Chapter 8
Balancing an Intuitive-Experimental Approach with Mathematical Rigour: A Case Study of an Experienced and Competent Mathematics Teacher in a Singapore Secondary School

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Abstract This chapter reports a case study of an experienced and competent mathematics teacher teaching Angle Properties of Circles to a class of Secondary Three students in the Express course of study. Geometry in the school curriculum serves as a good platform for inducting students into the rigour of mathematical thinking through deductive reasoning, and the world of deductive mathematical arguments in the form of mathematical proof, which forms the common language of mathematicians worldwide. It is this rigour and discipline that students usually encounter much difficulty with. Quite contrary to our stereotyped image of a traditional geometry lesson, the teacher used a variety of approaches to enrich the lesson. She used a series of scaffoldings to lead the students from inductive exploration through discovery activities to deductive reasoning and the formalism of writing of reasoning in geometry, juggling between her belief on the importance of discovery learning and the curriculum requirement of deductive reasoning in geometry. It was interesting to us that the teacher, in transiting from students' exploration to identifying the geometric properties, made use of rich visual imagery related to circle properties to develop in her students the concept images associated with the geometry property. Through the use of visuals to facilitate her students' learning, effort was made to ensure her students truly understood the geometrical properties and used the properties in working with problems. Deductive reasoning was introduced in the lesson closure portion of the lesson to stress the interconnectedness across the various geometrical properties. The stages that the teacher went through in guiding the students from the intuitive-experimental stage to the deductive reasoning resonates with the van Hiele levels of students' learning of geometry. The teacher highlighted during the interview about her conscious attempt to achieve a balance between an intuitive-experimental

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approach to facilitate her students' learning and maintaining mathematical rigour that is required of the geometry strand in the Singapore school mathematics curriculum.

Keywords Teaching geometry · van Hiele levels · Mathematical reasoning · Concept image

8.1 Introduction

The authors (hereafter, first person pronoun) are part of the project team (see Chapter 2) that examined the enactment of the secondary school mathematics curriculum in Singapore schools. In this chapter, we report a case study of an experienced and competent mathematics teacher teaching Angle Properties of Circles to a class of Secondary Three students in the Express course of study. Our stereotyped image of a typical geometry lesson is one that is full of deductive mathematical reasoning culminating in rigorous mathematical proofs; such lessons are difficult and boring to laypeople of mathematics. What we observed in this series of lessons was quite contrary to our preconceived idea of a geometry lesson. The teacher, whose students were the upper-bound of average ability, used a variety of approaches, ranging from an intuitive-experimental approach to the rigorous deductive approach. How these various approaches unfolded in the first lesson on Angle Properties of Circles is the focus of this chapter. Of interest are how the various geometry concepts were skillfully developed and connected through the approaches in different parts of the lesson. The teacher was fully cognisant of the syllabus requirement and her belief about the importance of student engagement.

8.2 Teaching of Geometry in Schools

Geometry has been recognised by mathematicians as an ideal vehicle to introduce students to "axiomatics" because of its "esthetic appeal" (Coxeter & Greitzer, 1967). One of the main goals of teaching mathematics has always been to facilitate students to develop deductive reasoning. Geometry seems to fit this goal perfectly (Ayalon & Even, 2010; Herbst, 2002). It is thus not surprising that our preconceived idea of a traditional secondary school geometry lesson is usually one in which students are expected to prove theorems. Mathematical proofs are usually seen by students as the "rules of the games", which is the essence of mathematics and therefore the core of academic mathematician's daily practice.

The International Commission on Mathematical Instruction (ICMI), in preparation for the study on "Perspectives on the Teaching of Geometry for the 21st Century", challenged academics to re-think the teaching of geometry, especially in the recent decades with the advent of technology and geometry teaching aides (ICMI, 1995). ICMI (1995) invited discussion among academics whether geometry teaching at the schools should take the form of an "intuitive" approach, or a "formalised" approach, or perhaps a mixture of both approaches with a gradual shift from an intuitive to a formalised approach "as the age of students and the school level progresses".

A geometry lesson using an intuitive approach of teaching geometry is in direct contrast to the traditional image (and even the objective) of a geometry lesson. Associating with an intuitive approach of teaching geometry, one is likely to think of computer-based learning environment such as the environment of the Dynamic Geometry (DG). The justification of intuitive approach is based on existing education literature on the positive impact of computer and technology on student learning. Studies have shown that computer environments such as that of a DG can stimulate learners to link their intuitive notions and formal aspects of mathematical knowledge (e.g. Sutherland, 1998; Sutherland, Olivero, & Weeden, 2004). DGs enable learners to manipulate objects by clicking, dragging, and measuring the objects in order to discover mathematical relationships. Researchers have studied how teachers can provide appropriate scaffolding for student learning through the use of appropriate pre-designed files (e.g. Leung, 2011).

In the mathematics curriculum document provided by the Singapore Ministry of Education (MOE) (2012), the underpinning theoretical principle in teaching of secondary school geometry was explicitly stated as:

> The learning of *Geometry* at this stage [i.e. at the secondary level] should adopt an *intuitive* and *experimental* approach. This approach is based on van Hiele's theory of geometry learning which advocates exploration and discovery through hands-on activities. (MOE, 2012, p. 32)

Using van Hiele's theory as the guiding principle, Leong and Lim-Teo (2008) identified that the greatest challenge of a secondary school mathematics teacher is to raise their students' view from Level 1 (which is a purely visually driven mode) to "one that focuses on their geometrical properties" (Leong & Lim-Teo, 2008, p. 121). We were interested to know: How do experienced and competent teachers conduct geometry lessons in Singapore mathematics classrooms?

