Mathematics Education – An Asian Perspective

Berinderjeet Kaur
Yew Hoong Leong *Editors*

Mathematics Instructional Practices in Singapore Secondary Schools

Springer

Mathematics Education – An Asian Perspective

Series Editors

Mathematics Education – An Asian Perspective facilitates high quality publications on rigorous aspects of mathematics education in Asia. This will be achieved by producing thematic books that capture knowledge and practices on mathematics education in Asia from both the insider and outsider perspectives. The series helps to establish a much needed Asian perspective to mathematics education research in the international landscape. Over the last decade or so, several international comparative studies have shed light on systems of schooling that were otherwise not very much sought after. Several educational systems in Asia, in particular East Asia have consistently produced stellar outcomes for mathematics in both TIMSS and PISA despite the fact that both studies measure achievement in mathematics in distinct ways that are very much orthogonal to each other, while other Asian systems have not been able to replicate the same level of success. Though one may occasionally chance upon a publication on some aspect of mathematics education in Asia, there appears to be in general a dearth of publications on mathematics education in Asian countries from the perspectives of scholars from Asia. Hence it is apparent that there is a gap in the availability of knowledge on mathematics education from the region in the international space.

This series has a wide scope with emphasis on relevancy and timeliness. It encompasses the general trends in educational research such as theory, practice and policy. Books in the series are thematic and focus both on macro and micro topics. An example of a themed book on a macro topic could be one on "School mathematic curricula – An Asian perspective" while a themed book on a micro topic could be one on "The pedagogy of 'simultaneous equations' in Asian classrooms".

Researchers interested in authoring or editing a book for the series can contact the series editors:

Berinderjeet Kaur: berinderjeet.kaur@nie.edu.sg Catherine Vistro-Yu: cvistro-yu@ateneo.edu

Book proposals for this series may also be submitted to Editor: Grace Ma Email: grace.ma@springer.com

More information about this series at http://www.springer.com/series/11679

Berinderjeet Kaur · Yew Hoong Leong
Editors

Mathematics Instructional Practices in Singapore Secondary Schools

Springer

Editors
Berinderjeet Kaur
National Institute of Education
Nanyang Technological University
Singapore, Singapore

Yew Hoong Leong
National Institute of Education
Nanyang Technological University
Singapore, Singapore

ISSN 2366-0155 ISSN 2366-0163 (electronic)
Mathematics Education – An Asian Perspective
ISBN 978-981-15-8955-3 ISBN 978-981-15-8956-0 (eBook)
https://doi.org/10.1007/978-981-15-8956-0

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Series Editors' Introduction

The sixth volume of the book series Mathematics Education: An Asian Perspective, entitled, *Mathematics Instructional Practices in Singapore Secondary Schools*, edited by Berinderjeet Kaur and Yew Hoong Leong, is an outcome of a research project. The project examined how mathematics teachers enacted the school mathematics curriculum in Singapore secondary schools.

How have mathematics teachers in Singapore secondary schools been narrowing the gap between high and low performers, and how has Singapore managed to steadily improve the performance of their lower performing students, as evidenced from recent PISA scores? This volume provides much food for thought by bringing to the fore the "how" and "why" with detailed explanations, in particular how experienced and competent teachers in Singapore secondary schools engage their students across the spectrum of academically low performing students to the high ability ones.

As noted by Professor Hung, in the foreword, written in a clear and engaging style, the chapters embedded in this volume describe and discuss the "DNA" of mathematics classrooms as practised in Singapore. The authors show how this "DNA" of sound mathematics instruction occurs, with instructional scaffolding within the work assignments given to students to engage in. On the surface, these instructional "DNA sequences" appear as "drill and practice", but below the surface, these "DNA sequences" shows a sophisticated interweaving of mathematics problems/concepts which build upon each other, and which scaffolds conceptual understanding, yet achieving procedural fluency. Not all students are able to master these sequences as quickly as others, and this volume also illustrates and discusses how teachers teaching students with lower achievement scores can be motivated by teachers who give them positive confidence attitudes in tackling mathematics concepts and problem solving.

There is no doubt that the volume provides a rich source of information and analyses from a scholarly insider's view. This volume adds to the availability of knowledge about mathematics education, in a high performing nation in Asia for the international audience. We hope researchers will find it a valuable resource, and for all, an enjoyable read.

Singapore, Singapore Berinderjeet Kaur
Quezon City, Philippines Catherine Vistro-Yu

Foreword

Singapore Mathematics is a well-known brand. Singapore Mathematics has been exported to countries such as the United States, the United Kingdom, Canada, Israel, and the Philippines. Thus, it comes as no surprise that this is a long awaited volume on instructional practices in mathematics classrooms in Singapore. Singapore has consistently performed well on TIMSS and PISA benchmarks whether in mathematics or in problem solving over the last decade. However, these reports have largely described and discussed the "what" or state of affairs locally. They do not delve deeply into the "how" and "why" local students perform well instructionally.

How have our teachers been narrowing the gap between high and low performers, and how has Singapore managed to steadily improve the performance of our lower performing students, as evidenced from recent PISA scores? This volume aims to answer all these questions by bringing to the fore the "how" and "why" with detailed explanations, in particular how our experienced and competent teachers in instructional settings engage their students across the spectrum of academically low performing students to the high ability ones.

Perhaps the most pertinent question concerning Singapore Mathematics is: what are the "secrets" of good mathematics instruction? In this volume, the authors "unveil" these "secrets" by delving into detailed analysis of instructional moves by teachers and with students that are examined and documented within the enacted curriculum and not just the planned curriculum. Teachers are able to enact the planned curriculum in sophisticated moves amidst a classroom of students which are typically above a class size of 30. This is achieved by a combination of good grounding philosophies in conceptual understandings, good instructional moves made in classrooms that engage the learners, and well-designed instructional resources (both adopted within class time and also as homework).

Written in a clear and engaging style, the chapters embedded in this volume describe and discuss the "DNA" of mathematics classrooms as practised in Singapore. The authors show how this "DNA" of sound mathematics instruction occurs, with instructional scaffolding within the work assignments given to students to engage in. On the surface, these instructional "DNA sequences" appear as "drill and practice", but below the surface, these "DNA sequences" shows a sophisticated interweaving of mathematics problems/concepts which build upon each other, and which scaffolds conceptual understanding, yet achieving procedural fluency. Not all students are able to master these sequences as quickly as others, and this volume also illustrates and discusses how teachers teaching students with lower achievement scores can be motivated by teachers who give them positive confidence attitudes in tacking mathematics concepts and problem solving.

The Singapore system is renowned for its stellar performances by students in Mathematics. This volume not only contains the *Practice* that is found in classrooms, it shows the *Research* that goes alongside it. The Research-Practice Nexus is manifested by the teachers' abilities both in mathematics concepts, design for learning, and the enacted execution which involves assessment for learning feedback cycles to students. Teacher learning in professional development and research is an integrated part of the whole process. NIE, as the national teacher education and learning institution, plays a significant role in coordinating all of the above efforts, actualising the research-practice nexus. Singapore Mathematics is thus the orchestrated result of the cumulation of efforts by researchers, teachers, teacher leaders, and the organisations that make up the teaching and learning ecology in Singapore, and this volume brings together a representative team to provide a comprehensive review of Singapore Mathematics as it is currently practiced in schools.

Professor David Wei-Loong Hung
Dean, Office of Education Research
National Institute of Education, Nanyang
Technological University, Singapore, Singapore

Preface

The purpose of this preface is to provide some information overview that will help the readers better appreciate the coherence of this book. When you come across some terms while reading particular chapters that you are unfamiliar with, we advise that you first return to this preface—there is a high chance that you may get some immediate pointers here.

Throughout the book, there are references to the various courses of study at the secondary levels in Singapore: Integrated Programme (IP), Express, Normal (Academic) (N(A)), and Normal (Technical) (N(T)). Details of these streams can be found in Sect. 1.2.2 in Chap. 1. In this first chapter, there is a broad sketch of the Singapore mathematics curriculum. International readers would also be familiar with the pentagonal representation of the curriculum framework as shown in Fig. 1.2. A number of the subsequent chapters make references to this pentagonal diagram. In particular, each of Chaps. 4–8 focuses on one side of the pentagon—Concepts, Skills, Metacognition, Attitudes, Processes—respectively.

There are also repeated references in the chapters of Teacher X (where X is an integer between 1 and 30, inclusive). An overview of the mathematics topic and the duration of instruction for each of these teachers is found in Table 2.3 in Chap. 2. This is a chapter to come back to when the reader comes across cursory mentions about Phases of the project, the 30 experienced and competent teachers, the survey, etc. in the later chapters and wishes to locate these pieces of information or data within the overall methodological approach undertaken in the project.

We have sequenced the chapters in this book according to this flow of thought: Chap. 1 gives the international reader a broad overview of Singapore education system and the Singapore mathematics curriculum; Chap. 2 focuses on the project from which the rest of the findings in the subsequent chapter draw upon; Chap. 3 provides a summary of the findings of the project centring around "the instructional core"; Chaps. 4–8 report on classroom enactment of the Singapore mathematics curriculum along each of the "sides" of the pentagon representation; Chap. 9 focuses in particular on the "Mathematics Talk" of the teachers; although—as the title of the book indicates—the scope of this volume is on the instructional practices of teachers, we think it appropriate to include a snippet from students' perspective in Chap. 10. Chaps. 11–14 shift the object of inquiry to "instructional materials" as a way to study enactment. Chap. 15 addresses a topic which is of ongoing interest—the use of technology in teaching mathematics. Chaps. 16 and 17—the former from the "outsider" perspective and the latter looking from the "inside"—round up the volume by taking a more reflective stance as the authors seek to bring the findings of previous chapter together.

For a volume of this size, we understand that a typical reader might not read from cover to cover. So we provide here some guidelines for readers with different starting points of interest.

"I wish to know the most fundamental core of Singapore mathematics teachers on which they build other features of instructional practice": Start with Chap. 3 on the "instructional core"; you may then zoom-in to other characteristics of practice build around this core.

"I wish to know what matters most to Singapore mathematics teachers": Start by reading Chaps. 16 and 17. You may find specific areas of interest that will lead you to other chapters in the volume.

"I have never stepped into a Singapore mathematics classroom before. I wish to be able to envisage how teachers carry out their lessons". Start with the case studies reported in Chaps. 8, 13, or 14. Some specifics of these cases may link you to other features of the project.

Or, simply browse the content page and look for titles of the chapters that catch your attention. Start with these first—hopefully, you will be led from one chapter to the next naturally as your interest follows the coherent narrative in this book.

Enjoy the read!

Singapore, Singapore

Berinderjeet Kaur
Yew Hoong Leong

Acknowledgements

The research reported in this book draws on the project (OER 31/15BK) funded by the Education Research Funding Programme, National Institute of Education (NIE), Nanyang Technological University, Singapore. The research was carried out in accordance with the Institutional Review Board Approval (IRB-2016-03-024) granted by the Nanyang Technological Research Integrity and Ethics Office.

The authors of the chapters in this book who are members of the research team that carried out the project are indebted to all the teachers and students from secondary schools in Singapore who participated in the project. The views expressed in the chapters are the authors and do not necessarily represent the views of NIE.

A special note of thanks to Chin Sze Looi for the editorial assistant work in the final stages of preparation for this book.

Contents

Editors and Contributors

About the Editors

Berinderjeet Kaur is a Professor of Mathematics Education at the National Institute of Education in Singapore. She holds a Ph.D. in Mathematics Education from Monash University in Australia. She has been with the Institute for the last 30 years and is one of the leading figures of Mathematics Education in Singapore. In 2010, she became the first full professor of Mathematics Education in Singapore. She has been involved in numerous international studies of Mathematics Education and was the Mathematics Consultant to TIMSS 2011. She was also a core member of the MEG (Mathematics Expert Group) for PISA 2015. She is passionate about the development of mathematics teachers and in turn the learning of mathematics by children in schools. Her accolades at the national level include the public administration medal in 2006 by the President of Singapore, the long public service with distinction medal in 2016 by the President of Singapore and in 2015, in celebration of 50 years of Singapore's nation building, recognition as an outstanding educator by the Sikh Community in Singapore for contributions towards nation building.

Yew Hoong Leong is an Associate Professor at the National Institute of Education, Nanyang Technological University. He began his academic career in mathematics education with the motivation of improving teaching by grappling with the complexity of classroom instruction. Along the journey, his research has broadened to include mathematics problem solving and teacher professional development. Together with his project teammates, they developed "Realistic Ambitious Pedagogy" and its accompanying plan of action—the "Replacement Unit Strategy".

Contributors

Lu Pien Cheng National Institute of Education, Nanyang Technological University, Singapore, Singapore

Berinderjeet Kaur National Institute of Education, Nanyang Technological University, Singapore, Singapore

Ngan Hoe Lee National Institute of Education, Nanyang Technological University, Singapore, Singapore

Yew Hoong Leong National Institute of Education, Nanyang Technological University, Singapore, Singapore

Kit Ee Dawn Ng National Institute of Education, Nanyang Technological University, Singapore, Singapore

Khiok Seng Quek National Institute of Education, Nanyang Technological University, Singapore, Singapore

Liyana Safii National Institute of Education, Nanyang Technological University, Singapore, Singapore

Wee Tiong Seah Melbourne Graduate School of Education, The University of Melbourne, Melbourne, VIC, Australia

Eng Guan Tay National Institute of Education, Nanyang Technological University, Singapore, Singapore

Tin Lam Toh National Institute of Education, Nanyang Technological University, Singapore, Singapore

Wei Yeng Karen Toh National Institute of Education, Nanyang Technological University, Singapore, Singapore

Cherng Luen Tong National Institute of Education, Nanyang Technological University, Singapore, Singapore

Lai Fong Wong National Institute of Education, Nanyang Technological University, Singapore, Singapore

Joseph B. W. Yeo National Institute of Education, Nanyang Technological University, Singapore, Singapore

Kai Kow Joseph Yeo National Institute of Education, Nanyang Technological University, Singapore, Singapore

Chapter 1
Overview of the School System and School Mathematics Curriculum in Singapore

Berinderjeet Kaur and Yew Hoong Leong

Abstract This chapter introduces the school system and school mathematics curriculum in Singapore. A brief overview of the system of schooling from the primary to the secondary years is provided. It also introduces the general features of the school mathematics curriculum that are relevant for one to appreciate the enactment of it particularly in the secondary schools that unfolds in Chapters 3–15. The framework of the school mathematics curriculum that has been steadfast since 1990 is elaborated. In addition to the aims and goals of the school mathematics curriculum the interconnected nature of mathematics courses that cater to the diverse abilities of students in the primary and secondary schools is also described. Lastly, the achievement of Singapore students in benchmark studies such as Trends in International Mathematics and Science Study (TIMSS) and Programme for International Student Assessment (PISA) is briefly reviewed.

Keywords School system · School mathematics curriculum framework · Syllabuses · TIMSS · PISA · Singapore

1.1 Introduction

Singapore is an island, with an area of 719.1 square kilometres. The population is approximately 5.5 million of which more than one million are foreigners working in the country. The GDP per capita as of December 2019 was Singapore Dollars $81,500 (which is the equivalent of 58,680 USD or 52, 160 Euro based on the rates of currency exchange on 22 June 2020). The two largest budget items of the government expenditure are Defence and Education, respectively. It is apparent that

B. Kaur (✉) · Y. H. Leong
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: berinderjeet.kaur@nie.edu.sg

Y. H. Leong
e-mail: yewhoong.leong@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_1

people are the only natural resource of Singapore and the nation spares no effort to actualize its simple objective of education that is:

> … to educate a child to bring out his greatest potential so that he will grow into a good man and a useful citizen. (Lee, 1979, p. iii)

In Singapore, education is also a key enabler of social mobility and the system provides equal opportunity for every child by:

- ensuring that no child is deprived of educational opportunities because of their financial situations;
- leveraging on the school system to provide more support for families from poorer backgrounds;
- investing in pre-school education targeted at children with families with poorer backgrounds; and
- investing in levelling up programmes in primary schools that attempt to level up academically weaker students in both English and Mathematics, so as to improve their foundations for future learning (Heng, 2012).

In Singapore, education for primary, secondary and tertiary levels is mostly supported by the state. All institutions, private and public, must be registered with the Ministry of Education (MOE). English is the language of instruction in all public schools and all subjects are taught and examined in English except for the "Mother Tongue" language paper. While "Mother Tongue" generally refers to the first language internationally, in Singapore's education system it refers to the second language as English language is the language of schooling and taken as a first language. Education takes place in three stages: "Primary education", "Secondary education" and "Post-secondary education". Detailed and most current information on Singapore's Education System is available at https://www.moe.edu.sg/. The following sections provide the reader with briefs about the school system and school mathematics curriculum.

1.2 The School System

1.2.1 Primary School

In Singapore, students start primary school in the year they turn 7 years of age. The school year in Singapore starts in the month of January. Every child receives a 6 year compulsory primary school education made up of a 4 year foundation stage and a 2 year orientation stage. The primary school curriculum provides children with a strong foundation in subject disciplines such as languages, humanities and the arts and mathematics and science; knowledge skills focussing on thinking and communication skills; and character development. Subject-based banding begins in Primary 5 and continues till Primary 6. It provides greater flexibility for children as they can take a combination of standard and/or foundation subjects depending on their strengths.

This helps the child focus on and stretch his potential in the subjects (standard) he is strong in, while building up the fundamentals in the subjects (foundation) in which he needs more support. For foundation, subjects support is available in the form of smaller class size, where teachers focus on helping students close gaps in their deficits and progress at a pace that is suited to their needs. The decision to take a foundation subject is often based on a child's achievement in the subject at the end of Primary 4 with close consultations by a school and the parents/guardians of the child. At the end of 6 years of primary school, students take the Primary School Leaving Examination (PSLE). The subjects tested in the PSLE are English Language/Foundation English Language, Mother Tongue Language/Foundation Mother Tongue, Mathematics/Foundation Mathematics and Science/Foundation Science. They may also take an optional subject that is Higher Mother Tongue Language. Generally, students who take the Foundation subjects progress to the Normal (Technical) course of study while the rest progress to the Express course and the Normal (Academic) course in secondary school.

1.2.2 Secondary School

Following 6 years of primary schooling, learning at secondary schools is tailored to different abilities. The PSLE is a placement examination. The score obtained by the student in the PSLE and other indicators such as special talent and/or interest helps teachers and parents guide students in taking an appropriate course of study at a secondary school. There are three courses of study at the secondary school. They are the Express Course (including the Integrated Programme (IP)), Normal (Academic) (N(A)) Course and the Normal (Technical) (N(T)) Course. Students who are academically the most able are in the Express course while the least academically able ones are in the Normal (Technical) course. The academically average ability students are in the Normal (Academic) course of study. Table 1.1 shows the enrolments of Secondary 1 students by course of study in the past 5 years (2014–2018).

It is apparent from Table 1.1, for the period 2014–2018, that the percentage of students in the Express course of study ranged from 62.2 to 64.0. The percentage of girls in the course of study ranged from 50.8 to 51.8. For the same period, the percentage of students in the N(A) and N(T) courses ranged from 23.0 to 25.4 and 12.4 to 13.1, respectively. The percentage of girls in the N(A) and N(T) courses ranged from 45.7 to 47.9 and 37.1 to 39.8, respectively.

While a student may be initially placed in a particular course based on his ability to cope with the learning pace and style, there are opportunities at every stage for him or her to make a lateral transfer to another course if it is more suited to his or her interests and abilities. Although it is possible for students to switch between courses of study at Secondary 1–3 levels, it is more common for students in Secondary 1 to switch courses of study at Secondary 2 level. Usually students from the Normal (Academic) course switch to the Express course but in some rare instances the reverse also does happen, i.e. students from the Express course go onto the Normal (Academic) course

Table 1.1 Secondary one enrolment by course (2014–2018)

Course of study	Sex		2014	2015	2016	2017	2018
Express	All	N (%)	27,490 (64.0)	26,736 (63.3)	24,613 (62.2)	24,475 (62.5)	24,432 (62.5)
	Female	N (%)	13,963 (50.8)	13,841 (51.8)	12,568 (51.1)	12,471 (51.0)	12,575 (51.5)
Normal (Academic)	All	N (%)	9,873 (23.0)	9,972 (23.6)	10,033 (25.4)	9,559 (24.5)	9,663 (24.7)
	Female	N (%)	4,713 (47.7)	4,556 (45.7)	4,795 (47.8)	4,576 (47.9)	4,575 (47.3)
Normal (Technical)	All	N (%)	5,606 (13.0)	5,509 (13.1)	4,904 (12.4)	4,948 (12.7)	4,991 (12.8)
	Female	N (%)	2,080 (37.1)	2,191 (39.8)	1,899 (38.7)	1,859 (37.6)	1,914 (38.3)
Total	All	N (%)	42,969 (100)	42,217 (100)	39,550 (100)	38,982 (100)	39,086 (100)
	Female	N (%)	20,756 (48.3)	20,588 (48.8)	19,262 (48.7)	18,906 (48.5)	19,064 (48.8)

Source of data: education statistics digest (MOE, 2015, 2016, 2017, 2018a, 2019)

in Secondary 2. Students can also take specific subjects at an academically-higher level in upper secondary. For example, if a student is in the N(T) course, he or she may be able to take some subjects at N(A) level. Figure 1.1 shows an overview of the pathways and possible lateral transfers among the courses of study. As Singapore's education system is continuously evolving, a detailed and most up to date outline of the pathways is available at https://www.moe.edu.sg/.

From Fig. 1.1, it is also apparent that students in the Express Course of study take the General Certificate of Education (Ordinary Level) (GCE O-Level) examination after 4 years of secondary schooling. However, students who are in the IP, which provides an integrated 6 year Secondary and Junior College education, do not take the GCE O-Level examination. Their 6 years of schooling culminates in General Certificate of Education (Advanced Level), International Baccalaureate or other diploma qualifications. The IP is for academically strong students who can benefit from programmes that provide broader learning experiences. The IP aims to stretch their potential in non-academic aspects that are beyond the formal academic curriculum. Schools that offer the IP admit students in Secondary 1. Students in the Express Course can also join in the IP at Secondary 3.

Students in the N(A) course of study take the General Certificate of Education (Normal(Academic) Level) (GCE N(A)-Level) examination after 4 years of secondary schooling. Based on their results in the GCE N(A)-Level examinations they may continue with another year of secondary school and take the GCE O-Level examination at the end of their fifth year in a secondary school or continue with their post-secondary education at a polytechnic or Institute of Technical Education (ITE). Students in the N(T) course of study take the General Certificate of

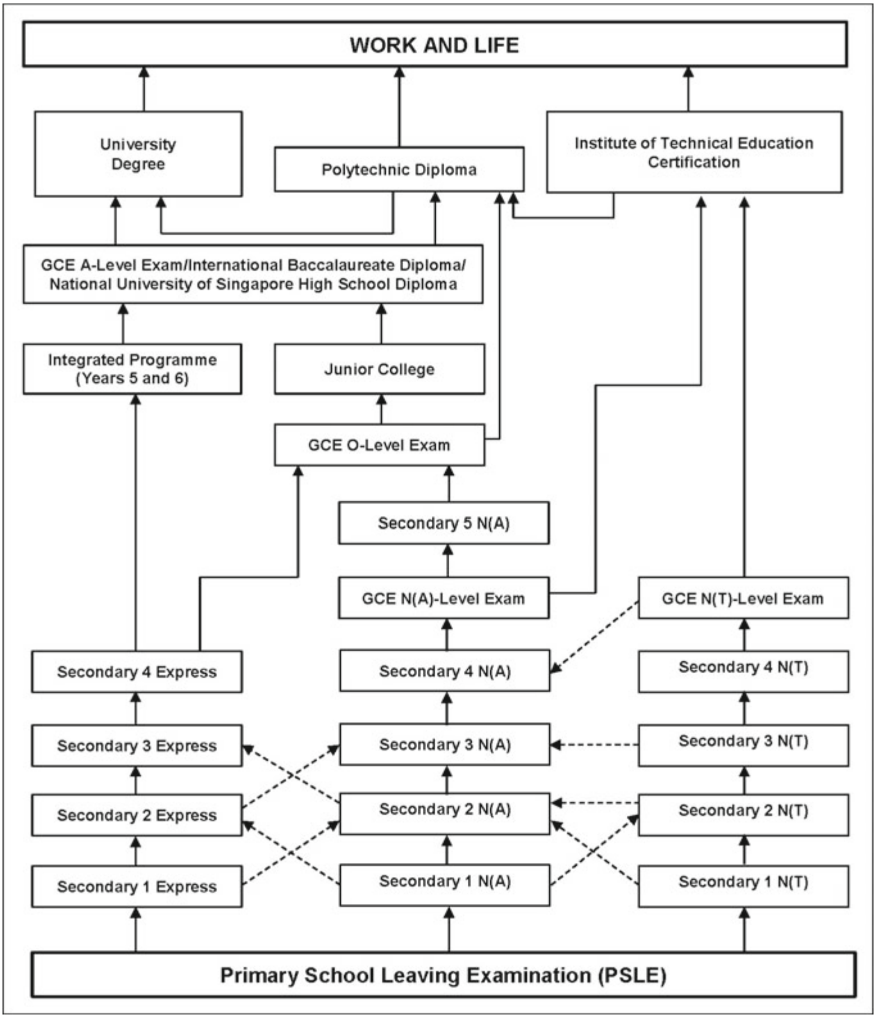


Fig. 1.1 An overview of the pathways and possible lateral transfers among the courses of study

Education (Normal(Technical) Level) (GCE N(T)-Level) examination after 4 years of secondary schooling. Based on their results in the GCE N(T)-Level examinations they may continue with another year of secondary school and take the GCE N(A)-Level examination at the end of their fifth year in a secondary school or continue with their post-secondary education at ITE. In brief the:

- Integrated Programme (IP) is a 6-year course that leads to the GCE A-Level examination or International Baccalaureate.
- Express is a 4-year course that leads to the GCE O-Level examination.

- Normal (Academic) is a 4-year or 5-year course that leads to the GCE N(A)-Level examination in Year 4 and, for eligible students the GCE O-Level examination in Year 5.
- Normal (Technical) is a 4-year course that leads to the GCE N(T)-Level examination.

As shown in Fig. 1.1, there are diverse pathways for students of all abilities to realise their potential and attain desired qualifications.

1.3 School Mathematics Curriculum

Mathematics is a core subject of the school curriculum across the primary and secondary years of schooling. A detailed history of the school mathematics curriculum is available in Kaur (2019). In the following sub-sections we briefly introduce the framework of the school mathematics curriculum that was developed in 1990, and salient features of the primary school and secondary school syllabuses.

1.3.1 Framework of the Singapore School Mathematics Curriculum

The central focus of the mathematics curriculum across the primary and secondary schools is the development of mathematical problem-solving competency. Supporting this focus are five inter-related components—concepts, skills, processes, metacognition and attitudes. The framework, shown in Fig. 1.2, has been steadfast for the last three decades (MOE, 2018b).

According to the MOE (2012, 2018b) syllabus documents problems may come from everyday contexts or future work situations, in other areas of study, or within mathematics itself. They include straightforward and routine tasks that require selection and application of the appropriate concepts and skills, as well as complex and non-routine tasks that requires deeper insights, logical reasoning and creative thinking. General problems solving strategies, e.g. Polya's four steps to problem solving and the use of heuristics, are important in helping one attempt non-routine tasks systematically and effectively.

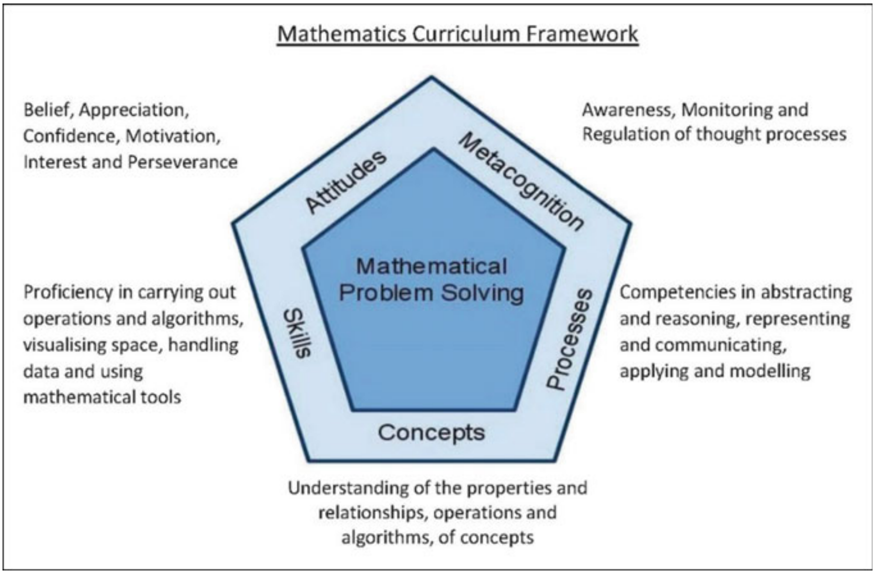


Fig. 1.2 Singapore school mathematics curriculum framework (MOE, 2018b, p. 10)

1.3.2 Primary Mathematics Curriculum

The aims of the primary mathematics curriculum are for students to:

- acquire mathematical concepts and skills for everyday use and continuous learning in mathematics;
- develop thinking, reasoning, communication, application and metacognitive skills through a mathematical approach to problem solving; and
- build confidence and foster interest in mathematics.

The primary mathematics syllabus assumes no formal learning of mathematics. However, early numeracy skills such as matching, counting, sorting, comparing and recognising simple patterns are useful in providing a good grounding for students to begin learning at Primary 1 (P1). The mathematics syllabus for all students from Primary 1 to Primary 4 (P1–4) is the same (MOE, 2012). However, in Primary 5 and Primary 6 (P5–6) there is differentiation in the content for Mathematics and Foundation Mathematics. The P5–6 mathematics syllabus continues the development of the same in P1–4, while the P5–6 Foundation Mathematics syllabus revisits some of the important concepts and skills in the P1–4 syllabus. The new concepts and skills introduced in the Foundation Mathematics is a subset of the Mathematics syllabus. Figure 1.3 shows content for Mathematics and Foundation Mathematics at the Primary 5 level for the topic Decimals.

Primary 5 - Decimals	
Mathematics	***Foundation Mathematics***
1.1 multiplying and dividing decimals (up to 3 decimal places) by 10, 100, 1000 and their multiples without calculator 1.2 converting a measurement from a smaller unit to a larger unit in decimal form, and vice-versa • kilometres and metres • metres and centimetres • kilograms and grams • litres and millilitres 1.3 solving word problems involving the four operations	1.1 notation, representations and place values (tenths, hundredths, thousandths) 1.2 comparing and ordering decimals 1.3 converting decimals to fractions 1.4 converting fractions to decimals when the denominator is a factor of 10 or 100 1.5 rounding decimals to • the nearest whole number • 1 decimal place • 2 decimal places 2.1 adding and subtracting decimals (up to 2 decimal places) without calculator 2.2 multiplying and dividing decimals (up to 3 decimal places) by 10, 100, 1000 2.3 converting a measurement from a smaller unit to a larger unit in decimal form, and vice-versa • kilometres and metres • metres and centimetres • kilograms and grams • litres and millilitres 2.4 solving word problems involving addition and subtraction

Fig. 1.3 An extract of content for decimals for primary 5 mathematics and foundation mathematics (MOE, 2012)

It is apparent from Fig. 1.3 that for P5 Foundation Mathematics syllabus items 1.1–2.1 are part of the P4 mathematics syllabuses that are re-visited and items 2.2–2.4 are a subset of the P5 Mathematics syllabus. Note that in item 2.2 the Foundation Mathematics students are allowed to use calculators and they only multiply and divide decimals (up to 3 decimal places) by 10, 100, 1000 unlike those doing Mathematics who multiply and divide decimals (up to 3 decimal places) by 10, 100, 1000 and their multiples without calculator. Similarly as shown in item 2.4 for Foundation Mathematics, students only solve word problems involving addition and subtraction while those doing Mathematics solve word problems involving the four operations.

1.3.3 Secondary Mathematics Curriculum

The goals of secondary mathematics education are twofold. One is to ensure that all students will achieve a level of mastery of mathematics that will enable them to function effectively in everyday life. The other is to provide those with an interest and ability in mathematics to learn more mathematics so that they can pursue mathematics or mathematics-related course of study in the next stage of their education. There are five syllabuses in the secondary mathematics curriculum catering to the needs, interests and abilities of students. They are:

- O-Level Mathematics (also commonly known as Elementary Mathematics)
- O-Level Additional Mathematics
- N(A)-Level Mathematics
- N(A)-Level Additional Mathematics
- N(T)-Level Mathematics.

The O-Level Mathematics and O-Level Additional Mathematics syllabuses are for the Express course students. The O-, N(A)- and N(T)-Level Mathematics syllabuses provide students with the core mathematics knowledge and skills in the context of a broad-based education. At the upper secondary levels (Secondary 3 and 4), students who are interested in mathematics may be offered Additional Mathematics as an elective. This prepares them better for courses of study that require mathematics. The specific aims of each syllabus are derived from the following broad aims:

- Learning and applying concepts and skills to solve problems, including those in contexts;
- Developing process and metacognitive skills through a mathematical approach to problem solving; and
- Inculcating positive attitudes towards mathematics.

Figure 1.4 shows an overview of the connected nature of the school mathematics curriculum from the primary to the secondary schools.

Tables 1.2 and 1.3 show the topics covered in the mathematics syllabuses for Number and Algebra and the detailed content for the topic: Functions and Graphs respectively, for the three courses of study. It is apparent from Table 1.2 that the N(T) Level Mathematics syllabus is a subset of the N(A) Level one and the N(A) Level Mathematics syllabus is a subset of the O-Level one. It is also apparent from Table 1.3 that for a topic the depth of content is graduated with each being a subset of the other with the O-level one being the largest set while the N(T)-Level one being the smallest. The O-Level Mathematics syllabus builds on the Standard Mathematics Syllabus in the primary school. The N(A)-Level Mathematics syllabus is a subset of O-Level Mathematics, except that it re-visits some of the topics in the Standard Mathematics Syllabus. The N(T)-Level Mathematics syllabus builds on the Foundation Mathematics Syllabus. It is also obvious from Tables 1.2 and 1.3 that gaps exist between the curriculums of the courses. Therefore, when lateral transfers, as shown in Fig. 1.1, do take place, bridging of knowledge is undertaken by teachers in schools during additional curriculum time.

The relationship between the O-Level Additional Mathematics syllabus and the N(A)-Level Additional Mathematics syllabus is somewhat different. Though the N(A)-Level Additional Mathematics syllabus is also a subset of the O-Level Additional Mathematics syllabus the detailed content of the topics remain the same in both the courses of study. As shown in Fig. 1.5, for the O-Level course of study, there are 6 topics, while for the N(A)-Level course of study there are two bridging topics and another 4 topics. The bridging topics fill the gap in the strand Number and Algebra that exists between the O-Level Mathematics and N(A)-Level Mathematics syllabuses.

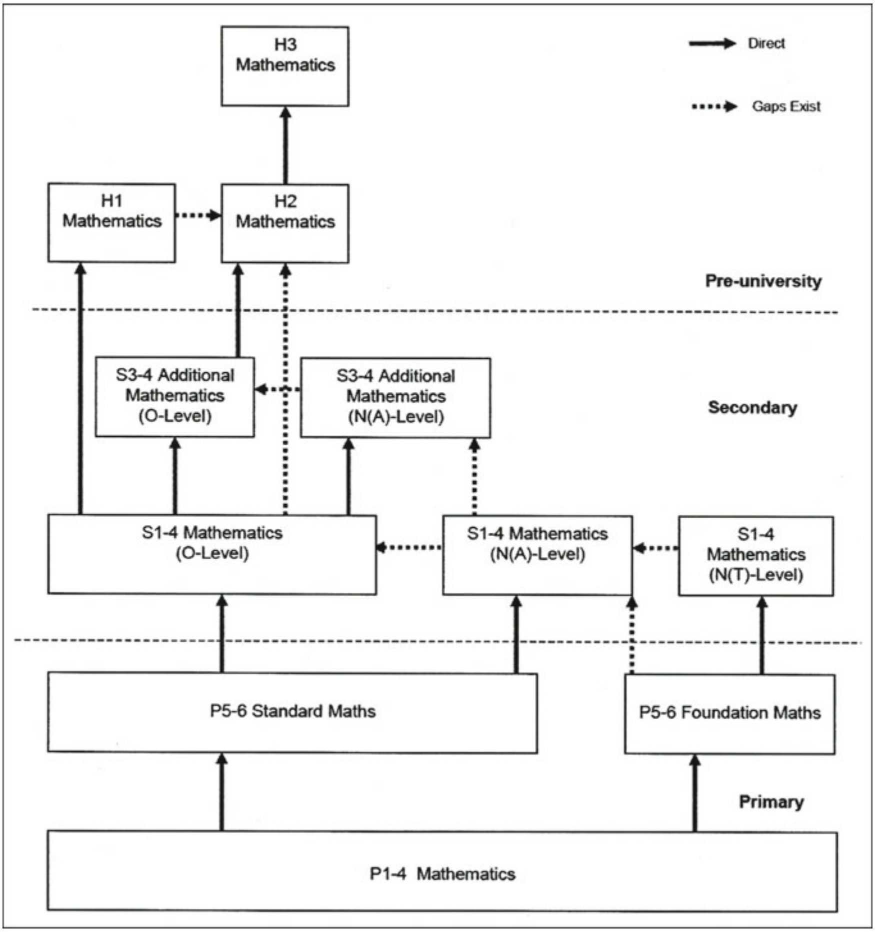


Fig. 1.4 An overview of the connected mathematics syllabuses

1.4 Concluding Remarks

The goals of the education system are shaped by the needs of Singapore for its economic survival. As part of the school curriculum, the study of mathematics has been critical since the late 1950s. It is a compulsory school subject, which takes into consideration the differing abilities and needs of students. It provides differentiated pathways and choices to support every learner in order to maximise their potential.

Students from across all the three courses of study, according to the demographic of student population at the respective grade levels, participate in all benchmark studies that Singapore participates in. For the Trends in International Mathematics and Science Study (TIMSS) students from Singapore have consistently ranked among the top three for both grades 4 and 8 in mathematics since 1995, for the past six cycles

Table 1.2 Topics in number and algebra for the three courses of study (MOE, 2018b)

Strand: Number and algebra	Course of study		
Topic	O-level	N(A) level	N(T) level
Numbers and their operations	x	x	x
Ratio and proportion	x	x	x
Percentage	x	x	x
Rate and speed	x	x	x
Algebraic expressions and formulae	x	x	x
Functions and graphs	x	x	x
Equation			x
Equations and inequalities	x	x	
Set language and notation	x		
Matrices	x		

Table 1.3 Detailed content for the topic: functions and graphs for the three courses of study (MOE, 2018b)

	Course of study		
Topic: Functions and graphs	O-level	N(A) level	N(T) level
Cartesian coordinates in two dimensions	x	x	x
Graph of a set of ordered pairs as a representation of a relationship between two variables	x	x	x
Linear functions ($y = \mathrm{a}x + \mathrm{b}$) and quadratic functions ($y = \mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$)	x	x	x
Graphs of linear functions	x	x	x
The gradient of a linear graph and the ratio of the vertical change to the horizontal change (positive and negative gradients)	x	x	x
Graphs of quadratic functions and their properties: – Positive and negative coefficient of x^2 – Maximum and minimum points – Symmetry	x	x	x
Sketching the graphs of quadratic functions given in the form: – $y = \pm (x - p)^2 + q$ – $y = \pm (x - a)(x - b)$	x		
Graphs of power functions of the form $y = \mathrm{a}x^n$, where $n = -2$, -1, 0, 1, 2, 3, and simple sums of not more than three of these	x	x	
Graphs of exponential functions $y = k\mathrm{a}^n$, where a is a positive integer	x	x	
Estimation of the gradient of a curve by drawing a tangent	x	x	

O-Level Additional Mathematics	N(A)-Level Additional Mathematics

• Quadratic functions
• Equations & inequalities
• Surds
• Polynomials & partial fractions
• Binomial expansions
• Exponential & logarithmic functions
	• Bridging topics
	• Functions and Graphs
	  • Sketching the graphs of quadratic functions given in the form:
	  - $y = \pm (x - p)^2 + q$
	  - $y = \pm (x - a)(x - b)$
	• Equations and Inequalities
	  • Solving linear inequalities in one variable, and representing the solution on the number line
	• Quadratic functions
	• Equations & inequalities
	• Surds
	• Polynomials & partial fractions

Fig. 1.5 The O-level and N(A)-level additional mathematics topics (MOE, 2018c)

(1995, 1999, 2003, 2007, 2011 and 2015). In addition, for the Programme in International Student Assessment (PISA), 15 year olds from Singapore have ranked in the top two positions in mathematics for the past three cycles (2009, 2012 and 2015). The achievement of Singapore students in benchmark studies such as TIMSS and PISA (Kaur, Zhu, & Cheang, 2019) affirm that the school mathematics curriculum is robust and in tandem with global trends. The consistent and commendable achievement of the students also show that the enactment of the curriculum places emphasis on mastery learning and problem solving. Chapters 3–15 of this book provide insights into the enactment of the secondary school mathematics curriculum in Singapore schools.

References

Heng, S. K. (2012, February 8). *Keynote speech by Minister of Education Heng Swee Kiat*, at the CSIS (Centre for Strategic and International Studies) Singapore Conference. Washington, DC.

Kaur, B. (2019). Overview of Singapore's education system and milestones in the development of the system and school mathematics curriculum. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 13–33). Singapore: Springer.

Kaur, B., Zhu, Y., & Cheang, W. K. (2019). Singapore's participation in international benchmark studies—TIMSS, PISA and TEDS-M. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 101–137). Singapore: Springer.

Lee, K. Y. (1979). Letter in response to the report on the Ministry of Education by Dr. Goh and his team. In K. S. Goh & The Education Study Team, *Report on the Ministry of Education 1978* (p. iii). Singapore: Singapore National Printers.

Ministry of Education. (2012). *Primary mathematics and teaching syllabus*. Singapore: Author.

Ministry of Education. (2015). *Education statistics digest*. Singapore: Author.

Ministry of Education. (2016). *Education statistics digest*. Singapore: Author.
Ministry of Education. (2017). *Education statistics digest*. Singapore: Author.
Ministry of Education. (2018a). *Education statistics digest*. Singapore: Author.
Ministry of Education. (2018b). *2020 Secondary mathematics syllabuses*. Singapore: Author.
Ministry of Education. (2018c). *Additional mathematics syllabuses: Secondary 3 to 4: Express course & normal (academic) course*. Singapore: Author.
Ministry of Education. (2019). *Education statistics digest*. Singapore: Author.

Berinderjeet Kaur is a Professor of Mathematics Education at the National Institute of Education in Singapore. She holds a Ph.D. in Mathematics Education from Monash University in Australia. She has been with the Institute for the last 30 years and is one of the leading figures of Mathematics Education in Singapore. In 2010, she became the first full professor of Mathematics Education in Singapore. She has been involved in numerous international studies of Mathematics Education and was the Mathematics Consultant to TIMSS 2011. She was also a core member of the MEG (Mathematics Expert Group) for PISA 2015. She is passionate about the development of mathematics teachers and in turn the learning of mathematics by children in schools. Her accolades at the national level include the public administration medal in 2006 by the President of Singapore, the long public service with distinction medal in 2016 by the President of Singapore and in 2015, in celebration of 50 years of Singapore's nation building, recognition as an outstanding educator by the Sikh Community in Singapore for contributions towards nation building.

Yew Hoong Leong is an Associate Professor at the National Institute of Education, Nanyang Technological University. He began his academic career in mathematics education with the motivation of improving teaching by grappling with the complexity of classroom instruction. Along the journey, his research has broadened to include mathematics problem solving and teacher professional development. Together with his project teammates, they developed "Realistic Ambitious Pedagogy" and its accompanying plan of action—the "Replacement Unit Strategy".

Part I
Global Features of Practice

Chapter 2
A Study of the Enacted School Mathematics Curriculum in Singapore Secondary Schools

Berinderjeet Kaur, Eng Guan Tay, Tin Lam Toh, Yew Hoong Leong, and Ngan Hoe Lee

Abstract A study of the enacted secondary school mathematics curriculum in Singapore schools, was a programmatic research project at the National Institute of Education (NIE) funded by the Ministry of Education (MOE) in Singapore through the Office of Education Research (OER) at NIE. The project had two aims. The first was to document how experienced and competent teachers enacted the school mathematics curriculum in secondary schools. It did this by examining: (i) pedagogies adopted by experienced and competent mathematics teachers when enacting the curriculum, and (ii) experienced and competent teachers' use of instructional materials for the enactment of the curriculum. The second was to establish how uniform these adopted pedagogies and use of instructional materials by experienced and competent teachers were practised in the mathematics classrooms of Singapore schools. The project had two phases. The first was the video-segment and the second was the survey-segment. The survey-segment was dependent on the findings of the video-segment. The video-segment documented the pedagogy of experienced and competent secondary mathematics teachers while the survey-segment helped to establish how uniform the pedagogy of experienced and competent teachers was in the mathematics classrooms of Singapore schools. Thirty experienced and competent mathematics teachers and their students participated in the first phase, while another 691 mathematics teachers from across the schools in Singapore participated in the second phase. Data collected have been subjected to purposeful analysis, using appropriate methods. The following

B. Kaur (✉) · E. G. Tay · T. L. Toh · Y. H. Leong · N. H. Lee
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: berinderjeet.kaur@nie.edu.sg

E. G. Tay
e-mail: engguan.tay@nie.edu.sg

T. L. Toh
e-mail: tinlam.toh@nie.edu.sg

Y. H. Leong
e-mail: yewhoong.leong@nie.edu.sg

N. H. Lee
e-mail: nganhoe.lee@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_2

chapters in the book, present evidence-based portraits of mathematics teaching and learning in Singapore secondary schools.

Keywords Secondary school mathematics curriculum · Experienced and competent mathematics teachers · Complementary accounts methodology · Teacher-intended curriculum · Secondary schools · Singapore

2.1 Introduction

This chapter, the second in the book, describes a programmatic research project that examined the teaching and learning of mathematics in Singapore schools. The large data set collected has been analysed by members of the research team. Some of the data, analysis and findings are presented in the next 13 chapters of this book. Chapter 3 describes the instructional core that drives the enactment of the secondary school mathematics curriculum, while Chapters 4–10 present aspects of the enactment of the curriculum and Chapters 11–15 are on the tasks and tools used for the enactment. The last two Chapters 16 and 17, attempt to illuminate the "what and how" of mathematics instruction in Singapore secondary schools. Chapter 16 provides an outsider's perspective about the pedagogy of mathematics teachers in Singapore schools. It does so using the analogy of expresso machines and the baristas who make coffee. The machines may be the same but the baristas make the difference. The final chapter, Chapter 17, provides an insider's perspective. It draws on the "pentagon"—the framework of the school mathematics curriculum and illuminates the interactions of its components that result in mathematics instruction in Singapore secondary schools. It also deepens insights about The Singapore Paradox—"examination-driven teaching" that has resulted in Singapore students stellar achievements in benchmark studies like Trends in International Mathematics and Science Study (TIMSS) and Programme in Student Assessment (PISA).

The programmatic research project, henceforth referred to as "the project", had two aims. The first was to document how experienced and competent teachers enacted the school mathematics curriculum in secondary schools. It did this by examining: (i) pedagogies adopted by experienced and competent mathematics teachers when enacting the curriculum, and (ii) experienced and competent teachers' use of instructional materials for the enactment of the curriculum. In the context of the project, an experienced and competent mathematics teacher in Singapore secondary schools was one who had taught the same course of study for a minimum of five years and was recognised by the school or cluster of schools as a competent teacher who has developed an effective approach to teaching mathematics. The second was to establish how uniform these adopted pedagogies and use of instructional materials by experienced and competent teachers were practised in the mathematics classrooms of Singapore schools. Shaped by the research interests of a group of colleagues in the Mathematics and Mathematics Education (MME) Academic Group at the National Institute of Education (NIE) in Singapore, the project was part of the CORE

Research Programme of the Office of Education Research (OER) at NIE. It was a special focus project of system studies in pedagogical and educational outcomes. It focused on understanding what goes on and what works in Singapore's classrooms—more specifically, the instructional core (City, Elmore, Fiarman, & Teitel, 2009). The instructional core comprises

> the teacher and the student in the presence of content … it is the relationship between the teacher, the student, and the content – not the qualities of any one of them by themselves – that determines the nature of instructional practice, [even though] each … has its own particular role and resources to bring to the instructional process. (City et al., 2009, pp. 22–23)

It was about the interactions between secondary school mathematics teachers and their students, as it is these interactions that fundamentally determine the *nature* of the actual mathematics learning and teaching that take place in the classroom. It also examined the content through the instructional materials used—their preparation, use in classroom and as homework. Such studies are crucial for the Ministry of Education (MOE) in Singapore and schools to gain a better understanding of what works in the instructional core in their classrooms and schools. This is critical for the development of their education system.

2.2 Conceptual Framework

The conceptual framework of the project was framed through a purposeful review of literature that had three parts. The first part reviewed the findings of the CORE 2 research conducted by David Hogan and colleagues at NIE from 2006 to 2012 concerning mathematics lessons in Singapore secondary classrooms. The second part reviewed a model of curriculum enactment and illuminated the concept of "teacher-directed" curriculum that guided the research in the project. Lastly, selected literature on teaching of mathematics were reviewed as this foregrounded the concept of pedagogy in the enacted curriculum.

2.2.1 What Did the Findings of CORE 2 Tell Us About Mathematics Teaching and Learning in Singapore Secondary Mathematics Classrooms?

As part of the CORE 2 research led by David Hogan, the quality of the enacted curriculum in Secondary 3 (Grade 9) mathematics lessons in Singapore was assessed using criteria and standards identified by Hattie in Visible Learning (2012). More than 1000 Secondary 3 students in 30 schools drawn from a representative random stratified sample of Secondary schools and 31 mathematics teachers from the Express and Normal (Academic) Courses of study were involved in the study. Data were gathered from student surveys, video-records of lessons, and post-lesson teacher interviews.

The findings of the research specific to secondary three mathematics lessons were as follows:

(i) Teachers focused more on procedural knowledge than conceptual knowledge and only engaged students in domain-specific knowledge practice in about a third of the instructional time of a typical lesson. Of the domain-specific knowledge practices, knowledge representation was emphasised. They also found that epistemic talk—systematic talk about knowledge that is critical to visible teaching and learning and to enhancing student understanding and skill formation—was lacking in the lessons. There was also lack of formative monitoring that could make student learning visible. Instead, procedural learning support was evident as teachers often helped with the "how to do" steps (Hogan, Kwek, et al., 2013).

(ii) Students were engaged in doing performative tasks (77.3%) more often than knowledge building tasks (22.7%) (Hogan, Towndrow, et al., 2013). A performative task mainly entails the use of lower order thinking skills such as recall, comprehension and application of knowledge while a knowledge building task calls for higher order thinking skills such as synthesis, evaluation and creation of knowledge.

(iii) There was a dominant performative orientation of pedagogical practice in Singapore (Hogan, Chan, et al., 2013, p. 100) and this may explain Singapore's stellar performance in international studies.

While the findings of the CORE 2 research provided some insights about the widespread orientation of our secondary school mathematics classroom teaching and learning, they did not inform us about what our experienced and competent teachers do when compared to the broad base of teachers studied in CORE 2. It is also not possible to infer how the "performative orientation" has contributed to our students' performance in PISA studies. Do our experienced and competent teachers engage students in metacognition, an essential element of twenty-first-century competencies—Civic Literacy, Global Awareness and Cross-Cultural Skills; Critical and Inventive Thinking; Communication, Collaboration and Information Skills—as envisioned by the Singapore Ministry of Education (MOE, n.d.)? How does the prescribed curriculum of the Ministry of Education for mathematics translate into teacher plans and classroom actions of experienced and competent teachers? The current project built on the findings of CORE 2, to study the pedagogies adopted by experienced and competent secondary mathematics teachers when enacting the curriculum. Findings reported in this book and elsewhere (see Appendix) have provided mathematics educators, curriculum developers and policymakers with much-valued insights about the "the best that takes place in our secondary mathematics classrooms" from the perspectives of both teachers and their students.

2.2.2 Curriculum Enactment Process

In the context of this project, teacher-intended curriculum represented plans of the teacher about what to teach and how he/she planned to teach it; teacher enacted curriculum represented what is taught during the lesson; and designated curriculum was the prescribed (official) curriculum by the MOE, in terms of syllabuses and guidelines. In our conceptualisation of the curriculum enactment process we drew upon the visual model created by Remillard and Heck (2014) shown in Fig. 2.1. Kaur (2014) in her review of research on the enactment of school mathematics curriculum in Singapore has noted that the model shown in Fig. 2.1 was rigorous for use in researching the curriculum enactment process in Singapore as it linked the official and operational curriculum in mathematics classrooms.

The model showed that as teachers drew on the designated curriculum (which in the case of the project is the Mathematics Syllabus for Secondary Schools [MOE, 2012]) along with other resources (particularly instructional materials) to design instruction they created what we referred to as "teacher-intended" curriculum in the context of the project. It included the interpretation and decisions teachers made to envision and plan instruction. Remillard and Heck (2014) noted that this form of curriculum was difficult to document as part of it existed in the most detailed form in the teacher's mind. Nevertheless, detailed teacher plans and post-lesson video-stimulated interviews with the teachers offered an opportunity to capture the teacher-intended curriculum and its enactment succinctly.

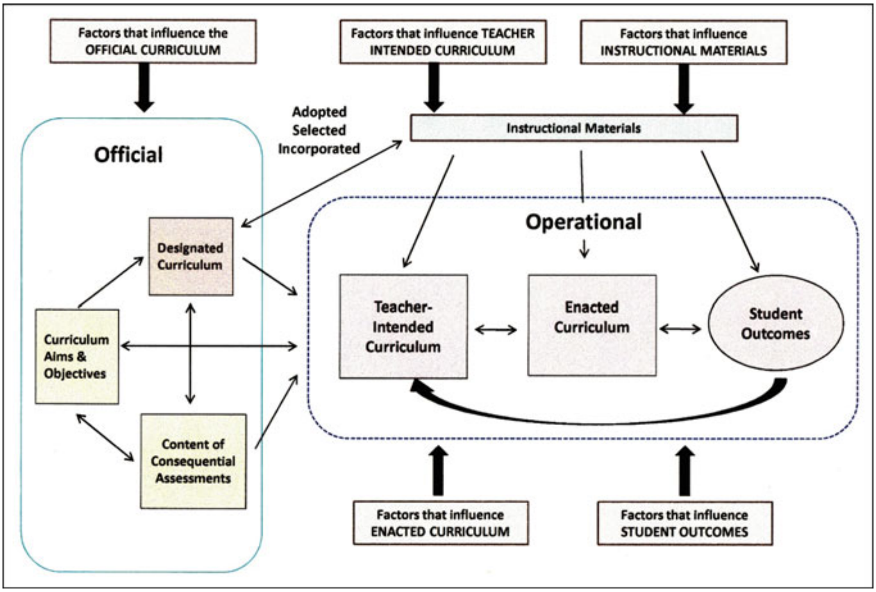


Fig. 2.1 Model of the curriculum enactment process (Remillard & Heck, 2014, p. 709)

Despite its importance, the enacted curriculum was multifaceted and difficult to measure and study. The number of potential features were numerous and at times even difficult to define let alone measure. Nevertheless some prominent dimensions that had been studied centred around the mathematics, the pedagogical moves and the use of resources and tools (Remillard & Heck, 2014). The following elaborates each of the above dimensions further.

(i) The mathematics—this refers to the content and nature of the mathematics topics and practices that are emphasised and valued. For example, Hiebert et al. (2003) and Stigler, Gonzalez, Kawanaka, Knoll, and Serrano (1999) study of how mathematics content was presented to students considered features like demonstration, practice, recall of concepts, conceptual connections and proof. Boaler and Staples (2008), Eisenmann and Even (2009), and Stein, Grover, and Henningsen (1996) added an additional focus on the level of cognitive expectations.

(ii) The teacher's pedagogical moves—this refers to teacher's actions, both intentional and unintentional, that shape what mathematics is addressed, including how it is represented and investigated. Teacher moves also influence how classroom interactions are structured, the kinds of interactions that are valued, and which tools and resources are used during instruction. In a review of research on the teacher's role in mathematics discourse, Walshaw and Anthony (2008) identified three distinct roles that teachers play to shape mathematics classroom discourse: (i) identifying and drawing out specific mathematical ideas, (ii) fine-tuning the mathematical language and conventions used, and (iii) shaping mathematical argumentation as it develops. For this project we built on earlier studies of instructional cycles (Seah, Kaur, & Low, 2006) and content learning discourse (Kaur, 2013) conducted in Singapore. The instructional cycles comprised combinations of segments such as [D]—whole class demonstration; [S]—seatwork (student work); and [R]—whole class review of student work. The content learning discourse was dominated by teacher talk and student listening.

(iii) The use of resources and tools—this refers to physical, technological, linguistic and cognitive tools that might be used during instruction by both teacher and students. Tools included instructional resources, like textbooks, as well as concrete resources like calculators, computers and manipulatives such as AlgeCards and algebra-tiles. In Singapore, tools are often introduced into the classroom through teachers' moves. This influences how the mathematics is represented and forms of student engagement, as well as the nature of the classroom interactions. For example, Leong, Ho, and Cheng (2015) showed how AlgeCards helped students factorise quadratic expressions meaningfully and Kaur, Low, and Seah (2006) studied the role of textbook in two grade eight mathematics classrooms.

2.2.3 *Perspectives of Mathematics Teaching*

Teaching is a cultural activity (Stigler & Hiebert, 1999) and there are varying Eastern and Western perspectives about mathematics teaching. Two significant dichotomies that exist between the perspectives of the West and East are: (i) the product versus process dichotomy, and (ii) the rote learning versus meaningful learning dichotomy (Leung, 2001). Anthony and Walshaw (2009) recognised that classroom teaching is a complex activity and that the classroom learning community is neither static nor linear. Based on their research on the Western perspective of mathematics teaching, they offered ten principles of effective pedagogy, among which were: (i) arranging for learning—mathematics learning experiences, (ii) mathematical communication with a focus on mathematical argumentation, (iii) mathematical tasks that influenced how students came to view, develop, use and made sense of mathematics, and (iv) tools and representations that supported students' thinking.

The three decades of research by Schoenfeld (2011) in the United States on mathematical problem-solving and mathematics instruction affirmed that moment-to-moment decision making in teaching could be modelled as a function of teachers' resources (especially knowledge), orientations (especially beliefs) and goals. He advocated that the five dimensions of mathematically powerful classrooms are: (i) the mathematics context; (ii) cognitive demand; (iii) access to mathematical content; (iv) agency, authority and identity; and (v) uses of assessment.

Kaur (2009) in her study of grade eight mathematics lessons in the east (Singapore), in which she juxtaposed student and teacher perceptions about effective lessons found that these lessons had the following characteristics:

(i) Whole class demonstration (exposition) where the teacher explained clearly the concepts and steps of procedures; made complex knowledge easily assimilated through demonstrations, use of manipulatives, real-life examples and introduced new knowledge;
(ii) Seatwork and out of class assignments where the teacher gave clear instructions related to mathematical activities for in-class and after classwork; provided interesting activities for students to work on individually or in small groups; provided sufficient practice tasks for preparation towards examinations; and
(iii) Review and feedback where the teacher reviewed past knowledge and used student work or group presentations to give feedback to individuals or the whole class.

From the findings of Kaur (2009), it was apparent that there was emphasis on the development of skills in Singapore classrooms, but to say that understanding was not emphasised could not be confirmed. Though algorithms lead to proficiency of skills, they can also contribute to understanding as exemplified by Fan and Bokhove (2014). Fan and Bokhove had aptly illustrated how algorithms were powerful in-roads for conceptual understanding with their three-level model of learning:

1. Cognitive level 1: knowledge and skills—involved direct teaching where teachers may tell, demonstrate, engage students in drill and practice and or remediation to correct their mistakes.
2. Cognitive level 2: understanding and comprehension—involved explaining where teachers explained the why of the steps in the algorithm and perhaps even why it worked; involved justifying where teachers engaged students to make sense of how the algorithm was derived logically or even prove it; involved making connections where teachers helped students connect the algorithm with their past knowledge.
3. Cognitive level 3: evaluation and construction—involved guided exploration where teachers created learning activities for students to explore and obtain the algorithm; followed by open exploration where teachers created learning activities for students to explore and obtain the algorithm.

2.2.4 Summary

From the findings of CORE 2, we knew there was a dominant performative orientation of pedagogical practice involving student classroom activities and discourse in Singapore. However, from these findings we were unable to infer if the performative orientation also pervaded the classrooms of experienced and competent secondary mathematics teachers in Singapore. Moreover, the dominant use of performative mathematical tasks and performative orientation of classroom pedagogy alone could not explain the success of Singapore students in PISA. There was a need to examine how experienced and competent secondary mathematics teachers in Singapore enacted the school mathematics curriculum, so that we had knowledge about the upper bound of pedagogies adopted by secondary mathematics teachers in Singapore schools; thereby illuminating the potential of the prescribed curriculum by the MOE.

For the last 30 years, the framework for school mathematics curriculum had placed emphasis on five factors: concepts, skills, processes, metacognition and attitudes (MOE, 2012) that contributed towards the primary goal of teaching mathematics in Singapore schools which was mathematical problem-solving. How had this emphasis shaped the practice of our mathematics teachers? If we mapped the pedagogy of our mathematics teachers against the five dimensions of mathematically powerful classrooms advocated by Schoenfeld (2011), what were the outcomes? If we looked deeper at why Singapore teachers engaged their students in working with algorithms or homework or how they used their mathematics textbooks for learning, what could we infer about how mathematics is being learnt? All the above thoughts shaped our foci of the project.

In the first phase we investigated: (i) pedagogies adopted by experienced and competent mathematics teachers when enacting the curriculum, and (ii) experienced and competent secondary school mathematics teachers' use of instructional materials for the enactment of the curriculum. In the second phase we established how uniform these adopted pedagogies and use of instructional materials by experienced and competent teachers were practised in the mathematics classrooms of Singapore secondary schools.

2.3 Research Design

The project had two phases. The first was the video-segment and the second was the survey-segment. The survey-segment was dependent on the findings of the video-segment. The video-segment documented the pedagogy of experienced and competent secondary mathematics teachers while the survey-segment helped to establish how uniform the pedagogy of experienced and competent teachers was in the mathematics classrooms of Singapore schools. We detail the phases in the following sub-sections.

2.3.1 Phase 1: Video-Segment of the Project

The video-segment of the study adopted the complementary accounts methodology developed by Clarke (1998, 2001), a methodology which was widely used in the study of classrooms across many countries in the world as part of the Learner's Perspective Study (Clarke, Keitel, & Shimizu, 2006). This methodology recognises that only by seeing classroom situations from the perspectives of all participants (teachers and students) can we come to an understanding of the motivations and meanings that underlie their participation. It also facilitates practice-oriented analysis of learning.

2.3.1.1 Method

A three-camera (teacher camera, student camera, whole class camera) approach, shown in Fig. 2.2, was used to collect data.

The teacher camera captured all gestures, tools and equipment the teacher used in the lesson. The student camera kept in view two to four students, the focus students, who were sitting adjacent to each other and focused on their actions during the lesson. Every lesson had a different group of focus students. The whole class camera captured the corporate behaviour of the class and was set at the front looking at the class such that it represented the "teacher's-eye view" of the class. Sequences of lessons for a complete topic, ranging from three to ten lessons spanning instruction time between 210 and 570 min, taught by the participating teachers were recorded.

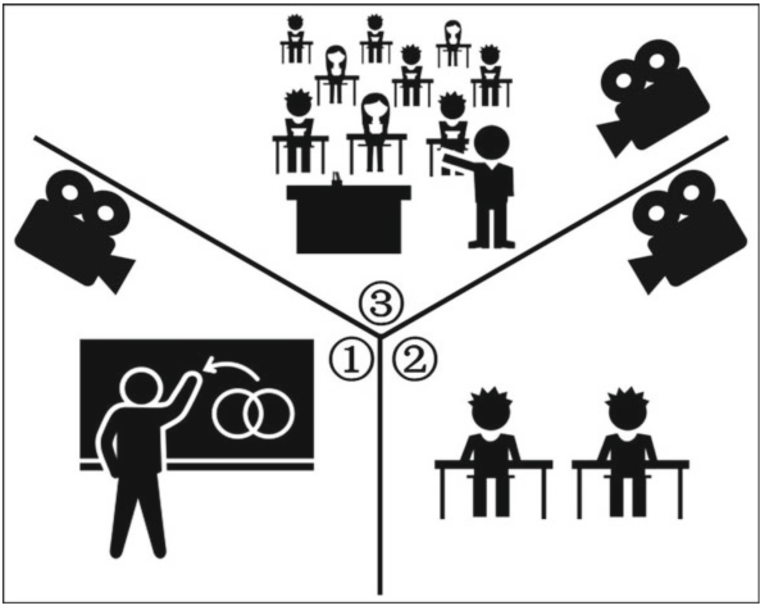


Fig. 2.2 Three-camera approach

In addition, all the written work done by the focus students during and following the lesson and instructional materials used by the teachers were also documented. Researchers also kept field notes of every lesson recorded.

Post-lesson video-stimulated interviews were held for the teachers and focus students. The teacher was interviewed once before he/she enacted the sequence of lessons and two to three times while enacting the lessons, with the last interview at the end of the enactment. Figures 2.3 and 2.4 show the interview prompts that guided the pre- and during enactment of lessons interviews. The teacher's plan for the sequence of lessons was the main stimulus for the pre-interview while instructional materials used during the lesson and video-records of the lesson were the stimuli for during- and post-lesson enactment interviews.

The focus students were also interviewed individually. The instructional materials used during the lesson and the video-record of the lesson were the stimuli for the student interviews. Figure 2.5 shows the prompts that guided the interview.

- Please share with me your goals for the planned sequence of lessons.
  You may include both content and non-content goals.
- Please share with me what mathematical goals you intend to achieve for each set of materials that you will be using.
- Which of these do you consider "ambitious" or challenging goals?
- How do you plan to achieve your goals?
- How do you plan to implement some of these instructional materials that you have mentioned earlier?
- Are there special features you have put in place for the instructional materials that you think might help students attain the mathematical goals of your lesson?
- How would you rate the level of challenge (1 to 5; with 5 being the most challenging) that each of these tasks presents to your students?
- Why do you rate them this way?
- Are there any specific difficulties you anticipate that some of your students may have with some of the instructional materials?
- Please share with us your plans to help your students with these difficulties.
- What other mathematics programmes do your students participate in outside your mathematics lessons?
- Could you tell me more about these programmes which help your students learn mathematics?
- On the average, what would be the amount of time you would expect your middle progress students to spend on mathematics homework each week?

Fig. 2.3 Prompts for teacher interview prior to enactment of his/her sequence of lessons

- Please choose a lesson that you'd like to talk with me about.
  - What were your goals for this lesson?
  - You may include both content and non-content goals.
  - Did you use all the materials that you had intended to use for the lesson?
- Do you think you have achieved your goals that you set for the lesson?
- How were those goals achieved?
- What is the most 'ambitious' or challenging thing you did in the lesson?
  - How do you think it went?
- Do you think your students have achieved these goals?
- Can you share with me what the highs and lows of the lesson were?

<u>Commence use of video</u>

- Please fast forward to any parts of the video that you think illustrate how you achieved the goals you've shared with me just now.
  - Is there a part of the lesson that you like best? Please show me the video segment of it.
  - Can you explain a little more why you like this part best?
- How would you rate your lesson today?
  - What are some of the words you would use to describe your lesson today?

<u>Prompt for the last interview only carried out after the sequence of lessons has been enacted</u>

- Looking back at the sequence of lessons, did you make any changes to the instructional materials you shared with me during the first (pre-enactment) interview, such as add, remove, modify or adapt?
  - Elaborate the change(s). Why were they made?
- Did the change(s) help you fulfil the intended goals better? How?

Fig. 2.4 Prompts for teacher interview during and after enactment of his/her sequence of lessons

Pre-interview tasks:
Interviewer to assure student that whatever transpires during the interview is solely for the purpose of research and will not be shared with his/her mathematics teacher.
Student is given a mathematical task to do before the commencement of interview [the task is sought from the teacher by the interviewer and is often similar to one that the teacher and students worked on during the lesson]

Interview:
A video-record of the lesson and all materials used during the lesson are at hand during the interview. Student is told that the interview is about the lesson.

- Please tell me what you think the lesson was about?
- Please list out 3 main things you have learnt from the lesson. You can refer to any of these materials from the lesson to help you.
- What did the teacher do to help you learn [this]?
  [Interviewer to point to a specific artefact from the lesson]
- Which of the materials used are most helpful to you?
- How did those materials help you? Why?
- How did you feel about the materials?
- On your part, what did you do to help yourself understand what the teacher was showing/explaining?
- Share with me if there was any part of the lesson that was challenging for you.
  - Why was it challenging?
  - How did you feel about it?
- Can you share with me what the highs and lows of the lesson were?

Commence use of video:

- Please fast forward to select the parts of the video that you think shows how you learnt [this].
- Is there a part of the lesson that you liked best?
- Can you show me the video segment?
- Could you help me to understand why you liked this part best?
- What are some of the words you would use to describe the mathematics lesson today?
- On the average, how much time do you usually spend on mathematics homework each week?
- Do you have any other help in your learning of mathematics when you are not in class/school?
- Does your school have any other programme/activities to help you learn mathematics outside lesson time?
- Which materials are useful in helping you to learn and perform well besides those that are used in your mathematics lessons?

Fig. 2.5 Prompts for focus student's interview

2.3.1.2 Participants

In this phase of the study, which was the video-segment, 30 experienced and competent teachers from across the four courses of study (the most academically able students are in the Integrated Programme while the least able are in the Normal [Technical] course of study) participated. These were teachers deemed as "good mathematics teachers" in their respective schools. They had at least five years of mathematics teaching experience, were recognised by their schools/cluster as competent teachers who have developed an effective approach of teaching mathematics and were keen to participate in the study. These teachers were nominated by their respective school leaders and the research team followed up on the nominations and

Table 2.1 Number of teacher participants in video-segment of the study

Course of study	Gender N (%)		Years of mathematics teaching N (%)			Total N (%)
	Male	Female	5–9	10–20	More than 20	
Integrated programme	1 (3.3)	3 (10.0)	1 (3.3)	2 (6.7)	1 (3.3)	4 (13.3)
Express	3 (10.0)	7 (23.3)	1 (3.3)	3 (10.0)	6 (20.0)	10 (33.3)
Normal (Academic)	3 (10.0)	5 (16.7)	2 (6.7)	2 (6.7)	4 (13.3)	8 (26.7)
Normal (Technical)	5 (16.7)	3 (10.0)	2 (6.7)	4 (13.3)	2 (6.7)	8 (26.7)
Total	12 (40.0)	18 (60.0)	6 (20.0)	11 (36.7)	13 (43.3)	30 (100)

interviewed the teachers. A strict requirement for participation in the study was that the teacher had to teach the way he/she did all the time, i.e. no special preparation was expected.

Within the scope of the research project due to limitations of time and funding, the research team capped the number of teachers at 30. Following the search for experienced and competent teachers who were willing to participate, the team managed to recruit four from the Integrated Programme (IP), ten from the Express course, eight from the Normal (Academic) (N(A)) course and eight from the Normal (Technical) (N(T)) course. The sample of teacher participants in this phase of the project was not constrained by the demographic of teachers teaching the respective courses of study. The sampling was purposeful as we needed teachers who were willing to participate in the study. Table 2.1 shows the number of teachers from each course of study and profile of the teachers. Sixty percent of the teachers were female and 80% of them had ten or more years of mathematics teaching experience.

By virtue of the design of the study all the students in the classes of the 30 teachers participated in it. However, only the focus students were interviewed. As the sample of the teachers was purposeful, it follows that the proportions of student participants were not in sync with the student demographic in the courses of study. In addition, assent from students and consent from their parents was needed for them to participate as focus students and be interviewed. This was challenging for two main reasons. The first was that not many students were keen to have their interviews recorded and secondly the interviews were conducted after school hours on the same day of the recordings. In some sense, there was very little control the research team had on the numbers that were interviewed. Table 2.2 shows the numbers of students by course of study who participated in the study and were interviewed. Almost half of the student participants were interviewed as they were the focus students.

Table 2.2 Number of student participants in video-segment of the study

Course of study	Students who participated *N* (%)			Students interviewed *N* (%)		
	Gender		Total	Gender		Total
	Male	Female		Male	Female	
Integrated programme	40 (4.7)	70 (8.2)	110 (12.9)	23 (2.7)	44 (5.1)	67 (7.8)
Express	148 (17.3)	150 (17.6)	298 (34.9)	81 (9.5)	57 (6.6)	138 (16.1)
Normal (Academic)	122 (14.3)	95 (11.1)	217 (25.4)	80 (9.4)	52 (6.1)	132 (15.5)
Normal (Technical)	152 (17.8)	77 (9.0)	229 (26.8)	77 (9.0)	33 (3.9)	110 (12.9)
Total	462 (54.1)	392 (45.9)	854 (100)	261 (30.6)	186 (21.7)	447 (52.3)

2.3.1.3 Data Collected

During this phase, i.e. the video-segment of the project, a total of 209 lessons enacted by the 30 experienced and competent teachers were video-recorded. Data was collected over a period of two years, i.e. from the start of the second semester of the school year in 2016 (June) till the end of the first semester of the school year in 2018 (May). Table 2.3 shows an overview of the 30 teachers' lessons and the spread of the lessons across the courses of study and year levels. Sequences of lessons for a topic were recorded and therefore it was not possible to mandate that a certain number of lessons be recorded for every teacher who participated. As such, the number of lessons recorded per course of study varies, ranging from 33 (15.8%) for the Integrated Programme to 64 (30.6%) for the Express course. Nevertheless, the sequences of lessons were adequate for the purpose of documenting the respective teachers' instructional practices for mathematics. In addition, to a large extent, common pedagogies in the classrooms of these experienced and competent teachers, irrespective of the courses of study, supported the intent of drawing on the findings of the video-segment to shape the survey for the survey-segment of the project as intended.

Interviews with the focus students, and interviews (at the start, in the midst and after the sequence of lessons) with the teachers were also video-recorded. Artefacts, comprising materials used by the teachers for instruction and student work, were also copied and digitised.

Table 2.3 Metadata of the video-recorded lessons

	Number of classes (Number of lessons) (%)					
	Secondary					
Course of study Teacher (T) (year level, no of lessons, instructional time in minutes)—topic	1	2	3	4	5	Total
Integrated programme Teacher 21 (*Sec 2, 7, 420)—Quadratic equations and graphs Teacher 12 (Sec 3, 8, 480)—Logarithms Teacher 13 (Sec 3, 10, 500)—Quadratic graphs and inequalities Teacher 17 (Sec 3, 8, 400)—Trigonometry (Sine & Cosine rule, 3D)	0 (0) (0)	1 (7) (3.4)	3 (26) (12.4)	0 (0) (0)	0 (0) (0)	4 (33) (15.8)
Express course Teacher 06 (Sec 2, 3, 210)—Pythagoras' theorem Teacher 20 (Sec 2, 6, 330)—Probability and statistics Teacher 05 (Sec 3, 3, 210)—Angle properties of circles Teacher 08 (Sec 3, 8, 570)—More about quadratic equations Teacher 15 (Sec 3, 6, 400)—Application of trigonometry Teacher 22 (Sec 3, 7, 420)—Solving quadratic equations Teacher 01 (Sec 4, 9, 540)—Vectors Teacher 03 (Sec 4, 5, 455)—Geometric proofs Teacher 10 (Sec 4, 9, 480)—Differentiation Teacher 27 (Sec 4, 8, 495)—Vectors	0 (0) (0)	2 (9) (4.3)	4 (24) (11.5)	4 (31) (14.8)	0 (0) (0)	10 (64) (30.6)
Normal (Academic) Teacher 16 (Sec 1, 7, 480)—Volumes and surface areas of solids Teacher 18 (Sec 1, 6, 460)—Simple algebra Teacher 19 (Sec 3, 7, 420)—Trigonometric ratios of acute angles Teacher 26 (Sec 3, 8, 400)—Coordinate geometry Teacher 11 (Sec 4, 7, 400)—Circular measure Teacher 28 (Sec 4, 7, 550)—Differentiation and application Teacher 29 (Sec 4, 6, 465)—Probability Teacher 02 (Sec 5, 10, 565)—Vectors	2 (13) (6.2)	0 (0) (0)	2 (15) (7.2)	3 (20) (9.6)	1 (10) (4.8)	8 (58) (27.8)

(continued)

Table 2.3 (continued)

	Number of classes (Number of lessons) (%)					
	Secondary					
Course of study Teacher (T) (year level, no of lessons, instructional time in minutes)—topic	1	2	3	4	5	Total
Normal (Technical) Teacher 30 (Sec 1, 6, 420)—Angles on a straight line Teacher 14 (Sec 2, 8, 400)—Volume, surface area—prism, cylinder Teacher 04 (Sec 3, 5, 350)—Simultaneous equations Teacher 07 (Sec 4, 8, 540)—Cumulative frequency Teacher 09 (Sec 4, 7, 390)—Volume, surface area—pyramid, cone Teacher 23 (Sec 4, 6, 430)—Volume, surface area—pyramid, cone Teacher 24 (Sec 4, 6, 390)—Pythagoras' theorem and Trigonometry Teacher 25 (Sec 4, 8, 420)—Trigonometric ratios of acute angles	1 (6) (2.9)	1 (8) (3.8)	1 (5) (2.4)	5 (35) (16.7)	0 (0) (0)	8 (54) (25.8)
Total	3 (19) (9.1)	4 (24) (11.5)	10 (70) (33.5)	12 (86) (41.1)	1 (10) (4.8)	30 (209) (100)

Key—*Sec 2 means Secondary 2 which is year eight of schooling

2.3.2 Phase II: Survey-Segment of the Project

2.3.2.1 Instrument

Findings from Phase I provided inputs for the survey, the instrument, used in this phase of the project. An online survey comprising three parts, A, B and C, was constructed. Part A was about the pedagogical structure of lessons and student-teacher interaction. Part B was about the enactment of the five aspects of the Singapore school mathematics curriculum framework (MOE, 2012) represented by the five sides of the "pentagon" in the framework; and Part C was about instructional materials.

Part A of the survey first sought inputs from participants specific to an instructional core, detailed in Chapter 3, which emerged from the analysis of the lessons enacted by the experienced and competent teachers. Briefly, the instructional core comprises three components: D [Development], S [Student Work] and R [Review of Student Work]. D refers to instruction that either develops a concept or introduces a skill, S refers to student work that is either done by students in class or at home, individually or in groups and R refers to review of student work that monitors student understanding. Examples of these aspects of instruction are shown in Fig. 2.6. The next part had 60 items with 36 on teacher actions and 24 on student actions. Every item was tagged to an aspect of the instructional core (D, S or R), a type of mathematics talk (learning talk or teaching talk) and model of instruction (Traditional instruction, Direct instruction, Teaching for understanding or Co-regulated learning strategies). Mathematics talk is elaborated in Chapter 9, but again very briefly mathematics talk from the perspectives of the teacher is teaching talk while that from the perspectives of the student is learning talk. Examples of these talks are shown in Fig. 2.6. As for models of instruction, Traditional instruction is teacher-centred with a focus on rote learning and memorisation. In the context of Asian classrooms it is often associated with drill and practice (Biggs & Watkins, 2001; Hogan, Chan, et al., 2013; Leung, 2006). Direct instruction

Survey items	Never/ Rarely	Sometimes	Frequently	Mostly/ Always
Reflecting on my lessons for the course (which I have chosen to do this survey on), I …				
use examples and non-examples to engage students in discussion to make sense of a concept [Development, Teaching talk – discussion, Teaching for Understanding]	o	o	o	o
provide students with directed guidance (ask close-ended questions) when they face difficulty with a mathematics task they are doing, focusing them on the concept/skill necessary to do the task [Student work, Teaching talk – recitation, Direct instruction]	o	o	o	o
ask direct questions to stimulate students' recall of past knowledge/check for understanding of concepts being developed in the lesson [Review, Teaching talk – recitation, Traditional instruction]	o	o	o	o
Reflecting on my lessons for the course (which I have chosen to do this survey on), I get my students to …				
provide answers or solutions (without any explanations) to my questions [Development, Learning talk - narrate, Traditional instruction]	o	o	o	o
ask questions when they do not understand [Development, Learning talk – question, Direct instruction]	o	o	o	o
teach/explain to another classmate while doing individual assigned seatwork [Student work, learning talk – explain, Teaching for Understanding]	o	o	o	o
review their mistakes and identify possible causes by themselves [Review, learning talk – evaluate, Co-regulated learning strategies]	o	o	o	o

Fig. 2.6 Items in part A of the survey

involves an explicit step-by-step strategy, often teacher-centred, with checks for mastery of procedural or conceptual knowledge (Good & Brophy, 2003; Hattie, 2003; Hogan, Chan, et al., 2013). Teaching for understanding places student learning at the core. Teacher facilitates, monitors and regulates student learning through student-centred approaches (Good & Brophy, 2003; Hogan, Chan, et al., 2013; Perkins, 1993). Co-regulated learning strategies involves self-directed learning, self-assessment and peer-assessment (Black, Harrison, Lee, Marshall, & Wiliam, 2003; Hogan, Chan, et al., 2013; Wiliam, 2007). Examples of actions from all four models of instruction are shown in Fig. 2.6.

The second part of the survey comprised five components which were: (i) introducing/developing a concept/formula/property, (ii) structure of worked examples and class practice, (iii) mathematical processes and metacognition, (iv) attitudes towards mathematics and (v) phases of lesson. Figure 2.7 shows examples of items from the five components.

Part C of the survey was a chronologically grounded one on instructional materials and it was customised for teachers from the different courses of study. Figures 2.8, 2.9, and 2.10 show examples of items in this part of the survey.

Survey items	Never/ Rarely	Sometimes	Frequently	Mostly/ Always
When introducing/developing a concept/formula/property, I explain the concept/formula/property to the whole class, asking students questions along the way	o	o	o	o
During my lessons the structure of worked examples and class practice can be as follows. I explain the solutions of a few worked examples before students go on with the practice questions.	o	o	o	o
To foster mathematical processes (reasoning, communications, connections, thinking skills, heuristics, applications) and metacognition, I get my students to check for reasonableness of their answers after solving a problem.	o	o	o	o
Attitudes towards mathematics I build students' confidence in doing mathematics by starting with tasks that students can do before progressing to more difficult tasks.	o	o	o	o
Phases of lessons I typically structure the different phases of a lesson or across lessons within a topic as follows: Introduction → Development → Consolidation → Conclusion	o	o	o	o

Fig. 2.7 Items in part B of the survey

The following is a list of reference materials. Rank the materials in order of usefulness, 9 being the one most useful to you.									
	1	2	3	4	5	6	7	8	9
Main textbook*	o	o	o	o	o	o	o	o	o
Supplementary textbook(s)	o	o	o	o	o	o	o	o	o
Main workbook**	o	o	o	o	o	o	o	o	o
Supplementary workbook(s)	o	o	o	o	o	o	o	o	o
School-based resource(s)	o	o	o	o	o	o	o	o	o
Commercial materials	o	o	o	o	o	o	o	o	o
Online resources	o	o	o	o	o	o	o	o	o
MOE-produced resources	o	o	o	o	o	o	o	o	o
Others	o	o	o	o	o	o	o	o	o

Fig. 2.8 Survey item in part C on reference materials (*Textbook and **Accompanying workbook adopted by the school for mathematics instruction and students buy their own copies)

Instructional Materials Recall that • Reference materials are resources that you refer to when you prepare for your lesson. • Instructional materials are what you bring into the classroom and that you actually use. We would like to find out the relationship between your reference materials and your instructional materials. Please select the option that is most applicable to you.
My instructional materials are exactly the same as my reference materials.
Sometimes, I adapt/modify my instructional materials from my reference materials.
Frequently, I adapt/modify my instructional materials from my reference materials.
Almost always, I adapt/modify my instructional materials from my reference materials.

Fig. 2.9 Survey item in part C on instructional materials

2.3.2.2 Method and Data Collected

Teachers with three or more years of mathematics teaching experience from all secondary schools in Singapore were invited to participate in the survey. The survey was online and completed by the teachers individually. Data was collected over a period of three months (September till November) in the second semester of the school year in 2018. The participants were asked to reflect on their lessons for the course (which they had chosen to do this survey on), and respond on a Likert Scale of 1 (Never/Rarely) to 4 (Mostly/Always) and also give qualitative responses when asked to do so. Therefore, the survey collected teacher responses that were both quantitative and qualitative. A Likert Scale comprising four points was used deliberately so that teachers doing the survey were pushed to form an opinion as a “neutral option” was absent. Options 1 (Never/Rarely), 2 (Sometimes), 3 (Frequently) and 4 (Mostly/Always) mirrored frequencies ranging from up to 25, 50, 75 and 100% of the times respectively.

At the start of the survey, participants indicated the course of study (Integrated Programme (IP), Express, Normal (Academic) (N(A)) or Normal (Technical) (N(T))

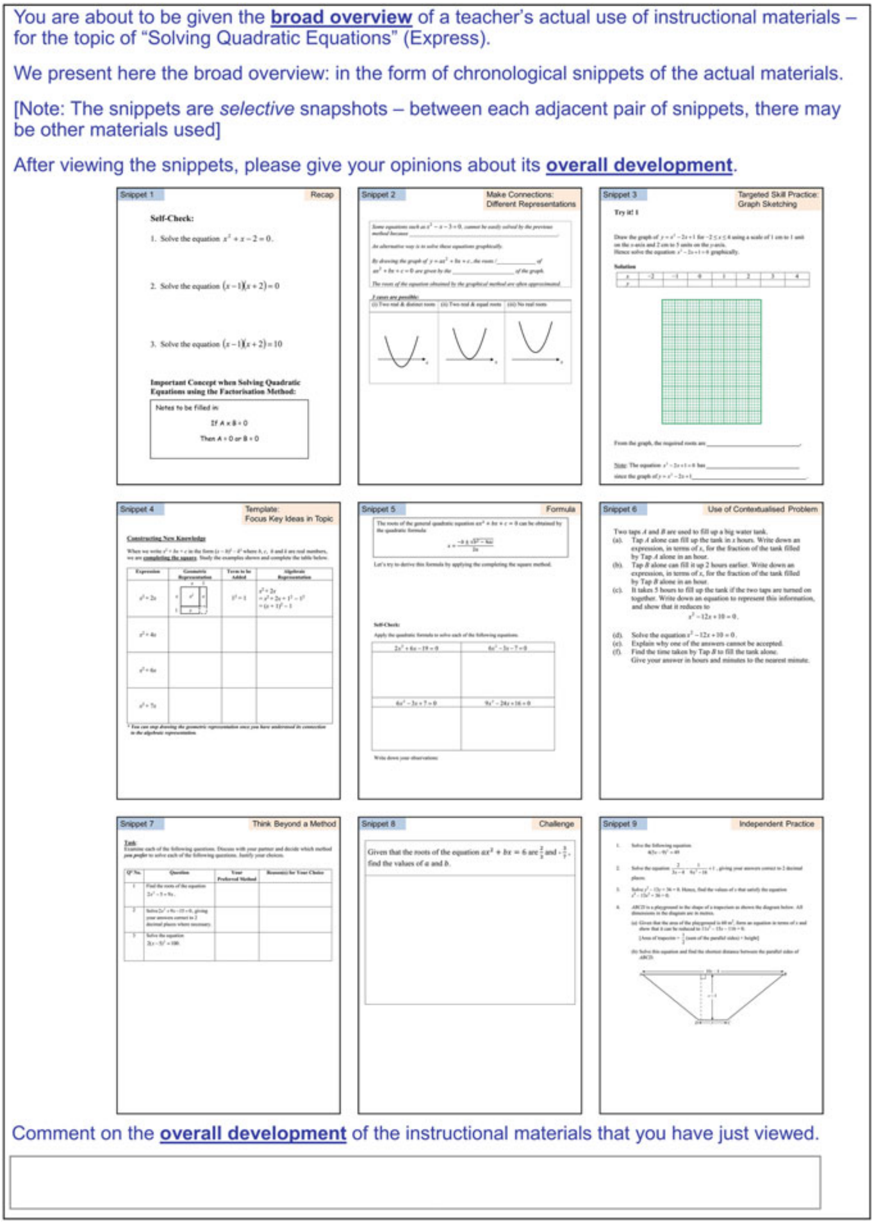


Fig. 2.10 Sample item in part C on teacher's actual use of instructional materials (*Note* The nine snippets are for illustration only and so they are not readable)

Table 2.4 Number of teacher participants in survey-segment of the study

Course of study	Gender N (%)		Years of mathematics teaching N (%)			Total N (%)	%
	Male	Female	3–9	10–19	20 or more		
Integrated programme	21 (3.1)	37 (5.5)	17 (2.5)	22 (3.3)	19 (2.8)	58 (8.6)	64.7
Express	135 (19.9)	245 (36.2)	168 (24.8)	139 (20.5)	73 (10.8)	380 (56.1)	
Normal (Academic)	66 (9.7)	85 (12.6)	69 (10.2)	53 (7.8)	29 (4.3)	151 (22.3)	35.3
Normal (Technical)	46 (6.8)	42 (6.2)	51 (7.5)	30 (4.4)	7 (1.0)	88 (13.0)	
Total	268 (39.6)	409 (60.4)	305 (45.0)	244 (36.0)	128 (18.9)	677 (100)	100

and subject (Mathematics or Additional Mathematics) they were doing the survey on. For parts A and B of the survey, the items were the same for all participants. However, for part C this was not the case as the survey items were specific to instructional materials by course of study and subject. Therefore, participants either attempted survey items on Additional Mathematics or Mathematics (Express) or Mathematics (N(A)/N(T)).

2.3.2.3 Participants

In this phase of the study which was the survey-segment, 691 participants (teachers) completed the survey. In the preliminary screening of the data, some responses were removed as they did not meet the requirements of the survey. The data of 677 teachers were used for subsequent analyses. Table 2.4 shows the profile of these teachers.

Forty percent of the teachers were male while 60% were female. This was representative of the demographic of the teacher population in secondary schools which was 36% males and 64% females (MOE, 2018). In addition, for the representation by course of study, almost 65% for the IP and Express course and 35% for the N(A) and N(T) courses was also coherent with the demographic of the student population in secondary schools which was 64 and 36% respectively for the IP and Express course and N(A) and N(T) courses (MOE, 2018). Forty-five percent of the teachers had more than three but less than ten years of mathematics teaching experience while the remaining 55% had more than ten years of the same experience. In the video-segment of the study, 80% of the teachers had more than ten years of mathematics teaching experience. The difference in participation percent by this group of teachers in the two phases of the study is not surprising as experienced and competent teachers generally take years of teaching to hone their pedagogies.

Appendix

Kaur, B. (2018). Students' voice: Perspectives of how their teachers facilitate learning of mathematics. In F.-J. Hsieh (Ed.), *Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education* (Vol. 1, pp. 73–82). Taipei, Taiwan: Taiwan Association for Mathematics Education.

Kaur, B., Lee, N. H., Ng, K. E. D., Yeo, B. W. J., Yeo, K. K. J., & Safii, L. (2019, July). Instructional strategies adopted by experienced secondary teachers when enacting the Singapore school mathematics curriculum. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research impacting practice (Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia)* (pp. 86–89). Perth: MERGA.

Kaur, B., Tay, E. G., Toh, T. L., Leong, Y. H., & Lee, N. H. (2018). A study of school mathematics curriculum enacted by competent teachers in Singapore secondary schools. *Mathematics Education Research Journal, 30*(1), 103–116.

Kaur, B., Tay, E. G., & Tong C. L. (2019). The instructional core of mathematics lessons in Singapore secondary schools. In M. Graven, H. Venkat, A. Essien, & P. Vale (Eds.), *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 4–55). Pretoria, South Africa: PME.

Kaur, B., & Toh, W. Y. K. (2019). Students' perspectives of good mathematics lessons, homework and how their teachers facilitate learning of mathematics. In T. L. Toh, E. G. Tay, & B. Kaur (Eds.), *Mathematics education in Singapore* (pp. 269–285). Singapore: Springer Nature.

Kaur, B., Toh, T. L., Lee, N. H., Leong, Y. H., Cheng, L. P., Ng, K. E. D., Yeo, K. K. J., Yeo, J. B. W., Wong, L. F., Tong, C. L., Toh, W. Y. K., & Safii, L. (2019). *Twelve questions on mathematics teaching—Snapshots from A Study of the Enacted School Mathematics Curriculum in Singapore*. Singapore: National Institute of Education.

Kaur, B., Toh, T. L., & Toh, W. Y. K. (2018). Opportunity to learn mathematics afforded by two competent teachers in their lessons. In E. Bergqvist, M. Osterholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, p. 81). Umeå, Sweden: PME.

Kaur, B., Wong, L. F., & Chew, C. K. (2018). Mathematical tasks enacted by two competent teachers to facilitate the learning of vectors by grade ten students. In P. C. Toh & B. L. Chua (Eds.), *Goals, tasks, and activities for mathematics instruction* (pp. 49–66). Singapore: World Scientific.

Kaur, B., Wong, L. F., & Toh, T. L. (2017). A framework to examine the mathematics in lessons of competent mathematics teachers in Singapore. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 41–48). Singapore: PME.

