Fuzzy Cellular Automata Model for Discrete Dynamical System Representing Spread of MERS and COVID-19 Virus

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Abstract Dynamical system is the mathematical model for computing changes over time of any physical, biological, economic or social phenomena. Usually discrete dynamical system is described mathematically by difference equations and the solution of such difference equation gives the exact value of the changing variable over time. Another widely used computation model that predicts the trend of the dynamical system is Cellular Automata. Crisp Cellular Automata model which makes use of exactly measurable variables and parameters have been studied widely. However, our practical experience tells us that getting exact measurement of any physical, biological, economic or social phenomena is difficult, if not impossible. The inexactness arising due to imprecision or vagueness is called fuzzy uncertainty and was introduced by Zadeh [22]. The discrete dynamical system where the measurements of variables and/or parameters are imprecisely defined are modelled by *Fuzzy Difference Equations* or *Fuzzy Cellular Automata*. We have found fuzzy triangular number solutions of fuzzy one dimensional first order finite difference equation and corresponding fuzzy cellular automata model. This technique have been used to find a fuzzy cellular automata model for the dynamical system representing MERS and COVID-19 virus spread. The model so obtained reveals the trend of growth and gradation of the infection.

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1 Introduction

Spread of any virus, results in the change of the number of persons infected by the corresponding virus. The total population being large it becomes almost impossible to distinguish the critically infected and mildly infected persons precisely. Hence fuzzy numbers are used to quantify the critical/mildly infected population. We know a dynamical system representing growth may be modelled by a first order linear difference equation when the variables are crisp. We have formulated fuzzy first order difference equation and found the solutions to include fuzzy variables.

Cellular Automata (CA) model of a dynamical system reveals the trend of change of the variable studied in the corresponding dynamical system. So we have designed the fuzzy CA model for growth of infected population. Application of fuzzy CA modelling to MERS and COVID-19 virus spread which is an imprecise dynamical system is also included in this chapter.

Cellular Automata model was introduced by von Neumann and Ulam [16, 18] for designing self replicating systems which later saw applications in Physics, Biology and Computer Science.

Neumann conceived a CA as a two-dimensional mesh of finite state machines called cells which are locally interconnected with each other. Each of the cells change their states synchronously depending on the states of some neighbouring cells (for details see [17, 18] and references therein). The local changes of each of the cells together induce a change of the entire mesh. Later one dimensional CA, i.e a CA where the elementary cells are distributed on a straight line was studied. Stephen Wolfram's work in the 1980s contributed to a systematic study of one-dimensional CA, providing the first qualitative classification of their behaviour [19, 20].

The applications of discrete fuzzy dynamical systems have been studied by many authors, including Barros et al. in the setting of theoretical aspects and ecological applications [1], in asymptotic stability of attractors [2]. Fuzzy Cellular Automata (FCA) models have been studied by Cattaneo et al. [10], Basu et al. [3], Betel et al. [5].

Buckley et al. (see [6–9]) solved second order linear constant coefficient difference equation of the form

$$y(k+2) + ay(k+1) + by(k) = g(k) \tag{1}$$

for $k = 0, 1, 2, \ldots$ where a, b are constants with $b > 0$ and $g(k)$ continuous for $k \geq 0$ having initial conditions $y(0) = \tilde{\gamma}_0$ and $y(1) = \tilde{\gamma}_1$ where $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are triangular fuzzy numbers. The chapter reports existence of three different types of solution namely classical solution ($\tilde{y}_t^C$), extension principle solution ($\tilde{y}_t^E$) and intervel arithmetic solution ($\tilde{y}_t^I$).

Section 2 is devoted to fundamental results used in this chapter. In Sect. 3 we report our work on solutions of one-dimensional fuzzy first order finite difference equation (FFDE). Crisp CA model and Fuzzy CA (FCA) model for FDE and FFDE were studied by us and are included in Sect. 4. We have designed FCA models which are temporally hybrid representing the spread of MERS and COVID-19 virus in Sect. 5.

2 Basic Concepts

2.1 Cellular Automaton

Cellular Automaton(CA) is a computation model of a dynamical system where the smallest computation unit is a *finite state semi automaton*. Thus a CA is a finite dimensional network of finite state semi automaton known as 'cells'.

The mathematical definition of a finite state semi automaton is given as:

Definition 2.1 A **Finite State Semi Automaton** (*abbrev.* **FSSA**) is a three tuple $A = \{Q, X, \mu\}$, where,

- Q is a finite set of memory elements sometimes referred as internal states
- X is the input alphabet
- $\mu : Q \times X \to Q$, is the rule by which an internal state on encountering an input alphabet changes to another internal state. μ is also called transition function.

Thus a CA is a computation model where finite/countably infinite number of cells are arranged in an ordered n-dimensional grid. Each cell receives input from the neighbouring cells and changes according to the transition function. The transitions at each of the cells together induce a change of the grid pattern [15].

Here we have considered only synchronous homogeneous one-dimensional CA. A typical one-dimensional CA is given below.

…	…	A_{i-1}	A_i	A_{i+1}	…	…

Fig. A
A typical grid of a one-dimensional CA

A CA does not have any external input and hence is self-evolving. However the different possible combinations of the state of a cell at any ith grid point along with the states of its adjacent cells can be considered as inputs for the cell at the ith grid point.

Each cell works synchronously leading to evolution of the entire grid through a number of discrete time steps. If the set of memory elements of each FSSA is $\{0, 1\}$ then a typical pattern evolved over time t (represented along horizontal axis) may be as shown in Table 1.

A formal definition of a CA [14] is given below:

Definition 2.2 Let Q be a finite set of memory elements also called the **state set**. The memory elements of the cells belonging to the set Q are placed on an ordered line.

Table 1 C_t is the configuration of the CA(represented along vertical axis) at time t

Grid Position (i)	Time			
	$t = 0$	$t = 1$	$t = 2$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_{i+2}	0	0	1	...
A_{i+1}	0	1	0	...
A_i	1	0	1	...
A_{i-1}	0	1	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Configuration →	C_0	C_1	C_2	...

A **global configuration** is a mapping from the group of integers $\mathbb{Z}$ to the set Q given by $C : \mathbb{Z} \to Q$.

The set $Q^{\mathbb{Z}}$ is the set of all global configurations where $Q^{\mathbb{Z}} = \{C | C : \mathbb{Z} \to Q\}$. A mapping $\tau : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ is called a **global transition function**.

A **CA** (denoted by $\mathcal{C}^Q_\tau$) is a triplet $(Q, Q^{\mathbb{Z}}, \tau)$ where Q is the finite state set, $Q^{\mathbb{Z}}$ is the set of all configurations, τ is the global transition function.

Remark 1 For a particular state set Q and a particular global transition function τ a triple $(Q, Q^{\mathbb{Z}}, \tau)$ denoted by $\mathcal{C}^Q_\tau$ defines the set of all possible cellular automata on (Q, τ). However, the evolution of a CA at times is dependent on the initial configuration (starting configuration) of the CA. A particular CA $\mathcal{C}^Q_\tau(C_0) \in \mathcal{C}^Q_\tau$ is defined as the quadruple $(Q, Q^{\mathbb{Z}}, \tau, C_0)$ such that $C_0 \in Q^{\mathbb{Z}}$ is the initial configuration of the particular CA $\mathcal{C}^Q_\tau(C_0)$.

At any time t, configuration $C_t \in Q^{\mathbb{Z}}$ and $\tau(C_t) = C_{t+1}$.

With reference to Table 1, $C_0 = \ldots 001000\ldots$; $\tau(C_0) = \tau(\ldots 0100\ldots) = \ldots 1010\ldots = C_1$; $\tau(C_1) = \ldots 0101\ldots = C_2$ etc.

CA defined above have the same global transition function $\tau : Q^{\mathbb{Z}} \to Q^{\mathbb{Z}}$ for all time t. However there are a special class of CA called *temporally hybrid CA* where the global transition function varies over time. The formal definition is given with reference to Definition 2.2.

Definition 2.3 A **Temporally Hybrid CA** (denoted by $\mathcal{C}^Q_{\tau_t}$) is a triplet $(Q, Q^{\mathbb{Z}}, \{\tau_t\})$ where Q is the finite state set, $Q^{\mathbb{Z}}$ is the set of all configurations, τ_t is a global transition function.

Evolution of a CA is mathematically expressed by the global transition function. However, this global transition is induced by transitions of the cells at each grid point of the CA. The transition of the state of the cell at the ith grid point of a CA at a particular time, depends on the state of the ith cell and its adjacent cells. These adjacent cells constitute the neighbourhood of the i^{th} cell. The transition of the cell at each grid point is called local transition.

Definition 2.4 For $i \in \mathbb{Z}, r \in \mathbb{N}$, let $S_i = \{i - r, \ldots, i - 1, i, i + 1, \ldots, i + r\} \subseteq \mathbb{Z}$. S_i is the neighbourhood of the ith cell. r is the radius of the neighbourhood of a cell.

It follows that $\mathbb{Z} = \bigcup_i S_i$
A restriction from $\mathbb{Z}$ to S_i induces the following:

1. Restriction of C to c_i is given by $c_i : S_i \to Q$; and c_i may be called **local configuration** of the i^{th} cell.
2. Restriction of $Q^{\mathbb{Z}}$ to Q^{S_i} is given by $Q^{S_i} = \{c_i | c_i : S_i \to Q\}$; and Q^{S_i} may be called the **set of all local configurations** of the ith cell.

The mapping $\mu_i : Q^{S_i} \to Q$ is known as a **local transition function** for the ith automaton having radius r. Thus, $\forall i \in \mathbb{Z}$, $\mu_i(c_i) \in Q$. So, if the local configuration of the ith cell at time t is denoted by c_i^t, then $\mu_i(c_i^t) = c_i^{t+1}(i)$.

Remark 2 If $\tau(C) = C^*$ then $C^*(i) = \tau(C)(i) = \mu_i(c_i)$. So we have,

1. $C_{t+1}(i) = \tau(C_t)(i) = \mu_i(c_i^t) = c_i^{t+1}(i)$
2. $\tau(C) = \ldots \mu_{i-1}(c_{i-1}).\mu_i(c_i).\mu_{i+1}(c_{i+1}) \ldots$
3. If all $\mu_i' s$ are identical then the CA is **homogeneous**.
4. For a temporally hybrid CA the $\mu_i' s$ are time dependent.