Other than the various generic pedagogical principles outlined in the secondary mathematics syllabus document, the Singapore Ministry of Education (MOE) (2012) does not prescribe precise delivery methods that teachers should adopt for their classroom instruction. However, the syllabus documents contain a list of learning experience statements (which are phrased as "Students should have opportunities to …") parallel to the syllabus content to be covered. A segment of the geometry syllabus document for Secondary Three Express course of study is shown in Fig. 8.1. The left-hand column delineates the content to be covered during the lessons while the right-hand column contains the learning experience statements.

The learning experience statements in the right-hand column of Fig. 8.1 highlight the processes learners need to experience in acquiring the corresponding content in the left-hand column. As illustrated in Fig. 8.1, the topic Angle Properties of Circles has two main emphases on the learning experience:

G3. Properties of circles	Students should have opportunities to:
3.1 symmetry properties of circles • Equal chords are equidistant from the centre • The perpendicular bisector of a chord passes through the centre • Tangents from an external point are equal in length • The line joining an external point to the centre of the circle bisects the angle between the tangents 3.2 angle properties of circles • Angle in a semicircle is a right angle • Angle between tangent and radius of a circle is a right angle • Angle at the centre is twice the angle at the circumference • Angles in the same segment are equal • Angles in opposite segments are supplementary	(a) Use paper folding to visualise symmetric properties of circles, e.g. the perpendicular bisector of a chord passes through the centre. (b) Use GSP or other dynamic geometry software to explore the properties of circles, and use geometrical terms correctly for effective communication.

Fig. 8.1 An extract of part of the syllabus content for Secondary Three Geometry (MOE, 2012)

1. The opportunity for students to experience geometry through manual activities such as paper folding, and technology such as the use of a DG software (e.g. Geometers' Sketchpad or GSP) to discover geometrical properties; and
2. The opportunity for students to use correct mathematical terms in geometry for effective communication. As these are general guidelines, the actual activities are not specified here and are left for teachers to interpret and enact in the classroom. Thus, teachers are faced with enactment decisions, especially when they see the need to fill in the "gaps" in order to enact the lessons (Kim & Atanga, 2013).

It was also interesting to note that the learning experience column of the syllabus document in Fig. 8.1 suggests the use of technology, and education research seems to suggest that teachers are generally resistant to the use of technology for various reasons (Polly, 2014). Our combined classroom experience also seems to suggest that some "experienced" teachers might not be very receptive to the use of technology for mathematics classroom instruction. Thus, we were excited to observe how teachers enact geometry lessons based on the newly introduced learning experience which suggests the use of technology as part of teaching and learning.

8.3 The Case Study

8.3.1 Method

The teacher in our study is Teacher 5 in her early 50s. She met the criteria of an "experienced and competent teacher" as she had more than five years of teaching mathematics experience for a course of study, in this case the Express course. In addition, the local education community and her school leaders also recognised her as a good mathematics teacher. She is a Lead mathematics teacher, one who is entrusted with the responsibility to develop fellow teachers in classroom practice. At the time of our study, she had been teaching mathematics in Singapore schools for more than 20 years, of which 15 years were in the school where we conducted the study.

The class that Teacher 5 taught was Secondary Three in the Express course of study. It had 14 boys and 28 girls. Teacher 5 described the class as a highly motivated group of students who took interest in learning mathematics actively. In addition to doing the core mathematics subject, known as Elementary Mathematics in Singapore, the students were also reading Additional Mathematics, a more advanced mathematics subject offered to higher ability students at the secondary level. During the interview, Teacher 5 commented that she had used various innovative approaches in engaging the students from this class. In designing her lessons for the class, she was mindful that her students needed a more rigorous treatment of mathematics in preparation for Additional Mathematics.

The sub-topic of geometry that Teacher 5 taught, and which is the focus of the case study described here, is Angle Properties of Circles in Elementary Mathematics. This sub-topic, as shown in section 3.2 of Fig. 8.1, covered four main properties:

- (Property 1) Angle at the centre of a circle is twice the angle at the circumference. (P1)
- (Property 2) Angle in a semicircle is a right angle. (P2)
- (Property 3) Angles in the same segment are equal. (P3)
- (Property 4) Angles in opposite segments are supplementary (add up to 180 degrees). (P4)

Teacher 5 completed teaching this sub-topic in three one-hour lessons. The first lesson was an introduction to the above four angle properties of a circle. Following which, she engaged her students in solving typical geometry problems and writing of short proofs in the second and third lessons. What captured our attention about her teaching was her selection of the instructional methods that she used during her introduction of the angle properties of the circles in the first of the three lessons. Avoiding the two extremes of totally using deductive approach or intuitive approach, she used a good mix of strategies by tapping on both approaches. She engaged her students to “discover” the geometrical properties of circles through the use of DG. This was followed by application of the “discovered” properties to do mathematical tasks of varying cognitive demand. In the lesson closure, she consolidated the lesson using a more deductive approach, showing the close connection across the four geometrical properties. The following sections detail and discuss the lesson.

8.3.2 Data

8.3.2.1 Lesson Observation and Video Analysis

A researcher sat throughout all the three lessons that Teacher 5 used to teach the sub-topic Angle Properties of Circles. The lessons were video-recorded using the Complementary Accounts Methodology first proposed by Clarke (1998, 2001). You may refer to Chapter 2 for details. The teacher's exposition and the teacher's conversation with students during the three lessons were transcribed. At the end of each lesson, the teacher and the focused students were interviewed to triangulate the data collected through the video-recordings of the lesson. The interviews were audio-recorded and transcribed.

8.3.2.2 Instructional Material Used by the Teacher

It is a common practice for mathematics teachers in Singapore to design their own instructional materials based on existing teaching resources available for the teachers and students. Teacher 5 used a variety of resources for her teaching: (1) she developed her own instructional material to supplement her teaching; and (2) she selected a variety of questions from various textbooks. She designed four exploratory activity worksheets to scaffold students' discovery of the above four geometrical properties of a circle.