Kaur, B., Wong, L. F., Toh, W. Y. K., & Tong, C. L. (2019). Mathematical vocabulary: A bridge that connects ideas. In T. L. Toh & J. B. W. Yeo (Eds.), *Big ideas in mathematics* (pp. 275–294). Singapore: World Scientific.

Lee, N. H., Ng, K. E. D., Yeo, B. W. J., Yeo, K. K. J., Kaur, B., Toh, T. L., Leong, Y. H., & Tay, E. G. (2018, November). *Pedagogies adopted by experienced secondary school mathematics teachers in relation to the intended curriculum.* A paper presented at Symposium: Pedagogies Adopted by Experienced Secondary School Mathematics Teachers. Paper presented at ERAS-APERA International Conference 2018, Singapore.

Leong, Y. H., Cheng, L. P., & Toh, W. Y. K. (2018). Designing practice examples to support mathematics reasoning. In *Proceedings of the 8th ICMI-East Asia Regional Conference of Mathematics Education* (Vol. 2, pp. 38–45). Taipei, Taiwan: Taiwan Association for Mathematics Education.

Leong, Y. H., Cheng, L. P., & Toh, W. Y. K. (2018). Design of mathematics instructional materials: A blank space. *Maths Buzz, 18*(1), 4–6. Singapore: Association of Mathematics Educators.

Leong, Y. H., Cheng, L. P., & Toh, W. Y. K. (2019). Formula: To derive or not to derive? *Maths Buzz, 19*(1), 2–5. Singapore: Association of Mathematics Educators.

Leong, Y. H., Cheng, L. P., & Toh, W. Y. K. (2019, July). Chronologically-grounded survey. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research impacting practice (Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia)* (pp. 90–93). Perth: MERGA.

Leong, Y. H., Cheng, L. P., Toh, W. Y. K., Kaur, B., & Toh, T. L. (2019). Example sequences to help students recognise and apply formula: The Singapore situation. *The Mathematics Educator, 18*(1&2), 23–48.

Leong, Y. H., Cheng, L. P., Toh, W. Y. K., Kaur, B., & Toh, T. L. (2019). Making things explicit using instructional materials: A case study of a Singapore teacher's practice. *Mathematics Education Research Journal, 31*(1), 47–66.

Leong, Y. H., Cheng, L. P., Toh, W. Y., Kaur, B., & Toh, T. L. (in press). Teaching students to apply formula using instructional materials: A case of a Singapore teacher's practice. *Mathematics Education Research Journal.*

Leong, Y. H., & Kaur, B. (2019). The enacted school mathematics curriculum. In T. L. Toh, E. G. Tay, & B. Kaur (Eds.), *Mathematics education in Singapore* (pp. 55–65). Singapore: Springer Nature.

Leong, Y. H., Kaur, B., Lee, N. H., & Toh, T. L. (2019). Exemplary characteristics of mathematics teachers. In T. L. Toh, E. G. Tay, & B. Kaur (Eds.), *Mathematics education in Singapore* (pp. 385–404). Singapore: Springer Nature.

Ng, K. E. D. (2018). *A constructivist approach to the teaching of arc lengths, sector areas and radian measures.* A paper presented at Symposium: Pedagogies Adopted by Experienced Secondary School Mathematics Teachers. Paper presented at ERAS-APERA International Conference 2018, Singapore.

Toh, T. L., Kaur, B., Tay, E. G., Lee, N. H., & Leong, Y. H. (2019, July). A study of school mathematics curriculum enacted by teachers in Singapore secondary schools. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research*

impacting practice (Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia) (pp. 82–85). Perth: MERGA.

Toh, W. Y. K., Leong, Y. H., & Cheng, L. P. (2019, July). *Making connections with instructional materials*. Paper presented at 42nd Mathematics Education Research Group of Australasia Conference, Perth, Australia.

Tong, C. L., Tay, E. G., Kaur, B., Quek, K. S., & Toh, T. L. (2019, July). Singapore secondary mathematics pedagogy: The DSR DNA. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research impacting practice (Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia)* (pp. 94–98). Perth: MERGA.

Yeo, J. B. W. (2018). Engaging the hearts of mathematics learners. In P. C. Toh & B. L. Chua (Eds.), *Mathematics instruction: Goals, tasks and activities* (pp. 115–132). Association of Mathematics Educators 2018 Yearbook. Singapore: World Scientific.

Yeo, K. K. J. (2018, November). *Mathematical practices enacted in the lessons: A case study of teaching quadratic equations.* A paper presented at Symposium: Pedagogies Adopted by Experienced Secondary School Mathematics Teachers. Paper presented at ERAS-APERA International Conference 2018, Singapore.

References

Anthony, G., & Walshaw, M. (2009). Characteristics of effective teaching of mathematics: A view from the west. *Journal of Mathematics Education, 2,* 147–164.

Biggs, J., & Watkins, D. (2001). *Teaching the Chinese learner: Psychological and pedagogical perspectives.* Hong Kong: The University of Hong Kong, Comparative Education Research Centre.

Black, P., Harrison, C., Lee, C., Marshall, B., & Wiliam, D. (2003). *Assessment for learning: Putting it into practice.* Milton Keynes: Open University Press.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record, 110*(3), 608–645.

City, E. A., Elmore, R. F., Fiarman, S. E., & Teitel, L. (2009). *Instructional rounds in education: A network approach to improving teaching and learning.* Cambridge, MA: Harvard Education Press.

Clarke, D. J. (1998). Studying the classroom negotiation of meaning: Complementary accounts methodology. Chapter 7 in A. Teppo (Ed.), *Qualitative research methods in mathematics education*, monograph number 9 of the *Journal for Research in Mathematics Education* (pp. 98–111). Reston, VA: NCTM.

Clarke, D. J. (Ed.). (2001). *Perspectives on practice and meaning in mathematics and science classrooms.* Dordrecht, the Netherlands: Kluwer Academic Press.

Clarke, D. J., Keitel, C., & Shimizu, Y. (Eds.). (2006). *Mathematics classrooms in twelve countries: The insider's perspective* . Rotterdam: Sense Publishers.

Eisenmann, T., & Even, R. (2009). Similarities and differences in the types of algebraic activities in two classes taught by the same teacher. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 152–170). New York: Routledge.

Fan, L., & Bokhove, C. (2014). Rethinking the role of algorithms in school mathematics: A conceptual model with focus on cognitive development. *ZDM Mathematics Education, 46,* 481–492.

Good, T. L., & Brophy, J. E. (2003). *Looking in classrooms*. New York: Allyn & Bacon.

Hattie, J. (2003, October). *Teachers make a difference: What is the research evidence? Distinguishing expert teachers from novice and experienced teacher*. Paper presented at the Building Teacher Quality: What does the research tell us ACER Research Conference, Melbourne, Australia.

Hattie, J. (2012). *Visible learning for teachers: Maximizing impact on learning*. London: Routledge.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., … Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 video study*. Washington, DC: U.S. Department of Education, National Centre for Education Statistics.

Hogan, D., Chan, M., Rahim, R., Kwek, D., Aye, K. M., Loo, S. C. … Luo, W. (2013). Assessment and the logic of instructional practice in secondary three English and mathematics classrooms in Singapore. *Review of Education, 1,* 57–106.

Hogan, D., Kwek, D., Towndrow, P., Rahim, R. A., Tan, T. K., Yang, H. J., & Chan, M. (2013). Visible learning and the enacted curriculum in Singapore. In Z. Deng, S. Gopinathan, & C. Lee (Eds.), *Globalization and the Singapore curriculum* (pp. 121–149). Dordrecht: Springer.

Hogan, D., Towndrow, P., Chan, M., Kwek, D., & Rahim, R. A. (2013). *CRPP Core 2 research program: Core 2 interim final report* . Singapore: National Institute of Education.

Kaur, B. (2009). Characteristics of good mathematics teaching in Singapore grade 8 classrooms: A juxtaposition of teachers' practice and students' perception. *ZDM Mathematics Education, 41,* 333–347.

Kaur, B. (2013). Participation of students in content-learning classroom discourse: A study of two grade 8 mathematics classrooms in Singapore. In B. Kaur, G. Anthony, M. Ohtani, & D. Clarke (Eds.), *Student voice in mathematics classrooms around the world* (pp. 65–88). Rotterdam: Sense Publishers.

Kaur, B. (2014). Enactment of school mathematics curriculum in Singapore: Whither research. *ZDM Mathematics Education, 46,* 829–836.

Kaur, B., Low, H. K., & Seah, L. H. (2006). Mathematics teaching in two Singapore classrooms: The role of the textbook and homework. In D. Clarke, K. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve classrooms* (pp. 99–115). Rotterdam: Sense Publishers.

Leong, Y. H., Ho, W. K., & Cheng, L. P. (2015). Concrete-pictorial-abstract: Surveying its origins and charting its future. *The Mathematics Educator, 16*(1), 1–19.

Leung, F. K. S. (2001). In search of an East Asian identity in mathematics education. *Educational Studies in Mathematics, 47*(1), 35–51.

Leung, F. K. S. (2006). Mathematics education in East Asia and the West: Does culture matter? In F. Leung, K. D. Graf, & F. Lopez-Real (Eds.), *Mathematics education in different cultural traditions: A comparative study of East Asia and the West* (pp. 21–46). New York: Springer.

Ministry of Education. (n.d.). Retrieved April 4, 2017, from https://www.moe.gov.sg/education/education-system/21st-century-competencies.

Ministry of Education. (2012). *The teaching and learning of 'O' Level, N(A) Level & N(T) Level mathematics*. Author: Singapore.

Ministry of Education. (2018). *Education statistics digest 2018*. Singapore: Ministry of Education.

Perkins, D. (1993). Teaching for understanding. *American Educator, 17*(3), 8, 28–35.

Remillard, J. T., & Heck, D. J. (2014). Conceptualising the curriculum enactment process in mathematics education. *ZDM Mathematics Education, 46,* 705–718.

Schoenfeld, A. (2011). *How we think: A theory of goal-orientated decision making and its educational applications*. New York: Routledge.

Seah, L. H., Kaur, B., & Low, H. K. (2006). Case studies of Singapore secondary mathematics classrooms: The instructional approaches of two teachers. In D. Clarke, K. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve classrooms* (pp. 151–165). Rotterdam: Sense Publishers.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used din reform classrooms. *American Educational Research Journal, 33*(2), 455–488.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap*. New York: The Free Press.

Stigler, J. W., Gonzalez, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS videotape classroom study: Methods and findings from an exploratory research project on eighth-grade mathematics instruction in Germany, Japan and the United States*. Washington, DC: U.S. Department of Education, National Centre for Education Statistics.

Walshaw, M., & Anthony, G. (2008). The role of pedagogy in classroom discourse: A review of recent research into mathematics. *Review of Educational Research, 78,* 516–551.

Wiliam, D. (2007). Keeping learning on track: Classroom learning and the regulation of assessment. In K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1053–1098). Charlotte, NC: Information Age Publishing.

Berinderjeet Kaur is a Professor of Mathematics Education at the National Institute of Education in Singapore. She holds a Ph.D. in Mathematics Education from Monash University in Australia. She has been with the Institute for the last 30 years and is one of the leading figures of Mathematics Education in Singapore. In 2010, she became the first full professor of Mathematics Education in Singapore. She has been involved in numerous international studies of Mathematics Education and was the Mathematics Consultant to TIMSS 2011. She was also a core member of the MEG (Mathematics Expert Group) for PISA 2015. She is passionate about the development of mathematics teachers and in turn the learning of mathematics by children in schools. Her accolades at the national level include the public administration medal in 2006 by the President of Singapore, the long public service with distinction medal in 2016 by the President of Singapore and in 2015, in celebration of 50 years of Singapore's nation building, recognition as an outstanding educator by the Sikh Community in Singapore for contributions towards nation building.

Eng Guan Tay is an Associate Professor and Head in the Mathematics and Mathematics Education Academic Group of the National Institute of Education at Nanyang Technological University, Singapore. Dr Tay obtained his PhD in the area of Graph Theory from the National University of Singapore. He has continued his research in Graph Theory and Mathematics Education and has had papers published in international scientific journals in both areas. He is co-author of the books *Counting, Graph Theory: Undergraduate Mathematics*, and *Making Mathematics Practical*. Dr Tay has taught in Singapore junior colleges and also served a stint in the Ministry of Education.

Tin Lam Toh is an Associate Professor and currently the Deputy Head of the Mathematics and Mathematics Education Academic Group in the National Institute of Education, Nanyang Technological University of Singapore. He obtained his PhD from the National University of Singapore in 2001. A/P Toh continues to do research in mathematics as well as mathematics education. He has published papers in international scientific journals in both areas.

Yew Hoong Leong is an Associate Professor at the National Institute of Education, Nanyang Technological University. He began his academic career in mathematics education with the motivation of improving teaching by grappling with the complexity of classroom instruction. Along the journey, his research has broadened to include mathematics problem solving and teacher professional development. Together with his project teammates, they developed "Realistic Ambitious Pedagogy" and its accompanying plan of action—the "Replacement Unit Strategy".

Ngan Hoe Lee is an Associate Professor at the National Institute of Education (NIE). He taught Mathematics and Physics in a secondary school before becoming a Gifted Education Specialist at the Ministry of Education. At NIE, he teaches pre- and in-service as well as postgraduate courses in mathematics education and supervises postgraduate students pursuing Masters and PhD degrees. His publication and research interests include the teaching and learning of mathematics at all levels—primary, secondary and pre-university, covering areas such as mathematics curriculum

development, metacognition and mathematical problem solving/modelling, productive failure and constructivism in mathematics education, technology and mathematics education, and textbooks and mathematics education.

Chapter 3
The Instructional Core That Drives the Enactment of the School Mathematics Curriculum in Singapore Secondary Schools

Berinderjeet Kaur, Eng Guan Tay, Cherng Luen Tong, Tin Lam Toh, and Khiok Seng Quek

Abstract A study of mathematics lessons enacted by 30 experienced and competent mathematics teachers in Singapore secondary schools shows that an instructional core drives mathematics lessons. Teachers enact their instructional objectives through micro-instructional objectives that draw on three main components, viz. Development [D], Student Work [S] and Review of Student Work [R]. A lesson often comprises of one or more cycles of instruction depending on the number of objectives. A cycle comprises combinations of D, S and R. The survey data collected from 677 secondary school mathematics teachers affirmed the hypothesised instructional core of mathematics lessons. Further analysis of the survey items showed that it is not possible to simply label actions of mathematics teachers as student-directed, teacher-directed, fluency or conceptual orientated. Rather, they are amalgams of these. Four factors that appear to aptly encompass actions of the teachers are: (i) student-centred in-class learning, (ii) teaching and practice for fluency, (iii) teacher-led conceptual learning, and (iv) teacher-guided student self-directed learning. The instructional core comprising the DSR cycle may be said to be the DNA of mathematics lessons in Singapore secondary schools.

B. Kaur (✉) · E. G. Tay · C. L. Tong · T. L. Toh · K. S. Quek
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: berinderjeet.kaur@nie.edu.sg

E. G. Tay
e-mail: engguan.tay@nie.edu.sg

C. L. Tong
e-mail: cherngluan.tong@nie.edu.sg

T. L. Toh
e-mail: tinlam.toh@nie.edu.sg

K. S. Quek
e-mail: khiokseng.quek@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_3

Keywords Instructional core · Instructional objectives · Micro-instructional objectives · DSR cycle · Teaching moves · Learning moves · Mathematics teachers · Secondary schools · Singapore

3.1 Introduction

The instructional core of any lesson underpins interactions between the teacher, the student and the content. Through a myriad of activities designed by teachers, the instructional core facilitates and connects these interactions while enacting the curriculum in schools. As noted by City, Elmore, Fiarman, and Teitel (2009),

> … it is the relationship between the teacher, the student, and the content – not the qualities of any one of them by themselves – that determines the nature of instructional practice, [even though] each … has its own particular role and resources to bring to the instructional process. (pp. 22–23)

Like the DNA of the human body, an instructional core unfolds the character of a lesson. The DNA of a human body is made up of building blocks. Similarly, mathematics lessons can also comprise instructional components (building blocks) that constitute the interactions between teacher, student and content, and characterise them.

Several past studies (Chang, Kaur, Koay, & Lee, 2001; Ho & Hedberg, 2005; Hogan et al., 2013; Kaur & Loh, 2009; Kaur & Yap, 1998) on mathematics classroom instruction reported in Leong and Kaur (2019) have led to portraits of mathematics teaching being teacher-centred and focussed on procedural fluency. However, none of these studies had attempted to unpack aspects of teacher and student actions that drive mastery learning. This chapter attempts to do this by presenting data from the project described in Chapter 2, which involved a study of mathematics lessons enacted by 30 experienced and competent teachers and survey data of another 677 mathematics teachers from secondary schools in Singapore. The question that guides the investigation reported in this chapter is, "What is the instructional core that drives the enactment of secondary school mathematics?"

3.2 An Instructional Core That Drives Mathematics Lessons of Experienced and Competent Teachers

In the first phase of the project, as noted in Chapter 2, sequences of lessons of 30 experienced and competent mathematics teachers were documented using the complementary accounts methodology developed by Clarke (2001). Lessons of four teachers, Teacher 1, Teacher 21, Teacher 24 and Teacher 29, that were representative of the 30 were analysed in depth. The first author, who was also the lead principal investigator of the project, was involved in documenting the classroom practice of the four teachers. Her observations of similar patterns of enactment by the four teachers

Table 3.1 Metadata of the four teacher

Teacher	Course of study	Sex	Age group	Number of years of teaching mathematics experience	Status of teacher
Teacher 21	Integrated Programme	F	50–59	15–20	Lead teacher
Teacher 1	Express	F	40–49	20–25	Lead teacher
Teacher 29	Normal (Academic)	F	50–59	20–25	Senior teacher
Teacher 24	Normal (Technical)	M	50–59	15–20	Senior teacher

Table 3.2 Details of lessons of the four teachers

Teacher	Course of study	Secondary level taught	Topic	Number of lessons	Total instructional time for topic (minutes)
Teacher 21	Integrated programme	2	Quadratic equations and graphs	7	420
Teacher 1	Express	4	Vectors	9	540
Teacher 29	Normal (Academic)	4	Probability	6	465
Teacher 24	Normal (Technical)	4	Pythagoras theorem and trigonometry	6	390

Key—*Secondary 2 is year 8 of schooling

led her to select their lessons for an in-depth study. Therefore, it may be said that the sample was a convenient one. Table 3.1 shows the profile of the four teachers.

The four teachers each had at least 15 years of mathematics teaching experience in Singapore secondary schools. Teacher 21 and Teacher 1 were Lead teachers while Teacher 29 and Teacher 24 were Senior teachers. A Lead teacher is one who is nationally recognised for his or her teaching competency and is trusted with the charge of developing fellow teachers in their school and the nation. A Senior teacher is one who is locally recognised for his or her teaching competency and is trusted with the charge of developing junior teachers in the school. Prior to becoming a Lead teacher, one has to be a Senior teacher. Table 3.2 shows details of the lessons taught by the four teachers.

It is apparent from Table 3.2 that the teachers taught different topics, with three of them teaching Secondary 4 (Grade 10) and one Secondary 2 (Grade 8). The instructional sequences ranged from six to nine lessons. The least instruction time documented for a topic was 6.5 h in duration.

A grounded theory approach similar to the one described in Seah, Kaur, and Low (2006) was adopted. The analysis revealed that mathematics lessons enacted by the teachers were driven by very specific instructional objectives, such as:

(i) verify and state Pythagoras theorem, and
(ii) find the unknown sides of a right-angled triangle using Pythagoras theorem.

To achieve these objectives, teachers detailed micro-instructional objectives. When enacting the micro-instructional objectives an instructional core appeared to drive the teaching and learning of mathematics in their lessons. The components of the instructional core that encapsulate interactions between the teacher, student and content are:

D →	Teacher develops concepts/demonstrates skills/engages students in activities to explore concepts
S →	Teacher sets students work to do [students apply the concepts/practise skills]
R →	Teacher reviews student work, drawing the attention of the whole class to errors, misconceptions, correct solutions, good presentations, etc.

Through the varied combinations of D, S and R, as shown in Episodes 3.1 and 3.2, teachers enacted micro-instructional objectives to achieve their instructional objectives. Episode 3.1 shows that to achieve the objective "To verify and state Pythagoras theorem", a teacher went through six micro-instructional objectives that involved components of the instructional core, D, S and R. Episode 3.2 further exemplifies how the teacher achieved the next objective in a similar manner.

Episode 3.1

Objective: To verify and state Pythagoras theorem			
Cycle 1	Core*	Micro-instructional objective	Teacher/student activities
	R	Review past knowledge	Teacher drew a right-angled triangle and reviewed its properties with input from students
	D	Verify Pythagoras theorem	Teacher explained the activity (guided investigation) and gave out the activity kits
	S	Verify Pythagoras theorem	Students carried out the activity verifying that the sum of the areas of the squares on the longest side of the triangle = sum of the area of the squares on the other two sides of the triangle
	R	Verify Pythagoras theorem	With inputs from students related to the activity, teacher formalised the relationship between the sides of a right-angled triangle

(continued)

(continued)

Objective: To verify and state Pythagoras theorem			
Cycle 1	Core*	Micro-instructional objective	Teacher/student activities
	R	Develop the vocabulary related to the sides of a right-angled triangle	Teacher labelled the sides of the right-angled triangle on the board; named the longest side as the hypotenuse and drew the attention of the whole class to the 'new' word ("must use it and spell it correctly")
	R	State Pythagoras theorem	With inputs from students, teacher wrote the theorem on the board: $a^2 = b^2 + c^2$ (where a is the length of the hypotenuse)

*Components of instructional core

Episode 3.2

Objective: To find the length of the hypotenuse given the other two sides of a right-angled triangle			
Cycle 2	Core*	Micro-instructional objective	Teacher/student activities
	D	Demonstrate how to find the length of hypotenuse of a right-angled triangle given the other two sides	Teacher demonstrated an example of how to find the hypotenuse of a right-angled triangle with sides 3 cm and 4 cm
	S	Engage students in applying new knowledge and skill building	Students worked individually and found the hypotenuse of a given right-angled triangle. Teacher walked around the class noting student work for "review"
	R	Monitor student understanding	Teacher drew on samples of student work (correct and incorrect solutions) and invited inputs from students. For the incorrect solutions, errors and their causes were identified. As some computational errors were due to incorrect use of the calculator, teacher did a quick review of how to find squares and square roots using a calculator
	S	Reinforce—application of new knowledge and skill building	Students worked individually and found the hypotenuse of a given right-angled triangle. Teacher walked around the class noting student work for "review"

(continued)

(continued)

Objective: To find the length of the hypotenuse given the other two sides of a right-angled triangle			
Cycle 2	Core*	Micro-instructional objective	Teacher/student activities
	R	Monitor student understanding	Teacher drew on samples of student work (mostly correct solutions) and invited inputs from students. Teacher highlighted correct and logical presentations

*Components of instructional core

A lesson often comprised of one or more cycles of instruction depending on the number of objectives. A cycle comprised combinations of D, S and R such as R-D-S-R-R-R or D-S-R-S-R as shown in Fig. 3.1.

After analysing the lessons of the four teachers (Teacher 21, Teacher 1, Teacher 29 and Teacher 24), the lessons of the other 26 experienced and competent teachers that participated in the video-segment were reviewed by three members of the project team and it was found that similar cyclic patterns were present in their lessons too. Our findings have led us to conjecture that the DNA of mathematics lessons in the classrooms of secondary school mathematics teachers in Singapore is as shown in Fig. 3.2.

Figure 3.2 posits that the DNA of mathematics lessons is made up of the instructional components, D [Development], S [Student Work] and R [Review of Student Work] (akin to the building blocks of the human DNA), and that these constitute the instructional core that drives the teaching and learning of mathematics in Singapore secondary schools. The D component develops a concept or introduces a skill. Teachers may show, tell, explain or guide students to uncover/make sense of new concept(s). They may also introduce and demonstrate skill(s). The S component always follows D and involves students working on mathematical task(s) during classwork, homework or assessment. Students may do the work individually or in groups. The tasks involve application of new knowledge or skill building that was

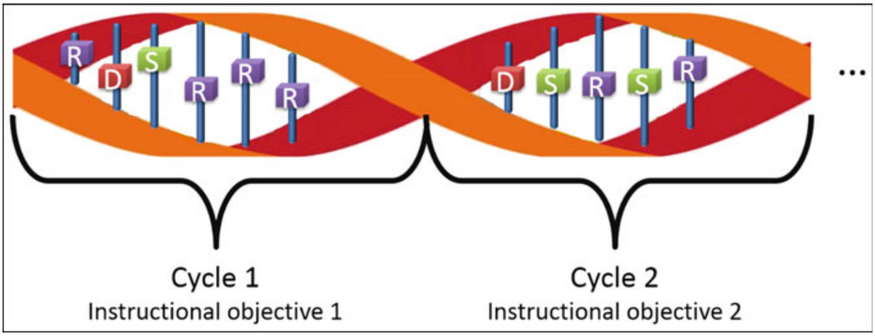


Fig. 3.1 Instructional cycles

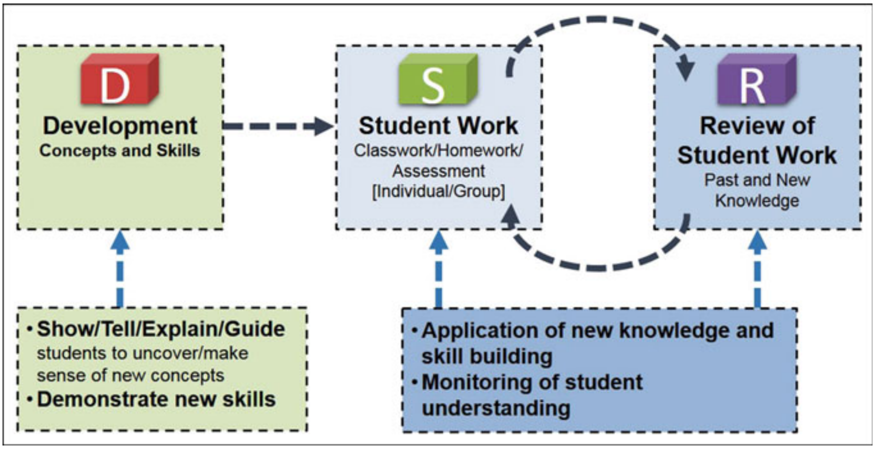


Fig. 3.2 DNA of mathematics lessons comprising the instructional core D, S and R

developed in the D component. The R component, a critical one, is where student understanding is monitored. Almost always, student work during the S component is used for whole class discussion during the R component. Good presentations are showcased, alternative solutions are discussed and erroneous solutions are examined and corrected with inputs from the class. This component also includes the review of past knowledge that is necessary for the lesson or a task that students are set to do. Many a time when a teacher is not satisfied with students' grasp of a concept or proficiency of a skill, the teacher engages students is more rounds of S $\rightarrow$ R $\rightarrow$ S $\rightarrow$ R … until he/she is satisfied with students' mastery of knowledge.

Just like the human DNA, combinations of actions from the components, D, S and R produce the varied types of lessons that teachers enact to achieve their instructional objectives. It appears that each instructional objective is achieved through an instructional cycle and a lesson may have one or more such cycles depending on the objectives of a lesson.

3.3 What Were Mathematics Teachers' Perceptions About the Instructional Core in General?

In the second phase of the project, a survey was conducted for 691 mathematics teachers from secondary schools in Singapore. Data from only 677 teachers were valid for analysis. The first part of this survey was based on the instructional core that was apparent in the lessons of experienced and competent mathematics teachers in secondary schools. Teachers who participated in the survey were introduced to the definitions of D, S and R as in Sect. 3.2 of the chapter. They were also shown the same examples of lesson objectives and their enactment through micro-instructional objectives comprising components, D, S and R (as shown in Episodes 3.1 and 3.2). Lastly, they were asked if their lessons: (i) were also guided by lesson objectives that were enacted through micro-instructional objectives, and (ii) comprised the instructional components D, S and R. Table 3.3 shows the survey items and respective responses.

The data in Table 3.3 shows that the instructional core was prevalent in the classrooms of mathematics teachers in secondary schools and affirmed the hypothesised instructional core of mathematics lessons. 94.8% of teachers who responded to the survey agreed that their lessons were guided by lesson objectives that are enacted through micro-instructional objectives. Thirty-five participants (5.2%) responded, “No, their lesson objectives were not enacted through micro-instructional objectives”. Table 3.4 shows the distribution of these 35 teachers across the courses of study.

It is apparent from Table 3.4 that across all the four courses of study between 4 and 5.8% of teachers claimed that their lessons objectives were not enacted through micro-instructional objectives. This affirms that the “No” response was not specific to any course of study. Figure 3.3 shows a sample of responses given by these 35 teachers. From the responses given by the teachers it was apparent that some teachers were actually enacting their lessons through micro-instructional objectives while others did not comprehend the term “micro-instructional objectives” or used different lenses to view their instructional practices.

Table 3.3 also shows that 96.8% of the teachers who did the survey recognised that the instructional components D, S and R were present in their lessons. However 22 teachers (3.2%) did not. Table 3.5 shows the distribution of these 22 teachers across the courses of study.

Table 3.3 Survey items and responses on the instructional core

Survey item	Response (%)	
	Yes	No
Are your lessons also guided by lesson objectives that are enacted through micro-instructional objectives?	94.8	5.2
Do your lessons also comprise the instructional components D, S and R?	96.8	3.2

Table 3.4 Distribution of the 35 teachers across the courses of study

Course of study	Teachers who responded to the survey		Teachers who responded “No”	
	N	%	*N*	%
Integrated programme	58	8.6	3	5.2
Express	380	56.1	22	5.8
Normal (Academic)	151	22.3	6	4.0
Normal (Technical)	88	13.0	4	4.5
Total	677	100	35	5.2

T(A) – Express
For each instructional objective, my lesson usually follow similarly to DSR.
Teacher develop concepts (through explorations or demonstrations)
Teacher to go through worked examples and students to practice
Homework will be assigned or mini quizzes to monitor and check students understanding of concept taught
Think I do not really pen down the micro-instructional objectives but the process of lesson implementation may include such objectives.

T(B) – Integrated Programme
For lessons on new concepts, students begin with reading and/or working on a problem or situation that is related to the new concept, but using their existing knowledge. Students arrive at the concept based on guided questions on a worksheet. My role is to help them arrive at the conclusion and clarify any assumptions. Then we move on to worked examples where they reinforce their understanding. The worked examples get progressively harder. So students mainly discover the concepts without me showing them.

T(C) – Normal (Academic)
Not really clear what is meant by 'micro-instructional objectives'. My lessons are guided by how well I know my class, their learning styles etc. I do not have a fix set of guidelines. I believe in varying the activities in my lessons in order to excite the students with what I have to teach. One lesson can be IT-based lesson, the other lesson can be a chalk-and-talk lesson, next lesson I can have Kahoot! Or activities such as paper-cutting. It also depends on what time are my lessons. If my lesson happens to be the last period, or in the afternoon, I need to have more activity-based lessons.

T(D) – Normal (Technical)
I have lesson objectives which comprise of i) what they will learn at the end of the lesson, ii) list of tasks the students will be doing during the lessons. I will also monitor understanding using Exit Pass (1 or 2 questions). But I do not have micro-objectives.

Fig. 3.3 Sample of responses from the 35 teachers

Table 3.5 Distribution of the 22 teachers across the courses of study

Course of study	Teachers who responded to the survey		Teachers who responded D, S and R were not present in their lessons	
	N	%	*N*	%
Integrated programme	58	8.6	2	3.4
Express	380	56.1	13	3.4
Normal (Academic)	151	22.3	5	3.3
Normal (Technical)	88	13.0	2	2.3
Total	677	100	22	3.2

It is apparent from Table 3.5 that across all the four courses of study between 2.3 and 3.4% of teachers claimed that the instructional components, D, S and R, were not present in their lessons. This again affirms that such a response was not specific to any course of study. Figure 3.4 shows a sample of responses given by these 22 teachers. From the responses given by the teachers, it was apparent that the instructional components, D, S and R, were present in their lessons. As the terminology of the

T(E) – Integrated Programme
We adopt the flipped classroom approach, where we get the students to view some videos then raise questions that come to their minds after viewing. Then those who are ready, just start doing the questions, those not clear will listen to worked examples being discussed in class before attempting the questions.

T(F) – Express
We do mostly carry out the DSR approach, but we now aim to have more student centred learning. Students have ownership of their learning through self-discovery or collaborative learning. However, the sequence would also include DSR.

T(G) – Normal (Academic)
Before D S R, explain motivation for new topic/ recap prior knowledge.

T(H) – Normal (Technical)
1. *Teacher introduce a hook to gain students' interest in that topic and also help students to see the relevance of the topic in relation to other subjects or the real world.*
2. *Teacher guide students to develop concept through meaningful activities and exploration*
3. *Teacher consolidate and demonstrate worked examples*
4. *Students practice the skills learnt*
5. *Teacher conduct class discussion to discuss alternative methods of solving the problem and also model good presentation*
6. *Set formative assessment*
7. *Conclude the lesson*

Fig. 3.4 Sample of responses from the 22 teachers

instructional components was new to many, they failed to contextualise them and claimed that they were absent in their lessons.

The next 60 items in the first part of the survey were the teaching and learning actions apparent in the classrooms of the 30 experienced and competent teachers that participated in the first phase of the project. Thirty-six of the items were on what the teachers did in class and another 24 were on what the teachers asked their students to do in class or as homework. The items were representative of the three instructional components, D (Development), S (Student Work) and R (Review of Student Work). Three sample items, that were part of the first 36, are as follows:

- I focus on mathematical processes (such as compare and contrast, logical reasoning) to facilitate the development of concepts or student understanding [D]
- I provide students with sufficient questions from textbooks/workbooks/other sources to practise so as to develop procedural fluency [S]
- I provide feedback to individuals for in-class work and homework to serve as information and diagnosis so that students can correct their errors or improve [R].

Teachers responded to the items on a Likert scale of 1 (Never/Rarely) to 4 (Mostly/Always).

In this chapter, we discuss the findings based on the analysis of the first 36 items. At the onset of the analysis, three items that focused on the use of resources were removed as we found that they did not fit into the scales we constructed. A detailed account of the analysis is found in Tong, Tay, Kaur, Quek, and Toh (2019). A Principal Component Analysis with both Varimax and Promax rotation methods resulted in four factors that considered 32 of the items. Another item, "I only progress to the next objective of the lesson when I am confident my students have grasped the one before" was also removed as it did not fit in the final four factors that resulted. The four factors are:

- Student-centred in-class learning (11 items)
- Teaching and practice for fluency (9 items)
- Teacher-led conceptual learning (7 items)
- Teacher-guided student self-directed learning (5 items).

Figures 3.5, 3.6, 3.7, and 3.8 show teacher actions in each of the four factors that appears to underlie teacher moves in mathematics lessons in Secondary schools.

Our justifications of the four factors that appear to underlie teacher moves in secondary mathematics classrooms of Singapore schools are as follows.

Instead of bifurcating into student-centred versus teacher-directed learning, or fluency versus conceptual learning, we find that these aspects are mixed and matched into four amalgams. The first is **student-centred in-class learning**. Teachers are student-centred both in the development phase (they ask questions to encourage reasoning, and build on students' responses) as well as in the seatwork phase (they provide students with probing guidance (open-ended questions), and walk around the class noting students' work that would be used to provide class feedback later). The second is **teaching and practice for fluency** . These all fall under items in Fluency subscales. Examples of these are using "I do, we do, you do" strategy

Factor 1 - Student-centred in-class learning

Reflecting on my lessons for the course (which I have chosen to do this survey on) I …

- ask students to recall past knowledge
- ask direct questions to stimulate students' recall of past knowledge/check for understanding of concepts being developed in the lesson
- ask questions to encourage reasoning and speculation, not just to elicit right answers
- use examples and non-examples to engage students in discussion to make sense of a concept
- focus on mathematical processes (such as compare and contrast, logical reasoning) to facilitate the development of concepts or student understanding
- lead whole class discussion (with guided questions) to facilitate the development of concepts
- exchange ideas with students on how to solve a problem
- ask students open-ended questions and allow them to build on one another's responses to develop concepts or clarify their understanding
- build on students' responses rather than merely receiving them
- provide students with probing guidance (open-ended questions about their thinking and why they are considering certain approaches) when they face difficulty with a mathematical task they are doing
- walk around the class noting students' work that I would draw on to provide the class feedback during whole class review when they are doing work at their desks

Fig. 3.5 Student-centred in-class learning scale

Factor 2 - Teaching and practice for fluency

Reflecting on my lessons for the course (which I have chosen to do this survey on) I …

- use "I do, We do, You do" strategy:
  - Demonstrate how to apply a concept/carry out a skill on the board [I do]
  - Demonstrate again using another similar example but with inputs from students [We do]
  - Ask students to do a similar question by themselves [You do]
- emphasise basic facts/steps for students to memorise them
- provide students with sufficient questions from textbooks/workbooks/other sources to practise so as to develop procedural fluency
- use exposition (teacher at the front talking to whole class) to explain mathematical ideas, facts, generalisations
- get students to automatise steps leading to a solution through repetitive exercises
- engage students in practising past exam papers
- provide students with directed guidance (ask close-ended questions) when they face difficulty with a mathematical task they are doing, focusing them on the concept/skill necessary to do the task
- tell students how to do it when they face difficulty with a mathematical task they are doing
- walk around the class and provide students with between desk instruction (i.e. help them with their difficulties) when they are doing their work at their desks

Fig. 3.6 Teaching and practice for fluency scale

Factor 3 - Teacher-led conceptual learning

Reflecting on my lessons for the course (which I have chosen to do this survey on) I …

- focus on mathematical vocabulary (such as factorise, solve) to help students adopt the correct skills needed to work on mathematical tasks
- explain what exemplary solutions of mathematics problems must contain (logical steps and clear statements and/or how marks are given for such work during examinations)
- encourage students to show me their work and review their progress for mathematics
- provide feedback to individuals for in-class work and homework to serve as information and diagnosis so that students can correct their errors or improve
- provide collective feedback to whole class for common mistakes and misconceptions related to in-class work and homework
- review student performance by providing the class detailed comments on tests and examinations

Fig. 3.7 Teacher-led conceptual learning scale

Factor 4 - Teacher-guided student self-directed learning

Reflecting on my lessons for the course (which I have chosen to do this survey on) I …

- help students identify strategies that would help them achieve their learning goals for mathematics
- get students to set their own learning goals for mathematics at the beginning of each school term/semester
- get students to make a plan to revise their work and correct their mistakes
- get students to work with peers to make a plan for revision and correction of mistakes
- get students to grade their own mathematics work (with the marking scheme/rubric provided and teach them how to use it)

Fig. 3.8 Teacher-guided student self-directed learning scale

Table 3.6 Component correlation matrix

	1	2	3
2	0.252		
3	0.507	0.361	
4	0.391	0.127	0.216

Legend
1- student-centred in-class learning
2- teaching and practice for fluency
3- teacher-led conceptual learning
4- teacher-guided student self-directed learning

during the development phase, and engaging students in practising past year exams. Next is **teacher-led conceptual learning**. Again these all fall under items in Fluency subscales but interestingly they are extracted under a different factor. Looking more closely at the items, we can understand why. Whereas the second factor emphasises fluency through thoughtful practice, this third factor emphasises fluency through conceptual understanding. Some items of this factor are focusing on mathematical vocabulary during the development phase, and helping students identify strategies during the review phase. The final factor is **teacher-guided student self-directed learning**. Indeed, students need guidance to revise on their own outside the classroom. For example, getting students to set their own learning goals and working with their peers to make a plan for revision and correction of mistakes. Thus, these moves are attempts by the teacher to ensure that learning takes place outside the classroom.

It is reasonable to believe in the East Asian context that fluency learning and conceptual learning are not mutually exclusive, nor student-centred learning and teacher-directed learning. For this reason, we chose Promax rotation in our Principal Components Analysis to see the correlations between the factors. Indeed, from Table 3.6, the four factors all have pairwise positive correlations. In particular, "student-centred in-class learning" has significant correlations[1] with "teacher-led conceptual learning" (0.507) and "teacher-guided student self-directed learning (0.391), and some correlation with "teaching and practice for fluency" (0.252). These correlations reinforce the connection between the D component, where the teacher demonstrates, and the S component, where the student works on his/her own understanding.

The discussion above gives us a clearer picture of how the Development-Student Work-Review of Student Work cycle plays out in Singapore classrooms. Data shows that these moves within these phases are generally enacted in the classroom. Interestingly, the data also shows that underlying these moves are student-centred considerations towards fluency and conceptual understanding.

[1] Tabachnik and Fidell (2013, p. 651) suggest that the existence of correlations "around 0.32 and above" warrant oblique rotation.

3.4 Concluding Remarks

Leung (2001) noted that in East Asian mathematics classrooms:

> Instruction is very much teacher dominated and student involvement minimal. … [Teaching is] usually conducted in whole group settings, with relatively large class sizes. … [There is] virtually no group work or activities, and memorization of mathematics is stressed … [and] students are required to learn by rote. … [Students are] required to engage in ample practice of mathematical skills, mostly without thorough understanding. (Leung, 2001, pp. 35–36)

The data presented in this chapter affirms that the teaching and learning of mathematics in Singapore secondary school go well beyond drill and practice, a stereotype of Asian Mathematics classrooms. Mathematics instruction in Singapore secondary school classrooms may be said to be guided by an instructional core, comprising three components that encapsulate interactions between teacher and student and content. These components are D—Development, S—Student Work and R—Review of Student Work.

References

Chang, A. S. C., Kaur, B., Koay, P. L., & Lee, N. H. (2001). An exploratory analysis of current pedagogical practices in primary mathematics classrooms. *The NIE Researcher, 192,* 7–8.

City, E. A., Elmore, R. F., Fiarman, S. E., & Teitel, L. (2009). *Instructional rounds in education: A network approach to improving teaching and learning*. Cambridge, MA: Harvard Education Press.

Clarke, D. (2001). *Perspectives on practice and meaning in mathematics and science classrooms.* Dordrecht: Kluwer.

Ho, K. F., & Hedberg, J. G. (2005). Teachers' pedagogies and their impact on students' mathematical problem solving. *Journal of Mathematical Behavior, 24,* 238–252.

Hogan, D., Chan, M., Rahim, R., Kwek, D., Aye, K. M., Loo, S.C., … Luo, W. (2013). Assessment and the logic of instructional practice in Secondary 3 English and Mathematics classrooms in Singapore. *Review of Education, 1,* 57–106.

Kaur, B., & Loh, H. K. (2009). *Student perspective on effective mathematics pedagogy: Stimulated recall approach study*. Singapore: National Institute of Education.

Kaur, B., & Yap, S. F. (1998). *Kassel project (NIE-Exeter Joint Study) third phase.* Singapore: National Institute of Education.

Leung, F. K. S. (2001). In search of an East Asian identity in mathematics education. *Educational Studies in Mathematics, 47*(1), 35–41.

Leong, Y. H., & Kaur, B. (2019). The enacted school mathematics curriculum. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 55–65). Singapore: Springer.

Seah, L. H., Kaur, B., & Low, H. K. (2006). Case studies of Singapore secondary mathematics classrooms: The instructional approaches of two teachers. In D. J. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The insider's perspective* (pp. 151–165). Rotterdam: Sense Publishers.

Tabachnik, B. G., & Fidell, L. S. (2013). *Using multivariate statistics* (6th ed.). Upper Saddle River, NJ: Pearson.

Tong, C. L., Tay, E. G., Kaur, B., Quek, K. S., & Toh, T. L. (2019, July). Singapore secondary mathematics pedagogy: The DSR DNA. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics education research impacting practice (Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia)* (pp. 94–98). Perth: MERGA.

Berinderjeet Kaur is a Professor of Mathematics Education at the National Institute of Education in Singapore. She holds a Ph.D. in Mathematics Education from Monash University in Australia. She has been with the Institute for the last 30 years and is one of the leading figures of Mathematics Education in Singapore. In 2010, she became the first full professor of Mathematics Education in Singapore. She has been involved in numerous international studies of Mathematics Education and was the Mathematics Consultant to TIMSS 2011. She was also a core member of the MEG (Mathematics Expert Group) for PISA 2015. She is passionate about the development of mathematics teachers and in turn the learning of mathematics by children in schools. Her accolades at the national level include the public administration medal in 2006 by the President of Singapore, the long public service with distinction medal in 2016 by the President of Singapore and in 2015, in celebration of 50 years of Singapore's nation building, recognition as an outstanding educator by the Sikh Community in Singapore for contributions towards nation building.

Eng Guan Tay is an Associate Professor and Head in the Mathematics and Mathematics Education Academic Group of the National Institute of Education at Nanyang Technological University, Singapore. Dr Tay obtained his PhD in the area of Graph Theory from the National University of Singapore. He has continued his research in Graph Theory and Mathematics Education and has had papers published in international scientific journals in both areas. He is co-author of the books *Counting, Graph Theory: Undergraduate Mathematics*, and *Making Mathematics Practical*. Dr Tay has taught in Singapore junior colleges and also served a stint in the Ministry of Education.

Cherng Luen Tong is a research assistant at the National Institute of Education. He is a trained Engineer from National Technological University. He made his mid-career switch to the teaching profession and taught mathematics in secondary schools for more than 10 years. Educational research arena was an uncharted path. The journey as a research assistant helped him grow interested in mathematics problem solving as well as statistical analysis.

Tin Lam Toh is an Associate Professor and currently the Deputy Head of the Mathematics and Mathematics Education Academic Group in the National Institute of Education, Nanyang Technological University of Singapore. He obtained his PhD from the National University of Singapore in 2001. A/P Toh continues to do research in mathematics as well as mathematics education. He has published papers in international scientific journals in both areas.

Khiok Seng Quek is Senior Lecturer at the Psychology and Child & Human Development (PCHD) Academic Group. He has served as the Sub-Dean of the Postgraduate Diploma in Education and as a consultant on teacher education in several NIE overseas projects. Prior to joining the PCHD Academic Group, Dr. Quek worked for about ten years as lecturer at the Mathematics and Mathematics Education Academic Group, where he taught a range of mathematics education courses at pre-service, in-service and graduate levels. He was also the coordinator for the Teaching of Primary Mathematics courses in the Diploma in Education and BA/BSc (Ed) programmes.

Part II
Enactment of the Intended Curriculum

Chapter 4
Learning Opportunities to Promote Conceptual Understanding in Singapore Secondary School Mathematics Instruction

Kai Kow Joseph Yeo

Abstract In the teaching of mathematics, experienced and competent teachers in Singapore secondary schools go beyond merely teaching facts, skills and conceptual structures. They use a repertoire of teaching approaches. Instructional activities in their lessons show a very systematic choice of variation and clear focus of teaching a specific concept. As teaching plays a major role in shaping students' learning opportunities, the opportunity for students to engage in constructing concepts depends largely on the nature of interaction generated by the teacher's pedagogical moves. This chapter discusses learning opportunities enacted by two experienced and competent teachers when they introduce concepts to students or engage students in constructing concepts in well-structured lessons. Distilling the learning opportunities requires a careful analysis of the classroom events that are instructionally guiding students' learning of mathematical concepts. Specifically representations and attending explicitly to concepts can facilitate conceptual learning among learners. In this chapter drawing on three excerpts of classroom instructions we discuss how two experienced and competent mathematics teachers create learning opportunities when they introduce concepts to students or engage students in constructing concepts. The teachers used their planned frames and enacted teacher-directed lessons that engaged students in making sense of concepts. Though the lessons were teacher-directed, the student–teacher classroom discourse was integral for the creation of knowledge by the students.

Keywords Learning opportunities · Experienced and competent mathematics teachers · Mathematical concepts · Classroom instructions

K. K. J. Yeo (✉)
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: kaikow.yeo@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_4

4.1 Introduction

In every mathematics lesson, the teacher interprets the curriculum and translates it into mathematical practices to be used in the classroom. While the teacher's teaching style can stimulate students' learning, it is their pedagogical content knowledge and understanding of the subject that develops or limits the students' mathematical understandings. Furthermore, teachers should aim to incorporate student-centred activities in their teaching, as well as carefully considering the learners' profile to ensure the instruction is appropriate. However, teachers are not automatons and vary in their knowledge, understanding and skills, even in the areas in which they are competent. As classroom teaching plays such a crucial role in students' learning, researchers have attempted to characterise the nature of the classroom teaching that maximises students' learning opportunities (Brophy & Good, 1996; National Academy of Education, 1999). In this chapter, we discuss students' learning opportunities provided by two experienced and competent teachers when they introduced concepts, a key component of the Singapore school mathematics curriculum framework (see Chapter 1, Fig. 1.2), to students and engaged them in constructing concepts in a well-structured lesson.

4.2 Teaching Mathematics That Promotes Conceptual Understanding

Ball, Lubienski, and Mewborn (2001) noted that 'what teachers and students are able [to] do together with mathematics in classrooms is at the heart of mathematics education' (p. 433). This underscored the importance of the roles that teachers and students play in mathematics classrooms. In particular, the teacher is not only responsible for explaining how to perform certain mathematical techniques, but must also understand and explain the concepts behind the techniques. The following sections explore two constructs of teaching mathematics that promote conceptual understanding: (i) using representations, and (ii) attending explicitly to mathematical concepts through classroom interactions.

4.2.1 Representations in the Learning of Mathematical Concepts

For many years, the principle of multiple representations has attracted much attention among mathematics educators. Representation is one of the critical constructs in research on the teaching and learning of mathematics (Cobb, Yackel, & Wood, 1992; Goldin, 1998; Janvier, 1987; Perkins & Unger, 1994; Vergnaud, 1997). It is suggested that in the teaching and learning of mathematics, students should be exposed to different modes of external representation. Cleaves (2008) classified six categories of external mathematical representations: numerical/tabular, pictorial, graphical, verbal, symbolic (equations or expressions) and physical/concrete. Furthermore, representations are often considered as an approach to form conceptual understanding. Lesh, Post, and Behr (1987) stressed that a student who 'understands' a mathematical concept 'can (1) recognise the idea embedded in a variety of qualitatively different representational systems, (2) flexibly manipulate the idea within given representational systems, and (3) accurately translate the idea from one system to another' (p. 36). In addition, Lesh and colleagues also indicated that the ability to translate between numerous representations of the same concept is seen as an indication of conceptual understanding and should also be an objective for teaching. A student has grasped concepts or constructed concepts if he could communicate through the use of external representations. As utilising only a single form of external representation cannot embody an abstract concept completely, it is necessary to have more than one external representation for each concept to help students formulate a well-rounded understanding. Worthwhile external representations alone do not guarantee students' learning. They are essential, but not sufficient, for effective mathematics instruction because worthwhile external representations may not be implemented as intended.

The National Council of Teachers of Mathematics's (NCTM) Standards (2000) indicated that a representation is not only a product (a picture, a graph, a number or a symbolic expression) but also a process, a vehicle for developing an understanding of a mathematical concept and communicating about mathematics. Language and pictures are viewed as two different categories of representation and communication of mathematical concepts, with essential differences in regard with their informational content, structure and usability (Schnotz, 2002), which complement each other. An instructional issue for teachers is determining how many modes of representations need to be taught at various levels of mathematics instruction. In addition, students need to become fluent in their use if they want to succeed in expressing and understanding mathematical ideas with correctness and precision. Many research studies have revealed that all these decisions are related to a mathematics teacher's level and depth of content knowledge (Fennema & Franke, 1992; Ma, 1999) and teaching skills, as well as his or her philosophy, values and beliefs of mathematics and mathematics teaching (Bishop, 1991; Chin, 1995; Ernest, 1989; Thompson, 1992).

Researchers and mathematics educators have argued that an understanding of multiple representations has many benefits in mathematics learning and teaching. Firstly, Kirwan and Tobias (2014) reported that no single representation is superior to another in all settings because each has its objective in highlighting different mathematical attributes or relationships in context. Furthermore, Dreher, Kuntze, and Lerman (2016) also pointed out that the interaction of different representations is essential for the development of a proper concept image. Secondly, the ability to use multiple representations and translate among these models is a significant process in extending students' mathematical understanding (Fennell & Rowan, 2001; Goldin & Shteingold, 2001). However, students are likely to experience challenges in establishing the relations among different representations. Thus, in the introduction of mathematical concepts, teachers need to assist students to move between representations as it supports the development of deeper conceptual understanding. While using a variety of representations is crucial to students' understanding of mathematics concepts and the relationship among them, each mode of representation provides only limited information and 'stresses some aspects and hides others' (Dreyfus & Eisenberg, 1996, p. 268). Thirdly, students' errors could also reflect their struggle with a certain representation, but not necessarily a lack of conceptual understanding underlying the problem (Flevares & Perry, 2001). Therefore, multiple representations which can reinforce each other are frequently needed for the development of a suitable concept image (Ainsworth, 2006; Elia, Panaoura, Eracleous, & Gagatsis, 2007; Even, 1990; Tall, 1988; Tripathi, 2008).

According to Duval (2006), while mathematics teaching and learning focuses on the use of representations, the conversion between representations is the main obstacle in the development of students' conceptual understanding. The National Council of Teachers of Mathematics (NCTM, 2000) encourages teachers and students to use multiple representations during mathematics instruction. It states that all students should 'create and use representations to organise, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; and use representations to model and interpret physical, social, and mathematical phenomena' (NCTM, 2000, p. 67).

4.2.2 Attending Explicitly to Mathematical Concepts

Students can achieve conceptual understandings of mathematics if teaching attends explicitly to mathematical concepts—to connections among mathematical concepts, procedures and ideas (Gamoran, 2001; Hiebert, 2003; Hiebert & Grouws, 2007). Brophy (1999) described such teaching as infused with coherent, structured, and connected discussions of the important ideas of mathematics. Mathematics teachers could attend explicitly to mathematical concepts when they draw students' attention to connections among mathematical ideas and representations. This could be conducted in a cohesive and well-structured manner. According to Hiebert and Grouws (2007), this may include a range of teachers' practices, such as those described below:

> Attending to concepts … could include discussing the mathematical meaning underlying procedures, asking questions about how different solution strategies are similar to and different from each other, considering the ways in which mathematical problems build on each other or are special (or general) cases of each other, attending to the relationships among mathematical ideas, and reminding students about the main point of the lesson and how this point fits within the current sequence of lessons and ideas. (p. 383)

Additionally, responding to students' talk in mathematically acceptable language can create opportunities for students to develop connections between language and conceptual understanding. In a recent study, Goos (2004) detailed how a secondary school mathematics teacher developed his students' mathematical thinking through scaffolding the processes of inquiry. The teacher 'call[ed] on students to clarify, elaborate, critique, and justify their assertions. The teacher structured students' thinking by leading them through strategic steps or linking ideas to previously or concurrently developed knowledge' (p. 269). In a series of lesson episodes, Goos showed evidence of how the teacher could pull students 'forward into mature participation in communities of mathematical practice' (p. 283), until they were able to engage independently with mathematical concepts. This was supported by Carpenter, Franke, and Levi (2003) who argued that the very nature of mathematics presupposes that students cannot learn mathematics with understanding without engaging in discussion. Through inter-conceptual and intra-conceptual connections, students develop conceptual understanding as they make sense of various mathematical ideas, their connections and applications (Lee, Ng, & Lim, 2019).

The above discussions evidently demonstrate the intricacies of the relationships between developing students' conceptual understanding, representations and attending explicitly to concepts. Although some mathematics educators propose that developing a connection between conceptual understanding and representations is not necessarily of great difficulty or concern for teachers, the reality that unfolds in mathematics classroom would suggest otherwise. Thus, in this chapter, I draw on lesson excerpts to illustrate how two experienced and competent teachers introduce concepts to students and engage them in constructing concepts. In analysing the classroom situations, I will focus on the following two important constructs of the classroom instruction: Representations and Attending explicitly to concepts (Cai, 2003; Hiebert & Grouws, 2007; Lesh et al., 1987), to explicate the learning opportunities that teachers provide to foster students' conceptual understanding through teacher–student interactions.

4.3 Facilitation of Conceptual Understanding by Two Experienced and Competent Mathematics Teachers

This chapter draws on data from the video-segment of the project detailed in Chapter 2 and illuminates how two experienced and competent mathematics teachers, Teacher 8 and Teacher 26, facilitated learning opportunities to promote conceptual understanding in their lessons. Teacher 8 is a female teacher who has taught mathematics for the last 20 years. She is a Lead mathematics teacher who is nationally recognised for her teaching competency and is trusted with the charge of developing fellow teachers in the school and the nation. Teacher 26 is a male teacher who has taught mathematics for the last 10 years. Both Teacher 8 and Teacher 26 are senior members of the school staff who often mentored junior teachers and were looked upon by the leadership of the schools as experienced and competent teachers. The profile of the students of Teacher 8 and Teacher 26 are distinct. Teacher 8 had 39 students in her Year 9 Express course class while Teacher 26 had 21 students in his Year 9 Normal (Academic) course class. The mathematical ability of students in the class of Teacher

26 was slightly below average as they were from the 40th percentile of their cohort, and those in the class of Teacher 8 were from the 50th percentile of their cohort. These percentiles refer to the whole of Year 9 students regardless of course of study.

4.4 Analysis of the Lesson for Teacher 8

Excerpt 1 and 2 demonstrate how experienced and competent Teacher 8 explained the concept and properties of quadratic equations to the whole class, asking students questions along the way.

4.4.1 Properties of Quadratic Equations

In Excerpt 1, Teacher 8 (T8) introduced the quadratic equation to her class by employing two kinds of representations: symbolic and verbal. Teacher 8 first presented and wrote the general form of quadratic equation, $ax^2 + bx + c = 0$, on the whiteboard. Next she wrote the equation $\frac{3}{x^2} + 2x - 1 = 0$ on the board. She then asked her students if the equation was quadratic? Teacher 8 immediately compared the two equations and demonstrated to students how to identify the respective coefficients (a, b and c) in the second equation.

Excerpt 1

T8: So what exactly is a quadratic equation? Let's get this clear everybody. A quadratic equation is an equation that looks like this (T8 points to '$ax^2 + bx + c = 0$' on the whiteboard) correct?

...

T8: $ax^2 + bx + c = 0$, where 'a', 'b' and 'c' are constants. They are fixed numbers, okay? a cannot be 0. That's the definition right? Okay, but it's not just enough to see that there's a 2 (two) here (T8 circles the power 2 in ax^2 in the quadratic equation on the whiteboard with her finger)

T8: Okay, like this. [T8 circles the x^2 in $\frac{3}{x^2} + 2x - 1 = 0$ on the whiteboard with her finger].

T8: Because you need to write it [T8 points to $3x^{-2}$ in $3x^{-2} + 2x - 1 = 0$ on the whiteboard).

T8: What is the power of, the highest power of x, okay? Has to be 2 for it to be quadratic. And this [T8 points to the x^{-2} in $\frac{3}{x^2} + 2x - 1 = 0$ on the whiteboard], when it's $\frac{3}{x^2}$, power of x is not 2. Understand or not? So if it's not 2, then not quadratic, okay? Can or not? Any questions?

This segment of the lesson, which lasted about five minutes, prepared the students to determine what equations can be considered quadratic equations by focusing on the exponent of x^2. The example, $\frac{3}{x^2} + 2x - 1 = 0$ where the x^2 was deliberately placed as a reciprocal, led the students to focus and compare a part of the equation. The example guided the students to notice that for the term $\frac{3}{x^2}$, the power of x was not 2 as compared to the general form of a quadratic equation, thus the equation could not be classified as a quadratic. Furthermore, the teacher's style of explanations and gestures belongs to a style which led the class to follow a specific path closely. The students were directed to inspect different terms and coefficients of the quadratic equation in a specific sequence (coefficient of x^2, then coefficient of x, and finally constant term) and they were not encouraged to skip any term and coefficients.

To make sure her students were familiar with and understood the concept of a quadratic equation, Teacher 8 went further to ask the class to justify why the coefficient a cannot be equal to zero in Excerpt 2. A student gave the answer that it will be a linear equation.

Excerpt 2

T8: So why cannot be, why cannot be 0? Why 'a' cannot be 0? [Teacher writes 'a $\neq$ 0' on the whiteboard]. Anybody can tell me?

T8: Because the definition say 'a' cannot be 0. Why ah? 'a', 'b' and 'c' are constants. 0 is a constant what, it's a fixed number, but why cannot be 0?

S1: Then it will be a linear equation.

T8 Then it will be a linear equation. Okay. So what is linear?

T8: [Teacher points to 'a' in ax^2 + bx + c = 0 on the whiteboard] So if 'a' is 0 what will happen to zero times x^2, what will happen to the x^2?

S2: Zero.

T8: Becomes 0 right? So you will only be left with bx + c = 0. Okay, and Student 1 (S1) says this is called linear equation.

When Teacher 8 posed the question: 'Why 'a' cannot be 0'? her intention was to encourage the students to examine the general form of quadratic equation, ax^2 + bx + c = 0, and provided justifications for the conditions of the value of a, b and c. Student 1's (S1) utterance, 'then it will be a linear equation', in response to Teacher 8's questions was difficult to interpret. To ensure that all the students were able to move smoothly from the verbal representation to symbolic representation of linear equation, Teacher 8 continued to probe further to ensure how the symbolic representation of a linear equation, bx + c, was derived. Although it was not possible to tell if all the students had understood the properties of quadratic equations, the teacher's explanation of general form of linear equation, bx + c = 0, and general form of quadratic equation, ax^2 + bx + c = 0, were clearly evident.

4.4.2 Learning Opportunities for Conceptual Understanding

In this subsection, I present the learning opportunities for fostering students' conceptual understanding through teacher–student interactions of Teacher 8's class. The discussion focuses on the two excerpts to illustrate Teacher 8's pedagogical moves to create learning opportunities for students. Teacher's *pedagogical moves* refers to teacher's actions, both intentional and unintentional, that shape what mathematics is addressed, including how it is represented and investigated.

In Excerpt 1, Teacher 8's objective was to explain and illustrate the properties of quadratic equations. Teacher 8 facilitated discussions that helped students negotiate the meaning of verbal and symbol representations of quadratic equations and advance their common understandings of the phenomena being studied. Teacher 8 explicitly attended to concepts by addressing and discussing properties of quadratic equations. She also assisted the students to observe connections between the coefficients of x^2, x and the constant term. Although she could have engaged students in drilling basic facts until a conceptual foundation of quadratic equations was developed, she instead created a learning opportunity for students to determine whether $\frac{3}{x^2} + 2x - 1 = 0$ could be considered an example of a quadratic equation. To convince the students, she rewrote $\frac{3}{x^2} + 2x - 1 = 0$ as $3x^{-2} + 2x - 1 = 0$. She challenged students by giving a counterexample, going beyond learning facts, helping them to learn to think mathematically how the symbolic form of quadratic equation was represented. It is great that the teacher used the counterexample to highlight the properties of quadratic equation. However, Teacher 8's switching back and forth between the general form of quadratic equation and a counterexample may be difficult for some students to follow. Nevertheless, such learning opportunities of switching between the general form of quadratic equation and a counterexample was evident in Excerpt 1 which promote conceptual understanding. One feature is that teachers and students pay explicit attention to concepts; the other is that students themselves wrestle with important mathematics (Hiebert & Grouws, 2007).

In Excerpt 2, Teacher 8 posed thinking questions to students such as 'why'? 'what will happen' and 'what if'? during the lesson. Teacher 8 used an initiating move and a series of sustaining moves to support students in constructing an argument about 'why "a" cannot be 0'? The initiating move, 'why "a" cannot be 0'? served to open up an opportunity for Student 1 (S1) and Student 2 (S2) to engage in co-constructing an argument. Meanwhile, the sustaining moves (e.g. 'So if "a" is 0 what will happen to zero times x^2, what will happen to the x^2'?) served to first acknowledge the truth in S1's initial contributions was based on efficiency or empirical evidence. In addition, the sustaining moves also pushed students for more, sending the message that these early justifications were partially correct. This provided opportunities to stimulate students to think and to make use of logical deduction. To ensure more learning opportunities, Teacher 8 played an important role to promote mathematical understandings through the 'orchestration' of whole class discussions where students actively participated by making explicit their thinking and by listening to contributions made by classmates. By facilitating these discussions, Teacher 8 was also able to monitor the understanding of individual students. This kind of teaching emphasised mathematical reasoning and promoted much classroom discourse and interaction between the teacher and students, as well as between the students themselves. Furthermore, such instruction provided opportunities for students to engage in mathematical practices such as making connections between quadratic and linear equation, understanding representations of quadratic and linear equation, communicating their thinking, justifying their reasoning and critiquing arguments.

4.5 Analysis of the Lesson for Teacher 26

In Excerpt 3, experienced and competent Teacher 26 brought students' attention to derive the formula for the distance between two points, (x_1, y_1) and (x_2, y_2), in a Cartesian plane. Teacher 26 guided the whole class to discover the formula by using Pythagoras' theorem through questioning.

4.5.1 Distance Between Two Points in a Cartesian Plane

Teacher 26 (T26) introduced the concept of the distance between two points in a Cartesian plane to his class by employing three kinds of representations: verbal, symbolic and graphical. He first presented Fig. 4.1 on the whiteboard and labelled the coordinates, *A* and *B*. He then asked students to identify a relationship in a right-angled triangle and guided the students to derive the formula for the distance between two points in a Cartesian plane, asking students questions along the way.

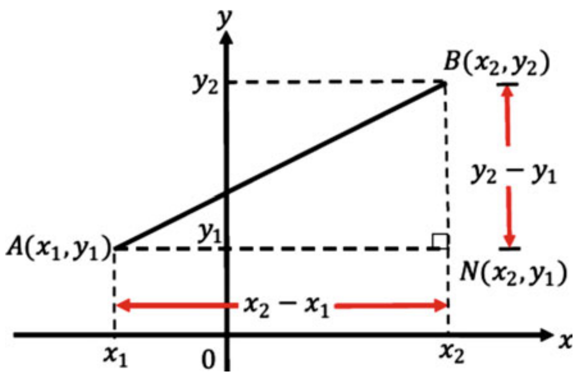


Fig. 4.1 Distance between two points in a Cartesian plane shown on the whiteboard

Excerpt 3

T26: I would like to explain to you how this formula comes about. What type of triangle is this?

S1: Right-angled triangle.

T26: What is the relationship in a right-angled triangle? Pythagoras theorem.

T26: For the x-axis, and the y-axis. The distance between them. Okay, these are general letters, x and y. Numbers are given, $y_2 - y_1$, what length would you define? Length of BN or AN?

S26: BN.

T26: BN is $y_2 - y_1$. Do you see, the deriving?

T26: How do I write AN?

S3: $x_2 - x_1$.

T26: And to the left-hand side (of the equation) you have the longest side, hypotenuse, that's what we need to find. Is this okay? (Teacher projects the length of line segment formula $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ in the notes onto the screen).

S4: Square root.

T26: We want AB only so we put the square root on both sides.

In Excerpt 3, Teacher 26 made a deliberate attempt to ask questions to encourage the students to examine the graphical representation of the line AB and its coordinates. In order to derive the distance formula in the Cartesian plane, Teacher 26 led the students to focus on how the coordinates of A, B and N were labelled and represented. The intention of Teacher 26 was to promote mathematical understandings through the 'orchestration' of whole class discussions where students actively participated by making explicit their thinking and by listening to contributions made by their classmates. Similar to Teacher 8, these whole class discussions also allowed Teacher 26 to monitor the understanding of individual students. To guide the students in the derivation of the formula, Teacher 26 continued to probe further to assist the students to observe how symbolic representation and graphical representation were related.

4.5.2 Learning Opportunities for Conceptual Understanding

In this subsection, I present the learning opportunities provided by Teacher 26 for fostering students' conceptual understanding through teacher–student interactions. The discussion focuses on Excerpt 3 to illustrate Teacher 26's pedagogical moves to create opportunities for students to learn.

In Excerpt 3, Teacher 26's instructional strategy was to develop students' representational competence by facilitating discussions that helped students negotiate the meaning of graphical, verbal and symbol representations of the distance between two points, (x_1, y_1) and (x_2, y_2), in a Cartesian plane. An important characteristic of representational competence is the ability to switch or translate from one representation to another. Two important types of translations need to be developed—translations between different modes of representations (e.g. from a graphical to an equation) and translations within a specific mode of representation (e.g. from one visual model to another, such as examining the right-angled triangle in a Cartesian plane). Teacher 26 had engaged students in dialogue to make explicit the connections among representations. From Excerpt 3, Teacher 26 provided these opportunities by asking students to alternate or reverse directionality in making connections among representation (e.g. for the x-axis, and the y-axis. The distance between them. These are general letters, x and y. Numbers are given, $y_2 - y_1$, what length would you define? Length of BN or AN?). This activity is a critical step to engage teachers and students in doing mathematics while making the role of representation explicit through the discussions about representations. Teacher 26 had created an important learning opportunity for students to use graphical representation so as to help students to make sense of mathematical symbols and notations and to avoid rote memorisation of procedures. This teacher-directed lesson shows that it is possible for both teachers and students to pay explicit attention to concepts in ways other than providing definitions or stating the formula for the distance between two points in a Cartesian plane. As students engaged in mathematical discourse using the graphical, verbal and symbol representations, the learning environment created an opportunity for students to internalise the concepts through teacher–student interactions. The analysis presented here provides one example of how to attend to concepts through a discussion grounded in questions created by teachers so that students have the opportunity to wrestle with mathematical concepts. In fact, Teacher 26 taught systematically so that the students developed the skills of representing and handling flexibly mathematical knowledge of distance between two points in a Cartesian plane.

While teachers attempt to develop students' understanding and flexibility of the use of multiple representations, they must make explicit to students the advantages of choosing one representation over another. For example, symbolic notation is typically used mainly as a way to summarise the given information in the problem and make it readily accessible, while a diagram or a graphical representation is more appropriate for exploration. The compact character of the former and the flexible nature of the latter allow for these different uses, thus students need to learn to attend to the characteristics of different representations to take advantage of their capacity to understand and solve problems. Although it was not evident in Teacher 26's instruction, representations can be used to support connections, reasoning, communication and problem-solving. Without promoting these mathematical ideas and verbalising them in class discussion, the rich potential of learning can be lost. Classroom discourse helped students clarify their thinking and bridge representations with important mathematical learning. Unfortunately, Teacher 26 missed the opportunity to ask students to identify similarities and differences among representations. If Teacher 26 had asked students the following question: Explain how you would find the distance between points $A(1, 4)$ and $B(13, 9)$. This would have provided an opportunity for students to connect abstract and pictorial representations and delay the step-by-step algorithm until after examining the meaning of the number manipulations. The students would observe that $(13 - 1)^2$ is the square of the difference in x-coordinates of A and

B and is always positive. The same can be said about $(9 - 4)^2$ as well. With such number manipulations, the Normal (Academic) course students in Teacher 26's class would observe for themselves why the formula remains the same for any coordinates of *A* and *B*, in any quadrant. These types of discussions direct attention to essential features of the underlying structure of mathematical ideas and support Normal (Academic) course students' abilities to recognise and utilise those structures in solving problems. However, it could be that Teacher 26 paid too much attention to helping students master the formula for the distance between two points, hence he neglected to discuss the similarities and differences among representations.

4.6 Concluding Remarks

In this chapter, I have analysed two well-structured lessons to examine the opportunities for students' learning. During whole class teaching, which is the D (Development) component of the instruction core presented in Chapter 3, the two teachers played an active role in expounding mathematical concepts mainly through the use of representations as their teaching tools. The teacher structured the representations and also attended explicitly to concepts so as to help students to develop the concepts. The lessons of both the teachers were teacher-directed but student centred. Students' responses to the teacher's questions were the focus for assimilation of knowledge by the students. The two experienced teachers had planned and delivered the lesson in such a way that students had little room to think independently. Instead, students mainly followed the teacher's 'planned frame' to learn what was determined prior to the lesson by the teacher. Learning opportunities depend on the types of representations and the teacher's questions in orchestrating the classroom discourse. To be effective in the classroom, students should have more opportunities to articulate their mathematical ideas and justify their answers (Hiebert & Wearne, 1993; Lampert, 1990).

Finally, in teaching and learning, representation can play a dual role, performing as both instructional tools and learning tools. As Lamon (2001) reported, representation can be 'both presentational models (used by adults in instruction) and representational models (produced by students in learning), which can play significant roles in instruction and its outcomes' (p. 146). Researchers and mathematics educators have long tried to develop a conceptually rich understanding of what effective mathematics teaching is and how to cultivate it in order to maximise learning opportunities for all students.

References

Ainsworth, S. (2006). A conceptual framework for considering learning with multiple representations. *Learning and Instruction, 16,* 183–198.

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433–456). Washington, DC: American Educational Research Association.

Bishop, A. J. (1991). Mathematical values in the teaching process. In A. J. Bishop, S. Mellin-Olsen, & J. van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching* (pp. 195–214). Dordrecht, the Netherlands: Kluwer.

Brophy, J. E. (1999). *Teaching (Education practices series no. 1)*. Geneva: International Bureau of Education. Retrieved December 6, 2019, from https://www.iaoed.org/downloads/prac01e.pdf.

Brophy, J. E., & Good, T. L. (1996). Teacher behaviour and student achievement. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 328–375). New York: Macmillan.

Cai, J. (2003). What research tells us about teaching mathematics through problem solving. In F. Lester (Ed.), *Research and issues in teaching mathematics through problem solving* (pp. 241–254). Reston, VA: National Council of Teachers of Mathematics.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.

Chin, C. (1995). *Mathematics teachers' beliefs, their classroom practices and influences on student learning: Four case studies.* Unpublished Doctor of Philosophy thesis, University of Cambridge, Cambridge, UK.

Cleaves, W. P. (2008). Promoting mathematics accessibility through multiple representations jigsaws. *Mathematics Teaching in the Middle School, 13*(8), 446–452.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*(1), 2–33.

Dreher, A., Kuntze, S., & Lerman, S. (2016). Why use multiple representations in the mathematics classroom? Views of English and German preservice teachers. *International Journal of Science and Mathematics Education, 14*(2), 363–382.

Dreyfus, T., & Eisenberg, T. (1996). On different facets of mathematical thinking. In R. J. Sternberg & T. Ben-Zeev (Eds.), *The nature of mathematical thinking* (pp. 253–284). Mahwah, NJ: Lawrence Erlbaum Associates.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61,* 103–131.

Elia, I., Panaoura, A., Eracleous, A., & Gagatsis, A. (2007). Relations between secondary pupils' conceptions about functions and problem solving in different representations. *Journal of Science and Mathematics Education, 5*(3), 533–556.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249–254). London: Falmer.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*(6), 521–544.

Fennell, F. S., & Rowan, T. (2001). Representation: An important process for teaching and learning mathematics. *Teaching Children Mathematics, 7*(2), 88–92.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147–164). New York: Macmillan.

Flevares, L. M., & Perry, M. (2001). How many do you see? The use of nonspoken representations in first grade mathematics lessons. *Journal of Educational Psychology, 93*(2), 130–145.

Gamoran, A. (2001). Beyond curriculum wars: Content and understanding in mathematics. In T. Loveless (Ed.), *The great curriculum debate: How should we teach reading and math?* (pp. 134–162). Washington, DC: Brookings Institution Press.

Goldin, G., & Shteingold, N. (2001). System of mathematical representations and development of mathematical concepts. In F. R. Curcio (Ed.), *The roles of representation in school mathematics* (pp. 1–23). Reston, VA: National Council of Teachers of Mathematics.

Goldin, G. A. (1998). Representational systems, learning, and problem solving in mathematics. *Journal of Mathematical Behaviour, 17*(2), 137–165.

Goos, M. (2004). Learning mathematics a classroom community of inquiry. *Journal for Research in Mathematics Education, 35*(4), 258–291.

Hiebert, J. (2003). What research says about the NCTM standards. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 5–23). Reston, VA: National Council of Teachers of Mathematics.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Greenwich, CT: Information Age.

Hiebert, J., & Wearne, D. (1993). Instructional task, classroom discourse, and students' learning in second grade. *American Educational Research Journal, 30,* 393–425.

Janvier, C. (1987). *Problems of representation in the teaching and learning of mathematical problem solving*. Hillsdale, NJ: Erlbaum.

Kirwan, J. V., & Tobias, J. M. (2014). Multiple representations and connections with the Sierpinski triangle. *The Mathematics Teacher, 107*(9), 666–671.

Lamon, S. J. (2001). Presenting and representing: From fractions to rational numbers. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 146–165). Reston, VA: National Council of Teachers of Mathematics.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27,* 29–63.

Lee, N. H., Ng, W. L., & Lim, L. G. P. (2019). The intended school mathematics curriculum. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 35–53). Singapore: Springer Singapore.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33–40). Hillsdale, NJ: Lawrence Erlbaum.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Hillsdale, NJ: Erlbaum.

National Academy of Education. (1999). *Recommendations regarding research priorities: An advisory report to the National Educational Research Policy and Priorities Board*. Washington, DC: National Academy of Education.

National Council of Teachers of Mathematics. (2000). *The principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Perkins, D. N., & Unger, C. (1994). A new look in representations for mathematics and science learning. *Instructional Science, 2,* 1–37.

Schnotz, W. (2002). Towards an integrated view of learning from text and visual displays. *Educational Psychology Review, 14*(1), 101–120.

Tall, D. (1988). Concept image and concept definition. In J. de Lange & M. Doorman (Eds.), *Senior secondary mathematics education* (pp. 37–41). Utrecht: OW & OC.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). New York: Macmillan.

Tripathi, P. (2008). Developing mathematical understanding through multiple representations. *Mathematics Teaching in the Middle School, 13*(8), 438–445.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 1–28). Hove: Psychology Press.

Kai Kow Joseph Yeo is a Senior Lecturer in the Mathematics and Mathematics Education Academic Group at the National Institute of Education, Nanyang Technological University, Singapore. Before joining the National Institute of Education in 2000, he held the post of Vice Principal and Head of Mathematics Department in secondary schools. As a mathematics educator, he teaches pre- and in-service as well as postgraduate courses in mathematics education and supervises postgraduate students pursuing Masters degrees. His publication and research interests include mathematical problem solving at the primary and secondary levels, mathematics pedagogical content knowledge of teachers, mathematics teaching in primary schools and mathematics anxiety.

Chapter 5
Teaching Practices That Promote Mastery in Mathematics Learning in Singapore Secondary School Classrooms

Ngan Hoe Lee and Liyana Safii

Abstract This chapter discusses the findings from an examination of how the teaching practices demonstrated by experienced and competent secondary mathematics teachers compare with the intended Singapore School Mathematics Curriculum in developing mastery in learning. In the first section of the chapter, we examine how experienced and competent mathematics teachers provide opportunities for students to develop and gain mastery in mathematics learning. Of particular interest are the ways the phases of lesson as well as the use of worked examples and class practice tasks are structured during the lessons to promote such kind of mastery in learning. We found that experienced and competent teachers tend to employ cycle(s) of lesson development and lesson consolidation during their teaching. They also tend to explain the solution of one or a few worked example(s) before providing students with opportunities to independently put into practice their learning on other related problems. In the second section, we examine, by drawing on the survey data, how teachers across Singapore in general compare with these practices adopted by the experienced and competent teachers. It appears that teachers across Singapore generally adopt rather similar teaching approaches to promote mastery in mathematics learning.

Keywords Singapore school mathematics curriculum · Secondary mathematics · Mastery in learning · Phases of lesson · Worked example

5.1 Introduction

In this chapter, we consolidate the data and findings from the project (detailed in Chapter 2) which focuses on the teaching practices of mathematics teachers. We first

N. H. Lee (✉) · L. Safii
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: nganhoe.lee@nie.edu.sg

L. Safii
e-mail: liyana.safii@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_5

document practices employed by experienced and competent secondary teachers to promote mastery in learning in their lessons. We do this through an analysis of the video-recorded data from Phase 1 of the project. Next we draw on the survey data from Phase 2 of the project and identify the practices employed by Singapore mathematics teachers in general. This allows us to compare the findings and identify any gaps between their practices.

5.2 Mastery in Learning

There are evidences that promoting mastery in learning provides a channel for teachers to determine areas of improvement in student learning, could positively affect students' learning attitude (Changeiywo, Wambugu, & Wachanga, 2011), as well as learning performance (Wambugu & Changeiywo, 2008). So, what does it mean to promote mastery in learning?

> If teachers want their students to focus on mastery of content and tasks, they need to allow students to work on tasks repeatedly, without penalties, until they achieve mastery. (Guskey & Anderman, 2013, p. 22)

Mastery in learning has its roots in Bloom's (1968) model. Bloom's mastery learning model was built on the consideration that learners possess different learning needs, such as time and learning pace, and focuses on formative assessments and immediate feedback that shape their progression in learning. The model posits that learners are able to thrive in their learning if they learn under appropriate conditions. As such, an integral part of the model is the division of curriculum content into smaller units to ensure that learners acquire the pre-requisite knowledge or skills for a particular learning unit before he/she is able to progress to the next learning unit.

The key characteristics of the mastery learning approach involve clear identification of learning goals and objectives, and providing formative assessment to ascertain students' level of mastery (Bloom, 1968) (Fig. 5.1). The formative assessment serves as a channel for teachers to ascertain students' progress in learning, supplement student learning with timely feedback, and intervene with corrective measures such as additional practice or time for learning. Students who have yet to demonstrate mastery have to be evaluated on the same concepts and skills through another formative assessment. This cycle should be repeated until students are able to perform at least 80% of their mastery level (Anderson, 2000). For students who have demonstrated mastery, teachers should allocate additional enrichment tasks that stretch their thinking beyond the basic learning goals prescribed for a particular unit. With this coherent instructional cycle, it is evident that Bloom's model places importance on cycles of assessing and reviewing to help students achieve mastery in learning.

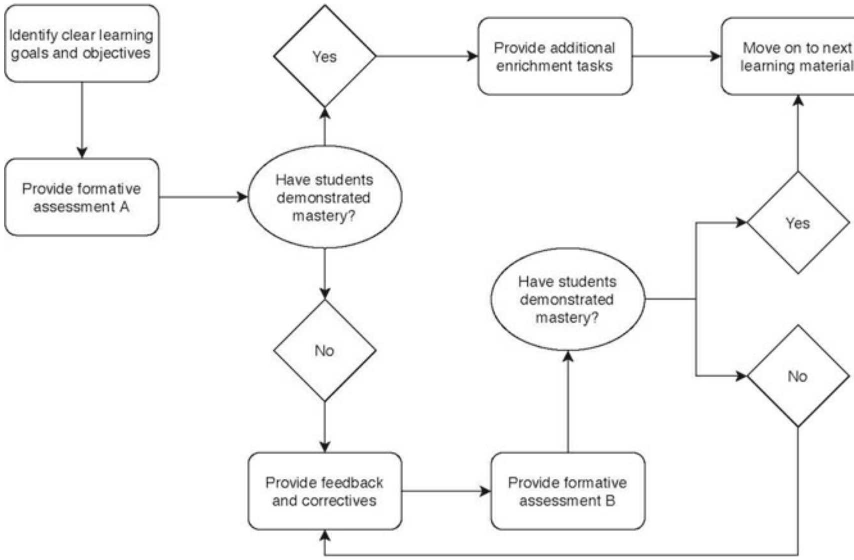


Fig. 5.1 Teaching practices that promote mastery learning approach (Adapted from Bloom, 1968)

5.3 Mastery in Learning and the Singapore School Mathematics Curriculum

The Singapore School Mathematics Curriculum Framework's (SSMCF) (Ministry of Education, 2012, p. 14) central focus is mathematical problem solving. In enabling students to develop their ability in mathematical problem solving, the Framework places emphases on the five inter-related components—conceptual understanding, skills proficiency, mathematical processes, attitudes, and metacognition (see Chapter 1, Fig. 1.2). Mathematical skills refer to "numerical calculation, algebraic manipulation, spatial visualisation, data analysis, measurement, use of mathematical tools, and estimation" (Ministry of Education, 2012, p. 15). To develop skills proficiency, it is also advocated that mathematics lessons should be well-structured so as to:

(a) Develop student mathematical skills that goes beyond procedural application and
(b) Give students sufficient opportunities to practise and use these skills such that they develop fluency in mathematical application.

Here, there is an emphasis on the teaching of mathematics skills such that the skills are not understood as merely procedures to be applied but understood with underlying mathematical principles (Ministry of Education, 2012). In other words, the learning goals in mathematics go beyond instrumental understanding (i.e. the "how" of procedural skills); mathematics should be taught with relational understanding (i.e. the "why" of procedural skills) (Skemp, 1987).

In addition, the SSMCF has framed three phases of learning to guide teachers in supporting student learning in the classroom: Readiness, Engagement, and Mastery (Ministry of Education, 2012, p. 22). In facilitating students' *readiness* to learn, teachers are encouraged to check on students' prior knowledge, introduce motivating contexts and create an environment that promotes productive and purposeful learning. As students require different forms of *engagement* in the learning of mathematics, teachers are recommended to use various forms of instructions in their lessons such as activity-based learning, teacher-directed inquiry, and direct instruction. For students to attain *mastery* of concepts and skills, teachers should supplement student learning with a wide range of approaches which include motivated practice for application of knowledge and skills, reflective review of their progression in learning, as well as extended learning that stretches their potential in problem solving. In other words, in the context of mastery in learning mathematics, the Singapore Mathematics Curriculum promotes relational understanding, especially in the development of mathematical skills proficiency, and promotes a variety of formative assessments that includes motivated practice for application of knowledge and skills.

In this chapter, we will present two rather commonly observed practices that experienced and competent Singapore mathematics teachers undertake to address mastery in learning mathematics—the structuring of the phases of a lesson to promote relational understanding, and the structuring of worked examples and class practice to promote motivated practice for application of knowledge and skills. We will also examine how these teaching practices fit directly into Bloom's mastery in learning model.

5.3.1 Phases of Lesson

Gagne (1985) argued that for cognitive strategies to be learned and internalised, there must be a chance to practice developing new solutions to problems, and the learner must be exposed to a credible role model or persuasive arguments. He also outlined the following nine instructional events and corresponding cognitive processes:

- gaining attention (reception),
- informing leaners of the objective (expectancy),
- stimulating recall of prior learning (retrieval),
- presenting the stimulus (selective perception),
- providing learning guidance (semantic encoding),
- eliciting performance (responding),
- providing feedback (reinforcement),
- assessing performance (retrieval), and
- enhancing retention and transfer (generalisation).

Gagne's nine instructional events exemplify and put into practice the instructional design that demonstrates the key characteristics in mastery in learning put forth by Bloom mentioned earlier. These nine instructional events also parallel many of the nine Teaching Actions advocated by the Singapore Ministry of Education (2018) when teachers enact their lessons:

- activating prior knowledge,
- arousing interest,
- encouraging learner engagement,
- exercise flexibility,
- providing clear explanation,
- pacing and maintaining momentum,
- facilitating collaborative learning,
- using questions to deepen learning, and
- concluding the lesson.

While Gagne's nine instructional events are hierarchical in nature, the Ministry of Education's nine teaching actions includes general good pedagogical practices that may occur throughout the lesson. As a means to help mathematics teachers to better structure mathematics lessons, whereby the norm for the duration of each lesson is about one hour, Lee (2009) proposed a four-phase lesson structure model that encapsulates most of these key elements: Introduction, Development, Consolidation, and Conclusion.

The model, as shown diagrammatically in Fig. 5.2, is used to guide pre-service mathematics teachers in Singapore to plan lessons. It recognises that planning for a well-structured lesson that involves a logical sequence of mathematics teaching is essential in providing students with a coherent learning process. Teachers begin the lesson by first developing students' readiness through the use of motivating contexts or assessing students' pre-requisite knowledge (*Introduction*) before engaging students in current learning by providing learning guidance with respect to the relevant knowledge and skills to fulfil the objectives of the lesson (*Development*).

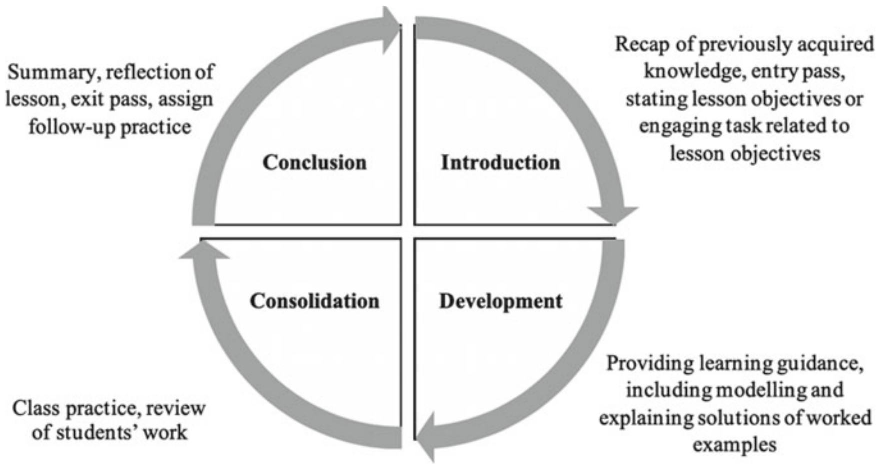


Fig. 5.2 Phases of mathematics lesson (Adapted from Lee, 2009)

Four-Phase Lesson Structure Model	Components D, S, R	Instructional Approach
Introduction	R	Teacher reviews students' prior knowledge
	D	Teacher engages students in creating or appreciating mathematical ideas/concepts
Development	D	Teacher develops students' knowledge on mathematical ideas/concepts
	D	Teacher demonstrates to students skills that are relevant to mathematical ideas/concepts
Consolidation	S	Teacher assigns students with individual or group work within or outside the lesson for application of concepts and practice of skills
	R	Teacher reviews work done by students (e.g., written work, work done on whiteboard, assessment)
Conclusion		

Fig. 5.3 Connection between four-phase lesson structure and the instructional components D, S, and R

The lesson then extends into the *Consolidation* phase in which teachers assign class practice and review students' work to develop fluency in mathematical application. The lesson then concludes with a closure of the lesson or setting up the stage for the subsequent lesson (*Conclusion*). In particular, the Development and Consolidation phases promote procedural fluency through opportunities for motivated practice of learnt knowledge and skills that are built on relational understanding. In other words, mastery in learning mathematics is achieved through the Development and Consolidation phases. It is useful to also note that this four-phase lesson structure model also maps into the three phases of learning embodied in the SSMCF (Readiness, Engagement and Mastery).

In Chapter 3, the authors introduced the D, S, R (D = Development, S = Student work, R = Review of student work) components of an instructional core that drives mathematics instruction in Singapore secondary schools. These components manifest the micro-lesson objectives that teachers enact while achieving their lesson objectives. In contrast, the four-phase lesson structure model discussed here is guided by a lesson structure model. It is possible to map the instructional components: D, S, and R within each phase of a lesson. Figure 5.3 exemplifies the connection between the components and the four-phase lesson structure model.

5.3.2 *Worked Example and Class Practice Task Nexus*

Worked examples "comprise the specification of a problem, the solution steps, and the final solution itself" (Renkl, Stark, Gruber, & Mandl, 1998, p. 90). For teaching and learning of mathematics, the use of worked examples as part of classroom instruction draws on the basis that assigning independent tasks to students during the initial stage of learning might not be productive as students have not yet attained a good grasp of newly introduced concepts and its application (Renkl, 2014, 2017). As a result, students are not able to draw on their repertoire of knowledge to activate effective problem-solving strategies and eventually may depend on strategies that are unproductive and overload their cognitive resources. One way to reduce such extraneous cognitive load is to allow students to learn using worked examples in which step-by-step solutions are provided. In providing step-by-step solutions, students are able to devote all available cognitive resources to focus on the context of the worked example and its solution, and construct problem-solving schemas for such problems (Pachman, Sweller, & Kalyuga, 2014). Students do this by learning how to derive solutions by modelling what others do, say, or write, and storing them into their long-term memory (Sweller, Ayres, & Kalyuga, 2011), and constructing their own explanations on deriving the solutions (Renkl, 2017). These schemas could later be activated for problem solving.

In fact, worked examples viewed as teacher modelling, as advocated by Gagne's 5th instructional event—guide learning, to show how to write "it correctly" in mathematics contributes to the third type of understanding that Skemp (1987) postulated as "formal/logical understanding" in the learning of mathematics. According to Skemp, formal/logical understanding is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning. Thus, worked examples, together with the teaching for relational and instrumental understanding, need to be included under the lesson Development phase so that the teacher sufficiently develops all three types of understanding in relation to the intended learning for the lesson.

There are evidences of the benefits of worked examples and a majority of these studies are centred on the *example-problem pairs* as a substitute for learning solely through problem-solving tasks (Renkl, 2014; Van Gog & Rummel, 2010). In *example-problem pairs* studies, students are provided with solutions of worked examples and then immediately with similar problems to be solved. In a classic example of *example-problem pairs*, Sweller and Cooper (1985) observed that not only did the worked examples help students solve problems faster, students also made fewer mathematical errors and subsequently became more efficient in problem solving. This could be contributed by the activation of easily retrievable schemas which were developed when students were exposed to solutions of worked examples.

