now using transformation $X = a(x - kt)$.

Equation (89) is of the form

$$-Ba\frac{du}{dX} + A\frac{u^2}{2} - ku = 0, \tag{90}$$

where $u = u(X)$.

The sine–Gordon equation is

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = m^2 sinu, \tag{91}$$

where $u = u(x, t)$ and using the transformation $X = a(x - ct)$. Equation (92) becomes

$$\frac{d^2 u}{dX^2} = \frac{m^2 sinu}{a^2(1 - k^2)} \tag{92}$$

Again introducing the new variable $\omega(X) = u(X)$, $\frac{d\omega}{dX} = sin\omega$, $\sin\omega(X) = sech(X)$, $\cos\omega(X) = tanh(X)$ and using some integral calculation, the predicated solution of (92) is

$$u(\omega) = A_0 + \sum_{j=1}^{n} \cos^{j-1}(A_j \cos\omega + B_j sin\omega) \tag{93}$$

Let as assuming the solution of Eq. (90) of type (93). Now taking the derivatives of (93) and substituting the derivative values in (90). Then in order to balance the two-term equation it has j = 1.

Then Eq. (93) becomes

$$u(\omega) = A_0 + A_1 \cos\omega + B_1 \sin\omega \tag{94}$$

Now taking the derivatives of (94) and substituting in (90) then Eq. (90) becomes

$$\begin{aligned}
&-kA_0 + \frac{A}{2}A_0^2 + \frac{A}{2}A_1^2 - kA_1\cos\omega + AA_0A_1\cos\omega - kB_1\sin\omega \\
&+AA_0B_1\sin\omega - \frac{A}{2}A_1^2\sin^2\omega + \frac{A}{2}B_1^2\sin^2\omega + BA_1a\sin^2\omega \\
&+AA_1B_1\cos\omega\sin\omega - BB_1a\cos\omega\sin\omega = 0
\end{aligned} \tag{95}$$

Now equating the coefficients of (95) to get the value of A_0, A_1, B_1 and k. Here

$$A_1 = \frac{Ba}{A}, B_1 = \pm i\frac{Ba}{A},$$

Then

$$u(x,t) = A_0 + \frac{aB}{A}\sqrt{1 - sech^2 X} \pm i\frac{Ba}{A}sech. \tag{96}$$

Equation (96) is the solution of Eq. (89).

12 Solutions of Schamel–K-dV Equation by $\frac{G'}{G}$ Method

Consider the simplest form of Schamel–K-dV equation as

$$\frac{\partial u}{\partial t} + Au^{\frac{1}{2}}\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} = 0, \tag{97}$$

where A and B are arbitrary coefficients and $u = u(x,t)$.

Now using the wave transformation $X = x - kt$, where k is constant.

Then Eq. (97), becomes

$$-k\frac{dU}{dX}+AU^{1/2}+B\frac{d^3U}{dX^3}=0, \tag{98}$$

where $U = U(X)$. Now integrating Eq. (98), then we have

$$-kU+\frac{2}{3}AU^{3/2}+B\frac{d^2U}{dX^2}+c=0, \tag{99}$$

where c is the integration constant.

Let us consider $U^{1/2}(X) = V(X)$ Eq. (99) becomes

$$\Rightarrow -kV^2+\frac{2}{3}AV^3+2B\left(\frac{dV}{dX}\right)^2+2BV\frac{d^2V}{dX^2}+c=0 \tag{100}$$

Let us assume the solution of (100) of the form

$$V(X)=\sum_{i=0}^{n}a_i\left(\frac{G'}{G}\right)^i \tag{101}$$

where $G = G(X)$ satisfies the the second-order differential equation

$$G''+\lambda G'+\mu G=0 \tag{102}$$

where λ and μ are constants.