Definition 2.5 If for a particular CA, $|Q| = 2$ so that we can write $Q = \{0, 1\}$, then the CA is said to be a **binary CA** or a **Boolean CA**.

2.2 Fuzzy Set, Fuzzy Number, α-cut

Definition 2.6 A **universal set** S is defined as a collection of elements or objects in the universe of discourse. The universal set may be finite, countable or uncountable.

A fuzzy set is a subset of the universal set whose boundary cannot be precisely defined.

Definition 2.7 If S is the universal set then a **fuzzy set** $\widetilde{A}$ in S is a set of ordered pairs: $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x)) | x \in \widetilde{A} \subseteq S\}$.$\mu_{\widetilde{A}}(x)$ is called the **membership function** or grade of membership of x in $\widetilde{A}$ and is given by $\mu_{\widetilde{A}} : \widetilde{A} \to [0, 1], x \in \widetilde{A}$.

Definition 2.8 A fuzzy number is a convex and normalized fuzzy subset of set of real numbers $\mathbb{R}$.

Definition 2.9 **Triangular fuzzy number** is a fuzzy number represented with three points as follows

$$\widetilde{A} = (a_1/a_2/a_3)$$

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & \text{if } x < a_1, \\ \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2} & \text{if } a_2 \le x \le a_3 \\ 0 & x > a_3 \end{cases} \tag{2}$$

Definition 2.10 Given a fuzzy set $\widetilde{A}$ in S and any real number $\alpha \in [0, 1]$, then the *α-cut*, denoted by $\widetilde{A}[\alpha]$ is the crisp set $\widetilde{A}[\alpha] = \{x \in \widetilde{A} \mid \mu_{\widetilde{A}}(x) > \alpha\}$.

Thus on setting the left and right reference functions of $\widetilde{A}$ as $\alpha = \frac{x-a_1}{a_2-a_1}$ and $\alpha = \frac{a_3-x}{a_3-a_2}$, it follows that

$$\widetilde{A}[\alpha] = [a_1 + (a_2 - a_1)\alpha, a_3 - (a_3 - a_2)\alpha]$$

Multiplication of two fuzzy numbers is defined as follows:

Definition 2.11 For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, if for some $\alpha \in (0, 1]$, we have, $\tilde{A}[\alpha] = [a_1(\alpha), a_2(\alpha)]$ and $\tilde{B}[\alpha] = [b_1(\alpha), b_2(\alpha)]$ then their product is (see [9]),

$$\tilde{A}[\alpha].\tilde{B}[\alpha] = [c_1(\alpha), c_2(\alpha)]$$

where, $c_1(\alpha) = \min\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}$ and $c_2(\alpha) = \max\{a_1(\alpha)b_1(\alpha), a_1(\alpha)b_2(\alpha), a_2(\alpha)b_1(\alpha), a_2(\alpha)b_2(\alpha)\}$.

2.3 Fuzzy Cellular Automaton

A fuzzy CA is a generalization of Boolean CA defined as follows:

Definition 2.12 A one-dimensional **fuzzy CA (FCA)** is a one-dimensional CA where the local transition function is a fuzzy transition function. So the formal definition is as follows:

An **FCA** (denoted by $\mathcal{C}^{\mathcal{F}}$) is a four-tuple $(\tilde{Q}, \tilde{Q}^{\mathbb{Z}}, \tilde{f}, \tilde{C}_0)$, where,

- $\tilde{Q} \subset [0, 1]$ is the state set
- $\tilde{Q}^{\mathbb{Z}}$ is the set of all configurations
- $\tilde{f}$ is the local transition function which is fuzzy in nature such that if r be the radius of the neighbourhood then $\tilde{f} : \widetilde{Q^{2r+1}} \to \tilde{Q}$
- $\tilde{C}_0$ is the initial configuration which is a fuzzy number.

Table 2 Fuzzy transition rule for Wolfram code 200

$c_{i-1}c_ic_{i+1}$	μ	$\tilde{f}$
000	0	$1-\min(\neg c_{i-1}, \neg c_i, \neg c_{i+1})$
001	0	$1-\min(\neg c_{i-1}, \neg c_i, c_{i+1})$
010	0	$1-\min(\neg c_{i-1}, c_i, \neg c_{i+1})$
011	1	$\min(\neg c_{i-1}, c_i, c_{i+1})$
100	0	$1-\min(c_{i-1}, \neg c_i, \neg c_{i+1})$
101	0	$1-\min(c_{i-1}, \neg c_i, c_{i+1})$
110	1	$\min(c_{i-1}, c_i, \neg c_{i+1})$
111	1	$\min(c_{i-1}, c_i, c_{i+1})$

The local transition function $\tilde{f}$ is a fuzzification of Boolean function.

Disjunctive Normal Form-fuzzification of the transition fuction of a classical Boolean CA gives a **Fuzzy transition function**. The Boolean operators AND, OR, NOT in the DNF expression of the Boolean rule can be fuzzified using different fuzzy operators. Here we have replaced $(a \wedge b)$ by $\min(a, b)$, $(a \vee b)$ by $\max(a, b)$ and $\neg a$ by $(1 - a)$ (see [11]).

For Example: Let us consider the Boolean local transition function for some ith cell to be **Rule 200** of Wolfram (see [21]) as shown in (Table 2).

The DNF for RULE 200 is

$$(c_{i-1} \wedge c_i \wedge c_{i+1})_{(111)} \ \vee \ (c_{i-1} \wedge c_i \wedge \neg c_{i+1})_{(110)} \ \vee \ (\neg c_{i-1} \wedge c_i \wedge c_{i+1})_{(011)}$$

Thus fuzzifucation of the DNF gives the fuzzy transition rule $\tilde{f}$ for Wolfram code 200 as

$$\max\{\min(c_{i-1}, c_i, c_{i+1})_{(111)}, \min(c_{i-1}, c_i, 1 - c_{i+1})_{(110)}, \min(1 - c_{i-1}, c_i, c_{i+1})_{(011)}\}$$

3 Fuzzy First-Order Difference Equation & Its Solution

Buckley found solutions of fuzzy second order difference equation. In this section we restrict our discussion to fuzzy first order linear difference equation and could give fuzzy triangular number solution using extension principle method, interval arithmetic method and classical solution as introduced by Buckley.

Let us consider a first order linear difference equation for $t = 0, 1, 2, \ldots$ of the form

$$y_{t+1} = \lambda y_t \tag{3}$$

where λ is a constant.

If we fuzzify the crisp equation (3) and solve, we are attempting to get the classical solution $\tilde{y}_t^C$. When we first solve Eq. (3) and then fuzzify the crisp solution we obtain $\tilde{y}_t^E$ (solution using extension principle) and $\tilde{y}_t^I$ (α cuts and interval arithmetic). Buckley established the fact that $\tilde{y}_t^C$, $\tilde{y}_t^E$ $\tilde{y}_t^I$ and may differ if more than one fuzzy numbers are used. However, they produce same result if a fuzzy number appears only once in the fuzzy expression. For a difference equation, too often the classical solution fails to exist.

3.1 Solution of FFDE When λ Is Constant and $\tilde{y}_0$ Is a Triangular Fuzzy Number

Fuzzification of FDE (3) gives the following FFDE

$$\tilde{y}_{t+1} = \lambda \tilde{y}_t \tag{4}$$

The solution of FFDE (4) will be as follows:

$$\tilde{y}_t = \lambda^t \tilde{y}_0 \tag{5}$$

A triangular fuzzy solution of the FDE (3) at time t is a **classical solution** denoted by $\tilde{y}_t^C$ if it satisfies FFDE (4) and is a triangular fuzzy number. For some $\alpha \in (0, 1]$, α*-cut* operation on $\tilde{y}_t^C$ gives

$$\tilde{y}_t^C[\alpha] = [y_{t1}(\alpha), y_{t2}(\alpha)] \tag{6}$$

where, $y_{t1}(\alpha) = \lambda^t y_{01}(\alpha)$ and $y_{t2}(\alpha) = \lambda^t y_{02}(\alpha)$.

Now, $\tilde{y}_t^C$ is a triangular fuzzy number if,

$$\frac{\partial y_{t1}(\alpha)}{\partial \alpha} > 0, \ \frac{\partial y_{t2}(\alpha)}{\partial \alpha} < 0, \ \text{and } y_{t1}(1) = y_{t2}(1) \ \ \forall t \geq 1$$

Theorem 3.1 *An FFDE given by* $\tilde{y}_{t+1} = \lambda \tilde{y}_t$ *will have a classical solution* $\tilde{y}_t^C$ *provided* $\lambda > 0$.

Proof Let $\tilde{y}_0 = (a_1/a_2/a_3)$. Then $a_1 \leq a_2 \leq a_3$.

If $\tilde{y}_t^C$ is a classical solution of the given FFDE, then we get

$$\tilde{y}_t^C = \lambda^t \tilde{y}_0. \tag{7}$$

Therefore, for some $\alpha \in (0, 1]$, α*-cut* on the classical solution will be

$$\tilde{y}_t^C[\alpha] = \lambda^t \tilde{y}_0[\alpha], \quad \text{for } t = 0, 1, 2, \ldots, \tag{8}$$

$$So, \quad y_{t1}(\alpha) = \{a_1 + (a_2 - a_1)\alpha\}\lambda^t \ ; \ y_{t2}(\alpha) = \{a_3 - (a_3 - a_2)\alpha\}\lambda^t$$

$$\text{i.e.,} \quad \frac{\partial y_{t1}(\alpha)}{\partial \alpha} = (a_2 - a_1)\lambda^t > 0 \ ; \ \frac{\partial y_{t2}(\alpha)}{\partial \alpha} = -(a_3 - a_2)\lambda^t < 0$$

$$\text{and,} \quad y_{t1}(1) = \{a_1 + (a_2 - a_1).1\}\lambda^t = a_2\lambda^t = \{a_3 - (a_3 - a_2).1\}\lambda^t = y_{t2}(1)$$

Hence all the conditions for the existence of a classical solution is satisfied provided $\lambda > 0$. □

A solution to FDE (3) is

$$y_t = \lambda^t y_0 \tag{9}$$

Fuzzification of (9) is

$$\tilde{y}_t = \lambda^t \tilde{y}_0 \tag{10}$$

Solution (10) is an **extension principle solution** denoted by $\tilde{y}_t^E$ if it is a triangular number which is possible provided,

$$\frac{\partial \tilde{y}_t}{\partial \tilde{y}_0} > 0 \ \forall t \geq 1 \ or \ \frac{\partial \tilde{y}_t}{\partial \tilde{y}_0} < 0 \ \forall t \geq 1$$

Let $\tilde{y}_0 = (a_1/a_2/a_3)$. Then for some $\alpha \in (0, 1]$, α-*cut* operation gives a crisp closed bounded interval such that $\tilde{y}_0[\alpha] = [y_{01}(\alpha), y_{02}(\alpha)]$ where,

- $y_{01}(\alpha) = a_1 + (a_2 - a_1)\alpha$ is a monotonic increasing function of α
- $y_{02}(\alpha) = a_3 - (a_3 - a_2)\alpha$ is a monotonic decreasing function of α
- $y_{01}(1) = y_{02}(1)$.