However, there exists a number of caveats to the effectiveness of worked examples. The use of worked examples is beneficial only in the initial stage of student learning (Kalyuga, Chandler, Tuovinen, & Sweller, 2001; Tuovinen & Sweller, 1999) and when students are encouraged to make sense of the solutions to worked examples on their own (Renkl 2014; Rittle-Johnson, Loehr, & Durkin, 2017). As such, additional support that encompasses processes such as the basis of using certain approaches or the selection of appropriate strategies needs to be supplemented in order to provide students with a more coherent learning process (Van Gog, Paas, & Van Merriënboer, 2004). In fact, the quality of self-constructed explanations in learning can be enhanced with some instructional support (Rittle-Johnson et al., 2017). For instance, supplementing worked examples with *instructional explanations* has been proven to be beneficial for learning (Wittwer & Renkl, 2010), though its effectiveness is limited to the initial learning process (Bokosmaty, Sweller, & Kalyuga, 2015). Furthermore, Kalyuga et al. (2001) argued that for students who have acquired the necessary learning, interpreting a worked example may be redundant and impose a greater cognitive load instead.

Studies have also shown that the effectiveness of worked examples can be optimised through the use of *completion or fading strategy* which considers students' active role in constructing knowledge in relation to a particular problem (e.g. Berthold, Eysink, & Renkl, 2009; Renkl, 2014). Adopting such strategy means that there is room for students to actively rationalise the solutions, albeit partially, rather than providing students with fully worked-out examples, by omitting certain steps in the solutions provided. In fact, removing the last step from the solution could be more effective in developing students' knowledge of a problem than the use of *example-problem pairs* (Atkinson, Renkl, & Merrill, 2003; Renkl, Atkinson, Maier, & Staley, 2002). Under such circumstances, students are encouraged to construct their own explanations while being supported with prompts.

It appears that there are prerequisites to the effectiveness of worked examples; worked examples have to be accompanied by a problem to solve, and incorporated into the lesson in a way that engages students to construct their own explanations or partial solutions.

In the Singapore context, the advocated use of "I do, we do, you do" strategy, which basically refers to tasks that the teacher demonstrates (teacher modelling), tasks that the teacher brings the whole class through with guidance (guided practice), and tasks that require students to attempt independently (independent practice), exemplifies the way the worked example and class practice task nexus is being played out in these abovementioned ways in the classroom contexts.

5.4 Promoting Mastery in Learning in Singapore Secondary Mathematics Classrooms

This section reports the data from the two phases of the project. The first involved 30 experienced and competent teachers, while the second involved again the 30 experienced and competent teachers and another 647 secondary mathematics teachers across Singapore. The teacher participants were made up of teachers who taught the four courses of study offered under the Singapore secondary education—Integrated Programme (IP), Express, Normal (Academic) (N(A)) and Normal (Technical) (N(T)). Details on the four courses of study are provided in Chapter 1, Sect. 1.2. Though the same 30 experienced and competent teachers were involved in the first and second phase of the project, the distribution of these teachers in each course of study differs slightly in both phases (see Tables 5.1 and 5.4, and Tables 5.6 and 5.7). This could be attributed to the teachers teaching more than one course of study in

Table 5.1 Use of Development-Consolidation cycles as observed in lessons conducted by experienced and competent teachers

Lesson structure	Percentage of participants				
	IP ($n = 4$)	Express ($n = 10$)	N(A) ($n = 8$)	N(T) ($n = 8$)	Total ($N = 30$)
My lesson is typically structured this way: Introduction, Development, Consolidation, Conclusion	25	60	88	38	57
My lesson is typically structured this way: Introduction, Development 1, Consolidation 1, Development 2, Consolidation 2, Conclusion (or similar, i.e. more than one cycle of Development-Consolidation)	25	20	38	25	27
I only have one Introduction at the beginning of the chapter and one Conclusion at the end of the entire chapter. There are mainly Development and Consolidation cycles in most of the lessons	50	30	13	25	27

their school. In addition, in the second phase, the experienced and competent teachers were not told to do the survey for the same course of study for which their lessons were recorded in the first phase of the study.

Two hundred and nine lessons conducted by the experienced and competent teachers were used to document the types of approaches adopted by experienced and competent teachers when promoting mastery in learning in the secondary mathematics classrooms. A grounded approach was adopted by the team of researchers to construct a coding scheme for the coding of the video-recorded lessons. The aim of the coding process was to consolidate the types of instructional approach adopted by the experienced and competent teachers, without noting their frequency of usage. Thus, the video-recorded lessons were coded for the different instructional approaches that were observed throughout the series of lessons recorded for each experienced and competent teacher. Subsequent observations of an instructional approach that has already been coded were not coded again.

5.4.1 Phases of Lesson

5.4.1.1 Video-Recorded Lessons

The video-recorded lessons were coded for the phases of lesson that the experienced and competent teachers adopted in their lessons. The lessons were coded according to whether they, for instance, include the use of single or multiple cycles of the Development-Consolidation phases as shown in Table 5.1.

The video data revealed that 57% of the experienced and competent teachers delivered their lessons through a single Development-Consolidation cycle, while 27% of them adopted multiple development-consolidation cycles within a lesson. 27% of these teachers used multiple Development-Consolidation cycles throughout a series of lessons taught for a particular mathematics unit. It was also observed that teachers who taught the IP course (50%) tended to deliver their lessons through multiple Development-Consolidation cycles throughout a series of lessons taught for a particular mathematics unit over other lesson structures. On the other hand, teachers who taught the Express (60%), N(A) (88%), and N(T) (38%) courses of study tended to use single Development-Consolidation cycle within a mathematics lesson over other lesson structures. It should however, be pointed out that more care should be exercised in interpreting these percentages due to the small sample size of teacher participants for each of the course of study.

The findings suggested though that despite the diversity in the nature of lessons, a common feature of these lessons is that the activities which define the Development and Consolidation phases were used as platforms for teachers to teaching students relationally and to provide ample opportunities for teachers to assess how well students are able to apply what they have learned, i.e. a good level of practice of formative assessment to go with teaching for relational understanding.

Table 5.2 illustrates how the phases of lessons unfold in a Pythagoras' Theorem lesson taught by an experienced and competent teacher, Teacher 6. The teacher was observed adopting a single Development-Consolidation cycle in his lesson that includes the following key features:

- **Set the stage for learning**. Check on students' previous learning and clarify the learning goal for the lesson.
- **Introduce new body of knowledge and convey expectations**. Explain a new mathematical concept and several worked examples.
- **Check for understanding and review**. Provide students with an opportunity to attempt a problem on their own, and conduct a review of student work by bringing students' attention to the quality of their peers' solutions.

Table 5.2 Activities characterised in Teacher 6's lesson on Pythagoras' Theorem

Phase	Activity
Introduction	1. Teacher recaps conditions for Pythagoras Theorem that were taught previously 2. Teacher informs students of the lesson objectives (i.e. to prove that a particular triangle is a right-angled triangle)
Development	1. Teacher explains the converse of Pythagoras Theorem 2. Teacher explains the solutions to three worked examples
Consolidation	1. Teacher assigns one practice question for students to attempt 2. Teacher selects some students to share their answers on the whiteboard and uses the students' answers to explain the solution to the practice question
Conclusion	1. Teacher recaps the main ideas that were taught in the lesson and assigns homework on Pythagoras Theorem

Table 5.3 Activities characterised in Teacher 17's lesson on Trigonometry

Phase	Activity
Introduction	1. Teacher assigns students with entry card on Sine Rule 2. Teacher explains solutions to the entry card
Development	1. Students work with their peers to discover the Cosine Rule using Edmodo (an online learning platform) by themselves
Consolidation	1. Teacher assigns three practice questions for students to attempt
Development	1. Teacher checks answers for the discovery of Cosine Rule, and explains the concept of Cosine Rule
Consolidation	1. Teacher checks and explains solutions to the three practice questions that students attempted
Conclusion	1. Teacher assigns practice questions as homework, and briefly recaps the concept of Cosine Rule

- **Bring the lesson to closure**. Connect the knowledge introduced in the lesson and assign additional practice questions to extend student learning.

In another lesson, Teacher 17 was observed adopting multiple cycles of Development and Consolidation phases. As illustrated in Table 5.3, Teacher 17 demonstrated similar lesson features as Teacher 6 which includes the following phases:

- **Set the stage for learning**. Check on students' previous knowledge.
- **Engage students with new body of knowledge**. Engage students in discovering a mathematical concept by themselves.
- **Check for understanding**. Extend student learning through a few mathematical problems that they attempt by themselves.
- **Provide teacher input on new body of knowledge**. Follow up on student learning by providing teacher's explanation of the student self-discovered mathematical concept.
- **Review student learning**. Review student work.
- **Bring the lesson to closure**. Summarise the learning for the day and assign additional problems for practice.

For both teachers, there is an evident use of Development-Consolidation cycle(s) to convey learning expectations—learning relationally, and providing opportunities for formative assessment to check on students' level of mastery. In other words, the way these teacher participants structured the phases of their lesson is one that exemplifies Bloom's concept of teaching for mastery learning.

5.4.1.2 Teacher Survey

The experienced and competent teachers were asked to reflect on their teaching practices and indicate how often they adopt the lesson structures that were observed in their lessons (as in Table 5.1). The self-reported survey data are consolidated in Table 5.4.

Based on the responses for "frequently" and "mostly/always", the survey data showed that 60 and 54% of the experienced and competent teachers used single or multiple Development-Consolidation cycle(s), respectively within a lesson. In addition, 64% of these teachers frequently or almost/always used multiple Development-Consolidation cycles throughout a series of lessons taught for a particular mathematics unit. The survey data also suggest that the three lesson structures were used fairly consistently by experienced and competent teachers across all courses of study. As compared to the other courses of study (at least 55% in each course), it was observed that comparatively fewer teachers who teach the N(T) course (33%) tend to deliver their lessons through a single Development-Consolidation cycle within a lesson. On the other hand, fewer teachers who teach the IP (25%) used multiple Development-Consolidation cycles within a lesson as compared to teachers in the other three courses (at least 50% in each course). However, due to the small sample of experienced and competent teachers in each course of study, care should be taken when interpreting these results.

A comparison of the data drawn from video-recorded lessons and the teacher survey revealed some discrepancies between these teachers' observed practice and their perceived use in the self-reported in the survey (see Tables 5.1 and 5.4). In particular, comparatively more experienced and competent teachers appeared to claim that they made use of multiple Development-Consolidation cycles within a lesson and within a series of lessons taught for a mathematics unit.

Other survey respondents ($N = 647$) were also asked to reflect on their teaching practices and indicate how often they adopt the lesson structures that were observed in lessons taught by the experienced and competent teachers (as in Table 5.1). Based on the responses for "frequently" and "mostly/always", the survey data revealed that 70 and 61% of the survey respondents adopted the single or multiple Development-Consolidation cycle(s), respectively within one lesson, and 51% of them used multiple Development-Consolidation cycles throughout a series of lessons taught for a particular mathematics unit (Table 5.5). The data also showed that the three lessons structures were used fairly consistently across all courses of study. In addition, the survey data of these survey respondents appears to be comparatively consistent with the data drawn from the survey responses of the experienced and competent teachers (see Tables 5.4 and 5.5).

Table 5.4 Use of Development-Consolidation cycles as self-reported by experienced and competent teachers

Lesson structure	Course of study	Percentage of participants				
		Never/rarely	Sometimes	Frequently	Mostly/always	Total (frequently and mostly/always)
My lesson is typically structured this way: Introduction, Development, Consolidation, Conclusion	All	10	30	53	7	60
	IP	0	25	50	25	75
	Express	9	18	64	9	73
	N(A)	11	33	56	0	56
	N(T)	17	50	33	0	33
My lesson is typically structured this way: Introduction, Development 1, Consolidation 1, Development 2, Consolidation 2, Conclusion (or similar, i.e. more than one cycle of Development-Consolidation)	All	3	43	41	13	54
	IP	0	75	25	0	25
	Express	0	46	27	27	54
	N(A)	11	22	56	11	67
	N(T)	0	50	50	0	50
I only have one Introduction at the beginning of the topic and one Conclusion at the end of the entire topic. There are mainly Development and Consolidation cycles in most of the lessons	All	13	23	41	23	64
	IP	50	0	0	50	50
	Express	9	27	46	18	64
	N(A)	12	22	33	33	66
	N(T)	0	33	67	0	67

Note $N = 30$. n(IP) = 4, n(Express) = 11, n(N(A)) = 9, n(N(T)) = 6

Table 5.5 Use of development-consolidation cycles as self-reported by survey participants

Lesson structure	Course of study	Percentage of participants				
		Never/rarely	Sometimes	Frequently	Mostly/always	Total (frequently and mostly/always)
My lesson is typically structured this way: Introduction, Development, Consolidation, Conclusion	All	6	24	47	23	70
	IP	6	22	54	18	72
	Express	5	27	46	22	68
	N(A)	7	21	44	28	72
	N(T)	6	19	54	21	75
My lesson is typically structured this way: Introduction, Development 1, Consolidation 1, Development 2, Consolidation 2, Conclusion (or similar, i.e. more than one cycle of Development-Consolidation)	All	7	32	44	17	61
	IP	6	39	46	9	55
	Express	7	33	43	17	60
	N(A)	9	31	47	13	60
	N(T)	10	23	49	18	67
I only have one Introduction at the beginning of the topic and one Conclusion at the end of the entire topic. There are mainly Development and Consolidation cycles in most of the lessons	All	15	34	38	13	51
	IP	17	28	42	13	55
	Express	14	31	39	16	55
	N(A)	16	40	34	10	44
	N(T)	12	39	40	9	49

Note $N = 647$. $n(\text{IP}) = 54$, $n(\text{Express}) = 369$, $n(\text{N(A)}) = 142$, $n(\text{N(T)}) = 82$

5.4.2 Worked Example and Class Practice Task Nexus

5.4.2.1 Video-Recorded Lessons

The video-recorded lessons were also coded for the way these experienced and competent teachers structured the worked example and class practice task nexus during their lessons. The lessons were coded according to whether it involved the explanation of the worked example before or after class practice as shown in Table 5.6.

On the one hand, the video data revealed that 73 and 50% of the experienced and competent teachers explained the solution of one or a few worked example(s), respectively before assigning class practice to students. On the other hand, 33% of these teachers were observed assigning class practice without explaining any worked examples, and 20% of them tended to explain a few worked examples without assigning class practice to students. None of the experienced and competent teachers were observed to allocate class practice before explaining solutions to worked examples.

Teachers who taught the IP course tended to explain one worked example before class practice (50%). 50% of IP teachers were also observed to assign class practice without explaining any worked examples to their students. This is aligned with the observation made by Kalyuga et al. (2001) as they felt that more knowledgeable learners may benefit more from problem solving than from worked examples because of redundancy posed by worked examples. In contrast, teachers who taught the Express course of study tended to use worked examples in their lessons; 70% of the teachers explained one worked example before class practice while 70% of them explained a few worked examples before class practice. This was similarly observed among teachers who teach the N(T) course of study; 88% of the teachers explained

Table 5.6 Use of worked example and class practice task nexus as observed in lessons conducted by experienced and competent teachers

Approach	Percentage of participants				
	IP ($n = 4$)	Express ($n = 10$)	N(A) ($n = 8$)	N(T) ($n = 8$)	Total ($N = 30$)
I explain the solution of one worked example before students go on with the practice questions	50	70	75	88	73
I explain the solutions of a few worked examples before students go on with the practice questions	25	70	38	50	50
I explain the solutions of a few worked examples without students working on practice questions in class	0	30	25	13	20
I ask students to do practice questions without showing any worked examples	50	20	38	38	33
I ask students to do practice questions first, before showing worked examples	0	0	0	0	0

one worked example before assigning students with class practice while 50% of them explained a few worked examples before assigning students with class practice. On the other hand, 75% of the teachers who teach the N(A) course of study were observed to explain one worked example before class practice.

Figure 5.4 shows how an experienced and competent teacher, Teacher 17, used a worked example to bring students' attention to two Sine Rule formulas and how the different formulas could be applied based on the context of the mathematical problem. For instance, the teacher made it explicit that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ would work well with problems that involve an unknown angle, while $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ would be more appropriate if the problem requires one to approach it without any information on the sides of the triangle. In this case, Teacher 17 weighed the possible ways to approach the worked example, and helped students make appropriate selection and application of problem-solving techniques.

In another lesson, Teacher 19 was observed showing a short video clip of real-life applications of trigonometric ratios to the class. Teacher 19 proceeded with the lesson by assigning students with a problem to solve without explaining any worked examples. As seen in Fig. 5.5, the problem involves the application of previously taught concepts and skills on Trigonometry.

Teacher 17: Now let's take a look at the formula, the Sine Rule. This is called the Sine Rule. What is this Sine Rule all about?

[Teacher writes $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ and $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ on the whiteboard]

Teacher 17: You stop, pause, and you take a look at these two. Yeah, we have arrived, these two relationships. Okay these are known as the Sine Rule. Now, they are both talking about the same thing but there are instances where I would prefer this formula...

[Teacher points to $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ on the whiteboard]

Teacher 17: Over this formula.

[Teacher points to $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ on the whiteboard]

Teacher 17: Okay when do I use this, and when do I use the other one, the reciprocal relationship? Okay let's take a look. As long as I have an equation.

[Teacher circles $\frac{\sin A}{a} = \frac{\sin B}{b}$ and $\frac{a}{\sin A} = \frac{b}{\sin B}$ on the whiteboard]

Teacher 17: I just take any two will suffice, equation. For this case, $\frac{\sin A}{a} = \frac{\sin B}{b}$, my take is this. If I want to find the angle, from this relationship, from this equation, I can straightaway make sin A the subject right? I just have to multiply by 'a', then I get $\sin A = \frac{\sin B}{b} \times a$. So if I want to find an unknown angle, I will tend to use this.

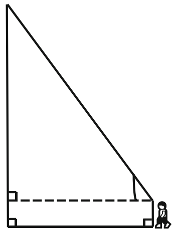
[Teacher points to $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ on the whiteboard]

Teacher 17: Because algebraically, it's advantageous to me. Okay? Otherwise, the formula is the same thing. Now, as for this other one…

[Teacher points to $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ on the whiteboard]

Teacher 17: $\frac{a}{\sin A} = \frac{b}{\sin B}$, if I want to find an unknown side, finding 'a', making small 'a' the subject will be very easy for me. It will be $\frac{b}{\sin B} \times \sin A$. Okay, so this one is good for finding unknown sides.

Fig. 5.4 Use of worked example by Teacher 17 to teach selection and application of appropriate problem-solving techniques



In the diagram, B shows a bird flying above a point A on the horizontal ground AD. The distance of AD is 80 m. A boy's eye level, E, is 1.7 m above the ground. His line of sight to the bird makes an angle of 54° with the horizontal. Find the height of the bird above the ground.

Fig. 5.5 Problem involving the use of trigonometric ratios assigned by Teacher 19

The above exemplars illustrate how the worked example and class practice nexus was adopted by two teachers in their lessons. Teacher 17 was explicit in conveying learning expectations by using a worked example to teach students the application of the Sine Rule before she proceeded to assign students with class practice of similar nature. On the other hand, without explaining solutions to any worked examples, Teacher 19 assigned a problem for students to solve as the concept and skills required of the task has already been taught in previous lessons.

5.4.2.2 Teacher Survey

In the self-reported survey, the 30 experienced and competent teachers were asked to reflect on their teaching practices and indicate how often they adopted the structures of worked example and class practice task nexus that was observed in their lessons (as in Table 5.6). The data are consolidated in Table 5.7.

Based on the responses for "frequently" and "mostly/always", the survey data showed that 40 and 64% of the experienced and competent teachers either explained the solution of one or a few worked example(s) before assigning students with class practice, respectively. In addition, 14% of the experienced and competent teachers reported that they asked students to attempt practice questions before showing worked examples, while 3% claimed that they asked students to attempt practice question without showing any worked examples. None of the teachers reported that they frequently or mostly/always explained the solutions of a few worked examples without assigning class practice. Comparison across the four courses of study based on the responses for "frequently" and "mostly/always" revealed some differences. As compared to teachers who teach the IP (50%) and Express (64%) courses, there were comparatively fewer teachers in the N(A) (22%) and N(T) (17%) courses who indicated that they explained the solution of one worked example before assigning students with practice questions. In contrast, there were lesser IP teachers (25%)

Table 5.7 Use of worked example and class practice task nexus as self-reported by experienced and competent teachers

Approach	Course of study	Percentage of participants				
		Never/rarely	Sometimes	Frequently	Mostly/always	Total (frequently and mostly/always)
I explain the solution of one worked example before students go on with the practice questions	All	27	33	13	27	40
	IP	25	25	25	25	50
	Express	18	18	18	46	64
	N(A)	33	45	11	11	22
	N(T)	33	50	0	17	17
I explain the solutions of a few worked examples before students go on with the practice questions	All	3	33	43	21	64
	IP	0	75	0	25	25
	Express	9	18	55	18	73
	N(A)	0	33	45	22	67
	N(T)	0	33	50	17	67
I explain the solutions of a few worked examples without students working on practice questions in class	All	80	20	0	0	0
	IP	100	0	0	0	0
	Express	64	36	0	0	0
	N(A)	89	11	0	0	0
	N(T)	83	17	0	0	0
I ask students to do practice questions without showing any worked examples	All	80	17	3	0	3
	IP	100	0	0	0	0
	Express	82	18	0	0	0
	N(A)	78	22	0	0	0
	N(T)	66	17	17	0	17

(continued)

Table 5.7 (continued)

Approach	Course of study	Percentage of participants				
		Never/rarely	Sometimes	Frequently	Mostly/always	Total (frequently and mostly/always)
I ask students to do practice questions first, before showing worked examples	All	63	23	11	3	14
	IP	75	0	0	25	25
	Express	64	27	9	0	9
	N(A)	56	33	11	0	11
	N(T)	66	17	17	0	17

Note $N = 30$. $n(\text{IP}) = 4$, $n(\text{Express}) = 11$, $n(\text{N(A)}) = 9$, $n(\text{N(T)}) = 6$

who indicated that they explained the solutions of a few worked examples before students work on solving practice questions as compared to teachers in the other three courses (at least 65% in each course). It was also observed that the only experienced and competent teacher who indicated that they assigned students with practice questions without showing any worked examples was a teacher who taught the N(T) course (17%). These results however need to be interpreted with caution as the sample size in each course of study is small.

A comparison of the data drawn from video-recorded lessons and the teacher survey again revealed some discrepancies between the experienced and competent teachers' use of the various approaches relating to the worked example and class practice task nexus as observed in their lesson, and their perceived use in the self-reported survey (see Tables 5.6 and 5.7). In particular, comparatively fewer experienced and competent teachers appeared to self-report their use of a worked example before class practice and worked example without class practice, and assignment of class practice without explaining worked examples. This suggests that teachers might not be aware that they adopt these approaches in their lessons as often as they think, and as in most self-report instrument, the survey is still subjected to participants' bias of a "more politically correct" response.

Other survey respondents ($N = 647$) were also asked to reflect on their teaching practices and indicate how often they adopted the structures of worked example and class practice task nexus that were observed in lessons taught by the experienced and competent teachers (as in Table 5.6). Based on the responses for "frequently" and "mostly/always", the survey data revealed that 59 and 69% of these teachers either explained the solution of one or a few worked example(s) before assigning students with class practice, respectively (Table 5.8). In addition, 8% of these teachers reported that they explained the solutions of a few worked examples without assigning class practice, 7% of them asked students to attempt practice questions before showing worked examples, and 3% of the teachers them asked students to attempt practice questions without showing any worked examples. Moreover, the survey data suggests that all five structures of worked example and class practice task nexus were consistently adopted across all course of study.

This data also appears to be comparatively consistent with the data drawn from the survey responses of the experienced and competent teachers (Table 5.7). Interestingly, relatively more survey respondents (59%) reported that they explain the solution of one worked example before assigning practice questions for students to attempt than the experienced and competent teachers (40%) on a frequently and mostly/always basis.

Table 5.8 Use of worked example and class practice task nexus as self-reported by survey participants

Approach	Course of study	Percentage of participants				
		Never/Rarely	Sometimes	Frequently	Mostly/Always	Total (Frequently and Mostly/Always)
I explain the solution of one worked example before students go on with the practice questions	All	11	30	32	27	59
	IP	8	39	33	20	53
	Express	10	29	34	27	61
	N(A)	13	24	33	30	63
	N(T)	16	36	22	26	48
I explain the solutions of a few worked examples before students go on with the practice questions	All	5	26	44	25	69
	IP	8	33	48	11	59
	Express	6	28	42	24	66
	NA	5	22	44	29	73
	NT	3	16	46	35	81
I explain the solutions of a few worked examples without students working on practice questions in class	All	71	21	6	2	8
	IP	74	20	4	2	6
	Express	72	22	5	1	6
	NA	67	24	8	1	9
	NT	72	14	7	7	14
I ask students to do practice questions without showing any worked examples	All	77	20	2	1	3
	IP	80	18	2	0	2
	Express	76	22	2	0	2
	NA	80	16	3	1	4
	NT	80	15	1	4	5

(continued)

Table 5.8 (continued)

Approach	Course of study	Percentage of participants				
		Never/Rarely	Sometimes	Frequently	Mostly/Always	Total (Frequently and Mostly/Always)
I ask students to do practice questions first, before showing worked examples	All	64	29	5	2	7
	IP	61	33	6	0	6
	Express	66	30	3	1	4
	NA	62	27	8	3	11
	NT	63	26	7	4	11

Note $N = 647$. $n(\text{IP}) = 54$, $n(\text{Express}) = 369$, $n(\text{N(A)}) = 142$, $n(\text{N(T)}) = 82$

5.5 Conclusion

This chapter provided some insights into the instructional approaches adopted by experienced and competent mathematics teachers in Singapore to promote mastery in learning in the classroom. The findings provide affirmation that in general, experienced and competent teachers are well-guided by the intended curriculum in their promotion of mastery in learning of mathematics. By conducting well-structured lessons that are defined by closely knitted single or multiple cycle(s) of the Development-Consolidation phases and the deliberate and mindful structuring of worked examples and class practice to effectively link the two phases, the experienced and competent teachers are able to teach for relational understanding with ample opportunities for formative assessment to check for students' ability to apply their learnt knowledge and skills in the mathematics class.

It is encouraging to observe that teachers across Singapore generally employ the similar approaches in their lessons. However, as in many other uses of self-report instruments, the survey data should be interpreted with caution. While the collection and analysis of the video-recorded lessons may be more time-consuming and resource-demanding, they provided more accurate insights to the practices of the teachers, and allowed us to make better sense of the self-reported data.

The findings also bear important implications for instructional and educational practices. Given the criteria that was used to select the sample of teachers for the first phase of the study, the practices enacted by these experienced and competent teachers have demonstrated a certain level of success in their classroom teaching. The insights to their practice in promoting mastery in learning of mathematics provided a repertoire of instructional approaches that could be promoted both at the pre- and in-service level of mathematics teacher education.

References

Anderson, J. R. (2000). *Learning and memory: An integrated approach* (2nd ed.). New York: Wiley.

Atkinson, R. K., Renkl, A., & Merrill, M. M. (2003). Transitioning from studying examples to solving problems: Effects of self-explanation prompts and fading worked-out steps. *Journal of Education & Psychology, 95,* 774–783.

Berthold, K., Eysink, T. H., & Renkl, A. (2009). Assisting self-explanation prompts are more effective than open prompts when learning with multiple representations. *Instructional Science, 37,* 345–363.

Bloom, B. S. (1968). Learning for mastery: Instruction and curriculum. Regional education laboratory for the Carolinas and Virginia, topical papers and reprints, number 1. *Evaluation Comment, 1*(2), 1–12.

Bokosmaty, S., Sweller, J., & Kalyuga, S. (2015). Learning geometry problem solving by studying worked examples: Effects of learner guidance and expertise. *American Educational Research Journal, 52*(2), 307–333.

Changeiywo, J. M., Wambugu, P. W., & Wachanga, P. W. (2011). Investigations of students' motivation towards learning secondary school physics through mastery learning approach. *International Journal of Science and Mathematics Education, 9*(6), 1333–1350.

Gagne, R. (1985). *The conditions of learning* (4th ed.). New York: Holt, Rinehart & Winston.
Guskey, T. R., & Anderman, E. M. (2013). In search of a useful definition of mastery. *Educational Leadership, 71*(4), 19–23.
Kalyuga, S., Chandler, P., Tuovinen, J., & Sweller, J. (2001). When problem solving is superior to studying worked examples. *Journal of Educational Psychology, 93,* 579–588.
Lee, N. H. (2009). Preparation of schemes of work and lesson plans. In P. Y. Lee & N. H. Lee (Eds.), *Teaching secondary school mathematics—A resource book* (2nd ed., pp. 337–356). Singapore: McGraw-Hill Education.
Ministry of Education. (2012). *O-Level mathematics teaching and learning syllabus.* Singapore: Ministry of Education, Curriculum Planning and Development Division.
Ministry of Education. (2018, October 11). *The Singapore teaching practice.* Retrieved September 24, 2019, from https://www.moe.gov.sg/about/singapore-teaching-practice.
Pachman, M., Sweller, J., & Kalyuga, S. (2014). Effectiveness of combining worked examples and deliberate practice for high school geometry. *Applied Cognitive Psychology, 28,* 685–692.
Renkl, A. (2014). Toward an instructionally oriented theory of example-based learning. *Cognitive Science, 38,* 1–37.
Renkl, A. (2017). Learning from worked-examples in mathematics: Students relate procedures to principles. *ZDM Mathematics Education, 49,* 571–584.
Renkl, A., Atkinson, R. K., Maier, U. H., & Staley, R. (2002). From example study to problem solving: Smooth transitions help learning. *Journal of Experimental Education, 70,* 293–315.
Renkl, A., Stark, R., Gruber, H., & Mandl, H. (1998). Learning from worked-out examples: The effects of example variability and elicited self-explanations. *Contemporary Educational Psychology, 23,* 90–108.
Rittle-Johnson, B., Loehr, A. M., & Durkin, K. (2017). Promoting self-explanation to improve mathematics learning: A meta-analysis and instructional design principles. *ZDM Mathematics Education, 49*(4), 599–611.
Skemp, R. (1987). *The psychology of learning mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.
Sweller, J., Ayres, P., & Kalyuga, S. (2011). *Cognitive load theory*. New York: Springer.
Sweller, J., & Cooper, G. A. (1985). The use of worked examples as a substitute for problem solving in learning algebra. *Cognition and Instruction, 2*(1), 59–89.
Tuovinen, J., & Sweller, J. (1999). A comparison of cognitive load associated with discovery learning and worked examples. *Journal of Educational Psychology, 91*(2), 334–341.
Van Gog, T., Paas, F., & Van Merriënboer, J. J. G. (2004). Process-oriented worked examples: Improving transfer performance through enhanced understanding. *Instructional Science, 32,* 83–98.
Van Gog, T., & Rummel, N. (2010). Example-based learning: Integrating cognitive and social-cognitive research perspectives. *Educational Psychology Review, 22,* 155–174.
Wambugu, P. W., & Changeiywo, J. M. (2008). Effects of mastery learning approach on secondary school students' physics achievement. *Journal of Mathematics, Science & Technology Education, 4*(3), 293–302.
Wittwer, J., & Renkl, A. (2010). How effective are instructional explanations in example-based learning? A meta-analytic review. *Educational Psychology Review, 22,* 393–409.

Ngan Hoe Lee is an Associate Professor at the National Institute of Education (NIE). He taught Mathematics and Physics in a secondary school before becoming a Gifted Education Specialist at the Ministry of Education. At NIE, he teaches pre- and in-service as well as postgraduate courses in mathematics education and supervises postgraduate students pursuing Masters and PhD degrees. His publication and research interests include the teaching and learning of mathematics at all levels—primary, secondary and pre-university, covering areas such as mathematics curriculum development, metacognition and mathematical problem solving/modelling, productive failure and

constructivism in mathematics education, technology and mathematics education, and textbooks and mathematics education.

Liyana Safii has been a research assistant at the National Institute of Education at Nanyang Technological University, Singapore for five years. She holds a Bachelor of Arts in Psychology from Murdoch University where she graduated as a valedictorian and was a recipient of the Vice Chancellor's Commendation for Academic Excellence Award. Her research interests include classroom practices in primary schools, especially in self-assessment and feedback.

Chapter 6
Facilitation of Students' Metacognition: Some Insights Gleaned from Mathematics Classrooms in Singapore Secondary Schools

Kit Ee Dawn Ng, Ngan Hoe Lee, and Liyana Safii

Abstract Metacognition has been an active field of research since the 1970s. Proponents operationalised the detection of metacognition in various ways. Teacher education research into how use of metacognitive strategies can be developed in students is crucial as there is compelling evidence that metacognition has direct implications on problem-solving success. Drawing on video-recorded lessons from Phase 1 of the project, this chapter first presents the strategies used by 30 experienced and competent teachers when facilitating students' metacognition. Four main strategies were found to be used, among others. Findings reveal that about 43% of these teachers encouraged students to compare different ways in solving a problem. In addition, 40% of the teachers allocated time in class for students to reflect and monitor their learning, while 30% of the same teachers required students to check on the reasonableness of their solutions. Less than 27% of the teachers, however, provided opportunities for students to assess their understanding of the problem before solving it. Next, two case-study teachers working with two mathematically diverse classrooms (i.e. mathematically strong versus mathematically challenged) are discussed with respect to their choice of instructional strategies towards activating students' metacognition. Finally, implications on teacher education, teaching, learning, and future research are drawn.

Keywords Experienced and competent mathematics teachers · Case-study · Metacognition · School mathematics curriculum · Secondary schools · Teacher education

K. E. D. Ng (✉) · N. H. Lee · L. Safii
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: dawn.ng@nie.edu.sg

N. H. Lee
e-mail: nganhoe.lee@nie.edu.sg

L. Safii
e-mail: liyana.safii@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_6

6.1 Metacognition

Coined by Flavell (1976), metacognition is defined as "one's *knowledge* concerning one's own cognitive processes and products or anything related to them" (p. 232). Flavell also articulated two preliminary-related components of metacognition: "active *monitoring*" and "consequent *regulation*", emphasising the importance of the "orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective" (p. 232). Metacognitive knowledge was further unpacked in Flavell's (1979, 1981, 1987) theoretical model for capturing the thinking processes of a person through actions and interactions between four aspects of a cognitive enterprise: cognitive goals, cognitive actions, metacognitive knowledge, and metacognitive experiences. Metacognitive knowledge is about *awareness* of how factors (i.e. person, task, and strategy) act and interact to influence the outcome of the cognitive endeavour. Foundational research in metacognition during mathematical problem-solving have mainly focused on analysing the nature of use of metacognitive knowledge within monitoring and regulatory processes adopted by the solver (see Garofalo & Lester, 1985; Schoenfeld, 1985a). Until recently, more attention was paid to research into metacognitive experiences and this consists of: "online task-specific knowledge" as well as "feelings, judgments or estimates", which can mainly be "non-conscious, non-analytic inferential processes" (Efklides, 2006, p. 3). Collectively, Flavell (1985) had espoused that metacognitive knowledge, metacognitive experiences, and cognitive behaviour are constantly interacting with one another during the course of a cognitive task (p. 108). Metacognition has been an active, exciting field of international research since it was first introduced as a construct in the 1970s.

Metacognition as a ground-breaking construct and subsequent pivotal research on the role metacognition plays in mathematical problem-solving paved the way for incorporating metacognition into the Singapore School Mathematics Curriculum Framework in 1990. Since then, metacognition has been one of five key components towards achieving the main goal of mathematical problem-solving in the Curriculum (see Chapter 1, Fig. 1.2). According to the current Singapore mathematics curriculum documents (Ministry of Education [MOE], 2018), metacognition involves a person reflecting on his or her thinking. Metacognition is further unpacked to include "awareness of, and the ability to control one's thinking processes, in particular the selection and use of problem-solving strategies. It includes monitoring and regulation of one's own thinking and learning" (p. 11). In other words, metacognition is now perceived to have three components: *awareness*, *monitoring* and *regulating* (Lee, Ng, & Yeo, 2019). The component of awareness also includes how cognizant is one about his or her own affective responses towards a problem (MOE, 2018).

Nonetheless, there are several challenges facing researchers in capturing metacognition. The most pertinent is that metacognition exists as an internal process within the self and is hence covert and unobservable. Furthermore, the distinction between cognition and metacognition has to be operationalised during analysis of task-based behaviours. In addition, when metacognition can occur in practice can also make a difference. Desoete and Roeyers (2002) articulated the difference between online and offline metacognition and this can be useful in classifying the various methods researchers have used to capture metacognition. This classification is also a useful lens to examine teachers' instructional strategies in metacognition. Online metacognition refers to real-time metacognitive processes being activated as the person attempts the task at hand. In contrast, offline metacognition comprises "prediction" and "evaluation" (Desoete & Roeyers, 2002, p. 123). Predictive metacognitive processes can occur when there is prior assessment of task objectives, effort to be put in, and task difficulty leading to an estimation of task outcome and efficacy *before* the task was engaged in. Evaluative metacognitive processes can occur *after* the task was completed, where retrospective reflections take place. An example of an online method for capturing metacognitive practices are think aloud protocols (e.g. Schoenfeld, 1985b) where the person talks out loud his or her thinking processes whil doing the task. One advantage of this is that there is direct immediate record of metacognitive processes and behaviours as they occur. Verbal and video records of the think aloud can be helpful for a more comprehensive interpretation of the metacognition within the context. However, it will be a tedious and time-consuming analysis process, particularly for larger sample sizes. Self-report questionnaires (e.g. Wong, 2007) are commonly used as an offline method for capturing metacognitive practices with the obvious advantage of efficient large scale data collection. Nevertheless, as discussed in Jacobse and Harskamp (2012), there can be item interpretation issues as well as incomplete or inaccurate reports due to memory lapse during questionnaire attempts.

6.2 Teachers' Metacognitive Instructional Strategies in Mathematics Classrooms

Research (e.g. Goos, 2002; Ng, 2008) revealed the need for problem solvers to move between monitoring and *appropriate* regulatory attempts to achieve a more efficient and effective solution process. Hence, the role of teachers is multi-fold to build students' metacognitive habits. Firstly, the development of students' metacognitive awareness and strategies has to be alongside with knowledge on when and how to use these strategies. Secondly, there has to be opportunities for students to reflect on how a problem-solving attempt can be done more efficiently, examining related macro managerial decision-making informed by effective monitoring-regulatory cycles. Finally, provision of various problem types (including non-routine and open-ended problems) serves as platforms for solution discussions and enactment of think-aloud processes, allowing novice problem solvers to learn from the experts.

There exists various intervention models of teachers' metacognitive instructional strategies in mathematics classrooms. Many of which have focused on providing a framework with clear interacting stages to guide students in activating their metacognitive processes (see Foong, 1993; Hacker, Kiuhara, & Levin, 2019; Lee, Yeo, & Hong, 2014; Raymond, Gunter, & Conrady, 2018; Teong, 2003). Metacognitive strategies embedded within the frameworks can be explicitly taught, demonstrated, or discussed during problem-solving lessons. For example, as reported in Lee et al. (2019), the Problem Wheel (see Lee, 2008) framework used a series of question prompts to help students make sense of the basic structure of a mathematical problem. In addition, the question prompts also provided the means for students to become aware of their existing knowledge, strategies, and related problem-solving experiences for understanding the problem at hand and its context. Through the prompts, students are reminded to select their strategies, monitor their use of strategies, and engage in appropriate regulatory behaviours during the problem-solving process. Nonetheless, such intervention models are often done on research timelines and are not sustained nor expanded at post-research. In Singapore, despite metacognition being incorporated in the Mathematics Curriculum Framework since the 1990s, there is still a lack of concerted efforts to promote teacher instructional strategies on metacognition in mathematics classrooms.

One main reason for this is that metacognition is a difficult construct to grasp. Ng, Lee, Seto, Loh, and Chen (2016) found that mathematics teachers from three primary schools had misconceptions of what metacognition was, confusing the construct with other thinking processes (e.g. critical thinking, higher order thinking) and teaching approaches (e.g. engaged learning, making thinking visible). The same study also found through video-recorded lessons and teacher surveys that some teachers claimed to have developed students' metacognition in their classes when there was either little evidence observed or there were misinterpretations of teacher practice. The research by Ng and her colleagues is one of the few Singapore research on teacher metacognitive instructional strategies. More can still be done to understand existing practices in Singapore schools pertaining to the development of students' metacognitive strategies in mathematics classrooms, across levels and courses of study.

This brings to mind another crucial piece of the puzzle: teacher education on metacognition and its associated research. Teacher education on metacognitive instructional strategies is important because there is compelling evidence that the presence of metacognition and the nature of its use have direct implications on problem-solving success (see Desoete & De Craene, 2019; Lee et al., 2018; Loh & Lee, 2019). Moreover, there is also some evidence (as cited above) that teachers need to have a clear conception of what metacognition is and how to develop metacognitive strategies in their students. Knowledge of existing practices in Singapore schools will help chart the way for more targeted and systematic teacher education on metacognitive instructional strategies.

Drawing on the data of the project (see Chapter 2), this chapter first presents an analysis of the strategies used by 30 experienced and competent teachers when facilitating students' metacognition. Next, two case-study teachers working with two mathematically diverse classrooms (i.e., mathematically strong versus mathematically challenged) are discussed with respect to their choice of instructional strategies towards activating students' metacognition. Finally, implications on teacher education, teaching, learning, and future research are drawn.

6.3 Data Analysis Methods

Video-records of lessons of the 30 experienced and competent teachers, who participated in Phase 1 of the project (see Chapter 2, Sect. 2.3 for details), were scanned for strategies teachers used to facilitate metacognition in their lessons. Four strategies were revealed (see Table 6.1). Each strategy appeared to be indicative of a component of metacognition. In the next section, we will present the use and comparisons of the strategies by the 30 experienced and competent teachers across levels and courses of study in terms of percentages.

Based on the video-recorded lessons, four case-study teachers, Teacher 30, Teacher 22, Teacher 17 and Teacher 11, teaching the Normal (Technical) (N(T)), Express, Integrated Programme (IP) and Normal (Academic) (N(A)) courses of study respectively were selected from the 30 experienced and competent teachers for further investigations. The teachers were selected by convenient purposive sampling out of the 30 teachers. Three of the case-study teachers were assigned to the first author to follow through during the data collection process. The remaining one was selected based on field notes and video data which showed identified attempts at using metacognitive instructional strategies by the research team. It was evident that there were differences between how the four case-study teachers promoted students' use of metacognition during their lessons. Only Teacher 17 and Teacher 11 were selected for focus in this chapter because they exemplified the use of metacognitive instructional strategies in mathematically contrasting classrooms. Teacher 17 worked with a mathematically strong class (Secondary 3 IP) while Teacher 11 taught in a mathematically challenged class (Secondary 4 N(A)). The types of metacognitive instructional strategies the two case-study teachers used and factors contributing to

Table 6.1 Teachers' metacognition instructional strategies identified from the 30 competent teachers gleaned from video-recorded lessons

No.	Strategy	Component of metacognition
1	I get my students to check on their understanding of a problem before they commence on solving a problem	Awareness
2	I get my students to compare different ways of solving a problem	Awareness
3	I get my students to reflect on their learning (including helping them monitor their own learning)	Monitoring
4	I get my students to check for reasonableness of their answers after solving a problem	Regulation

their chosen instructional strategies will be discussed in the following sections. For deeper insights, it was necessary to examine how these strategies were enacted within the lessons because there is fluidity between when and how the three components of metacognition are activated in students and the teachers' corresponding facilitation moves.

For the teacher cases, the following data were coded and analysed using grounded theory methods (Strauss & Corbin, 1990, 1998): (a) transcripts of lesson videos, (b) transcripts of teacher interviews, (c) lesson materials, and (d) student work artefacts. Field notes were recorded and used to aid contextual interpretation of the data. An analysis of the data sets listed provided insights into students' online and offline metacognition identified and facilitated by the teacher in real time. Grounded theory methods adopt an inductive process to generate theories grounded from data to explain a process, an action, or an interaction (Creswell, 2014). Grounded theory methods require a rigorous process of analysis where researchers engage in iterative cycles weaving empirical checks into the analytic process. This results in a progressively more focused flow of analysis leading to coding categories which explain and theorise the essence of the phenomenon being examined (Bryant & Charmaz, 2007). We adopted the reformulated grounded theory methods advocated by Strauss and Corbin (1990, 1998) because while we recognise the importance of deriving analytical categories from data, true to the nature of phenomenon, we also acknowledge that researchers can exercise theoretical sensitivity when looking through the lens of grounded theory in data analysis and interpretation. Theoretical sensitivity refers to perspectives about the phenomenon in which researchers bring into the research. This includes initial focus of empirical investigation based on beliefs, worldview, assumptions, and relevant literature review. Three sequential coding processes were outlined by Strauss and Corbin (1990): open, axial and selective coding. In the process of generating analytical codes for the data, we also engaged in constant comparative analysis (Glaser, 1992), working towards theoretical saturation of codes.

During the open coding process, data is interpreted within the context. Analytical codes are generated to categorise segments of data. Such codes are represented by a short name that simultaneously summarises and accounts for each piece of data. Two types of analytical codes were used: in vivo codes (actual words/phrases in the data used as codes) or active codes (meanings/interpretations of the phrases used as codes). In vivo and active codes were combined, recategorised, and/or reworded during the iterative process of empirical checks across different data sources and interpretations. The coding process was informed by the components of metacognition (i.e. awareness, monitoring, and regulation) and teachers' metacognitive instructional strategies synthesised from research. Table 6.2 shows excerpts from a preliminary coding framework to elicit factors affecting teachers' choice of metacognitive instructional strategies.

Table 6.2 A preliminary framework to elicit factors affecting teachers' choice of instructional approaches

Active codes generated during the open-coding process	Raw data used for coding [Segments in bold were used to form codes]	Explanation on how the active codes were derived
Activating students' offline regulation	Teacher Interview: **Teacher 17 (IP)**: The other thing I wanted to do is, this thing about the area of triangles. ***By now they would have learned all the different ways of calculating the area of triangle.*** Primary school is $\frac{1}{2} \times base \times height$, then earlier on in coordinate geometry they have learned the surveyor's formula, using the coordinates. Then after that, this lesson, they will learn $\frac{1}{2}ab\ \sin C$. Then at the end they will learn Heron's formula. ***So I'm going to ask them also, to look at the affordances, compare*** **I**: Compare the different ways of actually calculating the area of triangles **Teacher 17 (IP)**: Yeah **I**: That's nice. So that's compare and contrast. So would you be anticipating certain difficulties or you think they will be able to… **Teacher 17 (IP)**: ***They, I think not all of them may be able to tell you the affordances***	During the interview, the teacher articulated her awareness of what the students know about the various formulae to find the area of a triangle. The teacher made plans for students to compare and contrast the different formulae, examining the conditions of use of each formula. During the lesson, she encouraged open discussions about the appropriateness of use of these formulae using a real-world problem. The students were prompted to analyse the potentials and limitations of each formula in view of the given information in the problem. By doing so, the teacher is encouraging her students to engage in offline regulation—i.e. regulating the use of the various formulae for different problem situations

(continued)

Table 6.2 (continued)

Active codes generated during the open-coding process	Raw data used for coding [Segments in bold were used to form codes]	Explanation on how the active codes were derived
Promote self-monitoring of learning by students	Teacher Interview: **Teacher 11 (NA)**: Okay for today's lesson basically is for them to have more practice with the formula, using the formula. At the same time ***my goal was also to close the gap, for those students who were absent.*** Ya, so it's important that, you know, ***I have to make sure that they can be, they can be back, asap. In terms of, you know the level that they are at***, ya, so, to, bridge that gap la. Ya. And so that's why at the start of it, I actually did a review ah **Teacher 11 (NA)**: Yes, usually I will do that, ***because to be fair to those who were absent, they were not in the discussion, so they may not know,*** and knowing these kids, they don't read their textbook also, even though they know that we're gonna start on Chapter 4. Ya, so I thought the best way, you know, so it serves two purposes. ***While I'm checking on the students, you know, their understanding of yesterday's lesson, at the same time I'm able to introduce, you know, the lesson that they missed***	Teacher built-in a structure within all her lessons to allow time for students to reflect on their previous learning by using mini whiteboards This is obvious when the teacher interview data is triangulated with the lesson videos The teacher wanted students to record in writing what they've learnt and look at the progress they've made during her lesson; encouraging self-monitoring of learning

6.4 Metacognitive Instructional Strategies Used by the 30 Experienced and Competent Mathematics Teachers Gleaned from Video-Recorded Lessons

Table 6.3 shows that 43.3% of the 30 experienced and competent teachers facilitated students' metacognition by inviting students to compare different ways of solving the problem during the lesson. This provided opportunities for students to become *aware* of the collective repertoire of knowledge and skills which they could draw upon for the same problem within the class. Almost similarly, 40% of the experienced teachers encouraged students to reflect on their learning during their lessons. This implied that there was some form of students' self-*monitoring* of their learning facilitated by the teachers. About one-third of the teachers requested students to check for reasonableness of their answers after solving a problem. This suggested that many of the experienced teachers might not have provided ample time in class for students to assess their problem-solving attempts and subsequently perform appropriate *regulatory* actions in view of these assessments, working towards a higher chance of problem-solving success. In addition, only 26.7% of the experienced teachers required students to assess their understanding of the problem before they start working on it. Overall, it was revealed that less than half of the experienced and competent teachers have used metacognitive instructional strategies in their mathematics lessons.

Across all four courses of study, it was surfaced that all the experienced and competent teachers who taught IP classes in the sample had requested students to compare different ways of solving the problem. The percentages of teachers teaching Express, N(A), and N(T) classes who used the same approach were apparently less. Experienced and competent teachers teaching Express classes appeared to have encouraged students to reflect on their learning more, compared to other approaches such as

Table 6.3 Proportion of experienced and competent teachers who used the various types of metacognitive instructional strategies identified from video-recorded lessons

Metacognitive instructional strategies	Percentage of teachers				
	IP ($n = 4$)	Express ($n = 10$)	N(A) ($n = 8$)	N(T) ($n = 8$)	Total ($N = 30$)
I get my students to check on their understanding of a problem before they commence on solving a problem	25.0	40.0	37.5	0.0	26.7
I get my students to compare different ways of solving a problem	100.0	40.0	37.5	25.0	43.3
I get my students to reflect on their learning (including helping them monitor their own learning)	50.0	50.0	37.5	25.0	40.0
I get my students to check for reasonableness of their answers after solving a problem	25.0	20.0	37.5	37.5	30.0

checking for reasonableness of answers. On the other hand, the same percentage of experienced teachers for Normal (Academic) classes in the sample seemed to have used all four metacognitive instructional strategies.

Although not captured by the four strategies listed in Table 6.3, it was discovered from video records that Teacher 22 attempted to "model" her own metacognition when solving problems during the lesson. She was teaching the solving of quadratic equations by factorisation in a Secondary 3 Express mathematics classroom. The teacher demonstrated how she used "think aloud" to articulate her awareness, monitoring, and regulation of her thinking processes. She explicitly highlighted that she wanted the students to listen to her thinking process to learn not only how to solve the equation, but also how to solve it efficiently. For instance, the teacher verbalised her thinking as she attempted a problem by asking herself two questions: (a) Is this a quadratic equation? (b) Do I need to expand the given equation? The teacher highlighted that these questions guided her in thinking about the method that could be used to solve the equation in an efficient way. Indeed, the teacher was trying to articulate out loud her macro tactical decision-making processes as she assessed "her" repertoire of knowledge and skills while putting herself in the shoes of her students. Her choice of methods was goal-oriented, deliberate, and considered in view of this repertoire. She was just one step away from direct facilitation of her students' metacognitive behaviours. This teacher-case paved the way for a more in-depth investigation of other teacher-cases who enacted various metacognitive instructional strategies to directly facilitate students' metacognition. We present two such teacher-cases in two mathematically diverse classrooms.

6.5 Teacher 11 in a Mathematically Challenged Class

Teacher 11 has more than 20 years of teaching experience at the point of research. She was very experienced in teaching mathematics in N(A) classes. She conducted seven lessons on the topic of arc lengths, sector areas, and radian measures for the Secondary 4 N(A) class recorded in the research. Trigonometry is an important topic in school mathematics. Weaker students often find the learning of arc lengths, sector areas, and radian measures challenging, particularly the transition from degree to radian measures. According to Moore (2012), quantitative reasoning plays a central role in students' trigonometric understandings. Teacher 11 had been teaching the same class since the year before and she was cognizant about her students' lack of confidence and self-esteem in mathematics and their need for constant "reminders" or "revision" of what had been taught previously to draw connections to new learning outcomes. Since her first encounter with the class, she had adopted different questioning techniques and activities to help students construct their understanding of the mathematical concepts and quantitative relationships instead of telling the students directly. She allowed time for students to share their reflections, understanding, and mathematical reasoning during her lessons. It took her a long time to build such a learning culture. Her students were encouraged to discuss in class and she gave everyone a chance to show their work during her lesson segments, using effective questioning to elicit students' thinking.

Teacher 11 was detected to have adopted all four metacognitive instructional strategies outlined in Table 6.3 across her seven lessons. She often encouraged her students to *reflect on their learning (including helping them monitor their own learning)* using mini whiteboards, as shown in the teacher post-lesson interview excerpt (Table 6.4) below. In this lesson, the teacher had asked each student to record what he or she had learnt about Arc Lengths so that each student could actively monitor and reflect on his or her learning progress thus far. She also used such records to understand the needs of her students for this lesson. Throughout the topic, as teacher interview data were triangulated with lesson videos and field notes, it was observed that Teacher 11 had a built-in structure within all her lessons to allow time for students to reflect on their previous learning by using mini whiteboards or by

Table 6.4 Excerpt from Teacher 11's interview after Lesson 2 (Arc Length)

Person	Line	Contents
Teacher 11:	1	Okay for today's lesson basically is for them to have more practice
	2	with the formula, using the formula. At the same time ***my goal was***
	3	***also to close the gap, for those students who were absent***. Ya, so it's
	4	important that, you know, ***I have to make sure that they can be, they***
	5	***can be back, asap. In terms of, you know the level that they are at,***
	6	ya, so, to, bridge that gap la. Ya. And so that's why at the start of it, I
	7	actually did a review ah
Interviewer:		Mmm, yes, yes, so in a sense, you did the review and you had this mini whiteboard, I noticed you had a mini whiteboard thing, at the beginning right, where they did, and you actually asked those people who were absent, to write what they remembered about the circle. And those who were present yesterday to write what they learned about, from yesterday's lesson. So, was it really, was it something that you usually do, in terms of how you want to link it back to the previous lesson?
Teacher 11:	8	Yes, usually I will do that, ***because to be fair to those who were absent,***
	9	***they were not in the discussion, so they may not know***, and knowing
	10	these kids, they don't read their textbook also, even though they know
	11	that we're gonna start on Chapter 4. Ya, so I thought the best way, you
	12	know, ***so it serves two purposes. While I'm checking on the students,***
	13	***you know, their understanding of yesterday's lesson, at the same time***
	14	***I'm able to introduce, you know, the lesson that they missed***
Interviewer:		So how did it work, you find, based on today, this strategy?
Teacher 11:	15	I'm happy with it, actually I'm happy with it, because I could see that
	16	most of them could at least remember the arc length. Although there
	17	were some students who could remember even more, you know, so I
	18	was so, impressed

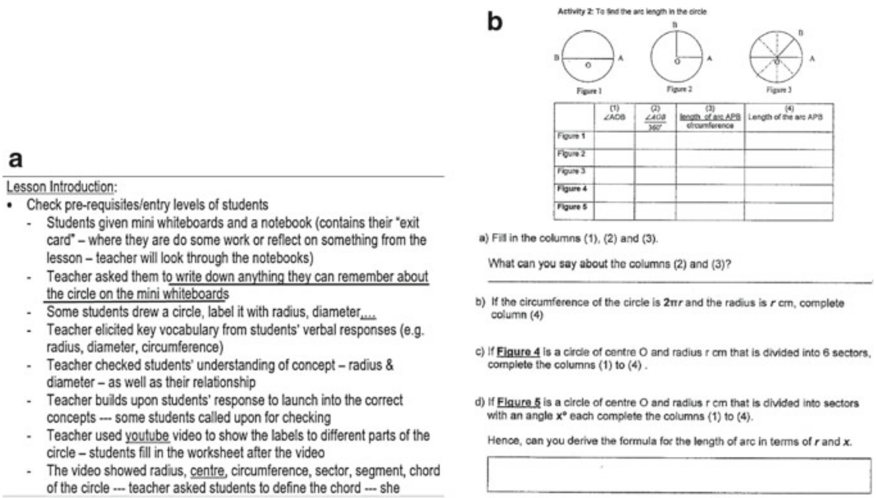


Fig. 6.1 **a** Field notes of Lesson 1 (Introduction of Arc Length) by Teacher 11. **b** Student Worksheet from Lesson 1 (Introduction of Arc Length) by Teacher 11

individual sharing based on targeted questioning. It could be interpreted from the data that one contributory factor which prompted the teacher to do so was her goal to promote *self-monitoring of learning* (Table 6.4, Lines 2 to 5, 8 to 14) in view of the class profile and her teaching style.

In addition, detected only through lesson observations, Teacher 11 had consciously activated her students' *awareness of relevant prior knowledge* before connections were explicitly made to a new concept such as Arc Length by targeted questioning. Figure 6.1a shows field notes of a lesson where Teacher 11 tried to draw out related prior knowledge before moving on to revise on the necessary prerequisites for understanding the Arc Length formula. Figure 6.1b shows the worksheet used to guide classroom discussion about the quantitative reasoning behind the derivation of the Arc Length formula using the formula for circumference of a circle.

6.6 Teacher 17 in a Mathematically Strong Class

Teacher 17 had about 15 years of teaching experience at the point of research and had served the same school throughout. She taught eight lessons on the topic of Trigonometry (trigonometric ratios of obtuse angles, sine rule and cosine rule) in a Secondary 3 IP class consisting of students who were mathematical strong and motivated. Teacher 17 had been teaching advanced mathematics in most of her classes. The culture and climate in the class being observed was very different than that of Teacher 11's. Teacher 17 preferred to challenge her class with higher order mathematical tasks which involved proofs and deductive thinking. She expected her students to read notes on their own before the lesson and to focus on mathematical communication (i.e. using correct mathematical terms, formulae, working steps) and reasoning during class discussions when students clarified their learning. Although her class was normally fast-paced and task-oriented, she allocated time for discussions and cross-checking of solutions.

Teacher 17 also adopted all four metacognitive instructional strategies outlined in Table 6.3 across her eight lessons. She often encouraged her students to *reflect on their learning (including helping them monitor their own learning)* and *compare different ways of solving a problem.* Table 6.5 shows an excerpt from a post-lesson teacher interview after a lesson on the area of triangle formula ($\frac{1}{2}ab\ \sin C$). During the lesson, the teacher gave a real-world problem and encouraged her students to compare the different formulae on the area of triangle they had previously learnt so as to decide on which was the most appropriate for use on the given problem. She wanted her students to assess the conditions in which each of these formulae could be most effectively used. Here, it was observed that Teacher 17 had skilfully weaved in questions to prompt students to reflect on their learning within purposefully facilitated comparisons of the different formulae for appropriateness and effectiveness in the problem-solving scenario. Throughout all her lessons for the topic, there were deliberate attempts by the teacher to factor in time during her lesson to help students reflect and monitor on their use of the various formulae. One contributory factor towards Teacher 17's decision to enact the metacognitive instructional strategies could be her goal to activate students' *offline regulation of their problem -solving attempts* (Table 6.5, Lines 7 to 8, 10, 15 to 17, 22 to 23) towards a more efficient and effective choice of formula.

In another lesson, Teacher 17 followed up from a previous challenging problem (given as homework) where students had to work on without a diagram to help them. She asked students to *check on their understanding of the problem before they commenced on solving a problem.* Then, she provided opportunities for students to discuss what they understood from the problem and the various possibilities of interpreting the problem using different diagrammatic representations. Here, Teacher 17's choice of the problem and her facilitation moves to get her students to deliberately return to examining their understanding of the problem were due to her goal to emphasise the *importance of assessing the understanding of the problem* (particularly the given conditions and information in the problem) so that students could consciously select the most appropriate methods to solve it. Table 6.6 shows Teacher 17's rationale for this emphasis (Lines 4 to 8, 9 to 11).

Table 6.5 Excerpt from Teacher 17's interview after Lesson 5 (Area of Triangle)

Person	Line	Contents
Teacher 17:	1	The other thing I wanted to do is, this thing about the area of
	2	triangles. By now they would have learned all the different ways of
	3	calculating the area of triangle. Primary school is $\frac{1}{2} \times base \times height$,
	4	then earlier on in coordinate geometry they have learned the
	5	surveyor's formula, using the coordinates. Then after that, this lesson,
	6	they will learn $\frac{1}{2}ab\ \sin C$. Then at the end they will learn Heron's
	7	formula. ***So I'm going to ask them also, to look at the affordances,***
	8	***compare***
Interviewer:		Compare the different ways of actually calculating the area of triangles
Teacher 17:	9	Yeah
Interviewer:		That's nice. So that's compare and contrast. So would you be anticipating certain difficulties or you think they will be able to…
Teacher 17:	10	They, I think ***not all of them may be able to tell you the affordances***
Interviewer:		Okay, so not all of them will be able to tell you what is so special about these formulas for certain cases…
Teacher 17:	11	I think I will tweak the question, maybe I will give you a plot of land
	12	okay. So, area of a triangle, how would you calculate the area of this
	13	plot? It might sound like that rather than what are the… Or I could go,
	14	what are some ways you can find the area of triangle? Okay look at
	15	the plot of land. ***What formula would you use? Because I think***
	16	***anything involving angles, if you are slightly off by 1°, the distance***
	17	***is large, the error will be…***
Interviewer:		In terms of error as well
Teacher 17:	18	The error, the error will be quite large. But then again I'm not sure if
	19	they can come to that because it's about life experience, if your
	20	distance is large and your angle is a bit off. So that's why if you have
	21	all the distances ready, the Heron's formula is quite nice $\frac{1}{2}ab\ \sin C$ is
	22	quite nice but ***provided you can find the perpendicular distance***
	23	***because sometimes it's not able to***
Interviewer:		So it will be, in the sense you have it a little bit fluid in the sense right?
Teacher 17:	24	Yes
Interviewer:		How far you can go, how in-depth they would want to go into the topic, you have to play by ear as the eight lesson progress
Teacher 17:	25	Yes

Table 6.6 Excerpt from Teacher 17's interview after Lesson 7 (Solving Trigonometry Problems)

Person	Line	Contents
Teacher 17:	1	Example 12 yesterday was, tough example because this diagram
	2	wasn't given. To draw this diagram is really difficult. It's very
	3	difficult. This is really testing, it's really testing them a lot of things
	4	because ***if they are not careful they might have assumed that A, O,***
	5	***B points are collinear points. I did pre-empt that, so I drew all the***
	6	***wrong triangles that they possibly have for Example 12. I asked***
	7	***them you know, what's wrong with drawing it this way, how do you***
	8	***know it's wrong.*** I did O in the centre, A, B at the side or O and A, B…
Interviewer:		Different possibilities
Teacher 17:	9	Yeah different possibilities, so ***ask them what's wrong with this. So***
	10	***if this cannot be the diagram, what***
	11	***does it imply about A, O, B?*** So yeah, I'm happy that one of the girls
	12	said they cannot be collinear

6.7 Implications and Future Directions

In this chapter, we examined four metacognitive instructional strategies that 30 experienced and competent mathematics teachers in Singapore secondary schools were found to have enacted, drawing upon video-recorded lessons during Phase 1 of the Project. It was discovered that not all of the 30 teachers used the strategies on a sustained, systematic, and purposeful basis like the two teacher-cases (Teacher 11 and Teacher 17) presented in this chapter.

Several observations stand out from the lessons conducted by Teacher 11 and Teacher 17 in terms of whether and how metacognitive instructional strategies were enacted in different classrooms, perhaps not limited to the teaching and learning of mathematics. Firstly, teachers can use metacognitive instructional strategies in mathematically diverse classroom settings, but attempts to weave in such strategies during lessons hinge on how teachers can harness elements of students' profile towards a conducive climate of use. For example, Teacher 11 had incorporated the need of her students to be "reminded" of their previous learning to her use of mini whiteboards, promoting students' self-monitoring of learning. In contrast, Teacher 17 activated students' regulatory behaviours constantly in her lessons as the "extra push" because while many of her students were able to draw connections between what they have learnt, she also wanted them to be critical in their choice of solution methods. Secondly, time needs to be spent on building students' habit of mind for metacognition. This is embedded within the classroom culture co-created by both teachers and students. For example, both Teachers 11 and 17 were observed to have consistently built-in time during lessons to listen to and discuss students' responses to their use of metacognitive instructional strategies. Thirdly, the enactment of metacognitive instructional strategies can develop students' online and offline metacognition-in-practice. This can be observed from Tables 6.4, 6.5 and 6.6. Fourthly, facilitation of students' metacognition is not limited to only problem-solving lessons. For example, Fig. 6.1a and b show Teacher 11's attempts to activate her students' awareness of relevant prior knowledge during the introductory lesson to the concept of Arc Length. Finally, the priority placed by teachers to promote students' metacognition is crucial for the implementation of metacognitive instructional strategies. Such a priority governs the teacher's classroom goals and prompts the teacher to consciously allocate time for the enactment of the metacognitive instructional strategies towards more productive student learning and a conducive classroom climate. This can be observed from the two teacher-cases presented here. In particular, Teacher 17 had purposefully chosen certain examples to be discussed in class to reinforce the importance of assessing the understanding of the problem.

The findings presented in this chapter provide added dimensions to future directions for teacher education on metacognition. Teacher education on metacognition has to work on at least two interconnected fronts: (a) educating teachers about what metacognition is (see Ng et al., 2016) and the various metacognitive instructional strategies, and (b) inculcating a climate of consistent, sustained use of these instructional strategies in mathematics classrooms. In addition, the rich data from video-recorded lessons, particularly those exemplified in the two teacher-cases, also provide an opening for more research to be done focusing on supporting teachers in a sustained enactment of metacognitive instructional strategies. Perhaps an in-depth, wider, and more comprehensive study focusing on teachers' understanding and use of metacognitive instructional strategies will also be needed for more insights into the inhibiting as well as favourable factors contributing to teachers' use of metacognitive instructional strategies in a sustained manner. Findings from such a study will be useful in planning purposeful teacher professional development programmes on metacognition for sustained implementation in mathematics classrooms. In essence, in the arena of metacognition in Singapore, the work has only just begun for mathematics educators.

References

Bryant, A., & Charmaz, K. (Eds.). (2007). *The SAGE handbook of grounded theory*. London: Sage.

Creswell, J. W. (2014). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research* (4th ed.). Harlow: Pearson Education.

Desoete, A., & Roeyers, H. (2002). Off-line metacognition: A domain-specific retardation in young children with learning disabilities? *Learning Disability Quarterly, 25*(2), 123–139.

Desoete, A., & De Craene, B. (2019). Metacognition and mathematics education: An overview. *ZDM Mathematics Education, 51*(4), 565–575. https://doi.org/10.1007_s11858-019-01060-w.

Efklides, A. (2006). Metacognition and affect: What can metacognitive experiences tell us about the learning process? *Educational Research Review, 1*(1), 3–14.

Flavell, J. H. (1976). Metacognitive aspects of problem solving. In L. B. Resnick (Ed.), *The nature of intelligence* (pp. 231–235). Hillsdale, NJ: Lawrence Erlbaum.

Flavell, J. H. (1979). Metacognition and cognition monitoring: A new area of cognitive—Developmental inquiry. *American Psychologist, 34*(10), 906–911.
Flavell, J. H. (1981). Cognitive monitoring. In W. P. Dickson (Ed.), *Children's oral communication skills* (pp. 33–60). New York: Academic Press.
Flavell, J. H. (1985). *Cognitive development*. Englewood Cliffs, NJ: Prentice-Hall.
Flavell, J. H. (1987). Speculation about the nature and development of metacognition. In F. E. Weinert & R. H. Kluwe (Eds.), *Metacognition, motivation and understanding* (pp. 21–29). Hillsdale, NJ: Lawrence Erlbaum.
Foong, P. Y. (1993). Development of a framework for analysing mathematical problem solving behaviours. *Singapore Journal of Education, 13*(1), 61–75.
Garofalo, J., & Lester, F. K., Jr. (1985). Metacognition, cognitive monitoring and mathematical performance. *Journal for Research in Mathematics Education, 16*(3), 163–176.
Glaser, B. (1992). *Basics of grounded theory analysis*. Mill Valley, CA: Sociology Press.
Goos, M. (2002). Understanding metacognitive failure. *Journal of Mathematical Behavior, 21*(3), 283–302.
Hacker, D. J., Kiuhara, S. A., & Levin, J. R. (2019). A metacognitive intervention for teaching fractions to students with or at-risk for learning disabilities in mathematics. *ZDM Mathematics Education, 51*(4), 601–612. https://doi.org/10.1007_s11858-019-01040-0.
Jacobse, A. E., & Harskamp, E. G. (2012). Towards efficient measurement of metacognition in mathematical problem solving. *Metacognition and Learning, 7*(2), 133–149. https://doi.org/10.1007_s11409-012-9088-x.
Lee, N. H. (2008). *Enhancing mathematical learning and achievement of secondary one normal (academic) students using metacognitive strategies*. Unpublished doctoral dissertation, Nanyang Technological University, Singapore.
Lee, N. H., Yeo, J. S. D., & Hong, S. E. (2014). A metacognitive-based instruction for Primary Four students to approach non-routine mathematical word problems. *ZDM Mathematics Education, 46*(3), 465–480.
Lee, N. H., Ng, K. E. D., Seto, C., & Loh, M. Y. (2018). Some potential and challenges in examining teachers' metacognitive instructional strategies using video technology. In L. H. Xu, G. Aranda, & D. Clarke (Eds.), *Video-based research in education: Cross-disciplinary perspectives*. London: Routledge.
Lee, N. H., Ng, K. E. D., & Yeo, B. W. J. (2019). Metacognition in the teaching and learning of mathematics. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 241–268). Singapore: Springer.
Loh, M. Y., & Lee, N. H. (2019). The impact of various methods in evaluating metacognitive strategies in mathematical problem solving. In P. Liljedahl & M. Santos-Trigo (Eds.), *Mathematical problem solving: Current themes, trends, and research* (pp. 155–176). Switzerland: Springer, Cham.
Ministry of Education [MOE]. (2018). *2020 Secondary Mathematics syllabuses*. Singapore: Author.
Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. In R. Mayes, R. Bonillia, L. L. Hatfield, S. Belbase, & S. Lee (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (pp. 75–92). Laramie, WY: University of Wyoming Press.
Ng, K. E. D. (2008). *The role of metacognition in collaborative mathematical knowledge application during interdisciplinary project work*. Paper presented at the Asia-Pacific Educational Research Association Conference: Educational research for innovation and quality in education—Policy and pedagogical engagements across contexts, National Institute of Education, Singapore.
Ng, K. E. D., Lee, N. H., Seto, C., Loh, M. Y., & Chen, S. (2016). *Teachers' conceptions of metacognition: Some preliminary findings from Singapore primary schools*. Paper presented at the British Educational Research Association Annual Conference 2016, University of Leeds, Leeds, United Kingdom.
Raymond, K., Gunter, M., & Conrady, K. (2018). Developing communication and metacognition through question generating. *Mathematics Teaching in the Middle School, 23*(5), 276–281.

Schoenfeld, A. H. (1985a). Metacognitive and epistemological issues in mathematical understanding. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 361–379). Hillsdale, NJ: Lawrence Erlbaum.

Schoenfeld, A. (1985b). Making sense of "outloud" problem solving protocols. *Journal of Mathematical Behavior, 4*(2), 171–191.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory, procedures and techniques*. Thousand Oaks, CA: Sage.

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage.

Teong, S. K. (2003). The effect of metacognitive training on mathematical word-problem solving. *Journal of Computer Assisted learning, 19*(1), 46–55.

Wong, K. Y. (2007). *Metacognitive awareness of problem solving among primary and secondary school students*. Paper presented at the Centre for Pedagogy and Practice Conference: Redesigning pedagogy, culture, knowledge, and understanding, National Institute of Education, Singapore.

Kit Ee Dawn Ng is a Senior Lecturer with the Mathematics and Mathematics Education Academic Group at the National Institute of Education, Singapore. She holds a PhD in mathematics education from University of Melbourne, Australia. She teaches in a wide range of pre-service and in-service programmes at both primary and secondary levels as well as postgraduate courses. She has received teaching excellence commendation awards from the National Institute of Education and research awards from Mathematics Education Research Group of Australasia and University of Melbourne. Her in-service courses, invited keynotes and workshops are aligned with her research interests. These include the use of real-world tasks (e.g. problems in real-world contexts, applications and mathematical modelling) in the teaching and learning of mathematics, fostering students' metacognition and mathematical reasoning, and school-based assessment practices. Dr Dawn Ng has published in journals, books and conference proceedings to share her research. She also contributes in editorial boards of journals. Dr Dawn Ng is currently wearing an additional hat of Assistant Dean for the Postgraduate Diploma in Education Programme with the Office of Teacher Education at the National Institute of Education.

Ngan Hoe Lee is an Associate Professor at the National Institute of Education (NIE). He taught Mathematics and Physics in a secondary school before becoming a Gifted Education Specialist at the Ministry of Education. At NIE, he teaches pre- and in-service as well as postgraduate courses in mathematics education and supervises postgraduate students pursuing Masters and PhD degrees. His publication and research interests include the teaching and learning of mathematics at all levels—primary, secondary and pre-university, covering areas such as mathematics curriculum development, metacognition and mathematical problem solving/modelling, productive failure and constructivism in mathematics education, technology and mathematics education, and textbooks and mathematics education.

Liyana Safii has been a research assistant at the National Institute of Education at Nanyang Technological University, Singapore for five years. She holds a Bachelor of Arts in Psychology from Murdoch University where she graduated as a valedictorian and was a recipient of the Vice Chancellor's Commendation for Academic Excellence Award. Her research interests include classroom practices in primary schools, especially in self-assessment and feedback.

Chapter 7
Cultivation of Positive Attitudes by Experienced and Competent Mathematics Teachers in Singapore Secondary Schools

Joseph B. W. Yeo

Abstract In this chapter, I report how 30 experienced and competent Singapore mathematics teachers tried to cultivate positive attitudes in their students and some possible factors that might have influenced the teachers' choice of instructional approaches. The video recording of 209 lessons of the 30 teachers were analysed and it was found that most of the teachers teaching lower-ability students attempted to build their students' confidence by starting with tasks that they could do before progressing to more difficult tasks, and to encourage their classes to persevere and to do well in mathematics. Meanwhile, most of the teachers teaching higher-ability students tended to focus on helping their students appreciate the relevance of mathematics by bringing in real-life examples and/or applications. Only a minority of the teachers tried to make lessons interesting by using mathematics-related resources or telling non-mathematics-related jokes. There was also evidence of a student benefiting from watching a funny Korean drama which the teacher skilfully linked to the learning of mathematics. On closer analysis of the data from classroom observations and interviews with the teachers and focus students, it was discovered that two factors appeared to influence the teachers' choice of the types of positive attitudes to develop in their students: the abilities of their students and the beliefs of the teachers on what mathematics is. The chapter concluded that other local teachers could emulate the practices of the 30 teachers to build confidence in their students and to encourage them to persevere, and that all teachers could perhaps pay more attention to helping their classes appreciate the relevance of mathematics and to make their lessons more interesting.

Keywords Attitudes · Interest · Appreciation · Confidence · Perseverance

J. B. W. Yeo (✉)
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: josephbw.yeo@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_7

7.1 Introduction

The Singapore School Mathematics Curriculum Framework (see Chapter 1, Fig. 1.2) was first adopted as the framework for mathematics curriculum in 1990 (Ministry of Education, 1990). The central focus of the framework is mathematical problem solving and it consists of five main components: concepts, skills, processes, metacognition and attitudes. Although the sub-components have undergone some minor revisions, the Framework is still relevant today (Ministry of Education, 2018).

As mentioned in Chapter 2, the first aim of the programmatic research project was to document how experienced and competent teachers enacted the school mathematics curriculum in secondary schools. In this chapter, I will report how 30 of these teachers tried to cultivate positive attitudes in their students according to the above Mathematics Curriculum Framework. It will focus on four out of the five types of desired attitudes, namely, interest, appreciation, confidence and perseverance. The last type of attitudes related to beliefs was not investigated because no teachers were observed addressing this aspect in their lessons. According to the syllabus document (Ministry of Education, 2012), a student has developed desired attitudes for the learning of mathematics if he/she shows interest and enjoyment in learning mathematics, appreciation of the beauty and power of mathematics, confidence in using mathematics and perseverance in solving mathematical problems. The study only investigated how the 30 teachers had tried to imbue such attitudes in their students, as it is beyond the scope of this study to examine whether the students had developed the desired attitudes. Although we also surveyed 677 mathematics teachers on how they imbued desired attitudes among their students, it is beyond the scope of this chapter to report the analysis of the survey data.

The research questions for this chapter are:

(1) How many of the 30 experienced and competent teachers tried to cultivate positive attitudes among their students?
(2) Which types of positive attitudes did the experienced and competent teachers attempt to imbue in their students and how did they do it?
(3) What are some possible factors that might have influenced the teachers' respective approaches in imbuing desired attitudes in their students?

7.2 Literature Review

7.2.1 Definitions of Attitudes and Affect

The study of attitudes is very complicated, partly because there is no common agreement on the definitions of terms, and partly because affective constructs are more difficult to describe and measure than cognition (McLeod, 1992). From a social psychological viewpoint, an '*attitude is a psychological tendency that is expressed by evaluating a particular entity with some degree of favour or disfavour*' (Eagly &

Chaiken, 1993, p. 1, emphasis in original) and when Rosenberg (1956) introduced the concept of attitudinal affect, it became a widespread practice to differentiate the affective part of attitudes from its cognitive and behavioural components (Schimmack & Crites, 2005). According to Haddock and Huskinson (2004), the affective component of attitudes refers to 'feelings or emotions associated with an attitude object' (p. 36) and an 'attitude object can be anything a person discriminates or holds in mind' (Bohner & Wänke, 2002, p. 5), such as concrete objects (e.g. pizza), abstract ideas (e.g. freedom of speech) or people. In other words, according to most social psychologists, attitudes include beliefs, behaviour and affect (emotions), although some of them (e.g. Simon, 1982) suggested using the word 'affect' as the more general term to include beliefs, attitudes and emotions.

In mathematics education, researchers used to follow the same viewpoint as most social psychologists. For example, Hart (1989) used the word 'attitude' towards an object to include beliefs about the object (cognitive component), emotional reactions to the object (affective component) and behaviour towards the object (behavioural component), and many research studies at that time also use 'attitude' to include beliefs about mathematics and self (McLeod, 1992). However, McLeod (1992) decided to follow Simon's (1982) suggestion of using the word 'affect' as the more general term to include beliefs, attitudes and emotions. Since then, many mathematics education researchers (e.g. Di Martino & Zan, 2011; Grootenboer & Marshman, 2015; Leder & Forgasz, 2006; Philipp, 2007; Tasmir & Tirosh, 2009) have followed suit and use the word 'affect' as the more general term to include beliefs, attitudes, emotions and even values. According to Furinghetti and Pehkonen (2002), ambiguity in terminologies is a known problem in research of affect in mathematics education. However, with proper definitions this issue of ambiguity can be significantly alleviated (Hannula, 2015). In this chapter (except for this section on Literature Review), the word 'attitudes' will be used as the more general term to include beliefs, interest, appreciation, confidence and perseverance, partly because this word 'attitudes' was used as one of the five components in the Singapore Mathematics Curriculum Framework described in Sect. 7.1.

7.2.2 Measurements of Attitudes

Since attitudes (not emotions) are latent, they cannot be observed directly (Krosnick, Judd, & Wittenbrink, 2005). Therefore, we can only measure attitudes that are revealed in overt responses, such as the traditional direct self-report method which asks the participants to describe their attitudes. This can be in the form of questionnaire items or interviews (Aiken, 1970; Leder & Forgasz, 2006). One disadvantage of self-reporting is that the participants may not be willing to describe themselves accurately (Krosnick et al., 2005). Other implicit measurements of attitudes include unobtrusive observation of behaviour, reaction to structured stimuli, performance on tasks, and physiological reactions (e.g. heart rate, pupil dilation and galvanic skin responses) (Kulm, 1980; Leder & Forgasz, 2006). However, the limitations of implicit measurement techniques are that an altitude may not always produce the same behaviour to an attitude object, and there may be other factors that produce the same response as the attitude to be measured (Krosnick et al., 2005).

7.2.3 Research on Attitudes

Krosnick et al. (2005) claimed that ‘attitude measurement is pervasive’ (p. 21) because social psychologists regularly measure attitudes when studying their causes, how they change and their effect on cognition and behaviour, even as early as the early 1900s (Allport, 1935). But research on affect (emotions and feelings) by social psychologists began much later in the mid or late 1900s (Forgas, 2001).

As for mathematics education, although there were quite a number of studies on attitudes (including beliefs) towards mathematics in the late 1990s, McLeod (1992) believed that ‘research on affect [attitudes, beliefs and emotions] in mathematics education continues to reside on the periphery of the field’ (p. 575). For example, in the International Handbook of Mathematics Education (Bishop, Clements, Keitel, Kilpatrick, & Laborde, 1996), which consists of 1358 pages, affect was only mentioned in the chapter on assessment (Clarke, 1996) under ‘assessing affect’ for slightly over one page, with no mention of attitudes, beliefs or emotions. Leder and Forgasz (2006) also observed that the number of Psychology in Mathematics Education (PME) conference papers on affect seems to have decreased somewhat in the early 2000s and that research on affective variables reported at PME conferences mostly reflects studies undertaken by other mathematics education researchers. Nevertheless, there is a recent proliferation of research in beliefs in mathematics education (e.g. Leder, Pehkonen, & Törner, 2002; Maaβ & Schlöglmann, 2009; Pepin & Roesken-Winter, 2015).

Most research on the affective domain in mathematics education tends to focus on finding out students’ existing attitudes and their effect on other variables such as test performance (Aiken, 1970; Leder & Forgasz, 2006; McLeod, 1992), and students’ and teachers’ beliefs (Leder et al., 2002; Maaβ & Schlöglmann, 2009; Pepin & Roesken-Winter, 2015). In Singapore, there were not many research studies on affective variables. Those that did followed the same trend as overseas research: relationship between attitudes (or anxiety) and mathematics achievement (or problem solving) (e.g. Ng-Gan, 1987; Yeo, 2004); and students’ and teachers’ beliefs (e.g. Kay, 2003; Tan, 2011). There were very few intervention studies on changing students’ attitudes (Yeo, 2018). An example of the latter is a study on the effect of an exploratory computer-based instruction on students’ conceptual knowledge, procedural knowledge and affective variables (Yeo, 2003), although the focus of the research was primarily on their learning and knowledge, rather than on their attitudes.

7.2.4 Nature of Attitudes

Attitudes can be formed and changed by interactions with people and the environment (Bohner & Wänke, 2002). When a person receives external information, he or she may form a new judgement based on this information and his or her prior knowledge, which is then stored in their memory (Albarracín, Johnson, Zanna, & Kumkale, 2005). Attitudes include both evaluative representations of these judgements in memory as well as judgements that a person forms online or on the spot. According to Bohner and Wänke (2002), there is some evidence that attitudes may in part be genetically influenced. Because attitudes can be changed, it is worthwhile to study what teachers do to imbue desired attitudes in their students. Therefore, the present study can add to existing research because it examined how 30 experienced and competent teachers tried to cultivate positive attitudes in their students.

7.3 Research Design

The research design for the collection of data for this chapter has been outlined in Chapter 2. In this section, I will briefly describe how the data were analysed to answer the research questions for this chapter. The 209 lessons of the 30 experienced and competent teachers were examined to pick up episodes of the teachers trying to imbue desired attitudes in the classroom. Then these episodes were classified according to the sub-components of attitudes in the Singapore Mathematics Curriculum Framework described in Sect. 7.1. The transcripts of the researchers' interviews with the teachers and with the focus students were also analysed to triangulate the data obtained from the lesson observations. The findings will now be presented.

7.4 Findings and Discussion

It was discovered that most of the 30 experienced and competent teachers tried to cultivate four sub-categories of attitudes in the classroom using the following instructional strategies:

- building students' confidence in doing mathematics by starting with tasks that they could do before progressing to more difficult tasks;
- encouraging their classes to persevere and to do well in mathematics;
- helping their students appreciate the relevance of mathematics by showing real-world examples and/or applications; and
- making lessons interesting by using mathematics-related resources (such as amusing mathematics videos) and/or telling non-mathematics-related jokes or stories.

An example of each of the above instructional strategies will be presented first, followed by an in depth analysis of the findings.

7.4.1 Examples of Different Instructional Strategies in Cultivating Positive Attitudes

7.4.1.1 Confidence

Figure 7.1 shows an example of progressively difficult tasks given to a Secondary 4 Normal (Technical) class by Teacher 9 to build her students' confidence. Questions 1 and 2 are basic questions (Level 1): given the height of a cone and its base radius or diameter, find its volume; Questions 3 and 4 are reverse questions (Level 2): given the volume of a cone and its base radius, find its height; and Questions 5 and 6 are application questions (Level 3). The level numbers were assigned by the researchers based on the difficulty level of the questions.

7.4.1.2 Perseverance

The following transcript is an example of how Teacher 3 encouraged his Secondary 4 Express class to persevere and to do well in mathematics. As the teacher was teaching a difficult Additional Mathematics topic on *Proofs of Plane Geometry*, and his class was the weakest Express class in the school, his intention was to urge his students to at least attempt to attain some marks for this kind of examination questions and not to give up:

> Ok, this question during O-level, is going to be worth 6 to 8 marks. Did I say everyone must score full marks? What is my aim? You must earn some marks. Nobody is going to give up on this question. Nobody is going to get zero. So long as you attempt to write some property, ok, show some understanding of how you can filter the question itself, it's already very good. Alright?

7.4.1.3 Appreciation

Figure 7.2 shows a task in a worksheet given by Teacher 21 to her class of Secondary 2 Integrated Programme students. There were pictures of five real-world objects or phenomena in the task and the question was to identify which of them was in the shape of a parabola. The teacher not only tried to help her class appreciate the relevance of mathematics in real life, but she was also concerned that her students might wrongly model any 'parabola-looking' curve in the real world with the equation of a quadratic function. Through this task, the teacher taught her students that some of these curves were not parabolas but catenaries (the equation of a catenary is a hyperbolic cosine function), e.g. the famous Gateway Arch in St. Louis is not a parabola but a catenary.

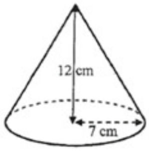
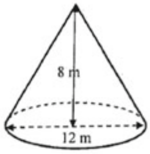
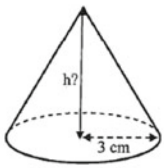
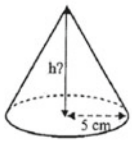
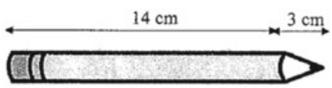
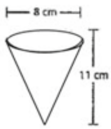
Level of Difficulty	Sample Questions
Level 1	1. A cone has a base radius of 7 cm and a height of 12 cm. Calculate the volume of the cone. [to 3.s.f] 
	2.  There is an error in the workings. **Circle** the error and **write down the correct** working and answer. Volume of cone $= \frac{1}{3} \times \pi \times 12^2 \times 8$ $= 1206\ m^3$ *Write your working & answer:*
Level 2	3. The volume of a cone is 66 cm^3. The base radius of the cone is 3 cm. Calculate h, the height of the cone. [to 3.s.f] 
	4. The volume of a cone is 225 cm^3. The base radius of the cone is 5 cm. Calculate h, the height of the cone. [to 3.s.f] 
Level 3	5. Find the <u>**volume**</u> of a pencil with a **radius of 0.5 cm**, a cone of height of 3 cm, and a cylinder height of 14 cm. [It's just like Example 3] [to 3.s.f] 
	6. <Optional> You have 37 850 cm^3 of lemonade to sell. Each customer uses one paper cup shown in the diagram. How many paper cups do you need? Ans: 206 *[Hint: find the vol of a paper cup first]*  

Fig. 7.1 An example of progressively difficult tasks given by Teacher 9 to her students

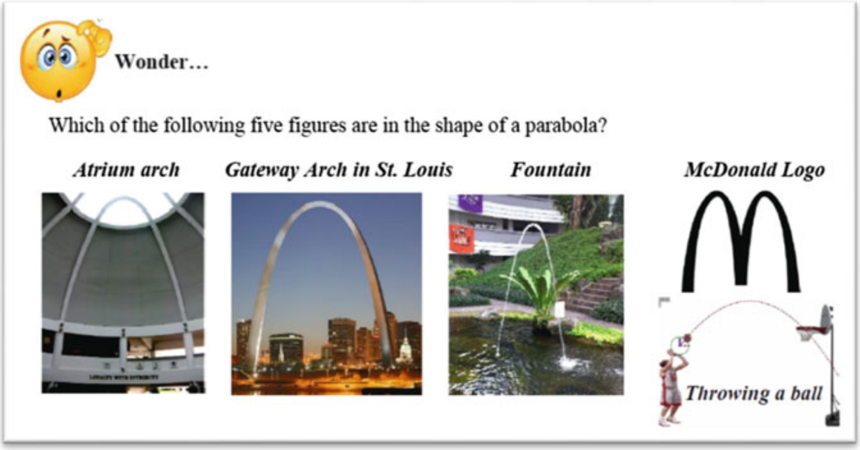


Fig. 7.2 Real-life examples and counter examples of parabolas in a worksheet given by Teacher 21 to her students

7.4.1.4 Interest (Non-mathematical Jokes or Stories)

The following transcript is an example of how Teacher 27 used non-mathematics-related jokes or stories to make lessons interesting. The teacher was teaching a Secondary 4 Express class on vectors. While talking about the starting and ending point of a resultant vector, he was giving an example of the starting point being their school and the ending point being their home. Then he asked which student stayed the furthest from the school.

Teacher 27:	Who stays the furthest away from school? I don't know, are there anyone who stay at Pasir Ris?
Student A:	Student B
Teacher 27:	JB? [JB stands for Johor Bahru, a town in Malaysia just north of Singapore]
[The class breaks into laughter]	
Teacher 27:	You stay at JB? Okay, so it's a classic example right? So Student B stays at JB
[Teacher marks JB on the class whiteboard]	
Teacher 27:	So what time do you wake up? Very early?
Student B:	3
Teacher 27:	3 o'clock. 3 am. 3 am to come to school, but you are never late right? I don't see you in the latecomer list, but there are some people staying next block, can come late
[The class breaks into laughter]	
Teacher 27:	Alright, I also stay next block, but I don't come late
[Some students laugh]	

The reader may not interpret the above excerpt to be particularly humorous, as slapstick humour and situational jokes typically lose their comedic quality when narrated. However, the two focus students for the lesson, who were interviewed separately after the lesson, said that this part (the teacher's jokes) was the high of the lesson.

7.4.1.5 Interest (Mathematics-Related Resources)

Teacher 6 had recently taught her Secondary 2 Express class the concept of Pythagoras' Theorem and her students had practised some questions on using the theorem to find the unknown length of a side of a right-angled triangle. These questions were pure mathematics questions without any contexts. In the following lesson, instead of using word problems (questions with fabricated contexts) from the textbook to demonstrate how Pythagoras' Theorem could be applied, the teacher decided to show her class a 10-minute video containing snippets of a two-episode Korean drama show (with English subtitles). The drama was a comedy about a girl called Dan Bi who somehow travelled back in time to ancient Korea and helped a king solve a mathematics problem using Pythagoras' theorem. The whole class found the drama funny because of the situational jokes and slapstick humour, but there was little about this video that focused on the application of the theorem.

However, the teacher designed three problems with contexts that continued the storyline in the drama for her students to solve in class using Pythagoras' theorem (see Fig. 7.3). In the first problem, the king shot a deer and wanted Dan Bi to record in history what he had done. This involved a simple application of Pythagoras' Theorem to find some information. The second problem continued the storyline: to celebrate the king's success in shooting the deer, the palace was having a celebration and Dan Bi was assigned to hang up a banner. This involved a more complicated problem of dividing a trapezium into a rectangle and a right-angled triangle before Pythagoras' Theorem could be applied. The third problem continued with the cooks needing to retrieve water for the well to prepare a feast for the celebration. The pulley system that Dan Bi designed required an even more complex application of Pythagoras' Theorem to solve. Not only did the teacher make use of the storyline to make the problems more interesting, she also designed them from simple to more difficult levels.

After the students had completed the three problems in class and the teacher had gone through the solutions, she gave them some word problems from the textbook as homework. To her students, the contexts of these textbook questions were not as interesting as the three problems done in class. However, one of the students, who seldom handed in homework punctually, unexpectedly handed in this set of homework on time. During an interview with the teacher, the following was what she said of this student:

> So through these two lessons you can see that at least she tried, because half the times I have to call her to submit her assignments. You know every time, I have to tell her, 'Eh, you have to submit your work, assignment is very late'. But this time I realised that the moment

The King wants Jang Dan Bi to complete all the tasks stated below. You are going to assist Dan Bi to solve the King's tasks.

Task 1:

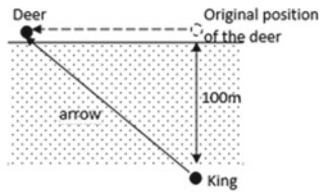
During one hunting trip, the King saw a deer on the opposite shore of a river that is 100m wide. The deer ran along the shore at a speed of 20m/s.

After 8 seconds, the arrow hit the deer.