Teacher 5 designed one "exploratory activity" worksheet to correspond to each of the four angle properties of circles in this sub-topic. A sample of the worksheet for Property (P1) is shown in Fig. 8.2. Each worksheet consists of three portions:

(A) Instruction to explore the property using a DG software (exemplified by instructions Steps 1 and 2 below);
(B) Instruction to guide the students to discover the properties and to complete the statement; (exemplified by instruction Step 3 and the boxed statement for student to complete); and
(C) Three practice questions which involve direct application of the discovered results in (B) (exemplified by instruction Step 4 and the questions that follow).

As illustrated in Worksheet 2 above, Teacher 5 provided very clear instructions for her students on the steps to access the online version of the worksheet (in the right column of Fig. 8.2). Steps 1 and 2 in the worksheet provided the students the procedure to access the online sketchpad operating in a DG environment. In the online sketchpad, students were provided the opportunity to click and drag to observe the geometrical property. The providence of the opportunity to allow users to click and drag in order to observe the invariant property (the angle at the centre is twice the angle at the circumference) amidst an arbitrary variation of conditions (varying the sizes of the circles, the point on the circle, etc.). Step 3 brought the users back to the focus of this worksheet to discover the relation between the angle at the centre of a circle and that at the circumference of the circle.

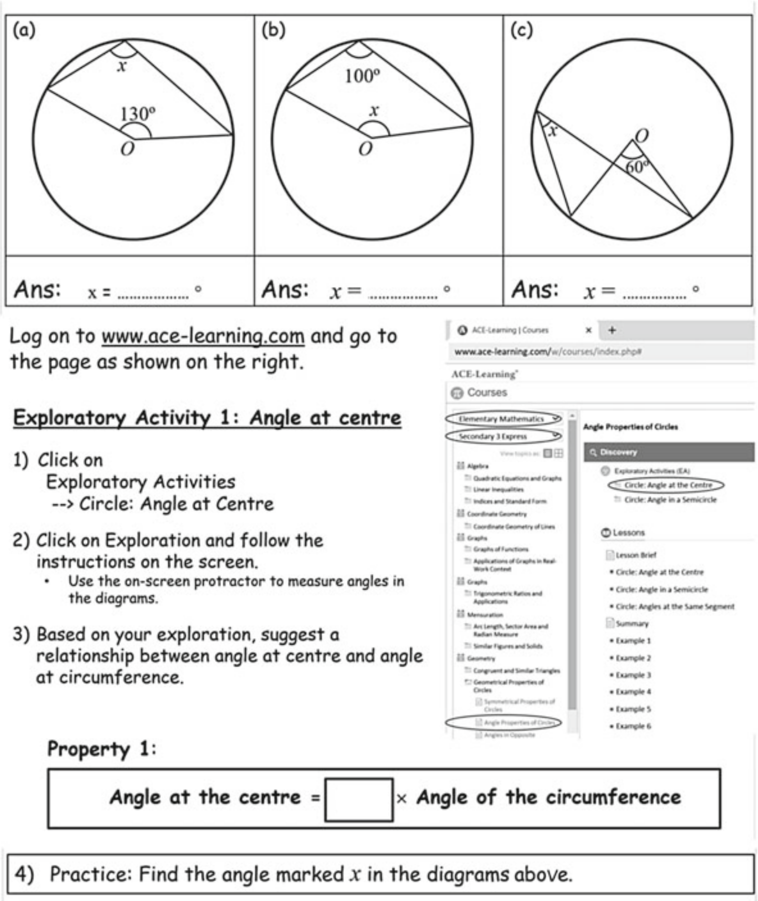

Fig. 8.2 Sample of a worksheet activity designed by Teacher 5

Step 4 of the worksheet immediately provided an immediate consolidation of the concepts by engaging the students to apply this property to three basic questions. These questions focus on an easy application of the property, checking the students' sound understanding (or lack) of the property introduced in the worksheet. The same four-step structure (as summarised in Fig. 8.3) applied for the other three worksheets for this sub-topic. Teacher 5 confirmed that this was the general structure that she would use to teach the other sub-topics of geometry in the syllabus.

In addition to the worksheets, Teacher 5 compiled a set of geometry questions from both the textbook used by the school, and questions from other textbooks and workbooks (not adopted by the school). We also noted that Teacher 5 did not use the textbook for direct classroom instruction. Teacher 5 confirmed that the textbook mainly served as the source of challenging mathematics questions and useful ideas for classroom instructions.

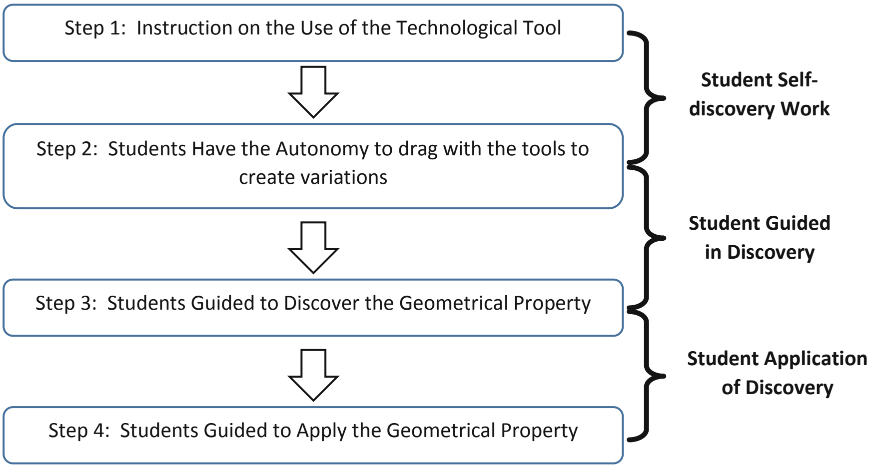

Fig. 8.3 The sequence of introducing a geometrical property used by Teacher 5

8.3.3 Analysis of the Data

The transcripts of the lessons and the teacher interview were studied in conjunction with the video-recordings of Teacher 5's lessons. In this chapter, as discussed in the preceding sections, we focus on the first of the three lessons.