Further, for some $\alpha \in (0, 1]$, α-*cut* operation on $\tilde{y}_t$ gives

$$\tilde{y}_t[\alpha] = \lambda^t \tilde{y}_0[\alpha] \tag{11}$$

So, α-*cut* operation on $\tilde{y}_t^E$ gives

$$\tilde{y}_t^E[\alpha] = [y_{t1}^E(\alpha), y_{t2}^E(\alpha)]$$

where,

- $y_{t1}^E(\alpha) = min\{\lambda^t y_0 \mid y_0 \in \tilde{y}_0[\alpha]\} \ = \ \lambda^t(a_1 + (a_2 - a_1)\alpha)$
- $y_{t2}^E(\alpha) = max\{\lambda^t y_0 \mid y_0 \in \tilde{y}_0[\alpha]\} \ = \ \lambda^t(a_3 - (a_3 - a_2)\alpha)$.

Remark 3 For an FFDE given by $\tilde{y}_t = \lambda^t \tilde{y}_0$ we get $\tilde{y}_t^C[\alpha] \ = \ \tilde{y}_t^E[\alpha]$

Theorem 3.2 *An FFDE given by $\tilde{y}_t = \lambda^t \tilde{y}_0$ will have an extension principle solution $\tilde{y}_t^E$ provided $\lambda > 0$.*

Proof If $\tilde{y}_t^E$ is an extension principle solution of the given FFDE, then it holds that either

$$\frac{\partial \tilde{y}_t}{\partial \tilde{y}_0} = \lambda^t > 0 \, \forall t \geq 1 \; or, \; \frac{\partial \tilde{y}_t}{\partial \tilde{y}_0} = \lambda^t < 0 \, \forall t \geq 1$$

Hence the condition for the existence of an extension principle solution is satisfied provided $\lambda > 0$. $\square$

A solution to an FFDE at time t is an **interval arithmetic solution** denoted by $\tilde{y}_t^I[\alpha] = [y_{t1}^I(\alpha), y_{t2}^I(\alpha)]$ if for some $\alpha \in (0, 1)$, $\tilde{y}_t^I[\alpha]$ is an interval which is possible provided

$$\forall t \geq 0, \; y_{t1}^I(\alpha) \neq y_{t2}^I(\alpha)$$

where,

- $y_{t1}^I(\alpha) = \lambda^t y_{01}(\alpha) = \lambda^t(a_1 + (a_2 - a_1)\alpha)$
- $y_{t2}^I(\alpha) = \lambda^t y_{02}(\alpha) = \lambda^t(a_3 - (a_3 - a_2)\alpha)$

For $\alpha = 1$, $\tilde{y}_t^I$ reduces to the point solution y_t.

Theorem 3.3 *An FFDE given by $\tilde{y}_t = \lambda^t \tilde{y}_0$ will have an interval arithmetic solution $\tilde{y}_t^I$ provided $\lambda \neq 0$ and for $\alpha \in (0, 1)$, $y_{01}(\alpha) < y_{02}(\alpha)$.*

Proof If $\tilde{y}_t^I$ is an interval arithmetic solution of the given FFDE, if

$$\lambda^t y_{01}(\alpha) \neq \lambda^t y_{02}(\alpha)$$

$$\Leftrightarrow \lambda \neq 0 \; and \; y_{01}(\alpha) \neq y_{02}(\alpha)$$

Since $\alpha \in (0, 1)$ and $a_1 < a_2 < a_3$, from the definition of $\tilde{y}_0[\alpha]$ it follows that

$$y_{01}(\alpha) < y_{02}(\alpha)$$

Hence the theorem. $\square$

Remark 4 Since the solution of FFDE $\tilde{y}_t = \lambda^t \tilde{y}_0$ has only one fuzzy number $\tilde{y}_0$, an **interval arithmetic solution** of this FFDE at any time t denoted by $\tilde{y}_t^I$ will produce the same result as that of $\tilde{y}_t^E$ (see [8, 9]).

Remark 5 Clearly, if $\tilde{y}_t$ be a solution of FFDE $\tilde{y}_t = \lambda^t \tilde{y}_0$ at time t, then for some $\alpha \in (0, 1]$,

$$\tilde{y}_t[\alpha] = \tilde{y}_t^c[\alpha] = \tilde{y}_t^E([\alpha] = \tilde{y}_t^I[\alpha]$$

3.2 Solution of FFDE When y_0 Is Constant and the Parameter λ Is a Fuzzy Triangular Number

In this section we will fuzzify the FDE $y_{t+1} = \lambda y_t, t = 0, 1, 2 \ldots$ by considering the parameter λ to be fuzzy, and denote it by $\tilde{\lambda}$ where $\tilde{\lambda}$ is a triangular fuzzy number. If y_0 be the crisp initial value then we obtain FFDE which maybe written as

$$\tilde{y_1} = \tilde{\lambda} y_0 \tag{12}$$

$$\tilde{y}_{t+1} = \tilde{\lambda} \tilde{y}_t \quad for\ t = 1, 2 \ldots \tag{13}$$

Thus recursively we get,

$$\tilde{y}_t = \tilde{\lambda}^t y_0 \tag{14}$$

where $\tilde{\lambda^t}$ is a triangular shaped fuzzy number since product of two triangular fuzzy numbers is a triangular shaped fuzzy number (see [9]).

Let $\tilde{\lambda} = (b_1/b_2/b_3)$ such that $b_1 < b_2 < b_3$. Then for some $\alpha \in (0, 1]$, α-*cut* operation gives a crisp closed bounded interval such that $\tilde{\lambda}[\alpha] = [\lambda_1(\alpha), \lambda_2(\alpha)]$ where,

- $\lambda_1(\alpha) = b_1 + (b_2 - b_1)\alpha$ is a monotonic increasing function of α
- $\lambda_2(\alpha) = b_3 - (b_3 - b_2)\alpha$ is a monotonic decreasing function of α
- $\lambda_1(1) = \lambda_2(1)$.

Clearly, if $\alpha \neq 1$, $\lambda_1(\alpha) < \lambda_2(\alpha)$.
Now, $\lambda_1(\alpha) \geq 0 \Leftrightarrow$ either $b_1 \geq 0$ or $|b_1| < (b_2 - b_1)\alpha$.
Again, $\lambda_2(\alpha) \leq 0 \Leftrightarrow$ either $b_3 \leq 0$ or $|b_3| < (b_3 - b_2)\alpha$.

We have found three types of solutions, namely, (i) the classical solution (ii) the extension principle solution and (iii) the interval arithmetic solution.

A solution to an FFDE at time t is a **classical solution** denoted by $\tilde{y}_t^c$ if it is a triangular shaped fuzzy number. Further, for some $\alpha \in (0, 1]$, α-*cut* operation on $\tilde{y}_t^c$ gives

$$\tilde{y}_t^c[\alpha] = [y_{t1}(\alpha), y_{t2}(\alpha)] \tag{15}$$

where, $y_{t1}(\alpha) = (\lambda_1(\alpha))^t y_0$ and $y_{t2}(\alpha) = (\lambda_2(\alpha))^t y_0$.

Theorem 3.4 *An FFDE given by $\tilde{y}_t = \tilde{\lambda}^t y_0$ will have a classical solution $\tilde{y}_t^c$ provided either $y_0 > 0$ and $\lambda_1(\alpha) > 0$ or $y_0 < 0$ and $\lambda_2(\alpha) < 0$ where $\tilde{\lambda}[\alpha] = [\lambda_1(\alpha), \lambda_2(\alpha)]$.*

Proof Let $\tilde{\lambda} = (b_1/b_2/b_3)$. Then $b_1 < b_2 < b_3$.

If $\tilde{y}_t^c$ is a classical solution of the given FFDE, then we get

$$\tilde{y}_t^c = \tilde{\lambda}^t y_0. \tag{16}$$

Therefore, for some $\alpha \in (0, 1]$, α-*cut* on the classical solution will be

$$\tilde{y}_t^c[\alpha] = (\tilde{\lambda}[\alpha])^t y_0, \quad for\ t = 1, 2, \ldots, \tag{17}$$

$$So, \quad y_{t1}(\alpha) = \{b_1 + (b_2 - b_1)\alpha\}^t y_0 \ ; \ y_{t2}(\alpha) = \{b_3 - (b_3 - b_2)\alpha\}^t y_0$$

$$Thus, \quad \frac{\partial y_{t1}(\alpha)}{\partial \alpha} = t\{b_1 + (b_2 - b_1)\alpha\}^{t-1}(b_2 - b_1)y_0 \ ,$$

$$\frac{\partial y_{t2}(\alpha)}{\partial \alpha} = -t\{b_3 - (b_3 - b_2)\alpha\}^{t-1}(b_3 - b_2)y_0$$

$$and, \quad y_{t1}(1) = \{b_1 + (b_2 - b_1).1\}^t y_0 = b_2^t y_0 = \{b_3 - (b_3 - b_2).1\}^t y_0 = y_{t2}(1)$$

Now, $\frac{\partial y_{t1}(\alpha)}{\partial \alpha} > 0$ and $\frac{\partial y_{t2}(\alpha)}{\partial \alpha} < 0$ if

- Case I : $y_0 > 0$ and $\{b_1 + (b_2 - b_1)\alpha\} > 0$
  Moreover, $\{b_1 + (b_2 - b_1)\alpha\} > 0 \Rightarrow \{b_3 - (b_3 - b_2)\alpha\} > 0$ since $b_3 > b_2 > b_1$.
- Case II : $y_0 < 0$ and $\{b_3 - (b_3 - b_2)\alpha\} < 0$
  Moreover, $\{b_3 - (b_3 - b_2)\alpha\} < 0 \Rightarrow \{b_1 + (b_2 - b_1)\alpha\} < 0$ since $b_1 < b_2 < b_3$.