The King wanted his hunting trip to be recorded in history, hence he needed the following information. Dan Bi would need to help him to calculate:

(a) the distance the deer travelled

(b) the distance the arrow travelled



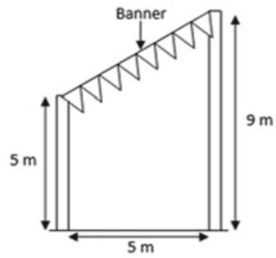
Task 2:

To celebrate the King's victory in hunting the deer, the palace is having a celebration.

The guards are going to put up a decorative banner between two poles.

One of the vertical poles is 5m tall and the other is 9m tall and both of them are 5m apart. Dan Bi is assigned to assist the guards with the amount of materials used to make the banner.

Calculate the minimum length of the banner required.



Task 3:

The cooks required water to cook up a feast for the celebration.

Dan Bi came up with a plan to help them retrieve the water from the well by constructing a pulley system.

The rope ran through two pulleys that are 5m above the ground as shown in the diagram.

Find the distance between the two pulleys.

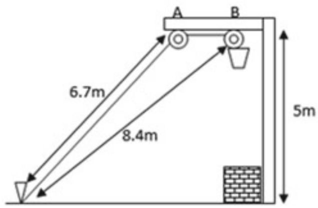


Fig. 7.3 Three problems based on storyline of Korean drama designed by Teacher 6 for her students

> I asked the work to be in, the work is actually handed in. And the quality of the work, okay, it's better than her norm. So that's why I said, probably that video actually motivates them a lot.

It seems that the student was somehow motivated by the Korean drama to solve the three tasks in class and her routine homework questions promptly. But it was observed that most of the students were laughing at the non-mathematics-related situational jokes and slapstick humour in the video, and not the part where Dan Bi was solving the problem using Pythagoras' Theorem. Therefore, what made most of the class interested initially was not the mathematics part of the drama but the non-mathematics-related jokes and stories. And the three tasks given by the teacher were interesting only because of the storyline that continued from the drama. Hence, we should not underestimate the power of non-mathematics-related resources to generate interest and motivate students to learn mathematics.

However, there is a big difference between the non-mathematics-related jokes in the drama used by the teacher here and the non-mathematics-related jokes told by the teacher teaching vectors described earlier in this section: in the latter, there was no direct link to the learning of mathematics (although some students may still be motivated to learn the subject, more research needs to be done to ascertain whether this is true); but in the former, the teacher was able to link the drama to the learning of mathematics through the three tasks which she had designed for her class to do. Therefore, whenever possible, teachers should try to link what interests the students to the learning of mathematics because the main purpose of making lessons interesting is not to make students laugh but to provide conducive opportunities for them to learn mathematics (Yeo, 2018; Yeo, Choy, Lim, & Wong, 2019).

7.4.2 LOVE Mathematics Framework to Engage the Hearts of Mathematics Learners

One common problem with making lessons interesting that teachers face is that it is not possible to make every lesson interesting or every part of a lesson interesting. Yeo (2018) and Yeo et al. (2019) proposed that teachers should do it often enough: at least once in most lessons. As with the case of Teacher 6 teaching Pythagoras' Theorem, she conducted three one-hour lessons on this topic in which the first and last lessons were not interesting. The first 15 min of the second lesson was also not interesting, until she showed the video and gave the three tasks for the students to do in class which was enough to make students interested in the topic. But that was enough to make students interested in the topic. As mentioned earlier, one student was motivated enough to hand in her homework on routine (not interesting) questions on time. Moreover, both students, who were interviewed separately after the lesson, said that the high point of the lesson was watching the video. One of them said:

> I think the video that she showed was the most helpful … It tells us how we should use it … and also it is in a fun and entertaining way.

Another common problem with making lessons interesting is that what fascinates one student may not appeal to another because interest is a subjective construct since different people have different tastes. So the key is to use a variety resources to interest different students or different groups of students. Yeo (2018) and Yeo et al. (2019) have addressed these concerns by developing the *LOVE Mathematics* framework for engaging the hearts of mathematics learners. LOVE Mathematics stands for 'Linking Opportunities in a Variety of Experiences to the learning of Mathematics'. The framework consists of three components:

- Variety of Experiences: Use different resources to provide a variety of experiences to interest different students;
- Opportunities: Do it often enough, at least once in most lessons, since it is not possible to make every part of every lesson interesting; and
- Linking these opportunities to the learning of mathematics: The main purpose to engage the hearts of students is still not to make them laugh but to learn mathematics, so whenever possible, we should make use of these opportunities to link them to the learning of the subject.

After looking at some exemplars that most of the 30 experienced and competent teachers used to imbue desired attitudes in their students and the LOVE Mathematics framework, I will now analyse in more detail how many of these teachers across the four courses of study attempted to develop types of desired attitudes among their students.

7.4.3 Types of Positive Attitudes Cultivated by the 30 Teachers Across the Four Courses of Study

Table 7.1 shows the number (and percentage) of the 30 experienced and competent teachers who attempted to infuse desired attitudes in their students according to the four courses of study: Integrated Programme (IP), Express, Normal (Academic) (N(A)) and Normal (Technical) (N(T)). The abilities of the students are generally higher for the IP course than students in the Express course, which in turn are higher than those in the N(A) course; while the abilities of the students in the N(T) course are generally the lowest. The reader can refer to Chapter 2 for a more detailed description of these four courses of study. In addition, because the last sub-component of making lessons interesting consisted of two different instructional approaches, the separated data are presented in Table 7.1.

We observed from Table 7.1 that most of the teachers (26 out of 30, or 86.7%) had tried to cultivate positive attitudes in their students. On closer analysis, most of the teachers (20 out of 30, or 66.7%) focused on building students' confidence in doing mathematics by starting with tasks that students could do before progressing to more difficult tasks, followed by 50% of the teachers encouraging the class to persevere and to do well in mathematics. Only 36.7% of the teachers (11 out of 30) had attempted to help students appreciate the relevance of mathematics by showing real-life examples and/or applications. The least common priority among the teachers (6 out of 30, or 20%) was to make lessons interesting for their students. Interestingly, there were slightly more teachers (4 teachers) who made lessons interesting by telling non-mathematics-related jokes or stories than

Table 7.1 Use of instructional approaches when cultivating positive attitudes for the learning of mathematics as observed in lessons conducted by the 30 experienced and competent teachers

Instructional approach	Number (and percentage) of teachers				
	IP ($n = 4$)	Express ($n = 10$)	N(A) ($n = 8$)	N(T) ($n = 8$)	Total ($N = 30$)
Attempting to cultivate any positive attitudes in the students	2 (50%)	8 (80%)	8 (100%)	8 (100%)	26 (86.7%)
Building students' confidence in doing mathematics by starting with tasks that students can do before progressing to more difficult tasks	0 (0%)	6 (60%)	8 (100%)	6 (75%)	20 (66.7%)
Encouraging the class to persevere and to do well in mathematics	1 (25%)	5 (50%)	5 (62.5%)	4 (50%)	15 (50%)
Helping students appreciate the relevance of mathematics by showing real-life examples and/or applications	2 (50%)	2 (20%)	4 (50%)	3 (37.5%)	11 (36.7%)
Making lessons interesting by telling non-mathematics-related jokes or stories, and/or using mathematics-related resources	0 (0%)	2 (20%)	2 (25%)	2 (25%)	6 (20%)
Making lessons interesting by telling non-mathematics-related jokes or stories	0 (0%)	1 (10%)	2 (25%)	1 (12.5%)	4 (13.3%)
Making lessons interesting by using mathematics-related resources (such as funny mathematics videos)	0 (0%)	1 (10%)	1 (12.5%)	1 (12.5%)	3 (10%)

those (3 teachers) who did this by using mathematics-related resources, including a teacher who did both.

Across the four courses of study, it is observed that all the teachers teaching the N(T) and N(A) courses (which are for lower-ability students) and 8 out of the 10 Express teachers (i.e. 80%) had attempted to cultivate positive attitudes in their students, but only 2 of the 4 IP teachers (i.e. 50%) had done the same. For the N(T), N(A) and Express classes, most of the teachers had focused on building students' confidence and encouraging the class to persevere, followed by helping students appreciate the relevance of mathematics and making lessons interesting. But for the IP course of study (which is for higher-ability students), the focus of the teachers who imbued desired attitudes is more on helping students appreciate the relevance of mathematics. In fact, only one of the four IP teachers had tried to encourage her class to persevere and to do well in mathematics on only one occasion in all her seven one-hour lessons that were observed over more than two weeks, i.e. encouraging their students did not seem to be a high priority among IP teachers.

7.4.4 Possible Factors Influencing Teachers' Approaches in Imbuing Desired Attitudes

From the above analysis, it appears that one factor that might have influenced the teachers' instructional approaches in cultivating which kind of positive attitudes is the abilities of the students whom they were teaching in their respective course of study. For weaker students, their teachers tended to build their confidence and encourage them to persevere in their studies. However, for students with higher abilities, their teachers were more inclined to help them appreciate the relevance of mathematics. This was further confirmed by interviews with the teachers. For example, Teacher 6 who showed the Korean drama (see Sect. 7.4.1) said that her type of students (who were from a weak Express class) needed motivation to solve more difficult mathematical problems, thus she used the video to provide the link to real life and to entice her class to solve the problems. The following shows part of a transcript of an interview with the teacher.

Interviewer:	So, what is your purpose for showing them this video?
Teacher 6:	It's actually to, to entice them to be interested in doing mathematics because sometimes when you realise that, when you keep on practising and they don't see how it can be linked, it is very difficult. So we want to see, eh, ancient times people are already using Pythagoras' theorem, right now you are also learning. And probably ancient times scholars took 3 days to finish, by then within a few minutes they already solved it using Pythagoras' theorem. Because, my class, my class I think they need this kind of motivation, because some of them will fall into a world of their own very easily. So we wanted them to, you know, entice them to this kind of thing, different kind of activity for them to do, erm, so after this, what they will do is, the king has a series of problems, so they will try to solve the king's problems by Pythagoras' theorem, whatever video is posed to them, by the king

Another factor that might have influenced the teachers' instructional approaches in cultivating which kind of positive attitudes is the beliefs of the teachers. For example, Teacher 3 who told his students not to get zero marks for the O-level Additional Mathematics examination question on proofs of plane geometry (see Sect. 7.4.1) believed that mathematics is about resilience, and so he tried to encourage his class not to give up on such examination questions but to score at least a few marks. The following shows part of a transcript of an interview with the teacher.

(continued)

(continued)

Teacher 3:	And my rationale to them is I'm teaching this topic in a way where I do not want them to skip the question. They may not be able to do the entire question, but I'm very certain that they are able to, like, understand what the word problem is about… to be able to apply techniques or even right now … certain theorems or concepts which they've learned to secure a few marks. So my ultimate goal for teaching this topic is no student should get zero out of 6 or 8 marks. … Of course the main thing I think in mathematics is also about resilience, don't give up easily. Ya. So as much as I think this is a difficult topic, there is a high tendency that there is a lot of students in looking at this question and they will entirely give up. But I think if we were to tackle this question in a more structured and sequential manner, I think, ah, when the students feel a little bit more comfortable, they are more assured, I think, they are more willing to try the questions. So I'm quite certain in that sense, boosting the confidence of students, and making sure that they do not give up easily

7.5 Conclusion

In conclusion, building students' confidence and perseverance in mathematics were of high priority amongst a large proportion of the 30 experienced and competent teachers. To do so, they eased students into more difficult tasks by beginning with easier questions and encouraged students to at least attempt to demonstrate their understanding for harder problems if they were not capable of answering them in full. Helping students to appreciate the relevance of mathematics and making lessons interesting were of lower priority for the 30 teachers. However, there was evidence to suggest that showing real-life examples and/or applications that may be of interest to students could help them to feel more motivated to engage and consequently increase their efforts to study the subject. While it is not possible to make every lesson interesting for students, teachers should not underestimate the power of non-mathematical resources to engage the hearts of students, although they must remember to link the activities to the learning of mathematics whenever possible. The LOVE Mathematics framework on linking opportunities in a variety of experiences to the learning of mathematics, as described in Sect. 7.4.2, can serve as a viable framework in the professional development of teachers in engaging the hearts of mathematics learners.

References

Aiken, L. R., Jr. (1970). Attitudes towards mathematics. *Review of Educational Research, 40,* 551–596.

Albarracín, D., Johnson, B. T., Zanna, M. P., & Kumkale, G. T. (2005). Attitudes: Introduction and scope. In D. Albarracín, B. T. Johnson, & M. P. Zanna (Eds.), *The handbook of attitudes* (pp. 3–19). Mahwah, NJ: Erlbaum.

Allport, G. W. (1935). Attitudes. In C. M. Murchison (Ed.), *Handbook of social psychology* (pp. 789–844). Reading, MA: Clark University Press.

Bishop, A. J., Clements, K., Keitel, C., Kilpatrick, J., & Laborde, C. (Eds.). (1996). *International handbook of mathematics education.* Dordrecht, The Netherlands: Kluwer Academic Press.
Bohner, G., & Wänke, M. (2002). *Attitudes and attitude change.* East Sussex, UK: Psychology Press.
Clarke, D. (1996). Assessment. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 327–370). Dordrecht, The Netherlands: Kluwer Academic Press.
Di Martino, P., & Zan, R. (2011). Attitude towards mathematics: A bridge between beliefs and emotions. *ZDM Mathematics Education, 43*(4), 471–482.
Eagly, A. H., & Chaiken, S. (1993). *The psychology of attitudes.* Orlando, FL: Harcourt Brace Jovanovich.
Forgas, J. P. (Ed.). (2001). *Handbook of affect and social cognition.* Mahwah, NJ: Erlbaum.
Furinghetti, F., & Pehkonen, E. (2002). Rethinking characterizations of beliefs. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education* (pp. 39–58). Dordrecht: Kluwer.
Grootenboer, P., & Marshman, M. (2015). *Mathematics, affect and learning: Middle school students' beliefs and attitudes about mathematics education.* Australia: Springer.
Haddock, G., & Huskinson, T. L. H. (2004). Individual differences in attitude structure. In G. Haddock & G. R. Maio (Eds.), *Contemporary perspectives on the psychology of attitudes* (pp. 35–56). East Sussex, UK: Psychology Press.
Hannula, M. S. (2015). Reaction to section 2: The relevance of affective systems and social factors: A commentary. In B. Pepin & B. Roesken-Winter (Eds.), *From beliefs to dynamic and affect systems in mathematics education: Exploring a mosaic of relationships and interactions* (pp. 269–277). New York: Springer.
Hart, L. E. (1989). Describing the affective domain: Saying what we mean. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 37–45). New York: Springer-Verlag.
Kay, H. G. (2003). *An exploratory study of the belief systems and instructional practices of four primary school mathematics teachers*. Unpublished master's thesis, National Institute of Education, Nanyang Technological University, Singapore.
Krosnick, J. A., Judd, C. M., & Wittenbrink, B. (2005). The measurement of attitudes. In D. Albarracín, B. T. Johnson, & M. P. Zanna (Eds.), *The handbook of attitudes* (pp. 21–76). Mahwah, NJ: Erlbaum.
Kulm, G. (1980). Research on mathematics attitude. In R. J. Shumway (Ed.), *Research in mathematics education* (pp. 356–387). Reston, VA: National Council of Teachers of Mathematics.
Leder, G. C., & Forgasz, H. J. (2006). Affect and mathematics education: PME perspectives. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 403–427). Rotterdam, The Netherlands: Sense Publishers.
Leder, G. C., Pehkonen, E., & Törner, G. (Eds.). (2002). *Beliefs: A hidden variable in mathematics education.* Dordrecht, The Netherlands: Kluwer Academic Publishers.
Maaβ, J., & Schlöglmann, W. (Eds.). (2009). *Beliefs and attitudes in mathematics educations: New research results.* Rotterdam, The Netherlands: Sense Publishers.
McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York: Macmillan.
Ministry of Education. (1990). *Mathematics syllabus.* Singapore: Author.
Ministry of Education. (2012). *Mathematics syllabus: Secondary one to four*. Singapore: Curriculum Planning and Development Division.
Ministry of Education. (2018). *2020 Secondary Mathematics syllabuses.* Singapore: Author.
Ng-Gan, L. C. (1987). *Relationship between secondary school students' mathematics attitude and achievement*. Unpublished master's thesis, National University of Singapore, Singapore.

Pepin, B., & Roesken-Winter, B. (Eds.). (2015). *From beliefs to dynamic and affect systems in mathematics education: Exploring a mosaic of relationships and interactions*. New York: Springer.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second Handbook of research on mathematics teaching and learning*. Charlotte, NC: Information Age Publishing & National Council of Teachers of Mathematics.

Rosenberg, M. J. (1956). Cognitive structure and attitudinal affect. *Journal of Abnormal and Social Psychology, 53*(3), 367–372.

Schimmack, U., & Crites, S. L., Jr. (2005). The structure of affect. In D. Albarracín, B. T. Johnson, & M. P. Zanna (Eds.), *The handbook of attitudes* (pp. 397–435). Mahwah, NJ: Erlbaum.

Simon, H. A. (1982). Comments. In M. S. Clark & S. T. Fiske (Eds.), *Affect and cognition* (pp. 333–342). Hillsdale, NJ: Erlbaum.

Tan, L. S. (2011). *Mediational analyses of motivational beliefs, affect, engagement, and achievement among high-ability and average-ability students in Singapore's mathematics classrooms*. Unpublished doctoral thesis, National Institute of Education, Nanyang Technological University, Singapore.

Tasmir, P., & Tirosh, D. (2009). Affect, subject matter knowledge and pedagogical content knowledge: The case of a kindergarten teacher. In J. Maaβ & W. Schlöglmann (Eds.), *Beliefs and attitudes in mathematics educations: New research results* (pp. 19–31). Rotterdam, The Netherlands: Sense Publishers.

Yeo, J. B. W. (2003). *The effect of exploratory computer-based instruction on secondary four students' learning of exponential and logarithmic curves*. Unpublished master's thesis, National Institute of Education, Nanyang Technological University, Singapore. [QA20.C65 Yeo; http://hdl.handle.net/10497/1330; MEd].

Yeo, J. B. W. (2018). Engaging the hearts of mathematics learners. In P. C. Toh & B. L. Chua (Eds.), *Mathematics instruction: Goals, tasks and activities*. Yearbook 2018 of the Association of Mathematics Educators (pp. 115–132). Singapore: World Scientific.

Yeo, J. B. W., Choy, B. H., Lim, L. G. P., & Wong, L. F. (2019). Innovative pedagogical practices. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 165–193). Singapore: Springer.

Yeo, J. K. K. (2004). *An exploratory study of secondary two students' mathematics anxiety and mathematical problem solving*. Unpublished doctoral thesis, National Institute of Education, Nanyang Technological University, Singapore.

Joseph B. W. Yeo is a Lecturer in the Mathematics and Mathematics Education Academic Group at the National Institute of Education, Nanyang Technological University, Singapore. He is the first author of the *think! Mathematics* textbooks (previously called *New Syllabus Mathematics* textbooks) used in many secondary schools in Singapore. His research interests are on innovative pedagogies that engage the minds and hearts of mathematics learners. These include an inquiry approach to learning mathematics, ICT and motivation strategies to arouse students' interest in mathematics (e.g. catchy maths songs, amusing maths videos, witty comics and intriguing puzzles and games). He is also the Chairman of Singapore and Asian Schools Math Olympiad (SASMO) Advisory Council, and the creator of Cheryl's birthday puzzle that went viral in 2015.

Chapter 8
Balancing an Intuitive-Experimental Approach with Mathematical Rigour: A Case Study of an Experienced and Competent Mathematics Teacher in a Singapore Secondary School

Tin Lam Toh and Berinderjeet Kaur

Abstract This chapter reports a case study of an experienced and competent mathematics teacher teaching Angle Properties of Circles to a class of Secondary Three students in the Express course of study. Geometry in the school curriculum serves as a good platform for inducting students into the rigour of mathematical thinking through deductive reasoning, and the world of deductive mathematical arguments in the form of mathematical proof, which forms the common language of mathematicians worldwide. It is this rigour and discipline that students usually encounter much difficulty with. Quite contrary to our stereotyped image of a traditional geometry lesson, the teacher used a variety of approaches to enrich the lesson. She used a series of scaffoldings to lead the students from inductive exploration through discovery activities to deductive reasoning and the formalism of writing of reasoning in geometry, juggling between her belief on the importance of discovery learning and the curriculum requirement of deductive reasoning in geometry. It was interesting to us that the teacher, in transiting from students' exploration to identifying the geometric properties, made use of rich visual imagery related to circle properties to develop in her students the concept images associated with the geometry property. Through the use of visuals to facilitate her students' learning, effort was made to ensure her students truly understood the geometrical properties and used the properties in working with problems. Deductive reasoning was introduced in the lesson closure portion of the lesson to stress the interconnectedness across the various geometrical properties. The stages that the teacher went through in guiding the students from the intuitive-experimental stage to the deductive reasoning resonates with the van Hiele levels of students' learning of geometry. The teacher highlighted during the interview about her conscious attempt to achieve a balance between an intuitive-experimental

T. L. Toh (✉) · B. Kaur
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: tinlam.toh@nie.edu.sg

B. Kaur
e-mail: berinderjeet.kaur@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_8

approach to facilitate her students' learning and maintaining mathematical rigour that is required of the geometry strand in the Singapore school mathematics curriculum.

Keywords Teaching geometry · van Hiele levels · Mathematical reasoning · Concept image

8.1 Introduction

The authors (hereafter, first person pronoun) are part of the project team (see Chapter 2) that examined the enactment of the secondary school mathematics curriculum in Singapore schools. In this chapter, we report a case study of an experienced and competent mathematics teacher teaching Angle Properties of Circles to a class of Secondary Three students in the Express course of study. Our stereotyped image of a typical geometry lesson is one that is full of deductive mathematical reasoning culminating in rigorous mathematical proofs; such lessons are difficult and boring to laypeople of mathematics. What we observed in this series of lessons was quite contrary to our preconceived idea of a geometry lesson. The teacher, whose students were the upper-bound of average ability, used a variety of approaches, ranging from an intuitive-experimental approach to the rigorous deductive approach. How these various approaches unfolded in the first lesson on Angle Properties of Circles is the focus of this chapter. Of interest are how the various geometry concepts were skillfully developed and connected through the approaches in different parts of the lesson. The teacher was fully cognisant of the syllabus requirement and her belief about the importance of student engagement.

8.2 Teaching of Geometry in Schools

Geometry has been recognised by mathematicians as an ideal vehicle to introduce students to "axiomatics" because of its "esthetic appeal" (Coxeter & Greitzer, 1967). One of the main goals of teaching mathematics has always been to facilitate students to develop deductive reasoning. Geometry seems to fit this goal perfectly (Ayalon & Even, 2010; Herbst, 2002). It is thus not surprising that our preconceived idea of a traditional secondary school geometry lesson is usually one in which students are expected to prove theorems. Mathematical proofs are usually seen by students as the "rules of the games", which is the essence of mathematics and therefore the core of academic mathematician's daily practice.

The International Commission on Mathematical Instruction (ICMI), in preparation for the study on "Perspectives on the Teaching of Geometry for the 21st Century", challenged academics to re-think the teaching of geometry, especially in the recent decades with the advent of technology and geometry teaching aides (ICMI, 1995). ICMI (1995) invited discussion among academics whether geometry teaching at the schools should take the form of an "intuitive" approach, or a "formalised" approach, or perhaps a mixture of both approaches with a gradual shift from an intuitive to a formalised approach "as the age of students and the school level progresses".

A geometry lesson using an intuitive approach of teaching geometry is in direct contrast to the traditional image (and even the objective) of a geometry lesson. Associating with an intuitive approach of teaching geometry, one is likely to think of computer-based learning environment such as the environment of the Dynamic Geometry (DG). The justification of intuitive approach is based on existing education literature on the positive impact of computer and technology on student learning. Studies have shown that computer environments such as that of a DG can stimulate learners to link their intuitive notions and formal aspects of mathematical knowledge (e.g. Sutherland, 1998; Sutherland, Olivero, & Weeden, 2004). DGs enable learners to manipulate objects by clicking, dragging, and measuring the objects in order to discover mathematical relationships. Researchers have studied how teachers can provide appropriate scaffolding for student learning through the use of appropriate pre-designed files (e.g. Leung, 2011).

In the mathematics curriculum document provided by the Singapore Ministry of Education (MOE) (2012), the underpinning theoretical principle in teaching of secondary school geometry was explicitly stated as:

> The learning of *Geometry* at this stage [i.e. at the secondary level] should adopt an *intuitive* and *experimental* approach. This approach is based on van Hiele's theory of geometry learning which advocates exploration and discovery through hands-on activities. (MOE, 2012, p. 32)

Using van Hiele's theory as the guiding principle, Leong and Lim-Teo (2008) identified that the greatest challenge of a secondary school mathematics teacher is to raise their students' view from Level 1 (which is a purely visually driven mode) to "one that focuses on their geometrical properties" (Leong & Lim-Teo, 2008, p. 121). We were interested to know: How do experienced and competent teachers conduct geometry lessons in Singapore mathematics classrooms?

Other than the various generic pedagogical principles outlined in the secondary mathematics syllabus document, the Singapore Ministry of Education (MOE) (2012) does not prescribe precise delivery methods that teachers should adopt for their classroom instruction. However, the syllabus documents contain a list of learning experience statements (which are phrased as "Students should have opportunities to …") parallel to the syllabus content to be covered. A segment of the geometry syllabus document for Secondary Three Express course of study is shown in Fig. 8.1. The left-hand column delineates the content to be covered during the lessons while the right-hand column contains the learning experience statements.

The learning experience statements in the right-hand column of Fig. 8.1 highlight the processes learners need to experience in acquiring the corresponding content in the left-hand column. As illustrated in Fig. 8.1, the topic Angle Properties of Circles has two main emphases on the learning experience:

G3. Properties of circles	Students should have opportunities to:
3.1 symmetry properties of circles • Equal chords are equidistant from the centre • The perpendicular bisector of a chord passes through the centre • Tangents from an external point are equal in length • The line joining an external point to the centre of the circle bisects the angle between the tangents 3.2 angle properties of circles • Angle in a semicircle is a right angle • Angle between tangent and radius of a circle is a right angle • Angle at the centre is twice the angle at the circumference • Angles in the same segment are equal • Angles in opposite segments are supplementary	(a) Use paper folding to visualise symmetric properties of circles, e.g. the perpendicular bisector of a chord passes through the centre. (b) Use GSP or other dynamic geometry software to explore the properties of circles, and use geometrical terms correctly for effective communication.

Fig. 8.1 An extract of part of the syllabus content for Secondary Three Geometry (MOE, 2012)

1. The opportunity for students to experience geometry through manual activities such as paper folding, and technology such as the use of a DG software (e.g. Geometers' Sketchpad or GSP) to discover geometrical properties; and
2. The opportunity for students to use correct mathematical terms in geometry for effective communication. As these are general guidelines, the actual activities are not specified here and are left for teachers to interpret and enact in the classroom. Thus, teachers are faced with enactment decisions, especially when they see the need to fill in the "gaps" in order to enact the lessons (Kim & Atanga, 2013).

It was also interesting to note that the learning experience column of the syllabus document in Fig. 8.1 suggests the use of technology, and education research seems to suggest that teachers are generally resistant to the use of technology for various reasons (Polly, 2014). Our combined classroom experience also seems to suggest that some "experienced" teachers might not be very receptive to the use of technology for mathematics classroom instruction. Thus, we were excited to observe how teachers enact geometry lessons based on the newly introduced learning experience which suggests the use of technology as part of teaching and learning.

8.3 The Case Study

8.3.1 Method

The teacher in our study is Teacher 5 in her early 50s. She met the criteria of an "experienced and competent teacher" as she had more than five years of teaching mathematics experience for a course of study, in this case the Express course. In addition, the local education community and her school leaders also recognised her as a good mathematics teacher. She is a Lead mathematics teacher, one who is entrusted with the responsibility to develop fellow teachers in classroom practice. At the time of our study, she had been teaching mathematics in Singapore schools for more than 20 years, of which 15 years were in the school where we conducted the study.

The class that Teacher 5 taught was Secondary Three in the Express course of study. It had 14 boys and 28 girls. Teacher 5 described the class as a highly motivated group of students who took interest in learning mathematics actively. In addition to doing the core mathematics subject, known as Elementary Mathematics in Singapore, the students were also reading Additional Mathematics, a more advanced mathematics subject offered to higher ability students at the secondary level. During the interview, Teacher 5 commented that she had used various innovative approaches in engaging the students from this class. In designing her lessons for the class, she was mindful that her students needed a more rigorous treatment of mathematics in preparation for Additional Mathematics.

The sub-topic of geometry that Teacher 5 taught, and which is the focus of the case study described here, is Angle Properties of Circles in Elementary Mathematics. This sub-topic, as shown in section 3.2 of Fig. 8.1, covered four main properties:

- (Property 1) Angle at the centre of a circle is twice the angle at the circumference. (P1)
- (Property 2) Angle in a semicircle is a right angle. (P2)
- (Property 3) Angles in the same segment are equal. (P3)
- (Property 4) Angles in opposite segments are supplementary (add up to 180 degrees). (P4)

Teacher 5 completed teaching this sub-topic in three one-hour lessons. The first lesson was an introduction to the above four angle properties of a circle. Following which, she engaged her students in solving typical geometry problems and writing of short proofs in the second and third lessons. What captured our attention about her teaching was her selection of the instructional methods that she used during her introduction of the angle properties of the circles in the first of the three lessons. Avoiding the two extremes of totally using deductive approach or intuitive approach, she used a good mix of strategies by tapping on both approaches. She engaged her students to “discover” the geometrical properties of circles through the use of DG. This was followed by application of the “discovered” properties to do mathematical tasks of varying cognitive demand. In the lesson closure, she consolidated the lesson using a more deductive approach, showing the close connection across the four geometrical properties. The following sections detail and discuss the lesson.

8.3.2 Data

8.3.2.1 Lesson Observation and Video Analysis

A researcher sat throughout all the three lessons that Teacher 5 used to teach the sub-topic Angle Properties of Circles. The lessons were video-recorded using the Complementary Accounts Methodology first proposed by Clarke (1998, 2001). You may refer to Chapter 2 for details. The teacher's exposition and the teacher's conversation with students during the three lessons were transcribed. At the end of each lesson, the teacher and the focused students were interviewed to triangulate the data collected through the video-recordings of the lesson. The interviews were audio-recorded and transcribed.

8.3.2.2 Instructional Material Used by the Teacher

It is a common practice for mathematics teachers in Singapore to design their own instructional materials based on existing teaching resources available for the teachers and students. Teacher 5 used a variety of resources for her teaching: (1) she developed her own instructional material to supplement her teaching; and (2) she selected a variety of questions from various textbooks. She designed four exploratory activity worksheets to scaffold students' discovery of the above four geometrical properties of a circle.

Teacher 5 designed one "exploratory activity" worksheet to correspond to each of the four angle properties of circles in this sub-topic. A sample of the worksheet for Property (P1) is shown in Fig. 8.2. Each worksheet consists of three portions:

(A) Instruction to explore the property using a DG software (exemplified by instructions Steps 1 and 2 below);
(B) Instruction to guide the students to discover the properties and to complete the statement; (exemplified by instruction Step 3 and the boxed statement for student to complete); and
(C) Three practice questions which involve direct application of the discovered results in (B) (exemplified by instruction Step 4 and the questions that follow).

As illustrated in Worksheet 2 above, Teacher 5 provided very clear instructions for her students on the steps to access the online version of the worksheet (in the right column of Fig. 8.2). Steps 1 and 2 in the worksheet provided the students the procedure to access the online sketchpad operating in a DG environment. In the online sketchpad, students were provided the opportunity to click and drag to observe the geometrical property. The providence of the opportunity to allow users to click and drag in order to observe the invariant property (the angle at the centre is twice the angle at the circumference) amidst an arbitrary variation of conditions (varying the sizes of the circles, the point on the circle, etc.). Step 3 brought the users back to the focus of this worksheet to discover the relation between the angle at the centre of a circle and that at the circumference of the circle.

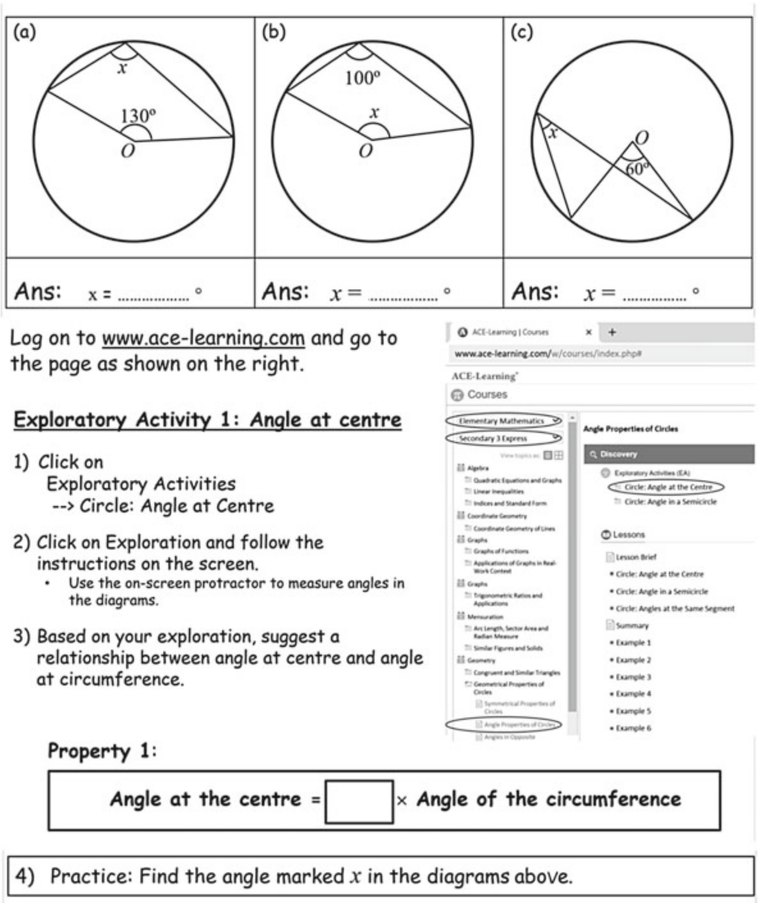


Fig. 8.2 Sample of a worksheet activity designed by Teacher 5

Step 4 of the worksheet immediately provided an immediate consolidation of the concepts by engaging the students to apply this property to three basic questions. These questions focus on an easy application of the property, checking the students' sound understanding (or lack) of the property introduced in the worksheet. The same four-step structure (as summarised in Fig. 8.3) applied for the other three worksheets for this sub-topic. Teacher 5 confirmed that this was the general structure that she would use to teach the other sub-topics of geometry in the syllabus.

In addition to the worksheets, Teacher 5 compiled a set of geometry questions from both the textbook used by the school, and questions from other textbooks and workbooks (not adopted by the school). We also noted that Teacher 5 did not use the textbook for direct classroom instruction. Teacher 5 confirmed that the textbook mainly served as the source of challenging mathematics questions and useful ideas for classroom instructions.

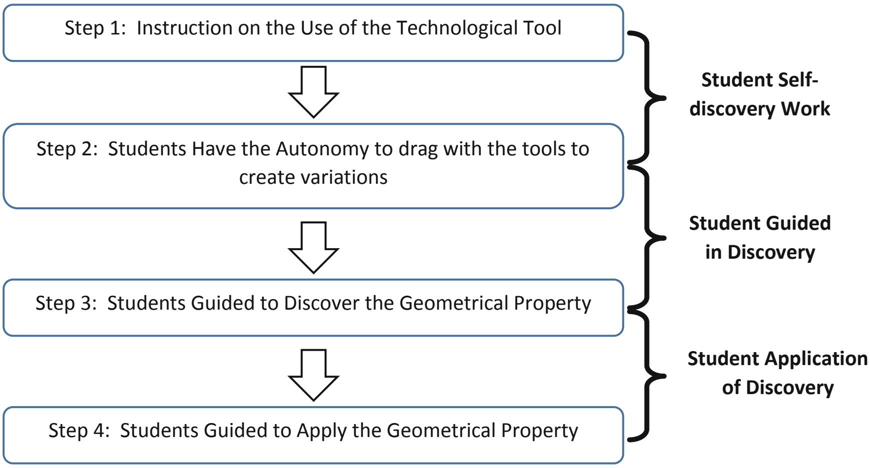


Fig. 8.3 The sequence of introducing a geometrical property used by Teacher 5

8.3.3 Analysis of the Data

The transcripts of the lessons and the teacher interview were studied in conjunction with the video-recordings of Teacher 5's lessons. In this chapter, as discussed in the preceding sections, we focus on the first of the three lessons.

The first lesson could be divided into three main segments:

(1) Lesson Introduction [00:00 to 00:11];
(2) Exploratory Activity [00:11 to 00:53]; and
(3) Lesson closure [00:53 to 00:58].

8.3.3.1 Lesson Introduction

In the Lesson Introduction, which lasted about eight minutes, Teacher 5 placed much emphasis on student understanding of the mathematical terms ("chord of a circle", which has been covered "the last time"). In the Lesson Introduction segment, Teacher 5 took the lead in providing the facilitation to get the students to focus on the concepts involved in this lesson.

Part of the transcript of the Lesson Introduction that follows illustrates our observation. We use the abbreviation T to represent Teacher 5 and S to represent (any) student (without identifying the student engaged in the discussion) in the class who participated in the discourse. Words appearing in square brackets [ ] refer to extrapolation or an interpretation of Teacher 5's speech and those in round brackets ( ) refer to the actions she performed in the lesson while she was involved in that part of the conversation.

Abridged transcript	Commentary
T[1]: … The last time what we did was [to study the properties] of a circle, remember, chords of a circle (began by drawing chords of circle on the whiteboard)… so today's objective is to find or use or understand angle properties (wrote "Objectives" and "Angle properties of a circle" on the whiteboard) :	Teacher introduced the lesson by building on their prior knowledge about the chord of the circle. This will be used in describing the angle on the circumference in today's sub-topic
T[2]: Yeah, chord properties we will revise tomorrow, along with this, so we have mixed questions [i.e. questions that require the combination of several sets of properties to solve]. But today we focus only on angle properties…	Teacher highlighted the focus of today's lesson
T[3]: You'd come across a circle, there's a circle, centre (drew a circle with a dot in the centre). You will see an angle like this – the two chords, meeting at one point on the circumference, ok. So, I have a circle with a centre here. This angle, what is special about this angle? :	Teacher demonstrated the angle that was formed by two chords meeting at a point on the circle
T[4]: On the circumference, and the angle that is formed on the circumference here … What is the arc subtending this angle (teacher wrote arc subtending). The word you see will be subtending (teacher underlined the word 'subtending'), this angle means?	Teacher highlighted the language associated with the angle subtended by the arc
T[5]: This angle is facing you in loose terms ah, if this is the angle formed at the circumference, this is called the arc (teacher drew Fig. 8.4), which is subtending the angle, right? It's facing there	Teacher introduced another way of associating the arc with the angle on the circumference

Developing in students the visual mode of the geometrical concept of an angle subtended by an arc was the highlight of the Lesson Introduction, with the teacher emphasis on the visual (Fig. 8.4). However, we also note that Teacher 5 did not merely establish a purely visually driven mode of the concept in her students. Instead, by using the visual mode of the concepts, she built up the defining characteristics of the geometrical concepts. The first concept: "the angle subtended by an arc" of a circle was built on the concept of the space formed by two chords which intersect on the circumference of a circle T[3]. Thus, the recap section at the beginning of the introduction section of the first lesson was selective on the chord in a circle T[1]; revision of the other properties of chords of a circle which were not relevant to the concept development in this lesson was shelved for subsequent lessons, as mentioned by Teacher 5 in T[2].

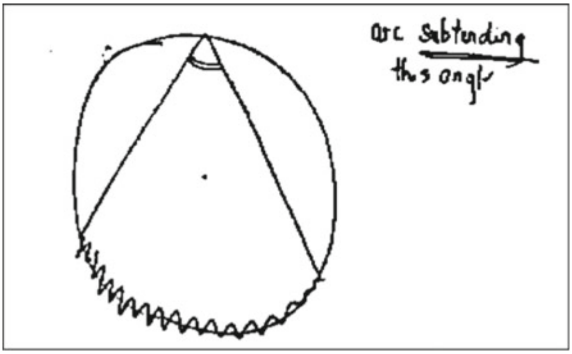


Fig. 8.4 Teacher 5's diagram on the whiteboard of angle subtended on the circumference

In the Lesson Introduction, we observed an interesting feature of Teacher 5's lesson: Teacher 5 was focused on getting the students to recognise the concepts and the precision of the terms used. She did not simply rest on students having seen the required angle, but each underlined term in "Angle subtended by an arc on the circumference" in relation to its visual representation. She skillfully switched between the geometrical properties and its visual representation to enable her students to link the concept and developing its concept image. We summarise this in Fig. 8.5.

The notion of the "arc" next served as an anchor to the next related concept of angle at the centre of the circle. Here, we observed that Teacher 5 repeatedly emphasised the word "arc" in preparation of the next concept of "angle at the centre of the circle". This is evident from the following transcript.

T: In exactly the same way, you will have another angle which will be formed at the centre (drew dotted line for angle at the centre, Fig. 8.6). Do you see both these angles, a (at the circumference) and b (at the centre), both are subtended by the same arc, correct? Both subtended by the same arc. Can? Because they are both made by this arc, so endpoints of this two angles are such that, they are made by this arc. Clear? So this is called angle subtended at the…

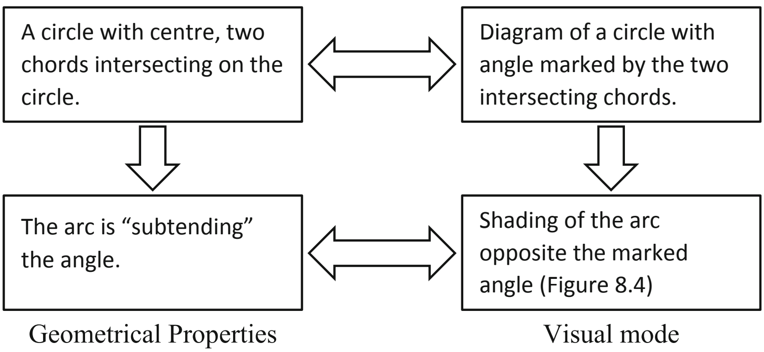


Fig. 8.5 Angle on a circle in both modes of using visuals and geometrical properties

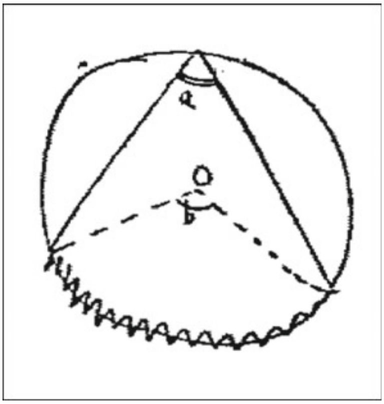


Fig. 8.6 Teacher 5 used the "arc" as the anchor between the two concepts of the angle at the centre and the angle on the circumference of a circle

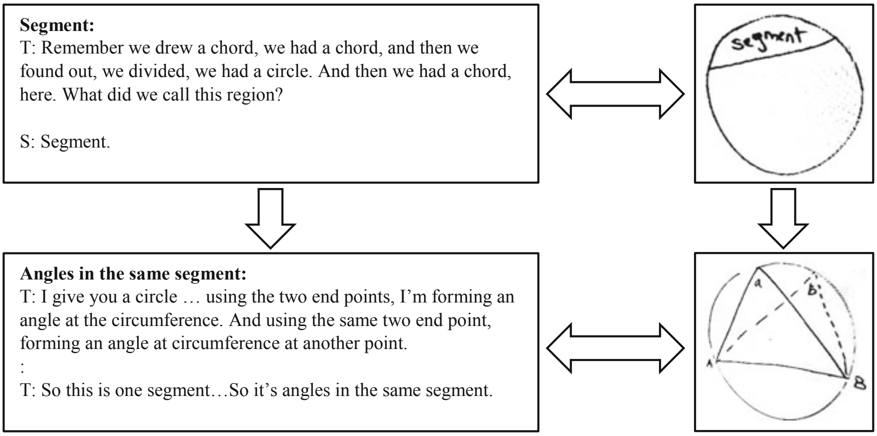


Fig. 8.7 Teacher 5's use of the same mixed mode of visuals and geometrical properties to introduce the concept of angles in the same segment

Here was the transition from the concept of a chord to an arc of a circle, which is the anchor concept for both angle at the centre and the angle at the circumference of a circle. The above was used with reference to Fig. 8.6.

A similar trend was observed when Teacher 5 next moved on to introduce the concept of angles in the same segment, as summarised in Fig. 8.7.

8.3.3.2 Exploratory Activity

In the exploratory activity segment [00:11] to [00:53] (which lasted 42 min), the students were engaged to work in pairs to discover the four angle properties of circles (P1 to P4) through the use of a DG software. Teacher 5 had designed four exploratory activity worksheets to be used in conjunction for exploration in this part of the lesson

(The sample activity Worksheet 2 was shown in Fig. 8.2). Each scaffolding worksheet consisted of three application problems on the related geometrical property. The three problems involved *immediate application* of the property and were of increasing level of complexity. We identified three key phases in the Main Lesson segment of the lesson:

Phase 1: Students' own exploratory work. Teacher 5 managed the students' progress of the discovery activity and addressed the individual students' concern (see below).

Abridged transcript	Commentary
T: OK, so, take the protractor and align it here, and how much is this angle? T: (to another student) Are you ok now? XXX T: (back to the first student) This [angle shown on the computer screen] is 140. So, that will be the angle at the centre S: But just now [my friend, i.e. another student] got 132 [on the screen] T: It's a different [angle, because these angles are] random[ly generated] [00:24:30] to [00:24:46]	Teacher 5 went to the individual students to get them to verify the angles that they had obtained on the screen, and to address the confusion that all the students got different angles as these figures were randomly generated from the system

Phase 2: Students' application of their discovery to solve three related problems. Here, an unexpected response from the student prompted Teacher 5 to address the students. Teacher 5 had wanted her students to apply the properties that they had discovered earlier; some students used the DG to construct the exact dimension of the diagram in the problems in order to determine the unknown.

Abridged transcript	Commentary
T: OK, look up here everyone. I think I see a few of you unable to understand the first part of the worksheet [i.e. the three practice questions printed on the first page of the activity worksheet]… Now, I don't want you to, for these three questions, I don't want you to use the diagram in the [*name of vendor's software*]. You know angle at the centre is two times the angle at circumference…	Teacher 5 brought across the objective of the questions is not to construct the exact diagram in the worksheet using the dynamic geometry software in order to find the unknown angle, but to apply the geometrical properties they had just discovered

At this phase, Teacher 5 consciously facilitated her students to link their earlier discovery to the associated geometrical properties when students appeared to have difficulty in solving the immediate application problems at Step 2:

Abridged transcript	Commentary
T: See this angle, at the centre. Same angle at the circumference. The arc is the same. So this is 60 [degrees], this one should be half, half. This is angle at the centre, this is angle at the circumference. Same like this one. The two angles, the one is up, so this is angle at the centre. 60, same two end points, giving you angle at the circumference. So double…	Teacher 5 consciously brought in the visuals to establish the similarity with the geometrical properties the student had discovered in the earlier activity

Phase 3: Teacher's explanation of the solution of the three practice questions. In this phase, Teacher 5 continued facilitating her students to work towards the answer by consciously relating the application problems to the geometrical properties that they had earlier discovered.

Abridged transcript	Commentary
T: This is the angle – this angle, this angle the one that is shaded is actually ok yes correct it is 2x. This is x, and that is 2x, remember this is 130 …. That is half, so whatever is your answer, divide [it] by two, you get the answer. How about this one? OK let's do it together : S: I don't understand T: See this angle at the centre? Same, angle at the circumference. The arc is the same. So this is …	Teacher 5 explained the first question in detail using the geometrical property, and invited the whole class to solve the next question Teacher 5 addressed the students' difficulty during the lesson

To us, what was the most impressive was that Teacher 5 facilitated her students to identify the meaning of the terms used to describe the geometrical properties with the associated geometrical diagram in emphasising the importance. In addition to identifying the "equal angles" in the geometrical statement that "Angles in the same segment are equal", she created the "segment" in the aforementioned geometrical statement (illustrated in the transcript below).

Abridged transcript	Commentary
T: Segment is the [region in the circle partitioned by] the chord, one of the arc, and one of the centre – er the one at the chord. This is a chord. This angle here, this angle here, they are equal… they are both in the same segment. S: How do I know it's the segment? T: Basically we are looking at its two points S: The same area on the … T: Yeah, on the segment, both going the same side…	Teacher 5 added in an additional chord to the geometrical diagram to show how the two angles initially called by angles subtended by the same arc to explain that indeed they are angles in the same segment

A part of the above diagrams is reproduced in Fig. 8.8.

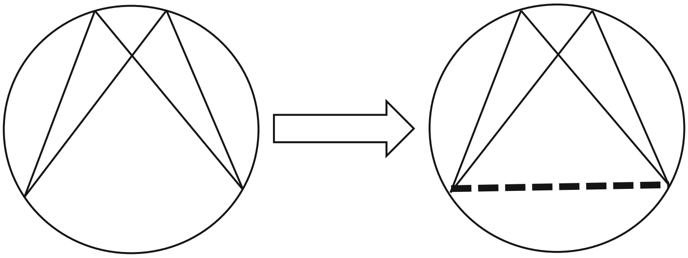


Fig. 8.8 A copy of the whiteboard writing by Teacher 5 who emphasised in addition to the two angles being equal, also stressed on the "same segments" that the two angles were located

Studies have shown that the pure constructivist approach of discovery learning on its own, which usually emphasises an extensive search of knowledge through problem solving, has a limitation in enhancing the learners' memory, and may in fact cause less learning (Rittle-Johnson, 2006). The other aspect of guiding the learners to pay attention to key knowledge that they have acquired is equally important to improve their understanding and ability to apply what they have learned (Kirschner, Sweller, & Clark, 2006). Here Teacher 5 has illustrated this very clearly as she skillfully incorporated three practice questions immediately after the scaffolding for discovering each geometrical property to focus the students' attention on the key geometrical properties.

Teacher 5 adopted a consistent structure in teaching each geometrical property to her students consisting of the three phases which were outlined above. She adopted a partial constructivist approach in getting her students to discover the properties through DG activity, with teacher intervention in helping students to focus on the key knowledge. The emphasis here is on students' *understanding* of the properties with the proof of the properties shelved to a later time. The proof of the properties was deferred; Teacher 5 emphasised much on *discovery* and *understanding and application* at this stage instead of *deductive proof* of the properties. We could summarise Teacher 5's instruction as consisting of the following cycle (Fig. 8.9) in getting her students to learn the four geometrical properties.

Kaur et al. (2019) proposed that an instructional core drives the teaching and learning of mathematics in the lessons of experienced and competent teachers in the research, which she called the DNA of mathematics lessons. She observed that the instructional core comprises a D-S-R (Development—Student Work—Review of Student Work) cycle. In the lesson of Teacher 5, we find that the cycle of instruction, in Fig. 8.9, used by Teacher 5 followed the D-S-R cycle. The Development phase in the geometry lessons we observed was the student discovery phase, in which Teacher 5's students had the opportunity to discover the geometrical properties. The Student Work phase we observed was their application of the newly discovered geometrical results to solve three mathematical tasks. Following this, the Review phase consisted of the teacher giving a direct exposition, with student participation, of the geometrical results in relation to the mathematical tasks just completed.

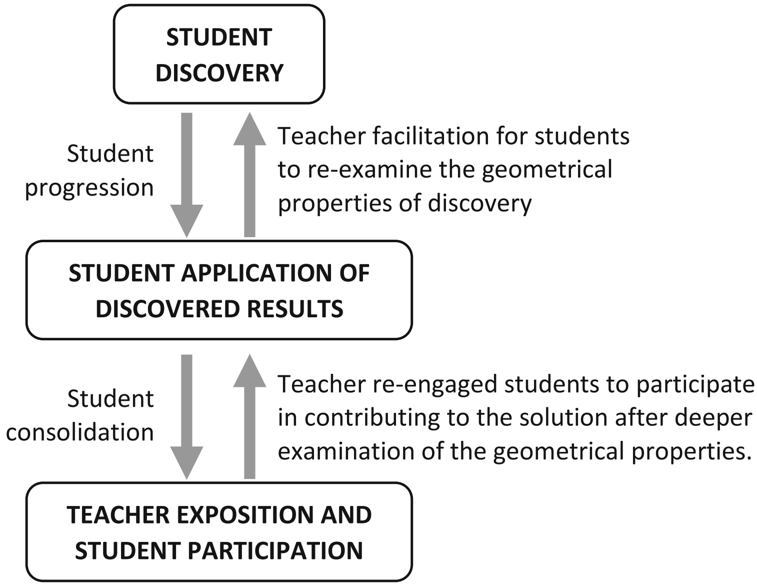


Fig. 8.9 Cycle of instruction in Teacher 5's lesson in developing each of the four angle properties of a circle

Though at times the D-S-R cycle could be teacher-centric, as the teacher may develop the lesson through demonstrations and explanations, Teacher 5's lessons show that the D-S-R structure is representative of both teacher-centric and student-focused developments. In student-focused developments the role of the teacher was then to serve as a guide to "value-add" to the student discovery by facilitating them to focus on the attributes of the geometrical properties explored by the students. The cycle of instruction, in Fig. 8.9, also depicts the development and consolidation phases of lessons as detailed in Chapter 5. Student discovery takes place during the *Development* phase. The application of discovered results by students accompanied by teacher exposition with inputs from students when reviewing student work takes place during the *Consolidation* phase. This phase aids in deepening conceptual knowledge of the students.

How important was this "discovery" part of the lesson to the Teacher 5? We transcribed our interview with her. In particular, when asked about her focus for the lessons during the teacher interview segment, considering both content and non-content goals, Teacher 5 highlighted that her [first] goal was for her students to discover the [geometrical] rules. The importance of engaging students to explore and self-discover was her main concern. This was reflected in the 20-minute interview with Teacher 5 during which she used the words "explore" and "discover" a total of 12 times. Part of the transcript is shown below.

> But for this particular topic [i.e. Geometry], I usually bring them to the computer lab to get them to explore first. So, my goal initially is to get them to go through the process of

> exploration. Self-discovery of the rules. So it's more of a deductive [should be "inductive"] approach. Because you see a few cases, in terms of how the properties play out. And then based on that, they are able to consolidate, which, so they're able to summarise, or conclude, that the relation between the angles is as what is being displayed.

The main objective of this first lesson was to engage her students in exploration and discovery of the geometrical properties using DG software. The worksheets that she had designed earlier served to provide the scaffold to serve this objective. People using a raft to cross the river will eventually discard the raft after they have crossed the river successfully. In the same way, Jones (2000) asserted that the first stage of engaging students in mathematical exploration, seen by mathematicians as lacking mathematical precision, is a crucial first step to mathematical explanations that will lead the students to next transcend such imprecise discovery (in the form of the software environment) to deductive geometric reasoning. From this lesson conducted by Teacher 5, what followed the discovery activity was not immediately followed by precise deductive mathematical proofs, but by three questions of immediate application of the geometry concepts. This ensured that her students had truly understood the geometrical properties just "discovered" by the students themselves. This was evident from the following part of the teacher talk.

> I don't want you to play with the diagram and match it with these three online [i.e. create the geometrical figures using the softwares]. So, it should be very fast, the page one. What about the second one, the same way. Just observe the relationship, then move to the next.

Deductive reasoning and proofs of the geometrical properties did not immediately follow the segment after the students' discovery of these properties. Rather, to ensure that her students had truly understood these properties was the most immediate activity after that.

8.3.3.3 Lesson Closure

Teacher 5's lesson closure for the first lesson was also of interest to us. Instead of merely reiterating the four main geometrical properties that had been covered in the lesson, she reiterated these four properties by using a semi-rigorous deductive approach to show the connectedness of the four geometrical properties. After restating Property (P1) (angle at the centre is twice the angle at the circumference) without proof, Teacher 5 demonstrated how Property (P3) (angle in the same segment are equal) is in fact a special case of Property (P1).

Teacher 5 started with a special angle of 100° at the centre of the circle and that of 50° at the circumference for two special cases. By deleting the angle at the centre, Teacher 5 showed that the two angles on the circumference of a circle are equal (to 50°). The sequence of what was shown on the whiteboard is presented in Fig. 8.10. She skillfully demonstrated that Property (P3) is indeed a special case of Property (P1).

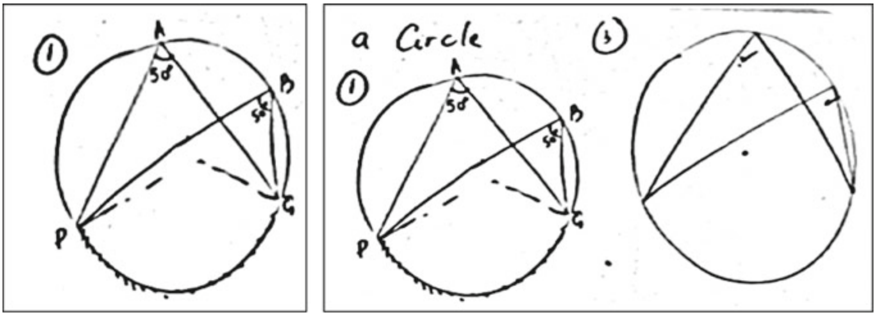


Fig. 8.10 The sequence of two drawings used by Teacher 5 to demonstrate that Property (P3) is a special case of Property (P1)

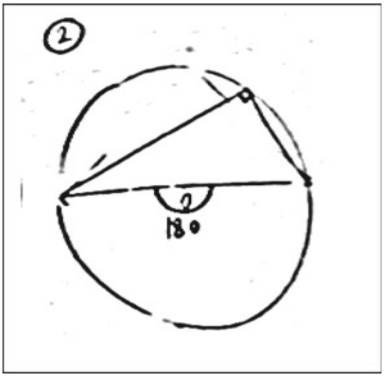
Abridged transcript	Commentary
T: All these [four points in Fig. 8.10] have the same two end points…. Look for these two points, the common points, see whether they are going to the centre, or the same two endpoints, on the same side, going towards the circumference, then they are connected. So I see these two endpoints here. PQ. I go to the centre, I see angle 100 [degrees]. If I go to the circumference, it will be how much? 50. OK so this is what we have seen. Same two points, going again to the circumference, it's 50	Teacher 5 referred to the left drawing of Fig. 8.10 to reinforce that both angles on the circumference are equal to 50 by applying Property (P1) twice
T: If I remove this [angle at the centre of the circle], do you realise that it looks like property number three? If I don't have the angle at the centre, basically you have again the same two endpoints,… Do you see the similarity?	Teacher 5 erased off the angle of the centre and convinced to students that both angles at the circumference are equal (i.e. Property (P3))

In a similar approach, Teacher 5 demonstrated that Property (P2) is also a special case of Property (P1) in Fig. 8.11.

Abridged transcript	Commentary
T: Property number 2, what was the property 2? … You have a diameter. And what did we find? When you see a diameter, [there are two] endpoints [on the two sides of the circle]. When you go to the circumference (teacher pointing to Fig. 8.11 on the whiteboard), you get a 90 degree, you get a 90 degree here, 90 here. This is also a special case of one. Have you realised that? This one, number 2, is a special case of 1 (Bell rang at this juncture)	Teacher 5 led her students to realise that the diameter can also be seen as having two points on the circle and subtending an angle of 180° at the centre of circle. Here the key message appeared to be that Property (P2) is also a special case of Property (P1)

(continued)

Fig. 8.11 The drawing used by Teacher 5 to demonstrate that Property (P2) is a special case of Property (P1)



(continued)

Abridged transcript	Commentary
T: OK, and [property] number 4, you see a quadrilateral, which I think most of you – in fact all of you are able to see the connections $a + b$ equals to, how much? 180 degrees. Also $c + d$ equals 180 degrees	As the lesson had ended, Teacher 5 did not continue to demonstrate that Property (P4) is also a corollary of Property (P1)

When asked what was an ambitious part of the three lessons on teaching this sub-topic on angle properties of a circle, she asserted that it was establishing the relation across the four angle properties of a circle. Her intention was to start off the second lesson by challenging them to derive a "proof" of Property (P4) from Property (P1). This was left as homework for her students as she ran out of time in the first lesson.

Abridged transcript of teacher interview	Commentary
T: Today actually frankly, I was not going for ambitious things, it would only come in tomorrow. Because today was getting them to just explore, understand the four rules. I – my ambitious part would only be that I just left it to them, ok the instructions are there, do it, so it's the first time, ok, no it's not the first time actually	Teacher 5 felt that the ambitious part of the lesson was to leave it for the students to discover the four geometrical properties of the circle through the activity worksheets
T: Actually I didn't use it, as a, as a, carry over you know, as a special case for this one. I gave it a separate activity. I gave it as a separate activity. And I try to later at the end, bring it together and see, this is the, mother property, and these two, you know, it just follows. So tomorrow, maybe I will even start off by asking, my ambitious part would be, why is it, can you just show me, can you just prove it, so we will start by proving this…	Teacher 5 planned for the students' discovery of the properties through inductive means and to introduce the proofs at the end of the lesson to show that the four properties could effectively be reduced to one property. She left the proof of Property (P4) to her students as homework

Teacher 5 was mindful of establishing the *connections* across the properties within this introductory lesson. As an after-note of the teacher interview, we were curious about how Property (P1) was left as the intuitive level without attempting to *prove* that the angle at the centre is twice the angle at the circumference deductively. Teacher 5 confirmed that as the proof of Property (P1) involves properties of triangles, she deliberately chose not to expand this proof for fear of distracting the students; her objective was to show to students the connections across these properties.

In lesson closure, she introduced two geometrical properties through a deeper approach of using deductive reasoning and challenged her students to derive Property (P4) as homework, and to attempt as many of the homework problems as possible before the second lesson. She obviously demonstrated lesson closure without closure, which was effective in her case compared to having a lesson with a neat closure with all issues resolved.

8.4 Discussion and Conclusion

It is apparent that Teacher 5 built a positive classroom culture by enabling her students to discover their own learning and equipped them with the crucial mathematical tools (i.e. the correct mathematical terms) for their discovery. Her lessons were well prepared with appropriate sequencing of appropriate activities, fully mindful of her students' capacity. To mathematics educators, the lesson was most impressive because she did not simply get students to "apply memorised procedures" (Schoenfeld, 2018, p. 499), but offers the conceptual richness of the mathematical concepts.

The general curriculum approach in Singapore (including mathematics) adopts Bruner's spiral approach, in that concepts and skills are re-visited iteratively at each higher level in order to ensure a coherent overall curriculum and a deep learning of the mathematical concepts. The various topics of geometry, as in the topics in the other major strands of mathematics, are distributed over all the years of secondary school mathematics education. The content in each level builds on the earlier levels as a foundation, which in turn serves as the foundation for the next higher level. Within this first lesson to the sub-topic Angle Properties of a Circle, we observed Teacher 5's attempt to use a "spiralling" that occurs within this lesson in introducing students to the geometrical properties by exploration, and in concluding the lesson by showing a deeper connection across these geometrical properties by a deductive reasoning approach.

It was interesting to observe that Teacher 5's lesson enactment resonates with van Hiele's levels of learning of geometry. She started with the Level 1 (Visual) by associating visuals with each geometrical entity during the lesson introduction (Figs. 8.4, 8.5, 8.6, and 8.7). This was followed by leading the students to Level 2 (Analysis) through engaging them in self-discovery activity of the geometrical properties using ICT. Immediately following this is students' direct application of the newly discovered properties to solve three problems for each problem, which corresponds to Level 3 (Relational). During lesson closure, she summarised the geometrical properties covered in this lesson using a deductive approach highlighting the relation between the geometrical properties—this corresponds very closely to Level 4 (Deduction). This forms a good starting point for her students in the next two geometry lessons in which Teacher 5 would emphasise deductive proofs. Although Teacher 5 did not explicitly articulate her consideration of van Hiele's levels of learning of geometry in her discussion, it is clear that this sequence of teaching was ingrained in her as she indicated that this is her "general approach" in teaching geometry.

References

Ayalon, M., & Even, R. (2010). Mathematics educators' views on the role of mathematics learning in developing deductive reasoning. *International Journal of Science and Mathematics Education, 8*(6), 1131–1154.

Clarke, D. J. (1998). Studying the classroom negotiation of meaning: Complementary accounts methodology. Chapter 7 in A. Teppo (Ed.), *Qualitative research methods in mathematics education*, monograph number 9 of the *Journal for Research in Mathematics Education*, Reston, VA: NCTM, 98–111.

Clarke, D. J. (Ed.). (2001). *Perspectives on practice and meaning in mathematics and science classrooms*. Dordrecht, Netherlands: Kluwer Academic Press.

Coxeter, H. S. M., & Greitzer, S. L. (1967). *Geometry revisited*. Washington, DC: Mathematical Association of America.

Herbst, P. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education, 33*(3), 176–203.

The International Commission on Mathematical Instruction. (1995). Perspectives on the teaching of geometry for the 21st century. *Educational Studies in Mathematics, 28*(1), 91–98.

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics, 44,* 55–85.

Kaur, B., Toh, T. L., Lee, N. H., Leong, Y. W., Cheng, L. P., Ng, K. E. D., … Safii, L. (2019). *Twelve questions on Mathematics teaching: Snapshots from a study of the enacted school mathematics curriculum in Singapore*. Singapore: Author.

Kim, O., & Atanga, N. A. (2013). Teachers' decisions on task enactment and opportunities for students to learn. In M. Martinez & A. Castro Superfine (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 66–73). Chicago, IL: University of Illinois at Chicago.

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential and inquiry-based teaching. *Educational Psychologist, 41*(2), 75–86.

Leong, Y. H., & Lim-Teo, S. K. (2008). Teaching of geometry. In P. Y. Lee (Ed.), *Teaching secondary school mathematics: A resource book* (pp. 117–142). Singapore: McGraw-Hill Education (Asia).

Leung, A. (2011). An epistemic model of task design in dynamic geometry environment. *ZDM: International Journal on Mathematics Education, 43*(3), 325–336.

Ministry of Education. (2012). *O-Level mathematics teaching and learning syllabus*. Singapore: Ministry of Education.

Polly, D. (2014). Elementary school teachers' use of technology during mathematics teaching. *Computers in the Schools, 31*(4), 271–292.

Rittle-Johnson, B. (2006). Promoting transfer: Effects of self-explanation and direct instruction. *Child Development, 77*(1), 1–15.
Schoenfeld, A. H. (2018). Video analyses for research and professional development: The teaching for robust understanding (TRU) framework. *ZDM Mathematics Education, 50*(3), 491–506.
Sutherland, R. (1998). Teachers and technology: The role of mathematical learning. In D. Tinsley & D. Johnson (Eds.), *Information and communication technologies in school mathematics* (pp. 151–160). London: Chapman & Hall.
Sutherland, R., Olivero, F., & Weeden, M. (2004). Orchestrating mathematical proof through the use of digital tools. In M. J. Hoines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (pp. 265–272). Bergen: Group for the Psychology of Mathematics Education.

Tin Lam Toh is an Associate Professor and currently the Deputy Head of the Mathematics and Mathematics Education Academic Group in the National Institute of Education, Nanyang Technological University of Singapore. He obtained his PhD from the National University of Singapore in 2001. A/P Toh continues to do research in mathematics as well as mathematics education. He has published papers in international scientific journals in both areas.

Berinderjeet Kaur is a Professor of Mathematics Education at the National Institute of Education in Singapore. She holds a Ph.D. in Mathematics Education from Monash University in Australia. She has been with the Institute for the last 30 years and is one of the leading figures of Mathematics Education in Singapore. In 2010, she became the first full professor of Mathematics Education in Singapore. She has been involved in numerous international studies of Mathematics Education and was the Mathematics Consultant to TIMSS 2011. She was also a core member of the MEG (Mathematics Expert Group) for PISA 2015. She is passionate about the development of mathematics teachers and in turn the learning of mathematics by children in schools. Her accolades at the national level include the public administration medal in 2006 by the President of Singapore, the long public service with distinction medal in 2016 by the President of Singapore and in 2015, in celebration of 50 years of Singapore's nation building, recognition as an outstanding educator by the Sikh Community in Singapore for contributions towards nation building.

Chapter 9
Meaningful Mathematics Talk That Supports Mathematics Learning in Singapore Secondary Schools

Lai Fong Wong, Berinderjeet Kaur, and Cherng Luen Tong

Abstract A wide variety of talk may occur within a mathematics lesson, but the mere presence of talk does not ensure that understanding follows—only meaningful mathematics talk can enhance learning. Talk may be used to convey meaning or to generate meaning. There is evidence to suggest that conceptual understanding is more likely to be associated with dialogic talk than with univocal discourse. We can examine mathematics talk from the perspectives of the teacher (teaching talk) and the students (learning talk) according to Alexander's dialogic teaching framework. Teaching episodes illustrate the kinds of mathematics talk (univocal and dialogic) enacted in the interactions between an experienced and competent teacher and his students. They show how the teacher uses the students' talk to generate meaning for both himself and the students, creates the learning moment by using students' responses as thinking devices, and thus provides opportunities for students to construct their own knowledge. The implications for mathematics teachers in Singapore secondary schools are discussed, as we acknowledge the reality of a teacher's classroom, which includes the competing demands of depth versus breadth in content coverage, students' differing abilities and interests, and time constraints.

Keywords Mathematics talk · Univocal · Dialogic · Thinking device · Generate meaning

---

L. F. Wong (✉) · B. Kaur · C. L. Tong
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: laifong.wong@nie.edu.sg

B. Kaur
e-mail: berinderjeet.kaur@nie.edu.sg

C. L. Tong
e-mail: chernglue n.tong@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_9

9.1 Introduction

The role of talk as central to knowledge building in mathematics classrooms has been recognised for many decades, and there have been a number of research that study the role of talk in supporting learning (e.g. Weaver, Dick, & Rigelman, 2005). Analyses of classroom talk have identified the provision of opportunity for students to voice and share ideas as an important component of learning that yields higher level of conceptual exchanges and leads to more robust learning (Alexander, 2004). However, the mere presence of talk does not ensure that understanding follows – only meaningful mathematics talk can enhance learning. The quality and type of talk are crucial to helping students think conceptually about mathematics (Kazemi & Stipek, 2009; Lampert, Blunk, & Pea, 1998; Nathan & Knuth, 2003; Van Zoest & Enyart, 1998).

The teacher's role is critical in how mathematics talk plays out in a mathematics classroom, and research reveals that teachers' instructional practices often give students little opportunity to talk, discuss, conjecture, reason, and justify. The Kassel project in 1995 on general features of mathematics instruction in Singapore classrooms reported that teachers "presented knowledge to the pupils as a class by telling and explaining" (Kaur, 1999, p. 195). The Learner's Perspective Study (LPS) in 2005 also revealed that "teachers played the most active role in expounding mathematical concepts and problem-solving skills" (Kaur, 2009, p. 340) and the most common interaction pattern was the initiation-response-feedback (IRF) discourse format (Sinclair & Coulthard, 1992) where the teacher asked a question, students responded and teacher gave feedback. In their study of nature of teacher questions (performative, procedural, and conceptual), Hogan, Rahim, Chan, Kwek, and Towndrow (2012) also noted that the prevalence of mundane IRF talk structure and that a substantial proportion of performative questions eventually lead on to procedural and explanatory talks, thus suggesting that Singapore mathematics classrooms provide limited opportunities for students to engage in rich classroom conversations.

9.2 Meaningful Mathematics Talk

Most research on mathematics talk anchor on two perspectives on teaching and learning: Vygotskian, and constructivism and socio-constructivism. A Vygotskian viewpoint suggests that teaching is beneficial when it "awakens and rouses to life those functions which are in a stage of maturing, which lie in the zone of proximal development" (Gallimore & Tharp, 1990, p. 177), and learning occurs when assistance is provided at opportune points in the learner's zone of proximal development. Thus, in a mathematics-talk learning classroom, both the teacher and students move through their own learning zones of proximal development as they assist one another in a recursive process of talking.

Constructivism suggests that students make sense of their learning by relating new information or ways of understanding to existing ideas or ways of thinking, and hence, actively constructs new understanding. Piaget and Inhelder (1969) pointed that new knowledge and experience can be *assimilated* when they fit comfortably into our existing schema; but when new ideas do not fit, we are forced to *accommodate* them by changing our schema, and that we sometimes resist. As cited in Atwood, Turnbull, and Carpendale (2010), "Piaget considered cooperative interaction especially conducive to learning because within conditions of cooperation individuals are more likely to share their perspectives with others, perspectives that can be questioned, affirmed, or revised" (p. 359), and Chapman's (1991) reconstruction of Piagetian theory supported that "the experience of interpersonal argumentation provides children with the need and the occasion to justify their assertions, ideally with arguments that have force even for persons who do not share the same perspectives" (p. 220). In other words, learning in schools is a social activity and the discussion of learning moves from the individual to the group.

This implies the need to set up a learning environment that encourages students to relate new ideas to existing ones in order to modify them, and together develop knowledge as a co-constructed activity of all classroom members, constituted in and through talk. Douglas Barnes (1992) advocated the idea that coming to terms with new knowledge requires working on understanding, which can most readily be achieved through talk because "the flexibility of speech makes it easy for us to try out new ways of arranging what we know, and easy also to change them if they seem inadequate" (Barnes, 2008, p. 5). According to him, two kinds of talk, exploratory and presentational, contribute to learning but each has a different place in the sequence of lessons.

There are many types of mathematics talk. In a research by Oregon Mathematics Leadership Institute (OMLI) that addressed the research question: *Can student achievement in mathematics be significantly improved by increasing the quantity and quality of meaningful mathematical discourse in mathematics classrooms?*, the team developed a Classroom Observation Protocol, specific to student talk. In this protocol, they define 9 types of discourse (see Fig. 9.1).

These types represent a continuum of the mathematics discourse desired in mathematics classrooms where students are thinking and talking about mathematics. The order of the discourse types represents the continuum of discourse in terms of increasing levels of cognitive demand. That is, giving a short right or wrong answer to a direct question represents the lowest level of cognitive demand and justifying mathematical ideas and procedures and making generalisations represent the highest levels.

According to Lotman (1988), talk may be used to convey meaning or to generate meaning. Wertsch (1991) used the term univocal and dialogic, respectively to represent these two functions. In a univocal talk, the listener receives the "exact" message that the speaker intends for the listener to receive, and once the speaker's intention has been conveyed, the talk ends. In contrast, in a dialogic talk, there is a give-and-take communication that extends beyond the conveyance of an exact message

Level	Definition	Explanation
1	Answering	A student gives a short answer to a direct question from the teacher or another student.
2	Making a Statement or Sharing	A student makes a simple statement or assertion, or shares his or her work with others and the statement or sharing does not involve an explanation of how or why. For example, a student reads what she wrote in her journal to the class.
3	Explaining	A student explains a mathematical idea or procedure by stating a description of what he or she did, or how he or she solved a problem, but the explanation does not provide any justification of the validity of the idea or procedure.
4	Questioning	A student asks a question to clarify his or her understanding of a mathematical idea or procedure.
5	Challenging	A student makes a statement or asks a question in a way that challenges the validity of a mathematical idea or procedure. The statement may include a counter example. A challenge requires someone else to reevaluate his or her thinking.
6	Relating	A student makes a statement indicating that he or she has made a connection or sees a relationship to some prior knowledge or experience.
7	Predicting or Conjecturing	A student makes a prediction or a conjecture based on their understanding of the mathematics behind the problem. For example, a student may recognize a pattern in a sequence of numbers or make a prediction about what might come next in the sequence or state a hypothesis a mathematical property they observe in the problem.
8	Justifying	A student provides a justification for the validity of a mathematical idea or procedure by providing an explanation of the thinking that led him or her to the idea or procedure. The justification may be in defense of the idea challenged by the teacher or another student.
9	Generalizing	A student makes a statement that is evidence of a shift from a specific example to the general case.

Fig. 9.1 OMLI Classroom Observation Protocol for student talk (Weaver et al., 2005)

leading to generation of meaning through dialogue as a "thinking device" (Lotman, 1988). There is evidence to suggest that conceptual understanding is more likely to be associated with dialogic talk than with univocal discourse (Knuth & Peressini, 2001; Wertsch & Toma, 1995).

Robin Alexander (2004) proposed that a different type of 'talk' is required within the classroom to stimulate students' thinking and learning, and he developed a pedagogical approach to classroom teaching known as 'dialogic teaching'. Dialogic teaching is teaching based on more equal dialogue between teachers and students and among students themselves. The principles of dialogic teaching provide a framework to develop purposeful and authentic learning activities. According to Alexander (2004), dialogic teaching harnesses the power of talk to stimulate and extend students' thinking, and advance their learning and understanding as students' talk is used as a thinking device. It helps the teacher more precisely to diagnose students' needs, frame their learning tasks, and assess their progress.

Dialogic teaching is not just any talk. It is as distinct from the question-answer and listen-tell routines of traditional teaching as it is from the casual conversation of informal discussion. Dialogic teaching draws on a broad repertoire of strategies and techniques—talk for everyday life, learning talk, teaching talk, and classroom organisation.

Students in dialogic classrooms do not just provide brief factual answers to ‘test’ or ‘recall’ type of questions, or answers that they think the teacher wants to hear. Instead they are engaged in a spectrum of strategies specific to learning (known as learning talk)—narrate, explain, analyse, speculate, imagine, explore, evaluate, discuss, argue, justify, and even ask questions of their own (Alexander, 2010). While Alexander did not provide further descriptions or explanations of these talk strategies in the literature, the following descriptors are used in our identification of learning talks occurred during the teaching episodes:

- Narrate: mere telling
- Explain: making an idea clear by providing more details
- Analyse: examine information in detail so as to explain and interpret it
- Speculate: predicting an outcome based on information provided
- Imagine: forming a supposition (of some idea not actually present)
- Explore: developing a concept through an investigation or finding alternatives
- Evaluate: forming an assessment or a judgement
- Discuss: talking about a topic in detail, taking into account different ideas
- Argue: exchanging or providing different views, with reasons in support
- Justify: showing or proving to be right or reasonable
- Ask questions of their own: (self-explained).

In Alexander’s dialogic teaching framework (2010), the spectrum of talk strategies specific to teaching (known as teaching talk) are:

- Rote: the drilling of facts, ideas, and routines through constant repetition;
- Recitation: the accumulation of knowledge and understanding through questions designed to test or stimulate recall of what has been previously encountered, or to cue pupils to work out the answer from clues provided in the question;
- Instruction/Exposition: telling the pupil what to do, and/or imparting information, and/or explaining facts, principles or procedures;
- Discussion: the exchange of ideas with a view to sharing information or solving problems; and
- Dialogue: achieving common understanding through structure, cumulative questioning and discussion which guide and prompt, reduce choices, minimise risk and error, and expedite ‘handover’ of concepts and principles.

According to Alexander (2010), rote, recitation, instruction, and exposition are frequently used, and they are probably the default modes of teaching talk. While there is always a place for these talk strategies, discussion and dialogue, which are less common, are what students need to experience much more frequently. By using discussion and dialogue, students do not merely listen and answer, but are empowered both cognitively and socially to think, engage, and take decisions about their learning.

Dialogic teaching requires interactions that encourage students to think, and to think in different ways; questions which invite much more than simple recall; answers which are justified, followed up and built upon rather than merely received; feedback which informs and leads thinking forward as well as encourages; contributions which are extended rather than fragmented; exchanges which chain together into coherent and deepening lines of enquiry; discussion and argumentation which probe and challenge rather than unquestioningly accept; professional engagement with subject matter which liberates classroom discourse from the safe and conventional; and classroom organisation, climate, and relationships which make all this possible (Alexander, 2010). Using Alexander's dialogic teaching framework, we can examine mathematics talk from the perspectives of the teacher (teaching talk) and the students (learning talk).

Teaching episodes in the next section will illustrate the kinds of mathematics talk (univocal and dialogic) enacted in the interactions between an experienced and competent teacher and his students. The various talk strategies specific to teaching and learning talks in Alexander's dialogic teaching framework are identified in these teaching episodes to illustrate how the teacher uses the students' talk to generate meaning for both himself and the students, creates the learning moment by using students' responses as thinking devices, and thus provides opportunities for students to construct their own knowledge.

9.3 Mathematics Talk Enacted by an Experienced and Competent Teacher

The teacher in focus is Teacher 27. An experienced and competent mathematics teacher, he is the Head of Mathematics Department in his school. He is in the age range of 40–49 years with 20–25 years of mathematics teaching experience. The lessons of Teacher 27 were selected for study of mathematics talk as they represented a comprehensive range of mathematics talk that was present in the lessons of the 30 experienced and competent teachers who participated in Phase 1 of the project. Teacher 27 taught his secondary 4 class, of 17 students in the Express course of study, the topic of Vectors that spanned 495 min of instruction time over a period of 8 lessons. In this section, the three teaching episodes illustrate the kinds of mathematics talk (teaching and learning talks) enacted in the interactions between Teacher 27 (T) and his students (S). The goal of the lesson was to develop student understanding of vectors and representations.

Episode 9.1

Line	Teaching episode	Teaching/learning talks
(1)	T: You may have heard of vectors when you're studying physics. Now, can you give me an example of what you already studied in physics which you understand as vectors? What did you already know about vectors?	Recitation

(continued)

(continued)

Line	Teaching episode	Teaching/learning talks
(2)	S: Gravity	Narrate
(3)	T: Gravity, what else? I'm not going to correct you now. I'm just letting you tell me what you understand about vectors. Tell me what else you know about vectors	
(4)	S: Got direction	Narrate
(5)	T: So vectors have direction. Is that correct? [Students nod.] Give me an example	Rote Exposition
(6)	S: Velocity	Narrate
(7)	T: Velocity. Does velocity have direction?	Exposition
(8)	S: Yes	
(9)	T: If I run towards Sean [pointing at Sean] at a speed of 4 km/h from here. Then I ask Hadi to run towards Sean also at a speed of 4 km/h from there [pointing at Hadi], are the two of us travelling at the same velocity? Hadi, let us run towards Sean now [Both T and Hadi move towards Sean.] We are both running towards Sean but are we running in the same direction?	Exposition
(10)	S: No. Towards the same direction, yes. Ay? So same direction?	Narrate
(11)	T: We are both running TOWARDS Sean but are we running in the same direction? I don't know. Yes or no, I'm not sure. You discuss	
	[Students discuss among themselves.]	
(12)	T: So, are Hadi and I running in the same direction?	
(13)	S: No	
(14)	T: We are not running in the same direction. Why?	Exposition
(15)	S: Because one person is pointed this way and the other person is pointed the other way	Explain
(16)	T: So we are running in different directions although we are both running at the same speed	Rote

Up to line 16 in Episode 9.1, the mathematics talk enacted is primarily univocal because the teacher's intention is to convey the message that vectors have directions. Teacher 27 ensures that his intended message for this lesson is adequately conveyed by using a live demonstration of two persons running at the same speed but in different directions. His focus thus far is on how well everyone understands his perspective rather than on making sense of the students'. The teaching talks invoked are Rote, Recitation, and Instruction/Exposition; while the learning talks invoked are Narrate and Explain.

However, the following Episode 9.2 reveals how the teacher carries on to leverage students' responses (Line 10 in Episode 9.1) as generators of meaning, illustrating the essence of a dialogic mathematics talk. The teaching talks invoked are still Recitation and Exposition; while the learning talks have included Justify.

Episode 9.2

Line	Teaching episode	Teaching/learning talks
(17)	T: But some of you have this idea that we're running in the same direction because we are running towards the same person? How to disprove that? How can we show that both of us are not running in the same direction?	Recitation
(18)	S: Bearings	Narrate
(19)	T: Bearings? Can you show me how?	Exposition
(20)	S: This is Sean. [S draws a point.] Teacher is running towards Sean in this direction. [S draws an arrow to the point.] Hadi is also running towards Sean in this direction. [S draws another arrow to the point.] North is in this direction. [S draws another arrow to denote North.] We measure the bearing of the two of you from Sean. So we can see that the two bearings are not the same	Justify

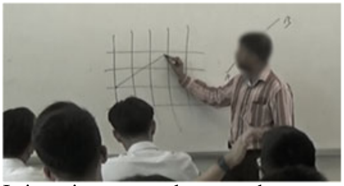
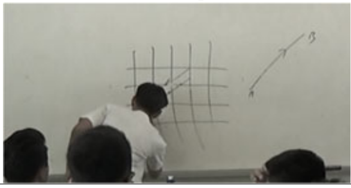
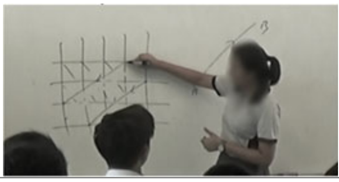
The following Episode 9.3 further illustrates a mathematics talk that embodies the dialogic characteristics.

Episode 9.3

Line	Teaching episode	Teaching/learning talks
(21)	T: Let's work in pairs. I want Partner A to draw any vector and label it $\vec{AB}$. Now Partner B, how are you going to draw a vector $\vec{PQ}$ such that $\vec{AB} = \vec{PQ}$, that is, to replicate exactly the same vector your partner has drawn? [Students discuss in pairs.]	Instruction
(22)	S: Use a protractor	Narrate
(23)	T: How to use a protractor to draw another vector that is equal to this vector?	Exposition
(24)	S: Draw another vector of same length and is parallel.	Narrate
(25)	T: So how do we make sure the two vectors are parallel?	Exposition
(26)	S: Use bearing. I use the protractor to measure the bearing like this	Explain
(27)	T: If you do not have a protractor, then how? Is there another way?	Dialogue
(28)	S: Use tracing paper	Imagine
(29)	T: That's a good idea. What if I make it difficult for you and say cannot use tracing paper? What other paper will you use?	Dialogue

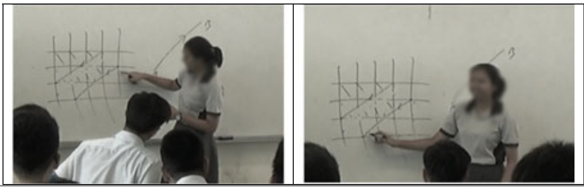
(continued)

(continued)

Line	Teaching episode	Teaching/learning talks
(30)	S: I draw horizontal lines like lines in the exercise book. I also draw vertical lines. Then I count how many lines and then draw my vector like this	Explore
(31)	T: Oh, that's smart! So if the vector is drawn on grid like this [T shows a vector drawn on grid]  Is it easier now to draw another vector that is equal to this? Shane, show us how. [S draws a vector on the board.] 	Discussion
(32)	T: How do the others ensure that the lengths of the two vectors are the same?	Dialogue
(33)	S: Use the boxes	Analyse
(34)	T: How to use the boxes?	Dialogue
(35)	S: The vector you draw is between 6 boxes, so you find another 6 boxes and draw the vector	Explain
(36)	T: I don't quite understand what you're saying. Can anyone help to explain?	Dialogue
(37)	S: The vector cuts across these 6 boxes. [S points the 6 boxes.] So I copy and draw my vector that cuts same 6 boxes like this. [S points the other 6 boxes.] 	Explain
(38)	T: Is everyone convinced that the two vectors are equal? How are you so sure that they are equal?	Dialogue

(continued)

(continued)

Line	Teaching episode	Teaching/learning talks
(39)	S: The vector is from here to here. [S points at initial and terminal points of the vector.] I start from here, it goes down by 2 boxes and then goes left by 3 boxes 	Analyse
(40)	T: So how are you counting?	Dialogue
(41)	S: Vertically and horizontally	Analyse
(42)	T: So if each box is a unit, the vector represents a movement of 3 units to the left and 2 units down	Rote
(43)	T: How can we express this vector in a form that represent 3 units to the left and 2 units down?	Discussion

Again, Teacher 27 first attempts to see/hear what the students understand of equal vectors and uses the students' talk to generate meaning for both himself and the students. He creates the learning moment by using student's response (line 30) to incept the idea of representing a vector horizontally and vertically. Rather than telling the class directly how a vector can be represented in a column vector, Teacher 27 turns to the whole class for inquiry and discussion. He prompts students to use peers' responses as thinking devices and provides opportunities for students to construct their own knowledge.

Teaching Episode 9.3 is primarily dialogic. A significant mark of Teacher 27's classroom is the degree to which the students took ownership of the learning situation. The student-generated responses that emerged during the lesson encouraged dialogues and discussions in a productive manner. The teacher encourages students to build ideas on the basis of one another's insights. Student collaboration is evident as students attempt to refine one another's ideas, help one another explain, and verify one another's claims. Teacher 27 does not attempt to convey a particular message by engaging his students in a specific approach. Instead, he is open to his students' ideas and allows his students to pursue approaches, that may be quite unexpected to him, to generate new mathematical understanding, and this is the essence of a dialogic mathematics talk.

9.4 Mathematics Talk in the Classrooms of Mathematics Teachers in General

As part of the survey, 677 teachers reflected on their lessons for a specific course of study—Integrated Programme (IP), Express, Normal (Academic) (N(A)), or Normal

(Technical) (N(T)), and indicated the frequency of their use of the kinds of teaching talk. Chapter 2 provides details about the different courses of study in Singapore secondary schools and also details of the survey. The aggregated data is shown in Table 9.1.

From the data in Table 9.1, we see that about 60% or less of the teachers for the IP course but 80% or more of the teachers for the Express/N(A)/N(T) courses frequently or mostly/always draw on Rote and Recitation; about 55% of the teachers for the IP course and more than 65% of the teachers for the other courses frequently or mostly/always draw on Instruction/Exposition; 80% or more of the teachers for the IP course and approximately 60–75% of the teachers in the other courses frequently or mostly/always draw on discussion and dialogue. It is apparent that teachers for the IP course draw less on the basic repertoire of teaching talk (rote, recitation, and instruction/exposition) but more on the larger oral repertoire (discussion and dialogue) as their students are of higher learning ability. Nonetheless, it is encouraging to see that more than 50% of the teachers for the other courses are also harnessing the power of dialogic teaching talk to engage students, stimulate and extend their thinking, and advance their learning and understanding.

In the survey, teachers were also asked to reflect on the kinds of learning talk they engaged their students in and indicate the frequencies. Table 9.2 shows the aggregated data.

The data in Table 9.2 informs the use of the basic repertoire of learning talk (Narrate and Explain). Less than 60% of teachers for all the courses frequently or mostly/always engage their students in Narrate; and about 70% of the teachers for all the courses, except N(T), frequently or mostly/always engage their students in Explain. However, on the use of the larger oral repertoire of learning talks, the teachers for the IP course have provided more opportunities for their students to develop their repertoire of learning talk (Speculate, Explore, Analyse, Evaluate, Discuss, Argue, Justify, and Question). In fact, 55% or less of the teachers for the Express/N(A)/N(T) courses frequently or mostly/always engage their students in learning talks, such as Explore, Evaluate, Discuss (except for Express), and Justify.