Now substituting the value of $\frac{d^2V}{dX^2}$, $\frac{dV}{dX}$, $V(X)$ in Eq. (100) balancing the highest order nonlinear term with highest order derivative we get $n = 2$. Then Eq. (101) becomes

$$V(X)=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2 \tag{103}$$

Now substituting the values of $V(X)$, $\frac{d^2V}{dX^2}$, $\frac{dV}{dX}$ in (100) and equating the coefficients of $\left(\frac{G'}{G}\right)^i$ to get the values of a_0, a_1, a_2, k we have

$$a_0=a_0,\ a_1=-30\frac{B}{A}\lambda,\ a_2=-30\frac{B}{A} \tag{104}$$

Substituting these values in the assuming solution and applying the transformation $U = V^2$ we have the following different types of solution of the form:

$$U(X)=\left[a_0-30\frac{B}{A}\lambda\left(\frac{G'}{G}\right)-30\frac{B}{A}\left(\frac{G'}{G}\right)^2\right]^2 \tag{105}$$

where

Case 1 When $\lambda^2 - 4\mu > 0$

$$\left(\frac{G'}{G}\right) = \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left(\frac{d_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}X + d_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}X}{d_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}X + d_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}X}\right) - \frac{\lambda}{2}$$

Case 2 When $\lambda^2 - 4\mu < 0$

$$\left(\frac{G'}{G}\right) = \frac{\sqrt{4m - \lambda^2}}{2}\frac{\left(-d_1 \sin \frac{\sqrt{4m-\lambda^2}}{2}X + d_2 \cos \frac{\sqrt{4m-\lambda^2}}{2}X\right)}{\left(d_1 \cos \frac{\sqrt{4m-\lambda^2}}{2}X + d_2 \sin \frac{\sqrt{4m-\lambda^2}}{2}X\right)} - \frac{\lambda}{2}$$

Case 3 When $\lambda^2 - 4\mu = 0$

$$\left(\frac{G'}{G}\right) = -\frac{\lambda}{2} + \frac{c_2 X}{c_1 + c_2 X}$$

13 Solutions of Schamel–K-dV Equation by Different Form of $\frac{G'}{G}$ Method

Consider the simplest form of Schamel–K-dV equation as

$$\frac{\partial u}{\partial t} + Au^{\frac{1}{2}}\frac{\partial u}{\partial x} + B\frac{\partial^3 u}{\partial x^3} = 0, \tag{106}$$

where A and B are arbitrary coefficients and $u = u(x, t)$.

Now using the wave transformation $X = x - kt$, where k is constant.

Then Eq. (106), becomes

$$-k\frac{dU}{dX} + AU^{1/2} + B\frac{d^3U}{dX^3} = 0, \tag{107}$$

where $U = U(X)$. Now integrating Eq. (107), then we have

$$-kU + \frac{2}{3}AU^{3/2} + B\frac{d^2U}{dX^2} + c = 0, \tag{108}$$

where c is the integration constant.

Let us consider $U^{1/2}(X) = V(X)$ Eq. (108) which becomes

$$\Rightarrow -kV^2 + \frac{2}{3}AV^3 + 2B\left(\frac{dV}{dX}\right)^2 + 2BV\frac{d^2V}{dX^2} + c = 0 \tag{109}$$

Let us assume the solution of (109) of the form

$$V(X) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i + \sum_{i=1}^{n} b_i \left(\frac{G'}{G}\right)^{-i} \tag{110}$$

where $G = G(X)$ satisfies the the second-order differential equation

$$G'' + \mu G = 0 \tag{111}$$

μ is constants.

Now substituting the value of $\frac{d^2V}{dX^2}$, $\frac{dV}{dX}$, $V(X)$ in Eq. (109) and balancing the highest order nonlinear term with highest order derivative we get $n = 1$. Then Eq. (110) becomes

$$V(X) = a_0 + a_1\left(\frac{G'}{G}\right) + b_1\left(\frac{G'}{G}\right)^{-1} \tag{112}$$

Now using Eq. (112) and its derivatives into Eq. (109) and equating the coefficients of $\left(\frac{G'}{G}\right)^i$, where $i = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6$ to find out the values of a_0, a_1, b_1, μ, b, c. By using Mathematica we got the values as follows

$$a_0 = \frac{3k}{2A},\quad a_1 = \frac{-6B}{A}, a_2 = 0, b_1 = 0, b_2 = 0, \mu = \frac{-k}{4B}, c = 0.$$

Substituting these values in the assuming solution and applying the transformation $U = V^2$, we have the following different types of solution of the form:

$$U(X) = \left[\frac{3k}{2A} - \frac{6B}{A}\left(\frac{G'}{G}\right)\right]^2 \tag{113}$$

where

Case 1 When $-\mu > 0$

$$\left(\frac{G'}{G}\right) = \sqrt{\mu}\left(\frac{d_2 \cosh\sqrt{-\mu}X + d_1 \sinh\sqrt{-\mu}X}{d_1 \cosh\sqrt{-\mu}X + d_2 \sinh\sqrt{-\mu}X}\right)$$