Hence all the conditions for existence of a classical solution are satisfied provided either $y_0 > 0$ and $\lambda_1(\alpha) > 0$ or $y_0 < 0$ and $\lambda_2(\alpha) < 0$. □

A solution to an FFDE at time t is an **extension principle solution** denoted by $\tilde{y}_t^E$ provided

$$\tilde{y}_t^{\ E} = \tilde{\lambda}^t y_0 \tag{18}$$

$$\frac{\partial \tilde{y}_t^{\ E}}{\partial \tilde{\lambda}} > 0, \ \forall t \geq 1 \ or \ \frac{\partial \tilde{y}_t^{\ E}}{\partial \tilde{\lambda}} < 0, \ \forall t \geq 1$$

Further, for some $\alpha \in (0, 1]$, α-*cut* operation on $\tilde{y}_t^E$ gives

$$\tilde{y}_t^E[\alpha] = [y_{t1}^E(\alpha), y_{t2}^E(\alpha)]$$

where,

- $y_{t1}^E(\alpha) = min\{\lambda^t y_0 \mid \lambda \in \tilde{\lambda}[\alpha]\} \ = \ (b_1 + (b_2 - b_1)\alpha)^t y_0$
- $y_{t2}^E(\alpha) = max\{\lambda^t y_0 \mid \lambda \in \tilde{\lambda}[\alpha]\} \ = \ (b_3 - (b_3 - b_2)\alpha)^t y_0$.

Theorem 3.5 *An FFDE given by* $\tilde{y}_t = \tilde{\lambda}^t y_0$ *will have an extension principle solution* $\tilde{y}_t^E$ *provided* $\lambda_1(\alpha) > 0$ *and* $y_0 \neq 0$ *where* $\tilde{\lambda}[\alpha] = [\lambda_1(\alpha), \lambda_2(\alpha)]$.

Proof If $\tilde{y}_t^E$ is an extension principle solution of the given FFDE, then it holds that either

$$\frac{\partial \tilde{y}_t^{\ E}}{\partial \tilde{\lambda}} > 0, \ \forall t \geq 1 \quad Or, \quad \frac{\partial \tilde{y}_t^{\ E}}{\partial \tilde{\lambda}} < 0, \ \forall t \geq 1$$

Now, $\frac{\partial \tilde{y}_t{}^E}{\partial \tilde{\lambda}} = t\tilde{\lambda}^{t-1}y_0$. Thus, $\forall t \geq 1$,

$$\frac{\partial \tilde{y}_t{}^E}{\partial \tilde{\lambda}} > 0 \ \ if \ \ \tilde{\lambda} > 0, y_0 > 0, \quad Or, \quad \frac{\partial \tilde{y}_t{}^E}{\partial \tilde{\lambda}} < 0 \ \ if \ \ \tilde{\lambda} > 0, y_0 < 0.$$

Consequently, $\tilde{\lambda} > 0 \Rightarrow \lambda_1(\alpha) > 0$.

Hence the condition for the existence of an extension principle solution is satisfied provided $\lambda_1(\alpha) > 0$ and $y_0 \neq 0$. □

An **interval arithmetic solution** of an FFDE at any time t is denoted by $\tilde{y}_t^I$. Further, for some $\alpha \in (0, 1]$, α-*cut* operation on $\tilde{y}_t^I$ gives

$$\tilde{y}_t^I[\alpha] = [\lambda_1(\alpha), \lambda_2(\alpha)]^t y_0 = [y_{t1}^I(\alpha), y_{t2}^I(\alpha)] \tag{19}$$

Theorem 3.6 *An FFDE given by* $\tilde{y}_t = \tilde{\lambda}^t y_0$ *will have an interval arithmetic solution* $\tilde{y}_t^I$ *for* $t \geq 1$ *provided* $y_0 \neq 0$ *and* $\lambda_1(\alpha) < \lambda_2(\alpha)$ *where* $\tilde{\lambda}[\alpha] = [\lambda_1(\alpha), \lambda_2(\alpha)]$.

Proof If $y_0 = 0$ or $\lambda_1(\alpha) = \lambda_2(\alpha)$ then the solution of the given FFDE will reduce to crisp constant and $\tilde{y}_t^I$ will not exist.

However, if $y_0 \neq 0$ and $\lambda_1(\alpha) < \lambda_2(\alpha)$ then $\tilde{y}_1^I = [\lambda_1(\alpha)y_0, \lambda_2(\alpha)y_0]$ and $\forall t \geq 2$, $\tilde{y}_t^I = [y_{t1}^I(\alpha), y_{t2}^I(\alpha)]$ obtained will be as follows:
Case I: $y_0 > 0$

- $\lambda_1(\alpha) \geq 0$
  $\tilde{y}_2^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)y_0, \lambda_2(\alpha)y_0] = [\lambda_1^2(\alpha)y_0, \lambda_2^2(\alpha)y_0]$
  $\tilde{y}_3^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1^2(\alpha)y_0, \lambda_2^2(\alpha)y_0] = [\lambda_1^3(\alpha)y_0, \lambda_2^3(\alpha)y_0]$
  Therefore by induction we get,
  $\tilde{y}_t^I = [\lambda_1(\alpha), \lambda_2(\alpha)]\tilde{y}_{t-1}^I = [\lambda_1^t(\alpha)y_0, \lambda_2^t(\alpha)y_0]$
- $\lambda_2(\alpha) \leq 0$
  $\tilde{y}_2^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)y_0, \lambda_2(\alpha)y_0] = [\lambda_2^2(\alpha)y_0, \lambda_1^2(\alpha)y_0]$
  $\tilde{y}_3^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_2^2(\alpha)y_0, \lambda_1^2(\alpha)y_0] = [\lambda_1^3(\alpha)y_0, \lambda_2^3(\alpha)y_0]$
  $\tilde{y}_4^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1^3(\alpha)y_0, \lambda_2^3(\alpha)y_0] = [\lambda_2^4(\alpha)y_0, \lambda_1^4(\alpha)y_0]$
  Therefore by induction we get,
  $\tilde{y}_t^I = [\lambda_1(\alpha), \lambda_2(\alpha)]\tilde{y}_{t-1}^I = [\lambda_2^t(\alpha)y_0, \lambda_1^t(\alpha)y_0]$ or $[\lambda_1^t(\alpha)y_0, \lambda_2^t(\alpha)y_0]$ according as t is even or t is odd.
- $\lambda_1(\alpha) < 0, \lambda_2(\alpha) > 0$ and $\lambda_1^2(\alpha) < \lambda_2^2(\alpha)$
  $\tilde{y}_2^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)y_0, \lambda_2(\alpha)y_0] = [\lambda_1(\alpha)\lambda_2(\alpha)y_0, \lambda_2^2(\alpha)y_0]$
  $\tilde{y}_3^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)\lambda_2(\alpha)y_0, \lambda_2^2(\alpha)y_0] = [\lambda_1(\alpha)\lambda_2^2(\alpha)y_0, \lambda_2^3(\alpha)y_0]$
  $\tilde{y}_4^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)\lambda_2^2(\alpha)y_0, \lambda_2^3(\alpha)y_0] = [\lambda_1(\alpha)\lambda_2^3(\alpha)y_0, \lambda_2^4(\alpha)y_0]$
  Therefore by induction we get,
  $\tilde{y}_t^I = [\lambda_1(\alpha), \lambda_2(\alpha)]\tilde{y}_{t-1}^I = [\lambda_1(\alpha)\lambda_2^{t-1}(\alpha)y_0, \lambda_2^t(\alpha)y_0]$
- $\lambda_1(\alpha) < 0, \lambda_2(\alpha) > 0$ and $\lambda_1^2(\alpha) > \lambda_2^2(\alpha)$
  $\tilde{y}_2^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)y_0, \lambda_2(\alpha)y_0] = [\lambda_1(\alpha)\lambda_2(\alpha)y_0, \lambda_1^2(\alpha)y_0]$
  $\tilde{y}_3^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)\lambda_2(\alpha)y_0, \lambda_1^2(\alpha)y_0] = [\lambda_1^3(\alpha)y_0, \lambda_1^2(\alpha)\lambda_2(\alpha)y_0]$

$\tilde{y}_4^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1^3(\alpha)y_0, \lambda_1^2(\alpha)\lambda_2(\alpha)y_0] = [\lambda_1^3(\alpha)\lambda_2(\alpha)y_0, \lambda_1^4(\alpha)y_0]$
Therefore by induction we get,
$\tilde{y}_t^I = [\lambda_1(\alpha), \lambda_2(\alpha)]\tilde{y}_{t-1}^I = [\lambda_1^{t-1}(\alpha)\lambda_2(\alpha)y_0, \lambda_1^t(\alpha)y_0]$ or $[\lambda_1^t(\alpha)y_0, \lambda_1^{t-1}(\alpha)\lambda_2^t(\alpha)y_0]$ according as t is even or t is odd.