The first lesson could be divided into three main segments:

(1) Lesson Introduction [00:00 to 00:11];
(2) Exploratory Activity [00:11 to 00:53]; and
(3) Lesson closure [00:53 to 00:58].

8.3.3.1 Lesson Introduction

In the Lesson Introduction, which lasted about eight minutes, Teacher 5 placed much emphasis on student understanding of the mathematical terms ("chord of a circle", which has been covered "the last time"). In the Lesson Introduction segment, Teacher 5 took the lead in providing the facilitation to get the students to focus on the concepts involved in this lesson.

Part of the transcript of the Lesson Introduction that follows illustrates our observation. We use the abbreviation T to represent Teacher 5 and S to represent (any) student (without identifying the student engaged in the discussion) in the class who participated in the discourse. Words appearing in square brackets [ ] refer to extrapolation or an interpretation of Teacher 5's speech and those in round brackets ( ) refer to the actions she performed in the lesson while she was involved in that part of the conversation.

Abridged transcript	Commentary
T[1]: … The last time what we did was [to study the properties] of a circle, remember, chords of a circle (began by drawing chords of circle on the whiteboard)… so today's objective is to find or use or understand angle properties (wrote "Objectives" and "Angle properties of a circle" on the whiteboard) :	Teacher introduced the lesson by building on their prior knowledge about the chord of the circle. This will be used in describing the angle on the circumference in today's sub-topic
T[2]: Yeah, chord properties we will revise tomorrow, along with this, so we have mixed questions [i.e. questions that require the combination of several sets of properties to solve]. But today we focus only on angle properties…	Teacher highlighted the focus of today's lesson
T[3]: You'd come across a circle, there's a circle, centre (drew a circle with a dot in the centre). You will see an angle like this – the two chords, meeting at one point on the circumference, ok. So, I have a circle with a centre here. This angle, what is special about this angle? :	Teacher demonstrated the angle that was formed by two chords meeting at a point on the circle
T[4]: On the circumference, and the angle that is formed on the circumference here … What is the arc subtending this angle (teacher wrote arc subtending). The word you see will be subtending (teacher underlined the word 'subtending'), this angle means?	Teacher highlighted the language associated with the angle subtended by the arc
T[5]: This angle is facing you in loose terms ah, if this is the angle formed at the circumference, this is called the arc (teacher drew Fig. 8.4), which is subtending the angle, right? It's facing there	Teacher introduced another way of associating the arc with the angle on the circumference

Developing in students the visual mode of the geometrical concept of an angle subtended by an arc was the highlight of the Lesson Introduction, with the teacher emphasis on the visual (Fig. 8.4). However, we also note that Teacher 5 did not merely establish a purely visually driven mode of the concept in her students. Instead, by using the visual mode of the concepts, she built up the defining characteristics of the geometrical concepts. The first concept: "the angle subtended by an arc" of a circle was built on the concept of the space formed by two chords which intersect on the circumference of a circle T[3]. Thus, the recap section at the beginning of the introduction section of the first lesson was selective on the chord in a circle T[1]; revision of the other properties of chords of a circle which were not relevant to the concept development in this lesson was shelved for subsequent lessons, as mentioned by Teacher 5 in T[2].

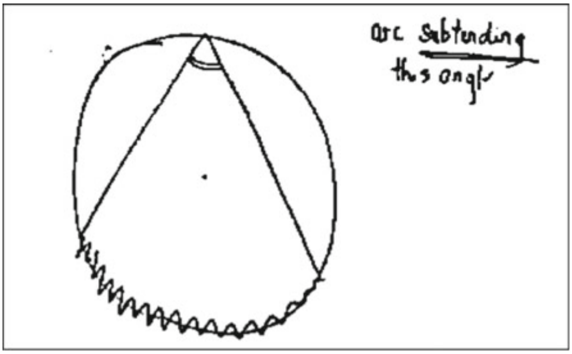

Fig. 8.4 Teacher 5's diagram on the whiteboard of angle subtended on the circumference

In the Lesson Introduction, we observed an interesting feature of Teacher 5's lesson: Teacher 5 was focused on getting the students to recognise the concepts and the precision of the terms used. She did not simply rest on students having seen the required angle, but each underlined term in "Angle subtended by an arc on the circumference" in relation to its visual representation. She skillfully switched between the geometrical properties and its visual representation to enable her students to link the concept and developing its concept image. We summarise this in Fig. 8.5.

The notion of the "arc" next served as an anchor to the next related concept of angle at the centre of the circle. Here, we observed that Teacher 5 repeatedly emphasised the word "arc" in preparation of the next concept of "angle at the centre of the circle". This is evident from the following transcript.