Case 2 When $-\mu < 0$

$$\left(\frac{G'}{G}\right) = \sqrt{\mu}\frac{\left(-d_1 \sin\sqrt{\mu}X + d_2 \cos\sqrt{\mu}X\right)}{\left(d_1 \cos\sqrt{\mu}X + d_2 \sin\sqrt{\mu}X\right)}$$

14 Solutions of Coupled Schamel–K-dV Equation by $\left(\frac{G'}{G}\right)$ Method

Consider the coupled Schamel–K-dV equation of the form

$$\frac{\partial u}{\partial t} + au^{\frac{1}{2}}\frac{\partial u}{\partial x} + bu\frac{du}{dx} + p\frac{\partial^3 u}{\partial x^3} = 0 \tag{114}$$

where a, b, and p are arbitrary coefficients. Now using the wave transformation $X = x - kt$, where k is constant. Then Eq. (114) becomes

$$p\frac{dU^3}{dX} - k\frac{dU}{dX} + aU^{\frac{1}{2}}\frac{dU}{dX} + bU\frac{dU}{dX} = 0 \tag{115}$$

Integrating Eq. (115), it becomes

$$p\frac{dU^2}{dX^2} - kU + \frac{2}{3}aU^{\frac{3}{2}} + \frac{1}{2}bU^2 + c = 0 \tag{116}$$

where c is the integration constant.

Let $U^{\frac{1}{2}} = V$ then Eq. (116) becomes

$$V\frac{d^2V}{dX^2} + \left(\frac{dV}{dX}\right)^2 - \frac{k}{2p}V^2 + \frac{a}{3p}V^3 + \frac{b}{4p}V^4 + \frac{c}{2p} = 0 \tag{117}$$

Let us assume the solution of (117) of the form

$$V(X) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i \tag{118}$$

where $G = G(X)$ satisfies the the second-order differential equation

$$G'' + \lambda G' + \mu G = 0 \tag{119}$$

where λ and μ are constants.

Balancing the highest order nonlinear term with highest order derivative of $V\frac{d^2V}{dX^2}$ and V^4, then it comes out that n = 1. Then

$$V(X) = a_0 + a_1\left(\frac{G'}{G}\right) \tag{120}$$

Now using Eq. (120) and its derivatives into Eq. (117) and equating the coefficients of $\left(\frac{G'}{G}\right)^i$, where $i = 0, 1, 2, 3, 4$ to find out the values of $a_0, a_1, \lambda, \mu, b, c$. By using

Mathematica, we got the values as follows:

$$a_0 = \frac{15}{8a}(k \pm 2\lambda\sqrt{pk}), \quad a_1 = \pm\frac{15\sqrt{pk}}{2a},$$
$$\lambda^2 - 4\mu = \frac{k}{4p}, b = \frac{-16a^2}{75k}, c = 0. \tag{121}$$

Substituting these values in the assuming solution and applying the transformation $U = V^2$ we have the following different types of solution of the form:

$$U(X) = \left[\frac{15}{8a}(k \pm 2\lambda\sqrt{pk}) \pm \frac{15\sqrt{pk}}{2a}\left(\frac{G'}{G}\right)\right]^2 \tag{122}$$

where

Case 1 When $\lambda^2 - 4\mu > 0$

$$\left(\frac{G'}{G}\right) = \frac{\sqrt{\lambda^2 - 4\mu}}{2}\left(\frac{d_2 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}X + d_1 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}X}{d_1 \cosh \frac{\sqrt{\lambda^2-4\mu}}{2}X + d_2 \sinh \frac{\sqrt{\lambda^2-4\mu}}{2}X}\right) - \frac{\lambda}{2}$$

Case 2 When $\lambda^2 - 4\mu < 0$

$$\left(\frac{G'}{G}\right) = \frac{\sqrt{4m - \lambda^2}}{2}\frac{\left(-d_1 \sin \frac{\sqrt{4m-\lambda^2}}{2}X + d_2 \cos \frac{\sqrt{4m-\lambda^2}}{2}X\right)}{\left(d_1 \cos \frac{\sqrt{4m-\lambda^2}}{2}X + d_2 \sin \frac{\sqrt{4m-\lambda^2}}{2}X\right)} - \frac{\lambda}{2}$$