Case II: $y_0 < 0$

- $\lambda_1(\alpha) \geq 0$
  $\tilde{y}_2^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)y_0, \lambda_2(\alpha)y_0] = [\lambda_2^2(\alpha)y_0, \lambda_1^2(\alpha)y_0]$
  $\tilde{y}_3^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_2^2(\alpha)y_0, \lambda_1^2(\alpha)y_0] = [\lambda_2^3(\alpha)y_0, \lambda_1^3(\alpha)y_0]$
  Therefore by induction we get,
  $\tilde{y}_t^I = [\lambda_1(\alpha), \lambda_2(\alpha)]\tilde{y}_{t-1}^I = [\lambda_2^t(\alpha)y_0, \lambda_1^t(\alpha)y_0]$
- $\lambda_2(\alpha) \leq 0$
  $\tilde{y}_2^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)y_0, \lambda_2(\alpha)y_0] = [\lambda_1^2(\alpha)y_0, \lambda_2^2(\alpha)y_0]$
  $\tilde{y}_3^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1^2(\alpha)y_0, \lambda_2^2(\alpha)y_0] = [\lambda_2^3(\alpha)y_0, \lambda_1^3(\alpha)y_0]$
  $\tilde{y}_4^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_2^3(\alpha)y_0, \lambda_1^3(\alpha)y_0] = [\lambda_1^4(\alpha)y_0, \lambda_2^4(\alpha)y_0]$
  Therefore by induction we get,
  $\tilde{y}_t^I = [\lambda_1(\alpha), \lambda_2(\alpha)]\tilde{y}_{t-1}^I = [\lambda_1^t(\alpha)y_0, \lambda_2^t(\alpha)y_0]$ or $[\lambda_2^t(\alpha)y_0, \lambda_1^t(\alpha)y_0]$ according as t is even or t is odd.
- $\lambda_1(\alpha) < 0, \lambda_2(\alpha) > 0$ and $\lambda_1^2(\alpha) < \lambda_2^2(\alpha)$
  $\tilde{y}_2^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)y_0, \lambda_2(\alpha)y_0] = [\lambda_2^2(\alpha)y_0, \lambda_1(\alpha)\lambda_2(\alpha)y_0]$
  $\tilde{y}_3^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_2^2(\alpha)y_0, \lambda_1(\alpha)\lambda_2(\alpha)y_0] = [\lambda_2^3(\alpha)y_0, \lambda_1(\alpha)\lambda_2^2(\alpha)y_0]$
  $\tilde{y}_4^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_2^3(\alpha)y_0, \lambda_1(\alpha)\lambda_2^2(\alpha)y_0] = [\lambda_2^4(\alpha)y_0, \lambda_1(\alpha)\lambda_2^3(\alpha)y_0]$
  Therefore by induction we get,
  $\tilde{y}_t^I = [\lambda_1(\alpha), \lambda_2(\alpha)]\tilde{y}_{t-1}^I = [\lambda_2^t(\alpha)y_0, \lambda_1(\alpha)\lambda_2^{t-1}(\alpha)y_0]$
- $\lambda_1(\alpha) < 0, \lambda_2(\alpha) > 0$ and $\lambda_1^2(\alpha) > \lambda_2^2(\alpha)$
  $\tilde{y}_2^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1(\alpha)y_0, \lambda_2(\alpha)y_0] = [\lambda_1^2(\alpha)y_0, \lambda_1(\alpha)\lambda_2(\alpha)y_0]$
  $\tilde{y}_3^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1^2(\alpha)y_0, \lambda_1(\alpha)\lambda_2(\alpha)y_0] = [\lambda_1^2(\alpha)\lambda_2(\alpha)y_0, \lambda_1^3(\alpha)y_0]$
  $\tilde{y}_4^I = [\lambda_1(\alpha), \lambda_2(\alpha)][\lambda_1^2(\alpha)\lambda_2(\alpha)y_0, \lambda_1^3(\alpha)y_0] = [\lambda_1^4(\alpha)y_0, \lambda_1^3(\alpha)\lambda_2(\alpha)y_0]$
  Therefore by induction we get,
  $\tilde{y}_t^I = [\lambda_1(\alpha), \lambda_2(\alpha)]\tilde{y}_{t-1}^I = [\lambda_1^t(\alpha)y_0, \lambda_1^{t-1}(\alpha)\lambda_2^t(\alpha)y_0]$ or $[\lambda_1^{t-1}(\alpha)\lambda_2(\alpha)y_0, \lambda_1^t(\alpha)y_0]$ according as t is even or t is odd.

Hence the theorem. □

3.3 Solution of an FFDE When λ Is Time Dependent and $\tilde{y}_0$ Is a Triangular Fuzzy Number

In this section we restrict our discussion to fuzzy first order nonlinear difference equation and its fuzzy solution.

Let us consider a first order nonlinear difference equation for $t = 0, 1, 2, \ldots$ of the form

$$y_{t+1} = \lambda(t)y_t \tag{20}$$

A solution of (20) is calculated by mathematical induction and is as follows:

$$\tilde{y_n} = \prod_{t=0}^{t=n-1} \lambda(t)\tilde{y_0} \tag{21}$$

3.4 Solution of an FFDE When λ Is Time Dependent and Is a Triangular Fuzzy Number Whereas y_0 Is Crisp

Let $\lambda\tilde{(t)} = (b_1(t)/b_2(t)/b_3(t))$, where $b_1(t) < b_2(t) < b_3(t)$.

From (20) y_t will be fuzzy for $t \geq 1$ and will be denoted by $\tilde{y_t}$. It is observed that $\tilde{y_1} = (b_1(0)y_0/b_2(0)y_0/b_3(0)y_0)$ is a fuzzy triangular number. So,

$$\tilde{y_1}[\alpha] = [b_1(0)y_0 + (b_2(0) - b_1(0))y_0\alpha, b_3(0)y_0(b_3(0) - b_2(0))y_0\alpha] \tag{22}$$

For $t \geq 2$ we can get an interval estimation solution (represented by α-*cut*) step by step as follows:

$$\tilde{y}^I_{t+1}[\alpha] = [y^I_{(t+1)1}(\alpha), y^I_{(t+1)2}(\alpha)] = [\lambda^t_1(\alpha), \lambda^t_2(\alpha)]\tilde{y}^I_t[\alpha] \tag{23}$$

So by using product of intervals [9] we get, $\tilde{y}^I_2[\alpha] = [b_1(1) + (b_2(1) - b_1(1))\alpha, b_3(1)(b_3(1) - b_2(1))\alpha][b_1(0)y_0 + (b_2(0) - b_1(0))y_0\alpha, b_3(0)y_0(b_3(0) - b_2(0))y_0\alpha]$.

4 CA Models for First Order FDE and FFDE

Here CA modelling have been formulated for dynamical systems represented by one-dimensional first order linear difference equations having time independent coefficients. We will use interval arithmetic solution of the FFDE to design the respective FCA.

4.1 First Order FDE and CA Models

Let a one dimensional linear discrete dynamical system be represented by the FDE (3). Here y_t is the state of the system at time t, and λ is the rate of evolution of the system from one time step to another .
If initially $y(t = 0) = y_0$, then we get

$$y_t = \lambda^t y_0, \quad for\ t = 1, 2, \ldots \tag{24}$$

If the initial phase point is real then all possible phase points are real.

Let the phase points be arranged in the increasing order of their values and be denoted in terms of variable 'x' as follows: For any real valued y_t, $\exists i \in \mathbb{Z}$, such that $y_t = x_i$,

$$y_{t+1} = \begin{cases} x_{i+1} & if \;\; y_t < y_{t+1} \\ x_{i-1} & if \;\; y_{t+1} < y_t \end{cases}$$

Let the phase space of the system be given by $X = \{x_i \mid i \in \mathbb{Z}\}$. The values of the phase points are such that for $i \in \mathbb{Z}$

$$\ldots < x_{i-2} < x_{i-1} < x_i < x_{i+1} < x_{i+2} < \ldots$$

Crisp CA models for FDE (3) have been designed (see [12, 13]) as follows: The ith cell of the CA representing the phase point x_i, is denoted by A_i and the state of the cell A_i at a particular time t is denoted by $A_i(t)$.

$A_i(t)$ is said to be in the ON stage or '1' state represented by a 'black' cell provided at time t the system is at x_i, otherwise it is in the OFF stage or '0' state represented by a 'white' cell in the following figures.

Here $y_0 > 0$ has been considered.

For different values of λ in the given FDE, different homogeneous CA models can be constructed as follows:

- Case 1: $\lambda = 1$
  The dynamical system becomes $y_{t+1} = y_t$. Here, $y_t = y_0, \forall t = 1, 2, \ldots$.
  The only phase point corresponding to y_0 is x_0 and it is represented by cell A_0. Hence we get a null boundaried (denoted by B), 1-celled CA following
  RULE 4$((1 - (c_{i+1} \vee c_{i+1})) \wedge c_i)$ of Wolfram code (Fig. 1a).
- Case 2: $\lambda > 1$
  Here we get a monotonically divergent system as

  $$y_0 < y_1 < y_2 < \cdots < +\infty$$

  Thus the phase points are $x_0, x_1, x_2, \ldots$ and correspondingly we get a countably infinite celled homogeneous CA following
  RULE 16$(c_{i-1} \wedge (1 - c_i) \wedge (1 - c_{i+1}))$ of Wolfram code (Fig. 1b).
- Case 3: $0 < \lambda < 1$
  Here we get a monotonically convergent system as

  $$y_0 > y_1 > y_2 > \cdots > 0$$

  Thus the phase points are $x_0, x_{-1}, x_{-2} \ldots$ and correspondingly we get a countably infinite celled homogeneous CA following
  RULE 2$((1 - c_{i-1}) \wedge (1 - c_i) \wedge c_{i+1})$ of Wolfram code (Fig. 1c).

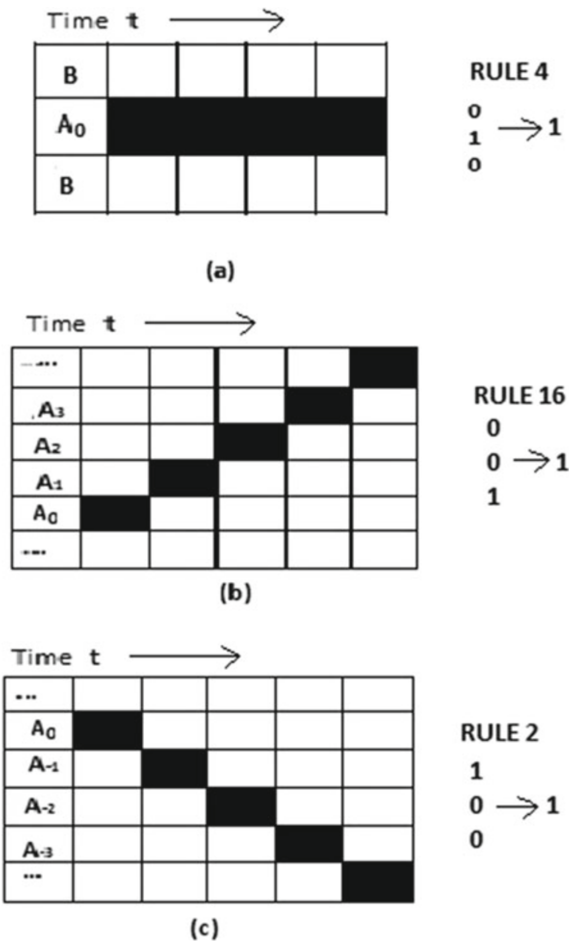

Fig. 1 Crisp CA models For $\lambda > 0$, $y_0 > 0$

4.2 *First Order FFDE and FCA Models*

On fuzzifying the FDE (3) given by, two cases arise.