9.5 Conclusion

The distinction between univocal and dialogic mathematics talks is at times difficult to discern. A mathematics talk can be a continuum between univocal and dialogic. Both univocal and dialogic can be appropriate forms of mathematics talk, depending on the instructional goals. However, instances of meaningful mathematics talk in which students are actively engaged in and are transforming one other's thinking are rare. Some challenges teachers faced when orchestrating meaningful mathematics talk include supporting students to make contributions that are productive to further the dialogue (Heaton, 2000; Staples, 2007); managing the mathematical direction that the mathematics talk takes (Jaworski, 1994; Sherin, 2002a; Silver & Smith, 1996);

Table 9.1 Use of teaching talk

Kind of talk/sample survey item	Course of study	% of respondents			
		Never/rarely	Sometimes	Frequently	Mostly/always
Rote • I use "I do, We do, You do" strategy • I emphasise basic facts/steps for students to memorise them • I provide students with sufficient questions to practise so as to develop procedural fluency • I ask students to recall past knowledge • I get students to automatise steps leading to a solution through repetitive exercises	IP	9.7	27.9	37.2	25.2
	Express	2.2	17.2	46.7	33.9
	N(A)	1.6	11.5	52.6	34.3
	N(T)	1.1	6.6	49.3	43.0
	All	2.5	15.5	47.6	34.4
Recitation • I ask direct questions to stimulate students' recall of past knowledge/check for understanding of concepts being developed in the lesson • I provide students with directed guidance (close-ended questions) when they face difficulty with a mathematical task they are doing, focussing them on the concept/skill necessary to do the task	IP	4.3	35.3	44.0	16.4
	Express	1.3	18.7	54.5	25.5
	N(A)	0.0	15.2	56.6	28.1
	N(T)	0.6	14.2	54.5	30.7
	All	1.2	18.8	54.1	26.0
Instruction/Exposition • I use the textbooks to introduce concepts/skills • I use exposition (teacher at the front talking to whole class) to explain mathematical ideas • I tell students how to do it when they face difficulty with a mathematical task they are doing • I explain what exemplary solutions of mathematics problems must contain • I ask questions to encourage reasoning, not just to elicit right answers	IP	15.9	27.6	41.4	15.2
	Express	6.1	27.1	45.5	21.4
	N(A)	4.5	29.4	50.2	15.9
	N(T)	4.1	29.1	43.6	23.2
	All	6.3	27.9	45.9	19.9

(continued)

Table 9.1 (continued)

Kind of talk/sample survey item	Course of study	% of respondents			
		Never/rarely	Sometimes	Frequently	Mostly/always
Discussion • I use examples and non-examples to engage students in discussion to make sense of a concept • I focus on mathematical processes (such as compare and contrast, logical reasoning) to facilitate the development of concepts or student understanding • I lead whole class discussion (with guided questions) to facilitate the development of concepts	IP	0.0	15.5	56.9	27.6
	Express	1.4	21.1	55.1	22.4
	N(A)	2.2	24.5	55.2	18.1
	N(T)	2.3	26.9	47.0	23.9
	All	1.6	22.2	54.2	22.1
Dialogue • I exchange ideas with students on how to solve a problem • I ask students open-ended questions and allow them to build on one another's responses to develop concepts or clarify their understanding • I build on students' responses rather than merely receiving them	IP	2.3	18.4	58.6	20.7
	Express	2.5	31.1	49.6	16.8
	N(A)	3.1	37.1	47.9	11.9
	N(T)	3.4	31.4	50.4	14.8
	All	2.7	31.4	50.1	15.8

Note Due to rounding errors the percentages do not always add up to 100

Table 9.2 Use of learning talk

Kind of talk/sample survey item	Course of study	% of respondents			
		Never/rarely	Sometimes	Frequently	Mostly/always
Narrate • I get my students to provide answers or solutions (without any explanation) to my questions • I get my students to practise a similar problem after I have shown them how to do it on the board • I get my students to state/list what they have learnt at the beginning/end of the lesson	IP	18.4	35.6	36.2	9.8
	Express	16.0	30.3	34.8	18.9
	N(A)	17.4	28.0	33.8	20.8
	N(T)	17.0	24.2	31.4	27.3
	All	16.6	29.4	34.3	19.6
Explain • I get my students to explain how their solutions or how their answers are obtained • I get my students to teach/explain to another classmate while doing individual assigned seatwork • I get my students to explain how they would correct an error or a misconception that I have put on the board	IP	2.3	27.6	54.6	15.5
	Express	2.1	26.7	55.4	15.8
	N(A)	3.8	29.4	54.1	12.8
	N(T)	23.1	50.4	24.2	2.3
	All	2.5	27.0	54.4	16.1
Speculate • I get my students to predict outcomes	IP	1.7	12.1	69.0	17.2
	Express	2.6	32.4	47.9	17.1
	N(A)	5.3	32.5	51.7	10.6
	N(T)	11.4	30.7	47.7	10.2
	All	4.3	30.4	50.5	14.8
Explore • I get my students to develop concepts through exploratory/investigate activities • I get my students to work collaboratively as a group on mathematical tasks and discuss the solution method before presenting to the whole class • I get my students to explore alternative solution methods for a problem besides the one I have shown on the board	IP	2.9	40.8	48.9	7.5
	Express	6.3	49.4	36.3	8.0
	N(A)	8.6	52.1	34.2	5.1
	N(T)	5.7	43.9	40.5	9.8
	All	6.5	48.5	37.5	7.5

(continued)

Table 9.2 (continued)

Kind of talk/sample survey item	Course of study	% of respondents			
		Never/rarely	Sometimes	Frequently	Mostly/always
Analyse • I get my students to analyse why a procedure (that I have shown on the board) works or why a solution method makes sense • I get my students to review their mistakes and identify possible causes by themselves	IP	0.9	19.8	62.1	17.2
	Express	2.1	25.4	56.6	15.9
	N(A)	4.6	29.8	52.0	13.6
	N(T)	2.8	23.9	52.3	21.0
	All	2.7	25.7	55.5	16.2
Evaluate • I get my students to compare multiple procedures/solution methods I have shown on the board and evaluate the efficiency, appropriateness, ease of use, or other advantages/disadvantages • I get my students to critique one another's work presented on board so as to improve their understanding of concepts or elegance in their presentation/solution	IP	2.6	29.3	50.9	17.2
	Express	3.8	40.4	43.3	12.5
	N(A)	6.3	43.7	41.7	8.3
	N(T)	4.5	42.0	41.5	11.9
	All	4.4	40.4	43.4	11.9
Discuss • I get my students to develop concepts together with me through class discussion • I get my students to discuss features of a problem that I have shown on the board	IP	1.7	25.9	62.9	9.5
	Express	4.5	36.6	47.2	11.7
	N(A)	5.3	45.0	45.0	4.6
	N(T)	8.5	41.5	36.4	13.6
	All	4.9	38.2	46.7	10.2
Argue • I get my students to offer alternative solution method(s) to a problem I have shown on the board	IP	0.0	24.1	56.9	19.0
	Express	0.3	33.7	53.2	12.9
	N(A)	3.3	47.7	41.7	7.3
	N(T)	1.1	37.5	45.8	15.9
	All	1.0	36.5	49.9	12.6

(continued)

Table 9.2 (continued)

Kind of talk/sample survey item	Course of study	% of respondents			
		Never/rarely	Sometimes	Frequently	Mostly/always
Justify • I get my students to justify why their solution(s) to a problem is different from the one I have put on the board • I get my students to defend and explain to classmate(s) why their approach/method to solve a problem is better	IP	4.3	41.4	49.1	5.2
	Express	8.0	39.1	42.9	10.0
	N(A)	11.6	40.7	43.4	4.3
	N(T)	12.5	33.5	41.5	12.5
	All	9.1	38.9	43.4	8.6
Question • I get my students to ask questions when they do not understand • I get my students to ask self/classmate questions to check their understanding • I get my students to ask questions (such as "what if") to probe further or for deeper understanding	IP	4.6	23.6	52.3	19.5
	Express	5.1	29.1	46.8	19.0
	N(A)	7.3	30.9	45.9	15.9
	N(T)	5.7	25.0	45.8	23.5
	All	5.6	28.5	46.9	19.0

Note Due to rounding errors the percentages do not always add up to 100 maintaining a 'common ground' which enables all students to follow the mathematical direction and to contribute appropriately (Staples, 2007); respecting the students' claims that are mathematically incorrect while trying to transform them and support the development of appropriate mathematical ideas (Chazan & Ball, 1999; Staples, 2007); seeing beyond one's own long-held and taken-for-granted mathematical ideas in order to hear and work with students' ideas (Heaton, 2000); creating appropriate norms for talking and interacting in the classroom (Cobb, 2000; Lampert, 2001); and most crucially, the teachers' sense of efficacy in anticipating and preparing for their role in instruction (Sherin, 2002b; Smith, 1996, 2000).

We also have to acknowledge the reality of a teacher's classroom, which includes the competing demands of depth versus breadth in content coverage, the students' differing abilities and interests, and time constraints. These factors often influence the learning goals, which in turn influence the kinds of mathematics talk. Thus, extensive mathematics talk may not be included in everyday lessons. Nonetheless, we encourage teachers to refrain from telling too much but to probe for students' ideas. Our data reveals that the mathematics talks enacted in our Singapore classrooms are often straddling between univocal and dialogic such that no one kind of talk is dominant over a significant period of time in a lesson. The two groups of univocal and dialogic talks are not mutually exclusive, and all kinds of talks have their place. We encourage teachers to continue to strive to engage the students in more dialogic mathematics talks so that they can acquire a deeper understanding of mathematics when they use their own responses, as well as those of their peers and teacher, as thinking devices.

References

Alexander, R. (2004). *Towards dialogic teaching: Rethinking classroom talk* (1st ed.). York: Dialogos.

Alexander, R. (2010). *Dialogic teaching essentials.* Cambridge: Cambridge.

Atwood, S., Turnbull, W., & Carpendale, J. I. (2010). The construction of knowledge in classroom talk. *The Journal of the Learning Sciences, 19*(3), 358–402.

Barnes, D. (1992). *From communication to curriculum.* Portsmouth, NH: Boynton.

Barnes, D. (2008). Exploratory talk for learning. In S. Hodgkinson & N. Mercer (Eds.), *Exploring talk in school: Inspired by the work of Douglas Barnes* (pp. 1–12). Thousand Oaks, CA: Sage.

Chapman, M. (1991). The epistemic triangle: Operative and communicative components of cognitive development. In M. Chandler & M. Chapman (Eds.), *Criteria for competence: Controversies in the conceptualization and assessment of children's abilities* (pp. 209–228). Hillsdale, NJ: Erlbaum.

Chazan, D., & Ball, D. L. (1999). Beyond being told not to tell. *For the Learning of Mathematics, 19*(2), 2–10.

Cobb, P. (2000). The importance of a situated view of learning to the design of research and instruction. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 45–82). Westport, CT.: Ablex.

Gallimore, R., & Tharp, R. G. (1990). Teaching mind and society: A theory of education and schooling. In L. C. Moll (Ed.), *Vygotsky and education: Instructional implications and* *applications of sociohistorical psychology* (pp. 175–205). Cambridge: Cambridge University Press.

Heaton, R. (2000). *Teaching mathematics to the new standards: Relearning the dance*. New York: Teachers' College Press.

Hogan, D., Rahim, R. A., Chan, M., Kwek, D., & Towndrow, P. (2012). Understanding classroom talk in secondary three mathematics classes in Singapore. In B. Kaur & T. L. Toh (Eds.), *Reasoning, communication and connections in mathematics: Yearbook 2012* (pp. 169–197). Association of Mathematics Educators.

Jaworski, B. (1994). *Investigating mathematics teaching: A constructivist enquiry*. London: Falmer Press.

Kaur, B. (1999). Mathematics classrooms of Singapore schools: Implications for fostering thinking and creativity. Paper presented at *8th Southeast Asian Conference on Mathematics Education (Technical papers): Mathematics for the 21st Century*. Philippines: Ateneo De Manila University.

Kaur, B. (2009). Characteristics of good mathematics teaching in Singapore grade 8 classrooms: A juxtaposition of teachers' practice and students' perception. *ZDM Mathematics Education, 41*(3), 333–347.

Kazemi, E., & Stipek, D. (2009). Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Journal of Education, 189*(1–2), 123–137.

Knuth, E., & Peressini, D. (2001). Unpacking the nature of discourse in mathematics classrooms. *Mathematics Teaching in the Middle School, 6*(5), 320.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven: Yale University Press.

Lampert, M., Blunk, M. L., & Pea, R. (Eds.). (1998). *Talking mathematics in school: Studies of teaching and learning*. Cambridge: Cambridge University Press.

Lotman, Y. M. (1988). Text within a text. *Soviet Psychology, 26*(3), 32–51.

Nathan, M. J., & Knuth, E. J. (2003). A study of whole classroom mathematical discourse and teacher change. *Cognition and Instruction, 21*(2), 175–207.

Piaget, J., & Inhelder, B. (1969). *The psychology of the child*. New York: Basic Books.

Sherin, M. G. (2002a). A balancing act: Developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education, 5*(3), 205–233.

Sherin, M. G. (2002b). When teaching becomes learning. *Cognition and Instruction, 20*(2), 119–150.

Silver, E. A., & Smith, M. S. (1996). Building discourse communities in mathematics classrooms: A worthwhile but challenging journey. In P. C. Elliott (Ed.), *Communication in mathematics, K-12 and beyond: 1996 yearbook* (pp. 20–28). Reston, VA: NCTM.

Sinclair, J., & Coulthard, M. (1992). Towards an analysis of discourse. In M. Coulthard (Ed.), *Advances in spoken discourse analysis* (pp. 1–34). London: Routledge.

Smith, J. P. (1996). Efficacy and teaching mathematics by telling: A challenge for reform. *Journal for Research in Mathematics Education, 27*(4), 387–402.

Smith, M. S. (2000). Balancing old and new: An experienced middle school teacher's learning in the context of mathematics instructional reform. *The Elementary School Journal, 100*(4), 351–375.

Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. *Cognition and Instruction, 25*(2–3), 161–217.

Van Zoest, L. R., & Enyart, A. (1998). Discourse, of course: Encouraging genuine mathematical conversations. *Mathematics Teaching in the Middle School, 4*(3), 150–157.

Weaver, D., Dick, T., & Rigelman, N. R. (2005). Assessing the quality and quantity of student discourse in mathematics classrooms. *Curriculum and Instruction Faculty Publications and Presentations*, 20.

Wertsch, J. V. (1991). *Voices of the mind: A sociocultural approach to mediated action*. Cambridge, Massachusetts: Harvard University Press.

Wertsch, J. V., & Toma, C. (1995). Discourse and learning in the classroom: A sociocultural approach. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 177–192). NJ: Hillsdale.

Lai Fong Wong has been a mathematics teacher for over 20 years. For her exemplary teaching and conduct, she was given the President's Award for Teachers in 2009. As a Head of Department (Mathematics), a Senior Teacher and then a Lead Teacher for Mathematics, she set the tone for teaching the subject in her school. Recipient of a Post-graduate Scholarship from the Singapore Ministry of Education she pursued a Master of Education in Mathematics at the National Institute of Education. Lai Fong is active in the professional development of mathematics teachers. Presently, she is involved in several Networked Learning Communities looking at ways to infuse mathematical reasoning, metacognitive strategies and real-life context in the teaching of mathematics. She is currently seconded as a Teaching Fellow in the National Institute of Education.

Berinderjeet Kaur is a Professor of Mathematics Education at the National Institute of Education in Singapore. She holds a Ph.D. in Mathematics Education from Monash University in Australia. She has been with the Institute for the last 30 years and is one of the leading figures of Mathematics Education in Singapore. In 2010, she became the first full professor of Mathematics Education in Singapore. She has been involved in numerous international studies of Mathematics Education and was the Mathematics Consultant to TIMSS 2011. She was also a core member of the MEG (Mathematics Expert Group) for PISA 2015. She is passionate about the development of mathematics teachers and in turn the learning of mathematics by children in schools. Her accolades at the national level include the public administration medal in 2006 by the President of Singapore, the long public service with distinction medal in 2016 by the President of Singapore and in 2015, in celebration of 50 years of Singapore's nation building, recognition as an outstanding educator by the Sikh Community in Singapore for contributions towards nation building.

Cherng Luen Tong is a research assistant at the National Institute of Education. He is a trained Engineer from National Technological University. He made his mid-career switch to the teaching profession and taught mathematics in secondary schools for more than 10 years. Educational research arena was an uncharted path. The journey as a research assistant helped him grow interested in mathematics problem solving as well as statistical analysis.

Chapter 10
The Enacted Curriculum—Students' Perspectives of Good Mathematics Lessons in Singapore Secondary Schools

Ngan Hoe Lee, Berinderjeet Kaur, and Liyana Safii

Abstract This chapter presents the characteristics of good mathematics lessons from the lens of typical secondary school students in Singapore. This chapter begins by examining the student perception in relation to the five inter-related problem-solving components embodied in the Singapore School Mathematics Curriculum Framework (SSMCF): concepts, skills, processes, metacognition and attitudes. Data from post-lesson student interviews which were stimulated by videos of the lesson revealed that the development of proficiencies in mathematics skills was most commonly emphasised in the "highs" of mathematics lessons while emphasis on metacognitive strategies was the least emphasised. This was true for all four courses of study (i.e. Integrated Programme, Express, Normal (Academic) and Normal (Technical)). The chapter further categorises the student data into teacher approaches and class activities that have been perceived by the students as the highs of mathematics lessons. While the perceived value for teacher approaches differ across all four courses of study, class practice and peer discussion were the most commonly cited class activities for all courses of study. Findings from the study provide important implications on the way to better engage students in the teaching and learning of mathematics.

Keywords Students' lens · Good mathematics lesson · Singapore · Secondary mathematics

N. H. Lee (✉) · B. Kaur · L. Safii
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: nganhoe.lee@nie.edu.sg

B. Kaur
e-mail: berinderjeet.kaur@nie.edu.sg

L. Safii
e-mail: liyana.safii@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_10

10.1 The Student Perspective

Classroom instructions are no longer teacher-centred. Students are increasingly playing an active role in classroom learning. The shift towards a collaborative participation of teachers and students suggests that the mechanisms underlying teaching and learning in the classroom cannot be construed only by examining the processes that encapsulates the teacher's participation in the classroom. In other words, "as learning is dependent upon the situations and circumstances in which it is engendered and the feelings these situations provoke in students, any attempt to improve mathematics teaching must take into account both teacher practice, student practice and their responses to each other's practice" (Kaur, 2008, p. 951). In relation to the learning of mathematics, this could mean that teaching and learning is perceived as "the product of interactions among the teacher, the students and the mathematics" (Kilpatrick, Swafford, & Findell, 2001). This implies that the students' perception and participation in the classroom should be emphasised along with the teachers' perception and participation in the classroom (Clarke, Keitel, & Shimizu, 2006).

Research on classroom instructions through the teacher's lens (e.g. teacher beliefs and perceptions) have been widely explored and detailed in the literature. However, the understanding of the teaching processes in the classrooms as experienced by the learners could also provide valuable insights on how teachers deliver their lessons. Ahmad and Aziz (2009) highlight that student perception plays an important role in research on classroom instructions as their perception is "coloured by challenging and interesting experiences that allow them to observe the learning and teaching behaviours more intimately than the teacher" (p. 19). This suggests that students' perceptions not only promote heightened awareness of their own classroom learning experiences and their teachers' classroom instructions, but also forms part of a feedback channel for teachers to reflect and improve on their classroom instructions (Ahmad & Aziz, 2009). The study of student perception thus can provide valuable contributions in the improvement of teaching and learning in the classroom.

Prior research have explored mathematics teaching through the learner's perspective, providing insights into what students consider valuable for their classroom learning. These studies are varied and include student perception on what constitutes good teaching, effective teaching or a good teacher (e.g. Attard, 2011; Kaur, 2009; Martinez-Sierra, 2014; Murray, 2011; Shimizu, 2009; Wang & Hsieh, 2017). Student perception gathered from these studies, however, have been mixed, possibly attributed by various social and cultural norms that underlie the educational system in different countries. Pang (2009) highlights that existing classroom instructions need to be studied in relation to these norms in order to understand the beliefs and values on which these practices are based upon. This is also emphasised in The Learner's Perspective Study (LPS), a large international comparative study on mathematics education which takes into account students' perceptions in the study of mathematics classrooms, learning and student outcomes around the world (Clarke, Keitel, & Shimizu, 2006). The researchers of the LPS note that the findings from the study showed how "culturally-situated are the practices of classrooms around the world and the extent to which students are collaborators with the teacher, complicit in the development and enactment of patterns of participation that reflect individual, societal and cultural priorities and associated value systems" (Clarke, Emanuelsson, Jablonka, & Mok, 2006, p. 1).

Singapore was also part of the LPS. It was discovered that Grade 8 (Year 2 in Singapore secondary school) students in Singapore perceived a good mathematics lesson as one where their teachers adopted some of the following classroom instructions (Kaur, 2009, p. 343):

1. Explaining mathematical concepts and demonstrating steps of procedures clearly
2. Showing demonstrations, or using manipulatives or real-life examples to make it easier for complex ideas to be understood
3. Reviewing previously taught knowledge
4. Introducing new knowledge
5. Giving individual or whole-class feedback using student individual work or group presentations
6. Giving clear instructions for activities that are expected to be completed during or after class
7. Providing students with opportunities to work on interesting activities individually or collaboratively in small groups
8. Allocating sufficient practices as part of exam preparation.

Drawing upon the same motivations that underlie the LPS, the current study examines students' learning experiences through their perspectives. We first detail the Singapore School Mathematics Curriculum Framework (SSMCF) to understand the context of mathematics teaching and learning in Singapore. We proceed to discuss the data and findings from one part of the project (detailed in Chapter 2) which examines Singapore secondary school students' perceptions of good mathematics lessons. These perceptions would be presented in the form of characteristics of good mathematics lessons, also referred to as the *highs* of the lessons. The highs of the lessons include moments of the lessons that the students feel would constitute part of a good lesson. These lesson characteristics would be analysed in relation to the five problem-solving components in the SSMCF (i.e. concepts, skills, processes, metacognition and attitudes). We also draw upon the data of students from four courses of study (i.e. Integrated Programme (IP), Express, Normal (Academic) (N(A)) and Normal (Technical) (N(T))) to help us understand the perceptions of students with diverse student learning profile. Details of the four courses of study are provided in Chapter 1, Sect. 1.2. The student data was examined from two perspectives—the teacher approach and class activity.

10.2 Mathematics Instruction in Singapore

10.2.1 Singapore School Mathematics Curriculum Framework

As briefly introduced in Chapter 1, mathematics instruction in Singapore is guided by a robust problem-solving framework for the teaching, learning and assessment of mathematics in the classroom. Known as the Singapore School Mathematics Curriculum Framework (SSMCF), the framework was developed in 1990, and has since undergone several changes and been an integral part of mathematics curriculum enactment in Singapore (Ministry of Education [MOE], 2012). The framework was constructed with the intention of providing teachers with directions to create a "more engaging, student-centred, and technology-enabled learning environment" as well as to "promote greater diversity and creativity in learning" (MOE, 2012, p. 17). The SSMCF (see Chapter 1, Fig. 1.2) draws upon five inter-related competencies that focus on mathematical problem solving: conceptual understanding, skills proficiency, mathematical processes, metacognition and attitudes, to develop students' ability in solving a wide range of problems including straightforward and routine tasks to complex and non-routine ones (MOE, 2018b). This is in line with Singapore Ministry of Education's (MOE) intention to equip students with twenty-first century competencies to prepare them for challenges brought about by the fast-changing world attributed by globalisation, shift in demographics and advancement in technology (MOE, 2018a). These twenty-first century competencies include skills such as critical and inventive thinking, and communication, collaboration and information skills. In the next section, we discuss the five components of SSMCF in further detail.

Introduce/construct mathematical concepts. Mathematical concepts in numbers, algebra, geometry, probability and statistics, and calculus are "connected and interrelated" (MOE, 2018b, p. 10). These concepts can be represented through numerical/tabular, pictorial, graphical, verbal, symbolic (equations or expressions) and physical/concrete (Cleaves, 2008). Goldin and Kaput (1996) postulate that students' comprehension of mathematical ideas is influenced by the mathematical representations that teachers use. In particular, conceptual understanding can be fostered through the use of multiple representations (Donovan & Bransford, 2005). Students who grasp a coherent understanding of mathematical concepts are able to "recognise the idea embedded in a variety of qualitatively different representational systems, flexibly manipulate the idea within given representational systems and accurately translate the idea from one system to another" (Lesh, Post, & Behr, 1987, p. 36). As such, teachers are encouraged to adopt a wide range of learning experiences that involve "hands on activities and the use of technological aids to help students relate abstract mathematical concepts with concrete experiences" (MOE, 2012, p. 15).

Develop proficiencies in mathematical skills . Mathematical skills include "carrying out the mathematical operations and algorithms and in visualising space, handling data and using mathematical tools" (MOE, 2018b, p. 10). Mathematical skills also comprise students' ability to use software in the learning and application of mathematics, especially in today's classroom settings where ICT tools are increasingly being incorporated into classroom learning. Bloom (1968) posits that for students to develop these mathematical skills, teachers should establish clear learning goals and complement student learning with formative assessments that serve as a medium for determining students' level of mastery. It is, however, important that mathematical skills are "taught with an understanding of the underlying mathematical principles and not merely as procedures" (MOE, 2012, p. 15). This means that the acquisition of both instrumental and relational understanding should be involved in the development of procedural fluency (Skemp, 1987). In other words, the acquisition of procedural skills should not just focus on the "how" but should also focus on the "why".

Emphasise on mathematical processes. Mathematical processes that are involved in the acquisition and application of mathematical knowledge require students to use certain skills. As identified in the SSMCF (MOE, 2018b, p. 11), these include:

1. *Abstracting and reasoning*—While abstraction is what makes mathematics powerful and applicable, justifying a result, deriving new results and generalising patterns involve reasoning;
2. *Representing and communicating*—Expressing one's ideas, solutions and arguments to different audiences involves representing and communicating and the use of notations in the mathematics language.
3. *Applying and modelling*—Applying mathematics to solve real-world problems often involves modelling, where reasonable assumptions and simplifications are made so that problems can be formulated mathematically, and where mathematical solutions are interpreted and evaluated in the context of the real-world problems.

These skills reflect the critical and inventive thinking competencies in the twenty-first Century Competencies Framework (MOE, 2018a). In particular, the skills required for mathematical problem solving could foster students' ability to "think critically" and "think out of the box" (MOE, 2018a). For students to develop proficiencies in such mathematical processes, teachers are encouraged to provide sufficient opportunities for students to engage in problem solving that involves complex and non-routine tasks (MOE, 2018b, p. 10).

Emphasise on metacognitive strategies. Metacognition, as defined by Flavell (1976), refers to "one's knowledge concerning one's own cognitive processes and products or anything related to them… Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective" (p. 232). Simply put, metacognition involves one's "awareness of, and the ability to control one's thinking processes, in particular the selection and use of problem-solving strategies" (MOE, 2018b, p. 12). These processes also involve students' ability to monitor and regulate their own thinking and learning. Metacognition, particularly in the learning of mathematics, in essence involves three facets—awareness, monitoring and regulating (Lee, Ng, & Yeo, 2019). To promote development of strategies that support metacognition, the SSMCF has advocated that teachers provide students with opportunities to "solve non-routine or open-ended problems" to provide opportunities for students to discuss their solutions, think aloud and reflect on what they are doing, keep track of how things are going and make changes when necessary (MOE, 2018b, p. 12).

Imbue desired learning attitudes. Attitudes towards mathematics learning reflect the affective facet of learning that includes one's "belief and appreciation of the value of mathematics, one's confidence and motivation in using mathematics, and one's interests and perseverance to solve problems using mathematics" (MOE, 2018b, p. 12). In line with Singapore's move to achieve balance between academic rigour and joy of learning, the Singapore MOE (2017) advocates that learning should promote students' discovery of their interests and passions, and love in the things that they do. In other words, learning should go beyond external motivations and achieving good grades. Teachers are recommended to incorporate fun learning experiences in the acquisition of knowledge and skills to instil the joy of learning among students. In particular, teachers are encouraged to use a wide range of resources to cater to varied student interest (variety), use these resources sufficiently (opportunity) and make connections between these resources and mathematics learning (linkage) (Yeo, 2018). These types of instructions are aimed at building students' desired attitudes towards the learning of mathematics.

10.3 Singapore Secondary School Students' Perspectives of Good Mathematics Lessons

To document students' perspectives of good mathematics lessons, data was collected through post-lesson video stimulated interviews that were conducted with 447 focus students. These focus students were students of the 30 experienced and competent teachers who were involved in the first phase of the project—the video segment—where their lessons were recorded (detailed in Chapter 2).

Two parts of the post-lesson student interviews which were stimulated by videos of the lesson were analysed when identifying characteristics of good mathematics lessons. These parts involved portions of the interview where the focus students were asked to identify the highs of a particular mathematics lesson in which they were the focus students. The highs of mathematics lessons were referred to as moments of the lessons that the students felt would constitute part of a good lesson. In particular, students were asked, "Can you share with me what the highs of this lesson were?" and were provided with the recorded video of that particular lesson. The recorded video served to help students in recalling how the lesson was taught. The students were instructed to fast forward the recorded lesson video to the parts of the lesson that they perceive to be the highs of the lesson.

A total of 636 responses were collected from this part of the interview (i.e. 108 from IP, 196 from Express, 194 from N(A) and 138 from N(T)). Most of the students shared at least one high moment of the lesson that they sat for. These responses were categorised into two perspectives: teacher approach and class activity. The findings will be presented in two parts. In the first part, we outline the teacher approaches and class activities in relation to the five problem-solving competencies as embodied in the SSMCF (i.e. concepts, skills, processes, metacognition and attitudes). In the second part, we delve into the types of teacher approaches and class activity that were valued by the students. The student interview data is also compared across the four courses of study.

10.3.1 Problem-Solving Competencies in the SSMCF

Analysis of the post-lesson student interviews revealed that the focus students perceived a variety of teacher approaches and class activities as the highs of mathematics lessons. Table 10.1 shows how commonly cited the teacher approaches and class activities were in relation to the five problem-solving competencies embodied in the SSMCF. The interview data revealed that the development of proficiencies in mathematics skills (42%) was most commonly emphasised in the highs of mathematics lessons, followed by emphasis on mathematical processes (27%), imbuement of desired learning attitudes (15%) and introduction of concepts to students or engagement of students in constructing concepts (12%). The emphasis on metacognitive strategies was the least emphasised (4%) in the highs of mathematics lessons.

The data also revealed that, generally, students across all four courses of study placed similar emphasis on these competencies. Regardless of the courses of study they were in, it appeared that students emphasise most on the development of proficiencies in mathematics skills (at least 40% for all courses of study) and least on metacognitive strategies (at most 5% for all courses of study) in the highs of mathematics lesson. As compared to students in other courses of study, students in the Express course (5%) appeared to place lesser emphasis on the introduction of

Table 10.1 Student perception of good mathematics lessons in relation to problem-solving competencies

Problem-solving competencies	Percentage of responses				
	IP (n = 108)	Express (n = 196)	N(A) (n = 194)	N(T) (n = 138)	Total (N = 636)
Skills	41	43	43	40	42
Processes	35	25	28	21	27
Attitudes	5	22	13	17	15
Concepts	14	5	11	22	12
Metacognition	5	5	5	0	4

concepts or engagement of students in constructing concepts in the highs of mathematics lessons. On the other hand, students in the IP course (5%) seemed to emphasise the imbuement of desired learning attitudes comparatively lesser than students in the other courses of study.

10.3.2 Teacher Approach and Class Activity

The teacher approaches and class activities were further examined to understand the nature of the highs of mathematics lessons, as identified by the students, as well as the reasons underlying their choices. Table 10.2 shows the percentages of responses for the different types of teacher approach and class activity that had been cited by the students. For the purpose of discussion, only the teacher approaches and class activities that recorded a frequency of at least 10 student responses, i.e. at least more than 1% of the total responses, would be discussed.

A comparison of the teacher approaches and class activities across all four courses of study revealed some similarities and differences. Class practice and peer discussion were commonly cited by students in all four courses of study as the highs of mathematics lessons. Apart from class practice and peer discussion, students in the IP course deemed the parts of the lessons where their teachers reviewed student work (12%) as the highs of mathematics lessons. On the other hand, students in the Express course tended to value teachers' attempt to make jokes (6%) and share alternative ways of solving problems (5%) during lessons. For students in the N(A) course, the teachers' attempt to review student work (9%) and explain how to solve a worked example (8%) were some of the high moments of the lessons. Six percents of students in the N(A) course also identified assessment for learning, such as use of the entry and exit cards, as the highs of the mathematics lessons. In addition, students in the N(T) course appeared to appreciate teachers' use of manipulatives when concepts were demonstrated in the lesson (7%).

With reference to the video recorded lessons, Table 10.3 details the reasons underlying the students' perceived value on the various teacher approaches and class activities.

As seen in our inferences made in Table 10.3, the students' reasons on their choice of teacher approaches and class activities add pedagogical value to mathematics lessons. It appears that the characteristics of mathematics lessons that students thought as important also reflect pedagogically sound practices. In other words, the students seemed to value the importance of pedagogically sound practices in the choice of teacher approaches and class activities that they had identified.

Table 10.2 Teacher approaches and class activities of good mathematics lessons as perceived by students in the different courses of study

Problem-solving competencies	Characteristics	Percentage of responses			
		IP ($n = 108$)	Express ($n = 196$)	N(A) ($n = 194$)	N(T) ($n = 138$)
Teacher approach					
Skills	Explaining how to solve a worked example • Convey expectations explicitly to students when explaining how to solve a worked example by showing the process of arriving at the solution and demonstrating the proper steps in the workings	–	–	8	–
Skills	Reviewing student work • Provide timely feedback • Provide opportunities for students to learn from strengths and weaknesses of their peers' work	12	–	9	–
Attitudes	Making jokes • Inject jokes into the lessons, making classroom learning experiences more enjoyable than otherwise	–	6	–	–
Concepts	Demonstrating a concept using manipulatives • Use manipulatives to enhance students' understanding of concepts, especially abstract concepts, through demonstration or visualisation of such concepts	–	–	–	7
Metacognition	Sharing alternative ways of solving problems • Encourage students to compare and contrast different problem-solving approaches, fostering their metacognitive strategies	–	5	–	–

(continued)

Table 10.2 (continued)

Problem-solving competencies	Characteristics	Percentage of responses			
		IP ($n = 108$)	Express ($n = 196$)	N(A) ($n = 194$)	N(T) ($n = 138$)
Class activity					
Skills	Assessment for learning • Opportunities provided for students to check their understanding of acquired knowledge and skills • Opportunities provided for students to monitor their progress in learning	–	–	6	–
Skills	Class practice • Opportunities provided for students to apply the knowledge and skills that they have learned through a variety of mathematical problems	21	29	16	23
Processes	Peer discussion • Opportunities provided for students to express their ideas/solutions and their underlying reasons or arguments that could sharpen their mathematical thinking through collaborative learning	11	7	8	8

Note The table only includes teacher approaches and class activities that recorded a frequency of at least 10 student responses

Table 10.3 Reasons underlying the students' perceived value on the various teacher approaches and class activities

Characteristic	Lesson context	Student interview response	Inferences
Explaining how to solve a worked example	Topic: Vectors Teacher 2 (T2) (N(A)) in Lesson 7 explained a worked example that involved the expression of the relationship between two vectors (i.e. $\overrightarrow{BC}$ and $\overrightarrow{DA}$) as an equation. The conditions of the vectors are such that they share the same length but have opposite directions. In his explanation, the teacher emphasised why $\left\|\overrightarrow{BC}\right\| = \left\|\overrightarrow{DA}\right\|$ cannot be a possible answer as this equation expresses a weak relationship. He added that the relationship between the two vectors should be expressed as $\overrightarrow{BC} = -\overrightarrow{DA}$ as this equation reflects the given conditions of the vectors (i.e. same length and opposite directions)	**Student (P13)**: So first I thought you can put like that. The magnitude, everything, but I learned that it's, you cannot put that. Because it's that, that, the two lengths was opposite direction, so you cannot put the magnitudes, so it (this part of the lesson) was high. I always put the line (the symbol for absolute value) one **Interviewer (I)**: You always put the magnitude **P13**: So I learned that you cannot put that. **Because the length was opposite, the arrow was opposite, one is showing below, one is showing on top, so you cannot put the magnitude** **I**: The magnitude not equal **P13**: Ya **I**: If they are pointing different directions **P13**: This was, wait ah. Ya, it's the length that is the same. But, the relation was opposite, the strong relation was opposite direction **I**: Ok. So by doing that, it doesn't tell the opposite direction **P13**: Ya **it is only telling that the length is the same. It doesn't tell that it is opposite. You need to put the negative sign** **I**: So at that point, you suddenly realise, it cannot… **P13**: Ya I suddenly realise that you cannot put that	The explanation of a worked example helps the student to understand how certain mathematical expressions can/cannot be expressed in mathematical problems

(continued)

Table 10.3 (continued)

Characteristic	Lesson context	Student interview response	Inferences
Reviewing student work	Topic: Logarithm Teacher 12 (T12) (IP) in Lesson 6 used students' work to explain the correct solutions to logarithmic equations which involved common and natural logarithms. While reviewing the students' solutions, the teacher highlighted common mistakes and discussed more efficient way(s) of solving such equations	**Student (P11)**: I would say the high point is when she starts pointing out some common mistakes that are made so that we can also **get a better understanding on the topic** because she starts going through the answers **then she wants to see whether we have different methods in solving the questions.** So she takes our workings and shows it on the projector, and if you make any mistakes, **she will tell us where we have gone wrong and how we should prevent (it)** **P11**: It's good to show us different ways of doing it because sometimes we may do much longer method when there's easier way to solve **Interviewer (I)**: So what did you learn from them? Why did you find (this moment) high? **P11**: Because if we know how to do the question in a faster method, then it will save more time. So have more time to work on other harder questions **I**: Then why do you say that it's a high? **P11**: Because it also help us get a good understanding on how to apply the different laws of logarithm. Then she uses our friend's workings then she writes it on the board to explain to us what we should not do 'cause my friend only derived at one solution but there were actually two because he made a wrong assumption on the question **I**: So when the teacher does this pointing out during your example, something wrong, and talking about it, this kind of discussion, how do you find it? **P11**: I find it very helpful because sometimes we also make this kind of mistakes in our test papers and we actually do not know where we went wrong. So if she goes through it like that we are able to know how to prevent ourselves from making these kind of mistakes. Maybe sometimes we make the wrong assumptions but if we understand the laws better then we will not make these kind of assumptions	The student finds value when the teacher uses their peers' work to explain how to obtain the correct solution to the problem; the student is then able to learn from mistakes made by the peers and also learn to use more efficient problem-solving approaches

(continued)

Table 10.3 (continued)

Characteristic	Lesson context	Student interview response	Inferences
Making jokes	Teacher 22 (T22) (Express) in Lesson 1 responded to student's mischievous comment by making a witty remark	**Student (P02)**: I like how she can give very witty remarks to those mischievous students. It kind of **makes it funny, interesting**, because like maths lesson can be a little boring and like, because it's the first lesson of the day and people are like "ha, so tired", but she can give witty remarks that kind of makes it like you kind of like, **that little laugh can give you a lot of adrenaline to enjoy the day** **Interviewer (I)**: Could you remember anything, any one of those witty remarks? **P02**: She gave it to many students, a few students so it's like, Student A. She likes to tease Student A because Student A is like very, like to give a lot of unnecessary comments which like make people like "omg", eye roll thing. So she gives a remark, he will keep quiet, it's kind of funny	Teacher's witty remarks make lessons enjoyable
Demonstrating a concept using manipulatives	<u>Topic: Mensuration</u> Teacher 9 (T9) (N(T)) in Lesson 6 had prepared a circle that was cut from a large piece of vanguard sheet for the lesson. While showing the students the circle she had prepared, she asked the students to think about whether a cone could be created from that particular circle. After eliciting responses from the students, the teacher demonstrated how a cone could be formed from a circle by cutting a minor sector from the circle. She showed the students that by cutting a minor sector from the circle, she was also able to form a major sector. Throughout the demonstration, the teacher emphasised on the part of the circle that she has cut as a minor sector, and the remaining part of the circle as a major sector. She concluded the demonstration by highlighting that cones (without a base) are created from sectors	**Student (P11)**: It's a high point because I can understand it. I understand when she told me that, she explained which one is minor and which one is major **Interviewer (I)**: So when you saw this do you find that it's a very good way to express it? **P11**: Yes **I**: Do you like it when teacher bring in manipulatives? **P11**: Yeah **I**: How come? **P11**: **So that I'm able to know more, better** **I**: Know more better, what do you mean by know more? **P11**: If she write it on whiteboard, I don't know which one is, like draw it out on whiteboard, I don't know what is minor, what is major. **When she cut down, I do understand that the small one is minor, and the big one is major. It's clearer**	The use of manipulatives helps the student in visualising concepts

(continued)

Table 10.3 (continued)

Characteristic	Lesson context	Student interview response	Inferences
Sharing alternative ways of solving problems	<u>Topic: Vectors</u> Teacher 1 (T1) (Express) in Lesson 3 explained a worked example that involved the simplification of the expression, $\overrightarrow{PQ} - \overrightarrow{PS}$, based on a given diagram. The teacher elicited answers from the students but realised that they were facing difficulties in tackling the problem as $\overrightarrow{PS}$ was expressed in negative form. To help the students, the teacher taught them to change the expression to $\overrightarrow{PQ} + \overrightarrow{SP}$ to avoid dealing with negative vectors.	**Student (P06)**: Like when she goes through the examples then you have to think through and yah, and she finds like the shortest possible methods so it might not be something that I already know of or the method that I use. So it's experience, because if in the future there are similar examples, then I will be able to do it in the shortest method… 'Cause I feel like I benefitted quite a lot from it. **Interviewer (I)**: In what way? **P06**: In a way that I managed to learn to, in a way **see this from a different point of view**, like I managed to, instead of really going through the vectors one by one**, I managed to cut short quite a few steps**	Sharing an alternative way to approach a problem teaches the student to be more metacognitive when solving problem
Assessment for learning	<u>Topic: Arc length, sector area and radian measures</u> Teacher 11 (T11) (N(A)) in Lesson 5 made her students complete an exit card before the lesson ended. The exit card required students to express i. a given angle in radian, in degree, and ii. a given angle in degree, in radian	**Student (P11)**: I think the exit card is quite helpful because we, in the end of the day, we try not to get people to help us, so it's basically on what we have learned, at the end of the day what we've catch as much as possible from her. Then as she mark our answers, **she understands that his understanding is this type, his level of, which part of his workings are weak, then from there she will slowly try to help us in our weak areas** **Interviewer (I)**: So you like doing exit card activity? **P11**: Yeah. **It just shows like we understand that, understand her teaching**, at least we also don't have to worry about different types of question that might be coming out	The exit card serves not just as an activity for the student to check his/her own understanding but also for teachers to identify and address students' weaknesses
Class practice	<u>Topic: Quadratic Equations</u> Teacher 21 (T21) (IP) in Lesson 1 made her students solve several quadratic equations (using the factorisation method) independently in class	**Student (P01)**: Solving the questions, because I'm not sure why actually but I just like to solve the equation and if I get the answer correct right, I will feel quite satisfied because **it shows that I'm making progress** **Interviewer (I)**: So it makes you feel good because you are making progress? **P01**: Yeah	Independent class practice allows the student to check their progression in learning

(continued)

Table 10.3 (continued)

Characteristic	Lesson context	Student interview response	Inferences
Peer discussion	Topic: Cumulative Frequency In the class of Teacher 7 (T7) (N(T)) in Lesson 4 students held discussions with their peers while attempting class practice tasks that were assigned to them. The tasks involved the use of a cumulative frequency graph which represents the examination marks of a group of students. Students were required to identify the students' marks in relation to concepts such as the median, lower quartile, 20th percentile and the interquartile range using the graph	**Student (P07)**: And the highest is, I think working with my friends to find out the same answer, so we get the same marks[a] together **Interviewer (I)**: So when you were saying that you get the same marks together, work together, how did you feel, why is it that there is a high? **P07**: For **me it's motivation for me to work harder**. Instead of working by myself, and **sometimes I do not have anyone to ask**, the teacher may be busy at times ah **I**: So which of the words to describe you, excited, happy? Achieved, motivated? **P07**: Motivated… I think instead of the teacher trying to explain the whole thing at once, he check with us if we still understand. So, so after he go through this, I work with my friends ah to find out which one, to confirm that which one is the, which part we supposed to use ah. **Because we also have different thinking ah. So we all use the same method, then we agree on each other's answer**	The student enjoys collaborative learning

[a] *Note* Marks here refer to a variable in the class practice task

10.4 Conclusion

The present study has enriched our understanding of how secondary students in Singapore consider a mathematics lesson to be a good one. In particular, the purpose of this chapter was to explore students' perceptions of valued teaching and learning experiences in mathematics classrooms, especially in relation to the context of mathematics instructions in Singapore and for students with various learning profiles.

The findings revealed insights on students' perception of good mathematics lessons in relation to the five problem-solving components embodied in the SSMCF. Students across all four courses of study appeared to be fairly consistent in what they considered as valuable aspects of mathematics lessons. In particular, students across all courses of study gave most priority to the proficiencies in mathematics skills and least priority to the emphasis of metacognitive strategies when considering the characteristics of good mathematics lessons. The lack of priority given to the emphasis of metacognitive strategies could be explained by the possible lack of perceived value in metacognitive strategies or students' lack of vocabulary to articulate their perceived value in relation to metacognitive strategies. The findings also revealed that as compared to other courses of study, students in the IP course gave lower priority to the imbuement of desired learning attitudes in mathematics lessons. This observed lack of priority could be attributed by IP students' self-sufficiency in cultivating the desired learning attitudes in the learning of mathematics. Moreover, with a climate that is heavily dependent on national examinations and placement, IP students might consider themselves to be already academically successful, and so do not place as much emphasis on developing interest or appreciation for mathematics. Thus, they might perceive the imbuement of desired learning attitudes in mathematics lessons as less necessary than students in other courses. The findings also showed that the preference for the use of manipulatives to demonstrate a concept appears to be distinctive of students in the N(T) course. Manipulatives are often used as a pedagogical resource tool to guide students in understanding abstract mathematical ideas through concrete experiences, especially for weaker students. Thus, the lack of priority given by students in other courses on the use of manipulatives could be attributed to lesser use of manipulatives in mathematics lessons taught by teachers in the IP, Express and N(A) courses.

The findings also highlighted eight key characteristics of good mathematics lessons identified by the students. Despite the difference in student learning profiles, it was observed that students across all courses of study appeared to value individual mathematical task attempts allocated in class (class practice) and the exchange of ideas with their peers (peer discussion). This suggests that students place importance on opportunities for mathematical application and checking for mastery of learning and skills, as well as collaborative learning. It is also interesting to note that five of these lesson characteristics—demonstrating a concept using manipulatives, assessment for learning, class practice, explaining how to solve a worked example and reviewing student work—are similar to the characteristics of good mathematics teaching observed in Kaur's (2009) study. In a nutshell, the characteristics of good mathematics lessons as viewed from the students' perspectives generally seem to resonate well with the framework that supports mathematics instructions in Singapore (i.e. the SSMCF). The students' perspectives provided an enhanced understanding of teaching and learning processes that occur in mathematics lessons as experienced by learners, and provided directions in better engaging our students in the teaching and learning of mathematics.

While the students perspectives of good mathematics lessons generally reflect classroom instructions advocated in the SSMCF, students appeared to be lacking in the ability to articulate what they deemed as important in the teaching and learning of mathematics or have a superficial awareness of mathematical strategies. Our findings thus call on teachers to provide support in the development of students' vocabulary that will help them to express clearly their needs or what is important to them in the teaching and learning of mathematics. For instance, teachers could provide more student exposure to the idea of metacognition as well as the teaching and learning of metacognitive strategies. In other words, the address of metacognition in the mathematics classroom may require a more deliberate rationalisation and articulation.

The findings also suggest that there could be value in emphasising heterogeneous grouping in mathematics lesson, as reflected in the students' perceived value in peer discussion during lessons. Piaget (1932) postulated that peer interaction has its own advantages; peer interaction helps students to identify and correct their misconceptions, and develop high-level cognitive architecture. Research on groupings in general, have been inconclusive as it appears that none of the group composition (i.e. homogenous or heterogeneous grouping) is equally advantageous for high, average and low-achieving students (e.g. Huang, 2009; Kaya, 2015; Saleh, Lazonder, & de Jong, 2007). However, low-achieving students seem to benefit from heterogeneous composition through motivation and stimulation from high-achieving students (Chang, Singh, & Filer, 2009). In particular, low-achieving students can benefit from better performance and higher motivation (Saleh, Lazonder, & de Jong, 2005).

References

Ahmad, F., & Aziz, J. (2009). Students' perception of their teachers' teaching of literature communicating and understanding through the eyes of the audience. *European Journal of Social Sciences, 7*(3), 17–26.

Attard, C. (2011). "My favourite subject is maths. For some reason no-one really agrees with me": Student perspectives of mathematics teaching and learning in the upper primary classroom. *Mathematics Education Research Journal, 23,* 363–377.

Bloom, B. S. (1968). Learning for mastery. Instruction and curriculum. Regional education laboratory for the Carolinas and Virginia, topical papers and reprints, number 1. *Evaluation Comment, 1*(2), 1–12.

Chang, M., Singh, K., & Filer, K. (2009). Language factors associated with achievement grouping in math classrooms: A cross-sectional and longitudinal study. *School Effectiveness and School Improvement, 20*(1), 27–45.

Clarke, D., Emanuelsson, J., Jablonka, E., & Mok, I. A. C. (2006). The learner's perspective study and international comparisons of classroom practice. In D. Clarke, J. Emanuelsson, E. Jablonka, & I. A. C. Mok (Eds.), *Making connections: Comparing mathematics classrooms around the world* (pp. 1–22). Rotterdam: Sense Publishers.

Clarke, D., Keitel, C., & Shimizu, Y. (2006). The learner's perspective study. In D. Clarke, C. Keitel, & Y. Shimizu (Eds.), *Mathematics classrooms in twelve countries: The insider's perspective* (pp. 1–14). Rotterdam: Sense Publishers.

Cleaves, W. P. (2008). Promoting mathematics accessibility through multiple representations jigsaws. *Mathematics Teaching in the Middle School, 13*(8), 446–452.

Donovan, S., & Bransford, J. (Eds.). (2005). *How students learn: History, mathematics, and science in the classroom.* Washington, DC: The National Academies Press.

Flavell, J. H. (1976). Metacognitive aspects of problem solving. In L. B. Resnick (Ed.), *The nature of intelligence* (pp. 231–235). Hillsdale, NJ: Lawrence Erlbaum.

Goldin, G. A., & Kaput, J. J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 397–430). Mahwah, NJ: Erlbaum.

Huang, M. (2009). Classroom homogeneity and the distribution of student math performance: A country-level fixed-effects analysis. *Social Science Research, 38*(4), 781–791.

Kaur, B. (2008). Teaching and learning of mathematics: What really matters to teachers and students? *ZDM—The International Journal on Mathematics Education*, *40*(6), 951–962.

Kaur, B. (2009). Characteristics of good mathematics teaching in Singapore grade 8 classrooms: A juxtaposition of teachers' practice and students' perception. *ZDM—The International Journal on Mathematics Education, 41*(3), 333–347.

Kaya, S. (2015). The effect of the type of achievement grouping on students' question generation in science. *The Australian Educational Researcher, 42*(4), 429–441.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Lee, N. H., Ng, K. E. D., & Yeo, J. B. W. (2019). Metacognition in the teaching and learning of mathematics. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 241–268). Singapore: Springer.

Lesh, R., Post, T., & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 33–40). Hillsdale, NJ: Lawrence Erlbaum.

Martinez-Sierra, G. (2014). Good mathematics teaching from Mexican high school students' perspective. *International Journal of Science and Mathematics Education, 12*(6), 1547–1573.

Ministry of Education. (2012). *O-Level mathematics teaching and learning syllabus.* Singapore: Author.

Ministry of Education. (2017, March 7). MOE FY 2017 committee of supply debate speech by Minister of Education (Schools) Ng Chee Meng. Retrieved from https://www.moe.gov.sg/news/speeches/moe-fy-2017-committee-of-supply-debate-speech-by-minister-of-education-schools-ng-chee-meng.

Ministry of Education. (2018a, October 11). 21st century competencies. Retrieved from https://www.moe.gov.sg/education/education-system/21st-century-competencies.

Ministry of Education. (2018b). *2020 Secondary mathematics syllabuses.* Singapore: Author.

Murray, S. (2011). Secondary students' descriptions of "good" mathematics teachers. *Australian Mathematics Teacher, 67*(4), 14–21.

Pang, J. S. (2009). Good mathematics instruction in South Korea. *ZDM—The International Journal on Mathematics Education, 41*(3), 349–362.

Piaget, J. (1932). *The language and thought of the child* (2nd ed.). London: Routledge & Kegan Paul.

Saleh, M., Lazonder, A. W., & de Jong, T. (2005). Effects of within-class ability grouping on social interaction, achievement, and motivation. *Instructional Science, 33*(2), 105–119.

Saleh, M., Lazonder, A. W., & de Jong, T. (2007). Structuring collaboration in mixed-ability groups to promote learning, verbal interaction and motivation of average-ability students. *Contemporary Educational Psychology, 32*(3), 314–331.

Shimizu, Y. (2009). Characterizing exemplary mathematics instruction in Japanese classrooms from the learner's perspective. *ZDM—The International Journal on Mathematics Education, 41* (3), 311–318.

Skemp, R. (1987). *The psychology of learning mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Wang, T. Y., & Hsieh, F. J. (2017). Taiwanese high school students' perspectives on effective mathematics teaching behaviours. *Studies in Educational Evaluation, 55,* 35–45.

Yeo, J. B. W. (2018). Engaging the hearts of mathematics learners. In P. C. Toh & B. L. Chua (Eds.), *Mathematics instruction: Goals, tasks and activities* (Association of Mathematics Educators 2018 Yearbook, pp. 115-132). Singapore: World Scientific.

Ngan Hoe Lee is an Associate Professor at the National Institute of Education (NIE). He taught Mathematics and Physics in a secondary school before becoming a Gifted Education Specialist at the Ministry of Education. At NIE, he teaches pre- and in-service as well as postgraduate courses in mathematics education and supervises postgraduate students pursuing Masters and PhD degrees. His publication and research interests include the teaching and learning of mathematics at all levels—primary, secondary and pre-university, covering areas such as mathematics curriculum development, metacognition and mathematical problem solving/modelling, productive failure and constructivism in mathematics education, technology and mathematics education, and textbooks and mathematics education.

Berinderjeet Kaur is a Professor of Mathematics Education at the National Institute of Education in Singapore. She holds a Ph.D. in Mathematics Education from Monash University in Australia. She has been with the Institute for the last 30 years and is one of the leading figures of Mathematics Education in Singapore. In 2010, she became the first full professor of Mathematics Education in Singapore. She has been involved in numerous international studies of Mathematics Education and was the Mathematics Consultant to TIMSS 2011. She was also a core member of the MEG (Mathematics Expert Group) for PISA 2015. She is passionate about the development of mathematics teachers and in turn the learning of mathematics by children in schools. Her accolades at the national level include the public administration medal in 2006 by the President of Singapore, the long public service with distinction medal in 2016 by the President of Singapore and in 2015, in celebration of 50 years of Singapore's nation building, recognition as an outstanding educator by the Sikh Community in Singapore for contributions towards nation building.

Liyana Safii has been a research assistant at the National Institute of Education at Nanyang Technological University, Singapore for five years. She holds a Bachelor of Arts in Psychology from Murdoch University where she graduated as a valedictorian and was a recipient of the Vice Chancellor's Commendation for Academic Excellence Award. Her research interests include classroom practices in primary schools, especially in self-assessment and feedback.

Part III
Tasks and Tools

Chapter 11
Singapore Secondary School Mathematics Teachers' Selection and Modification of Instructional Materials for Classroom Use

Lu Pien Cheng, Yew Hoong Leong, and Wei Yeng Karen Toh

Abstract This chapter examines how experienced and competent Singapore secondary school mathematics teachers select and modify materials for instructional practice. For the empirical section, we begin by analysing survey responses of 677 participants across a wide range of secondary schools to determine the extent of modification among teachers before identifying which instructional materials were used as reference materials in their modification. The findings showed that the teachers relied heavily on their school-based materials as reference materials. We next analyse the instructional materials of 30 experienced and competent teachers which reveal that the teachers' selection and modification of instructional materials were carried out in such a way as to integrate into their own instructional conceptions. The characteristics of the instructional materials that help teachers enact worthy instructional goals of teaching mathematics, such as making connections, reasoning, and challenge, were distilled from the 30 experienced and competent teachers' interview transcripts and their instructional materials.

Keywords Instructional materials · Task design · Task modification · Reference materials · Instructional design moves

L. P. Cheng (✉) · Y. H. Leong · W. Y. K. Toh
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: lupien.cheng@nie.edu.sg

Y. H. Leong
e-mail: yewhoong.leong@nie.edu.sg

W. Y. K. Toh
e-mail: karen.toh@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_11

11.1 Introduction

In this chapter and the next three chapters we turn to another aspect of the project (see Chapter 2 for details) that focus on Singapore mathematics teachers' use of instructional materials. In an earlier paper, we reported a teacher's use of instructional materials that he crafted to realise his goal of "making things explicit" (Leong, Cheng, Toh, Kaur, & Toh, 2019a). The paper also illuminated how the teacher selected and modified his instructional materials. In this chapter, we broaden our investigation to more Singapore secondary school mathematics teachers to: (i) gain deeper insight on the selection or modification of materials for instructional practice, and (ii) examine the characteristics of the instructional materials that help teachers enact worthy instructional goals of teaching mathematics.

11.2 Instructional Materials

Teachers are key to effective curriculum delivery. "The effectiveness of their curriculum delivery is connected to the quality of instructional materials (Ko & Sammons, 2014)" (as cited in Lashley, 2019, p. 2). Indeed, instructional materials are important mediators to connect teaching and learning. Not only are instructional materials resources designed to support or supplement instruction (Remillard & Heck, 2014), they are "one which is classroom-ready and that carries the teachers' actual instructional goals" (Leong, Cheng, & Toh et al., 2019a, p. 50). Instructional materials include textbooks, curriculum guides, descriptions of mathematical tasks, and instructional software (Remillard & Heck, 2014). It is important that instructional materials maintain high standards because "the standards of the instructional materials in the classroom for curriculum delivery directly impact the quality of the learning experiences" (Lashley, 2019, p. 3). However, designing high-quality instructional materials requires considerable thought in order to achieve the needed impact (Lashley, 2019).

11.2.1 Selection and Modification of Instructional Materials

"Textbooks and curriculum guides are the most common form of instructional materials used throughout the world and continue to play a critical role in national education systems" (Remillard & Heck, 2014, p. 707). Teachers also frequently develop their own materials (Steiner, 2018) and they seek after instructional materials that would address their students' learning needs. Their selection of instructional materials may be based on, for example, professional judgement and experience in selection of instructional materials (Bugler et al., 2017). They also added that criteria such as accuracy and visual appeal, alignment to standards and depth of knowledge, ease of use and support, and engagement and ability to meet student needs are also used for the selection of instructional materials. In order to improve the effectiveness of the selected and produced instructional materials, learners' interest and diversity should be considered (Lashley, 2019). Research on learning styles, and the design of instructional materials for flexibility, diversity, and balance can also be taken into consideration (Rowntree, 1992). In the selection of mathematical tasks, it is critical that the tasks selected to match the instructional objectives and that teachers recognise the nature of tasks in order to maximise learning opportunities afforded through different tasks (Lee, Lee, & Park, 2019).

"Teachers can use textbooks in any number of different ways, adapting and adding to them – or omitting some or all of any given activity (e.g. Grammatosi & Harwood, 2014; Gray, 2010; Menkabu & Harwood, 2014; Shawer, 2010)" (Harwood, 2017, p. 264). It is sometimes necessary for teachers to modify textbook tasks to respond to new curriculum standards or educational aspirations. For example, nurturing creativity is one of the essential twenty-first-century skills (Coil, 2013, 2014; Piirto, 2011) and creative thinking can be fostered through tasks designed for higher-order thinking (Kaur & Yeap, 2009). According to Lee et al. (2019), although creativity is explicitly addressed in the Republic of Korea's mathematics curricula, secondary school mathematics teachers did not feel the need for task modification (Kim & Kim, 2014) as many tasks in the middle and high school mathematics textbooks still require students to obtain correct answers by using procedures or algorithms (Kim & Kim, 2013). Lee (2017) reported that "few mathematics teachers design new tasks or adapt the tasks from textbooks to be appropriate for a high-level cognitive approach (Remillard 1999; Smith 2000; Stein, Grover, & Henningsen, 1996; Stigler & Hiebert 2004)" (p. 997).

Teachers also draw from a variety of resources or references to design their instructional materials. These reference materials, also known as base materials (Leong, Cheng, & Toh et al., 2019a), undergo some modification and selection process before the teachers morphed them into a form that is considered suitable for use in classroom work to advance their instructional goal. The teacher's modification of the textbook (reference materials) for his instructional materials could be to make things more explicit for his students (Leong, Cheng, & Toh, et al., 2019a). Three strategies were detailed in their report: (i) "Explicit-from" reference materials to fill gaps in the textbook content such as critical ideas and links between representations, (ii) "explicit-within" for students to revisit similar tasks in sequential units for skill consolidation and concept linkage, and (iii) "explicit-to" in order to direct students from the questions in the instructional materials to planned classroom enactments. Several other strategies for task modification have been reported in the literature. For example, Zaslavsky (1995), showed how to modify standard tasks that have only one correct answer into open-ended tasks that allow learners to explore more solutions to the tasks, to pose more questions, and to try various strategies. In the same vein, Yeo (2018) provided examples of modification of a textbook problem into a more open-ended task "that make[s] assumptions on the missing information" (p. 200). Lee, Lee, and Park (2016) reported three types of task modification strategies by pre-service teachers:

(i) *context modification* refers to modification by changing the context of tasks, making them student-friendly or diverse
(ii) *condition modification* refers to modification by adding, deleting, or transforming the conditions in tasks (Prestage & Perks, 2007). This can also be characterised by adding questions to remind students what they have learned, "changed the condition of the task to step questions to facilitate students with constructing mathematical concepts" (Lee et al., 2019, p. 979) and when students' cognitive level was considered, the condition of the task was modified to "provide the opportunity of inductive reasoning or informal justification" (p. 980).
(iii) *question modification* refers to modification by changing what students are required to answer. This can also be characterised by the opportunity "to facilitate students' reflective thinking" (Lee et al., 2019, p. 980). For example, including questions that require students to reflect whether their solutions have any meaning in real life. It included also questions that required learners to provide explanation about their solutions.

11.2.2 Characteristics of Instructional Materials

Many instructional materials have been published to respond to "new" curriculum standards over the years, "with the explicit intent of helping teachers and students enact reform-oriented subject matter and pedagogical goals" (Lloyd & Behm, 2005, p. 48). According to González, Estrada, and González (2017), *The Guide for Evaluating Teaching Materials and Development* reported in Travé, Pozuelos, Cañal, and Rodríguez (2016) is a tool that can be used to evaluate instructional materials in terms of six aspects:

(1) epistemological aspects of teaching material, e.g., material identifies school knowledge results from the "interaction between scientific and everyday knowledge" (p. 976)
(2) axiological aspects, e.g., "inclusion of cultural elements and promotion of respect for the environment" (p. 976)
(3) psychological aspects, e.g., takes into account the kind of learning promoted by the material and the role of previous knowledge
(4) pedagogical elements, e.g., considers:

- key competences
- objectives
- contents (e.g., organisation and connection with environment)
- methodology approach (e.g., non-directive, inquiry-based)
- activities (e.g., sequencing according to some structure; explanatory orientation that is applicable to textbook; theoretical–practical design that requires description, explanation, argument and requires diversity of sources of information and materials)

- assessment where evaluation is viewed "as a process of understanding and reflection for improving learning and teaching" (p. 977)

(5) teaching design, e.g., "based around the textbook with other complementary material" or "based around self-produced materials complemented by various other resources" (p. 978)
(6) professional development, e.g., "material promoted the design, development and evaluation of the syllabus from an enquiry-based perspective" (p. 982).

As seen from the literature review, much has been reported about the quality of instructional materials, but relatively less is reported about how teachers design these materials. If we assume that teachers do not usually create their instructional materials from scratch, it necessarily implies that they select and modify from reference materials. It will thus be interesting to examine these processes and mechanisms teachers engage in when they select and modify reference materials for their instructional materials. This is the focus for the rest of this chapter as we proceed to the empirical section.

11.3 Teachers' Reference Materials

We first report findings from four survey items completed by 677 experienced and competent Singapore secondary school mathematics teachers. The results of the survey items inform us of the most useful reference materials for the teachers, the reference materials that they based their modification upon, and the extent of modification among teachers.

11.3.1 Item 1

Item 1 of the survey requires the teacher respondents to rank the materials (e.g., main textbook, school-based material, etc.) in order of usefulness given a list of reference materials.

> Item 1: *The following is a list of reference materials. Rank the materials in order of usefulness, 9 being the one most useful to you.*

Our analysis of this survey item revealed that the main reference material which had the most influence on teachers was the main textbook, followed by school-based materials. As shown in Table 11.1, out of 677 respondents to the survey, 432 chose main textbook as what they consider as the most useful reference materials, followed by 143 (21%) who chose school-based materials. The main workbook supplements

Table 11.1 Most useful reference materials chosen by 677 survey participants

Reference materials	9	8	7	6	5	4	3	2	1	Total
Main textbook	432	117	55	18	7	7	7	11	23	677
Supplementary textbook(s)	6	93	73	91	112	113	104	65	20	677
Main workbook	16	143	95	97	102	80	66	66	12	677
Supplementary workbook(s)	3	10	27	80	90	137	144	122	64	677
School-based resource(s)	143	125	135	77	81	50	31	21	14	677
Commercial materials	5	33	82	81	98	90	115	133	40	677
Online resources	28	119	149	136	66	71	73	27	8	677
MOE-produced resources	8	29	51	79	100	94	92	190	34	677
Others	36	9	11	18	20	35	45	42	461	677

the main textbook and allows for more practice, assessment and development of problem-solving and thinking skills.

While we expect textbooks to be the main reference materials for teachers, because in Singapore, mathematics textbooks are "part of the official curriculum to the extent that they are incorporated into the designated curriculum through authorised selection or adoption processes" (Remillard & Heck, 2014, p. 710), we were surprised at how highly the teachers valued school-based materials as references. This appears to be a "new" finding as prior studies of this scale within Singapore had not revealed a similar significant preference for school-based materials. If so, this may indicate a quite recent phenomenon where school-based materials are gaining more influence on secondary mathematics teachers. While we did not anticipate how highly the teachers valued school-based materials (as shown in the finding from Item 1), we were aware—through noticing that many of the teachers we studied in Phase 1 of the research relied on school-based materials—that they were also used in a number of secondary schools. As such, we wanted to survey the type of school-based materials referred to across a broad range of schools. This is the purpose of Item 2.

11.3.2 Item 2

Item 2: *Do you use any school-based resource(s) as your reference materials? If yes, please specify.*

We counted and sorted teachers' responses to their use of school-based resources into three categories (Fig. 11.1), namely, (i) past year papers (ii) other school papers and (iii) within-school materials such as notes and teaching packages, lessons developed by the mathematics department. More than half of the respondents (410 out of 677) indicated the use of within-school materials. This seems to signal a shift

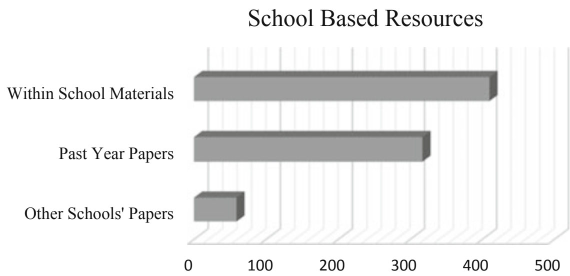


Fig. 11.1 Categories of school-based resources used by teachers as reference materials

towards supplementing externally designed materials (such as textbooks) with internally designed school-based materials. As schools rely more on their "in-house" expertise for instructional materials, what are some implications for teaching and research? For one, since the quality of instruction is largely influenced by the quality of the materials referred to, a study into the kind of actual within-school materials used by schools would be a productive inquiry. However, to date, there have been scarce research in this area within Singapore.

We then examined the reference materials that the teachers used for modifications and selection to design their own instructional materials by analysing Item 3.

11.3.3 Item 3

> Item 3: *What were used or modified from the reference materials for the design of your instructional materials (you may select more than one item).*

The results showed a variety of tasks (e.g., practice items, challenging items, diagrams, activity, worked examples, organisation of content(s)) that were being modified from the reference materials for teachers' design of instructional materials (Fig. 11.2). Practice examples were found to be the most frequently modified. In separate studies, we zoomed-into the design principles used by some of these teachers in crafting sequences of practice examples (Leong, Cheng, Toh, Kaur, & Toh, 2019b, in press).

Lastly, we examined Item 4 to determine the relationship between Secondary mathematics teachers' reference materials and instructional materials. "By instructional materials (IM), we mean materials that teachers bring into the classroom for instructional purposes, and in a form that is classroom-ready for students' access in the learning of mathematics" (Leong, Cheng, & Toh, 2019a, p. 90).

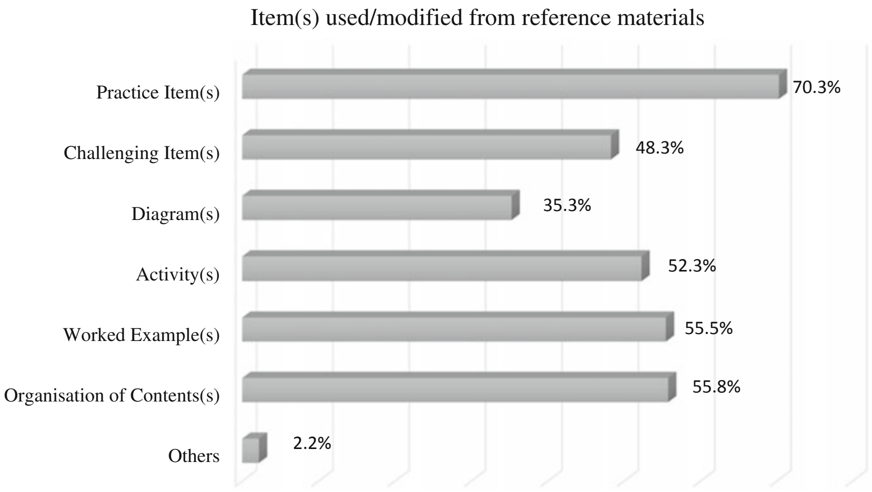


Fig. 11.2 Items used for modification and selection of instructional materials

11.3.4 Item 4

> Item 4: *What is the relationship between their reference materials and their instructional materials?*

Responses to this item were coded (see Table 11.2) as: (1) *Exactly the same* as reference materials, (2) *Sometimes* adaptation and modification were made, (3) *Frequently* adaptation and modification were made, and (4) *Mostly/Always* modifications were made. The results revealed that only 3.5% of the respondents did not make any adaptation and modifications from their reference materials, that is, 96.5% of the respondents made adaptations and modifications from their reference materials (see Fig. 11.3 for the graphical representation of this result). This means that a vast majority of Singapore secondary mathematics teachers do not view their duty as merely "lifting" items from reference materials to give to their students; rather, they see their role as necessarily one of mediation between the reference materials and

Table 11.2 Relationship between reference materials and instructional materials

	Code(s)	Frequency	Percentage (%)
Exactly the same	1	24	3.5
Sometimes	2	253	37.4
Frequently	3	262	38.7
Almost always	4	138	20.4
Total		677	

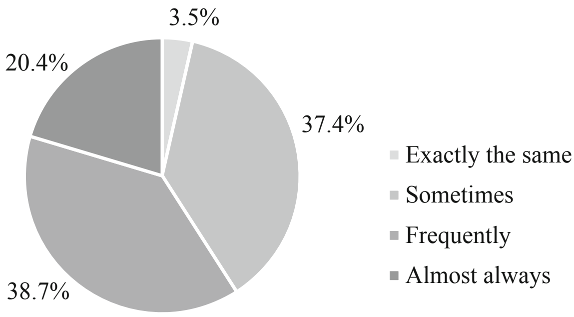


Fig. 11.3 Relationship between reference materials and instructional materials

student learning: they are required to value-add by modifying them. This leads to a natural question: how do teachers select and modify materials? This is the substance of the next section of this chapter.

11.4 Teachers' Strategies in Selection and Modification for Their Instructional Materials

We inquire into this aspect of the investigation through two research questions.

11.4.1 Research Question 1: How Do Singapore Secondary School Mathematics Teachers Select or Modify Materials for Instructional Practice?

11.4.1.1 Method

The 30 experienced and competent Singapore secondary school mathematics teachers who participated in the first phase of the project submitted their instructional materials they planned for the mathematics topics before they were interviewed (pre-module interview) and before any observations on their mathematics lessons using the planned instructional materials were made. From the instructional materials that they initially submitted, we were able to trace several examples of modification and selection from their reference materials, which was chiefly the main textbook (see Sect. 11.3.1, Item 1). This suggests that modification and selection of materials was done before enactment of the mathematics lesson (Stage 1 in Fig. 11.4).

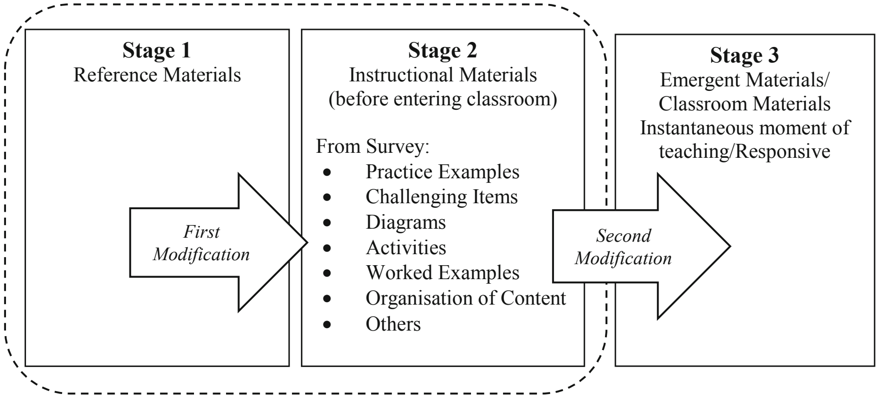


Fig. 11.4 Modification and selection before and during enactment of lesson

When asked whether the teachers have any special features that they have put in place, through their instructional materials, that will help them assess whether their students have attained the mathematical goals of the lesson, Teacher 1 said,

> It would be through certain questions. These are the questions that, if they are able to answer, that means, they will have learned what they are supposed to learn, that means, that *sub, that small content goals, smaller sub goals*. So they'll be like, so called particular questions, that, by doing, by going through these questions, if they are able to answer, that means they know, and then we can move on. Because if not, we probably have to *go back and think of other examples* [modification during enactment]. Either other examples, other ways of showing them, or they just need more practice questions. It really depends.

Teacher 9 said,

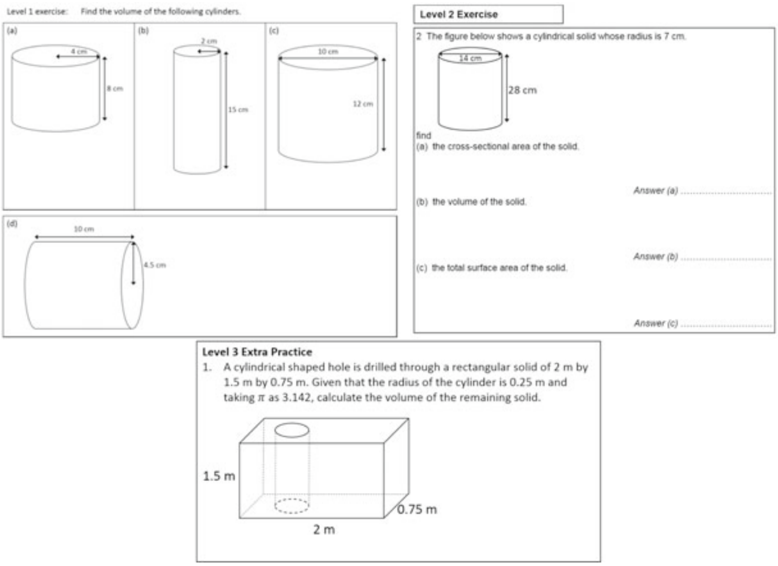
> So, besides the notes, and the worksheets, I give them quizzes, and if I find that the classes, not able to handle certain things like yesterday … *I may need - will recap*… So I *will prepare something* [modification during enactment] to, a very short recap to go through the… Not misconceptions, *the gaps that they still have.*

This suggests that modification and selection of materials were also made during the enactment of the planned lessons, at different junctions of the topic (Stage 2 in Fig. 11.4). During the enactment of the lesson, the teachers sometimes came up with examples on the spot to respond to the students' learning needs and we refer this as *emergent material* to meet the instantaneous moment of teaching (Stage 3 Fig. 11.4).

In this chapter, we analysed only modification of instructional materials before the enactment (First modification from Stage 1 to 2 in Fig. 11.4).

Phase 1: We begin by examining in detail the interview transcripts of two randomly selected exemplary teachers (Teacher 1 and 3) and looked for instances when they explicitly communicated their instructional design moves before triangulating with their instructional materials and reference materials. Here, we used reference materials as textbooks (mainly the teachers' school main textbooks) approved by Ministry of Education (MOE) in Singapore. The template in Table 11.3 serves as a way to

Table 11.3 Example of analysis to determine participants' modification and selection moves from reference materials (Textbook)

Interview Transcript
Teacher 14: *I'll say that my questions that I gave them it's actually from level 1, level 2,…* basically for *level 1 um it's really more on the, the simple one* like, even like identifying which one is a prism,… *another level 1 question is also to be able to do the direct questions* as well. *So level 2 will be a little bit more wordy … So I slowly build up…* I foresee that they might not be able to see, so that's why I give them the different orientation for them to get used to it… my building up to the volume of cylinder they will need to find the area of the, the 2D figure. So for area of 2D figure, I ask them to actually memorise like, to find the area of the circle is actually πr^2. So that's why … link it to the *r*. So the students will be more, I'll say, … they will relate better when it comes to radius … instead of diameter. So I give them the radius first, then after that the diameter … some of *the higher ability one, the HA students right they can finish this exercise pretty fast, so that's why my level 3 question is to actually stretch them.*
Reference Materials
We are unable to reproduce here the diagrams from page 254 of the source due to copyright reasons. Source: Toh, T. L. (2014). *Maths 360 Normal (Technical)* 2. Singapore: Marshall Cavendish Education.
Instructional Materials (Modified Tasks)


collate and organise the relevant data for the first two teachers. Based on the template, we derive categories of how these two teachers selected and modified reference materials for their instructional materials.

Phase 2: We broaden the analysis to five other teachers (randomly selected from the different tracks) using the categories by looking at their instructional materials to trace modification and selection from their reference materials. We continued our analysis of the instructional materials using Table 11.3. In summary, in this phase, (i) three of the instructional materials are from the Express, Teacher 1, 3, and 8; (ii) two from Normal (Technical), Teacher 9 and 14; (iii) one Normal (Academic), Teacher 11, and (iv) one from the Integrated Programme, Teacher 13.

Phase 3: In this phase of the analysis, we scanned through the rest of the teachers' instructional materials to confirm and refine the categories. We examined the instructional materials of 30 teachers.

11.4.1.2 Findings

Our analysis resulted in three categories that illuminate the modification and selection design moves made by the teachers: (i) *modified*, (ii) *new*, and (iii) *smoothened*. We elaborate and provide examples of the three modification and selection design moves below.

(i) Modified

Teachers modified and selected their materials from the textbooks for varied purposes. For example, the tasks in Fig. 11.5 were for Secondary 4 Express students on the topic of Vectors. Textbook item (iii) was modified to item (c) in the teacher's instructional material. The modified item (c) required more thinking on the part of the student as compared to item (iii) and thus the modification increased the cognitive demand of the tasks.

(ii) New

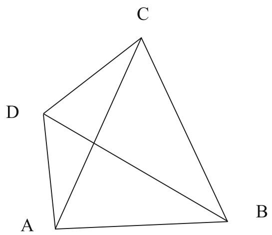
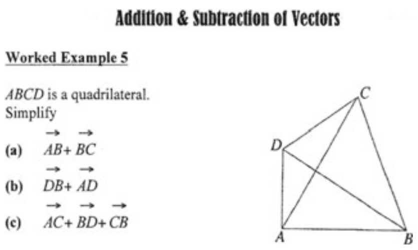
Reference Materials	Instructional Materials (Modified Tasks)
The diagram shows a quadrilateral ABCD. Simplify (i) $\overrightarrow{AB} + \overrightarrow{BC}$, (ii) $\overrightarrow{DB} + \overrightarrow{AD}$, (iii) $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$.  [Reproduced from Yeo, J., Teh, K. S., Loh, C. Y., & Chow, I. (2016). *New syllabus mathematics 4* (7th ed.). Singapore. With permissions from Shinglee Publishers Pte Ltd]	 **Addition & Subtraction of Vectors** Worked Example 5 *ABCD* is a quadrilateral. Simplify (a) $\overrightarrow{AB} + \overrightarrow{BC}$ (b) $\overrightarrow{DB} + \overrightarrow{AD}$ (c) $\overrightarrow{AC} + \overrightarrow{BD} + \overrightarrow{CB}$

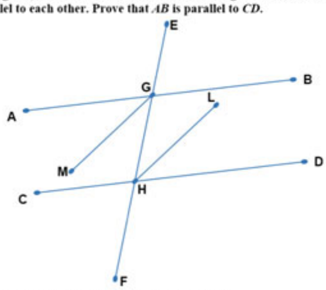
Fig. 11.5 Modification from textbook for Secondary 4 Express Vectors (Teacher 1)

<table>
<tr><th>Interview Transcript</th><th>Instructional Materials (New)</th></tr>
<tr><td>Interviewer: … what is one special feature that you put in place, based on the package, special features?

Teacher 3: I think it would be the step by step approach, sequential [manner). Because I think textbook they only teach you the concept. Based on the concept, they do not tell you where to begin with. So we tell them … So what are some of the key steps that you need to do, in writing out what are some of the theorems, before you even get started? So that, it sort of eases the students into the question itself.</td>
<td><u>Chapter 10: Proofs in Plane Geometry</u>

Name:____________________________( ) Date:__________

Class:__________

<u>Method 1 – Denoting the unknown (Basic Proofs in Plane Geometry)</u>
Eg 1 In the diagram, bisector GM and HL of alternate angles AGH and DHG respectively are parallel to each other. Prove that AB is parallel to CD.


Step 1: Mark all given information on diagram and deduce.
Step 2: State what you want to prove and outline your approach.
Step 3: Write out your proof step by step, giving reasons where necessary.

Let $\angle AGM = x$.
$\angle MGH =$ __________
$\angle MGH = \angle GHL =$ __________ (   )
$\angle GHL = \angle LHD =$ __________
$\angle AGH = \angle GHD =$ __________
$\therefore AB \parallel CD$ (   )</td></tr>
</table>

Fig. 11.6 New items for Secondary 4 Express geometrical proofs (Teacher 3)

Teachers also created new instructional materials which were clearly not from the teachers' school main textbook and the innovations were for varied purposes. An example is provided in Fig. 11.6.

It is clear from the teacher interview that the new materials are created to gradate the level of difficulty. This gradation also reflects the teacher's sensitivity and response to students' responses to pre-existing or available materials before class instruction. The teacher created new materials at specific junctures of the topic to fill learning gaps anticipated by the teacher for the group of students that he will be carrying out the instructions.

The next example (Fig. 11.7) illustrates new material created in order to facilitate the connections of mathematical concepts. Notice that the subheadings "Eliciting Prior Knowledge" in the new instructional material explicitly highlighted students' attention to recall area of a square and then connected this geometric representation to the perfect square expressions.

The purpose of this new material is to connect to completing the square method from geometric to algebraic representation.

(iii) Smoothened

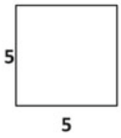
While the teachers we examined do indeed modify items and add new items—as described in the earlier sections—our further analysis shows that these two moves alone do not adequately explain the teachers' strategy of instructional design. When both actions are visible to us within the same tasks, there is smoothening of the

Instructional Materials (New)

2.3 Completing the Square Method

Eliciting Prior Knowledge

1. Draw a geometric representation of each of the following. The first one has been done for you.

5^2	7^2	$(x+1)^2$	$(x-2)^2$
			

2. Explain in words what each of the following represents with reference to its geometric representation. The first one has been done for you.

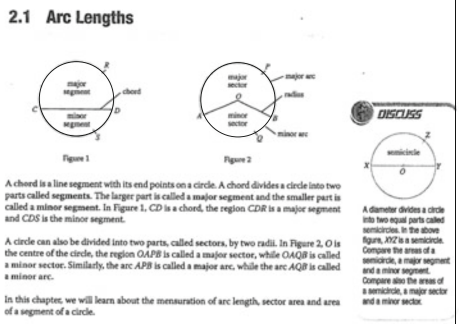
5^2	5^2 represents the area of a square with sides of 5 units in length.
7^2	
$(x+1)^2$	
$(x-2)^2$	

Fig. 11.7 New items for Secondary 3 Express quadratic equations (Teacher 8)

instructional materials. We define tasks as a series of work students are required to do organised around an ostensible goal, e.g., identify four types of angles. Figure 11.8 illustrates smoothening of instructional materials.

The teacher combined two figures in Sect. 2.1 of the textbook into one. This modification of diagrams (collapsing the number of diagrams) summarises several key terms for this topic and draws out key differences between the terms such as chord and radius. A table was created (new material) below the modified diagram in the teacher's instructional material to repeat some of the key terms such as radius, diameter, chord, arc, and sector. Not only that, the students are required to describe those terms in the space provided in the table. The students also have to draw the radius in the circle provided in the table, diameter in the circle provided in the next row of the table etc. The modifications made and the new material added appears as an "entity" rather than separate activities. In other words, the teacher also *smoothens* these components to provide continuity and connection in the students' learning. The reference materials are insufficient to help the targeted students learning and the teacher modifies, adds, and smoothens the learning materials to facilitate this learning transition. The teacher, Teacher 11, said during the interview for the topic arc length,

Reference Materials



2.1 Arc Lengths

A **chord** is a line segment with its end points on a circle. A chord divides a circle into two parts called **segments**. The larger part is called a **major segment** and the smaller part is called a **minor segment**. In Figure 1, *CD* is a chord, the region *CDR* is a major segment and *CDS* is the minor segment.

A circle can also be divided into two parts, called **sectors**, by two radii. In Figure 2, *O* is the centre of the circle, the region *OAPB* is called a **major sector**, while *OAQB* is called a **minor sector**. Similarly, the arc *APB* is called a **major arc**, while the arc *AQB* is called a **minor arc**.

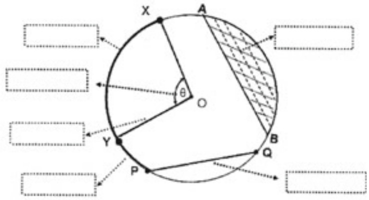
In this chapter, we will learn about the mensuration of arc length, sector area and area of a segment of a circle.

A diameter divides a circle into two equal parts called semicircles. In the above figure, *XYZ* is a semicircle. Compare the areas of a semicircle, a major segment and a minor segment. Compare also the areas of a semicircle, a major sector and a minor sector.

[Source: Chow, W. K. (2016). *Discovering mathematics Normal (Academic) 4A* (2nd ed.). Singapore: Star Publishing Pte Ltd]

Instructional Materials (Smoothened)

<u>Activity 1:</u>
Fill in the mathematical terms for the highlighted parts or shaded regions of the circle in the following diagram.



Parts of a circle	Description	Illustration
Radius		
Diameter		
Chord		
Arc		
Sector		

Fig. 11.8 Smoothened materials for Normal (Academic) Secondary 4 arc length (Teacher 11)

> From my years of experience … I find that students are not comfortable … in listing the radian … radian measure…my goal is really for them to be able to accept this radian measure and be able to use it … when it's required … they are not comfortable. So… that is something I would like to … make it easy for the students.

Figure 11.9 illustrates another example where smoothening of instructional materials can be observed from Teacher 11.

New rows for Fig. 4 and 5 in the teachers' instructional materials were added as compared to the textbook which provides rows for Fig. 1, 2, and 3. This provided more specific examples for students to observe patterns and relationships from the data generated for rows from Fig. 1 to 5 in the teacher's instructional materials to facilitate the generalisation process. The instructions in the teacher's material were added with each column labelled as (1), (2), and (3) to modify the instructions to the task into a form that was less wordy. Question (c) and (d) in the teachers' instructional material is a modification to Question 2 and 3 respectively in the textbook. Question (c) and (d) are less wordy and have a more direct approach towards the derivation of the formula for the length of arc in terms of r and θ as compared to Question 2. This suggests the modification and selection move to "remove unnecessary work". Once again, the modifications made and new material added appears as a coherent activity which clearly smoothens students' learning. Here, sensitivity towards students' responses to the existing materials before class and facilitating

Reference Materials

CLASS ACTIVITY 1

Objective: To find the arc length in a circle by considering it as a fraction of the circumference of the circle.

Express your answers in this activity in terms of π where appropriate.

1.

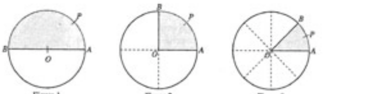


Figure 1 Figure 2 Figure 3

(a) Cut a circle of radius 6 cm from a piece of paper and fold it into 2 halves, 4 quadrants and 8 equal sectors as shown in the above figures, where O is the centre of the circle.

(b) Work together with the classmate sitting next to you.

(i) Observe the size of $\angle AOB$ in each figure formed.

(ii) What is the ratio of the length of arc APB to the circumference in each figure?

[Source: Chow, W. K. (2016). Discovering mathematics 4A, 2nd Edition, p. 39. With permissions from Star Publishing Pte Ltd]

(c) Fill in the following columns, $\angle AOB$, $\frac{\angle AOB}{360^\circ}$ and $\frac{Length\ of\ arc\ APB}{Circumference}$.

	$\angle AOB$	$\frac{\angle AOB}{360^\circ}$	$\frac{Length\ of\ arc\ APB}{Circumference}$	Length of arc APB
Figure 1	180°	$\frac{180^\circ}{360^\circ}=\frac{1}{2}$	$\frac{1}{2}$	
Figure 2				
Figure 3				

(d) What can you say about the fractions $\frac{\angle AOB}{360^\circ}$ and $\frac{Length\ of\ arc\ APB}{Circumference}$?

(e) Given that the circumference of the circle = 12π cm, express the length of the arc APB in each figure in terms of π. Using your answer, complete the table in (c).

2. In Figure 4, a circle O and radius r cm is divided into 6 equal sectors.

(a) Write down the circumference of the circle in terms of r.

(b) Express $\angle AOB$ as a fraction of 360°.

(c) Express arc APB as a fraction of the circumference of the circle.

(d) Hence find the length of arc APB in terms of r.

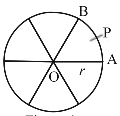


Figure 4

3. In Figure 5, AOB is a sector of the circle O and radius r cm, and $\angle AOB = x°$.

(a) Express $\angle AOB$ as a fraction of 360°in terms of x.

(b) Express arc APB as a fraction of the circumference of the circle.

(c) Hence derive a formula of expressing the length of arc APB in terms of r and x.

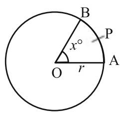


Figure 5

[Reproduced from Chow, W. K. (2016). Discovering mathematics 4A, 2nd Edition, p. 40. With permissions from Star Publishing Pte Ltd]

Instructional Materials (Modified Tasks)

Activity 2: To find the arc length in the circle

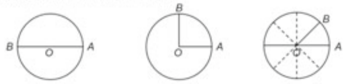


Figure 1 Figure 2 Figure 3

	(1) $\angle AOB$	(2) $\frac{\angle AOB}{360^\circ}$	(3) $\frac{Length\ of\ arc\ APB}{Circumference}$	(4) Length of the arc APB
Figure 1				
Figure 2				
Figure 3				
Figure 4				
Figure 5				

a) Fill in the columns (1), (2) and (3).
What can you say about the columns (2) and (3)?

____.

b) If the circumference of the circle is $2\pi r$ and the radius is r cm, complete column (4)

c) If **Figure 4** is a circle of centre O and radius r cm that is divided into 6 sectors, complete the columns (1) to (4).

d) If **Figure 5** is a circle of centre O and radius r cm that is divided into sectors with an angle $x°$ each, complete the columns (1) to (4).

Hence , can you derive the formula for the length of arc in terms of r and x.

Fig. 11.9 Smoothened materials for Normal (Academic) Secondary 4 formula for arc length (Teacher 11)

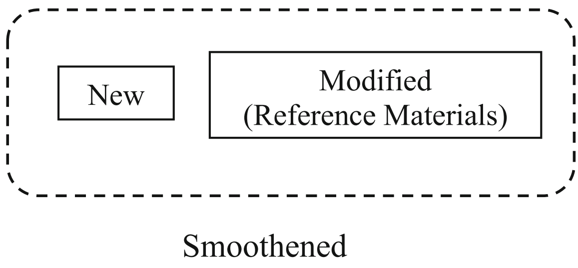


Fig. 11.10 Modification and selection instructional design moves

students' connections of mathematical ideas are evident from the modification and addition of new materials.

Figure 11.10 summarises the teachers' modification and selection of instructional design moves. One can modify without adding items in the instructional materials. Smoothening occurs when teachers modify and add new items within tasks in their instructional materials towards an ostensible goal. Our findings also reveal the teachers' selection and modification of instructional materials were carried out in a way to integrate with their conceptions of instruction, such as (i) increasing cognitive demand of task to raise students' level of thinking, (ii) gradating level of difficulty, (iii) being sensitive to students' responses to the existing materials before class, (iv) helping students to make connections, and (v) removing unnecessary work. Not only do teachers create their own instructional materials, they also modify from their reference materials and smoothen it to become a coherent unit for their students.

Out of the 30 experienced and competent teachers, 29 of them modified material from reference materials and 23 of them also inserted new materials. Out of this, 23 teachers modified, added, and smoothened the material.

11.4.2 Research Question 2: What Are the Characteristics of Instructional Materials That Will Help Teachers Enact Worthy Instructional Goals of Teaching Materials and Help Students Achieve Desirable Outcomes?

11.4.2.1 Phase 1

In order to investigate the characteristics of the instructional materials that help teachers enact worthy instructional goals of teaching mathematics and help students achieve desirable outcomes, we analysed interview transcripts of the 30 experienced and competent teachers to the interview questions, in particular, the goals articulated by the teachers for the following pre-module interview questions:

(i) Please share with us your goals for this series of lessons. You may include both content and non-content goals

(ii) Please share with us what mathematical goals you intend to achieve for each set of materials that you will be using to determine.

We also extracted instances when the teachers explicitly articulated their goals in the post-lessons interviews. We excluded affective goals in our analysis as this goes beyond the scope of the chapter. From the goals that the teachers articulated during their interviews, we locate examples of instructional materials that help teachers enact those goals and collated them in a table as shown in Table 11.4. From these two data sources, we elicited and coded the characteristics of the instructional materials. We collapsed our codes into categories as illustrated in Table 11.5.

We used the task analysis guide (lower-level demands, higher-level demands) by Stein, Smith, Henningsen, and Silver (2009) to determine whether the instructional materials were challenging. Tasks that were identified as having higher-level demands were those that required procedures with connections and doing mathematics.