T: In exactly the same way, you will have another angle which will be formed at the centre (drew dotted line for angle at the centre, Fig. 8.6). Do you see both these angles, a (at the circumference) and b (at the centre), both are subtended by the same arc, correct? Both subtended by the same arc. Can? Because they are both made by this arc, so endpoints of this two angles are such that, they are made by this arc. Clear? So this is called angle subtended at the…

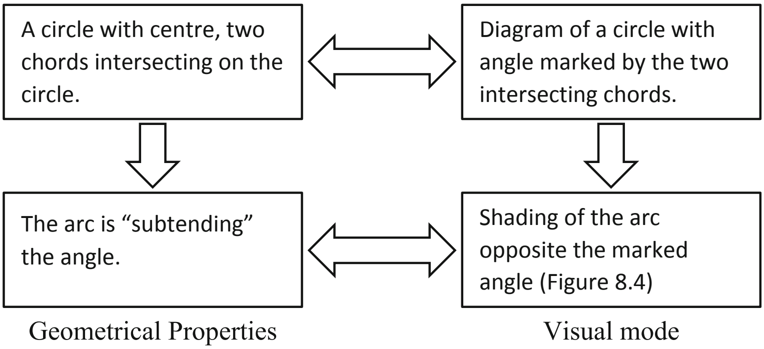

Fig. 8.5 Angle on a circle in both modes of using visuals and geometrical properties

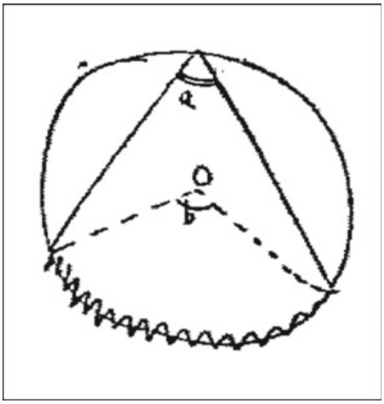

Fig. 8.6 Teacher 5 used the "arc" as the anchor between the two concepts of the angle at the centre and the angle on the circumference of a circle

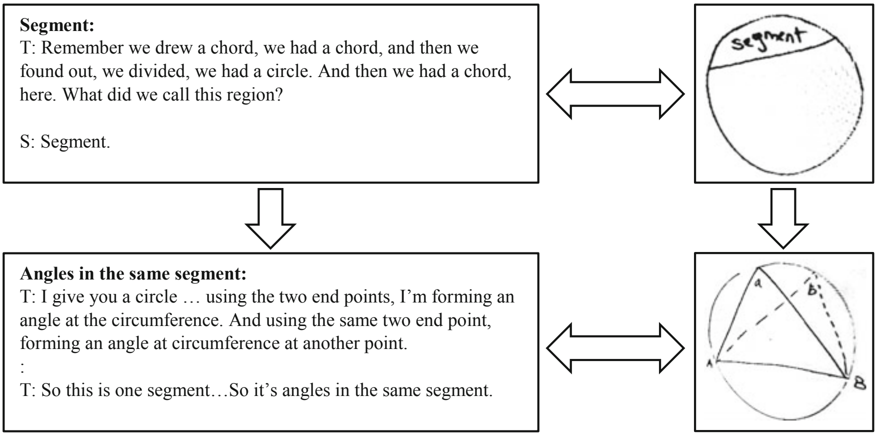

Fig. 8.7 Teacher 5's use of the same mixed mode of visuals and geometrical properties to introduce the concept of angles in the same segment

Here was the transition from the concept of a chord to an arc of a circle, which is the anchor concept for both angle at the centre and the angle at the circumference of a circle. The above was used with reference to Fig. 8.6.

A similar trend was observed when Teacher 5 next moved on to introduce the concept of angles in the same segment, as summarised in Fig. 8.7.

8.3.3.2 Exploratory Activity

In the exploratory activity segment [00:11] to [00:53] (which lasted 42 min), the students were engaged to work in pairs to discover the four angle properties of circles (P1 to P4) through the use of a DG software. Teacher 5 had designed four exploratory activity worksheets to be used in conjunction for exploration in this part of the lesson

(The sample activity Worksheet 2 was shown in Fig. 8.2). Each scaffolding worksheet consisted of three application problems on the related geometrical property. The three problems involved *immediate application* of the property and were of increasing level of complexity. We identified three key phases in the Main Lesson segment of the lesson:

Phase 1: Students' own exploratory work. Teacher 5 managed the students' progress of the discovery activity and addressed the individual students' concern (see below).

Abridged transcript	Commentary
T: OK, so, take the protractor and align it here, and how much is this angle? T: (to another student) Are you ok now? XXX T: (back to the first student) This [angle shown on the computer screen] is 140. So, that will be the angle at the centre S: But just now [my friend, i.e. another student] got 132 [on the screen] T: It's a different [angle, because these angles are] random[ly generated] [00:24:30] to [00:24:46]	Teacher 5 went to the individual students to get them to verify the angles that they had obtained on the screen, and to address the confusion that all the students got different angles as these figures were randomly generated from the system

Phase 2: Students' application of their discovery to solve three related problems. Here, an unexpected response from the student prompted Teacher 5 to address the students. Teacher 5 had wanted her students to apply the properties that they had discovered earlier; some students used the DG to construct the exact dimension of the diagram in the problems in order to determine the unknown.

Abridged transcript	Commentary
T: OK, look up here everyone. I think I see a few of you unable to understand the first part of the worksheet [i.e. the three practice questions printed on the first page of the activity worksheet]… Now, I don't want you to, for these three questions, I don't want you to use the diagram in the [*name of vendor's software*]. You know angle at the centre is two times the angle at circumference…	Teacher 5 brought across the objective of the questions is not to construct the exact diagram in the worksheet using the dynamic geometry software in order to find the unknown angle, but to apply the geometrical properties they had just discovered

At this phase, Teacher 5 consciously facilitated her students to link their earlier discovery to the associated geometrical properties when students appeared to have difficulty in solving the immediate application problems at Step 2:

Abridged transcript	Commentary
T: See this angle, at the centre. Same angle at the circumference. The arc is the same. So this is 60 [degrees], this one should be half, half. This is angle at the centre, this is angle at the circumference. Same like this one. The two angles, the one is up, so this is angle at the centre. 60, same two end points, giving you angle at the circumference. So double…	Teacher 5 consciously brought in the visuals to establish the similarity with the geometrical properties the student had discovered in the earlier activity

Phase 3: Teacher's explanation of the solution of the three practice questions. In this phase, Teacher 5 continued facilitating her students to work towards the answer by consciously relating the application problems to the geometrical properties that they had earlier discovered.