- Case-I: $\tilde{y}_0$ is a fuzzy triangular number and $\lambda > 0$ is a crisp constant.
  Here, the solution of FFDE (4) will be (5). And, for $\alpha \in (0, 1]$ we get the α-cut solutions of the form (11).
- Case-II: $\tilde{\lambda}$ is fuzzy triangular number and $y_0 > 0$ is a crisp initial value.
  Here, the solution of FFDE (13) will be (14). And, for $\alpha \in (0, 1]$ we get the α-cut solutions similar to the form (19).

If $\tilde{y}_0$ is real then all possible fuzzy phase points $\tilde{y}_t$ are real.

Let the fuzzy phase points be arranged in the increasing order of their values and be denoted in terms of variable $\tilde{x}$ as follows: For any fuzzy triangular number

$\tilde{y}_t$, $\exists i \in \mathbb{Z}$, such that
$\tilde{y}_t = \tilde{x}_i$,
$\tilde{y}_{t+1} = \begin{cases} \tilde{x}_{i+1} & if \;\; \tilde{y}_t < \tilde{y}_{t+1} \\ \tilde{x}_{i-1} & if \;\; \tilde{y}_{t+1} < \tilde{y}_t \end{cases}$
where each $\tilde{x}_i$ is a fuzzy triangular number.

The values of the fuzzy phase points are such that

$$\ldots < \tilde{x}_{i-1} < \tilde{x}_i < \tilde{x}_{i+1} < \ldots$$

Let $\tilde{x}_i = (c_{i1}/c_{i2}/c_{i3})$ where $c_{i1} < c_{i2} < c_{i3}$.
For some $\alpha \in (0, 1]$, α-cut operation on $\tilde{x}_i$ gives

$$\tilde{x}_i[\alpha] = [c_{i1} + (c_{i2} - c_{i1})\alpha, c_{i3} - (c_{i3} - c_{i2})\alpha]$$

Fuzzy CA (FCA) model for an FFDE are as follows: The ith cell of the FCA representing the α-cut $\tilde{x}_i[\alpha]$, is denoted by $\tilde{A}_i$ and the state of the cell $\tilde{A}_i$ at a particular time t is denoted by $\tilde{A}_i(t)$.

$\tilde{A}_i(t)$ is said to be in the ON stage provided at time t the system is within $\tilde{x}_i[\alpha]$ for $\alpha \geq 0.25$, otherwise it is in the OFF stage.

- Case 1: $\lambda = 1$
  The only fuzzy phase point is $\tilde{x}_0$ corresponding to $\tilde{y}_0$ and it is represented by $\tilde{A}_0$. If $\tilde{\lambda} \approx 1$ then also the variations in the values of the fuzzy phase points are negligible.
  Hence for both cases the FCA has transition function $min(1 - c_{i-1}, c_i, 1 - c_{i+1})_{(010)}$ from DNF-fuzzification of Wolfram's **RULE 4** (Fig. 2b).
- Case 2: $\lambda > 1, \tilde{y}_0 > 0$ or $\tilde{\lambda} > 1, y_0 > 0$
  Here for $\lambda > 1, \tilde{y}_0 > 0$ we get,

$$\tilde{y}_0 < \tilde{y}_1 < \tilde{y}_2 < \cdots < +\infty$$

  The corresponding fuzzy phase points are considered to be $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \ldots$ and the FCA will have cells $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2, \ldots$.
  Hence this FCA has transition function $min(c_{i-1}, 1 - c_i, 1 - c_{i+1})_{(100)}$ from DNF-fuzzification of Wolfram's **RULE 16**(Fig. 3b).
  For $\tilde{\lambda} > 1, y_0 > 0$, we get a fuzzy system from $\tilde{y}_1$ onwards such that,

$$\tilde{y}_1 < \tilde{y}_2 < \cdots < +\infty$$

  The corresponding fuzzy phase points are considered to be $\tilde{x}_1, \tilde{x}_2, \ldots$ and the FCA will be similar to (Fig. 3b).
- Case 3: $0 < \lambda < 1, \tilde{y}_0 > 0$ or $0 < \tilde{\lambda} < 1, y_0 > 0$
  Here for $0 < \lambda < 1, \tilde{y}_0 > 0$ we get,

$$\tilde{y}_0 > \tilde{y}_1 > \tilde{y}_2 > \cdots > 0$$

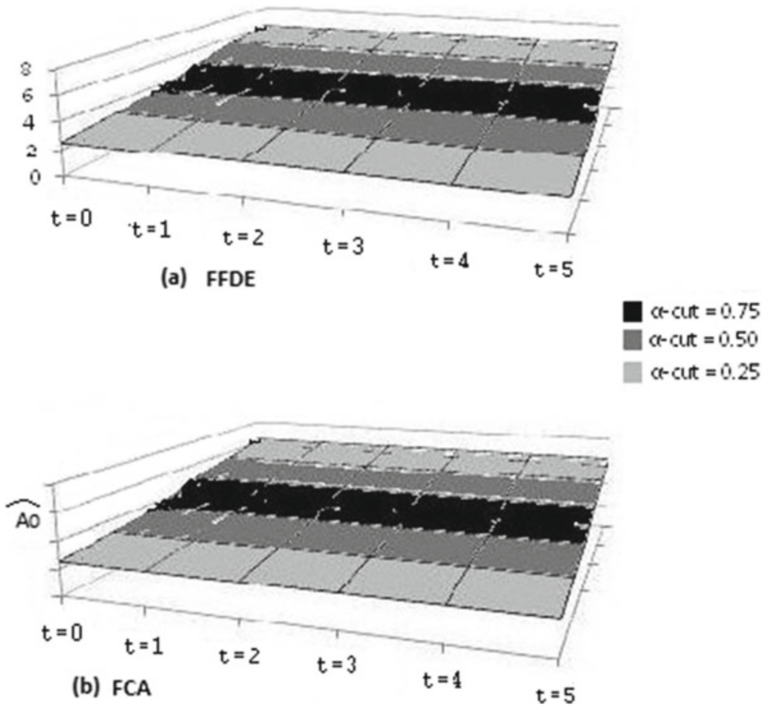

Fig. 2 **a** FFDE, **b** FCA For ($\lambda = 1, \tilde{y}_0 > 0$)

The corresponding fuzzy phase points are considered to be $\tilde{x}_0, \tilde{x}_{-1}, \tilde{x}_{-2}, \ldots$ and the FCA will have cells $\tilde{A}_0, \tilde{A}_{-1}, \tilde{A}_{-2}, \ldots$.
Hence this FCA has transition function $min(1 - c_{i-1}, 1 - c_i, c_{i+1})_{(001)}$ from DNF-fuzzification of Wolfram's **RULE 2**(Fig. 4b).
For $0 < \tilde{\lambda} < 1$, $y_0 > 0$, we get a fuzzy system from $\tilde{y}_1$ onwards such that,

$$\tilde{y}_1 > \tilde{y}_2 > \cdots > 0$$

The corresponding fuzzy phase points are considered to be $\tilde{x}_{-1}, \tilde{x}_{-2}, \tilde{x}_{-3}, \ldots$ and the FCA will be similar to (Fig. 4b).

5 Output and Results

We have made an estimation of the suspected number of virus-infected people having different infection levels, on the basis of the calculated infection rate of the virus.

It is known that the spread of any virus depends on the basic reproduction number (R_0) of the virus which indicates how contagious the virus is. In general, the health condition of a virus-infected person is categorized as critical or mild. For COVID-19

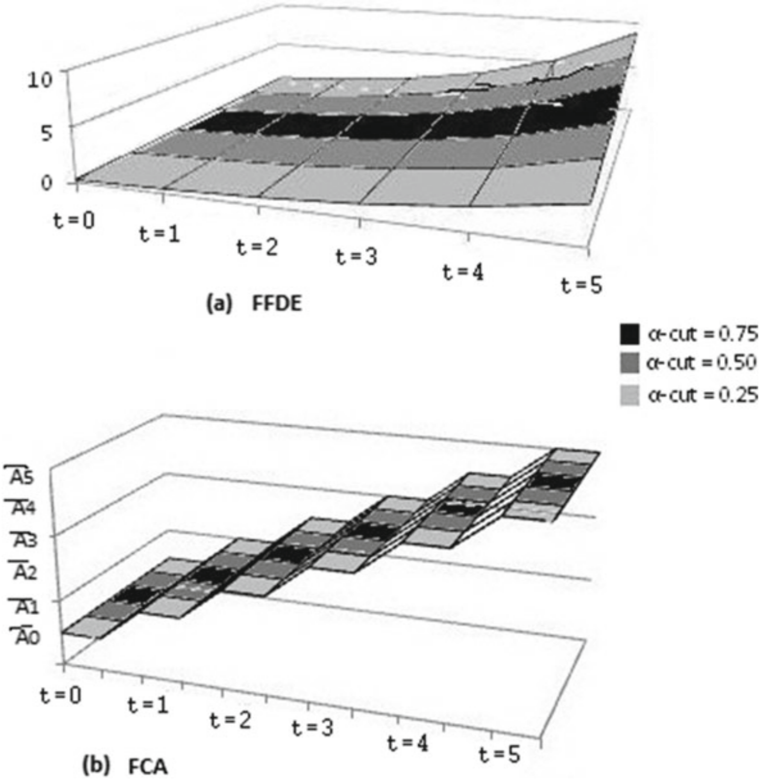

Fig. 3 **a** FFDE, **b** FCA For $(\lambda > 1, \tilde{y_0} > 0)$ Or $(\tilde{\lambda} > 1, y_0 > 0)$

virus, usually around 2% of the active cases have critical conditions while the rest 98% are mild [24]. Among the cases having mild conditions, there can be different levels of infection manifested by different grades of symptoms.

FCA model (which are temporally hybrid) representing the spread of MERS and COVID-19 virus, indicating a gradation of infection, have been designed here.