Table 11.4 Example of analysis process for characteristics of instructional materials

Code(s)	Extracts from interview transcripts	Instructional materials	Researchers' notes
Relate from one thing to another; connection Category: Making Connections	Teacher: … So for the volume itself, volume itself I would like to link to understand prism. Because prism was covered in Sec 2, then if they—I want them to be able to relate from one thing to the other. Then the prism and the pyramid the volume is actually related so I want them to see the connection. Even though the… The cover page on the examination, the formulas are given, but I want them to understand how the formulas come about. It make meaning … otherwise it's just throw them the formula, it won't make meaning to what they are learning	**<u>Activity:</u>** • Find out what is the **relationship** between the **volume of prism** and **volume of pyramid** • Find out the **formula for Volume of Pyramid**	Teacher planned to help students make the conceptual connections between the formulae for the computation of the volume of a prism to that for a pyramid. e.g. connect volume of pyramid to volume of prism learnt in Sec 2

Table 11.5 Sample of codes and categories for characteristics of instructional materials

Codes from interview transcripts and instructional materials	Categories
Link to the various forms, building on past knowledge, build up, linkage, refer to, relate	Make connections
Infer, reason out, justify, explain why	Reasoning
Higher order thinking, higher level questions, stretch, challenging questions, advanced questions, complex	Challenge
Quizzes, exit pass, entrance pass, assessment, check students' understanding	Assessment
Step by step, systematically, structure	Template
Procedural, formula, practice examples, exercise	Deliberate sequencing of examples
Context, real-life, applications	Context
ICT, videos, on the portal, software	ICT-related materials include *space in instructional materials to record* e.g. ICT explorations, making conjectures

11.4.2.2 Findings

Table 11.4 illustrates an example where the teacher's goal is to make conceptual connections between the formulae for the computation of the volume of a prism to that for a pyramid, i.e., connect volume of pyramid to volume of prism learnt in Secondary 2. The teacher's instructional material clearly reflected this goal. Figures 11.11, 11.12, 11.13, 11.14, 11.15, 11.16, and 11.17 provide examples of ostensible goals of what the teachers made explicit in the design of their instructional materials: *reasoning, challenge, assessment, template, deliberate sequencing of examples, context* and *ICT* to help students achieve desirable outcomes. Next, we tabulated the number of teachers who made each of the above goals explicit in their instructional design moves during the interviews (Table 11.6, Column 2).

Interview Transcript	**Instructional Materials**
Teacher 2: Then the one that you are referring to about the non-content goals right, I mean that will be more the mathematics disposition itself. I want to ... you know, I want a student *to be able to do reasoning* and be able to apply also problem-solving skills … particularly in the last part when they are supposed to use vectors to solve geometrical problems itself. So that will be a built up to it. *Again, in the process of learning itself, there will be a lot of reasoning involved.*	Exercise: Which of the following pairs are parallel and why? (i) a – b and b – a (ii) $a + \frac{1}{2}b$ and $4a + 2b$ (iii) $\binom{2}{4}$ *and* $\binom{-1}{-2}$ (iv) $\binom{2}{1}$ *and* $\binom{1}{2}$

Fig. 11.11 Reasoning as a goal for instructional materials from a Normal (Academic) Secondary 5 class on vectors (Teacher 2)

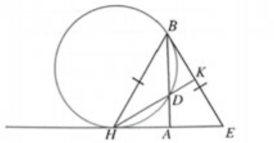
Interview Transcript	Instructional Materials
Teacher 3: I think whatever that I wanted to do during the lesson, more or less it is done. There were some additional practice. I know back in my mind, this topic is not very well received. But I have some motivated ones. So in my way, the motivated ones, they can do all the extra questions. But basically, everyone must attempt at least four similar questions. So that is the basic. But the better ones, *they are going to do the more advanced questions by attempting more.*	**9** In the figure, *HAE* is the tangent to the circle at *H*, *BH* = *BE* and *KH* is the angle bisector of ∠*BHE* and it cuts the circle at *D*. Given that *BD* produced meets *HE* at *A*, prove that **(i)** *HD* = *BD*, **(ii)** a circle can be drawn to pass through *A*, *D*, *K* and *E*.  [Source: Yeo, J., Teh, K. S., Loh, C. Y., & Chow, I. (2013). New syllabus additional mathematics 4 (9th ed.). Singapore. With permissions from Shinglee Publishers Pte Ltd]

Fig. 11.12 Challenge as a goal for instructional materials from an Express Secondary 4 class on geometrical proofs (Teacher 3)

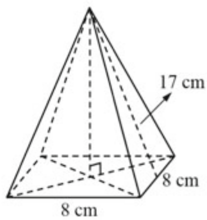
Interview Transcript	Instructional Materials (Quiz)
Teacher 9: … Besides the scaffolding and the examples …I also give them *quizzes.* Because quizzes they are actually quite motivated to have quizzes because when they have quizzes they like challenge each other, and they also have ... It's a form of *self-assessment for* them to say … *how did I do, am I better now or am I worse off.* So there's a bit of competition among the class itself. Then they also want to score well, because when they score well the self-esteem … So I like to give them quizzes, and the quizzes I give them is recurring, that means … *The topics I've covered many many weeks earlier I put them inside my quiz so it's like a recurring quiz* … So the questions keep repeating. And then once they … keep looking at the question they get accustomed to it.	10. (a) Draw the net of the square pyramid.  (b) Find the surface area of the pyramid.

Fig. 11.13 Assessment as a goal for instructional materials from a Normal (Technical) Secondary 4 class on volume and surface area- pyramid & cone (Teacher 9)

11.4.2.3 Phase 2

We also realised that there were many instances in the teachers' instructional materials that fit into some of the categories in Table 11.5, even though those goals were not articulated by the teachers during the interviews. For example, the categories *challenge, assessment, template, deliberate sequencing of examples, context,* and *ICT* are categories that are generally identifiable from the instructional materials. We apply the categories in Table 11.5 back to the teachers' instructional materials

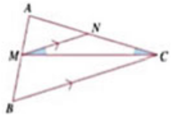
Interview Transcript	Instructional Materials (Quiz)
Teacher 3: … There are certain concepts. Ok. Concepts are actually stated in the textbook itself. Concepts for example similar triangles, midpoint theorem, as well as angles in alternate segment. These are the three major concepts that are being taught in this topic itself but because this is a topic that is not ah focusing on solving, it's focusing on proving on geometry, plane geometry. So the thing is that … I also added into the package itself what are some of the *step by step approaches* the students can start in identifying the question and then …I also shared with them some techniques we actually came up with ourselves so that is actually easier for students to actually do the proving.	Example 4  In the diagram, *MN* and *BC* are parallel lines, and $\angle NMC = \angle NCM$, prove that (a) ΔABC is similar to ΔAMN, (b) $NC \times AC = AN \times BC$. Step 1: Mark all given information on diagram and deduce. Step 2: State what you want to prove and outline your approach. Step 3: Write out your proof step by step, giving reasons where necessary. (a) $\angle BAC$ = __________ ( ) $\angle ABC$ = __________ ( ) $\angle ACB$ = __________ ( ) _______________________. (Proven) (b) (Hint: By similar triangle) _______________________. (Proven)

Fig. 11.14 Template as a goal for instructional materials from an Express Secondary 4 class on geometrical proofs (Teacher 3)

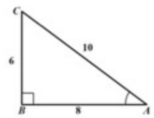
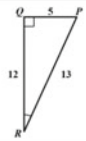
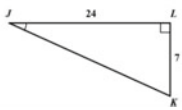
Interview Transcript	Instructional Materials (Quiz)
Teacher 19: ... I think it's more of uh, a lot of scaffolding, uh the scaffolding is there, and I think the questions is basically... progressive, difficulty level is progressive … it's not like I straightaway ask them to find uh the unknown sides and angles, but getting them at the first part, because, after 6.1 part (d), oh- I mean 6 point- before part (d), I'll get them to do the recognising of sides. So before they talk about finding the actual sine, cosine, tangent. Then after that from there, after they get more or less used to finding sine, cosine, tangent, the ratio for it, then they will move on to 6.2 which is finding the unknown sides …	**(D) Test Yourself!** 1. For the right-angled triangle shown, find a) sin *A* b) cos *A* c) tan *A*  2. For the right-angled triangle shown, find a) sin *P* b) cos *P* c) tan *R*  3. For the right-angled triangle shown, find a) sin *K* b) cos *J* c) tan *K* 

Fig. 11.15 Deliberate sequencing of examples as a goal for instructional materials from a Normal (Academic) Secondary 3 class on trigonometric ratio of acute angles (Teacher 19)

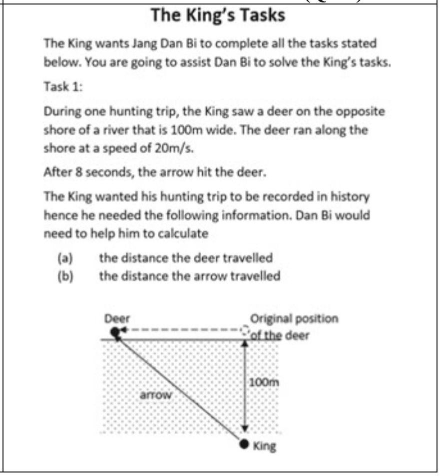
Interview Transcript	Instructional Materials (Quiz)
Teacher 6: …the goal of today's lesson is actually to er get them to apply the Pythagoras theorem and see actually how in real life context, how they can actually use Pythagoras theorem, ya. Interviewer: ... in your lesson, how did you go about achieving your goal? Teacher 6: So … what I did is actually edited the video, … to show them …a scene where the girl actually travel past time and how she actually uses Pythagoras theorem,… to actually help the king to solve the problem. And after that, they'll actually do some questions … to apply what they have learned in terms of the real life contexts.	 **The King's Tasks** The King wants Jang Dan Bi to complete all the tasks stated below. You are going to assist Dan Bi to solve the King's tasks. Task 1: During one hunting trip, the King saw a deer on the opposite shore of a river that is 100m wide. The deer ran along the shore at a speed of 20m/s. After 8 seconds, the arrow hit the deer. The King wanted his hunting trip to be recorded in history hence he needed the following information. Dan Bi would need to help him to calculate (a) the distance the deer travelled (b) the distance the arrow travelled

Fig. 11.16 Context as a goal for instructional materials from an Express Secondary 2 class on Pythagoras theorem (Teacher 6)

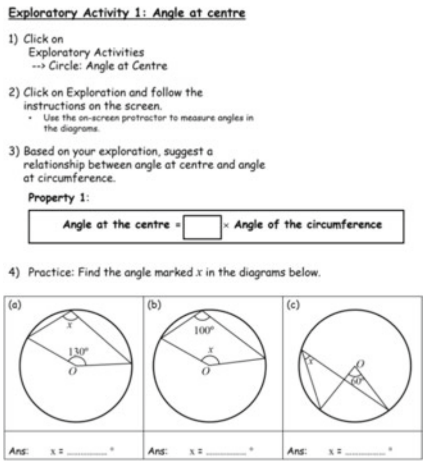
Interview Transcript	Instructional Materials (Quiz)
Teacher 5: … But …for this particular topic, I usually bring them to the computer lab to get them to explore first. So, my goal initially is to get them to go through the process of exploration. Self-discovery of the rules. So, it's more of a deductive approach. Because you don't see a few cases, in terms of how the properties play out. And then based on that, they are able to consolidate, which, so they're able to summarise, or conclude, that the relation between the angles is as what is being displayed.	Diagram is based on an activity that students have access to from a commercial online repository of materials.  Exploratory Activity 1: Angle at centre 1) Click on Exploratory Activities --> Circle: Angle at Centre 2) Click on Exploration and follow the instructions on the screen. • Use the on-screen protractor to measure angles in the diagrams. 3) Based on your exploration, suggest a relationship between angle at centre and angle at circumference. Property 1: Angle at the centre = ☐ × Angle of the circumference 4) Practice: Find the angle marked x in the diagrams below.

Fig. 11.17 ICT as a goal for instructional materials from an Express Secondary 3 class on angle properties of circles (Teacher 5)

Table 11.6 Characteristics of teachers' instructional materials

	Number of teachers (Interviews and instructional materials)	Number of teachers (Instructional materials only)
Challenge	21	27
Deliberate sequencing of examples	19	29
Making connections	18	21
Assessment	16	23
Support reasoning	12	19
Context	8	20
ICT	5	9
Template	4	13

to find their prevalence in the instructional materials. By doing so, we were trying to locate characteristics implicitly embedded into the design of the instructional materials. Table 11.6 (Column 1 and 3) summarises this result.

11.5 Discussion

In this chapter, we found that the textbook was the most useful reference material for teachers, followed by school-based materials. 96.5% of the respondents made adaptations and modifications from their reference materials. The teachers modified a variety of tasks, such as, practice items, challenging items, diagrams, activity, worked examples, organisation of content(s), from the reference materials when designing their instructional materials—with practice examples being the most frequently modified. From the finding above, it is clear to us that most of our teachers do not merely offload or adapt their reference materials into their instructional materials. Rather, they intentionally select materials that are suitable for their goals and make explicit efforts to coherently tie these in with new tasks that they construct for their students. In other words, the instructional materials were mediated through the goals of the teachers in a purposeful manner. This image of teachers' use of instructional materials was also depicted by Lee et al. (2019) as "active interpreter and user of textbook" (p. 966). The instructional design moves could be categorised into: (i) modified (ii) new, and (iii) smoothened. These instructional design moves are under-reported in the international literature and the examples that illuminate the design moves in this study can potentially provide a local-sensitive knowledge base for teacher professional development in designing quality instructional materials for effective teaching and learning.

González et al. (2017)—as reviewed earlier—provided us an overview of possible characteristics of instructional materials. Our findings are quite different as our study examined the actual moves that teachers pull together in their design of instructional materials. The teachers we studied reflected a number of the aspects reported in González et al. (2017)—e.g., psychological aspects in their deliberate sequencing of examples and pedagogical elements such as assessment—where all these are integrated together in their instructional design moves. "Challenge" is a characteristic in 27 out of the 29 instructional materials which suggests that most of the teachers made intentional effort to include challenging tasks in their instructional materials. In this same book, we have devoted Chapter 12 on challenging items where we elucidate all the connections between all these aspects. More than half of the instructional materials carry the characteristics of "support reasoning" and 'making connections'. This is not surprising as reasoning and making connections are two of the processes in the Singapore mathematics curriculum. Almost half of the instructional materials have templates and this interesting finding is reported in Leong, Cheng, and Toh (2019b).

Using the curriculum materials effectively includes not only being able to recognize and distinguish between high- and low-quality materials. Skilful selection and modification of instructional materials guided by clear goals of the teachers—in this study the characteristics inherent in the teachers' instructional materials—for classroom use are also critical. There has been a lot of interest recently in professional development research that draws upon task design and analysis, and instructional materials to develop teacher capacity (e.g. *Journal of Mathematics Teacher Education*, 2007, Volume 10). We see this study as a further contribution to this body of knowledge particularly suited for the local community.

References

Bugler, D., Marple, S., Burr, E., Chen-Gaddini, M., & Finkelstein, N. (2017). How teachers judge the quality of instructional materials: Selecting instructional materials, Brief 1—Quality. *WestEd*. Retrieved December 7, 2019, from https://www.wested.org/resources/selecting-instructional-materials-brief-1-quality/.

Coil, C. (2013). The application of common core state standards to foster creativity and curiosity. In J. B. Jones & L. J. Flint (Eds.), *The creative imperative: School librarians and teachers cultivating curiosity together* (pp. 151–164). Santa Barbara, CA: ABC-CLIO.

Coil, C. (2014). Creativity in an assessment driven environment. *Knowledge Quest, 42*(5), 48–53.

González, G. T., Estrada, F. J. P., & González, G. T. (2017). How teachers design and implement instructional materials to improve classroom practice. *Intangible Capital, 13*(5), 967–1043.

Grammatosi, F., & Harwood, N. (2014). An experienced teacher's use of the textbook on an academic English course: A case study. In N. Harwood (Ed.), *English language teaching textbooks: Content, consumption, production* (pp. 178–204). Basingstoke: Palgrave Macmillan.

Gray, J. (2010). The ELT coursebook as cultural artefact: How teachers censor and adapt. *ELT Journal, 54*(3), 274–283.

Harwood, N. (2017). What can we learn from mainstream education textbook research? *RELC Journal, 48*(2), 264–277.
Kaur, B., & Yeap, B. H. (2009). *Pathways to reasoning and communication in the primary school mathematics classroom.* Singapore: National Institute of Education.
Kim, M., & Kim, G. Y. (2013). The analysis of mathematical tasks in the high school mathematics. *School Mathematics, 15*(1), 37–59.
Kim, D., & Kim, G. Y. (2014). Secondary mathematics teachers' understanding and modification of mathematical tasks in textbooks. *School Mathematics, 16*(3), 445–469.
Ko, J., & Sammons, P. (2014). *Effective teaching* (Reprint of 2014 Report 978-1-909437-75-3). University of Oxford: Education Development Trust.
Lashley, L. (2019). A Reflective analysis of the selection and production of instructional material for curriculum delivery at the primary level in postcolonial Guyana. *Sage Open, 9*(2), 1–15.
Lee, E.-J., Lee, K.-H., & Park, M.-S. (2019). Developing preservice teachers' abilities to modify mathematical tasks: Using noticing-oriented activities. *International Journal of Science and Mathematics Education, 17*(5), 965–985.
Lee, K.-W. (2017). Convergent and divergent thinking in task modification: A case of Korean perspective mathematics teachers' exploration. *ZDM—The International Journal on Mathematics Education, 49*(7), 995–1008.
Lee, K.-W., Lee, E.-J., & Park, M.-S. (2016). Task modification and knowledge utilization by Korean prospective mathematics teachers. *Pedagogical Research, 1*(2), 1–13.
Leong, Y. H., Cheng, L. P., & Toh, W. Y. K. (2019a). Chronologically-grounded survey. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Mathematics Education Research Impacting Practice (Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia)* (pp. 90–93). Perth: MERGA.
Leong, Y. H., Cheng, L. P., & Toh, W. Y. K. (2019b). Formula: To derive or not to derive? *Maths Buzz, 19*(1), 2–5. Singapore: Association of Mathematics Educators.
Leong, Y. H., Cheng, L. P., Toh, W. Y. K., Kaur, B., & Toh, T. L. (2019a). Making things explicit using instructional materials: A case study of a Singapore teacher's practice. *Mathematics Education Research Journal, 31*(1), 47–66.
Leong, Y. H., Cheng, L. P., Toh, W. Y. K., Kaur, B., & Toh, T. L. (2019b). Example sequences to help students recognise and apply formula: The Singapore situation. *The Mathematics Educator, 18*(1&2), 23–46.
Leong, Y. H., Cheng, L. P., Toh, W. Y. K., Kaur, B., & Toh, T. L. (2019). Teaching students to apply formula using instructional materials: A case of a Singapore teacher's practice. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-019-00290-1.
Lloyd, G. M., & Behm, S. L. (2005). Preservice elementary teachers' analysis of mathematics instructional materials. *Action in Teacher Education (Association of Teacher Education), 26*(4), 48–62.
Menkabu, A., & Harwood, N. (2014). Teachers' conceptualization and use of the textbook on a medical English course. In N. Harwood (Ed.), *English language teaching textbooks: Content, consumption, production* (pp. 145–177). Basingstoke: Palgrave Macmillan.
Piirto, J. (2011). *Creativity for 21st century skills.* Dordrecht: Springer Science and Business Media.
Prestage, S., & Perks, P. (2007). Developing teacher knowledge using a tool for creating tasks for the classroom. *Journal of Mathematics Teacher Education, 10*(4–6), 381–390.
Remillard, J. T. (1999). Curriculum materials in mathematics education reform: A framework for examining teachers' curriculum development. *Curriculum Inquiry, 29*(3), 315–342.
Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM—The International Journal on Mathematics Education, 46* (5), 705–718.
Rowntree, D. (1992). *Exploring open and distance learning materials.* London: Kogan Page.
Shawer, S. F. (2010). Classroom-level curriculum development: EFL teachers as curriculum developers, curriculum-makers and curriculum-transmitters. *Teaching and Teacher Education, 26*(2), 173–184.

Smith, M. S. (2000). Balancing old and new: An experienced middle school teacher's learning in the context of mathematics instructional reform. *The Elementary School Journal, 100*(4), 351–375.
Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: *An analysis of mathematical tasks used in reform classrooms. American Educational Research Journal, 33*(2), 455–488.
Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2009). Implementing standards-based mathematics instruction: A casebook for professional development (2nd ed.). Reston, VA: New York: National Council of Teachers of Mathematics; Teachers College Press.
Steiner, D. (2018). Materials matter. *Learning Professional, 39*(6), 24–28.
Stigler, J. W., & Hiebert, J. (2004). Improving mathematics teaching. *Educational Leadership, 61*(5), 12–17.
Travé, G., Pozuelos, F. J., Cañal, P., & Rodríguez, F. P. (2016). Validación de un instrumento de análisis de materiales (AMADE). In P. Cañal, G. Travé, F. J. Pozuelos, et al. (Eds.), *La enseñanza sobre el Medio Natural y Social, Investigación y experiencias* (pp. 97–114). Sevilla: Díada.
Yeo, K. K. J. (2018). Extending textbook exercises into short open-ended tasks for primary mathematics classroom instruction. In P. C. Toh & B. L. Chua (Eds.), *Mathematics instruction: Goals, tasks and activities* (pp. 195–210). Singapore: World Scientific.
Zaslavsky, O. (1995). Open-ended tasks as a trigger for mathematics teachers' professional development. *For the Learning of Mathematics, 15*(3), 15–20.

Lu Pien Cheng is a Senior Lecturer in the Mathematics and Mathematics Education Academic Group at the National Institute of Education, Nanyang Technological University in Singapore. She received her PhD in Mathematics Education from the University of Georgia (U.S.) in 2006. She specialises in mathematics education courses for primary school teachers. Her research interests include the professional development of primary school mathematics teachers, task design in mathematics education and cultivating children's mathematical thinking in the mathematics classrooms.

Yew Hoong Leong is an Associate Professor at the National Institute of Education, Nanyang Technological University. He began his academic career in mathematics education with the motivation of improving teaching by grappling with the complexity of classroom instruction. Along the journey, his research has broadened to include mathematics problem solving and teacher professional development. Together with his project teammates, they developed "Realistic Ambitious Pedagogy" and its accompanying plan of action—the "Replacement Unit Strategy".

Wei Yeng Karen Toh worked as a research assistant at the National Institute of Education for seven years. She has a Bachelor of Science in Applied Mathematics from the National University of Singapore. She was awarded with a Distinction grade for the Post-Graduate Diploma in Education at the National Institute of Education. She taught in a few secondary schools as a mathematics and chemistry teacher for seven years. Her research interests include studying the history of mathematics, learning mathematical and pedagogical content knowledge, examining classroom practices and investigating teachers' designs of instructional materials. To advance her career in research, she pursued a Master of Education in Mathematics at the National Institute of Education.

Chapter 12
Use of Challenging Items in Instructional Materials by Singapore Secondary School Mathematics Teachers

Yew Hoong Leong, Lu Pien Cheng, and Wei Yeng Karen Toh

Abstract Do Singapore secondary mathematics teachers include challenging items regularly in their instructional materials, and if so, how do they help students engage productively with them? We attempted to inquire into these questions in the study. Using some of the actual challenging items used by the teachers in the first phase of the project, we design a chronologically grounded survey that aimed at inquiring the extent of use of challenging items among teachers across a wide range of secondary schools. The findings revealed that teachers' inclusion of challenging items range on average between "Sometimes" and "Frequently". Also, the picture that emerges from their written comments is one where teachers are sensitive to students' cognitive ability and affective disposition to the extent that deliberate planning and supportive mechanisms are the norm when they plan to use challenging items in the instructional materials.

Keywords Challenging items · Instructional materials · Chronologically grounded survey

12.1 Introduction

This chapter focuses on Singapore teachers' inclusion of challenging items in their design of instructional materials for mathematics classrooms. This area of study has been a domain of continual interest especially among mathematics education researchers. One approach taken in this tradition is by looking at the level of "cognitive demand"—a phrase widely attributed to Stein and her colleagues (Henningsen

Y. H. Leong (✉) · L. P. Cheng · W. Y. K. Toh
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: yewhoong.leong@nie.edu.sg

L. P. Cheng
e-mail: lupien.cheng@nie.edu.sg

W. Y. K. Toh
e-mail: karen.toh@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_12

& Stein, 1997; Stein, Smith, Henningsen, & Silver, 2000; Tekkumru Kisa & Stein, 2015)—of tasks and how this demand can be maintained through students' engagement of the task during mathematics lessons. As will be explicated in the later sections of this chapter, we will use their framework in analysing the level of cognitive demand of task.

In light of the high performance of Singapore students in international comparison tests such as TIMSS and PISA (Kaur, Zhu, & Cheang, 2019), it seems natural to ask, "Do Singapore teachers regularly provide their students with opportunities to access these challenging tasks? How do they help students to maintain engagement at a high level of cognitive demand with these tasks?"

As the design setup of the project is elaborated in Chapter 2, only a brief review that is related to our study is provided here. Recall in the first phase of the project there are two investigations: one on the enactment of teachers in the classroom, and the other on the teachers' design and use of instructional materials to fulfil their goals of enactment. The study reported in this chapter is located within the latter. By instructional materials, we mean the actual materials that teachers bring into the classroom for students to work on to attain the intended goals of learning. Within the context of this study, we investigate particularly those portions of the instructional materials that consist of challenging tasks for students.

We first carried out a depth-wise search for the types of challenging items that the 30 experienced and competent mathematics teachers in the first phase of the project included in their mathematics instructional materials. Some of these items—and their design principles—were extracted for the next, second, phase of the project. The selection of these challenging tasks to be included in the survey in the second phase is based on its fulfilment of the criteria as cognitively demanding, as will be explicated in Sect. 12.2.1. We now turn to the second phase of the project as the rest of the chapter will report findings from this latter phase.

12.2 Chronologically-Grounded Survey

The aim of the second phase is to examine the extent in which the design and use of challenging items by the teachers in Phase One were also shared or modified by other secondary mathematics teachers in Singapore. We administered a "chronologically-grounded survey" (CGS)—based on items uncovered in the first phase of work—for more than 600 secondary mathematics teachers across the full range of school-types in Singapore (Leong, Cheng, & Toh, 2019). In terms of realistic methodology, it is not feasible to proceed using the same research instruments and frames in the first phase—that is, video-recording of representative lessons with teacher- and student-interviews interspersed within these recorded lessons—at this larger scale. As such, we drew the data in this second phase from teachers' response to an online questionnaire.

How do we translate the design moves we extracted from the earlier teachers into survey items in such a way that other teachers who read the items could resonate with

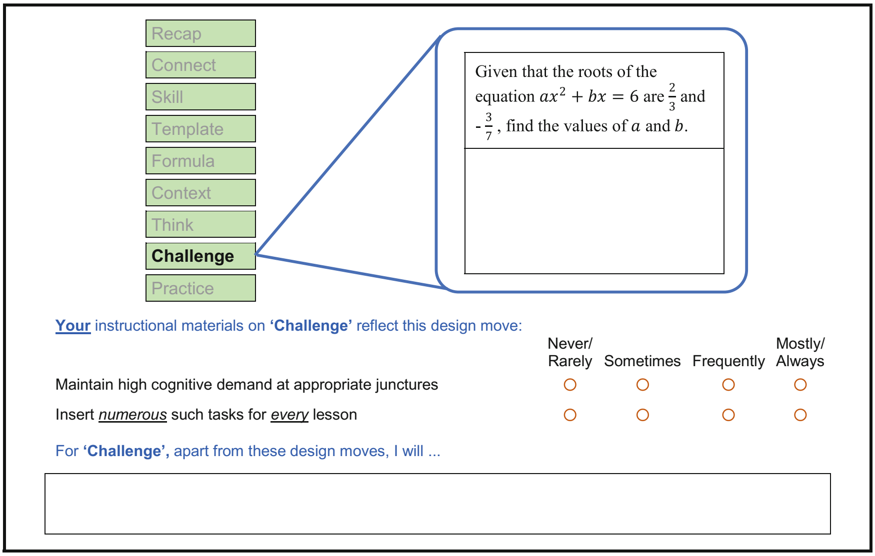


Fig. 12.1 An extract of an online screen-page in the second section of the questionnaire on a challenging item

its contextual meanings when responding to the questionnaire? CGS is an attempt at addressing this question. The CGS carried these features: (i) it was based on the actual instructional material used by teachers in Phase One; (ii) it presents (in the first section of the questionnaire) a broad sweep of how the material was used by showing chronological snippets of the material; (iii) the items (in the second section of the questionnaire) zoom-into specific moves used by the teacher in each snippet to elicit response; and (iv) the last section allows (optional) open comments. Figure 12.1 gives an extract of one such online screen-page in the second section of the questionnaire on a challenging item.

Figure 12.1 provides a concrete illustration of the four features: (i) the diagram at the top-right is an extract from a teacher's instructional material; (ii) the left column retains the chronological categories that were covered in the first section of the questionnaire; (iii) the response items below describe the zoomed-in design moves relevant to this particular part of the teacher's instructional material; and (iv) the free-response section at the bottom allows for additional comments triggered by the above sections.

We think that CGS has the potential of addressing these challenges that are common with survey design: (a) [Construct validity]. The use of authentic instructional materials and the chronological arrangement strengthens the teacher's closeness of interpretation to the intended meaning of the items. Singapore teachers are known to organise their instructional plans and routines along temporal lines.

Compared to "contextless" survey items, it is easier for teachers in the CGS environment to experientially connect to the chronological flow and contents of the instructional materials thus more readily respond to the items with a degree of mental resonance (or dissonance); (b) [research-practice link]. We did not craft the items based on some imaginary or purely theoretical starting point. Rather, as explained in the preceding section, we drew from the actual instructional materials used by the teachers in Phase One, and based on careful analyses, derived the characteristics of design that are "grounded" in practice. This rigorous process strengthens the closeness-to-practice within the research setup; (c) [belief-practice gap]. When teachers commit to, say, "frequently" to a response item, they are indicating an avowed belief towards the stated design move. It is acknowledged that there can be a significant gap between this avowed belief and actual design moves. With the items being "chronologically-grounded", there is a higher likelihood that teachers would not merely read these items in abstraction; rather, they would project mental imageries of how these sections of the instructional materials square (or not) with their lived professional experiences.

We crafted three sets of CGSs—each corresponding to respondents who teach primarily Additional Mathematics, Mathematics (Express), and Mathematics (Normal). In Singapore, Mathematics (Express) is the core mathematics subject for the majority of secondary students. Mathematics (Normal) is offered to students who would take an additional year to learn the same amount of content covered in 4 years in Mathematics (Express). Additional Mathematics is a separate mathematics subject offered to students who have the potential to pursue more advanced mathematics at the post-secondary levels. There is an earlier section of the survey where the respondents were asked for the secondary mathematics subject they teach primarily; the system will subsequently load the matching CGS according to the respondent's choice.

12.2.1 Challenging Tasks

Figure 12.1 shows the challenging task that was included in the Mathematics (Express) CGS. In determining whether a task is to be considered challenging, we adopted the framework developed by Stein and Smith (1998) with respect to the different levels of cognitive demand of tasks. Figure 12.2 shows an extract of their "Task Analysis Guide".

In Henningsen and Stein (1997), tasks were considered of "high level of cognitive demand" when they are categorised under "procedures-with-connections" or "doing mathematics". Similarly, we would consider a task "challenging" with respect to the target student group if it is classified under these categories.

For the task shown in Fig. 12.1, we categorise it as "doing mathematics" because it fulfils some of the descriptors listed in Fig. 12.2 under this category, such as "there is not a predictable well-rehearsed approach to the task", "require students to explore … the nature of mathematical concepts" (in this case, about roots of equation), "require

THE TASK ANALYSIS GUIDE

Lower-Level Demands

Memorisation Tasks

- involve either reproducing previously learnt facts, rules, formulae, or definitions OR committing facts, rules, formulae, or definitions to memory.
- cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.
- are not ambiguous—such tasks involve exact reproduction of previously seen material and what is to be reproduced is clearly and directly stated.
- have no connection to the concepts or meaning that underlie the facts, rules, formulae, or definitions being learned or reproduced.

Procedures Without Connections Tasks

- are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or placement of the task.
- require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it.
- have no connection to the concepts or meaning that underlie the procedure being used.
- are focused on producing correct answers rather than developing mathematical understanding.
- require no explanations, or explanations that focus solely on describing the procedure that was used.

Higher-Level Demands

Procedures With Connections Tasks

- focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
- suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.
- usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning.
- require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual links that underlie the procedures in order to successfully complete the task and develop understanding.

Doing Mathematics Tasks

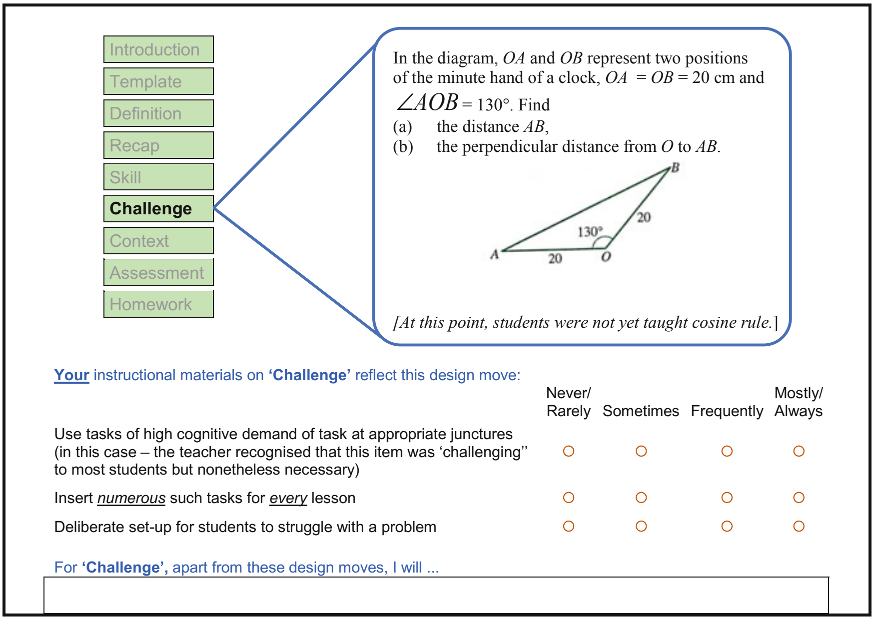
- require complex and non-algorithm thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example).
- require students to explore and understand the nature of mathematical concepts, processes, or relationships.
- demand self-monitoring or self-regulation of one's own cognitive processes.
- require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.
- require students to analyse the task and actively examine task constraints that may limit possible solution strategies and solutions.
- require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

Fig. 12.2 Extract of characteristics of mathematical tasks at each of the four levels of cognitive demand. Taken from Stein and Smith (1998)

students to access relevant knowledge and experiences and make appropriate use of them in working through the task", and "require students to analyse the task and actively examine task constraints that may limit possible solution strategies".

Figure 12.3 shows the challenging task in the Mathematics (Normal) CGS. Similar to the task as shown in Fig. 12.1, this task was also classified under "doing mathematics" as there is no prescribed method to follow to complete it. There is a need to explore auxiliary lines that would connect to relevant knowledge within the context of right-angled triangles, and then utilising the associated concepts and skills to solve the problem.

The challenging task for the Additional Mathematics CGS is shown in Fig. 12.4. We have classified it under "procedures-with-connections". At first look, the task appears to be one of mere procedural application of Chain Rule, and hence "procedures-without-connections". But a closer analysis reveals that careful re-representing of the function using the index notation with the appropriate use of



Introduction
Template
Definition
Recap
Skill
Challenge
Context
Assessment
Homework

In the diagram, *OA* and *OB* represent two positions of the minute hand of a clock, $OA = OB = 20$ cm and $\angle AOB = 130°$. Find
(a) the distance *AB*,
(b) the perpendicular distance from *O* to *AB*.

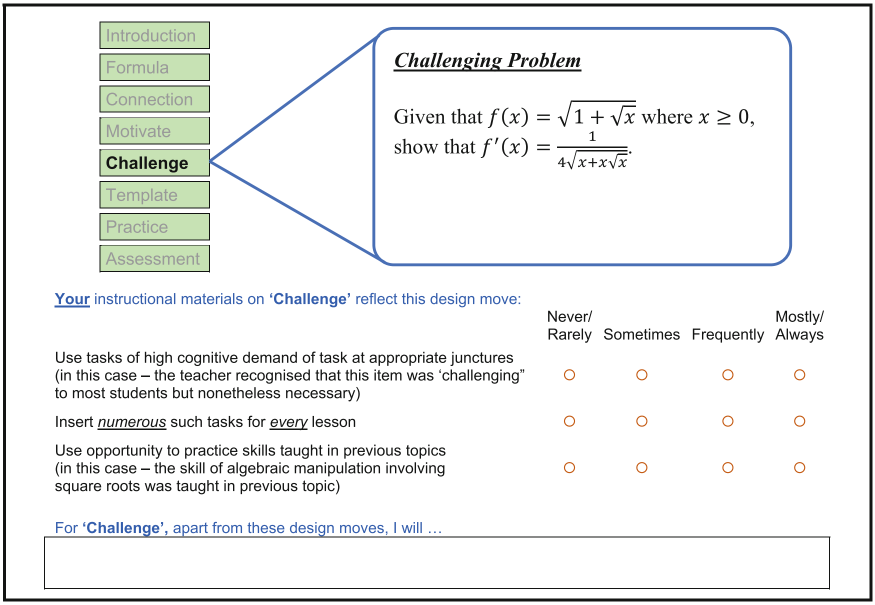
[At this point, students were not yet taught cosine rule.]

Your instructional materials on **'Challenge'** reflect this design move:

	Never/ Rarely	Sometimes	Frequently	Mostly/ Always
Use tasks of high cognitive demand of task at appropriate junctures (in this case – the teacher recognised that this item was 'challenging'' to most students but nonetheless necessary)	○	○	○	○
Insert *numerous* such tasks for *every* lesson	○	○	○	○
Deliberate set-up for students to struggle with a problem	○	○	○	○

For **'Challenge',** apart from these design moves, I will ...

Fig. 12.3 An extract showing the challenging item in the Mathematics (Normal) Grounded Chronological Survey (GCS)



Introduction
Formula
Connection
Motivate
Challenge
Template
Practice
Assessment

Challenging Problem

Given that $f(x) = \sqrt{1 + \sqrt{x}}$ where $x \geq 0$, show that $f'(x) = \frac{1}{4\sqrt{x + x\sqrt{x}}}$.

Your instructional materials on **'Challenge'** reflect this design move:

	Never/ Rarely	Sometimes	Frequently	Mostly/ Always
Use tasks of high cognitive demand of task at appropriate junctures (in this case – the teacher recognised that this item was 'challenging" to most students but nonetheless necessary)	○	○	○	○
Insert *numerous* such tasks for *every* lesson	○	○	○	○
Use opportunity to practice skills taught in previous topics (in this case – the skill of algebraic manipulation involving square roots was taught in previous topic)	○	○	○	○

For **'Challenge',** apart from these design moves, I will …

Fig. 12.4 An extract showing the challenging item in the Additional Mathematics Grounded Chronological Survey (GCS) brackets is necessary to correctly apply the Chain Rule. This means students need to "suggest pathways to follow that are broad general procedures that have close connections to underlying concepts", and "making connections among multiple representations [that] help to develop meaning". Moreover, after applying the Chain Rule, there is substantial manipulations required to work towards the form as given in the task—and in the process, drawing upon index-related calculations learnt in earlier Year levels. In other words, it "require[s] some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly." These descriptors fit those listed in Fig. 12.2 under "procedures-with-connections".

12.3 Data

Responses to Section (iii) of each of the CGSs, i.e. the items (in the second section of the questionnaire) that zoom-into specific moves used by the teacher in each snippet to elicit response, were coded according to this numerical assignment: 1, 2, 3, and 4 for each of Never/Rarely, Sometimes, Frequently, and Mostly/Always respectively. The common item across all the CGSs is "insert *numerous* such tasks for *every* lesson". This is an item that we have deliberately inserted to check the quality of response to this survey page (and more broadly, to the whole CGS). The logic is: if a respondent read this item carefully (and the underline and italics are meant to help them do so!), he/she is unlikely to express strong commitment to this statement. Thus, for respondents who indicate "4" for this item and other items in the same page without making a substantive remark under the free-response section, we assume he/she has not read the items on the page carefully and have thus removed their data from analysis in this study. Seven, six, and three of such data points were removed in this exercise from the CGS of the Additional Mathematics, Mathematics (Express), and Mathematics (Normal) respectively. The final respective number of respondents for each of the categories are 149 (23%), 284 (43%), and 227 (34%). The means for the quality-check item for these categories are 2.12, 1.90, and 1.94 respectively. That is, on average, the response to this item is "Sometimes". The means are the lowest across all the survey items within each of the category. We have therefore reason to interpret the survey data as arising mostly from respondents who have read the items carefully and provided an honest response.

We devoted substantial effort in Section (iv) of the CGSs—we think that, since the open-response section is optional, it takes extra effort to type in comments and if so it should indicate something substantive which the respondents want to contribute to. The numbers who wrote comments in this page of the CGSs were 61, 93, and 93 respectively. The percentages of those who provided comments in the respective categories were 41, 33, and 41%. When we first read the comments, they appear myriad and disconnected from one another. Since we did not have prescribed codes to use to categorise the responses, we proceeded initially with seeking to summarise each comment with one word—and treat each of these words as codes for the analysis of all the comments across all the three CGSs. This process yielded too many codes to be useful for further interpretation. We then re-examine these codes together with each comment with a view of combining some of these codes, and going beyond the surface meaning of the comment to the likely underlying instructional goals. To illustrate this, I present three sample comments:

- Will differentiate the instructions for students with different abilities
- Provide some hints to get the students started
- Ensure that if the problem is too challenging scaffolds or hints [be] given.

On the surface, the three comments appear to have nothing in common. The logic of linking them together flows along this direction: The first comment shows that the ostensible goal is about students' cognitive "ability" to handle task of high cognitive demand. The second comment does not look like it is about "ability". But, what is the reason for wanting to "provide hints"? (This question shows our way of drilling deeper in search of the underlying instructional goals). We think it has to do with the respondent's judgement that the item in itself is beyond his/her students' "ability" and thus the need to help them engage with the task through assistance in the form of hints. That this connection between the first two comments is not purely our theoretical speculation is strengthened by the third comment which explicitly links "hints" to "problem too challenging".

This process of reducing, comparing, and refining codes underwent numerous cycles of modifications. We also wanted to develop a common set of codes across the three CGSs for the purpose of comparison. The final set of codes are: task-affect, place of task, student ability, and other supporting features. Task-affect refers to the affective aspects of students in relation to challenging tasks. Place of task has to do with the positional placement of such challenging tasks within the context of the teacher's overall instructional vision. Student ability are references to the cognitive ability of the students with respect to challenging tasks. Other supporting features includes all other comments that pointed to ways to support the goal of challenging students to engage with these tasks productively. Where appropriate, we developed sub-codes to uncover further details about each area of consideration. Examples of responses that were placed under these codes and sub-codes will be shown in Sect. 12.4. We also do not view these codes as entirely separate in the sense that there is no logical relations between them; rather, we seek to draw links—where reasonable—so that we have an overall portrait of how Singapore secondary mathematics teachers think about and use challenging items in their design of instructional materials.

12.4 Findings

A similar item on "cognitive demand" is included in Section (iii) of each of the categories (see Figs. 12.1, 12.3, and 12.4) for the purpose of comparison across the categories. Minor adaptations in the phrasing were given to fit the context of each

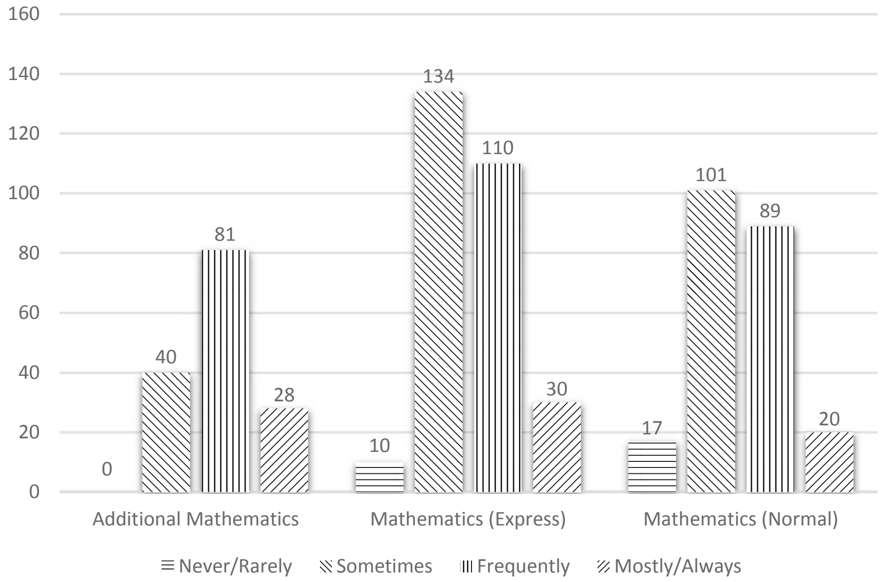


Fig. 12.5 Frequency graphs of responses to the cognitive demand item across categories

CGS to help teachers resonate better with the item. The responses for this item across the three categories are presented in Fig. 12.5.

The means for the Additional Mathematics, Mathematics (Express), and Mathematics (Normal) are 2.92, 2.56, and 2.49 respectively. The decreasing means also correspond to the decreasing levels of mathematical competencies associated with the respective subjects: as the mathematical "ability" level of students decrease, the teachers' commitment to setting tasks of higher cognitive demand decreases—which is expected. In the case of Additional Mathematics, the more pronounced higher commitment to more frequent use of challenging task may also have to do with the cognitive demand of the task. Compared to the challenging tasks in the other categories that belong to the higher category of "doing mathematics", the challenging task in the Additional Mathematics CGS is on the lower demand of "procedures-with-connections". As such, respondents may be more ready to commit to its increased frequency of use.

What may be surprising is the relatively high mean for Mathematics (Normal). This subject is taken by students who perform roughly at the 15–35th percentile range for the Year 6 common examination across all Singapore Primary schools. One may expect that the type of challenges presented to students who are offered this subject to be low. But based on the mean of 2.49, the average response to the frequency of use of such items is in the middle of "Sometimes" and "Frequently". This means that, on average, Singapore secondary students who read Mathematics (Normal) are still given challenging items to attempt on a sometimes-frequently basis. We do not have the data to determine what this category of sometimes-frequently translates to in terms of number per time period. But if it means, say, once-a-fortnight, over a prolonged period, it can still amount to substantial exposure to challenging mathematics, and thus an expectation to push oneself to meet the level of cognitive demand. For the rest of this section, will report the findings based on each of the identified codes.

12.4.1 Task-Affect

Table 12.1 shows the comments included under this code. The italicised portions are included by us to help the readers understand the reasons for coding them under the "affect" domain of the task.

There is an immediately noticeable difference under the task-affect code across the categories. The difference is not merely in the number of responses that attended to task-affect—it is unsurprising that teachers involved mainly in the teaching of Additional Mathematics do not instinctively worry about students' affect as it may be assumed that only more mathematically-proficient and thus more motivated students read Additional Mathematics. Neither is it surprising that more teachers of Mathematics (Normal) made explicit their attention to affect than teachers of Mathematics (Express). But there is also a difference in the response to students' affect. While the responses under Mathematics (Express) were primarily that of avoidance or reduction of challenging tasks so as not to discourage such students; in the case of Mathematics (Normal), almost all the responses stated specific strategies they would employ to help students' engagement with challenging tasks. These strategies include group work, placement of such tasks at a more suitable juncture, provision of incentives, scaffolds for solutions, and gradation of tasks that will aid in the gradual build-up of confidence.

There are dangers of generalising based merely on a small number of responses as shown in Table 12.1. But the differences are so striking that we think an inference has its place here: Due to a larger number of students who would struggle with challenging tasks in the Mathematics (Normal) group, as compared to the other groups, it is perhaps unsurprising that teachers of these student groups, over time, need to attend more to the affective needs of these students through devising a range of affect-enhancing strategies. In reality, however, this realisation can often lead to the easier way out "on the ground"—"since the students don't like these challenging tasks, then we don't give it to them". Instead, the responses show that these teachers accept that some students struggle affectively with these challenging tasks but at the same time they are committed to finding ways to help them engage with them. If this is indeed a widespread practice among Singapore Mathematics (Normal) teachers, it would partially explain the high performance of Singapore mathematics students in international tests: even our lower-progress students are regularly given and encouraged to engage with cognitively-demanding mathematics tasks.

Table 12.1 Comments coded under "task-affect"

Additional Mathematics	Mathematics (Express)	Mathematics (Normal)
[No entry]	• Majority of my students are weak, so cannot give too many of these type[s] of questions, as they will be *discouraged* • I feel it runs the risk of *alienating them* even further if they cannot cognitively manage the topic at easier levels. • Challenge is to *build confidence* through success • Challenge question can still vary in levels. Students *feel good* if they are able to solve a challenging question despite it being only Level 1 • Those who are grappling with foundations, I try *not to demoralise* them with such questions	• Provide *incentive/rewards* for the students who can attempt the tougher questions • Put [challenging task] at the end of the lesson instead of the middle as it might *disrupt their interest* in learning of all the concepts in the lesson. • Students *require more motivation* with more manageable task • *Build more confidence* by having easier examples first • If I include a challenging problem, I will definitely address it in class to *build my students confidence* in dealing with tough problems • I will put it as last question or last page of the notes so as *not to put off* students • Make sure that students are given some challenging questions to practise after the basic ones so as to challenge their thinking skills further and *boost their morale* further if they are able to solve them • I usually limit to a maximum of one of such question in a lesson so as *not to discourage* them • Provide scaffolds for students or hints if needed. Group work can be done so students do not feel like they are struggling alone. This makes them more likely to accomplish the task with *good motivation* • Not use such challenging questions as it will *deflate the confidence* of my students • Design challenge questions appropriate to students' ability (differentiated). Some high cognitive demand questions may just *demoralise* my students • I will use buddy mentoring to *level up the confidence* of low progress learners
	5 comments	12 comments

12.4.2 Place of Task

Similar to the previous category on task-affect, we have also included all the comments coded under place of task as shown in Table 12.2.

As can be seen in the responses, "place" as used in this code has a number of meanings. It is not restricted to a physical location (as in, its location in a set of instructional materials). It also includes temporal location—with respect to the point in time such tasks are introduced in a lesson or unit of lessons. In general, responses that refer to a kind of positioning of challenging items within the context of the teacher's vision of teaching are included in this category.

Relatively, there are fewer comments under this code for the Additional Mathematics group. This may mean that lesser Additional Mathematics teachers are conscious of the need to attend to placement of tasks. The common terms used—especially in the other two groups—were "optional", "homework", "at the end", and "if time permits". These terms—or their equivalent—are italicised in Table 12.2 for the ease of reference for readers. Taken together, the portrait of how teachers use challenging items is one of lower priority compared to other items of lower cognitive demand. They are thus brought in "optionally" and "if time permits" into regular classroom instruction. Perhaps, assigning them as "homework" also fits into this lower-priority scheme—students may not do them in class but motivated students would still be given the opportunity to attempt them as homework. But these items were seen to have a place "at the end" of a lesson (or a chapter), presumably because students would then be more ready—having learnt more relevant content and developed more related skills—to attempt them productively.

Upon closer scrutiny, there are also some differences between the Mathematics (Express) group and the Mathematics (Normal) group. There are five comments (bold in Table 12.2 for ease of reference) under the latter group that is of a different consideration from the lower-priority theme mentioned earlier. The foci of these comments were on the surrounding context and the intentional goal of such tasks. The first four comments mentioned the placing of challenging tasks within the context of other simpler tasks that act as build-up mechanisms to ready students for greater challenge. The last two comments (the 4th comment also belongs to the earlier set) stated the use of challenging tasks with a specific deliberate goal—one for building students' confidence (that is, if students can solve even challenging tasks, it will boost their confidence), and the other for twinning with a specific problem solving strategy. In other words, these comments focused on specific design considerations surrounding the challenging task so as to render it more effective for the instructional goal intended. These teachers do not see the challenging item "in isolation"; rather, they are intentionally conscious of how to place it within an instructional scheme to enhance purposeful instructional ends. It is interesting that none of these considerations were mentioned under the Mathematics (Express) group. Again, we can only surmise an inference: Teachers of Mathematics (Normal) classes are aware that challenging tasks are not naturally readily accepted by students; there is thus a need to intentionally build them into their instructional plan carefully.

Table 12.2 Comments coded under "place of task"

Additional Mathematics	Mathematics (Express)	Mathematics (Normal)
• Usually challenge questions are set in *assignments* rather than in class worksheets • Include two or three challenging questions *at the end* for the higher abilities students to try out • Give them higher order questions to try at *home* • They will come at the *later part*	• For revision packages, there are challenging questions inserted into them. For weekly revision worksheets, there is one standard and one challenging version *online* for those who wish to try • Such questions are usually reserved *at the end* … • Sometimes, these questions are given as *optional* [sic] *homework* questions or discussion questions during lesson • I will leave the challenge part to the *last part* of my worksheet … • Usually difficult tasks are left to *the end* of the entire topic • Challenging questions are inserted *at the end* of the worksheet to students who would like to challenge themselves • Make it *optional* for students to try out the challenging questions of different difficulty levels • Issue as *homework* to discuss the next day • It's good at certain juncture to train students to [refer to prior content] on their own to reinforce understanding • Also add in some *optional* challenges for students who are interested to explore and investigate • Include one such task as part of the *assignment* and bring it out for class discussion on the next day • I usually have this type of problems in test (as a bonus question) • Insert as bonus question in class test • Include in Quiz • If *times permits* … • Place this as an *optional* question for those students who are way ahead to start thinking about the questions • Usually I will get students to try all these questions *at home* and come back to class the next lesson to discuss their solutions • Make these challenging questions *optional* for higher ability students to attempt on their own	• Make the question *optional* for the students to do • Set it *at the end* of the lesson for students to think about it *at home* and come back to share with the class how to solve the problem • Place it *at the back* of the paper/worksheet instead of in between • Put it *at the end* of the lesson instead of the middle as it might disrupt their interest in learning of all the concepts in the lesson • Make it *optional* for slower students … • **Big gap between this Challenge from the previous snippet. Include more scaffolding** • **Build more confidence by having easier examples first** • **Not show this high level of question yet. Will show a more simple practice question** • Make it *optional* • **If I include a challenging problem, I will definitely address it in class to build my students confidence in dealing with tough problems. … Also, at this junction, it may not be appropriate as there is not enough practice examples given for basic questions** • … [J]ust pose such questions to the stronger ones • I will put it *as last question* or last page of the notes so as not to put off students • Normally insert challenges *at the end* of a topic as a graded assignment for students to do as *homework* • The challenge will be put at beginning of the lesson or *at the end* of lesson. … normally I don't put it in the middle because the flow of thought will be interrupted • Introduce *at the end* of topic • Only cover it when *time permits* • Although I always include challenging questions in my resources, they are usually placed *at the end* of my lesson package after I have finished teaching the entire chapter • Insert a few *at the end* of the chapter and do as classwork • **Usually … the question can either be targeted for learning specific problem solving heuristics in context or lead to a deeper appreciation of concept learnt** • Challenge at only appropriate topics or timings
4 comments	18 comments	19 comments

12.4.3 Student Ability

There were 19, 34, and 46 comments placed under the "student ability" code for Additional Mathematics, Mathematics (Express), and Mathematics (Normal) respectively. Again, the increasing numbers are not surprising when we consider that teachers of mathematically low-progress classes are more conscious of the task-ability gap. Since there are a total of 98 comments under this code, they are too many to be displayed in a table like the ones in Tables 12.1 and 12.2. We will instead report here on the main themes highlighted by these comments.

The common themes across the groups are "high-ability", "low-ability", "differentiation", "guidance", and "group work". The comments under this code stress on the likely "gap" that may occur between the ability of their students and the cognitive demand of the task. This is where the high-ability versus low-ability talk comes in: the assumption was that high-ability students are generally capable of attempting these tasks productively on their own and should be left alone to do so with limited extra resources from the teacher. Also, their cognitive needs should be attended to and so the inclusion of such challenging tasks are necessary for them; in contrast, the low-ability students would need far more resources from the teacher to make progress in these tasks. One way to help these students is to provide "differentiation"—give tasks that are gradated and the level of cognitive demand transparent to students, so that low-ability students will start off with easier questions and then move on to more challenging ones when they feel ready, or not at all. In any case, these students need a lot of "guidance" with challenging questions. A number of such guiding strategies were proposed in the comments: teacher guidance as and when student requires, guidance that are built into the task such as partial solutions or fill-in-the-blanks, or breaking down the task into scaffolding parts that would help lead to the final solution. Another common way to help these low-ability students is "group work", with the assumption that these students would more likely make progress in a group rather than being left alone to struggle with the task.

We do not detect significant differences for this code across the groups. There were, however, four comments in the Mathematics (Normal) group which were very specific. This level of specificity is not found in the comments of the other groups. We reproduce the comments here:

- Get students to rotate the paper. Hint the concepts they learnt in trigo ratio.
- Extend AO to the right, drop a perpendicular line from B, investigate the new triangle formed.
- I will change the order of the parts. I will begin with a similar triangle but with AB as the base.
- Start with a right-angled triangle, and using the correct trigo ratios to find the unknown side or angle followed by two right-angled triangles joined together before giving them the challenging questions

To us, this indicates a finer-grained specificity that may be instinctive among teachers of Mathematics (Normal) classes. Their experiences with particular difficulties students encounter in their classes may have afforded these teachers a response towards specific pinpointed techniques to help these students become unstuck when confronted with challenges.

12.4.4 Other Supporting Features

There were 36, 45, and 29 comments placed under the "other supporting features" code for Additional Mathematics, Mathematics (Express), and Mathematics (Normal) respectively. The common themes that emerge across the categories were types of challenging task, sources of challenging tasks, thinking skills to focus on, and teachers' involvement. On types of challenging task, there were specific suggestions of challenging tasks other than the ones shown in Figs. 12.1, 12.3, and 12.4, such as problems set in real-life context, application questions, and even spot-the-error tasks. On sources of challenging tasks, there were recommendations that such tasks can be taken from popular local online repositories, questions generated by students themselves, and specific pages from common textbooks. But most teachers pointed to past year exam papers that had challenging items. Presumably, these teachers preferred that students be familiarised with challenging questions that are closer to the type in high-stakes examination. For thinking or reasoning skills to focus on, there was a range of emphases: to provide the justification for solution steps, to establish connections among mathematical ideas from different topics, to not only solve the challenging problem but to also adapt/extend the problem so that similar solutions strategies can be applied to them, to attend to alternative solution strategies, and to communicate solutions to others—especially in presenting to fellow students. On the teachers' involvement, the focus was on teacher questioning and thus guiding students to obtain or understand the solution to the problems.

From these themes, we may form an overall portrait—one that focuses not just immediately on the challenging task presented but also the supporting setup to achieve the underlying goal: to help students habitually engage with challenging tasks. Indeed, to provide such tasks regularly, teachers need to know stable sources for them (that is, where to look for such tasks instead of the impractical expectation of teachers generating them on their own). Not only so, teachers are to be conscious in selecting different types of challenging tasks so as to expose students to a wide range of problem types. Also, it helps not only to focus on techniques that are task-specific; rather, there should be an explicit teaching of thinking or reasoning processes that cut across a range of challenging tasks. Finally, there is a recognition that the teachers' role—especially in design and actual scaffold of students through suitable prompts—is vital for sustaining students' engagement in challenging tasks.

Again, we noted that a number of comments from the Mathematics (Normal) group are unique to this group:

- *Discuss with students to check on their understanding* before seeking input on how they propose to solve the question

- *I get them to verbalise their thought process* and demonstrate the thinking process
- *Get students to notice* what is different from the rest of the questions the students have tried earlier and see if they can spot the extra step(s) that must be taken to get them to a procedure that they are familiar with
- *Get students to suggest* how they would approach the problem
- *Guide the students* to check their answer.

The common thread across these comments is the sensitivity towards the need to start with students and not just with the demands of the task. Those portions that reflected this commonality are italicised within the comments for ease of the readers' reference. We see again that some teachers who worked with lower-progress mathematics learners were particularly attuned to the needs of the learners.

12.5 Discussion and Conclusion

The findings from this study—when the strands are weaved together—provide a portrait of how Singapore secondary mathematics teachers think about the use of tasks of high cognitive demand for their students. It is not one where teachers randomly or arbitrarily insert these tasks to students as and when they appear in the teaching sequence; nor, perhaps others outside of Singapore would surmise: that of constantly "drilling" students to do repeatedly a series of such similar challenging tasks. [This might be one image of how Singapore students do so well in international comparison tests]. Rather, the picture that emerges is one of deliberate inclusion of challenging tasks in a way that takes into consideration the context and the students. In particular, the teachers were cognizant of the task demands in relation to the range of cognitive abilities of their students; as such, there needed to be careful planning of how the task could be engaged in a sustained way for all their students. Strategies included placement of the tasks at more suitable junctures of learning, differentiation of tasks for different students, teacher scaffolding, and structures such as group work to draw upon the resources of other students. The examination of the tasks were also an important consideration—so that the nature of the task matched the specific reasoning skills that were intended for the needs of the students. The selection of tasks—the sources and the potential in providing a variety of problem-types—were also mentioned as a critical process. Apart from these more cognitive aspects, managing the affect of students was also flagged as an important consideration, especially among the teachers of Mathematics (Express) and Mathematics (Normal) classes.

There is also evidence to suggest that some teachers of Mathematics (Normal) classes have developed greater sensitivity to the difficulties that these students have when confronted with challenging tasks. As such, they were more specific when offering strategies to help these students engage with the tasks—down to the types of gradating tasks leading to challenging tasks, how to (re-)word some of the tasks, and efforts to start with the incorporation of students' current ways of thinking about the task. As the number of comments of this nature are small, we should be careful not to generalise these observations. However, if they do represent the thinking of a larger group of Mathematics (Normal) teachers, it indicates a deliberate willingness to think of ways of regularly challenging these students who are viewed as "lower-ability" in mathematics with tasks of high cognitive demand.

The findings here also provide some illumination on the question we regularly get from international observers: Why do Singapore students do so well in PISA—which ostensibly also measures ability of solving tasks of high cognitive demand? It appears that the teachers take it as their responsibility (that is, they do not resist this notion) of regularly exposing their students—regardless of their ability in mathematics—to challenging tasks. But it is beyond mere exposure: the teachers take serious steps to ease these tasks into their instructional scheme so that students are more able to engage productively with these tasks—both cognitively and affectively. They are concerned that when these tasks are introduced in class, that there are also carefully-planned supportive features to heighten the chance of its acceptance by the students and of its being productively engaged by them. The goal seems to be a normalisation of students "rising up" to these tasks instead of "padding down" the tasks to suit the students "level".

In a global climate where the pressure seems to be that of reducing stress for students (especially of mathematics), the conventional wisdom is tending towards one where tasks ought to be always manageable by all students so that they would not lose interest in mathematics. But this stance is hardly supported empirically. Rather, it seems to us a slippery slope of an ever-increasing expectation of "making things easy" which would in the long run result in a watered-down mathematics syllabus that is ultimately not beneficial to the students. In contrast, the Singapore experience is one of commitment towards regular (not excessive) exposure of tasks of high cognitive demand as an appropriate cognitive challenge for students. Instead of lowering this expectation, efforts are placed at devising strategies to help students develop useful skills and habits to "rise up" to the challenges. The extant literature is still scarce in this area of helping and sustaining students' engagement with tasks of high cognitive demand. This study contributes in part to the awareness of the kinds of strategies that Singapore teachers use for this purpose. But much more can be done in research and in the codification of coherent strategies so that professional development work for teachers can be targeted to this area of international interest.

References

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524–549.

Kaur, B., Zhu, Y., & Cheang, W. K. (2019). Singapore's participation in international benchmark studies—TIMSS, PISA and TEDS-M. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 101–137). Singapore: Springer.

Leong, Y. H., Cheng, L. P., & Toh, W. Y. K. (2019). Chronologically-grounded survey. In G. Hine, S. Blackley, & A. Cooke (Eds.), *Proceedings of the 42nd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 90–93). Perth: MERGA.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework of reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*(4), 268–275.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards based mathematics instruction: A casebook for professional development*. New York, NY: Teachers College Press.

Tekkumru Kisa, M., & Stein, M. K. (2015). Learning to see teaching in new ways: A foundation for maintaining cognitive demand. *American Educational Research Journal, 52*(1), 105–136.

Yew Hoong Leong is an Associate Professor at the National Institute of Education, Nanyang Technological University. He began his academic career in mathematics education with the motivation of improving teaching by grappling with the complexity of classroom instruction. Along the journey, his research has broadened to include mathematics problem solving and teacher professional development. Together with his project teammates, they developed "Realistic Ambitious Pedagogy" and its accompanying plan of action—the "Replacement Unit Strategy".

Lu Pien Cheng is a Senior Lecturer in the Mathematics and Mathematics Education Academic Group at the National Institute of Education, Nanyang Technological University in Singapore. She received her PhD in Mathematics Education from the University of Georgia (U.S.) in 2006. She specialises in mathematics education courses for primary school teachers. Her research interests include the professional development of primary school mathematics teachers, task design in mathematics education and cultivating children's mathematical thinking in the mathematics classrooms.

Wei Yeng Karen Toh worked as a research assistant at the National Institute of Education for seven years. She has a Bachelor of Science in Applied Mathematics from the National University of Singapore. She was awarded with a Distinction grade for the Post-Graduate Diploma in Education at the National Institute of Education. She taught in a few secondary schools as a mathematics and chemistry teacher for seven years. Her research interests include studying the history of mathematics, learning mathematical and pedagogical content knowledge, examining classroom practices and investigating teachers' designs of instructional materials. To advance her career in research, she pursued a Master of Education in Mathematics at the National Institute of Education.

Chapter 13
Sequencing of Practice Examples for Mathematical Reasoning: A Case of a Singapore Secondary School Teacher's Practice

Lu Pien Cheng, Yew Hoong Leong, and Wei Yeng Karen Toh

Abstract Variation of examples is a common technique some teachers use in the design of their instructional materials. It is, however, not clear how mathematical reasoning can be supported through teachers' carefully selected examples. Through a case study of an experienced and competent Singapore secondary school mathematics teachers who emphasised "reasoning" as a specific goal of his instructional practice, we examine how practice examples were designed to target reasoning in the teaching of mathematics. In particular, the study unpacks how mathematical reasoning can be utilised as a glue in advancing a canonical technique alongside the development of supportive lesson routines. The findings showed the following four design principles (i) Deliberate use of examples to advance technique (ii) Advance technique through comparing, inferring and justifying (iii) Special cases to expose and target students' faulty reasoning undergirding the techniques they used (iv) Consolidate and formalise the reasoning in standard written form through whole class instructional segment.

Keywords Mathematical reasoning · Instructional materials · Practice examples

13.1 Introduction

We are part of the project team (see Chapter 2) that aims to distil the distinctive features of mathematics teaching in Singapore classrooms. In the course of our data collection, the manner in which examples were used by Teacher 13 particularly caught our attention. Teacher 13's use of examples was not merely for achieving fluency of technique; his other ostensible goal was to use the examples to "advance" (a term he

L. P. Cheng (✉) · Y. H. Leong · W. Y. K. Toh
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: lupien.cheng@nie.edu.sg

Y. H. Leong
e-mail: yewhoong.leong@nie.edu.sg

W. Y. K. Toh
e-mail: karen.toh@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_13

used) mathematical "reasoning"—a term he used very often in his interviews with us and in his classroom teaching. As we studied the numerous instances in which the examples were used to encourage students' reasoning, we found that Teacher 13's conception of these examples and how they were utilised in his classroom work presented aspects that are still unreported in the current literature.

13.2 Use of Examples

Before we examine the case of Teacher 13's use of examples to advance reasoning, we turn to the literature that relates to this focus of study. In particular, we review literature on task design, variation theory in the sequencing of tasks and practice examples.

13.2.1 Task Design

Mathematical tasks are important vehicles for building student capacity for mathematical thinking and reasoning (Stein, Grover, & Henningsen, 1996). The recently published ICMI Study 22 on *Task design in mathematics education* presents an up-to-date summary of relevant research about task design in mathematics education. While there are advances in knowledge in areas, such as multiple frameworks and sets of principles on task design—the design and implementation of task sequences is one of the key areas identified as still needing further research (Watson & Ohtani, 2015). In this study, we limit the discussion on tasks with the ostensible goal of students' reasoning. Smith and Stein's (1998) categorisation of tasks to different levels of cognitive demands where the extent to which students engage in thinking and reasoning differ in each level is relevant and influential. However, there is yet little research on how tasks or a sequence of tasks can be employed to help students advance their reasoning in the process of working through these tasks.

13.2.2 Sequences of Tasks and Variation Theory

In the design principles behind the design of task sequences, there is major contribution in the literature from Variation Theory (VT). In the design of task sequences, "VT focuses task designers on what varies and what remains invariant in a series of tasks in order to enable learners to experience and grasp the intended object of learning" (Kieran, Doorman, & Ohtani, 2015, p. 45). Watson and Mason (2006) claimed that "tasks that carefully display constrained variation are generally likely to result in progress in ways that unstructured sets of tasks do not, as long as learners are working within mathematically supportive learning environments" (p. 92). Their paper illustrates how differently controlled variations can help students make different generalisations and abstractions—generalisations as "sensing the possible variation in a relationship" and abstractions as "shifting from seeing relationships as specific to the situation, to seeing them as potential properties of similar situations" (p. 94).

Watson and Mason's (2006) study reported a rare venture within the tradition of VT-focusing on the mathematical content as the only "object of learning"- and they explain "generalisations" that are afforded by a particular way of sequencing tasks. This shift to "generalisations" brings this branch of VT-related research closer to our inquiry on mathematical reasoning. In fact, we do not find mathematical reasoning an explicit goal in much of the research reported under the banner of VT. Another recent rare connection of reasoning to VT was reported by Vale, Widjaja, Herbert, Bragg, and Loong (2017). They showed how justification or logical argument fits into this sequence of learning experiences by mapping the variation in children's reasoning (e.g., comparing and contrasting to generalise and identifies verifying) in number commonality problems. Their findings reiterated the importance of designing the task so that the action of comparing and contrasting guides student awareness of features that matter. Their paper, however, does not unpack the deliberate task sequencing for the purpose of advancing students' mathematical reasoning.

For the purpose of this study, instead of using "task" which is a very board category, we employ Teacher 13's use of "examples" which we take to be a special type of tasks. We think that examples are tools to provide variation to aid students' reasoning. Examples, especially "combination of several similar examples and further not-quite similar examples" are necessary for students "to work on a higher level" that leads to conceptual learning, fluency and accuracy (Watson & Mason, 2006, p. 97).

13.2.3 Practice Examples

> From a mathematical perspective, an example is often considered an object satisfying certain conditions (e.g., Alcock & Inglis, 2008; Watson & Mason, 2005), or a representative of a class (e.g., Mills, 2014; Zazkis & Leikin, 2007). Zaslavsky (2014) adds a requirement that the person using the example should be able to answer the question: "What is this an example of?" (Zaslavsky, 2019, p. 246)

Zodik and Zaslavsky (2008, p. 165) referred to examples as "a particular case of a larger class, from which one can reason and generalise". Their treatment of examples included non-examples that are "associated with conceptualisation and definitions, and serve to highlight critical features of a concept; as well as counter-examples that are associated with claims and their refutations" (p. 165). "Mathematical objects only become examples when they are perceived as examples of something: conjectures and concepts, application of techniques or methods …" (Goldenberg & Mason, 2008, p. 184). That is, examples may differ in their nature. An example of a concept such as rational number is different in nature from an example of how to carry out a procedure (Zodik & Zaslavsky, 2008). Examples can also differ in their purpose such as to illustrate how to find a common denominator of two proper fractions for adding fractions, or to illustrate it so as to generalise the procedure to algebraic fractions to solve more advanced equations (Zodik & Zaslavsky, 2008). Rowland (2008)'s work informed us of possible considerations in the choice and selection of examples; variation to be experienced by the learners in accordance to Marton and colleagues' (2004) Theory of Variation. Zodik and Zaslavsky (2008) also suggested some considerations when teachers select or generate examples.

Two different use of examples in teaching were reported by Rowland (2008). The first use is inductive—providing examples of something and the examples are "particular instances of the generality" (Rowland, 2008, p. 150). The second use is not inductive, but for the purpose of practice and is often referred to as "exercises". To illustrate this use: students learnt the procedures to find equivalent fractions and then rehearsed and worked on the "exercise" so that they could remember it through repetition and eventually developed fluency of the procedures. Examples used in this way was "for practice", also known as *practice examples* in that they are "vehicle for [students] to gain fluency with the algorithm" (Rowland, 2008, p. 158).

In this chapter, we define Teacher 13's sequence of practice examples as a carefully selected set of examples meant to fulfil his instructional goals. Reminded by Zaslavsky (2019), we specify or define practice examples by referring to the goals, "what is it an example of?". In Teacher 13's case, his practice examples were

1. For the purpose of practice to gain fluency by an underlying technique;
2. They are thus examples of the technique; and
3. As a set of vehicles to advance reasoning (This will be detailed in the analysis later).

13.3 Method

13.3.1 Context

Teacher 13 was identified as an experienced and competent teacher. An "experienced" teacher is one who has taught the same mathematics course at the same level for a minimum of five years; and "competent" selection is based on recognition by the local professional community as a teacher who is effective in teaching mathematics. Below, we summarise a number of factors about Teacher 13's practices that lends itself to an unpacking of his reasoning work:

1. Teacher 13's repeated reference to "reasoning" in interviews articulating comprehensively his goals for the tasks in his instructional materials and his classroom discourse determined our choice of Teacher 13 as a case study of mathematical reasoning—a characteristic feature of case study.

2. A full set of handouts (instructional material) for students use in class (hereafter referred to as "Notes") was developed with his colleagues before the start of the module.
3. He constantly made references between his goals, his actual activity in class, and his use of instructional materials. This enables us to study the interactions among these major pieces of his instructional processes.

13.3.2 Teacher 13's Class and Students

The class in which we recorded his teaching consisted of 20 students aged between 14 and 16. They were in their third year—akin to Year 9—of a six-year Integrated Programme (IP). In Singapore, there are various education paths students can choose to take after they complete six years of elementary school. Of which, students in the IP pursues a six-year course which integrates the four-year secondary and two-year pre-university education programmes.

13.3.3 Topic

The module that Teacher 13 taught was "Quadratic Graphs and Inequalities" over 10 lessons. The coverage includes from Ministry of Education (2012):

1. Solve quadratic equations in one variable using: (i) the general formula, (ii) completing the square and (iii) graphical method;
2. State the conditions for a quadratic equation to have: (i) two real roots, (ii) two equal roots and (iii) no real roots;
3. State the conditions for $ax^2 + bx + c$ to be always positive (or always negative); and
4. Solve quadratic inequalities, and represent the solution: (i) using a graph, and (ii) on the number line.

13.4 Data and Analysis

13.4.1 Data

The instructional materials used by Teacher 13 were mainly the Notes and questions from the textbooks (Yeo, Teh, Loh, & Chow, 2013; Yeo et al., 2015) (as homework) for the students. These instructional materials form the primary source of data. The next data source are the interviews conducted with Teacher 13—one pre-module interview before his lessons and three post-lesson interviews after three lessons he selected (Lessons 3, 6 and 8). All interviews were video recorded. The pre-module interview was conducted to mainly find out Teacher 13's instructional goals and how he designed and planned to utilise his instructional materials to fulfil his goals. Examples of the pre-module interview questions were:

- Please share with us some of the goals for this series of lessons. You can include both content and non-content goals.
- Are there any specific difficulties you anticipate that some of your students may have with some of the instructional materials?

The post-lesson interviews were conducted to find out if he had met his instructional objectives with the instructional materials he designed and planned to use. Examples of the questions were:

- What were the design principles you used in the instructional materials?
- What do you have in mind when you designed this/these item(s)? What learning experience(s) do you want your students to go through?

The third source of data is Teacher 13's enactment of his lessons in the module. We adopted non-participant observer roles during the course of our study—one researcher sat at the back of the class to observe Teacher 13's lessons—for the researcher to make relevant and specific references to his teaching actions when pursuing certain threads during the post-lesson interviews. A video camera is also placed at the back of the class to record Teacher 13's actions. A total of 10 lessons were video recorded for Teacher 13.

13.4.2 Analysis

We proceeded with our analysis illustrated in Fig. 13.1.

Stage 1: Identification of Units of Analysis of the Notes

From Teacher 13's lessons, we noticed that the tasks, in the form of *Examples*, he utilised from the Notes could be grouped into units according to the method applied to solve them. In this paper we present detailed analysis of two units which contained Teacher 13's most references to reasoning goals. They are:

- Unit 1: Examples and Practice Examples 5, 6(a), 6(b), 6(c) and 6(d)
- Unit 2: Examples and Practice Examples 10, 11 and 12

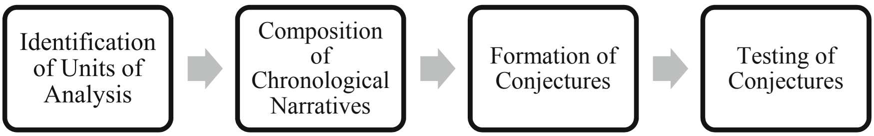


Fig. 13.1 Analysis procedure

We coded Teacher 13's classroom vignettes of the respective lessons and the responses he gave in the relevant interviews for these units of analysis. Some of the initial codes were "reasoning", "deducing", "comparing" and "thinking"—words Teacher 13 used often in his interviews and lessons. We then referred to Jeanotte and Kieran's (2017) conceptual model of mathematical reasoning in which they defined mathematical reasoning as "a process of communication with others or with oneself that allows inferring mathematical utterances from other mathematical utterances" (p. 7). Eventually, we consolidated our coded data into three categories, "comparing", "justifying" and "inferring". Comparing and justifying were taken from Jeannotte and Kieran. We included inferring as a third category, by this we mean a conscious guessing of another step in the deductive reasoning process. In other contexts, this is equivalent to "deducing"; we avoid this latter term as it has a more restrictive use in Jeanotte and Kieran (2017). These three categories will be elaborated in the *findings* section to support our claim that Teacher 13 utilised his tasks to develop his students' mathematical reasoning.

Stage 2: Composition of Chronological Narratives

For some of these selected units with related data on Teacher 13's enactment and interview comments, we crafted chronological narratives (CN), i.e. according to timeline for each of them. In each CN, we integrated pre-module interview transcriptions, post-lesson interview transcriptions, tasks in his Notes and his classroom vignettes. The process for the composition of CN for Unit 1, for instance, began when we examined his responses in a post-lesson interview. He commented that Example 6(d) was specifically selected so that students could apply some reasoning skills; and Example 6(c) was one that required students to be able to deduce from the graph they would sketch. Hence, we studied the video clip of the corresponding lesson segments carefully for the surrounding context. We noticed that the tasks—Examples 6(c) and Example 6(d)—that required reasoning were similar to Example 5 which Teacher 13 demonstrated to his students earlier in Lesson 5. This led us to re-examine Lessons 5 and 6 with a view of Teacher 13's agenda in helping students learn reasoning skills along the categories of comparing, inferring and justifying. We consolidated the evidence and organised them chronologically in a table. Table 13.1 shows the CN for Unit 1.

Stage 3: Formation of Preliminary Design Principles related to Mathematical Reasoning

We begin our intensive source of analysis to identify themes related to how Teacher 13 embedded reasoning in Unit 1. The themes were confirmed after several rounds of discussions with several members of the research team and that the uncovered design principles were supported by the data sources.

Stage 4: Testing of Preliminary Design Principles

In the final stage of analysis, we brought the design principles we conjectured in Stage 3 and checked it against all the other CNs. Through this process, we managed to refine the initial design principles into a form that contributes to theory generalisation. In the next section, we present our findings on the processes of analysis under Stages 3 and 4 by first detailing the CN on Unit 1, followed by another CN on Unit 2.

Table 13.1 Overview of the Chronological Narrative (CN) of Unit 1

Lesson no.	Time spent	Activity	Data (see Fig. 13.3 for Example 5 and 6)
1	8 min 41 s	Whole Class Instruction (Example 5 in Notes)	• Demonstrated and explained the method to solve Example 5 • Explained that Example 5 is a case in which the quadratic expression cannot be factorised "very nicely" • Presented solution using two different methods
	4 min 30 s	Table-to-Table Instruction (Example 6(a) in Notes)	• Walked around class to facilitate students' learning while they worked on Example 6(a)
	1 min 39 s	Whole Class Instruction (Example 6(a) in Notes)	• Presented and explained the method to solve Example 6(a) • Set Examples 6(b)—6(d) as homework
2	6 min 56 s	Table-to-Table Instruction (Example 6(b) in Notes)	• Walked around class to facilitate students' learning while they worked on Example 6(b) • Encouraged students to reason and think about their solutions • Emphasised that students had to present their solutions by using formal reasoning
	5 min 6 s	Whole Class Instruction (Example 6(b) in Notes)	• Stressed to students that they should learn to solve Example 6(b) by first understanding the undergirding reasoning for Example 5 • Probed students to think of alternative solutions so that they could compare the methods • Encouraged students to think about the methods they are applying
	6 min 48 s	Table-to-Table Instruction (Example 6(c) in Notes)	• Walked around class to facilitate students' learning while they worked on Example 6(c) • Encouraged students to make connections, infer, form conjectures, justify, validate and deduce • Emphasised that students had to present their solutions by using formal reasoning

(continued)

Table 13.1 (continued)

Lesson no.	Time spent	Activity	Data (see Fig. 13.3 for Example 5 and 6)
	7 min 31 s	Whole Class Instruction (Example 6(c) in Notes)	• Explained the undergirding reasoning behind Example 6(c) which is an unusual problem • Highlighted to students that they could "deduce" the solution without using the graph
2	4 min 26 s	Table-to-Table Instruction (Example 6(d) in Notes)	• Walked around class to facilitate students' learning while they worked on Example 6(d) • Encouraged students to make sense, make connections, infer, validate and reason • Stressed the importance of understanding the undergirding reasoning in Example 6(d)
	2 min 20 s	Whole Class Instruction (Example 6(d) in Notes)	• Presented and explained the solution for Example 6(d)
	39 min 49 s	Post-Lesson Interview (Lesson 06)	• Commented on his goals: Students need to: (i) know how to write formal reasoning; (ii) think about what they are doing • Commented that Examples 5 and 6 were designed to expose students to "unusual cases" • Mentioned that Examples 6(c) and (d) are special cases. • Articulated the design principles: tasks were designed to promote (i) thinking; (ii) reasoning

Evidences in support of these refined design principles are presented summarily in a third unit.

13.5 Results

13.5.1 Analysis of Unit 1: Researchers' Formation of Conjectures

Conjecture 1: Advance one more

In Unit 1, the technique that Teacher 13 taught the students can be summarised into these two steps:

Step 1: Obtain the (type of) roots (e.g., factorisation, quadratic formula, completing the square method)
Step 2: Use graphs to visualise the region that satisfy the inequalities

The technique was applied to Examples 3 and 4 (see Fig. 13.2).

To Teacher 13, Step 1 was merely a recall of the method of factorisation for all these items; his focus was on Step 2, how the roots of the quadratic equation were represented graphically as a visual to obtain the required regions that satisfy the required inequalities. From Teacher 13's post-lesson interview after Lesson 6,

> The first section [Example 3 & Example 4(a), (b), (c) as shown in Fig. 13.2] is easy to factorise [Step 1], just focus on using graphical method [Step 2]. Then *advance one more* … in Example 5. Right, what happens if we cannot factorise properly [Step 1] … (Post-Lesson Interview 2, 08:53)

Teacher 13's "*advance one more*" caught our attention. Since he mentioned that "*advance one more*" occurred first in Example 5, we studied the example and contrasted it against earlier examples. Figure 13.3 shows Examples 5 and 6.

On the surface, it seemed that *advance one more* refers to the development of the method (from factorisation to the quadratic formula) to find the roots to the

Example 3	Solve the inequality $2x^2 - 7x + 6 < 0$.
Example 4	Solve the following inequality using a graphical approach: (a) $x^2 - 4x + 3 > 0$ (b) $3x^2 - 4x - 7 \leq 0$ (c) $4 - x^2 < 0$

Fig. 13.2 Examples 3 and 4 in the Notes

Example 5	Solve the inequality $2x^2 + x - 4 > 0$. Solution: We observe (or check) that the expression $2x^2 + x - 4$ is not easily factorised. In this case, we have to find the x-intercepts using the quadratic formula. We present our working in this way:
Example 6	Solve the following inequalities, giving exact answers: (a) $x^2 + 4x - 7 \geq 0$ (b) $2x^2 < 5$ (c) $x^2 + 2x + 11 > 0$ (d) $3x^2 - 30x + 75 \leq 0$

Fig. 13.3 Examples 5 and 6 in the Notes

quadratic equations in Step 1. Step 2 where students use graphs to visualise the regions of the solution sets remains intact. Teacher 13's Notes pointed out explicitly that the expression given in Example 5 is not easily factorised (see Fig. 13.3).

Our analysis of Teacher 13's enactment of this portion of the notes shows that he carefully pointed out to students that Example 5 "is the case where you cannot factorise very nicely" (Lesson 5, 25:59). Hence, an adjustment of the method to solve the quadratic equation was needed which he demonstrated using the quadratic formula. After finding the roots of the quadratic equation, he reverted back to Step 2 and emphasised, "… then you can do the usual [Step 2]. Everything else will follow the previous procedure" (Lesson 5, 28:25). Our analysis of Teacher 13's lesson showed that the suite of examples in Example 6 was designed for students to practice their technique to solve the quadratic inequalities. He said "Example 6 gives you time to practice what to do when your quadratic expression is not very friendly - it doesn't factorise nicely. OK? So for simple ones, can you just try 6(a)". (Lesson 05, 34:00).

It becomes clear to us that Teacher 13's "*advance one more*" carried on to the "special cases" of Examples 6(c) and 6 (d), "… 6(c) and 6(d) are special cases as I said. … Example 4, they will say, Okay, I'm quite comfortable with the method [technique], now these are all the unusual cases …" (Post-Lesson Interview 2, 10:04). The special cases were situations when students usually conclude that there are no solutions to the inequalities because Step 1 results in either no real roots (Example 6(c)) or equal roots (Example 6(d)).

> Question 6(c) is a rather unusual question… your first instinct is to try to find *x*-intercepts, alright? And then you realise that it doesn't work … Now, if you are doing a quadratic equation, and you use the formula and you see this you say, No solution. … But that's not the case for this inequality …. (Lesson 6, 21:51).

Through Examples 6(c) and 6(d), he expected his students to modify their technique to solve quadratic inequalities by examining their incorrect conclusion—"no real roots" means no solution to the inequalities—by referring them back to the graphical representation. In other words, Teacher 13's "*advance one more*"—in the case of Example 6(c)—meant that the task required students to modify the learnt technique further to accommodate the "no real roots" situation. In particular, the conclusion "no real roots" in Step 1 does not prevent the use of graphical representation in Step 2 to obtain the required solution set for the inequalities.

Similarly, Example 6(d) presented yet another "special case" of "equal roots" which required yet another modification in the way students should use the technique, in this case, the graphical interpretation of $3(x - 5)^2 \leq 0$. Figure 13.4 gives a brief outline of how Teacher 13 "*advance one more*" in between examples:

- 6(a) $\to$ 6(b): included alternative method to solve equation
- 6(b) $\to$ 6(c): no solution in equation does not imply no solution for inequality
- 6(c) $\to$ 6(d): graphical representation for "equal roots" case

We noted that there was careful design of Examples 5 and 6. Teacher 13 was not just using examples merely to repeat practice the same procedures; he deliberately

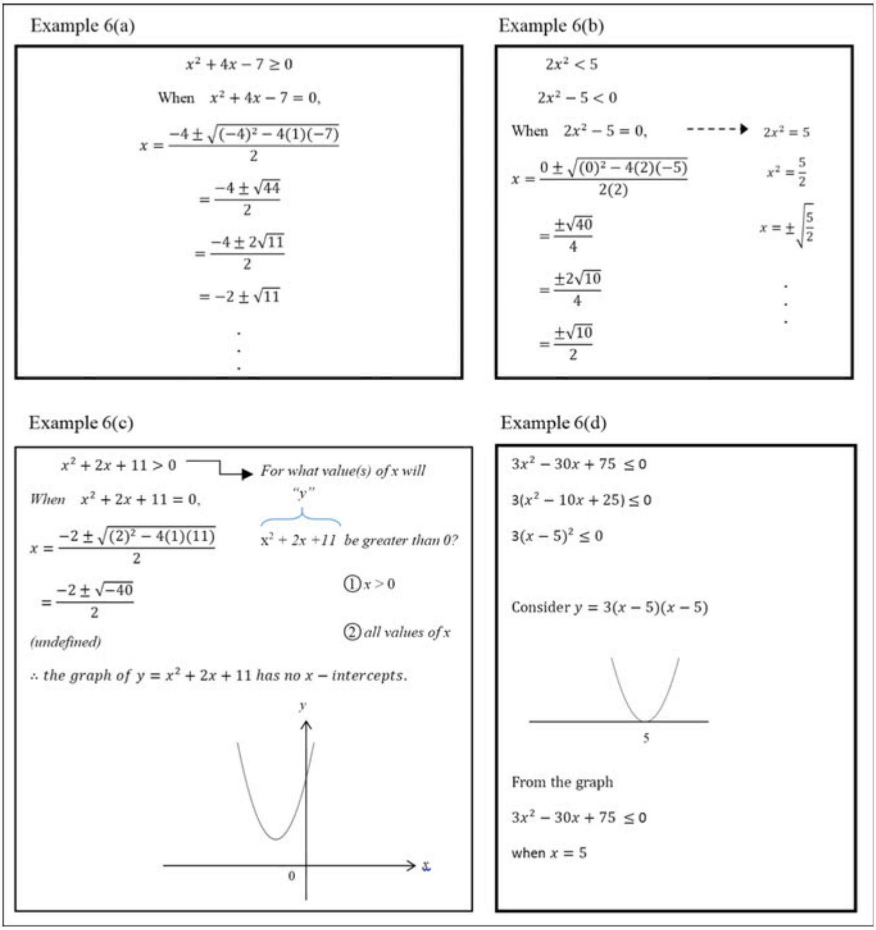


Fig. 13.4 Whiteboard work of Teacher 13 for Example 6(a) to 6(b) to 6(c) to 6(d)

used the sequence of examples to "*advance one more*". By this, he meant that, where appropriate, the next example provides the technique refinement/modification: "one more" step in the direction of a more comprehensive applicability of the technique to cases where students would normally not consider by themselves. Thus, our first conjecture is that Teacher 13 designed the sequence of his notes in such a way to achieve the design principle: (1) advance the technique.

Conjecture 2: Advance more through reasoning

Our analysis of Teacher 13's intention through his carefully selected examples in the Notes showed that he wanted his students to see each shift in the solution strategy through the examples and to learn to "*think* flexibly".

> Alright, there are different ways to actually solve this [writes on board as he speaks]. But 'cause yesterday's example [Example 5 & 6(a)] is [to] use quadratic [formula], then you just use quadratic formula, right? Have to be a little bit more *flexible* than that. (Lesson 6, 11:36)

> … if they keep using the same method, it indicates that they are not quite *thinking* about what they are doing, alright. … So I want to break them out of that mode… (Post-Lesson Interview 2, 14:23)

In particular, from Teacher 13's interview, we learn that the development of reasoning was intentionally built into the Notes: "we purposely build in a lot of reasoning type of steps for them to do". Teacher 13 elaborated this during an interview,

> … today the focus is on the non-standard examples [step 1 cannot be factorised] … So here is to promote *reasoning* in general, because here the-the basic idea they want to learn is if I can get the sketch of the graph [Step 2], I can use the graph to deduce a solution, […] This way we make sure that they know the thinking behind the particular graphical method [Step 2], and we put in all these parts to make sure that they are actually applying the *reasoning* behind the graphical method [Step 2]. (Post-Lesson Interview 2, 22:16)

In other words, "*advance one more*" was done through reasoning. We observed that for Example 5, in Teacher 13's Lesson 6, two methods were used to identify the roots of the quadratic expression $2x^2 + x - 4 = 0$ in Step 1. Figure 13.5 shows the two methods, namely, Method 1 (quadratic formula) and Method 2 (complete the square).