Abridged transcript	Commentary
T: This is the angle – this angle, this angle the one that is shaded is actually ok yes correct it is 2x. This is x, and that is 2x, remember this is 130 …. That is half, so whatever is your answer, divide [it] by two, you get the answer. How about this one? OK let's do it together : S: I don't understand T: See this angle at the centre? Same, angle at the circumference. The arc is the same. So this is …	Teacher 5 explained the first question in detail using the geometrical property, and invited the whole class to solve the next question Teacher 5 addressed the students' difficulty during the lesson

To us, what was the most impressive was that Teacher 5 facilitated her students to identify the meaning of the terms used to describe the geometrical properties with the associated geometrical diagram in emphasising the importance. In addition to identifying the "equal angles" in the geometrical statement that "Angles in the same segment are equal", she created the "segment" in the aforementioned geometrical statement (illustrated in the transcript below).

Abridged transcript	Commentary
T: Segment is the [region in the circle partitioned by] the chord, one of the arc, and one of the centre – er the one at the chord. This is a chord. This angle here, this angle here, they are equal… they are both in the same segment. S: How do I know it's the segment? T: Basically we are looking at its two points S: The same area on the … T: Yeah, on the segment, both going the same side…	Teacher 5 added in an additional chord to the geometrical diagram to show how the two angles initially called by angles subtended by the same arc to explain that indeed they are angles in the same segment

A part of the above diagrams is reproduced in Fig. 8.8.

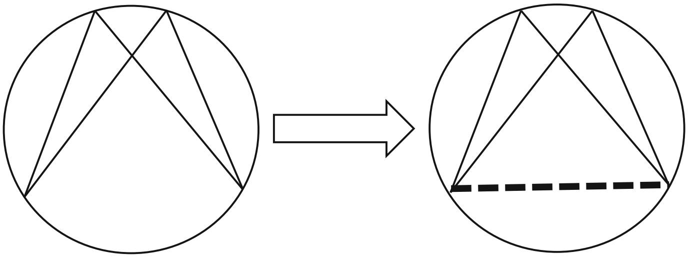

Fig. 8.8 A copy of the whiteboard writing by Teacher 5 who emphasised in addition to the two angles being equal, also stressed on the “same segments” that the two angles were located

Studies have shown that the pure constructivist approach of discovery learning on its own, which usually emphasises an extensive search of knowledge through problem solving, has a limitation in enhancing the learners’ memory, and may in fact cause less learning (Rittle-Johnson, 2006). The other aspect of guiding the learners to pay attention to key knowledge that they have acquired is equally important to improve their understanding and ability to apply what they have learned (Kirschner, Sweller, & Clark, 2006). Here Teacher 5 has illustrated this very clearly as she skillfully incorporated three practice questions immediately after the scaffolding for discovering each geometrical property to focus the students’ attention on the key geometrical properties.

Teacher 5 adopted a consistent structure in teaching each geometrical property to her students consisting of the three phases which were outlined above. She adopted a partial constructivist approach in getting her students to discover the properties through DG activity, with teacher intervention in helping students to focus on the key knowledge. The emphasis here is on students’ *understanding* of the properties with the proof of the properties shelved to a later time. The proof of the properties was deferred; Teacher 5 emphasised much on *discovery* and *understanding and application* at this stage instead of *deductive proof* of the properties. We could summarise Teacher 5’s instruction as consisting of the following cycle (Fig. 8.9) in getting her students to learn the four geometrical properties.

Kaur et al. (2019) proposed that an instructional core drives the teaching and learning of mathematics in the lessons of experienced and competent teachers in the research, which she called the DNA of mathematics lessons. She observed that the instructional core comprises a D-S-R (Development—Student Work—Review of Student Work) cycle. In the lesson of Teacher 5, we find that the cycle of instruction, in Fig. 8.9, used by Teacher 5 followed the D-S-R cycle. The Development phase in the geometry lessons we observed was the student discovery phase, in which Teacher 5’s students had the opportunity to discover the geometrical properties. The Student Work phase we observed was their application of the newly discovered geometrical results to solve three mathematical tasks. Following this, the Review phase consisted of the teacher giving a direct exposition, with student participation, of the geometrical results in relation to the mathematical tasks just completed.

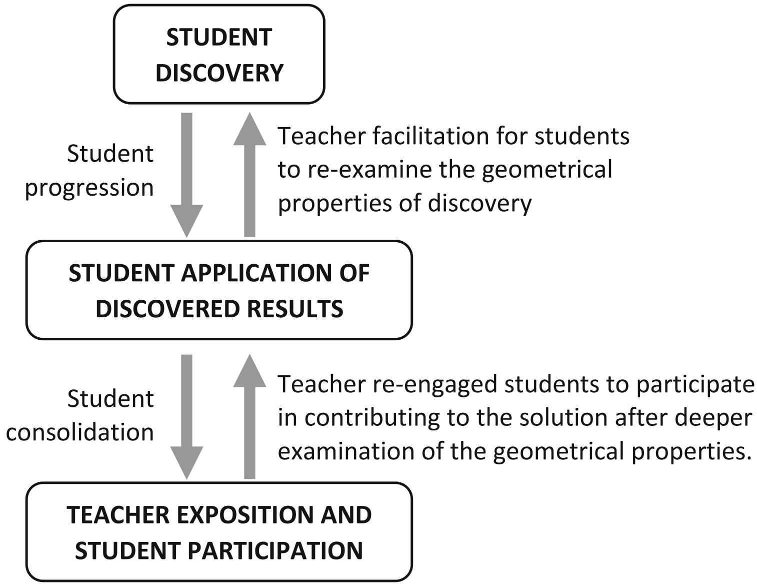

Fig. 8.9 Cycle of instruction in Teacher 5's lesson in developing each of the four angle properties of a circle

Though at times the D-S-R cycle could be teacher-centric, as the teacher may develop the lesson through demonstrations and explanations, Teacher 5's lessons show that the D-S-R structure is representative of both teacher-centric and student-focused developments. In student-focused developments the role of the teacher was then to serve as a guide to "value-add" to the student discovery by facilitating them to focus on the attributes of the geometrical properties explored by the students. The cycle of instruction, in Fig. 8.9, also depicts the development and consolidation phases of lessons as detailed in Chapter 5. Student discovery takes place during the *Development* phase. The application of discovered results by students accompanied by teacher exposition with inputs from students when reviewing student work takes place during the *Consolidation* phase. This phase aids in deepening conceptual knowledge of the students.