Case 3 When $\lambda^2 - 4\mu = 0$

$$\left(\frac{G'}{G}\right) = -\frac{\lambda}{2} + \frac{c_2 X}{c_1 + c_2 X}$$

15 Solutions of Coupled Schamel–K-dV Equation by Different Form of $\left(\frac{G'}{G}\right)$ Method

Consider the Coupled Schamel–K-dV equation of the form

$$\frac{\partial u}{\partial t} + au^{\frac{1}{2}}\frac{\partial u}{\partial x} + bu\frac{du}{dx} + p\frac{\partial^3 u}{\partial x^3} = 0 \tag{123}$$

where a, b and p are arbitrary coefficients. Now using the wave transformation $X = x - kt$, where k is constant. Then Eq. (123) becomes

$$p\frac{dU^3}{dX} - k\frac{dU}{dX} + aU^{\frac{1}{2}}\frac{dU}{dX} + bU\frac{dU}{dX} = 0 \tag{124}$$

Integrating Eq. (124), it becomes

$$p\frac{dU^2}{dX^2} - kU + \frac{2}{3}aU^{\frac{3}{2}} + \frac{1}{2}bU^2 + c = 0 \tag{125}$$

where c is the integration constant.

Let $U^{\frac{1}{2}} = V$ then Eq. (125) becomes

$$V\frac{d^2V}{dX^2} + \left(\frac{dV}{dX}\right)^2 - \frac{k}{2p}V^2 + \frac{a}{3p}V^3 + \frac{b}{4p}V^4 + \frac{c}{2p} = 0 \tag{126}$$

Let us assume the solution of (126) of the form

$$V(X) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i + \sum_{i=1}^{n} b_i \left(\frac{G'}{G}\right)^{-i} \tag{127}$$

where $G = G(X)$ satisfies the the second-order differential equation

$$G'' + \mu G = 0 \tag{128}$$

where μ is constants.

Balancing the highest order nonlinear term with highest order derivative of $V\frac{d^2V}{dX^2}$ and V^4, then it comes out that n = 1. Then

$$V(X) = a_0 + a_1\left(\frac{G'}{G}\right) + b_1\left(\frac{G'}{G}\right)^{-1} \tag{129}$$

Now using Eq. (129) and its derivatives into Eq. (126) and equating the coefficients of $\left(\frac{G'}{G}\right)^i$, where $i = 0, \pm 1, \pm 2, \pm 3, \pm 4$ to find out the values of a_0, a_1, b_1, μ, b, c. By using Mathematica, we got the values as follows:

$$\begin{aligned} a_0 &= \frac{15k}{8a}, \quad a_1 = \pm\frac{15\sqrt{pk}}{2a}, b_1 = \pm\frac{15k\sqrt{pk}}{64ap}, \\ \mu &= \frac{k}{32p}, b = \frac{-16a^2}{75k}, c = \frac{225k^3}{512a^2} \end{aligned} \tag{130}$$

Substituting these values in the assuming solution and applying the transformation $U = V^2$, we have the following different types of solution of the form:

$$U(X) = \left[\frac{15k}{8a} \pm \frac{15\sqrt{pk}}{2a}\left(\frac{G'}{G}\right) \pm \frac{15k\sqrt{pk}}{64ap}\left(\frac{G'}{G}\right)\right]^2 \tag{131}$$

where

Case 1 When $-\mu > 0$

$$\left(\frac{G'}{G}\right) = \sqrt{\mu}\left(\frac{d_2 \cosh\sqrt{-\mu}X + d_1 \sinh\sqrt{-\mu}X}{d_1 \cosh\sqrt{-\mu}X + d_2 \sinh\sqrt{-\mu}X}\right)$$

Case 2 When $-\mu < 0$

$$\left(\frac{G'}{G}\right) = \sqrt{\mu}\frac{\left(-d_1 \sin\sqrt{\mu}X + d_2 \cos\sqrt{\mu}X\right)}{\left(d_1 \cos\sqrt{\mu}X + d_2 \sin\sqrt{\mu}X\right)}$$

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