5.1 Fuzzy Model Representing Growth-Trend of the Number of Virus-Infected People Within a Short Span of Time

The R_0 value of any virus, can vary between different intervals of time which constitute a considerably larger period of time (reported in Sect. 5.2). However, within any short time interval the rise or fall in the number of the virus-infected individuals may apparently seem to occur at a constant rate, say ρ. The virus-infected population

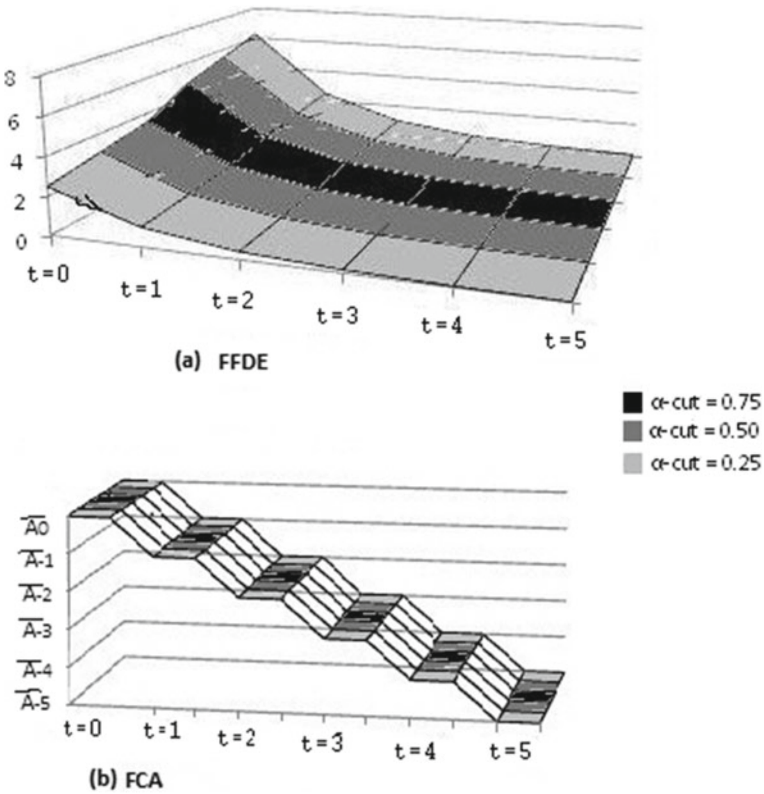

Fig. 4 **a** FFDE, **b** FCA For ($0 < \lambda < 1$, $\tilde{y}_0 > 0$) Or ($0 < \tilde{\lambda} < 1$, $y_0 > 0$)

grows or decays according as $\rho > 1$ or $\rho < 1$, and it almost remains at the same level if ρ is nearly 1.

An FFDE for the growth-trend of number of infected people within a short time, has been obtained by fuzzifying an FDE (of the form 3), given as:

$$y_{t+1} = \rho y_t, \quad \rho > 0, \; t = 0, 1, 2, \ldots \tag{25}$$

where, 'y_t' is the reported number of people being actively infected by the virus at time t. If the initial number of people reported to be actively infected be $y_0 > 0$, then the solution of (25) at time t, will be

$$y_t = \rho^t y_0 \tag{26}$$

- Case-I: $\tilde{y}_0 > 0$ is fuzzy and ρ is crisp.
  The solution of the corresponding FFDE $\forall t \geq 0$ will be $\tilde{y}_t = \rho^t \tilde{y}_0$. For some $\alpha \in (0, 1]$,

$$\tilde{y}_t[\alpha] = \rho^t \tilde{y}_0[\alpha] \tag{27}$$

- Case-II: $y_0 > 0$ is crisp and $\tilde{\rho}$ is fuzzy.
  Then $\tilde{y}_1 = \tilde{\rho} y_0$ and the solution of the corresponding FFDE $\forall t \geq 1$ will be $\tilde{y}_t = \tilde{\rho}^t y_0$. For some $\alpha \in (0, 1]$,

$$\tilde{y}_t[\alpha] = (\tilde{\rho}[\alpha])^t . y_0 \tag{28}$$

Let us suppose $\tilde{y}_t[\alpha]$ represents the suspected number of actively infected people provided $\alpha \geq 0.25$. We further assume that $\tilde{y}_t[\alpha]$ represents

- highly infected population if $\alpha \geq 0.75$
- moderately infected population if $0.50 \leq \alpha < 0.75$
- slightly infected population if $0.25 \leq \alpha < 0.50$.

Among the highly infected people, the cases which become critical can be indicated by $\alpha \geq 0.98$ (in case of COVID-19).

And at $\alpha = 1.0$, the α-cut operation on $\tilde{y}_t$, denoted by $\tilde{y}_t[1]$ coincides with the crisp value y_t (which indicates the reported active cases).

Let R_0 be the basic reproduction rate of a virus from any time interval (T) to the next time interval $(T + 1)$ where each interval has an n-day span. If ρ be the constant growth rate of the number of infected individuals (Y) at time T, for a period of n days, then

$$R_0 Y \approx (\rho)^n Y \Leftrightarrow R_0^{(1/n)} \approx \rho \tag{29}$$

5.1.1 FCA Model Representing COVID-19 Spread for a Short Period

The 2019 Novel Coronavirus (2019-nCoV or COVID-19) also known as Severe Acute Respiratory Syndrome Coronavirus 2 (SARS-CoV-2) was first reported towards the end of 2019 Wuhan city in the Hubei province of China. Around March 2020, COVID-19 emerged to be a world-wide pandemic. Then by the second week of April 2020, the curve for the number active cases in China almost flattened. At that time, in some countries like Germany, the number of active COVID-19 cases started falling since the growth-rate fell below 1. However in countries like India, USA, Singapore, the growth-rate being above 1, the number of active cases was still on the rise.

Here the trend of growth of number of COVID-19 patients for a short period from 11/04/2020 to 15/04/2020 for India, Germany and China has been depicted. The number of active cases y_t, at time t, has been recorded from [24]. The value of ρ has been calculated from (26) for $t = 4$.

(1) **India** having $\rho > 1$ from 11/04/20–15/04/20

The number of active cases on 11/04/2020 was 7189 and on 15/04/2010 was 10,440. Considering the scale of 1000:1, $y_0 = 7.189$ and $y_4 = 10.440$ gives $\rho =$

Table 3 $\tilde{y}_0 = (6.789/7.189/7.589)$; $\rho = 1.098$

Time t	$\tilde{y}_t[0.25]$	$\tilde{y}_t[0.50]$	$\tilde{y}_t[0.75]$
0(11/4)	[6.889,7.489]	[6.989,7.389]	[7.089,7.289]
1(12/4)	[7.564,8.223]	[7.674,8.113]	[7.784,8.003]
2(13/4)	[8.305,9.029]	[8.426,8.908]	[8.547,8.788]
3(14/4)	[9.119,9.914]	[9.252,9.781]	[9.384,9.649]
4(15/4)	[10.013,10.885]	[10.158,10.740]	[10.304,10.594]

1.098. Let $\tilde{y}_0 = (y_0 - 0.4/y_0/y_0 + 0.4)$ and $\tilde{\rho} = (\rho - 0.02/\rho/\rho + 0.02)$ be triangular fuzzy numbers.

Case-I: Let $\tilde{y}_0 = (6.789/7.189/7.589)$ be fuzzy and $\rho = 1.098$ be a crisp constant. Thus $\tilde{y}_0[\alpha] = [6.789 + 0.4\alpha, 7.589 - 0.4\alpha]$ and $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (27). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 3. Here, $\tilde{y}_0 < \tilde{y}_1 < \tilde{y}_2 < \tilde{y}_3 < \tilde{y}_4$.

Thus the fuzzy phase points are $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_4$. The corresponding FCA will have cells $\tilde{A}_0, \ldots, \tilde{A}_4$ and the ON states of these cells will be $\tilde{A}_0(0)$, $\tilde{A}_1(1)$, $\tilde{A}_2(2)$, $\tilde{A}_3(3)$, $\tilde{A}_4(4)$. This FCA will be as shown in (Fig. 5a).

Case-II: Let $y_0 = 7.189$ be crisp and $\tilde{\rho} = (1.078/1.098/1.118)$ be fuzzy. Thus $\tilde{\rho}[\alpha] = [1.078 + 0.02\alpha, 1.118 - 0.02\alpha]$ and $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (28). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 4. Here $\tilde{y}_1 < \tilde{y}_2 < \tilde{y}_3 < \tilde{y}_4$.

The corresponding FCA will be similar to as shown in (Fig. 5a).

(2) **Germany** having $\rho < 1$ from 11/04/20–15/04/20

The number of active cases on 11/04/2020 was 65181 and on 15/04/2020 was 58349. Considering the scale of 1000:1, $y_0 = 65.181$ and $y_4 = 58.349$ gives $\rho = 0.973$. Let $\tilde{y}_0 = (y_0 - 0.4/y_0/y_0 + 0.4)$ and $\tilde{\rho} = (\rho - 0.01/\rho/\rho + 0.01)$ be triangular fuzzy numbers.

Case-I: Let $\tilde{y}_0 = (64.781/65.181/65.581)$ and $\rho = 0.973$ be a crisp constant. Thus $\tilde{y}_0[\alpha] = [64.781 + 0.4\alpha, 65.581 - 0.4\alpha]$ and $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from *(27)*. The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in *Table* 5. Here, $\tilde{y}_0 > \tilde{y}_1 > \tilde{y}_2 > \tilde{y}_3 > \tilde{y}_4$

Thus the fuzzy phase points are $\tilde{x}_0, \tilde{x}_{-1}, \ldots, \tilde{x}_{-4}$. The corresponding FCA will have cells $\tilde{A}_0, \ldots, \tilde{A}_{-4}$ and the ON states of these cells will be $\tilde{A}_0(0)$, $\tilde{A}_{-1}(1)$, $\tilde{A}_{-2}(2)$, $\tilde{A}_{-3}(3)$, $\tilde{A}_{-4}(4)$. This FCA will be as shown in (Fig. 5b).

Case-II: Let $y_0 = 65.181$ be crisp and $\tilde{\rho} = (0.963/0.973/0.983)$ be fuzzy.

Thus $\tilde{\rho}[\alpha] = [0.963 + 0.01\alpha, 0.983 - 0.01\alpha]$ and $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (28). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 6. Here, $\tilde{y}_1 > \tilde{y}_2 > \tilde{y}_3 > \tilde{y}_4$.