How important was this "discovery" part of the lesson to the Teacher 5? We transcribed our interview with her. In particular, when asked about her focus for the lessons during the teacher interview segment, considering both content and non-content goals, Teacher 5 highlighted that her [first] goal was for her students to discover the [geometrical] rules. The importance of engaging students to explore and self-discover was her main concern. This was reflected in the 20-minute interview with Teacher 5 during which she used the words "explore" and "discover" a total of 12 times. Part of the transcript is shown below.

> But for this particular topic [i.e. Geometry], I usually bring them to the computer lab to get them to explore first. So, my goal initially is to get them to go through the process of

> exploration. Self-discovery of the rules. So it's more of a deductive [should be "inductive"] approach. Because you see a few cases, in terms of how the properties play out. And then based on that, they are able to consolidate, which, so they're able to summarise, or conclude, that the relation between the angles is as what is being displayed.

The main objective of this first lesson was to engage her students in exploration and discovery of the geometrical properties using DG software. The worksheets that she had designed earlier served to provide the scaffold to serve this objective. People using a raft to cross the river will eventually discard the raft after they have crossed the river successfully. In the same way, Jones (2000) asserted that the first stage of engaging students in mathematical exploration, seen by mathematicians as lacking mathematical precision, is a crucial first step to mathematical explanations that will lead the students to next transcend such imprecise discovery (in the form of the software environment) to deductive geometric reasoning. From this lesson conducted by Teacher 5, what followed the discovery activity was not immediately followed by precise deductive mathematical proofs, but by three questions of immediate application of the geometry concepts. This ensured that her students had truly understood the geometrical properties just "discovered" by the students themselves. This was evident from the following part of the teacher talk.

> I don't want you to play with the diagram and match it with these three online [i.e. create the geometrical figures using the softwares]. So, it should be very fast, the page one. What about the second one, the same way. Just observe the relationship, then move to the next.

Deductive reasoning and proofs of the geometrical properties did not immediately follow the segment after the students' discovery of these properties. Rather, to ensure that her students had truly understood these properties was the most immediate activity after that.

8.3.3.3 Lesson Closure

Teacher 5's lesson closure for the first lesson was also of interest to us. Instead of merely reiterating the four main geometrical properties that had been covered in the lesson, she reiterated these four properties by using a semi-rigorous deductive approach to show the connectedness of the four geometrical properties. After restating Property (P1) (angle at the centre is twice the angle at the circumference) without proof, Teacher 5 demonstrated how Property (P3) (angle in the same segment are equal) is in fact a special case of Property (P1).

Teacher 5 started with a special angle of 100° at the centre of the circle and that of 50° at the circumference for two special cases. By deleting the angle at the centre, Teacher 5 showed that the two angles on the circumference of a circle are equal (to 50°). The sequence of what was shown on the whiteboard is presented in Fig. 8.10. She skillfully demonstrated that Property (P3) is indeed a special case of Property (P1).

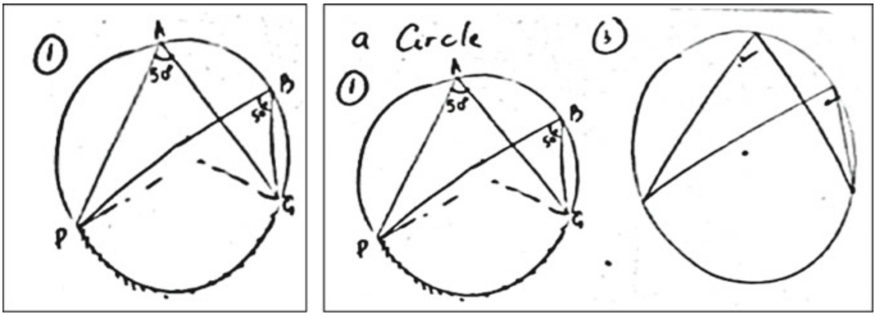

Fig. 8.10 The sequence of two drawings used by Teacher 5 to demonstrate that Property (P3) is a special case of Property (P1)

Abridged transcript	Commentary
T: All these [four points in Fig. 8.10] have the same two end points…. Look for these two points, the common points, see whether they are going to the centre, or the same two endpoints, on the same side, going towards the circumference, then they are connected. So I see these two endpoints here. PQ. I go to the centre, I see angle 100 [degrees]. If I go to the circumference, it will be how much? 50. OK so this is what we have seen. Same two points, going again to the circumference, it's 50	Teacher 5 referred to the left drawing of Fig. 8.10 to reinforce that both angles on the circumference are equal to 50 by applying Property (P1) twice
T: If I remove this [angle at the centre of the circle], do you realise that it looks like property number three? If I don't have the angle at the centre, basically you have again the same two endpoints,… Do you see the similarity?	Teacher 5 erased off the angle of the centre and convinced to students that both angles at the circumference are equal (i.e. Property (P3))

In a similar approach, Teacher 5 demonstrated that Property (P2) is also a special case of Property (P1) in Fig. 8.11.