He encouraged his students to think and choose the method they prefer. This act was intentional and engineered into the design of the Notes as shared by Teacher 13 during the interview,

> So again we give them the options, we don't tell the students, we try not to tell the students which way to do… This way, both ways are fine for us. So we give them option—you choose the option that you *think* you prefer, that you *think* you are more likely to succeed in. So that's the design principle. (Post-Lesson Interview 2, 11:10)

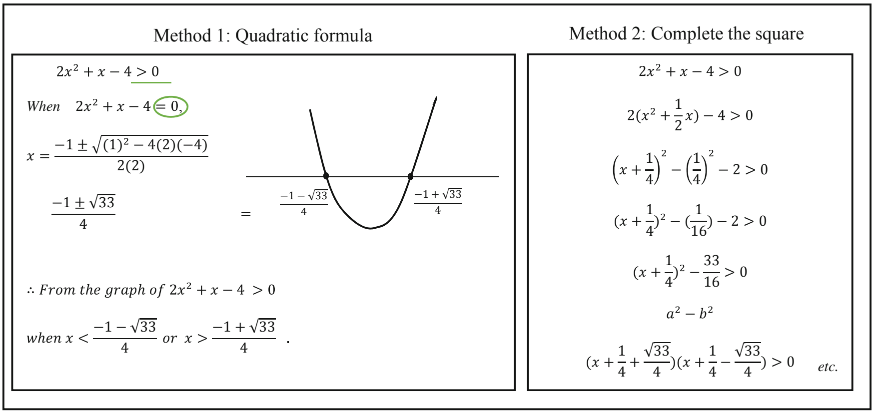


Fig. 13.5 Comparing methods in Example 5

The comparison of methods was also seen in Fig. 13.4, Example 6(b) and Example 6(c). From Teacher 13's interview "[Example 6(b)] they [are] used to illustrate that there are other methods [Step 1] we can use." He expected his students to compare the various methods to find the roots in Step 1 and decide on the easier method (see Figure 13.4). Clearly, the reasoning process of comparing was intentionally built into some of the examples.

In the vignette for Example 6(c) presented in Table 13.2, we unpack two other reasoning processes used by Teacher 13: justifying and inferring for students in Group A during their seatwork.

Table 13.2 Justifying and inferring with students in Group A for Example 6 (c) Lesson 6

Line	Speaker	Content
1	Teacher 13:	Part (c), how?
2	Student 2:	Part (c) no solution.
3	Teacher 13:	*No solution?*
4	Student 2:	Cause the discriminant is negative.
5	Teacher 13:	What does that mean?
6	Student 2:	0…
7	Teacher 13:	But this part- this calculation is to do what? You do this calculation is to?
8	Student 2:	To find the value of x.
9	Teacher 13:	To find the x-intercepts right? So no solution means what?
10	Student 2:	No x-intercepts.
11	Teacher 13:	No x-intercepts. Correct. *So what?* Therefore you can still carry on and answer this question. It just tells you there is no intercept. You are not- your task is not to find the x-intercepts, correct? Your task is to solve this inequality. Correct. No x-intercept. So what can you deduce from there? [***infer***]
12	Student 1:	It's not touching the x-axis. [***claim***]
13	Teacher 13:	It's not touching the x-axis, yah. So it's "floating", OK, you can say that.
14	Students:	[students replied simultaneously but inaudible]
15	Teacher 13:	OK, so does that help you answer this question?
16	Student 2:	*So what?* So x is more than 0, ah? [***infer, claim***]
17	Teacher 13:	Alright, because the idea is what? If I know the graph, I should be able to answer this inequality question right? So knowing that there's no x-intercept, means you know how to draw the graph. So can you draw the graph?
18	Student 2:	Oh, so this is above 0. [***claim***]
19	Teacher 13:	So, if you can draw the graph, then you can answer. What kind of x values makes the graph greater than 0? [***infer***]
20	Student 3:	But there's no answer. [***claim***]
21	Student 4:	x more than 0, ah? [***claim***]
22	Student 2:	x above 0. [***claim***]
23	Teacher 13:	*Is that true?* Ask yourself that question. *Is that true?* [***justify***]

Table 13.3 Justifying and inferring with students in Group B for Example 6 (c) Lesson 6

Line	Speaker	Content
1	Teacher 13:	No *x*-intercepts, that means? [***infer***] When $x^2 + 2x + 11 = 0$, $x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(11)}}{2}$ $= \frac{-2 \pm \sqrt{-40}}{2}$ (undefined)
2	Student 4:	That means it's either above- [***claim***]
3	Teacher 13:	That means cannot touch the *x*-axis at all right?
4	Student 4:	Yah, so it's above the thing [***claim***]
5	Teacher 13:	*How do you know* it's above? [***justify***]
6	Student 5:	Is it horizontal line?
7	Teacher 13:	*How do you know* it's above?… Wait, wait, one by one. *How do you know* it's above? [***pressing for justification***]

In Line 11, Teacher 13 wanted the students in Group A to infer what it meant when there were no *x*-intercepts. Student 1 claimed that the graph did not touch the *x*-axis in Line 12. Student 2 inferred and claimed that "*x* is more than 0" in Line 16. Student 2 made another claim that the graph "is above 0" in Line 18. The engagement with reasoning does not stop here as Teacher 13 required Student 2 to justify her claim, "*Is that true?*" in Line 23.

Table 13.3 presents another vignette for Example 6(c) where the two reasoning processes, justifying and inferring, were unpacked with another group (Group B) of students.

Here we noted that Teacher 13's other design principle is: (2) the technique was advanced through comparing, inferring and justifying. In the next two sections, we present the evidence for the next two uncovered design principles.

Conjecture 3: Expose and target students' faculty reasoning through carefully designed examples

One of the reasons why Teacher 13 built-in reasoning opportunities through the carefully selected examples in Example 6 was to deal with anticipated students' misconceptions (e.g., no solution in equation implies no solution for inequality).

> … or if there's a *misconception* that keeps popping up, I'll also bring them up. … Just now this question, the one where the solution is all real values [Example 6(c)], so many of them … will say, "Oh, no answer; cannot; no roots, because it's square root negative 40", and then they want to stop there. So I have to address that point … (Post-Lesson Interview 2, 15:47)

Revisiting the vignette in Table 13.2, from Line 3, we noticed that Teacher 13 picked up Student 2's faulty reasoning in Example 6(c). In Line 4–6 he had Student 2 confront the faulty reasoning, and in Line 6–19 Teacher 13 addressed and corrected Student 2's faulty reasoning. From Table 13.2, we see that Teacher 13 was not merely correcting wrong technique but also the faulty reasoning underlying it. Hence, in the design of his Notes, he incorporated examples where he could expose the faulty reasoning behind the wrong techniques. We therefore state that Teacher 13 utilised a third design principle: (3) special cases to expose and target students' faulty reasoning undergirding the techniques they used.

Conjecture 4: Presentation of solutions to reflect reasoning

Teacher 13 was concerned about presentations of the *written reasoning* in a way that is acceptable to the mathematical community and he required the students to present their solutions very clearly. He emphasised to his students that "you have to be very specific in your presentation so people understand what you are doing". (Lesson 5, 25:59). For example, in Fig. 13.6, Teacher 13 showed his reasoning in arriving at the correct solution on the whiteboard and pointed out the incorrect way of presenting the solution for Example 5.

> The working looks almost the same. [writes on board] But this is just so that you make it very clear that, [here, in the italicised text, the teacher is telling the students how they can write their justification] *I'm not trying to solve the inequality, here. I'm only considering what happens when it is equals to 0. So that I can draw my graph.* So that statement is to justify this set of working. So there's no confusion. (Lesson 5, 28:49)

Reflecting on our analysis of the first unit of analysis, we noticed that the white board appeared to be central in stringing together all the intended objectives by the teacher as expressed in earlier stated preliminary design principle:

(4) The whole class instructional segment was used to consolidate and formalise the reasoning in standard written form.

Figure 13.7 presents our provisional diagrammatic representational this stage of our analyses.

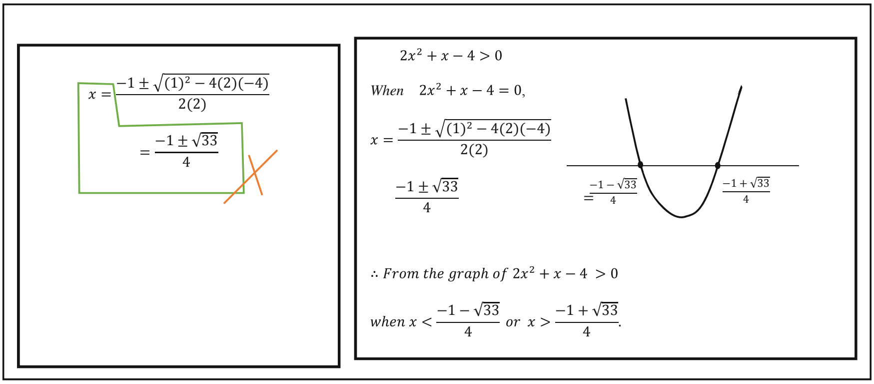


Fig. 13.6 Formal reasoning in the written form

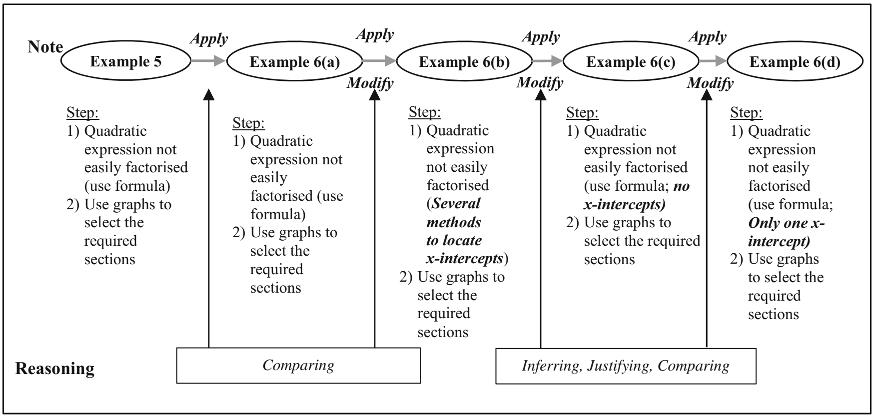


Fig. 13.7 An example of technique and reasoning applied within and across Examples 5 and 6

13.5.2 Analysis of Unit 2: Further Examination of the Conjectures

For Unit 2, the description is deliberately brief as we zoom straight into providing evidence on how the design principles in the previous section are supported or refined. The technique in Unit 2 involves the following steps:

1. Obtain the (expected) roots (same as Unit 1)
2. Algebraic deduction together with graphs to solve the inequality

13.5.3 Advance the Technique

Figure 13.8 shows the sequence of examples in this second unit of analysis. The preamble in the Notes shows that Unit 2 was intended to continue to develop students' techniques in solving quadratic inequalities using "algebraic approach".

In Step 2, Example 10 (see Fig. 13.8), students "use the algebraic deduction to solve the inequality" by considering two cases for $(2x - 3)(x - 2) > 0$.

Case 1: Make logical connections between $(2x - 3) > 0$ and $(x - 2) > 0$ and with the aid of number line conclude that $x > 2$.
Case 2: Make logical connections between $(2x - 3) < 0$ and $(x - 2) < 0$ and with the aid of number line conclude that $x < \frac{3}{2}$.

Putting the conclusions of the two cases together, $x > 2$ and $x < \frac{3}{2}$, and with the aid of number line, students solve the inequality $2x^2 - 7x + 6 > 0$ when $x < \frac{3}{2}$ *or* $x > 2$. As indicated in the Notes, the inequality stated in Example 10 is identical to Example 1. The cubic expression in Example 11 required the students to extend the method of factorisation (Step 1) into products of linear and quadratic expression

4. Solving Quadratic inequalities Using an Algebraic Approach

It is possible to solve quadratic inequalities using a purely algebraic approach. However, this can be quite tedious, so we will normally not use such a method except for more complex problems where a simple graphical approach cannot be applied directly.

Example 10 Solve the inequality $2x^2 - 7x + 6 > 0$ using an algebraic method.

Solution: This is the same problem as that in Example 1. Compare and contrast the graphical method and the algebraic method.

Step 1: We begin by factorising the expression $2x^2 - 7x + 6$:

$2x^2 - 7x + 6 > 0$

$\Rightarrow (2x - 3)(x - 2) > 0$

Step 2: Use the algebraic deduction to solve the inequality:

Example 11 By factorising the expression $x^3 - x^2 + x - 1$, solve the inequality $x^3 - x^2 + x - 1 \leq 0$

Example 12 By completing the square, show that $4x - x^2 - 7$ is negative for all real value of x.

Hence solve the inequality $\frac{3x^2 + 2x - 1}{4x - x^2 - 7} > 0$.

Fig. 13.8 Examples 10, 11 and 12 in the instructional materials

$$x^3 - x^2 + x - 1 = x^2(x-1) + (x-1) = (x-1)\left(x^2 + 1\right)$$

and subsequently apply a similar argument as Example 10 for Step 2. For Example 12, in Step 1, the numerator $3x^2 + 2x - 1$ can be factorised easily into $(3x - 1)(x + 1)$ but the denominator cannot be factorised into linear factors with rational coefficients. Hence, flexibility in thinking in the use of the method is required - completing the square was used instead. For Step 2, applying the algebraic approach, the numerators and denominators of $\frac{3x^2+2x-1}{4x-x^2-7}$ need to be considered separately (refinement and development of algebraic method).

- 10 → 11: extend technique to cubic inequalities
- 11 → 12: extend technique to rational inequalities

Examples 10, 11 and 12 thus supports Conjecture 1.

13.5.4 Advance Technique Through Comparing, Inferring and Justifying

As reflected in the instructions for Example 10 (see Fig. 13.8), the reasoning process of comparing was intentionally built into Example 10 for students to compare and contrast the graphical and the algebraic method. Teacher 13's video Lesson 8 showed him comparing the two methods (see Fig. 13.9).

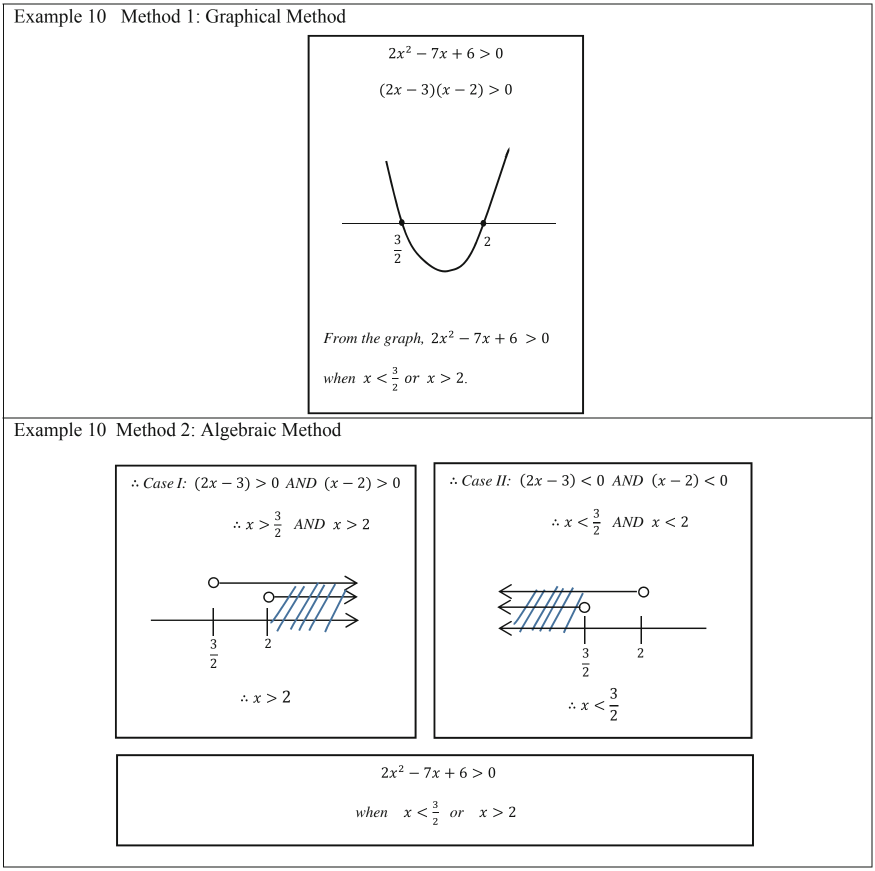


Fig. 13.9 Comparing methods in Example 10

We also observed his use of inferring to develop the algebraic method through Example 10 (see Table 13.4).

It is clear that Teacher 13 had reasoning in mind for Examples 11 and 12:

> Yeah so, I brought them to the algebraic *reasoning* for this, showed them how to obtain a solution and main thrust of the message is that we try not to use this method [Step 2] for basic quadratic inequalities,… But why we are talking about these kind of *reasoning* is because of certain questions leading to these two [referred to Examples 11 and 12]. So these are considered *special cases*. So, here… After they factorise one of these factors it's like, '$x^2 + 1$' I think. Ah, so have to guide them through the *reasoning* of if '$x^2 + 1$' is positive, therefore the other factor has to be negative and so on. … (Post-Lesson Interview 3, 05:21)

Examples 11 and 12 were the examples in the Notes that served as vehicles to pull along the development of mathematical reasoning as the technique was advanced.

Inferring was observed during the discussion of Example 11 in Teacher 13's Lesson 8. For example, during the class seatwork, Teacher 13 led his students to

Table 13.4 Inferring for Example 10 in Lesson 8

Line	Speaker	Content
1	Teacher 13:	Suppose I don't want to draw the graph for comparison [$(2x - 3)(x - 2) > 0$], for your knowledge, what can we do? So deduction here ah, going to ask you, I have two numbers, these are two things right, two object multiplied I get positive number, greater than zero. *What can you say about these two factors*? [***inferring***]
…	…	…
5	Teacher 13:	Both negative. OK so if I wanted to use an algebraic method, what I am going to deduce here? So I have Case 1, this $(2x - 3)$ is greater than zero and this $(x - 2)$ is greater than zero, alright. Or Case 2: Both negative. Alright, so this is carry on to reason OK, that if both are positive, each of these inequalities I can solve very quickly. [writes "therefore $x > 3/2$ and $x > 2$"] OK, so next I am going to ask you, if something must be greater than two, at the same time must be greater than 3/2, *then what condition can combine these two things?* [***inferring***]

Table 13.5 Justifying and inferring for Example 12 in Lesson 8

Line	Speaker	Content
1	Teacher 13:	Alright once you are able to *justify* that the expression is negative, right, then you want to ask yourself, "Then, how do I complete the rest of the question?" So I know $4x - x^2 - 7$ is negative, OK *then so what*? [***inferring***]

express the cubic expression in Example 11 as $(x - 1)(x^2 + 1)$. Next, he asked "… Then what can you deduce about those two factors?" so that his students can infer the polarity of the two factors for the cubic expression to be non-positive.

Inferring as well as justifying were observed in the discussion of Example 12 in Lesson 8. For example, in Table 13.5, Teacher 13 required his students to justify that $4x - x^2 - 7$ (the denominator) is negative. He also required his students to infer how knowing the denominator to be negative for all real values of x contribute to the overall polarity of $\frac{3x^2 + 2x - 1}{4x - x^2 - 7}$.

Other incidents of justifying were also observed in Example 12, e.g., as Teacher 13 interacted with a group of students (Group C). Teacher 13 said to the group, "What makes that for sure negative—may not be, right?… *must have a stronger argument than that*" (Lesson 8, 21:54). And again with students in Group C, "so how do you *justify this*, no matter what must be negative?" (22:38).

The built-in reasoning features of comparing, justifying and inferring to advance the technique were evident in Examples 10, 11 and 12, thus supporting, Conjecture 2.

Table 13.6 Example 11 in Lesson 8

Line	Speaker	Content	
1.	Teacher 13:	OK, if you have a negative number, you square root it is still pos- this square, alright x^2, OK, don't make this mistake ah, *don't say x^2 is always positive, because it can also be zero.* [***Exposing the faulty reasoning***] Alright. *So reason out, since, x^2 can be zero or positive ($x^2 \geq$ 0), cannot be negative, $x^2 + 1$ has to be positive, for all real x.* [***Addressing the faulty reasoning***] *OK. So I have this factor, $x^2 + 1$, it is always positive.* What can I deduce? Think along what you [just] told me just now. So if this is always positive, what is the deduction?…	E.g. 11 $x^3 - x^2 + x - 1 \leq 0$ $x^2(x-1) + 1(x-1) \leq 0$ $(x^2+1)(x-1) \leq 0$

13.5.5 Expose and Target Students' Faulty Reasoning Through Carefully Designed Examples

During the whole class discussion in Lesson 8, we observed Teacher 13 exposing and addressing students' faulty reasoning undergirding the technique they used for Example 11, thus supporting Conjecture 3 (see Table 13.6).

13.5.6 Consolidate and Formalise Reasoning in Written Form Through Whole Class Instructional Segment

When the students had difficulty writing the justification for Example 12, Teacher 13 referred students to his written justification on the board at the end of each example—thus, the importance of whole class discussion and Teacher 13's formal written reasoning on the board work to consolidate and support the development of method and reasoning through the sequence of examples. This supports Conjecture 4. In the last post-lesson interview, Teacher 13 said that "we need to train them how to write, how to *justify* that $x^2 + 1$ was positive [Example 11]" (25:04) and "I try to guide them how to write the justification". Teacher 13 was also seen to emphasise the importance of communicating written reasoning clearly for Example 12 "and *present properly*" (Lesson 8, 20:49).

Figure 13.10 summarises the technique and reasoning applied within and across Examples 10, 11 and 12 from our analysis of Unit 2.

We present the findings of the third unit in summary form in Appendix A. Figure 13.11 summarises the four design principles presented in this paper.

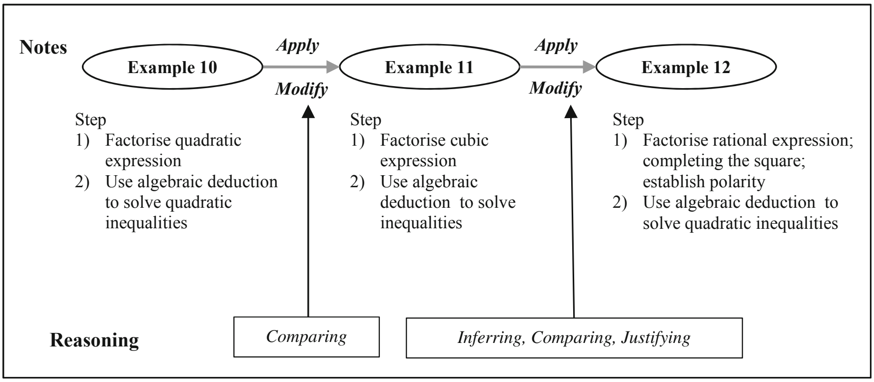


Fig. 13.10 An example of technique and reasoning applied within and across Examples 10, 11 & 12

(1) Deliberate use of examples to advance technique
(2) Advance technique through comparing, inferring and justifying
(3) Special cases to expose and target students' faulty reasoning undergirding the techniques they used
(4) Consolidate and formalise the reasoning in standard written form through whole class instructional segment

Fig. 13.11 Summary of the four design principles

13.6 Discussion

As shown in the analyses, the design of practice examples involves a complex interaction among the sequence of practice examples, reasoning and explication of the reasoning in Teacher 13's lessons. Reasoning was not merely an afterthought or add-on to the sequencing of examples in Teacher 13's Notes. He arranged his sequence of examples in such a way so that not only the technique of solving inequalities was developed; it was also amenable to reasoning moves supporting each tweak of the technique. He wanted students to reason along the whole trajectory of "twists and turns" from one example to the next. They were intended for students to advance the technique to increase their flexibility and mathematical reasoning as they practised the 2 steps in the technique.

Repeatedly, Teacher 13 mentioned that helping the students to think flexibly was his explicit goal of instruction. By this, he meant that as the students work through the examples, they would learn beyond a direct application of technique. Instead, they were expected to attend to necessary modifications to the technique and thus adapt flexibly through reasoning to suit the change in each proceeding example.

But reasoning was not merely embedded implicitly with the design of the Notes; it was also explicated overtly as an instructional goal for the students during the lessons. When they worked on the examples during seatwork, Teacher 13 repeatedly pressed students to reason out their steps. He drew on students' "raw reasoning"—including faulty reasoning—as he led whole class discussions following the seatwork to demonstrate sound mathematical reasoning in final written form. In the process of completing the examples, students were expected to develop their ability to attend to alternative solutions and utilise them flexibly in subsequent examples, in accordance to Watson and Mason (2006). Comparison of strategies were also encouraged both within an example and across examples. What came through strongly in the case of Teacher 13 was his use of reasoning *as a kind of glue* to provide adhesive in these ways:

1. Form the undergirding fallback to link the story of technique-tweaking across the examples he designed in his Notes;
2. Link the seatwork component of his classroom practice to the whole class discussion segment. In particular, he used students' raw reasoning attempts during seatwork as ingredients to present (and correct) reasoning in a way that sets the standard of reasoning norms that is acceptable to the community;
3. Binds the instructional materials closely with the actual classroom enactment—interaction between task design and pedagogies (Sullivan, Knott, & Yang, 2015).

For Point 3, we noted that it is not uncommon in actual classroom teaching of mathematics to find incoherence between the design of instructional materials and classroom enactment. For example, it is conceivable that the designed instructional materials and classroom enactment are driven by divergent goals so that classroom work goes down a different path from the intended trajectory embedded in the materials. In the case of Teacher 13, as his central focus was on reasoning, both the Notes and the classroom practices are drawn together to build around the central coherence of the reasoning process. The result was a tight materials–classroom link that was strengthened through a heavy emphasis on rigorous reasoning. Evidence of this tight link can be found in almost every "movement" across adjacent pairs of examples (e.g., Example 5 → Example 6(a)). When we combine the insights with the findings (they are also represented in Figs. 13.7, 13.10 and 13.11), we construct a visual depiction of Teacher 13's design work as shown in Fig. 13.12.

The centrality of mathematical reasoning in Teacher 13's plan and action serve as an organising frame to tie the various instructional pieces together. In other words, the model as illustrated in Fig. 13.12 has the potential of depicting teachers' ways of organising and connecting instructional components that matter to them—in the case of Teacher 13, it demonstrated his organisation around mathematical reasoning. In identifying the "glue" and the key adherents around it, we can begin to characterise teacher's fundamental instructional methods. A discussion of alternative models—around different glues and adherents—can be a productive way forward in teachers' inquiry towards quality mathematics teaching.

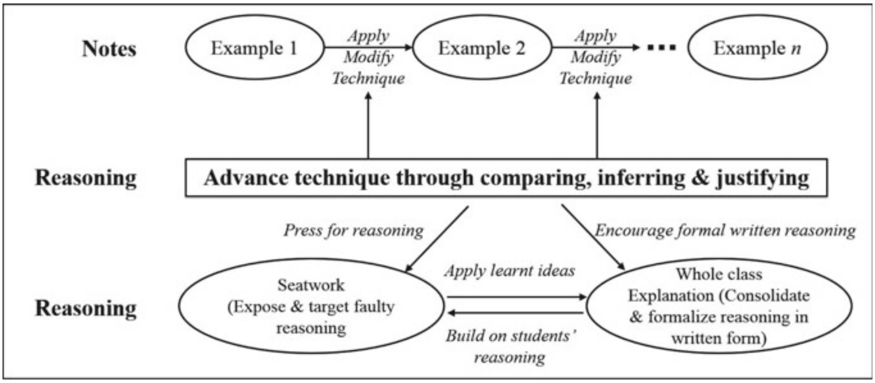


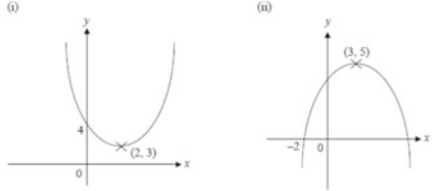
Fig. 13.12 An illustration of technique and reasoning applied across sequence of examples

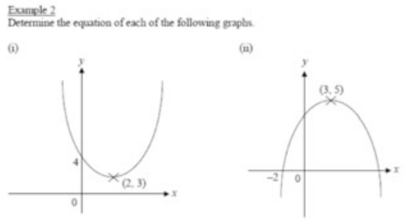
13.7 Conclusion

Developing students' mathematical reasoning is a core aspect of teaching and learning the mathematics classrooms (Ball & Bass, 2003; Boaler, 2010). To promote students' mathematical reasoning, students need to be provided challenging learning environments instead of lessons where students just solve exercises using well-known procedures (Mata-Pereira & Ponte, 2017). However, suitable tasks are not enough to ensure that students develop mathematical reasoning (Ball & Bass, 2003). It can thus be challenging to design tasks that place reasoning as its ostensible goal where the task must be *just right* to ensure that students develop reasoning. It is therefore, important to know the explicit task design moves which lead students to engage and advance in mathematical reasoning, and in what ways they may be used in the classroom. Teacher 13's case places reasoning as the centre piece (Fig. 13.12), as such, it illuminates how sequences of practice examples could be designed and enacted through careful and informed decisions to develop and advance students' reasoning. This paper, therefore, can provide the mathematics education research community another perspective at designing tasks.

Appendix A

Third Unit of Analysis

Sequence of Examples	Conjecture Supported	Brief Remarks
Example 1 Given the quadratic function $y = 2x^2 + 4x - 6$, (i) express the function in a completed-square form, (ii) express the function in a factorised form. Hence, identify the axis-intercepts and the turning point of the graph of $y = 2x^2 + 4x - 6$. Sketch the graph, indicating clearly the shape, line of symmetry, axis intercepts and turning point of the curve. Practice Question Sketch the graph of $y = 4\left(x - \frac{5}{2}\right)^2 - 16$, indicating clearly the axis-intercepts, turning point and line of symmetry. Example 2 Determine the equation of each of the following graphs. (i) (ii) 	**(1) the examples were sequenced to advance method**	• Example 1 was a standard question which Teacher 13 used to demonstrate the basic skills required of students in this topic. • The equation in Practice Example was written in the completed square form and could be factorised alternatively by applying difference of two squares. Teacher 13 used this Practice Examples to highlight this alternative solution to students. $y = 4\left(x - \frac{5}{2}\right)^2 - 16$ $= 2^2\left(x - \frac{5}{2}\right)^2 - 4^2$ • Teacher 13 used Example 2(i) to demonstrate the method to determine a quadratic equation given a graph with y-intercept and a turning point so students could learn to select the completed square form $y = a(x - h)^2 + k$ to form the equation. • Teacher 13 used Example 2(ii) so students could further learn to use line of symmetry $x = 3$—given the turning point and one x-intercept—to derive the other x-intercept, and then determine the equation of the graph.

 [00:17:33] T: OK, … turn to the next page. So if- I will discuss the first one, then you can try the second. … I give you a graph, it's a quadratic graph. … You want to figure out what is the equation of this graph, OK? … First ask yourself, **“What is given?” What information is given to you? Explicitly or implicitly. What do you see?** [T hears students' responses.] You see a turning point, yah. You see a *y*-intercept. One more thing? ... It's concave upwards, alright? Alright, three things. If it's concave upwards, you know coefficient of x^2 has to be positive. You know the turning point and you know the *y*-intercept. So, if I were to ask you: **"Which is a good form of equation to start with?" Would you start this equation off with the general form, the completed square form or the factorised form?** … … The question you want to ask is, "Amongst all the unknowns here, your *a*, *b*, *c*; *a*, *h*, *k*; *a*, *p*, *q*; that diagram, allows you to fill up the most for which form? ... Which one? Which one allows you to replace with numbers?” … Alright, completed square form allows me to change- to already write in terms of *h* and of *k,* since I know the turning point. So that is a good choice. So for this particular question ... OK, so [T writes on board]. **"Since (2, 3) is the turning point", alright, try and justify** [T writes on board]. So at least you know it's of this form. Alright? **It has to be *a* times *x* minus 2 square plus 3**. … Example 2(i) *Since (2, 3) is the turning point,* *let* $y = a(x-2)^2+3$	**(2) the sequence of Examples in the instructional materials pulled along the development of mathematical reasoning as the method was advanced** **(2.1) comparing** **(2.2) inferring** **(2.3) justifying**	• During the whole class instruction on Example 2(i) in Lesson 4, Teacher 13 demonstrated the approach to determine an equation given a graph. • From the lesson vignette beginning from [00:17:33.18], we observed Teacher 13 prompted students to *infer (2.2)* from the given information in the graph; *compare (2.1)* which of the three forms of equations they should select for this graph; and *justify (2.3)* their choice by writing a mathematical statement.
Guiding Student Alicia (pseudonym) during Seatwork Segment [00:29:22.01] T: OK, so second is this, alright. Now I want to find *a*, so what do I do? Just now what did I do? **I found a point so I can sub in and solve for *a*. So can I find a point to sub in?** [00:29:34.25] Alicia: **But do I need the *y*-intercept?**	**(3) the non-standard examples were designed to expose and address students’ faulty reasoning undergirding the methods they used**	• During the seatwork segment in Lesson 4, Teacher 13 guided Student Alicia (among others) with Example 2(ii). • From the interaction with Student Alicia, we observed how Teacher 13 guided her when she had the misconception that she had to use the *y*-intercept on the curve to solve for unknown *a* in the form $y = a(x-h)^2+k$.

[00:29:36.20] T: **Don't need to use y-intercept what. Any point will do, right?** As long as I have the x and the y-intercept, then a will be the only unknown already. **So what's a good point to use?** [00:29:46.22] Alicia: Sir, what is x? What is x? [00:29:52.04] T: **Look at your diagram. Which one gives you x and y coordinates?** [00:29:55.02] Alicia: Is it the line of symmetry? No right? [00:29:56.24] T: Which one haven't you used up till now? This one. Why not just use this one? [00:30:02.01] Alicia: **How to use?**		• Example 2(ii) was designed slightly differently from Example 2(i) so as to surface the concept to students that "the point" required to solve for unknown a need not be the y-intercept on the curve.
Consolidating Solution to Example 2(ii) during Whole Class Instruction [00:36:56.23] T: **Alright, can you see this set of working. What's the difference here? Look at this method here. OK, can you tell me what's the different method that's being used here?** [00:37:26.17] T [to whole class]: OK, can you see what the idea is here? **Notice the completed square form is not being used here. … Instead, somehow, factorised form popped up. How did that happen?** ... Alicia, can [you] say how this method works? … [to students] How did she get the "8" to appear? … What did she do? ... Example 2(ii) (Student's Work) $h = 3$ $k = 5$ *1st x-intercept = -2* *2nd x-intercept* $h = \frac{-2+x}{2}$ $3 = \frac{-2+x}{2}$ $6 = -2 + x$ $x = 8$ $y = a(x+2)(x-8)$ $5 = a(3+2)(3-8)$ $5 = -25a$ $a = -\frac{1}{5}$	**(4) the whole class instructional segment was used to consolidate and formalise the reasoning in standard written form**	• Teacher 13 consolidated the solution for Example 2(ii) by presenting a student's response.

References

Alcock, L., & Inglis, M. (2008). Doctoral students' use of examples in evaluating and proving conjectures. *Educational Studies in Mathematics, 69,* 111–129.

Ball, D., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27–44). Reston, VA: NCTM.

Boaler, J. (2010). The road to reasoning. In K. Brodie (Ed.), *Teaching mathematical reasoning in secondary school classrooms* (pp. v–vii). New York: Springer.

Goldenberg, P., & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics, 69*(2), 183–194. Retrieved from http://www.jstor.org/stable/40284542.

Jeanotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*(1), 1–16.

Kieran, C., Doorman, M., & Ohtani, M. (2015). Frameworks and principles for task design. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education*. *New ICMI Study Series.* Cham: Springer.

Marton, F., Runesson, U., & Tsui, B. M. (2004). The space of learning. In F. Marton & A. B. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3–40). Mahwah, NJ: Lawrence Erlbaum.

Mata-Pereira, J., & Ponte, J.-P. (2017). Enhancing students' mathematical reasoning in the classroom: Teacher actions facilitating generalization and justification. *Educational Studies in Mathematics, 96*(2), 169–186.

Mills, M. (2014). A framework for example usage in proof presentations. *The Journal of Mathematical Behavior, 33,* 106–118.

Ministry of Education. (2012). *Mathematics syllabus: Secondary one to four. Express course. Normal (Academic) Course*. Singapore: Author.

Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics, 69*(2), 149–163. Retrieved from http://www.jstor.org/stable/40284540.

Smith, M. C., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School, 3*(5), 344–350.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455–488.

Sullivan, P., Knott, L., & Yang, Y. (2015). The relationships between task design, anticipated pedagogies, and student learning. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education*. *New ICMI Study Series.* Cham: Springer.

Vale, C., Widjaja, W., Herbert, S., Bragg, L. A., & Loong, E. Y. (2017). Mapping variations in children's mathematical reasoning: The case of "What else belongs?". *International Journal of Science and Mathematics Education, 15,* 872–893.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ: Erlbaum.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91–111. https://doi.org/10.1207/s15327833mtl0802_1.

Watson, A., & Ohtani, M. (2015). *Task design in mathematics education*. *[electronic resource]: An ICMI study 22*: Cham, Switzerland : Springer.

Yeo, J., Teh, K. S., Loh, C. Y., & Chow, I. (2013). *New syllabus additional mathematics* (9th ed.). Singapore: Shing Lee Publishers.

Yeo, J., Teh, K. S., Loh, C. Y., Chow, I., Ong, C. H., & Liew, J. (2015). *New syllabus Mathematics 3* (7th ed.). Singapore: Shing Lee Publishers.

Zaslavsky, O. (2014). Thinking with and through examples. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36, v.1* (pp. 21–34). Vancouver, Canada: PME.

Zaslavsky, O. (2019). There is more to examples than meets the eye: Thinking with and through mathematical examples in different settings. *Journal of Mathematical Behavior, 53,* 245–255.

Zazkis, R., & Leikin, R. (2007). Generating examples: From pedagogical tool to a research tool. *For the Learning of Mathematics, 27,* 11–17.

Zodik, I., & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics, 69*(2), 165–182. Retrieved from http://www.jstor.org/stable/40284541.

Lu Pien Cheng is a Senior Lecturer in the Mathematics and Mathematics Education Academic Group at the National Institute of Education, Nanyang Technological University in Singapore. She received her PhD in Mathematics Education from the University of Georgia (U.S.) in 2006. She specialises in mathematics education courses for primary school teachers. Her research interests include the professional development of primary school mathematics teachers, task design in mathematics education and cultivating children's mathematical thinking in the mathematics classrooms.

Yew Hoong Leong is an Associate Professor at the National Institute of Education, Nanyang Technological University. He began his academic career in mathematics education with the motivation of improving teaching by grappling with the complexity of classroom instruction. Along the journey, his research has broadened to include mathematics problem solving and teacher professional development. Together with his project teammates, they developed "Realistic Ambitious Pedagogy" and its accompanying plan of action—the "Replacement Unit Strategy".

Wei Yeng Karen Toh worked as a research assistant at the National Institute of Education for seven years. She has a Bachelor of Science in Applied Mathematics from the National University of Singapore. She was awarded with a Distinction grade for the Post-Graduate Diploma in Education at the National Institute of Education. She taught in a few secondary schools as a mathematics and chemistry teacher for seven years. Her research interests include studying the history of mathematics, learning mathematical and pedagogical content knowledge, examining classroom practices and investigating teachers' designs of instructional materials. To advance her career in research, she pursued a Master of Education in Mathematics at the National Institute of Education.

Chapter 14
Designing Instructional Materials to Help Students Make Connections: A Case of a Singapore Secondary School Mathematics Teacher's Practice

Wei Yeng Karen Toh, Yew Hoong Leong, and Lu Pien Cheng

Abstract It is widely acknowledged that making connections is an important part of learning mathematics—instead of seeing mathematics as comprising merely isolated procedures to follow, it is desirable that students learn the distinctiveness of mathematics as being a tightly connected subject. In fact, the Singapore mathematics curriculum framework listed "connections" as part of mathematical processes—one of the five areas of major foci. In the study reported here, we look specifically at how an experienced and competent secondary mathematics teacher listed "making connections" as one of her ostensible principles in the design of the instructional materials for her lessons on quadratic equations. The method used can be summarised as one of progressive widening of the analytical lens: we started by conducting an in-depth examination of one unit of her instructional material to uncover the connecting strategies she built into it. Based on the strategies we uncovered, we widened the analysis to include its adjoining unit. From here, not only did we test the applicability of these strategies on the next unit, we also explored her design principles on how she connected between units. Finally, we further widened the lens of focus to the whole set of instructional material to study other connecting strategies she used across all the units in the material. The four design principles she used are: connections across multiple modes of representation, conceptual connections, temporal connections, and connections across different solution strategies. This teacher's design principles with respect to making connections challenge conventional stereotypes of how Singapore mathematics teachers carry out instruction—it is not merely repeated practice of unrelated procedures; rather, it is a careful structuring of instruction such that the underlying mathematical connections are made explicit. Not only so, the deliberate design was not just carried out in-class; it was, as reflected in the

W. Y. K. Toh · Y. H. Leong (✉) · L. P. Cheng
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: yewhoong.leong@nie.edu.sg

W. Y. K. Toh
e-mail: karen.toh@nie.edu.sg

L. P. Cheng
e-mail: Lupien.cheng@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_14

careful crafting of the instructional material, an intentional plan prior to the teaching of the unit. The principles used by the teacher hold useful lessons for mathematics teachers, especially within the context of teacher professional development. These are discussed towards the end of the chapter.

Keywords Making connections · Secondary mathematics · Instructional materials

14.1 Introduction

This chapter reports a case of how an experienced and competent secondary mathematics teacher, Teacher 8, designed her instructional materials to "link everything together" (Post-Lesson Interview after Lesson 5, 00:18:20). It is this remark, coupled with ten other phrases with equivalent meanings that she made throughout her interviews, that intrigues us and motivated us to embark on our inquiry. We are curious to investigate the way she designed her instructional materials to achieve this goal. When we examined her instructional materials, we found that she had designed some of her tasks for the purpose of explicating certain intended *connections*. Hence, we surmise that her conscious intents to create links within the topic is an undergirding design principle that she applied in crafting the instructional materials. During our inquiry, we also found other teachers in the project, though not as ubiquitous as compared to Teacher 8, that consciously included making connections in the design of instructional materials. This is shown in Table 11.6 of Chapter 11.

A quick scan of the literature shows that there are researchers who had examined how mathematics teachers can teach in an interconnected manner (e.g. English & Halford, 1995; Hill, Ball, & Schilling, 2008; Ma, 1999; Pepin & Haggarty, 2007; Sun, 2019). Pepin and Haggarty (2007) for instance, reported on English, French, and German lower secondary textbooks containing tasks which provide opportunities for students to learn mathematics through making connections. They asserted that if we assume that "learning with understanding is enhanced by making connections, then mathematical tasks should reflect this" (p. 1). And in Sun's (2019) study, she clarified how Chinese textbooks make connections between whole numbers and fractions. However, little is reported about how mathematics teachers design instructional materials with a deliberate goal of helping students see the mathematics they learn in a connected way. Therefore, we are motivated to uncover how Teacher 8 incorporated *connections* in the design of her instructional materials. We begin by reviewing some literature on the *connectionist perspective* and how mathematics teachers *make connections* before describing the details of the case study.

14.2 Teaching with a Connectionist Perspective

In their study on the standard algorithms for the four basic arithmetic operations, Raveh, Koichu, Peled, and Zaslavsky (2016) implemented a framework with a *connectionist perspective*. This perspective is built on the recommendations of mathematics education researchers who highlight the importance of teachers' competency in perceiving different interconnections among the mathematics topics they teach (e.g. English & Halford, 1995; Hill et al., 2008; Ma, 1999). Ma (1999) underscored the need for mathematics teachers to have a "thorough understanding" of mathematics. She stated that it is best for teachers to be able to make connections within mathematics with both "depth" and "breadth"—that is, to make connections within and across topics. Likewise, English and Halford (1995) emphasised the importance for mathematics teachers to know the connections within the curriculum so as to provide sufficient connections between mathematical procedural skills and conceptual knowledge in their lessons for students. This is so that students will be less prone to develop difficulties in their learning.

The connectionist perspective is traced back to Askew, Brown, Rhodes, Wiliam, and Johnson (1997) who wrote about three *orientations* that mathematics teachers generally possess: transmission; discovery; or connectionist. A teacher with a transmission orientation views mathematics as a series of facts and algorithms that must be imparted to students and he/she teaches in a didactic manner with an emphasis for students to attain procedural fluency in computational skills. A teacher with a discovery orientation views mathematics as pieces of constructed knowledge and he/she facilitates students' learning by encouraging them to explore solutions on their own. And a teacher with a connectionist orientation views mathematics as a linkage of concepts that he/she constructs collaboratively with students through discussions. These three orientations are "ideal" types and a typical teacher may possess a mixture of orientations.

A connectionist orientation is aligned to a commitment to both "efficiency" and "effectiveness" in mathematics—that is, that students become "numerate". A numerate student has the "awareness of different methods of calculation" and the "ability to choose an appropriate method" when he/she solves a problem. With this belief, a connectionist orientated teacher "emphasise[s] the *links* [emphasis added]" (p. 31) between various aspects of the mathematics curriculum so that students can acquire mathematical concepts that are related in tandem. Askew et al. (1997) described how a mathematics teacher taught a class of Year 6 students fractions, decimal fractions, percentages, and ratios in an integrated manner, rather than as separate topics. The students were given one value and they worked among the different forms of representations. As an evaluation, the teacher and students discussed the appropriate contexts in which each form of representation could be used.

Interestingly, from the transcriptions of the post-lesson interview after Teacher 8's fifth lesson, we find evidence that suggests that she is inclined to a connectionist orientation:

> [T]he big idea I was trying to drive at in this lesson was really this part: helping my students *link* the completing the square method and the [quadratic] formula because I think this [quadratic] formula is often taught as the teacher telling the students … (Post-Lesson 5 Interview, emphasis added, 00:02:59)

This motivated our study of Teacher 8 as a case of using instructional materials to support her connectionist agenda. However, we do not claim that she was aware of the connectionist theory. It is plausible that she had designed her instructional materials primarily to help her students make the connections within the topic better. In the next section, we list some specific strategies for making connections explicated in the literature. Some of these strategies were also employed by Teacher 8, and will be elaborated in the Findings section.

14.3 Making Connections

Mathematics teachers worldwide are encouraged to incorporate connections to deepen students' understanding of concepts (Fyfe, Alibali, & Nathan, 2017; Ma, 1999; Turner, 2015). For instance, the National Council of Teachers of Mathematics (NCTM, 2000) encourage students from Grades 9 through 12—between the ages of 14 and 19—to "develop an increased capacity to *link* mathematical ideas" (p. 354). Likewise, Singapore's mathematics curriculum framework advocates *connections* as one of the processes for proficient problem solving; and one of the aims of the secondary mathematics syllabus is to enable students to "*connect* ideas within mathematics …" (Ministry of Education, 2012, p. 8). This emphasis of making connections in mathematics is important because "mathematical meanings are developed by forging connections between different ways of experiencing and expressing the same mathematical ideas" (Healy & Hoyles, 1999, p. 60).

The specific strategies of making connections that we discuss in this chapter as described from the literature are: (1) connections across multiple modes of representations; (2) conceptual connections; (3) temporal connections; and (4) connections across different strategies to solve problems.

14.3.1 Connections Across Multiple Modes of Representations

Mathematical concepts are naturally abstract (De Bock, van Dooren, & Verschaffel, 2015). Thus, representations are made to communicate their meanings. However, according to Duval (2006), no single representation can entirely elucidate a mathematical concept so multiple modes of representations are required to help facilitate students' understanding. When multiple modes of representations are used, students are able to harness the different advantages each representation offers. Thus, many different modes of representations which complement each other are typically required for the development of an idea (e.g. Ainsworth, 2006; Elia, Panaoura, Eracleous, & Gagatsis, 2007; Tall, 1988). Studies have also shown that when teachers make connections across multiple modes of representations, they can facilitate greater understanding for students because they emphasise the conceptual connections (more in the next section) among the representations (Crooks & Alibali, 2014; Rittle-Johnson & Alibali, 1999). As an example, Dreher, Kuntze, and Lerman (2016) described vividly the use of multiple modes of representations for "$\frac{3}{4}$" such that students can have a comprehensive conceptual understanding of this fraction.

14.3.2 Conceptual Connections

Teachers can also help students learn mathematics in a connected way by helping them make conceptual connections. Leong (2012) described how connections can be made when a teacher extends the ideas students had learnt in a prior topic to a current one. He suggested that the ideas in the topic of "Special Quadrilaterals" which students learn in Year 7 in Singapore can be extended to the ideas in the topic of "Cyclic Quadrilaterals" which they will learn in Year 9. Similarly, teachers can lead students in Years 9 and 10 respectively to realise that the algorithm for computing the length of a line segment and the magnitude of a vector, respectively, is actually based on Pythagoras' Theorem which they would have learnt in Year 8. As such, it connects the concept of length of line segment on the Cartesian plane to the concept of Pythagoras' Theorem.

14.3.3 Temporal Connections

Even though a single lesson is frequently regarded as a unit for teaching and planning, teachers tend to take into consideration the planning for a topic as a module over a series of lessons. As stated by Leong (2012), "teachers think about the content suitable for a lesson in terms of what goes *before* and what is to come *after*" (p. 244). In the language of "connections", the components in a lesson will not only connect with one another within itself, but they will also be linked to what precedes and follows in prior and subsequent lessons. He described how a Year 9 topic in Singapore on "angle properties in a circle" is usually taught in an interconnected manner such that students could see the connections among the four theorems—(i) angle at the circumference is twice angle at the centre; (ii) angle at semicircle; (iii) angles in the same segment; and (iv) opposite angles in a cyclic quadrilateral—taught over a few lessons. From this example, he also highlighted that the underlying instrumental link for temporal connections is time—in the chronological sense of it. It is over time that the connections across the four theorems in the topic are made consistently.

Conceptual connections and temporal connections appear similar as both involve connecting prior knowledge to new knowledge. However, there is a difference. When teachers make conceptual connections of an idea, it need not be developed over a series of lessons bounded by a specific period, but when teachers make temporal connections of an idea, this idea is being morphed chronologically over several lessons within an extended duration of time.

14.3.4 Connections Across Different Methods to Solve Problems

Students can learn to make connections within mathematics by solving problems using different methods (Fennema & Romberg, 1999; Leikin & Levav-Waynberg, 2007; Toh, 2012). During this process, mathematical knowledge is constructed when students shift between representations, comparing methods, and connecting different concepts and ideas (Fennema & Romberg, 1999). Toh (2012) suggested that this could be achieved by teaching students to use different methods to solve the same problem. He illustrated his point by describing how the solutions to a rich problem can be used as a summary to link several topics together. He urged teachers to adopt this strategy so that students who perceive mathematics as a fragmented subject can learn to appreciate its connectedness.

14.4 Method

Teacher 8 was one of 30 experienced and competent teachers who participated in the first phase of the project detailed in Chapter 2. As mentioned briefly at the start of this chapter, the choice of Teacher 8 as a case study of making connections was predominantly because she articulated that one of her teaching goals was to "link everything together". In addition, other factors about Teacher 8's practices lends itself to a rich unpacking of her work—a characteristic feature of case study: (1) During interviews, she expressed comprehensively her objectives for many tasks. This allowed us to uncover her intentions behind the activities we recorded in her classroom; (2) she crafted a full set of handouts for students' use in class (hereafter referred to as "Notes") before the start of the module. In other words, her work generated a rich set of instructional materials on which to ground our study; (3) she constantly made references among her objectives, her actual activities in class, and her use of instructional materials. This allowed us to study the interactions among these major pieces of her instructional processes.

The class that Teacher 8 taught as resident teacher was a Year 9 Express class. It comprised 39 students whose age range was 14–16 years old. The module that she taught was "Quadratic Equations". The contents—as stipulated by the Ministry of

Education (2012)—that she had to cover were: (i) solving quadratic equations in one variable by (a) the use of formula, (b) completing the square for $y = x^2 + px + q$, and (c) the graphical method; (ii) solving fractional equations that can be reduced to quadratic equations; and (iii) formulating a quadratic equation in one variable to solve problems.

14.4.1 Data

Under instructional materials, Teacher 8 used mostly the set of Notes she designed for her students. This forms the first primary source of data. The next source of data is the set of transcripts of interviews we conducted with Teacher 8. We conducted one pre-module interview before she conducted her suite of lessons and three post-lesson interviews after Lessons 2, 5, and 8, based on her selection. All interviews were video recorded and transcribed verbatim. We designed an interview protocol with two sets of questions and prompts respectively for the pre-module interview and post-lesson interviews.

The pre-module interview was conducted to find out what Teacher 8's instructional goals were and how she had designed and planned to utilise her instructional materials to fulfil her goals. Some prompts in the pre-module interview were:

- Please share with me what mathematical goals you intend to achieve for this set of materials that you will be using.
- How different is this set of materials that you developed compared to those in the textbook?
- Are there any other specific instructional materials that you are going to prepare for this module?

The post-lesson interviews were conducted to find out if she had met her instructional objectives with the instructional materials she had designed and planned to use. Some of the questions were:

- Did you use all the materials that had you intended to use for the lesson?
- How did the materials help you achieve your goals for this lesson?

The third source of data is Teacher 8's enactment of her lessons in the module. We adopted non-participant observer roles during the course of our study. That is, one researcher sat at the back of the class to observe Teacher 8's lessons. This was so that the researcher would be able to make relevant and precise references to her teaching moves when pursuing some threads during the post-lesson interviews. A video camera was also placed at the back of the class to record Teacher 8's actions. We recorded a total of eight lessons. Three were 60-minute lessons while rest were 90-minute lessons.

14.4.2 Analysis of Data

We proceeded with the analysis along these stages:
Stage 1: Identification of units of analysis of the Notes
Each unit is a section in the set of Notes prepared by Teacher 8 (e.g. "Factorisation Method", "Graphical Method", "Completing the Square Method", "Quadratic Formula Method", "Thinking Activity", etc.). We coded the units according to the mathematical contents targeted in each unit. We matched the comments in Teacher 8's pre-module interview according to the references she made to these units. Together with the coded content, we were better able to verify the instructional objectives intended for each unit.
Stage 2: Composition of chronological narratives
For some of these selected units with rich related data on Teacher 8's enactment and interview comments, we crafted chronological narratives for each of them. These are narratives that coherently bring together related data sources for that particular unit of analysis. In each chronological narrative, we integrated several data sources—pre-module interview transcriptions, post-lesson interview transcriptions, tasks in her Notes, and her classroom vignettes. The chronological narrative for "Completing the Square Method", for instance, was composed by first examining the text in the pre-module interview. As we found her commenting at length about how she planned to develop the concept of "completing the square" with her Notes, we validated her intentions for designing the mathematical tasks and questions by examining the unit on "Completing the Square Method" in her Notes. After which, we proceeded to search the video recordings of the related lessons she conducted for evidence to corroborate her use of the instructional materials. We consolidated the evidence and organised them in a table. Table 14.1 presents the evidential ingredients for building the chronological narrative for "Completing the Square Method".
Stage 3: Strategies related to making connections
We begin specifically to look for the strategies that Teacher 8 used to make connections by closely examining the chronological narrative on "Completing the Square

Table 14.1 Main evidence leading to the building of the Chronological Narrative of the Unit on "Completing the Square Method"

Chronological sequence	Main data source	Description
1	Pre-Module Interview	• Explained rationale for the way she designed the tasks in the unit on "Completing the Square Method" • Explained how she planned to help her students connect their prior knowledge on perfect squares in lower secondary to the new knowledge on completing the square

(continued)

Table 14.1 (continued)

Chronological sequence	Main data source	Description
2	Lesson 3 Video Recording and Notes	• Elicited students' prior knowledge on perfect squares • Elicited students to illustrate pictorially the squares of 7, $(x + 1)$, and $(x - 2)$ on their Notes (as shown in Fig. 14.3) • Emphasised that students have to make sense of "perfect squares" algebraically and pictorially • Assigned students to express the squares of 7, $(x + 1)$, and $(x - 2)$ in words (as shown in Fig. 14.4) • Assigned students to work in groups to give examples of "perfect squares" • Conducted class discussion on the examples given by each group • Explained geometrically the meaning of $(a + b)^2$ and $(a - b)^2$ • Assigned students to work in pairs on Task A2 (as shown in Fig. 14.5) • Contrasted 120 to 121 by illustrating 120 as an "incomplete square" of side 11 on white board (as shown in Fig. 14.6) • Utilised table (as shown in Fig. 14.7) in Notes to help students connect the concept of completing the square algebraically and geometrically • Explained the first row of entry in the table for $x^2 + 2x$ by redrawing the diagram and relating to the algebraic expressions • Assigned students to complete the table as homework
3	Lesson 4 Video Recording and Notes	• Conducted class discussion for the homework assigned at the end of Lesson 3 • Stressed that students have to connect the geometrical representations to the algebraic expressions so as to gain conceptual understanding of the "completing the square method"

(continued)

Table 14.1 (continued)

Chronological sequence	Main data source	Description
4	Lesson 5 Video Recording and Notes	• Conducted class discussion to help students recall the algorithm for the "completing the square method" and generalise the theorem • Conducted class discussion on the practice items on p. 5 of the Notes to help students learn to apply the method • Assigned students to complete practice items on p. 6 to p. 8

Method". This chronological narrative was chosen as a first-entry study because it is one where Teacher 8 articulated that she "actually took great trouble to prepare [the] worksheets" (Pre-Module Interview, 00:03:17). This chronological narrative became an intensive source of analysis for emerging themes related to her strategies in making connections. We underwent many rounds of discussions, conjecturing, refuting, and re-conjecturing until we reached stability in agreement among the members of the research team (authors of this chapter)—where the purported strategies could be substantiated from all the data sources. Figure 14.1 shows the various units of analysis. It also highlights that the chronological narrative on "Completing the Square Method" is the first in the process of analysis. The report of this analysis is given in Sect. 14.5.1.

Stage 4: Confirmation and expansion of strategies

In the final stage of analysis, we examined the preliminary strategies we conjectured in Stage 3 to check it against two other chronological narratives following this process: we repeated the process of the analysis as in Stage 3 on "Quadratic Formula Method"; we then drew connections between these two adjacent units of analysis (this stage of analysis is presented in Sect. 14.5.2); finally these conjectures were further refined as we examined across a number of units of analysis (the next stage is presented in Sect. 14.5.3). These sequential phases of analysis are also presented diagrammatically in Fig. 14.2.

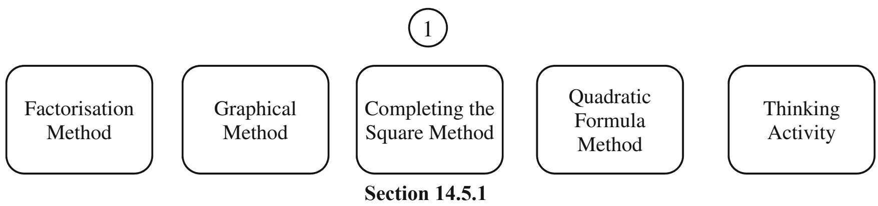


Fig. 14.1 Diagram showing different units of analysis and the first unit of analysis

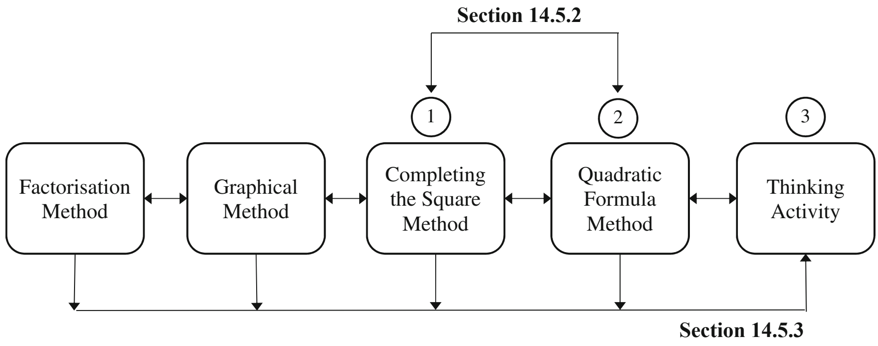


Fig. 14.2 Diagram illustrating analyses across different units of analysis

14.5 Findings

14.5.1 Making Connections Within a Unit

In the Notes that Teacher 8 prepared for the unit on "Completing the Square Method", she designed three sections which we labelled: A, B, and C. We focus our report on certain tasks in Sections A and B whereby she had applied one or more strategies to make connections to develop the completing the square method. Our analysis will exclude Section C as it comprises mainly of practice items.

When we first looked at Task A1, (as shown in Fig. 14.3), we were curious as to why it was designed in this manner. We noticed that in the first column, the top row had "5^2" written in it, and in the bottom row, there was a diagram of a square with sides of 5 units. We also noticed that throughout the four columns, the top row was presenting a kind of symbolic representation; and students were expected to produce a geometrical representation. The diagram in the first column had been provided to them as an example. It appeared that Teacher 8 designed in it such a way that students could revise the meaning of squares of numbers and then connect them to the geometrical meaning of squares with areas of given sides. We noticed her

A1. Draw a geometric representation of each of the following. The first one has been done for you.

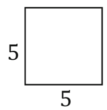
5^2	7^2	$(x+1)^2$	$(x-2)^2$
			

Fig. 14.3 Task A1 in Section A of Notes on "Completing the Square"

deliberate design for students to relate symbolic terms to geometric figures. Also, this activity extends to squares with sides that involve algebraic expressions. At this point, we saw explicitly what she meant by "to link everything together"—numeric to geometric; algebraic to geometric; and from numeric numbers (left) to symbolic algebra (right)—and found obvious pieces of evidence for her use of **the strategy to connect across multiple modes of representations**. In addition, from her pre-module interview, we verified her intention when she expressed that her insertions of diagrams were "so that they *have the algebraic* procedure and they also *have the pictorial* representation of what they are doing algebraically" (Pre-Module Interview, emphases added, 00:02:23).

Subsequently, we noticed that she intended to connect numeric, algebraic, and geometric representations in Task A1 to *words* in Task A2, as shown in Fig. 14.4. Her instructions clearly stated: "Explain in *words* what each of the following represents with reference to its *geometric* representation"; and the first column of the table are the same numeric and algebraic expressions as those in Task A1. The sample statement in the first row of the table also exemplifies how she expected her students to explain in terms of "area of a square".

We surmise that she had purposefully designed Task A2 such that students could learn to use words to connect to numbers; and algebraic expressions to their geometrical representations so that students can make connections across multiple modes of representations. We validated her intention to connect across multiple modes of representations from her pre-module interview transcript:

> [F]or the worksheets right … first I [will] elicit prior knowledge: "What does it mean when you square a *number*? What does it mean [when] you square the *algebraic expression*?" … [T]hen after that I [will] try to get them to write in *words so* they [can] get used to the math language. … [A]fter that, I [will] show them the *pictorial representation* … (Pre-Module Interview, 00:14:52, emphases added)

The second task which caught our attention was Task A6. From the way the task was designed, we infer that it was a continuation to help students make connections across

A2. Explain in words what each of the following represents with reference to its geometric representation. The first one has been done for you.	
5^2	5^2 represents the area of a square with sides of 5 units in length
7^2	
$(x+1)^2$	
$(x-2)^2$	

Fig. 14.4 Task A2 in Section A of Notes on "Completing the Square"

multiple representations. The table that Teacher 8 had drawn up as shown in Fig. 14.5 was to get her students to discern if the numeric and algebraic expressions in the first column (on the left) were perfect squares. She expected them to write in words in the third column the reasons for their conclusion. She had provided sample statements for the first and second numbers—81 and 120. She had planned for students to state whether the given expressions in the first column could be “expressed as k^2”.

However, upon closer analysis, we notice another strategy being used in this task—Teacher 8 intended to **help her students make conceptual connections**. She designed this task to help her students make sense of the concept of “incomplete squares” to the concept of “perfect squares”—which they had learnt previously in Years 6 or 7. She developed the concept of an “incomplete square” from the associated concept of “perfect squares” in her lesson by discussing the number “120” and highlighting the difference between “121” and “120”. She elicited from students that “121” was a perfect square—that is “11^2”—but “120” was not. To make a geometric connection to this number, she illustrated “120” as a square with sides 11, but was one that was “short of that little bit” (Lesson 3, 01:19:13). The diagram she drew on the board is shown in Fig. 14.6. She went on to ask her students: “If I want to make 120 into a perfect square, what shall I do?” (Lesson 3, 01:19:40). Her point was to show students that the concept of completing the square was to complete an ‘incomplete’ square by adding on a small square with a specific side. So for the case of “120”, she explained that she would have to add a small square of sides 1 to make “120” a complete square with sides 11; and then she said: “So this is the idea behind completing the square” (Lesson 3, 01:19:54).

Teacher 8’s attempts to make conceptual connections can also be seen from the entries in the fourth and fifth row in the first column. She selected “$x^2 + 2x + 1$” as an introductory example because it is the expanded quadratic expression of the most basic polynomial in the $(ax + b)^2$ form. And this expanded quadratic expression for $(x + 1)^2$ can be expressed in terms of a perfect square, “k^2”. The difference between “$x^2 + 2x + 1$” (in the fourth entry) and “$x^2 + 2x$” (in the fifth entry) is 1—just

Expression	Perfect Square Expression? (Yes/No)	Reason
81	Yes	Can be expressed as 9^2
120	No	Cannot be expressed as k^2, where k is an integer
100		
$x^2 + 2x + 1$		
$x^2 + 2x$		

Fig. 14.5 The first five rows of Task A6 in Section A of Notes

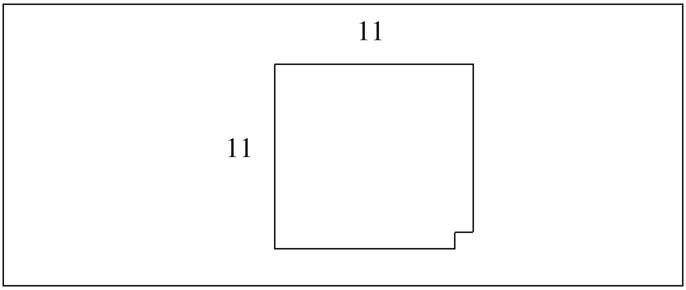


Fig. 14.6 Diagram which Teacher 8 drew on the whiteboard to illustrate 120

like the difference between "121" and "120". In other words, "$x^2 + 2x + 1$" is a 'perfect square' like "121" but "$x^2 + 2x$" is an 'incomplete square' that is short of a square of side 1, just like "120". She made this careful selection so as to help her students "construct the *new* knowledge [of an 'incomplete' square] by *connecting* to *prior* knowledge [of a perfect square]" (Pre-Module Interview, emphases added, 00:15:17).

Teacher 8's use of both strategies to make connections across multiple modes of representations and to make conceptual connections continues for the third time in Section B of her Notes. For this section, she created a table. As shown in Fig. 14.7, the table had four columns. We name them as Columns B1, B2, B3 and B4 (from left to right) for easy reference. Teacher 8 designed the table such that Column B1 is

When we write $x^2 + bx + c$ in the form $(x - h)^2 - k^2$ where b, c, h and k are real numbers, we are **completing the square**. Study the examples shown and complete the table below.

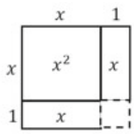
Expression	Geometric Representation	Term to be Added	Algebraic Representation
$x^2 + 2x$		$1^2 = 1$	$x^2 + 2x$ $= x^2 + 2x + 1^2 - 1^2$ $= (x + 1)^2 - 1$
$x^2 + 4x$			

Fig. 14.7 Table in Section B of Notes showing first two rows

for algebraic expressions; B2 for geometric representations; B3 for students to write down a "term to be added"; and B4 for algebraic representation.

Based on surface analysis of the table, we note from the entries in the first row how she had intended to let her students see that the algebraic expression $x^2 + 2x$ in B1 could be represented geometrically with a diagram as shown in B2. The diagram of an incomplete square in B2 was intended to help students visualise that if $x^2 + 2x$ were to be represented as a square with side "$x + 1$", there would be a missing corner. And this corner is actually a small square with side of 1 unit—that is, 1^2—to sensitise students to the need to add this 1^2 to "complete the square". This information is contained in B3. The algebraic working in B4 is the algebraic documentation of what goes on in B2 and B3.

Upon careful inspection of the entries, we realise that Teacher 8 designed the table to harness the connections she had made in the earlier tasks. There were links across multiple modes of representations—from algebraic to geometric (from B1 to B2), geometric to numeric (B2 to B3), and geometric plus numeric to algebraic (B2 + B3 to B4)—just like those in Tasks A1 and A2.

In addition, she carefully linked Section B to A by deliberately repeating the choice of $x^2 + 2x$ as the first entry. This was the same entry in the fifth row of Task A6. And the geometric representation of this expression—an "incomplete square" of sides $x + 1$—corresponded to the perfect square "$(x + 1)^2$" which is the third entry in both Tasks A1 and A2. Figure 14.8 explains how the algebraic working in Column B4 connects to the other tasks.

When we link up all the details in our analysis in this unit, we uncover aspects of how Teacher 8 planned to "link everything together" using her instructional materials. She crafted her tasks within this unit such that her students could make connections by progressing from stage to stage until they arrived at the concept for the completing the square method. The tasks in Section A set the background for what was to come

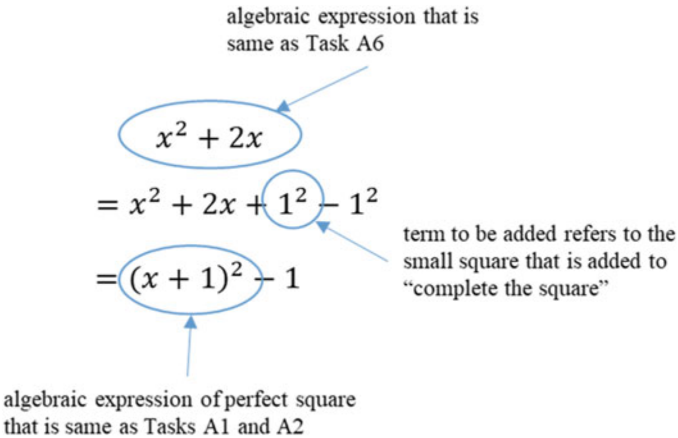


Fig. 14.8 Breakdown of the algebraic working

in the table in Section B. She designed it such that when her students completed the tasks in Section A, they would be prepared to make the conceptual connections to the tasks that progress across the columns in the table.

14.5.2 Making Connections Between Adjacent Units

We proceed to analyse the next unit on "Quadratic Formula Method". As details of the analysis process of a unit were given in the previous section, we will be brief in this section. The first page of this unit is shown in Fig. 14.9. The formula is in a text box, placed at the top of the page—occupying one-third of it—while two-thirds of the page is left blank. We did not fully understand how she intended to let her students "derive this formula by applying the completing the square method" until we uncover them from the transcriptions of her pre-module interview, post-module interview after Lesson 5 and that of Lesson 5.

From the videos recordings, we observed that Teacher 8 only started teaching the unit on Quadratic Formula in Lesson 5 after explaining three practice items from the earlier unit of Completing the Square: Solve (i) $x^2 - 16x - 4 = 0$, (ii) $x^2 + 5x + 4 = 0$, and (iii) $2x^2 + 15x + 10 = 0$. She had presented her solutions (with students' participation) in three separate columns on the whiteboard sequentially from left to right. After which, she erased only her written solutions for items (i) and (ii), and left the solution of (iii) on the extreme right column on the whiteboard. We reproduce her actual working steps on the whiteboard for Item (iii) in Fig. 14.10.

Quadratic Formula

The roots of the general quadratic equation $ax^2 + bx + c = 0$ can be obtained by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's try to derive this formula by applying the completing the square method.

Fig. 14.9 Task on first page of the unit on Quadratic Formula

$$2x^2 + 15x + 10 = 0$$

$$x^2 + \frac{15}{2}x + 5 = 0$$

$$x^2 + \frac{15}{2}x = -5$$

$$x^2 + \frac{15}{2}x + \left(\frac{15}{4}\right)^2 = -5 + \left(\frac{15}{4}\right)^2$$

$$\left(x + \frac{15}{4}\right)^2 = \frac{145}{16}$$

$$\frac{15}{4x +} = \pm\sqrt{\frac{145}{16}}$$

$$x = \pm\sqrt{\frac{145}{16}} - \frac{15}{4}$$

$$x = -0.740 \; or \; x = -6.76$$

Fig. 14.10 Actual working steps for item (iii)

Upon analysis of Lesson 5, we discovered that she had planned to make use of the numeric workings of item (iii) from the unit on completing the square to help her students cope with the abstract and complex algebraic manipulations they had to handle when they derive the quadratic formula from the general quadratic equation $ax^2 + bx + c = 0$. We found her telling her students: "I want to show you what exactly I am doing here [Item (iii)] with number coefficients [as it] is exactly the same way as what you are doing here [with the general quadratic equation] with algebraic coefficients" (Lesson 5, 00:42:50). And in the post-module interview after Lesson 5, we found her explanation for leaving the working steps of item (iii) by the right-hand side of the white board—she articulated that she had placed it "side by side" (Post-Lesson Interview after Lesson 5, 00:05:02) to the derivation steps so that students "can see the parallel" (Post- Lesson Interview after Lesson 5, 00:05:02). From this vignette, we observe once again how Teacher 8 helped her students learn a concept **by making connections across multiple modes of representations**.

Besides this, Teacher 8 had another objective for leaving two-thirds of the page blank for her students to derive the quadratic formula by applying the completing the square method. She had planned this because she did not wish for them to "just memorise [the quadratic formula] blindly" (Pre-Module Interview, 00:03:24) and apply on practice items or problems. She wanted them to **make the conceptual connections** between the completing the square method and the quadratic formula. She stressed this in her pre-module interview when she said she "want[ed] them to listen to the conceptual development" (Pre-Module Interview, 00:18:51) before memorising and applying the quadratic formula. We verified her plan when we observed how she established the conceptual connections in Lesson 5. She had asked students to make close reference to item (iii) on the right-hand side of the board to help them derive the quadratic formula from the general quadratic equation.

Subsequently, we found that Teacher 8 used a third strategy in making connection between two adjacent units of analysis—she also incorporated **temporal connections** in her development of the quadratic formula. She had cautiously timed her lessons such that she would demonstrate the derivation of the quadratic formula from the general quadratic equation immediately after she completed item (iii). We found her explanation for this design principle during her pre-module interview that substantiates our conjecture:

> I actually took great trouble to prepare this worksheet … to help them appreciate this idea of completing the square. Then after that right, I will go on to the quadratic formula … and I want to show them how… [the] *formula is derived from completing the square.* That's why I *sequenced* the worksheets in this order …. (Pre-Module Interview, 00:03:15, emphases added)

14.5.3 Making Connections Across Units

Teacher 8's goal of helping students make connections across units was to let them see the links across the whole topic of solving quadratic equation. This was observed in the third unit of analysis: "Thinking Activity". It was through this task sheet that we are able to observe how she "tie[d] everything together" (Pre-Module Interview, 00:02:28). She stressed in her pre-module interview that this task sheet was designed because her "ultimate goal [was] to help [students] appreciate the affordance and constraint of each method". And during the lesson when students were assigned to work on this task sheet, she explained to them that they were to "consolidate everything that [they] had learnt" (Lesson 7, 00:50:44) from the past few lessons.

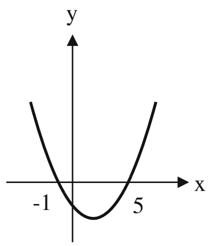
There were altogether five tasks—Task 1 to Task 5—in this "Thinking Activity" task sheet. As we found Task 1 and Task 3 particularly interesting, we focused our analyses on them.

As shown in Fig. 14.11, Teacher 8 presented the solutions for the quadratic equation $x^2 - 4x - 5 = 0$ using four different methods. She articulated explicitly in her pre-module interview that she had purposefully displayed the solutions "side by side instead of sequentially" (Pre-Module Interview, 00:14:52) so that students could "make comparisons" (Pre-Module Interview, 00:08:35). She stated clearly in the instructions for Task 1: "discuss the pros and cons of the method, and give suggestions on *when* the method should be used".

We surmise that her intention of presenting the solutions using all the four methods was to demonstrate to students that the same problem could be solved by more than one strategy. In addition, she had probably wanted her students to learn to apply the most suitable strategy to solve a problem, depending on its context. It seems clear to us that her goal was to **make connections across different methods**. We validated our conjecture with the evidence we found in her pre-module interview and lesson transcript. In her pre-module interview, she emphasised her objective for designing this "Thinking Activity":

You can use one of the following four methods to solve the quadratic equation $x^2 - 4x - 5 = 0$.

Method 1: By Factorisation	Method 2: By Completing the Square	Method 3: By using the Formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Method 4: By Graphical Method
$x^2 - 4x - 5 = 0$	$x^2 - 4x - 5 = 0$	$x^2 - 4x - 5 = 0$	
$(x+1)(x-5) = 0$	$(x-2)^2 - 2^2 - 5 = 0$	$a = 1, b = -4, c = -5$	
$(x+1) = 0$ or $(x-5) = 0$	$(x-2)^2 - 9 = 0$	$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)}$	
$x = -1$ or $x = 5$	$(x-2)^2 = 9$	$x = \frac{4 \pm \sqrt{16+20}}{2}$	
	$x - 2 = \pm 3$	$x = \frac{4 \pm 6}{2}$	
	$x = 3 + 2$ or $x = -3 + 2$	$x = 5$ or $x = -1$	
	$x = 5$ or $x = -1$		



From the graph,
$x = 5$ or $x = -1$

<u>Task 1</u>: Working in pairs, take turns to discuss each solution with your partner. Each person is to talk about 2 solutions. You should describe the method used, discuss the pros and cons of the method, and give suggestions on when the method should be used.

Fig. 14.11 Task 1 in "Thinking Activity" task sheet

> They are actually required to choose or identify key characteristics by themselves, they are supposed to *make comparisons*, they are supposed *to reason out why a method is more efficient than the other* … (Pre-Module Interview, 00:08:32.02, emphases added)

And during the lesson, she highlighted to students that they "must not rule out" (Lesson 7, 01:14:39) using another method even though they might prefer the factorisation method.

The other task that illustrated how Teacher 8 helped students make connections across units is Task 3 as shown in Fig. 14.12. In this activity, students were asked to select the "most efficient" method for each "question". Notice that Question 1 can be solved by any of the four methods but it would be most efficient to use the factorisation method. Subsequently, it would be most efficient to solve Question 2 using the quadratic formula method though it could also be solved by the completing the square method; and lastly, it would be most efficient to solve Question 3 by first dividing the equation by 2, then taking the square root for the equation, though one could also expand the left-hand side of the equation and then solve it by any of the other three methods. Nevertheless, one may also apply the quadratic formula method for every question without thinking about its "affordance and constraint". Hence, we infer that Teacher 8 had crafted Task 3 so that students could learn to regulate their

Task: Examine each of the following questions. Discuss with your partner and decide which method ***you prefer*** to solve each of the following questions. Justify your choices.

No.	Question	Your Preferred Method	Reason(s) for Your Choice
1	Find the roots of the equation $2x^2 - 5 = 9x$.		
2	Solve $2x^2 + 9x - 15 = 0$, giving your answers correct to 2 decimal places where necessary.		
3	Solve the equation $2(x-5)^2 = 100$.		

Fig. 14.12 Task 3 in "Thinking Activity" task sheet

understanding on the four methods and apply the most appropriate one for each question.

We think this task indeed requires students to think about the suitability of each method and not merely apply one method throughout mechanically. Teacher 8's responses in her post-lesson interview support our inference:

> [For] these particular set of task sheets, the content goal is really to help students to understand *when* they should be using which method. They need to have this appreciation for each [of the] different types of questions [where] some methods are more efficient than others .... That was the idea behind this worksheet. ... ... [W]hen they came to Task 3, they were *forced to make a choice* on which was their most efficient method, and I can see from their many responses that *many of them chose different methods.* (Post-Lesson Interview after Lesson 8, 00:07:48, emphases added)

In short, Teacher 8 helped her students make connections across units by providing a platform for them to engage in problem-solving with different methods. Through this activity, they were given the opportunity to appreciate the connectedness of the four methods and the conditions in which each was more appropriate.

14.6 Discussion

As mentioned in the beginning of the chapter, a mathematics teacher who conducts their lessons to help students perceive the connections across mathematics concepts views teaching mathematics within a connectionist orientation (Askew et al., 1997; Raveh et al., 2016). Based on our analysis of the instructional materials she created, we argue that Teacher 8 is an illustration of one who subscribes to the connectionist perspective. Her commitment to tight connection in her instruction is not merely a cursory one; as described in the previous section, she deliberately worked in various strategies of connection in the way she planned and carried out the lessons. Her commitment is extended to the way she designed her instructional materials. From the way she embedded intermodal links in her instructional materials, it is clear to us that she wanted to use the materials as an instrument to help her enact her goal of "link everything together" in her series of lessons. Yet, Teacher 8's connectionism is not limited to only one level of analysis. Her version of connectionism goes beyond a particular level as mentioned in the findings—she views connections within a unit, between adjacent units, and across all the units within the topic.

Second, Teacher 8 sequenced her instructional materials in such a way that her students could make temporal connections throughout the topic. For instance, students could link the unit on the factorisation method to the graphical method for finding the roots of a quadratic curve; they were also led to draw temporal links between the unit on completing the square method to and the unit on the graphical method for finding the maximum or minimum point of a quadratic curve; the same was also evident in the link between the unit on quadratic formula to the unit on graphical method for finding the discriminant of a quadratic curve. This careful sequencing reflected her conscious planning—evidence of the hypothetical learning trajectory she had constructed for her students. Moreover, to be able to plan lessons such that the units were so tightly connected requires vision that spans beyond the temporal boundaries of one or two lessons. To enact temporal connections as indicated in the lessons, one needs to project one's temporal horizons and hence connections across the content development over the *whole* unit. This, to us, calls into question of whether there is sufficient professional development work for teachers to conceive of planning at this scale.

Third, we think that Teacher 8's conception of connections across methods has implications to the development of students' problem-solving abilities—in particular, this metacognitive awareness of multiple strategies (and their respective affordances); that is, the consciousness of looking across different solution methods requires an executive function at work psychologically, and this mechanism to take executive control is a component of metacognition (Holton & Clarke, 2006; Schoenfeld, 1992). This link between her move of making connections and her intentions to highlight metacognitive regulation as part of problem-solving is underreported in the literature, although it was mooted a long time ago: "Can you derive the result differently? … One of the first and foremost duties of the teacher is not to give students the impression that mathematical problems have little *connection* with each other, and no *connection* at all with anything else" (Pólya, 1945, p. 15, emphases added).

14.7 Conclusion

Mathematics is a subject that is interconnected. In order for students to master the concepts, teachers need to help students relate one mathematical idea to another. Nonetheless, though many studies have examined how teachers make connections during their lessons, little is known about how teachers design their own instructional materials to help students make these connections. This chapter exemplifies a mathematics teacher in Singapore who not only advocates teaching in an interconnected way, she deliberately integrates connections within a set of instructional materials she designed for a topic. The most interesting characteristic we discovered from her instructional materials is that she does not only incorporate connections within a sub-section in a mathematics topic; taking a "global" view of the topic, she was able to insert numerous places at using different strategies to help students make connections between adjacent sub-sections; and even across all sub-sections.

As this teacher's instructional goals are embodied in her instructional materials explicitly, we can present a rich case of a teacher who helps students make connections via her instructional materials. However, as there is currently limited literature on how teachers design connections with instructional materials, more research work can be concentrated in this area. We think this Singapore teacher presents an interesting portrait of how "making connections" can be an organising principle in teachers' design of instructional materials for teaching mathematics. It is unclear at this stage if this represents the "Singapore portrait", however, we propose that this in an area worthy of further pursuit—to broaden and test the extent of whether this teacher's portrait to other mathematics teachers in Singapore and perhaps, even beyond.

Appendix

Section C on Page 5 of Teacher 8's Notes

Applying New Knowledge

C1. Solve the following equations

(a) $x^2 = 4$	(b) $(x + 1)^2 = 4$
(c) $(x - 2)^2 = 25$	(d) $(x - 4)^2 = 10$

C2. Given two equations $(x + 1)^2 = 4$ and $x^2 + 2x - 3 = 0$, how are they related? Which equation is easier to solve and why?

References

Ainsworth, S. (2006). A conceptual framework for considering learning with multiple representations. *Learning and Instruction, 16,* 183–198.

Askew, M., Brown, M., Rhodes, V., Wiliam, D., & Johnson, D. (1997). *Effective teachers of numeracy: Report of a study carried out for the Teacher Training Agency.* London: King's College, University of London.

Crooks, N. M., & Alibali, M. W. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental Review, 34,* 344–377.

De Bock, D., van Dooren, W., & Verschaffel, L. (2015). Students' understanding of proportional, inverse proportional, and affine functions: Two studies on the role of external representations. *International Journal of Science and Mathematics Education, 13*(Suppl. 1), 47–69.

Dreher, A., Kuntze, S., & Lerman, S. (2016). Why use multiple representations in the mathematics classroom? Views of English and German preservice teachers. *International Journal of Science and Mathematics Education, 14*(2), 363–382.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1/2), 103–131.

Elia, I., Panaoura, A., Eracleous, A., & Gagatsis, A. (2007). Relations between secondary pupils' conceptions about functions and problem solving in different representations. *Journal of Science and Mathematics Education, 5*(3), 533–556.

English, L. D., & Halford, G. S. (1995). *Mathematics education: Models and processes.* Hillsdale, NJ: Lawrence Erlbaum Associates Inc.

Fennema, E., & Romberg, T. A. (Eds.). (1999). *Classrooms that promote mathematical understanding*. Mahwah, NJ: Erlbaum.

Fyfe, E. R., Alibali, M. W., & Nathan, M. J. (2017, October). *The promise and pitfalls of making connections in mathematics*. Paper presented at the annual meeting of the North American chapter of the international group for the Psychology of Mathematics Education, Indianapolis, Indiana.

Healy, L., & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: Making connections with computers? *Mathematical Thinking and Learning, 1*(1), 59–84.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372–400.

Holton, D., & Clarke, D. (2006). Scaffolding and metacognition. *International Journal of Mathematical Education in Science and Technology, 37*(2), 127–143.

Leikin, R., & Levav-Waynberg, A. (2007). Exploring mathematics teacher knowledge to explain the gap between theory-based recommendations and school practice in the use of connecting tasks. *Educational Studies in Mathematics, 66*(3), 349–371.

Leong, Y. H. (2012). Presenting mathematics as connected in the mathematics classroom. In B. Kaur & T. L. Toh (Eds.), *Reasoning, communication and connections in mathematics* (pp. 239–260). Singapore: World Scientific.

Ma, L. (1999). *Knowing and teaching elementary mathematics.* Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Ministry of Education. (2012). *Mathematics syllabus: Secondary one to four. Express course. Normal (Academic) Course*. Singapore: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for teaching mathematics.* Reston: Author.

Pepin, B., & Haggarty, L. (2007). *Making connections and seeking understanding: Mathematical tasks in English, French and German textbooks*. Paper presented at the American Educational Research Association, Chicago.

Pólya, G. (1945). *How to solve it: A new aspect of mathematical method* (2nd ed.). Princeton, NJ: Princeton University Press.

Raveh, I., Koichu, B., Peled, I., & Zaslavsky, O. (2016). Four (algorithms) in one (bag): An integrative framework of knowledge for teaching the standard algorithms of the basic arithmetic operations. *Research in Mathematics Education, 18*(1), 43–60.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of math: Does one lead to the other? *Journal of Educational Psychology, 91,* 175–189.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). NY: Macmillan.

Sun, X. H. (2019). Bridging whole numbers and fractions: Problem variations in Chinese mathematics textbook examples. *ZDM Mathematics Education, 51,* 109–123.

Tall, D. (1988). Concept image and concept definition. In J. de Lange & M. Doorman (Eds.), *Senior secondary mathematics education* (pp. 37–41). Utrecht: OW & OC.

Toh, T. L. (2012). Reasoning, communication and connections in A-Level mathematics. In B. Kaur & T. L. Toh (Eds.), *Reasoning, communication and connections in mathematics* (pp. 129–147). Singapore: World Scientific.

Turner, P. (2015). Making connections. *Australian Senior Mathematics Journal, 29*(2), 51–61.

Wei Yeng Karen Toh worked as a research assistant at the National Institute of Education for seven years. She has a Bachelor of Science in Applied Mathematics from the National University of Singapore. She was awarded with a Distinction grade for the Post-Graduate Diploma in Education at the National Institute of Education. She taught in a few secondary schools as a mathematics and chemistry teacher for seven years. Her research interests include studying the history of mathematics, learning mathematical and pedagogical content knowledge, examining classroom practices and investigating teachers' designs of instructional materials. To advance her career in research, she pursued a Master of Education in Mathematics at the National Institute of Education.

Yew Hoong Leong is an Associate Professor at the National Institute of Education, Nanyang Technological University. He began his academic career in mathematics education with the motivation of improving teaching by grappling with the complexity of classroom instruction. Along the journey, his research has broadened to include mathematics problem solving and teacher professional development. Together with his project teammates, they developed "Realistic Ambitious Pedagogy" and its accompanying plan of action—the "Replacement Unit Strategy".

Lu Pien Cheng is a Senior Lecturer in the Mathematics and Mathematics Education Academic Group at the National Institute of Education, Nanyang Technological University in Singapore. She received her PhD in Mathematics Education from the University of Georgia (U.S.) in 2006. She specialises in mathematics education courses for primary school teachers. Her research interests include the professional development of primary school mathematics teachers, task design in mathematics education and cultivating children's mathematical thinking in the mathematics classrooms.

Chapter 15
Use of Technology by Experienced and Competent Mathematics Teachers in Singapore Secondary Schools

Joseph B. W. Yeo

Abstract This chapter reports how 30 experienced and competent Singapore mathematics teachers used technology inside and outside their classrooms. The first ICT (Information and Communication Technology) Masterplan in Education for Singapore was launched in 1997 to provide some comprehensive strategies to harness ICT for teaching and learning. Since then, there were quite a number of local research studies on the use of ICT in mathematics teaching. However, in 2019, it was noted that research interest in this area had dwindled after 2004. Therefore, this chapter provides a timely update on whether Singapore mathematics teachers still use technology in their teaching and if yes, how. The video recording of 209 lessons of the 30 teachers were analysed and it was found that 23 of them made use of ICT in various ways. The most common modes were the use of technology as a tool for students to explore and discover mathematics and as teacher aids to project various resources onto the screen. Fewer teachers made use of computer-assisted instructions (CAI) on the Internet. Teachers could emulate the practices of the 30 teachers to harness the affordances of technology as a tool for students to construct their own knowledge through interactive investigative activities with the help of graphing or dynamic geometry software.

Keyword Technology · ICT · Tool mode · Tutor mode · Investigation

15.1 Introduction

The first ICT (Information and Communication Technology) Masterplan in Education for Singapore was launched by the Ministry of Education (MOE) in 1997 to build a strong foundation to harness ICT for teaching and learning (MOE, 2020a). It called for core ICT training for all teachers, development of ICT infrastructure and support for all schools, and production of educational software and resources

J. B. W. Yeo (✉)
National Institute of Education, Nanyang Technological University, Singapore, Singapore
e-mail: josephbw.yeo@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_15

for relevant subjects. It was during this period that MOE bought the dynamic geometry software, the Geometer's Sketchpad (GSP), for every school to use for their mathematics lessons.

The second ICT Masterplan in Education was unveiled in 2002 (for the period 2003–2008) and it built on the first ICT Masterplan to seed innovation (MOE, 2020b). It called for a more pervasive use of ICT as a tool to customise education to meet the needs and abilities of students so as to support and develop lifelong learners. Some key priorities of the masterplan were to set baseline standards for students' learning experiences and teachers' ICT integration practises, give greater autonomy to schools to take full ownership of their ICT implementation through devolved ICT funds, and generate innovative practices through more recognition schemes.

The third ICT Masterplan in Education was launched in 2008 (for the period 2009–2014) to strengthen and scale up the use of ICT in the first two masterplans to harness ICT to transform learners (MOE, 2020c). The enabler goals were for school leaders to provide direction and create conditions to harness ICT for teaching and learning, for teachers to have the capacity to plan and deliver ICT-enriched learning experiences, and for the ICT infrastructure to support learning anytime and anywhere. The outcome goals were for students to develop competencies for self-directed and collaborative learning through the effective use of ICT as well as become discerning and responsible ICT users.

The fourth ICT Masterplan in Education (for the period 2015-present) built on the first three masterplans to deepen learning and sharpen practices in order to prepare students to be future-ready and responsible digital learners in the twenty-first century (MOE, 2020d). This was also in line with MOE's direction towards a values-based and student-centric education in 2012 (Kaur, 2019; Natarajan, Lim, & Cheah, 2018). The implementation of the masterplan focused on four areas: deeper integration of ICT in curriculum, assessment and pedagogy; sustained professional learning among school teams and learning communities; translational research, innovation and scaling in ICT-enabled pedagogies and practices; and a connected ICT learning ecosystem in terms of both physical and socio-cultural infrastructure.

A common theme among all the four ICT Masterplans is still the encouragement and strengthening of the effective use of technology in teaching and learning for the development of students. Since the first ICT Masterplan in Education was introduced in 1997, there was a proliferation of local research on the use of ICT in mathematics teaching, peaking in 2002 and started to dwindle after 2004 (Ng, Teo, Yeo, Ho, & Teo, 2019). Therefore, we only had a sense of what Singapore mathematics teachers were doing in the infusion of ICT in their classrooms before 2004, and after that, there was a dearth of local research. Hence, this research project on how 30 experienced and competent Singapore mathematics teachers enacted the school curriculum was timely because it provided an insight into how these teachers harnessed technology inside and outside their classrooms for teaching and learning the subject. Unlike other chapters which reported the findings from the survey of 677 mathematics teachers, it was beyond the scope of the same survey instrument to ask these teachers how they use ICT in their teaching.

Thus the research questions addressed in this chapter are:

(1) How many of the 30 experienced and competent teachers leveraged on technology in their lessons?
(2) How do the experienced and competent teachers infuse technology in their teaching?