Abridged transcript	Commentary
T: Property number 2, what was the property 2? … You have a diameter. And what did we find? When you see a diameter, [there are two] endpoints [on the two sides of the circle]. When you go to the circumference (teacher pointing to Fig. 8.11 on the whiteboard), you get a 90 degree, you get a 90 degree here, 90 here. This is also a special case of one. Have you realised that? This one, number 2, is a special case of 1 (Bell rang at this juncture)	Teacher 5 led her students to realise that the diameter can also be seen as having two points on the circle and subtending an angle of 180° at the centre of circle. Here the key message appeared to be that Property (P2) is also a special case of Property (P1)

(continued)

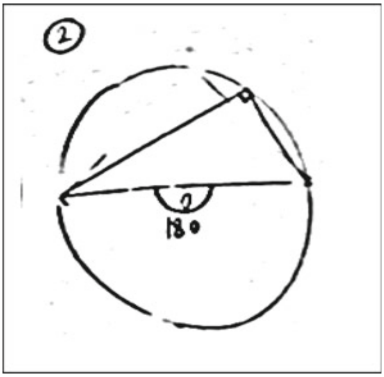

Fig. 8.11 The drawing used by Teacher 5 to demonstrate that Property (P2) is a special case of Property (P1)

(continued)

Abridged transcript	Commentary
T: OK, and [property] number 4, you see a quadrilateral, which I think most of you – in fact all of you are able to see the connections $a + b$ equals to, how much? 180 degrees. Also $c + d$ equals 180 degrees	As the lesson had ended, Teacher 5 did not continue to demonstrate that Property (P4) is also a corollary of Property (P1)

When asked what was an ambitious part of the three lessons on teaching this sub-topic on angle properties of a circle, she asserted that it was establishing the relation across the four angle properties of a circle. Her intention was to start off the second lesson by challenging them to derive a "proof" of Property (P4) from Property (P1). This was left as homework for her students as she ran out of time in the first lesson.

Abridged transcript of teacher interview	Commentary
T: Today actually frankly, I was not going for ambitious things, it would only come in tomorrow. Because today was getting them to just explore, understand the four rules. I – my ambitious part would only be that I just left it to them, ok the instructions are there, do it, so it's the first time, ok, no it's not the first time actually	Teacher 5 felt that the ambitious part of the lesson was to leave it for the students to discover the four geometrical properties of the circle through the activity worksheets
T: Actually I didn't use it, as a, as a, carry over you know, as a special case for this one. I gave it a separate activity. I gave it as a separate activity. And I try to later at the end, bring it together and see, this is the, mother property, and these two, you know, it just follows. So tomorrow, maybe I will even start off by asking, my ambitious part would be, why is it, can you just show me, can you just prove it, so we will start by proving this…	Teacher 5 planned for the students' discovery of the properties through inductive means and to introduce the proofs at the end of the lesson to show that the four properties could effectively be reduced to one property. She left the proof of Property (P4) to her students as homework

Teacher 5 was mindful of establishing the *connections* across the properties within this introductory lesson. As an after-note of the teacher interview, we were curious about how Property (P1) was left as the intuitive level without attempting to *prove* that the angle at the centre is twice the angle at the circumference deductively. Teacher 5 confirmed that as the proof of Property (P1) involves properties of triangles, she deliberately chose not to expand this proof for fear of distracting the students; her objective was to show to students the connections across these properties.

In lesson closure, she introduced two geometrical properties through a deeper approach of using deductive reasoning and challenged her students to derive Property (P4) as homework, and to attempt as many of the homework problems as possible before the second lesson. She obviously demonstrated lesson closure without closure, which was effective in her case compared to having a lesson with a neat closure with all issues resolved.

8.4 Discussion and Conclusion

It is apparent that Teacher 5 built a positive classroom culture by enabling her students to discover their own learning and equipped them with the crucial mathematical tools (i.e. the correct mathematical terms) for their discovery. Her lessons were well prepared with appropriate sequencing of appropriate activities, fully mindful of her students' capacity. To mathematics educators, the lesson was most impressive because she did not simply get students to "apply memorised procedures" (Schoenfeld, 2018, p. 499), but offers the conceptual richness of the mathematical concepts.

The general curriculum approach in Singapore (including mathematics) adopts Bruner's spiral approach, in that concepts and skills are re-visited iteratively at each higher level in order to ensure a coherent overall curriculum and a deep learning of the mathematical concepts. The various topics of geometry, as in the topics in the other major strands of mathematics, are distributed over all the years of secondary school mathematics education. The content in each level builds on the earlier levels as a foundation, which in turn serves as the foundation for the next higher level. Within this first lesson to the sub-topic Angle Properties of a Circle, we observed Teacher 5's attempt to use a "spiralling" that occurs within this lesson in introducing students to the geometrical properties by exploration, and in concluding the lesson by showing a deeper connection across these geometrical properties by a deductive reasoning approach.

It was interesting to observe that Teacher 5's lesson enactment resonates with van Hiele's levels of learning of geometry. She started with the Level 1 (Visual) by associating visuals with each geometrical entity during the lesson introduction (Figs. 8.4, 8.5, 8.6, and 8.7). This was followed by leading the students to Level 2 (Analysis) through engaging them in self-discovery activity of the geometrical properties using ICT. Immediately following this is students' direct application of the newly discovered properties to solve three problems for each problem, which corresponds to Level 3 (Relational). During lesson closure, she summarised the geometrical properties covered in this lesson using a deductive approach highlighting the relation between the geometrical properties—this corresponds very closely to Level 4 (Deduction). This forms a good starting point for her students in the next two geometry lessons in which Teacher 5 would emphasise deductive proofs. Although Teacher 5 did not explicitly articulate her consideration of van Hiele's levels of learning of geometry in her discussion, it is clear that this sequence of teaching was ingrained in her as she indicated that this is her "general approach" in teaching geometry.

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