15.2 Literature Review

Taylor (1980) classified the use of the computer in the school during the 1960s and 1970s into three modes: tutor, tool and tutee. In the tutor mode, the computer is the tutor and so students learn *from* the computer. This involves using computer-assisted instruction (CAI) to tutor and drill pupils in procedural skills. In the tool mode, the computer is the tool and so students learn *with* the computer. In those days, this involved using application software such as *LOGO* to explore mathematical ideas. Designed by Seymour Papert and others in 1967, the *LOGO* software consists of a turtle and the user explores geometrical concepts by moving the turtle around. In the tutee mode, the computer is the tutee and so students learn *through* teaching (e.g. programming) the computer. There are generally two approaches to the tutee mode. The first approach is to learn a programming language. But Jensen and Williams (1993) believed it was unwise to spend so much curriculum time studying a programming language that is constantly changing. The second approach is to learn some simple programming, like *LOGO*, and then programme the turtle to move around to explore mathematical ideas. Therefore, *LOGO* programming to explore concepts involves both the tutee and the tool modes.

In the 1990s, Jensen and Williams (1993) and Manoucherhri (1999) observed that the use of CAI to teach students mathematics was the most frequently used mode since the 1970s. With the advance of ICT when technology is no longer confined to just a desktop computer but there are handheld instruments (such as graphing calculators, tablets and smart phones), Internet and social media, are Taylor's three modes still relevant today? If one were to search the Internet for mathematics resources, one would find that most of the websites just contain non-interactive reading materials or videos of a person teaching. Lagrange, Artigue, Laborde, and Trouche (2003) called these resources "CAI on the Internet" (p. 255), which they "assumed to be the actual technological engagement of an average student, at least in developed countries" (ibid.). Even for flipped classrooms where students learn the materials before going to class for discussion, those materials are usually in the form of notes, PowerPoint slides or videos of the teacher teaching (Ng et al., 2019). All of these uses of technology are just Taylor's tutor mode. This suggests that the tutor mode is still very much prevalent even today.

However, many mathematics researchers seem to advocate the use of the tool mode. For example, Kaput (1992), in the *Handbook of Research on Mathematics Teaching and Learning*, talked about using the computer as a tool for exploration, e.g. virtual manipulative, dynamic geometry environments such as CABRI Geometry and the Geometer's Sketchpad (GSP), and probability, statistics and data modelling. Zbiek, Heid, Blume, and Dick (2007), in the *Second Handbook of Research on Mathematics Teaching and Learning*, wrote about the use of cognitive technological tools, such as microworlds, simulations and representational toolkits (which include graphing calculators, spreadsheets and computer algebra system), for exploring and constructing mathematical ideas.

In the same vein, Balacheff and Kaput (1996), in the *International Handbook of Mathematics Education*, wrote about computer-based learning environments in mathematics, such as microworlds, computer algebra systems (CAS), dynamic geometry software and mathematical and statistical modelling. Lagrange et al. (2003), in the *Second International Handbook of Mathematics Education*, found that most of the research on the use of ICT in mathematics education from 1994 to 1998 tended to focus on dynamic computer software or symbolic calculators, rather than on CAI on the Internet or on CD-ROM. In the *Third International Handbook of Mathematics Education*, there are four whole chapters dedicated to the use of ICT as a tool for modelling (Williams & Goos, 2013) and exploration in geometry (Sinclair & Robutti, 2013), algebra (Heid, Thomas, & Zbiek, 2013) and statistics (Biehler, Ben-Zvi, Bakker, & Makar, 2013). Some of the computer tools used are TI-Nspire (a CAS), GSP and ThinkerPlots (an interactive statistical software). Even in the chapter in the same handbook on learning with the use of the Internet, Borba, Clarkson, and Gadanidis (2013) stated that they "have chosen not to report on studies that are predominantly text based and/or use rapid response modes aimed mainly at testing students' abilities" (p. 700). Instead, they would rather "report on studies that seem to push the boundaries of how the Internet can be used creatively and with worth in mathematics education" (ibid.): all the examples given use some kinds of tools to explore, and nothing on CAI.

Based on the review of literature, it seems that there exists a dichotomy of how researchers would use technology as a tool to explore mathematics and how teachers and students actually use technology as the tutor via CAI. This is understandable because researchers usually prefer students to construct their own knowledge through investigative activities, according to the theory of constructivism (Ernest, 1994; von Glasersfeld, 1990) while direct instruction via CAI may not be so helpful to do so. It will be interesting to find out which mode is favoured by experienced and competent mathematics teachers in Singapore.

15.3 Research Design

The research design for the collection of data reported in this chapter has been outlined in Chapter 2. In this section, I will briefly describe how the data were analysed to answer the research questions for this chapter. The 209 lessons of the 30 experienced and competent teachers were examined to pick up episodes of the teacher teaching mathematics with the help of technology in the classroom. Because some teachers told their students to view their pre-recorded lessons at home or made reference to answering queries from their students through WhatsApp after school, the use of technology by the teachers also included those done outside the classroom. I will now present the findings.

15.4 Findings and Discussion

It was discovered that most of the 30 experienced and competent mathematics teachers infused technology inside and outside the classrooms using the modes described in Table 15.1. Out of the 30 teachers, 4 taught the Integrated Programme (IP), 10 taught the Express course, 8 taught the Normal (Academic) (N(A)) course and the remaining 8 taught the Normal (Technical) (N(T)) course. The reader can refer to Chapter 2 for a more detailed description of these four courses of study, but the abilities of the students are generally higher for the IP course than students in the

Table 15.1 Use of technology by the 30 experienced and competent teachers

Instructional Approach	Number (and Percentage) of Teachers				
	IP ($n = 4$)	Express ($n = 10$)	N(A) ($n = 8$)	N(T) ($n = 8$)	Total ($N = 30$)
Tutor mode: direct-instruction videos in YouTube or e-learning portals; or self-recorded direct-instruction videos uploaded onto YouTube	1 (25%)	2 (20%)	2 (25%)	0 (0%)	5 (16.7%)
Tool mode: exploration using interactive software or online templates such as Geometer's Sketchpad, GeoGebra, Desmos, TI-Nspire, Excel and algebra discs; or apps such as a sound meter	4 (100%)	3 (30%)	2 (25%)	2 (25%)	11 (36.7%)
Engagement mode: amusing videos to engage the hearts of students	0 (0%)	1 (10%)	0 (0%)	1 (12.5%)	2 (6.7%)
Assessment mode: practice and quizzes using software such as Kahoot, e-learning portals or school-created websites	0 (0%)	1 (10%)	0 (0%)	2 (25%)	3 (10%)
Teacher aids: projection of textbooks, e-books, PowerPoint slides, teacher's notes, questions and/or student work onto a screen with the help of laptop, iPads/Apple TV or visualiser (i.e. document camera)	1 (25%)	4 (40%)	2 (25%)	4 (50%)	11 (36.7%)
Student aids: to seek help for homework outside curriculum time, e.g. through WhatsApp	0 (0%)	1 (10%)	1 (12.5%)	0 (0%)	2 (6.7%)
No evident use of technology	0 (0%)	3 (30%)	1 (12.5%)	3 (37.5%)	7 (23.3%)

Express course, which in turn are higher than those in the N(A) course; while the abilities of the students in the N(T) course are generally the lowest.

From Table 15.1, we observed that 23 of the 30 teachers (76.7%) made use of technology in one way or another while 7 teachers (23.3%) did not use any form of technology at all. Of the 23 teachers who made use of technology, 10 of them used more than one mode (not shown in Table 15.1). If we exclude the 3 teachers who used only the visualiser and/or PowerPoint slides, then only 20 of the 30 teachers (66.7%) made use of more modern technology in or outside the classrooms. As far as teaching and learning with the help of technology is concerned, 5 teachers (16.7%) used the tutor mode and 11 teachers (36.7%) used the tool mode, out of which 2 of them (6.7%) utilised both tutor and tool modes. In other words, a total of 14 teachers (46.7%) used either the tutor or tool mode or both. None of the teachers used the tutee mode for teaching and learning.

As for the 7 teachers who did not use any form of technology, the topics taught by them at the time of the video recording were: differentiation (Teacher 10 and Teacher 28), vectors (Teacher 27), volume and surface area of prisms and cylinders (Teacher 14), trigonometric ratios of acute angles (Teacher 25), bearings and 3-dimensional problems using trigonometry (Teacher 15), and simultaneous linear equations (Teacher 4). Some of these topics did not lend themselves to the use of ICT or there may be other equally effective pedagogy (such as the use of concrete manipulative) to teach them, which may explain why these teachers did not engage in technology just for the sake of using ICT when it does not enhance student learning.

On closer analysis, all the 4 IP teachers (100%) used the tool mode (with one of them using the tutor mode to show some YouTube videos as well) while 25–30% of the teachers in each of the other 3 streams used the tool mode (with one Express teacher using the tutor mode as well). It seems that the tool mode is more popular with IP teachers, who teach high-progress learners. But it does not mean that teachers teaching the other three courses of study does not use the tool mode.

In addition, 11 of the 30 teachers (36.7%) used technology as an aid for themselves, mainly to project various resources and/or student work onto the screen via visualisers or laptops (all classrooms in Singapore have visualisers, or what some countries call document cameras; and all teachers are issued with a laptop). Only 3 teachers (10%) used technology for assessment, 2 teachers (6.7%) used it outside curriculum time for students to seek help for homework, and 2 teachers (6.7%) used it to engage the hearts of the students (by showing interesting mathematics videos). Let us now examine in more details what the teachers did for each of the various modes.

15.4.1 Tutor Mode

With the advance of the Internet and social media, the tutor mode does not change. Instead of installing a program or using a CD-ROM with CAI, teachers in Singapore can now use direct-instruction videos in YouTube or in an e-learning portal which their school has to subscribe, for their students to view and learn the contents. Among

Table 15.2 Use of tutor mode by 5 experienced and competent teachers

Types	Topics	No. of teachers (Course of Study)
Existing YouTube videos	• Parts of a circle • Laws of logarithm, mathematical constant e and natural logarithm	2 (1 N(A) and 1 IP)
Existing videos in an e-learning portal	• Hypotenuse of a triangle • Applications of trigonometry	2 (1 Express and 1 N(A))
Self-recorded videos posted on YouTube	• Proofs in plane geometry	1 (1 Express)

the 5 teachers who utilised the tutor mode, 2 of them used a video from YouTube to teach their students topics like parts of a circle (Teacher 11), the laws of logarithm, the mathematical constant e and natural logarithm (Teacher 12) (see Table 15.2). Another two teachers made use of a video in an e-learning portal to teach their students topics such as the hypotenuse of a triangle (Teacher 6) and applications of trigonometry (Teacher 19). Teacher 3 recorded himself teaching how to prove some geometrical properties and posted the series of videos on YouTube.

Most of the videos were viewed in the classrooms or computer labs, with the exception of the YouTube video on the mathematical constant e and natural logarithm, and some of the self-recorded videos on proofs in plane geometry, which the teachers expected their students to view at home. According to the annual survey on infocomm usage in household and by individuals for 2019, nearly 100% of resident households with school-going children have broadband (i.e. high-speed) Internet access at home, so it is not an issue for students to view some of these videos at home (Infocomm Media Development Authority, 2019, p. 8).

In other words, the Internet and social media did not change the tutor mode but they only make it easier for the tutor mode to be implemented by making the videos more readily available, and they also provide a platform for teachers to post their self-recorded direct-instruction videos for easy access by the students.

15.4.2 Tool Mode

The Internet also makes the tool mode more easily accessible. Before the World Wide Web was able to host interactive templates, mathematics software such as Geometer's Sketchpad (GSP) or Graphmatica had to be installed in the teacher's and students' desktop computers, and pre-designed templates or files had to be uploaded onto the computers. With the advance of technology, teachers and students can now use interactive online software or templates such as GeoGebra and Desmos without any pre-installation and they can use them on handheld devices such as iPad or mobile phones without having to go to a computer lab in the school. Secondary school students are not allowed to use graphing calculators in examinations, so they usually do not have access to graphing calculators for such purpose. Most of them will use their own mobile phones or the teacher will loan school iPads for them to use. There was also a plan by MOE for every student to get a laptop or tablet by 2028 (Ang, 2020), but this plan has been brought forward to 2021 (Ong, 2020) due to the need for home-based learning during the circuit breaker (or lockdown) to stem the spread of the coronavirus Covid-19.

Among the 11 teachers who used the Tool mode, almost half of them (i.e. 5 teachers) used a graphing software, such as Desmos and TI-Nspire, for students to investigate the properties of graphs of linear functions (Teacher 26), quadratic functions (Teacher 13 and Teacher 21) and logarithmic functions (Teacher 12), as well as the relationships between the sine and cosine ratios of acute and obtuse angles (Teacher 17) (see Table 15.3). An almost equal number of teachers (i.e. 4 teachers) made use of a dynamic geometry software, such as the Geometer's Sketchpad (GSP) and GeoGebra, for students to explore and discover angle properties of circles (Teacher 5), Pythagoras' Theorem (Teacher 6 and Teacher 24) and Cosine Rule (Teacher 17). One of the teachers, Teacher 20, utilised a statistical software, namely Excel, as a tool to compile students' experiment results of tossing a coin and compute the experimental probability of obtaining a Head. Two teachers tapped on an online interactive applet for different uses: one of them used an applet as a visualising tool to help students observe how many surfaces a triangular prism has (Teacher 23); the other, Teacher 18, used the AlgeDisc™ application in AlgeTools™ to explore the balancing of an equation using algebra discs (AlgeTools™ was created by the Ministry of Education of Singapore and has since been decommissioned). One of the teachers, Teacher 12, also used a sound meter as a tool for students to measure the sound intensity of certain activities, such as normal breathing, soft whisper and classroom noise. Out of the 11 teachers, two of them used more than one tool: Desmos and GSP, or Desmos and sound meter.

Figure 15.1 shows a Desmos template used by Teacher 17 in the IP for students to explore the relationship between the sine and cosine ratios of acute and obtuse angles (the tangent ratio of obtuse angles is not in the syllabus because students only need sine and cosine ratios of obtuse angles when applying sine rule and cosine rule respectively). Because the circle is a unit circle, the coordinates of the point on the circle are ($\cos a$, $\sin a$). The slider for the angle a allows the user to drag and change the value of a.

Figure 15.2 shows a GSP template used by the same teacher, Teacher 17, for her class to explore and discover cosine rule. The students would click and move each of the vertices of the triangle, and the measures of the angles and lengths of the triangle would change automatically and instantaneously. The values in the table would also change accordingly. Students would then observe that certain values in the table would always be equal regardless of how the triangle was changed. This would lead them to discover the cosine rule.

To summarise, most of the 11 teachers used either a graphing software or a dynamic geometry software. Three of the 11 teachers (Teacher 18, Teacher 23 and Teacher 26) used the tool purely for teacher demonstration while 7 of them (Teacher 5, Teacher 6, Teacher 12, Teacher 13, Teacher 20, Teacher 21 and Teacher 24) let their

Table 15.3 Use of tool mode by 11 experienced and competent teachers

Types	Topics	No. of teachers (Course of Study)
Graphing Software (e.g. Desmos, TI-Nspire)	• Investigate properties of graphs of linear functions • Investigate properties of graphs of quadratic functions • Investigate characteristics of graphs of logarithmic functions • IInvestigate relationships between the sine and cosine ratios of acute and obtuse angles	5 (4 IP and 1 N(A))
Dynamic Geometry Software (e.g. Geometer's Sketchpad, GeoGebra)	• Investigate angle properties of circles • Investigate to discover Pythagoras' Theorem • Investigate to discover Cosine Rule	4 (1 IP, 2 Express and 1 N(T))
Statistical Software (e.g. Excel)	• Teacher used Excel as a tool to compile students' experiment results of tossing a coin and compute experimental probability of obtaining a Head	1 (1 Express)
Online Interactive Applets	• IAs a visualising tool for students to observe how many surfaces a triangular prism has	1 (1 N(T))
AlgeDisc™ application in AlgeTools™	• This is an online interactive tool using algebra discs to balance an equation (it has since been decommissioned)	1 (1 N(A))
Sound Metre	• Measure sound intensity of certain activities such as normal breathing, soft whisper and classroom noise	1 (1 IP)

students use the software to investigate the mathematics. The last teacher, Teacher 17, did both: on two occasions, it was purely teacher demonstration; on another occasion, she let the students used GSP to explore cosine rule. Student-centred investigation was done either in the computer lab (by Teacher 5 and Teacher 6), or in the classroom with laptops or iPads provided by the school. Sometimes, the students had to use their own mobile devices for such activities.

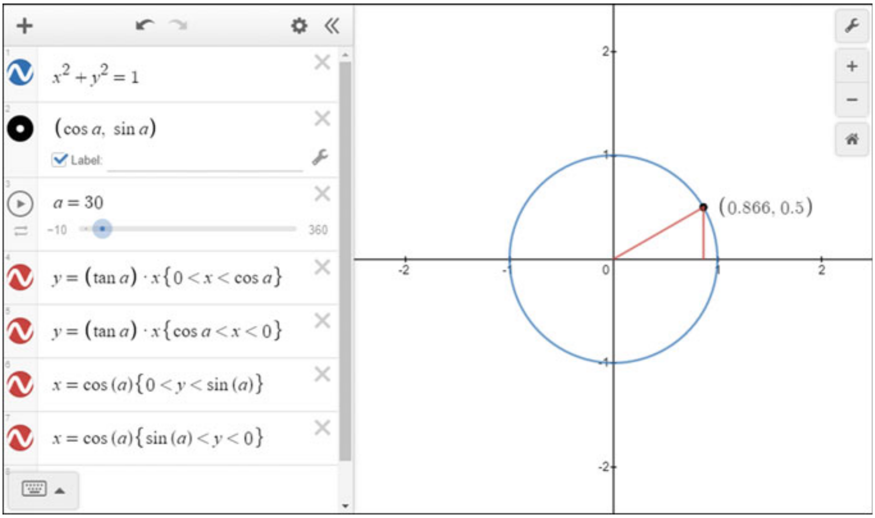


Fig. 15.1 Teacher 17's Desmos template for exploring the sine and cosine ratios of acute and obtuse angles

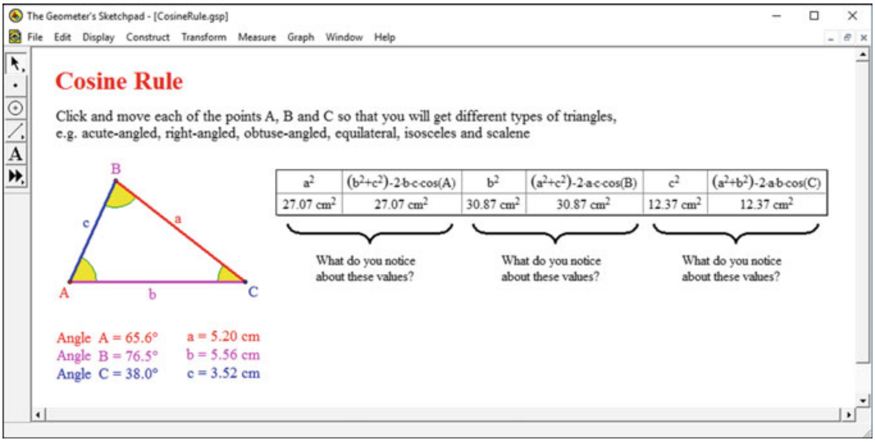


Fig. 15.2 Teacher 17's GSP template for students to explore and discover cosine rule

15.4.3 Engagement Mode

Two of the teachers, Teacher 9 and Teacher 6, did not just use videos to teach students mathematics but they use them to engage their hearts and to arouse their interest. Teacher 9 screened a video from YouTube in class on the Great Pyramid of Giza: the video did not teach students any mathematics but just provided some information about the pyramid such as its history and its dimensions. The teacher hoped to use this real-life example of a pyramid to motivate her Secondary 4 N(T) students to learn more about finding the volume and surface area of a pyramid.

Teacher 6 showed her Secondary 2 Express class a 10-minute video containing snippets of a Korean drama (with English subtitles) about a girl who managed to travel back in time to ancient Korea and helped the king solve a mathematics problem using Pythagoras' theorem. The whole class found the drama funny because of the slapstick humour and situational jokes, but there was really nothing much about the theorem. However, the teacher designed three problems with contexts that continued the storyline in the drama for her students to solve in class using Pythagoras' theorem. In other words, the teacher used the Korean drama to arouse the interest of her students to solve word problems involving the application of Pythagoras' theorem. For more details on the three problems and the outcome, please refer to Chapter 7 in this book.

15.4.4 Assessment Mode

Three teachers, Teacher 7, Teacher 20 and Teacher 30, used ICT for assessment. Teacher 7 used Kahoot, an app that allows students to answer multiple choice questions, to assess his Secondary 4 N(T) students' learning of the three types of averages, namely mean, median and mode. Teacher 20 used an e-learning platform which the school has subscribed, for his Secondary 2 Express students to take an online quiz on probability over the weekend. Teacher 30 asked her Secondary 1 N(T) class to take some quizzes on angles from an online portal in class using laptops loaned from the school.

15.4.5 Teachers' Aids

A significant proportion of the 30 teachers (11 teachers or 36.7%) also used ICT as aids for themselves. They were Teacher 1, Teacher 2, Teacher 3, Teacher 7, Teacher 8, Teacher 9, Teacher 16, Teacher 21, Teacher 22, Teacher 23 and Teacher 24. The main usage was to project various resources (e.g. textbooks, e-books, PowerPoint slides, teacher's notes, questions and/or student work) onto the screen with the help of their laptop/iPad or visualiser (i.e. document camera) so that their students could view them.

15.4.6 Students' Aids

Two of the teachers, Teacher 8 and Teacher 29, also used ICT as a means for students to seek their help outside curriculum time through the use of WhatsApp. This leverage of technology was not possible before the invention and proliferation of social media.

15.5 Conclusion

Slightly more than 75% of the 30 experienced and competent Singapore mathematics teachers harnessed the use of technology in their teaching in numerous ways. The most common modes were the use of ICT as a tool for students to learn mathematics through investigation (36.7%), and as teacher aids to project various resources onto the screen for students to see (36.7%). One sixth (or 16.7%) of the teachers also used ICT in the tutor mode for direct instruction. Across the four courses of study, IP teachers seem to favour the tool mode over the tutor mode more than the other teachers, although there were teachers in the other three courses who also used the tool mode. Technology may advance but for mathematics, it seems that the tool mode and the tutor mode do not change much. Also, these experienced and competent local teachers are using the tool mode as advocated by many researchers (as discussed in Sect. 15.2) more than the use of the tutor mode. Therefore, Singapore teachers could follow the examples of the 30 teachers to harness the affordances of technology as a tool for students to construct their own knowledge through interactive investigative activities with the help of a suitable graphing or dynamic geometry software.

References

Ang, J. (2020, March). Parliament: All secondary school students to have personal digital devices by 2028, $200 Edusave top-up to support purchase. *The Straits Times*. https://www.straitstimes.com/politics/parliament-all-secondary-school-students-to-have-personal-digital-devices-by-2028-200.

Balacheff, N., & Kaput, J. J. (1996). Computer-based learning environments in mathematics. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 469–501). Dordrecht, The Netherlands: Kluwer Academic Press.

Biehler, R., Ben-Zvi, D., Bakker, A., & Makar, K. (2013). Technology for enhancing statistical reasoning at the school level. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *Third international handbook of mathematics education* (pp. 643–689). New York: Springer.

Borba, M. C., Clarkson, P., & Gadanidis, G. (2013). Learning with the use of the Internet. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *Third international handbook of mathematics education* (pp. 691–720). New York: Springer.

Ernest, P. (1994). Social constructivism and the psychology of mathematics education. In P. Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and mathematical education* (pp. 62–72). London: The Falmer Press.

Heid, M. K., Thomas, M. O. J., & Zbiek, R. M. (2013). How might computer algebra systems change the role of algebra in the school curriculum? In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *Third international handbook of mathematics education* (pp. 597–641). New York: Springer.

Infocomm Media Development Authority. (2019). *Annual survey on infocomm usage in household and by individuals for 2019*. https://www.imda.gov.sg/-/media/Imda/Files/Infocomm-Media-Landscape/Research-and-Statistics/Survey-Report/2019-HH-Public-Report_09032020.pdf?la=en.

Jensen, R. J., & Williams, B. S. (1993). Technology: Implications for middle grades mathematics. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 225–243). New York: National Council of Teachers of Mathematics & Macmillan.

Kaput, J. J. (1992). Technology and mathematics education. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 515–556). Reston, VA: National Council of Teachers of Mathematics & MacMillan.
Kaur, B. (2019). Overview of Singapore's education system and milestones in the development of the system and school mathematics curriculum. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 13–33). Singapore: Springer.
Lagrange, J. B., Artigue, M., Laborde, C., & Trouche, L. (2003). Technology and mathematics education: A multidimensional study of the evolution of research and innovation. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second international handbook of mathematics education* (pp. 237–269). Dordrecht: The Netherlands: Kluwer Academic Publishers.
Manoucherhri, A. (1999). Computers and school mathematics reform: Implications for mathematics teacher education. *Journal of Computers in Mathematics and Science Teaching, 18*(1), 31–48.
Ministry of Education (MOE). (2020a). *Masterplan 1*. https://ictconnection.moe.edu.sg/masterplan-4/our-ict-journey/masterplan-1.
Ministry of Education (MOE). (2020b). *Masterplan 2*. https://ictconnection.moe.edu.sg/masterplan-4/our-ict-journey/masterplan-2.
Ministry of Education (MOE). (2020c). *Masterplan 3*. https://ictconnection.moe.edu.sg/masterplan-4/our-ict-journey/masterplan-3.
Ministry of Education (MOE). (2020d). *Masterplan 4*. https://ictconnection.moe.edu.sg/masterplan-4.
Natarajan, U., Lim, K., & Cheah, H. M. (2018). *Twenty years of Thinking Schools, Learning Nation (TSLN) vision: Reflections on Singapore's ICT Masterplans* (The HEAD Foundation Working Papers Series No. 2 / 2018).
Ng, W. L., Teo, B. C., Yeo, J. B. W., Ho, W. K., & Teo, K. M. (2019). Use of technology in mathematics education. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 313–348). Singapore: Springer.
Ong, Y. K. (2020). *Opening address by Mr Ong Ye Kung, Minister for Education at the 2020 Schools and Institutes of Higher Learning Combined Workplan Seminar*. https://www.moe.gov.sg/news/speeches/opening-address-by-mr-ong-ye-kung–minister-for-education-at-the-2020-schools-and-institutes-of-higher-learning-combined-workplan-seminar.
Sinclair, N., & Robutti, O. (2013). Technology and the role of proof: The case of dynamic geometry. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *Third international handbook of mathematics education* (pp. 571–596). New York: Springer.
Taylor, R. (Ed.). (1980). *The computer in the school: Tutor, tool, tutee*. New York: Teachers College Press.
von Glasersfeld, E. (1990). An exposition of constructivism: Why some like it radical. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics: Journal for Research in Mathematics Education Monograph Number 4* (pp. 19–29). Reston, VA: National Council of Teachers of Mathematics.
Williams, J., & Goos, M. (2013). Modelling with mathematics and technologies. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *Third international handbook of mathematics education* (pp. 549–569). New York: Springer.
Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169–1207). Charlotte, NC: Information Age Publishing & National Council of Teachers of Mathematics.

Joseph B. W. Yeo Yeo is a Lecturer in the Mathematics and Mathematics Education Academic Group at the National Institute of Education, Nanyang Technological University, Singapore. He is the first author of the think! Mathematics textbooks (previously called New Syllabus Mathematics textbooks) used in many secondary schools in Singapore. His research interests are on innovative

pedagogies that engage the minds and hearts of mathematics learners. These include an inquiry approach to learning mathematics, ICT and motivation strategies to arouse students' interest in mathematics (e.g. catchy maths songs, amusing maths videos, witty comics and intriguing puzzles and games). He is also the Chairman of Singapore and Asian Schools Math Olympiad (SASMO) Advisory Council, and the creator of Cheryl's birthday puzzle that went viral in 2015.

Part IV
Conclusion

Chapter 16
Framing the Portraits of Singapore Secondary Mathematics Pedagogy: An Outsider's Perspective

Wee Tiong Seah

Abstract Thirteen different research articles which report on a programmatic research on the enacted secondary school mathematics curriculum in Singapore have been perused in the preparation for this chapter. It considers the instructional practices associated with Singapore secondary mathematics teachers, and identifies possible contextual factors that facilitate these teachers' enactment of the mathematics curriculum, framed by the Social Cognitive Theory. These factors include teachers' content knowledge, trust in the leadership, students as disciples, societal valuing of excellence, and twenty-first century competency education. The role of teacher self-efficacy is also examined. An understanding of these contextual factors helps to frame the portraits of mathematics teaching and learning in Singapore secondary schools, and could also allow us to better assess how best to replicate particular instructional practices in other mathematics education systems. In particular, it appears that what works in practice reflects the harmonious interaction between teacher professionalism on the one hand, and policy and other contextual factors on the other hand, underlied by what individuals, institutions, and the society value now and over time.

Keywords Context · Experienced and competent teachers · Instructional practice · Secondary mathematics · Singapore

16.1 From Machine to Cup, from Plan to Lesson

Coffee is one of the world's most popular beverages, and some baristas are known for creating great cups of the liquid gold consistently. Yet, across different countries, these and other baristas are likely using similar or identical expresso machines. Thus, the acquisition of one of these expensive expresso machines does not guarantee great

W. T. Seah (✉)

Melbourne Graduate School of Education, The University of Melbourne, Melbourne, VIC, Australia
e-mail: wt.seah@unimelb.edu.au

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_16

coffee; the human barista is as important in the coffee-making process. For this reason, a skilful barista is considered to have mastered the science and art of making the perfect brew, and they are often celebrated and hotly sought-after.

Enacting a (mathematics) curriculum is no different. When Brown (2009) described teachers' work interpreting and enacting the curriculum as a design activity, he too was highlighting the creative process involved in interpreting and customising mathematics learning for students. An important component of this creative design work—whether one is creating a perfect cuppa or designing for effective mathematics teaching—is decision-making. Remillard (2018) distinguished between planned decisions and in-the-moment-design decisions [IMDDs], and both types of decisions contribute to the enacted curriculum. It should be noted that any IMDD made "is not necessarily a reflection of poor planning or underdeveloped resources. Rather, they reflect the substantive distinction between the written, planned, and enacted curriculum" (Remillard, 2018, p. 491).

It is well-known that Singapore students consistently produce world-leading results in major international mathematics assessment exercises such as the Trends in International Mathematics and Science Study [TIMSS] and Programme for International Student Assessment [PISA]. This achievement has understandably attracted much attention from educators and researchers across the world. Not only have Singapore students been demonstrating excellent mathematical knowledge and skills (the focus of TIMSS), but they have also proven themselves at applying these knowledge and skills to novel mathematical problem situations (the focus of PISA). While the rigorous and forward-thinking national mathematics curriculum can lay claim to much credit for this achievement (American Institutes for Research, 2005; Schleicher, 2018), just like those state-of-the-art expresso machines, the roles played by Singapore mathematics teachers—akin to the barista masters—in enacting the curriculum are equally significant. It is thus no wonder that the Australian Association of Mathematics Teachers, for example, has been organising Exchange Study Tours for Australian educators and teachers to interact with Singapore teachers face-to-face in classroom settings.

In this context, the publication of this book 'Mathematics Instructional Practices in Singapore Secondary Schools' has been timely. The book chapters drew upon the findings of an extensive programmatic research project conducted in Singapore in 2016—2018 to investigate the enacted secondary mathematics curriculum in Singapore schools (thereafter called the 'enactment project' in this chapter). The pedagogical practices of thirty experienced and competent teachers were examined, and for some of these practices, more than 600 other teachers were surveyed to determine the extent to which they were commonly employed in the Singapore secondary school mathematics education system.

However, just like it is often wondered why it is almost impossible to replicate the 'perfect cappuccino/latte' in certain countries, it is important to identify the features of different socio-political/institutional/personal contexts which either facilitate or impede a teacher's master strokes. Thus, the purpose of this chapter is to take advantage of the author's positioning as an outsider to look in at Singapore's educational scene, to identify features which affect how local teachers enact the curriculum in rich and effective ways, but which have not all been explicitly identified amongst the chapters of the book mentioned above. For readers in Singapore, this chapter might identify some contextual features which are otherwise invisible. For other readers, this chapter might provide us with a more rounded portrait of the mathematics instructional practices in Singapore secondary schools. In so doing, this chapter responds to the question: What contextual features facilitate Singapore secondary school teachers' enactment of the mathematics curriculum?

The author is also a privileged outsider to the Singapore mathematics education system. He was teacher-trained in Singapore and taught secondary school mathematics there for several years in the 1990s across the spectrum of perceived student abilities. This experience and contextual knowledge should add strength to the analysis reported in this chapter, for it would have enriched the ways in which the author has understood and interpreted the enactment project from the outside in. In particular, this cross-cultural professional experience sensitises the author to the place-based nature of mathematics teaching and learning.

As such, the discussion in this chapter is framed by the Social Cognitive Theory which is outlined in the next section. The contextual features identified can be categorised generally into personal and environmental factors. The personal factor which will be featured in this chapter will be teachers' content knowledge. As for the environmental factors, this chapter will focus on: trust in the leadership, students as disciples, societal valuing of *excellence*, and twenty-first century competency education. The role of teachers' self-efficacy will also be included.

16.2 Social Cognitive Theory

Albert Bandura's (1986) 'Social Cognitive Theory' regards an individual's behaviour and action as part of a three-component, dynamic, and reciprocal model in which personal factors [P], environmental factors [E], and behaviour [B] interact with and influence one another on an ongoing manner. In this model, B would refer to the Singapore teachers' enactment of the local mathematics curriculum. The individual—which in this chapter refers to any of the Singapore secondary mathematics teachers—is an active agent whose practice represents an enactment of the intended curriculum. The cognitive, affective, and conative aspects of a teacher's functioning would constitute P, examples of which include their mathematics content knowledge and pedagogical content knowledge, the affective dispositions, and the professional motivations. For example, we can see in Chapter 7 how teachers' cultivation of students' non-cognitive traits [B] were guided by both teachers' own beliefs [P] and teacher-perceived students' abilities [P]. These would interact with E, the environmental factors, in bidirectional ways. This may not be explicit in the chapter understandably, given that it was not the focus of the study. Yet, we can imagine, for examples, how the mathematics topic of the day [E] might affect the extent to which particular attitudes are included in the lesson plan [B], how the cultivation of *perseverance* [B] might modify a teacher's perception of a student's ability [P], and how the sustained and successful promotion of *confidence* amongst students in a class [B] would redefine the classroom context [E] for future mathematics teaching. What these mean in our discussion in this chapter is that

> knowledge of the factors, whether planned or fortuitous, that can alter the course of life paths provides guides for how to foster valued futures. At the personal level, it requires cultivating the capabilities for exercising self-directedness. These include the development of competencies, self-beliefs of efficacy to exercise control, and self-regulatory capabilities for influencing one's own motivation and actions. (Bandura, 1989, pp. 7–8)

Thus, this triadic reciprocal causation model highlights the role of teacher agency in enacting the curriculum. A key component of human agency is self-efficacy (Bandura, 1989), where

> self-judgments of operative capabilities function as one set of proximal determinants of how people behave, their thought patterns, and the emotional reactions they experience in taxing situations. (Bandura, 1989, p. 59)

There is then a particularly important function for teachers' self-efficacy as it regulates—and is regulated by—their personal factors, environmental factors, and decisions and actions. As much as the enactment of the curriculum involves decision-making before and during class (as was discussed above), this too is a function of a teacher's judgement of their own capacity to deliver what has been planned.

16.3 Teachers' Content Knowledge

A significant personal factor which affects a teacher's enactment of the curriculum is their own level of content knowledge. Without an excellent knowledge of mathematics, any teacher would not be aware of all the possibilities which are available for the creation, adaptation, and modification of instructional materials, for the solutions to challenging tasks, nor for the ways of simplifying mathematics explanations. More generally, as Ball, Thames, and Phelps (2008) asserted, "teachers must know the subject they teach. Indeed, there may be nothing more foundational to teacher competency" (p. 404). Similarly, the Teacher Education and Development Study in Mathematics [TEDS-M] adopts the view that "knowledge of content to be taught is a crucial factor in influencing the quality of teaching" (Tatto et al., 2008, p. 19).

In the project that this book is based on, there are many instances when teachers' command of the mathematics content knowledge had facilitated their interpretation and transformation of the mathematics curriculum. In Chapter 11, for example, we see in the mixed methods study with more than 600 teachers in Singapore how these teachers selected and modified instructional materials for classroom use. More than 96% of the teacher respondents revealed that they made adaptations and modifications to their reference materials such as textbooks and school-based resources. Indeed,

> a vast majority of Singapore secondary mathematics teachers do not view their duty as merely 'lifting' items from reference materials to give to their students; rather, they see their role as

> necessarily one of mediation between the reference materials and student learning: they are required to value-add by modifying them. (pp. 205–230)

Furthermore, the pedagogical considerations which guide the adaptations and modifications have led to a development of a range of teacher strategies, namely, 'modify', 'new', and 'smoothen'. Underlying the effective use of each of these strategies has to be good-enough knowledge of relevant mathematics. For example, when a teacher participant was observed to have modified a vectors addition item to increase the cognitive demand for their students, this teacher clearly understood the commutative law for vector addition to achieve their intention, rather than simply regarding vector addition as arranging vectors 'end-to-end'.

In addition, as reported in Chapter 12, students across all three ability streams in Singapore are regularly exposed to challenging tasks. Similarly, Chapter 14 described how a Singapore teacher (Teacher 8) employed a range of design principles towards creating instructional materials with the aim of promoting students' connection-making. The author has also often heard comments overseas that Singapore mathematics teachers are good at relating to their students by simplifying their explanations to their ability levels. All these teacher moves would not have been possible if the teachers involved do not know enough relevant mathematics content.

Singapore's participation in the International Comparative Study in Mathematics Teacher Training [ICSMTT] (Burghes, 2011) had shown that Singapore's pre-service secondary mathematics teachers' content knowledge was one of the best in the world, having ranked fourth. At the same time, the National Institute of Education, Singapore's provider of teacher education programmes, is also actively taking steps 'to ensure and/or develop such [content] knowledge in prospective as well as practising teachers' (Tay, Lim, Ho, & Toh, 2017, p. 130).

The significance of teacher content knowledge on teaching effectiveness is visible when we examine the practices of out-of-field mathematics teachers, for example. The shortage of qualified mathematics teachers in secondary schools in several countries such as the USA, Australia, and Ireland (Ní Riordain & Hannigan, 2009) have led to large groups of out-of-field mathematics teachers who generally lack relevant mathematics content knowledge and pedagogical content knowledge. For example, 21–38% of Years 7–10 mathematics classes in Australia are believed to have been taught by out-of-field teachers (Prince & O'Connor, 2018). Since a command of mathematics content knowledge is crucial to teacher quality in planning, facilitating, and assessing mathematics learning, then these out-of-field teachers will find mathematics teaching an extra challenge cognitively, since they need to interact with mathematics concepts and skills. McConney and Price (2009) reported how teachers who believed that they have some control over subjects they were teaching out-of-field also thought that they were better supported and more capable. This is perhaps why two professional development programmes catering to out-of-field teachers in Australia had prioritised acquisition of mathematics content knowledge over other forms of knowledge (Vale, 2010).

16.4 Trust in the Leadership

According to the 2020 Edelman Trust Barometer (Edelman, 2020), Singapore is one of only 7 out of 28 countries in which the government is trusted by a majority of its people. This is not unfounded. In less than a generation, the same government in Singapore had led a small island which has no natural resources through self-government to independence, and transformed it into a technologically advanced island state with one of the highest GDP in the world. For many Singapore teachers, their parents would have survived the early difficult days with the government, and as such, there is a sense of shared camaraderie. This trust in—and respect for—the government extends to the (mathematics) education system as well; Singapore's achievement in TIMSS and PISA since their respective inceptions would have further strengthened this trust in the country's education leadership. In the context of the enactment project, this trust thus constitutes a facilitating environmental factor which ultimately supports teachers' enactment of the curriculum.

In this positive environment of trust and professionalism, teachers, the teacher education provider, and the government can engage in productive dialogues which ultimately benefit the quality of teaching and learning. One area in which the education system would have benefitted from this productive relationship is the positive attitudes and mindsets which teachers, principals, policy makers, and educational researchers have for one another when they implement and evaluate new approaches to teaching (mathematics). This is not to say that teachers will enact in their practice what is 'given' to them without question. Rather, innovative pedagogical approaches which have been adapted for the Singapore context are more often assessed by the professional community with a more open mind. What we see as a result of this tripartite relationship is very often an informed and engaged implementation of new teaching approaches, which in turn facilitates higher probabilities of improving the quality of school mathematics education.

16.5 Students as Disciples

The same teacher teaching the same topic in different classes would not enact the curriculum identically, as the teacher attends to different student needs and demands. In other words, the enacted curriculum is co-constructed with students. Students thus constitute another set of environmental factors affecting curriculum enactment. While it was not the intention of the enactment project to focus on students, they do get mentioned since teachers' decisions and practice are intertwined with students' participation. The experienced and competent teacher featured in Chapter 9, Teacher 27, provided an example of how dialogic mathematics talk was able to take place with the engaged participation of his students. Yet, "instances of meaningful math talk in which students are actively engaged in and are transforming one other's thinking are rare" (pp. 163–181). Although the reasons for this phenomenon were not offered, understandably, the author wonders how the teachers felt when confronted with this phenomenon? Or, would this be expected in the student–teacher relationship in the classroom somewhat, so that the teachers' (lack of) response in turn reinforces the students' 'silence'? Indeed, to what extent has Singapore teachers' enactment of the mathematics curriculum taken into consideration (consciously or otherwise) the students' behaviour and expectations in class? How representative is the case in Chapter 4, when "students had little room to think independently. Instead, students mainly followed the teacher's 'planned frame' to learn what was prior determined by the teacher" (pp. 63–77).

Perhaps this reflects the cultural reality in Singapore, in which the student–teacher relationship continues to be perceived by the society as being essentially disciple-master in nature. In Confucius Heritage Cultures such as Singapore, the teacher is often regarded as the holder of knowledge, especially for subjects such as mathematics which is often perceived to have no allowance for 'grey areas'. In addition, Singapore scores highly along the 'power distance' dimension in Geert Hofstede's 6D Model (Hofstede Insights, 2020), which indicates a high "extent to which the less powerful members of institutions and organisations within a country expect and accept that power is distributed unequally" (Hofstede Insights, 2020). In the education setting, this would mean that we can expect to see students in Singapore schools respecting their teachers' authority, which in the local culture also implies not arguing back, not questioning, and not making disrespectful comments (see also Hogan, 2014).

16.6 21st Century Competency Education

One of the five inter-related components of the Singapore Mathematics Curriculum Framework (Singapore MoE, 2019) is named 'attitudes'. "Attitudes include one's belief and appreciation of the value of mathematics, one's confidence and motivation in using mathematics, and one's interests and perseverance to solve problems using mathematics" (Singapore MoE, 2019, p. 11). Thus, despite the label for the component being affective, it actually refers to the range of affective and conative—that is, non-cognitive—features of learning that are to be fostered amongst students. This is significant: even though 'productive dispositions' is one of the five identified intertwined strands of mathematical proficiencies in the influential report prepared by Jeremy Kilpatrick and his colleagues (US National Research Council, 2001), only a few countries appear to have incorporated it in their latest mathematics curriculum reform. The Australian Curriculum (ACARA, 2016), for example, features a set of four proficiency strands that are parallel to the four cognitive strands in the American report, but leaving out the 'productive dispositions' strand.

This expectation for teachers to cultivate 'desirable' affect and enabling conation amongst their mathematics students would have supported teachers' intentions and efforts to incorporate it in their instructional practice. In Chapter 7, the research with experienced and competent mathematics teachers revealed that they teach in ways which cultivate confidence, perseverance, appreciation, and interest. The fostering of student beliefs, however, was not observed. According to the research findings, the teacher's actions were not only supported by environmental factors, but also guided by the teachers' own beliefs as well as their perception of student ability, both personal factors.

It should be noted that Singapore's approach goes beyond making mathematics learning fun for students. The findings reveal that the non-cognitive traits that are promoted both deepen students' appreciation of the nature of mathematics (e.g. real-world applications), and develop students' personal competencies (e.g. perseverance). Similarly, in an earlier study, Toh, Cheng, Ho, Jiang, and Lim (2017) demonstrated how comics were used in mathematics lessons to promote interest AND to facilitate twenty-first-century skills. It seems that non-cognitive traits are not only developed to facilitate mathematics learning, but mathematics learning also has the responsibility of developing relevant traits amongst students as part of the country's holistic education process. As noted in a OECD (2019) report,

> the curriculum in Singapore … highlights that competencies are to be learnt with core values – care, integrity, respect, resilience, responsibility and harmony – at the centre of their learning framework. Singapore's Ministry of Education believes that 21st-century competencies are not learned in a vacuum, but in specific contexts …. These values are expected to be embedded into every subject. (p. 7)

16.7 Societal Valuing of *Excellence*

The Singapore culture values *excellence,* the spirit of which is best captured by a Hokkien dialect lexicon often used locally, 'zho sui sui' (literally: get it done beautifully and perfect). The term is found in the value statements and slogans of many institutions and companies. The Singapore healthcare system, one of the best in the world, has been described as delivering 'affordable excellence' (Haseltine, 2013). 'Excellence' is one of the six core values of the iconic Singapore Airlines. But it is not 'all policy and no action'. *Excellence* is a trait that has been built into the Singapore psyche, where 'okay is not good enough'. One will hardly ever hear anyone saying, 's/he will be alright!'.

The same trait is valued in the Singapore education system as well, in part due to the cultural influence of Confucianism in which academic excellence is revered. As Kaur (2004) asserted,

> [in relation to striving] towards excellence in the mathematics classroom [,] …. As the saying goes – good, better, best; never let it rest till good is better and better is BEST!, generally mathematics teachers in Singapore schools are poised to do the best they can for their pupils. (n.p.)

Thus, this creates an environment for Singapore teachers to expect nothing but the best of their teaching and of their students. When the teacher participants of the project this book is based on creatively designed or selected challenging tasks for their students (see Chapter 12), they were confident that their practice would be supported by the students' willingness to try and engage.

This cultural trait might explain why the mathematics pedagogical practice in Singapore continues to display a dominant performative orientation (see Chapter 2). Teaching for student mastery using worked examples and class practice was found to be not just a teacher move exercised by the experienced and competent teachers in Singapore, but one which defines the instructional practice of the 600+ teachers surveyed in the study noted in Chapter 5. This focus on mastery learning is indeed also an expression of the valuing of *excellence*.

The strive towards excellence does not need to come at the expense of developing other skills such as understanding. In the project this book is based on, while the teacher participants were engaged with teaching practices that promote mastery (see Chapter 5) and that introduce students to challenging tasks (see Chapter 12), they were also teaching in ways which promote conceptual understanding (see Chapter 4), mathematical reasoning (see Chapter 13), and connections (see Chapter 14).

16.8 Teachers' Self-Efficacy

Perhaps because the teacher participants in the Enactment Project were all considered experienced and competent, their instructional practice as reported in the various studies reflect high teacher self-efficacy. That is to say, to be able to plan and execute all that had been summarised in this chapter so far reflects teacher beliefs in their capabilities to exercise control over their respective enactment of the mathematics curriculum. Certainly, Singapore teachers' strong mathematics content knowledge, value alignment with the society, students' identity as disciple, and Singapore's achievement in TIMSS and PISA, are contextual features which would contribute to the teachers' self-efficacy beliefs to some extent. Since a teacher's identity as a professional is heavily related to their self-efficacy (Canrinus, Helms-Lorenz, Beijaard, Buitink, & Hofman, 2012), it is important that this particular form of teacher beliefs is nurtured and developed.

The high esteem with which the teaching profession is held in the Singapore culture (Dolton, Marcenaro, de Vries, & She, 2018; Tan & Liu, 2017) would also contribute to high teacher efficacy. In Singapore, teachers are respected for the crucial roles they play in educating the young generations, and this trust in their professionalism is expressed in the ways they are entrusted by the Ministry of Education to design and deliver lessons that cater to the capacity and needs of their own students. Schools and teachers are also encouraged and supported to take ownership over their professional learning, as reflected in such policies as 'Thinking Schools, Learning Nation' in the late 1990s and in the adoption of the Professional Learning Communities [PLC] (Hord, 1997) model in the early 2000s as the main means across all schools for teacher professional learning. For a teacher, the combination of such professional support, their trust in the leadership (see Sect. 16.4), and the respect they experience in school and in the society, all come together to strengthen their belief in their ability to teach and to enable student learning, that is, teacher efficacy.

Empirically derived knowledge regarding Singapore mathematics teachers' self-efficacy does not appear to exist, however. A study did find that Singapore secondary school teachers' self-efficacy was dependent on the educational streams of their respective students (Chong, Klassen, Huan, Wong, & Kates, 2010). Specifically, teachers in high-track schools displayed greater efficacy beliefs compared to their peers in other schools where there was a greater variety of student achievement types. This study did not break down its 222 teacher participants by the subjects they were teaching though, so it is not clear the extent to which a similar scenario might apply to mathematics teachers.

16.9 Conclusions

There is no doubt that teachers and educators around the world are interested to find out not just how Singapore organises her school mathematics curriculum, but also how Singapore teachers enact it in their instructional practices. The Enactment Project this book is based on has collected relevant data from a wide range of perspectives to shed light on how experienced and competent teachers have done this in their instructional practices, and also how representative these might be in the Singapore secondary teaching profession. This chapter complements these findings by suggesting how the teacher moves are to be understood in context, as unique teacher personal factors and environmental influences shape these moves while also being shaped by them. It identifies teachers' content knowledge, trust in the leadership, students as disciples, societal valuing of *excellence*, and twenty-first-century competency education as examples of such factors. The role of teacher self-efficacy is also examined.

Understanding these contextual factors allows us to better assess how best to replicate particular instructional practices elsewhere, amongst other possibilities. For example, foreign visitors to the Singapore mathematics classroom would notice pedagogical approaches and professional learning programmes that might be familiar in their home education systems, such as student-centred teaching (see Chapter 3 for details) and PLCs, and they may be left wondering what the secret ingredient to Singapore's success in school mathematics education is. It is hoped that the discussion in this chapter would remind these visitors that what works in practice reflects the harmonious interaction between teacher professionalism on the one hand, and policy and other contextual factors on the other hand, underlied by what individuals, institutions, and the society value now and over time.

Perhaps this is why, despite the best efforts of talented baristas elsewhere with the best espresso machines, there is something about latte in Melbourne, Australia which makes it one of the best—if not the best—in the world.

References

American Institutes for Research. (2005). *What the United States can learn from Singapore's world-class mathematics system (and what Singapore can learn from the United States): An exploratory study*. Retrieved from Washington, DC: https://files.eric.ed.gov/fulltext/ED491632.pdf.

Australian Curriculum Assessment and Reporting Authority (ACARA). (2016). *The Australian curriculum: F-10 Mathematics*. Retrieved from Sydney, Australia: https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/rationale/.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59,* 389–407.

Bandura, A. (1986). *Social foundations of thought and action: A social cognitive theory*. Englewood Cliffs, NJ: Prentice-Hall.

Bandura, A. (1989). Social cognitive theory. In R. Vasta (Ed.), *Annals of child development* (Vol. 6, pp. 1–60)., *Six theories of child development* Greenwich, CT: JAI Press.

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). NY: Routledge.

Burghes, D. (2011). *International comparative study in mathematics teacher training*. Reading, UK: CfBT Education Trust.

Canrinus, E. T., Helms-Lorenz, M., Beijaard, D., Buitink, J., & Hofman, A. (2012). Self-efficacy, job satisfaction, motivation and commitment: Exploring the relationships between indicators of teachers' professional identity. *European Journal of Psychology of Education, 27,* 115–132.

Chong, W. H., Klassen, R. M., Huan, V. S., Wong, I., & Kates, A. D. (2010). The relationships among school types, teacher efficacy beliefs, and academic climate: Perspective from Asian middle schools. *The Journal of Educational Research, 103*(3), 183–190.

Dolton, P., Marcenaro, O., de Vries, R., & She, P. O. (2018). *Global teacher status index 2018*. London: Varkey Foundation.

Edelman. (2020). *Edelman Trust Barometer 2020: Global report*. Retrieved from Chicago, IL: https://www.edelman.com/trustbarometer.

Haseltine, W. A. (2013). *Affordable excellence: The Singapore healthcare story*. Washington, DC: Brookings Institution Press.

Hofstede Insights. (2020). *Hofstede insights: Country comparison*. Retrieved from https://www.hofstede-insights.com/country-comparison/singapore/.

Hogan, D. (2014). *Why is Singapore's school system so successful, and is it a model for the West?* Retrieved from https://theconversation.com/why-is-singapores-school-system-so-successful-and-is-it-a-model-for-the-west-22917.

Hord, S. M. (1997). Professional learning communities: What are they and why are they important? *Issues about Change, 6*(1), 5–6.

Kaur, B. (2004). *Teaching of mathematics in Singapore schools*. Paper presented at the 10th International Congress on Mathematical Education, Copenhagen, Denmark.

McConney, A., & Price, A. (2009). *An assessment of the phenomenon of 'Teaching-out-of-field' in WA schools*. Perth, WA: Western Australian College of Teaching. http://researchrepository.murdoch.edu.au/10229/1/Assessment_of_Teaching_Out_of_Field_Final_Report_for_publication.pdf.

Ni Riordain, M., & Hannigan, A. (2009). *Out-of-field teaching in post-primary mathematics education: An analysis of the Irish context*. Sheffield, UK: National Centre for Excellence in Mathematics and Science Teaching and Learning.

OECD. (2019). *OECD future of education and skills 2030 conceptual learning framework: Attitudes and values for 2030*. Retrieved from Paris, France: http://www.oecd.org/education/2030-project/teaching-and-learning/learning/attitudes-and-values/Attitudes_and_Values_for_2030_concept_note.pdf.

Prince, G., & O'Connor, M. (2018). *Crunching the numbers on out-of-field teaching* . Retrieved from Victoria, VIC, Australia: https://amsi.org.au/media/AMSI-Occasional-Paper-Out-of-Field-Maths-Teaching.pdf.

Remillard, J. T. (2018). Mapping the relationship between written and enacted curriculum: Examining teachers' decision making. In G. Kaiser, H. Forgasz, M. Graven, A. Kuzniak, E. Simmt, & B. Xu (Eds.), *Invited lectures from the 13th International Congress on Mathematical Education, ICME-13 Monographs* (pp. 483–500). Cham, Switzerland: Springer.

Schleicher, A. (2018). Educating learners for their future, not our past. *ECNU Review of Education, 1*(1), 58–75.

Singapore Ministry of Education [MoE]. (2019). *Mathematics syllabuses: Secondary one to four*. Singapore.

Tan, O. S., & Liu, W. (2017). Championing the teacher factor. In O. S. Tan, W. C. Liu, & E. L. Low (Eds.), *Teacher education in the 21st Century*. Singapore: Springer.

Tatto, M., Ingvarson, L., Schwille, J., Peck, R., Senk, S., & Rowley, G. (2008). *Teacher education and development study in Mathematics (TEDS-M): Policy, practice, and readiness to teach primary and secondary mathematics*. *Conceptual framework*. East Lansing, MI: Teacher Education and Development International Study Center, College of Education, Michigan University.

Tay, E. G., Lim, S. K., Ho, W. K., & Toh, T. L. (2017). Preparing mathematics teachers in Singapore: The issue of mathematics content knowledge. In O. S. Tan, W. C. Liu, & E. L. Low (Eds.), *Teacher education in the 21st Century*. Singapore: Springer.

Toh, T. L., Cheng, L. P., Ho, S. Y., Jiang, H., & Lim, K. M. (2017). Use of comics to enhance students' learning for the development of the twenty-first century competencies in the mathematics classroom. *Asia Pacific Journal of Education, 37,* 437–452.

US National Research Council. (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: The National Academies Press.

Vale, C. (2010). Supporting 'out-of-field' teachers of secondary mathematics. *Australian Mathematics Teacher, 66*(1), 17–24.

Wee Tiong Seah is an Associate Professor at the Melbourne Graduate School of Education, The University of Melbourne, Australia. He is also an appointed Research Fellow of Zhejiang Basic Education Research Centre, China. He taught Mathematics and English Language in secondary schools in Singapore before embarking on an academic career in Monash University, followed by The University of Melbourne, both in Australia. In Australia, Wee Tiong teaches pre-, in-service as well as postgraduate courses in mathematics education and in research methodology, and supervises research students pursuing Masters and PhD degrees. His publication and research interests include the development of enabling valuing amongst students and teachers, self-efficacy, and the role of artificial intelligence programmes in complementing mathematics teachers' professional practice.

Chapter 17
Portraits of the Singapore Secondary School Mathematics Enactment: An Insider's Perspective

Yew Hoong Leong and Berinderjeet Kaur

Abstract This chapter summarises the contributions of the preceding chapters of this volume and other publications related to the project. We bring together the various facets given by these reports by painting two portraits of the enactment of Singapore secondary mathematics curriculum. The first portrait draws from the well-known pentagonal model of the Singapore mathematics curriculum, and it seeks to address the question, "Does this model accurately depict what really goes on in the Singapore mathematics classrooms?" The second portrait borrows from the journey metaphor in describing the goal-driven nature and the deliberate planning involved behind the "examination-oriented" "look" of classroom instruction. This addresses another question: "How do we explain the seeming paradox of high performance of Singapore students in international tests while maintaining what looks like traditional modes of instruction?" We conclude the chapter by reflecting on these two portraits in relation to the wider cultural context in Singapore.

Keywords Enactment · Instructional materials · School mathematics curriculum framework · Syllabuses · Singapore

17.1 Bringing the Various Facets Together

In the earlier chapters of this volume, the various authors—who are the team members of the project as described in Chapter 2—report findings on different aspects of the overall study. In this chapter, we attempt the challenging task of pulling these rich and diverse perspectives together. Using the metaphor of painting a portrait (in this case, the portrait of the enactment of Singapore secondary mathematics teachers), we seek to draw together these various reports as facets that contribute to a coherent

Y. H. Leong (✉) · B. Kaur
National Institute of Education, Nanyang Technological University, Nanyang Avenue, Singapore
e-mail: yewhoong.leong@nie.edu.sg

B. Kaur
e-mail: berinderjeet.kaur@nie.edu.sg

B. Kaur et al. (eds.), *Mathematics Instructional Practices in Singapore Secondary Schools*, Mathematics Education – An Asian Perspective,
https://doi.org/10.1007/978-981-15-8956-0_17

whole. We will also draw upon other publications arising from this same project that are not included in this volume.

At the same time, in the course of the project duration, we were frequently asked these two questions (both by educators within Singapore and international professionals looking into Singapore mathematics education):

- Is the (now internationally well-known) pentagon model, which is the framework of the Singapore school mathematics curriculum (see Fig. 1.2 in Chapter 1), really what goes on in the Singapore mathematics classrooms?
- How do you explain the "paradox" of Singapore's high performance in PISA with the still very traditional modes of teaching in Singapore mathematics classrooms?

Thus, in developing the enactment portrait, we also attempt to address these questions. In fact, we present two portraits by structuring the rest of this chapter according to each of these questions—a portrait that is centred around discussions of the pentagon; and a portrait that unwinds the "paradox".

17.2 Revisit of the Pentagon Model

Leong (2008) did a cursory comparison of the pentagon as intended and the pentagon as assessed. For our purposes, we find this pictorial comparison useful and it is included in Fig. 17.1.

The pentagon on the right side of Fig. 17.1 shows a rough picture of the components that are *directly* assessed in typical high-stakes examination papers in Singapore. Attitudes and Metacognition are not directly tested while a high percentage of the items in the papers tests Skills, less on Concepts, and even less on Processes. It is arguable—depending on how one defines "problem solving" and whether students consider "problems" set by teachers as indeed problems or more like routine exercises—whether or how much mathematical problem solving items are in these papers.

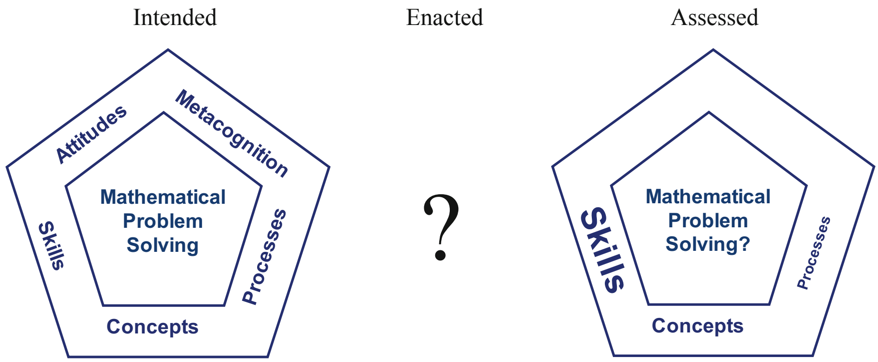


Fig. 17.1 Pentagon as intended and pentagon as assessed

Interestingly, this picture of distribution appears to match some students' assignment of value to what is important in mathematics lessons, as shown in the findings in Chapter 10. Leong (2008) asserted that, given the difference in emphases among the components in the pentagon between the official curriculum and the high-stakes examinations, it is not surprising that teachers feel "sandwiched" between these poles: do they teach more in accordance to the picture on the left—presumably, with roughly even distribution of emphasis on each of the components? Or, do they apportion their emphasis roughly according to that shown in the picture on the right? The Question Mark in the middle of Fig. 17.1 denotes this enactment dilemma of teachers. It is a question that this project sought to address too—how does enactment look like when juxtaposed between these two "models"?

The actual enactment picture is far more complex than what can be presented in a pentagon. Nevertheless, we begin to paint the picture by considering the respective components of the pentagon. Since Skills loom large (literally, in the diagram on the right), we shall begin the discussion on Skills.

It is clear that Singapore secondary mathematics teachers put a high priority on teaching Skills. They want students to be able to fluently carry out some standard methods as stipulated in the secondary mathematics syllabus—such as methods of solving quadratic equations (Chapters 13 and 14) and the use of the Pythagoras Theorem in solving right-angled triangles (Chapter 4). It is worth noting that Skills have been unjustly disparaged in literature that tends to equate Skills-talk with "merely drill-and-practise" mode of instruction. We think the importance of Skills has been given back its rightful position in mathematics teaching and learning when the influential report—commissioned by the American National Science Foundation—by Kilpatrick, Swafford, and Findell (2001) included "Procedural Fluency" as one of the core strands of their Mathematical Competence framework.

But the evidence in this project points towards this: Singapore secondary mathematics teachers do not think (nor enact the teaching) of Skills apart from the other relevant components of the pentagon. For example, in Chapter 4 of this volume, the description of the lesson on Pythagoras Theorem showed that the teacher devoted time not only in the application of the theorem, but also in its development. There was an emphasis in relating the theorem to other relevant mathematical concepts (such as its converse). In other words, the lesson trajectory is one where Skills and Concepts were developed together. Furthermore, neither is this a case of a one-off occurrence. Chapter 4 (on relating the formula of finding the distance of the line segment joining two points on the Cartesian Plane to Pythagoras Theorem) and Chapter 14 (on relating the quadratic formula to the method of Completing the Square) provide more illustrations of this Skills-Concepts co-development. Moreover, this intention that students learn Concepts alongside the development of Skills is not limited to the 30 experienced and competent teachers studied in the first phase of the project. In the second phase, where we collated the qualitative responses of 156 teachers from a broad spectrum of Singapore secondary schools, Leong et al. (2019b) reported, "[I]t is clear from the types of comments listed … that a substantial number of teachers were concerned not just with formula application but also with how it connected with related concepts in the topic" (p. 37).

In fact, this tight connection between Skills and Concepts is so ubiquitous in the teachers' lesson enactment and instructional materials that we posit that Skills-Concepts form a central axis in the organisation of their lessons—as is supported by all the case studies reported in this volume (Chapters 4, 8, 9, 13, and 14); and they build other components on this organisational axis.

One of the pentagon components that were intentionally built into this axis is Processes. In this regard, the case of Teacher 13 as described in Chapter 13 is particularly illuminating. The authors of the report depicted the teacher's move as one of using mathematical reasoning—one of the Processes in the pentagon—as a kind of "glue" to link the various solution methods and ideas together. Reinterpreted into our pentagon language, Teacher 13 wanted students to learn the various methods of solving different types of quadratic equations (Skills); he also wanted the students to learn the underlying conceptual underpinnings (and limitations) of each of the methods (Concepts). In addition, on top of this Skills-Concepts developmental axis, he sequenced his examples in such a way that mathematical reasoning was used as the main Process to link them together. While Teacher 13's case provided an explicit foregrounding of how Processes—such as reasoning—shaped the enactment of his lessons, it is by no means an isolated case.

In Chapter 8, Teacher 5 was also clearly concerned that the students experience authentic mathematical Processes in learning the geometrical theorems. The students were given the opportunity, through inductive processes afforded by Dynamic Geometry software, to conjecture and test findings; towards the end, proof of the theorems was done so that students see the deductive process of explaining what was discovered earlier. Teacher 27 in Chapter 9 also included mathematical Processes such as justification in his math talk. His interest was not only that students were able to carry out the procedures; he wanted them to reason out the steps as well. In Chapter 14, Teacher 8 wanted students to "link everything together"; in other words, her goal was not limited to students' ability to carry out each of the methods of solving quadratic equations; she also ostensibly aimed at students' ability to *connect* (a mathematical Process) the various strands of Skills and Concepts together.

There is also evidence to suggest that this desire to teach mathematics in a way that mathematical Processes come to the fore was not restricted to the cases mentioned in the previous paragraphs. Table 11.6 of Chapter 11 indicated that out of 30 experienced and competent teachers we studied under Phase One of the project, there were evidence of "making connections" in the instructional materials of 21 of these teachers; the number is 19 for "support reasoning". In Phase Two, where the survey was completed by more than 600 teachers, Table 9.2 of Chapter 9 shows that a significant proportion of the teachers indicated that they at least "frequently" require their students to use these Processes-related talks in their mathematics classrooms: "Explain" (70%), "Explore" (45%), "Analyse" (72%), "Evaluate" (55%), "Argue" (63%), "Justify" (52%).

The portrait with respect to the pentagon that is emerging up to this point of our discussion is this: the most explicit and visible parts of teachers' planning and enactment are the Skills and Concepts. Indeed, they are so closely thought of and enacted that we posit a Skills-Concepts organisational axis in the way teachers develop their lessons. Less visible, less ubiquitous, but wherever there are opportunities, teachers also bring on board relevant mathematical Processes in their instructional materials and classroom talk.

If Skills, Concepts, and Processes are thought of as the disciplinary and cognitive aspects of the mathematics teaching enterprise, then Attitudes and Metacognition can be thought of as more generic and affective—hence, perhaps seen by secondary mathematics teachers as ‘further away’ from their core business as *mathematics* teachers. Admittedly, these latter aspects are not as visible or conspicuous as the former.

Nevertheless, there is substantial evidence through our project findings to indicate that the teachers were highly cognisant of student Attitudes when they thought of how they structured their lessons. When Attitudes are mentioned, often the first thing that comes to mind is students’ interest in the subject—that is, “How can we make mathematics more ‘cool’ for the students?” Indeed, some of the teachers attended to this aspect of Attitudes. Where relevant, teachers included real-life examples in class partly to pique students’ interest in the subject. This was mentioned in Chapter 7. Table 11.6 of Chapter 11 indicated that 20 of the 30 experienced and competent teachers included “Context” (that is, related to everyday experiences) items in their instructional materials.

But the evidence we obtained in the project pointed to students’ *confidence* (not interest—at least, not directly) that was the teachers’ main attitudinal focus. In particular, teachers were careful to adjust their materials and examples so as to build up students’ confidence in doing mathematics. This ostensible goal of confidence-building was highlighted in Chapter 7. This is colluded by the findings related to the instructional materials that teachers designed. Table 11.6 of Chapter 11 shows that 29 out of 30 teachers practise “deliberate sequencing of examples” in their instructional materials. The follow-up study of this observation in Leong et al. (2019b) indicated that one major consideration in this “deliberate sequencing” was the managing of cognitive load of students. Above other design considerations, the 156 teacher-respondents in the survey generally placed “start off with easier items to build confidence” as top priority when thinking about how to sequence practise examples. This sensitivity towards students’ affect was also evident when teachers thought of exposing them to challenging items (see Chapter 12). The teachers described strategies they adopted to alleviate students’ sense of intimidation when confronted with challenging items.

Here, we pause for a while (from the work of portrait-painting of the pentagon) to reflect on the still-common “teacher-centred versus student-centred” talk, which was also briefly referred to in Chapter 3. Up to this point, we wonder how readers would place the Singapore secondary teachers’ enactment across this “dichotomy”. [We think this is a false dichotomy]. If we grant that the disciplinary aspects of the enterprise (that is, Skills, Concepts, and Processes) reveal how “teacher-centred” Singapore teachers are—in that, the agenda in these aspects are largely teacher-determined and teacher-directed, what can be said of this premium placed on building students’ confidence—is it “teacher-centred” or “student-centred”? Should we then say that, cognition-wise, Singapore secondary teachers are teacher-centred; and affect-wise, they are student-centred? These questions reveal how over-simplistic this dichotomous talk is. In reality, we think teachers see themselves as intellectual authority with respect to the subject, and this renders them leaders in the class when it comes to learning disciplinary norms; but this does not translate into a careless ignoring of students' learning needs—especially in the area of students' affect; the teachers consciously factor in the building of their confidence in the design of their instructional materials and in the enactment of classroom instruction.

Finally, we come to the last of the five components—Metacognition. If attitudinal matters are less visible (than cognitive ones), then metacognitive moves by teachers are even more hidden from direct view. Nevertheless, Chapter 6 of this volume highlights specific metacognitive strategies that were used by some of the experienced and competent teachers. In particular, a common strategy used was to encourage students to compare different solution methods as a way to learn and reflect upon the affordances and constraints of each method. This strategy was also evident in Teacher 13 (Chapter 13) and Teacher 8 (Chapter 14), although it was not clear if these teachers had the goal of teaching metacognition explicitly in mind or if they saw multiple solution strategies as part of the norms of doing authentic mathematics. Table 6.3 of Chapter 6 shows that more than a majority of the 677 teacher-respondents in the survey indicated that they at least "frequently" provided opportunities for their students to learn different solution methods. These teachers also indicated that they regularly help their students to "reflect" on their learning and methods. Suffice to say, there is much more scope in the integration of Metacognition in the teaching of mathematics beyond specific metacognitive strategies that are reported in this project. Many areas in this domain (such as, the relation between "generic" metacognition and domain-specific metacognition, teachers' metacognitive practices in their own learning of mathematics, the extent of teachers' modelling of authentic metacognitive practices in the classrooms) are as yet under-explored by teachers. At the time of writing, the Ministry of Education of Singapore is commissioning a sizeable research project on metacognition. For now, we think Metacognition is the least understood by most Singapore mathematics teachers among all the components in the pentagon. Where teachers carry out metacognitive practices in their classroom work, they were more *implicit* (and hidden under other goals of instruction) rather than at the foreground of their agenda.

Based on the summary of the various components of the pentagon, we give our portrait of the pentagon as enacted (in place of the Question Mark in Fig. 16.1) in Fig. 17.2.

The different font sizes of each of the components reflect roughly the weight we think teachers place in fulfilling them as goals of instruction. The bubbling up and the conjoining of them with a bold line segment show the tight Skills-Concept axis that is the central organisation frame in teachers' planning and enactment. Where relevant, Processes are brought into the development axis of Skills and Concepts (as shown by the arrow). Likewise, students' Attitudes are also considered when designing tasks. The different thickness of these arrows are meant to show the different levels of visibility—while Processes can appear at the foreground of teachers' work, in that they would explicate it as a goal of instruction (such as, reasoning, making

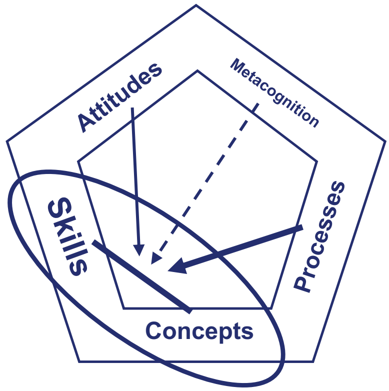


Fig. 17.2 Pentagon as enacted

connections), dealing with students' Attitudes are present but at the background of the teachers' considerations, in that they mentioned its importance usually only when asked. Metacognition comes into play implicitly, often not even intentionally by the teachers, and hence represented by the perforated arrow.

The reader will notice that mathematical problem solving, which is supposed to be at the heart of the pentagon, is not included in Fig. 17.2. Why is it missing? The answer depends largely on how "problems" are defined. According to the Singapore Ministry of Education (2020), "Problems … include straightforward and routine tasks … as well as complex and non-routine tasks … . General problem solving strategies e.g. Polya's 4 steps to problem solving and the use of heuristics, are important in helping one tackle non-routine tasks systematically and effectively" (p. 9). This quotation seems to keep the definition "open": while the first sentence presents "problems" to include all kinds of tasks, the second sentence, by the very reference to Polya—and hence his commitment to problem solving (e.g. Polya, 1945) as an enterprise that presents actual problems to students—would necessarily exclude routine exercises. We can hence interpret this paragraph to mean that, while "problems" can be understood in its broadest sense as any mathematical tasks posed to students, there should be an emphasis on "problems" that are truly problematic to students—so that they will learn problem solving strategies. For our current purpose, we take "problems" to refer to the latter category. In this volume, only Chapter 12 provides some indication of the state of mathematical problem solving conducted in secondary mathematics classrooms. If we take "challenging items" to be a proxy for "problems", then it appears that the picture is quite encouraging—a majority of the teacher-respondents indicated either "Sometimes" or "Frequently" when asked about the use of challenging items in their classes. The study does not go into the actual problem solving strategies—or the manner in which they were taught—to the students, and so we should interpret the frequency of use of challenging items with reservation. In fact, in other local projects that were specifically about helping teachers enact problem solving lessons (e.g. Ho et al., 2019, Leong et al., 2011, Toh et al., 2019), it was found that teachers require continual professional development to sustain problem solving as a regular activity in their classrooms. Nevertheless, we posit that in Singapore secondary mathematics classrooms, problem solving may not be as "elusive" as frequently claimed (e.g. Stacey, 2005).

17.3 The Singapore "Paradox"

Some authors write about an East Asian learner paradox (e.g. Biggs, 1996, Leung, 2001). Mok (2006) stated the paradox as "the apparent contradiction between the teaching methods and environment in East Asian schools (i.e. large classes, whole-class teaching, examination-driven teaching, focus on content rather than process, emphasis on memorisation, etc.) and the fact that East Asian students have regularly performed better than their Western counterparts in comparative studies" (pp. 131–132). We assume that, since Singapore is located in East Asia and fits roughly into the portrayal of traditional methods of teaching but with high mathematics performance, the paradox also applies to us.

First, we state the parts of Mok's caricature that we think do not apply to the Singapore context. We reject the description of "focus on content rather than process". As elaborated in the previous section and depicted in Fig. 17.2, Processes such as reasoning and making connections are explicitly interwoven into the Skills-Concepts development axis of the teachers' lessons. And this is not restricted to the experienced and competent teachers; teachers in our survey in Phase Two of the project professed their commitment to teaching a number of relevant Processes in their mathematics lessons. Moreover, Mok's use of "rather than" to juxtapose "content" and "process" presupposed that teachers have to choose one or the other—which is a false dichotomy. In reality, as we think is the case in most Singapore secondary mathematics teachers as reported in this volume, they intend to teach *both* content and processes. Chapter 8 and Chapter 14 provide compelling cases of how attending to both can be workable.

We also find "emphasis on memorisation" puzzling. What does this actually mean concretely? We take the case depicted in Chapter 8 as an example for the discussion here: Suppose Teacher 5, after she has developed the conceptual links among the various circle theorems, ask the students, "Would you like to learn an easy way to remember these theorems?" After which, she proceeded for 10 min to teach them quick ways to remember (or memorise) them. In this context, would we consider Teacher 5 to "emphasise on memorisation"? And even if we do consider it so, what is paradoxical about it—when the teacher follows up the content and process development with an effort to consolidate the learning by helping students to commit it to memory? Isn't this sound pedagogy? Our point here is: based on the evidence of our study, Singapore secondary mathematics teachers focus on both Skills *and* Concepts (see Fig. 17.2); there is no evidence to suggest that they focus on memorisation of procedures in isolation from or excessively over conceptual understanding. Where "emphasis" is rightly done, there is no paradox.

Having clarified the aspects of the caricature that does not apply to the Singapore context, we think the remaining descriptions are indeed shared by us. But as "large class size" and "whole class teaching" are more structural givens than conscious pedagogical decisions, we will restrict our discussion on "examination-driven teaching".

Indeed, all the 30 experienced and competent teachers who participated in the first phase of the project were "examination-driven", in the sense that they set the target of their students being able to do well for subsequent (especially, high-stakes) examinations as one of their major instructional goals of teaching mathematics; also, because of this goal, the instructional contents and student tasks they planned for their lessons take their reference from typical examination items. Insofar as textbooks items are trusted as indications of standard examination items, there is also this associated adherence to textbook items as official proxies of what is expected in examinations. During the teacher interviews in the project, it is not uncommon for teachers to justify their inclusions of certain tasks in the lessons due to their being "included in examinations" or "included in textbooks".

Before we proceed further into the heart of the paradox, we would like to make some related comments about being "examination-driven". Often, this term has been used to lambast teachers for being narrowly focused in their work of teaching. But we should check such excesses against this reality: In a system where students' examination results largely shape the career choices ahead of them, is it not socially responsible for teachers to take it as their primary role to help students attain the best they can in examinations? Conversely, would we think teachers are socially responsible to their charges if they ignore the importance of examinations for their students' social-economic future just so that they can pursue their own educational ideas about teaching mathematics?

Also, from an education-system point of view, is it really so bad that teachers have a clear and concrete goal to prepare their students for? Perhaps this is clearer when contrasted against an alternative option: there are no high-stakes examinations; teachers can do what they think is professionally expedient as mathematics teachers; the individuality of the teacher and his/her conscience become the guide for what "drives" the mathematics instruction in class. To be sure, this is the utopian vision of many teachers—that they be 'left alone' to do what they like; but seen as a system, there remains no concrete goal to bring the many teachers in the system together with a common vision and accountability of what is to be taught and to what degree of rigour is to be expected. At its worst, teachers lose a sense of purpose and direction to "drive" their teaching and students are worse-off for it. We like to imagine that every teacher, left to himself/herself, knows what is best to teach for the students over the long term. This assumption has yet to be proven *at scale*. If the alternative to being "examination-driven" is being non-driven (a prospect we think very likely although few talk about it), then the choice is clear.

In addition, one who is "examination-driven" need not be *solely* driven by examination in his/her instructional work. Examination-orientedness may only be one of the goals of teaching. As seen in many of the case studies in this volume (Chapters 4, 8, 9, 13, and 14), the teachers are capable of keeping a keen eye on preparing their students to be examination-ready while concurrently pursuing other worthy goals of teaching mathematics.

Another point about being "examination-driven" is the actual assessment content of these examinations. If the examinations that we have in mind comprised merely items of low cognitive demand, then being examination-driven in such a context will indeed result in a great disservice to the students. It will indeed descend into a training of mechanistic "drill and practise" automatons—a portrayal common in the literature that presents being examination-driven as "bad". But what if the "examination" to be "driven" towards typically consists of a significant number of items that are considered of high cognitive demand (see Chapter 12)? This seems to be the case in the Singapore context as depicted by the right pentagon in Fig. 17.1. For a teacher to be "examination-driven" in this context, he/she will have to regularly include items that are considered challenging to students so that what the students do in class approximates the kind of items they will encounter in subsequent examinations. From purely a content perspective, isn't this kind of examination-drivenness "good" for the students—do they get to engage with items of high cognitive demand regularly? In fact, we think the content of Singapore mathematics examinations partially explain the paradox: If Singapore secondary students, regardless of ability bands (see Chapter 12), regularly engage in challenging items in preparations for examinations, is it surprising that they will perform well in similar "examinations", such as the international tests presided by PISA?

17.4 The Heart of the Paradox

But, one may argue that simply giving challenging items to students does not necessarily result in good performance in these items. In other words, to unwind the paradox, we still cannot sidestep the heart of the issue: What actually takes place in these Singapore examination-driven classrooms that prepare the students so well for examinations such as the ones conducted by TIMSS and PISA?

The "examination-driven teaching" is a surface façade—and hence what usually catches the eye of a cursory observer. As a number of chapters in this volume has described, when we plunge beneath the surface, we find embellishments that may hold the keys to a reconsideration of the just-drilling-for-examinations first impressions.

In Chapter 3, the authors reported that Singapore secondary mathematics teachers commonly use or subscribe to the Development-Student work-Review (DSR) sequence in their instructional practice. What is particularly insightful to us is that these DSRs appear in cycles—and you can find a number of these cycles even within a short instructional episode that spans merely a few minutes. The teachers do not merely plan and execute at a broad-grained level; they also go meticulously into a mode of repeated small-step development, monitoring of students' learning, and consolidations—all at fine-grained levels that approximate the incremental steps in students' acquisition of knowledge. Translated to the examination-driven context, it means that the teachers do not merely give examination-type items (and some challenging ones) to the students and leave them to work on them; they execute detailed cycles of DSR moves to help students learn the requisite skills, concepts, and processes to be successful for these type of items. In the language of Chapter 12, teachers offered carefully planned "scaffolds" to help students gain access to and maintain engagement with these items (especially challenging items).

This careful fine-grained planning did not begin *in* the classroom. It was also conspicuous in the instructional materials developed by these teachers prior to their entering into the classroom. Chapters 11 to 14 of this volume report how the teachers thought about and carried out the design of their instructional materials. A common underlying thread among these teachers that stood out for us was how *deliberate* they were in weaving their agenda into the tasks they crafted for their students. In Chapter 13, Teacher 13 was deliberate through his instructional materials in building opportunities for students to use mathematical reasoning as they work on an item and move across items. In Chapter 14, Teacher 8 was deliberate through her instructional materials in helping students make connections among concepts and across solution methods. This deliberate weaving is also evident in the samples shown in Chapter 11—illustrations of how the teachers brought in "new" materials together with "modified" ones, and how these were "smoothened" so that they were presented as developmentally coherent to the students. This "deliberateness" challenges a narrow view of examination-orientedness—as if being "examination-oriented" necessarily results in a teaching mode where teachers just mindlessly hand out pages of examination questions for students to work on. In the case of the teachers in our study, there were deliberate and goal-driven efforts to transform textbook materials into carefully designed instructional materials that took into consideration the learning needs of their students.

Not only was the design deliberate, we were also fascinated at how much attention teachers put into very fine-grained levels of details which would not have caught the eye of most observers. Chapter 12 reports on the various strategies used by the teachers (particularly, teachers who specialise in the teaching of low-progress learners). The length to which teachers go—such as consideration for the placement of items, the actual scaffolds, the motivational prompts—shows the level of detail they thought about the items and how they are to be used in class. This attention to details applies to example sequencing as well. Leong et al. (in-press) reported how one of the experienced and competent teachers in this project (Teacher 10) carefully sequenced her practise examples in a way that affords variation (cf., Variation Theory) and took into consideration students' cognitive load (cf., Cognitive Load Theory). But hers was not an isolated case; it was found that this careful attention to example sequencing was common across most of the teachers surveyed in Phase Two of the project (Leong et al., 2019b). This finding is colluded by the report in Chapter 11 where evidence of "deliberate sequencing of examples" was found in 29 out of the 30 teachers.

In summary, we think that "examination-driven teaching" can be viewed at two levels of zoom: one that is broad-grained and another that depicts the work of teaching as bringing the students to "drive towards" examination requirements (shown in the left side of Fig. 17.3). As mentioned earlier, this is not an untrue portrait of the work of Singapore secondary mathematics teachers—in that they view preparation of students

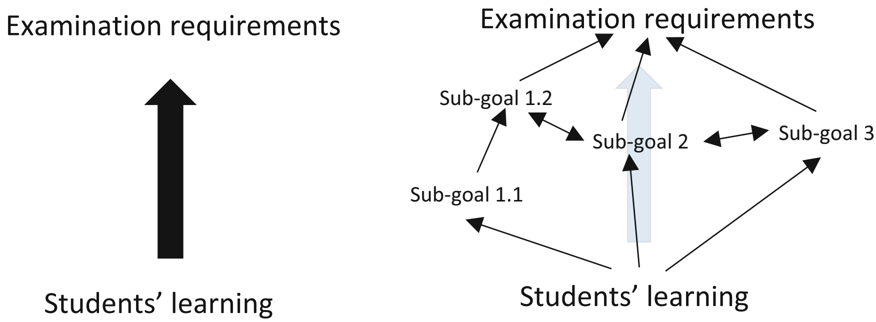


Fig. 17.3 Examination-driven teaching viewed in two levels of zoom

for examination as a key part of their role as teachers, and they take reference from examination items in their selection of content for their students. However, there is a second and a more fine-grained level of zoom which reveals, as uncovered in many chapters in this volume and which we summarised in the preceding paragraphs, a portrait which is more nuanced, consisting of sub-destinations and carefully planned manoeuvres (as illustrated in the right diagram of Fig. 17.3).

This more nuanced portrait in Fig. 17.3 is largely motivated by the case of Teacher 2 as reported in Leong, Cheng, Toh, Kaur, and Toh (2019a). In his instructional practice, we find many of the features that are discussed in this volume and so we use his case as an illustration to pull together the various facets already discussed.

Before he began to teach a module on Vectors, he designed a full set of instructional materials for the whole topic. Through analysing the materials and his responses during interviews, we found that he had the whole development of the topic mapped out—represented as paths and sub-goals in Fig. 17.3. He "drove" towards the examination requirements by bringing the students to various sub-goals which served as milestones (following the journey metaphor). He did so through the careful and deliberate planning of tasks which was coordinated with how he would implement them in class (some of these strategies we reviewed in the earlier paragraphs). At these milestones, he checked through formative assessments if the students met the requirements of the sub-goals, and where necessary, he would zoom-into fill particular gaps in students' knowledge. At suitable junctures, he would connect some of these various strands of knowledge (represented as different paths in the diagram) by helping students to work through tasks that required a coordination of these knowledge areas.

The diagram in Fig. 17.3 is a gross over-simplification of Teacher 2's (and many other Singapore secondary mathematics teachers') practice and as such does not do justice to the deliberateness and intensity of his (and their) work. But we constructed this portrait to uncover elements that are normally hidden in examination-driven talk. As soon as we see that being "examination-driven" is not incompatible with a pedagogy that attends to careful details of students' learning development in conjunction with the content trajectory and which produces deliberately designed tasks that are student-friendly, the "paradox" dissipates. To us, there is no paradox between a pedagogy that is relentlessly "driven" towards examination content goals and towards students' attainment of those goals, and their sub-goals and high performance in international comparison tests such as TIMSS and PISA. In fact, we think the former substantially *explains* the latter.

17.5 Conclusion

As editors of this volume, we used the task of writing this concluding chapter as an opportunity to reflect on Singapore mathematics education. It became clearer to us in the preparation of this chapter that Singapore mathematics education is a microcosm of Singapore herself. We think the ingredients which render Singapore "successful" are also the same ingredients that render Singapore (mathematics) education "successful" (if measured by the performance in international comparison tests). Interestingly, this perspective is shared by Seah (Chapter 16) as he viewed the findings in this volume from the broader lens of Singapore society's strive for excellence.

We have heard numerous visitors to Singapore making this comment (or its equivalent), "I don't know which 'box' to place Singapore in—it is neither East nor West, neither socialist nor capitalistic …" This might also be how readers of this volume feel, "Which pedagogical 'box' do I place Singapore mathematics education—it is neither teacher-centred nor student-centred, neither procedure-based nor concept-based …" If asked to use one word to describe Singapore (and concomitantly, Singapore mathematics education), most would choose "pragmatic". However, we prefer the term "eclectic". Our instinct as a people is not to merely follow the hollow theoretical models of others and assume that their avowed "success" would work for us. We have a habit of being open-minded: drawing upon the affordances of different models and mixing them—deliberately and experimentally—to see which configurations work best for us given our unique context. We think this deliberate eclecticism has been the underlying disposition for researchers and teachers to experiment with different pedagogical mixes all along, resulting in the portraits that we present today (and as illustrated in Fig. 17.2 and Fig. 17.3).

But the portraits will not stay "still"—it is a constantly "moving" portrait. Another feature embedded deep in the Singaporean psyche is this, "We cannot afford to stay still in this climate of tough global competition—we must keep working hard and constantly evolve in response to the changing challenges". This "working hard" and "being ready for changes" also account for the practices of the Singapore secondary mathematics teachers that we have reported in this volume. The picture presented is one where the teachers are meticulous and hard working in attending to their planning and design work as well as to the learning needs of the students. They are also prepared to change—to improve—where the implementations do not work according to plan; that is, they are not easily content with low performance of students and the status quo; rather, they are "driven"—in the positive sense of the term—to help students attain their potential.

But constant evolvement also means that teachers do not easily rest on their laurels. There are still areas for Singapore mathematics education to develop further. As Fig. 17.2 shows, there are components of the pentagon that should fill the agenda of teachers and researchers in the near future.

References

Biggs, J. B. (1996). Western misperceptions of the Confucian-heritage learning culture. In D. Watkins & J. Biggs (Eds.), *The Chinese learner: Cultural, psychological and contextual* (pp. 69–84). Hong Kong SAR, China: Comparative Education Research Centre and The Australian Council for Educational Research.

Ho, W. K., Yap, R. A. S., Tay, E. G., Leong, Y. H., Toh, T. L., Quek, K. S., et al. (2019). Understanding the sustainability of a teaching innovation for problem solving: A systems approach. In P. Liljedahl & M. Santos-Trigo (Eds.), *Mathematical problem solving: Current themes, trends and research* (pp. 339–360). Burnaby: Springer International Publishing.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Leong, Y. H. (2008). *Singapore mathematics teachers: Sandwiched between curriculum goals and examination goals.* Paper presented at International Congress on Mathematics Education 11, Monterrey, Mexico.

Leong, Y. H., Cheng, L. P., Toh, W. Y. K., Kaur, B., & Toh, T. L. (2019a). Making things explicit using instructional materials: A case study of a Singapore teacher's practice. *Mathematics Education Research Journal, 31*(1), 47–66.

Leong, Y. H., Cheng, L. P., Toh, W. Y. K., Kaur, B., & Toh, T. L. (2019b). Example sequences to help students recognise and apply formula: The Singapore situation. *The Mathematics Educator, 18*(1&2), 23–48.

Leong, Y. H., Cheng, L. P., Toh, W. Y. K., Kaur, B., & Toh, T. L. (In-press). Teaching students to apply formula using instructional materials: A case of a Singapore teacher's practice. *Mathematics Education Research Journal*, https://doi.org/10.1007/s13394-019-00290-1.

Leong, Y. H., Dindyal, J., Toh, T. L. Quek, K. S., Tay, E. G., & Lou, S. T. (2011). Teacher education for a problem-solving curriculum in Singapore. *ZDM: The International Journal on Mathematics Education, 43*(6–7), 819–831.

Leung, F. K. S. (2001). In search of an East Asian identity in mathematics education. *Educational Studies in Mathematics, 47,* 35–51.

Mok, I. A. C. (2006). Shedding light on the East Asian learner paradox: Reconstructing student-centredness in a Shanghai classroom. *Asia Pacific Journal of Education, 26* (2), 131–142.

Ministry of Education. (2020). *Mathematics (express/ normal (academic)) syllabuses.* Singapore: Author.

Polya, G. (1945). *How to solve it*. Princeton, NJ: Princeton University Press.

Stacey, K. (2005). The place of problem solving in contemporary mathematics curriculum documents. *Journal of Mathematical Behavior, 24*(3–4), 341–350.

Toh, T. L., Chan, C. M. E., Tay, E. G., Leong, Y. H., Quek, K. S., … Dong, F. M. (2019). Problem solving in the Singapore school mathematics Curriculum. In T. L. Toh, B. Kaur, & E. G. Tay (Eds.), *Mathematics education in Singapore* (pp. 141—164). Singapore: Springer.

Yew Hoong Leong is an Associate Professor at the National Institute of Education, Nanyang Technological University. He began his academic career in mathematics education with the motivation of improving teaching by grappling with the complexity of classroom instruction. Along the journey, his research has broadened to include mathematics problem solving and teacher professional development. Together with his project teammates, they developed "Realistic Ambitious Pedagogy" and its accompanying plan of action—the "Replacement Unit Strategy".

Berinderjeet Kaur is a Professor of Mathematics Education at the National Institute of Education in Singapore. She holds a Ph.D. in Mathematics Education from Monash University in Australia. She has been with the Institute for the last 30 years and is one of the leading figures of Mathematics Education in Singapore. In 2010, she became the first full professor of Mathematics Education in Singapore. She has been involved in numerous international studies of Mathematics Education and was the Mathematics Consultant to TIMSS 2011. She was also a core member of the MEG (Mathematics Expert Group) for PISA 2015. She is passionate about the development of mathematics teachers and in turn the learning of mathematics by children in schools. Her accolades at the national level include the public administration medal in 2006 by the President of Singapore, the long public service with distinction medal in 2016 by the President of Singapore and in 2015, in celebration of 50 years of Singapore's nation building, recognition as an outstanding educator by the Sikh Community in Singapore for contributions towards nation building.