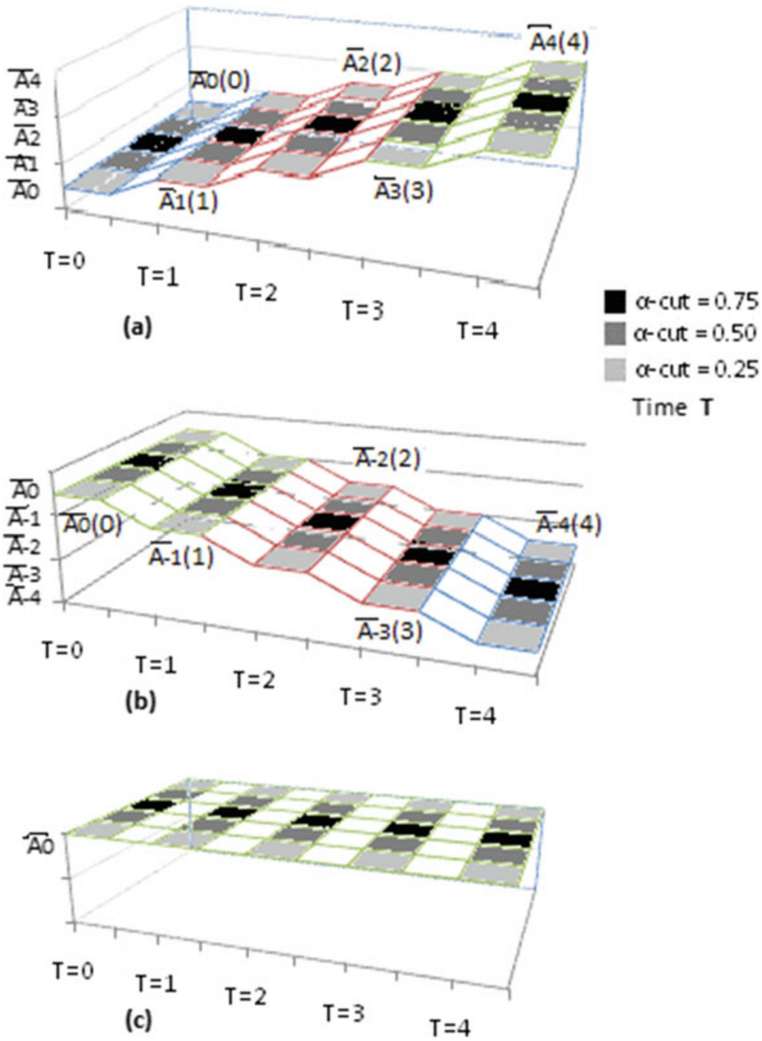

Fig. 5 FCA for short period COVID-19 growth-trend from 11/04/2020 To 15/04/2020 in **a** India With $\rho > 1$, **b** Germany With $\rho < 1$, **c** China With $\rho \approx 1$

The corresponding FCA will be similar to as shown in (Fig. 5b).

(3) **China** having $\rho \approx 1$ from 11/04/20–15/04/20

The number of active cases on 11/04/2020 was 1138 and on 15/04/2020 was 1107. Considering the scale of 1000:1, $y_0 = 1.138$ and $y_4 = 1.107$ gives $\rho \approx 1$. Let $\tilde{y}_0$ and $\tilde{\rho} = (\rho - 0.03/\rho/\rho + 0.03)$ be triangular fuzzy numbers.

Case-I: If $\tilde{y}_0$ be fuzzy and $\rho = 1.0$ be a crisp constant, then $\forall t,\ \tilde{y}_t = \tilde{y}_0$.

The only fuzzy point corresponding to $\tilde{y}_0$ is $\tilde{x}_0$.

Table 4 $y_0 = 7.189$; $\tilde{\rho} = (1.078/1.098/1.118)$

	$\tilde{\rho}[0.25]$= [1.083,1.113]	$\tilde{\rho}[0.50]$= [1.088,1.108]	$\tilde{\rho}[0.75]$= [1.093,1.103]
Time t	$\tilde{y}_t[0.25]$	$\tilde{y}_t[0.50]$	$\tilde{y}_t[0.75]$
1(12/4)	[7.786,8.001]	[7.822,7.965]	[7.858,7.929]
2(13/4)	[8.432,8.906]	[8.510,8.826]	[8.588,8.746]
3(14/4)	[9.132,9.912]	[9.259,9.779]	[9.387,9.647]
4(15/4)	[9.890,11.032]	[10.074,10.835]	[10.260,10.641]

Table 5 $\tilde{y}_0 = (64.781/65.181/65.581)$; $\rho = 0.973$

Time t	$\tilde{y}_t[0.25]$	$\tilde{y}_t[0.50]$	$\tilde{y}_t[0.75]$
0(11/4)	[64.881,65.481]	[64.981,65.381]	[65.081,65.281]
1(12/4)	[63.129,63.713]	[63.227,63.616]	[63.324,63.518]
2(13/4)	[61.425,61.993]	[61.519,61.898]	[61.614,61.803]
3(14/4)	[59.766,60.319]	[59.858,41.657]	[59.950,60.135]
4(15/4)	[58.153,58.690]	[58.242,40.533]	[58.332,58.511]

Table 6 $y_0 = 65.181$; $\tilde{\rho} = (0.963/0.973/0.983)$

	$\tilde{\rho}[0.25]$ = [0.965,0.980]	$\tilde{\rho}[0.50]$ = [0.968,0.978]	$\tilde{\rho}[0.75]$ = [0.970,0.975]
Time t	$\tilde{y}_t[0.25]$	$\tilde{y}_t[0.50]$	$\tilde{y}_t[0.75]$
1(12/4)	[62.90,63.877]	[63.095,63.747]	[63.226,63.551]
2(13/4)	[60.698,62.60]	[61.076,62.345]	[61.329,61.963]
3(14/4)	[58.574,61.348]	[59.122,60.973]	[59.489,60.414]
4(15/4)	[56.524,60.121]	[57.230,59.632]	[57.704,58.903]

Case-II: Let $y_0 = 1.138$ be crisp and $\tilde{\rho} = (0.97/1.0/1.03)$ be fuzzy. Thus $\tilde{\rho}[\alpha] = [0.97 + 0.03\alpha, 1.03 - 0.03\alpha]$ and $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (28). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 7. Here the variations in the values of $\tilde{y}_t$ in the stipulated period are negligible. The corresponding FCA for both cases will have only one cell $\tilde{A}_0$ as shown in (Fig. 5c).

5.2 *Fuzzy Model Representing Spread of Virus over a Considerably Large Period of Time*

It is known that the number of people infected by the virus grows or falls according as $R_0 > 1$ or $R_0 < 1$.

Table 7 $y_0 = 1.138$; $\tilde{\rho} = (0.97/1.0/1.03)$

	$\tilde{\rho}[0.25] =$ [0.977,1.022]	$\tilde{\rho}[0.50] =$ [0.985,1.015]	$\tilde{\rho}[0.75] =$ [0.992,1.007]
Time t	$\tilde{y}_t[0.25]$	$\tilde{y}_t[0.50]$	$\tilde{y}_t[0.75]$
1(12/4)	[1.112,1.163]	[1.121,1.155]	[1.129,1.146]
2(13/4)	[1.086,1.187]	[1.104,1.172]	[1.120,1.154]
3(14/4)	[1.061,1.215]	[1.088,1.190]	[1.111,1.162]
4(15/4)	[1.037,1.241]	[1.071,1.208]	[1.102,1.170]

An FFDE representing spread of virus population over a period of time, has been obtained by fuzzifying the FDE (of the form 20) given as:

$$y_{t+1} = R_{0(t)} y_t, \quad R_{0(t)} > 0, \; t = 0, 1, 2, \ldots \tag{30}$$

where, 'y_t' is the reported number of people being actively infected by the virus at time t, and '$R_{0(t)}$' is the basic reproduction number of the virus at time t. If the initial number of people tested to be actively infected be $y_0 > 0$, the following two cases are considered:

- Case-I: $\tilde{y}_0 > 0$ is fuzzy and $\forall t \geq 0$, $R_{0(t)}$ is crisp.
  The FFDE $\forall t \geq 0$ will be $\tilde{y}_{t+1} = R_{0(t)}\tilde{y}_t$ and for some $\alpha \in (0, 1]$,

$$\tilde{y}_{t+1}[\alpha] = R_{0(t)}\tilde{y}_t[\alpha] \tag{31}$$

- Case-II: $y_0 > 0$ is crisp and $\forall t \geq 0$, $\tilde{R}_{0(t)}$ is fuzzy.
  Then $\tilde{y}_1 = \tilde{R}_{0(0)} y_0$ and the FFDE $\forall t \geq 1$ will be $\tilde{y}_{t+1} = \tilde{R}_{0(t)}\tilde{y}_t$.
  For some $\alpha \in (0, 1]$,

$$\tilde{y}_1[\alpha] = (\tilde{R}_{0(t)}[\alpha]) y_0 \; ; \quad \forall t \geq 1, \tilde{y}_{t+1}[\alpha] = \tilde{R}_{0(t)}[\alpha].\tilde{y}_t[\alpha] \tag{32}$$

  $\tilde{R}_{0(t)}[\alpha].\tilde{y}_t[\alpha]$ has been computed using Definition (2.11).

Here $\tilde{y}_t[\alpha]$ represents the suspected number of actively infected people provided $\alpha \geq 0.25$ such that they are assumed to be as given in Sect. (5.1).

Through these FFDE we depict the spread of MERS virus and COVID-19 virus and hence design their corresponding FCA which are temporally hybrid.

5.2.1 FCA Model Representing Spread of MERS Virus

Middle East Respiratory Syndrome Coronavirus(MERS-CoV or MERS) was first reported in 2012 in Saudi Arabia. A significant outbreak of MERS have been observed in Saudi Arabia and South Korea in and around 2015. Though, at the onset of these

Table 8 $\tilde{y}_0 = (0.3/1/1.7)$; $R_{0(t)} = \frac{y_{t+1}}{y_t}$ be crisp with $R_{0(0)} = 5.7$

Time t	y_t	$R_{0(t)}$	$\tilde{y}_t[0.25]$	$\tilde{y}_t[0.50]$	$\tilde{y}_t[0.75]$
0	1	5.7	[0.47, 1.52]	[0.65, 1.35]	[0.82, 1.17]
1	5.7	3.5	[2.71, 8.69]	[3.70, 7.69]	[4.70, 6.70]
2	20	1.5	[9.47, 30.42]	[12.97, 26.93]	[16.46, 23.44]
3	30	2.5	[14.21, 45.63]	[19.45, 40.40]	[24.68, 35.16]
4	75	0.5	[35.53, 114.08]	[48.64, 100.99]	[61.71, 87.89]
5	37	0.3	[17.76, 57.04]	[24.32, 50.50]	[30.86, 43.95]
6	11	0.55	[5.33, 17.11]	[7.29, 15.15]	[9.26, 13.18]
Day 50	6	–	[2.93, 9.41]	[4.01, 8.33]	[5.09, 7.25]

outbreaks, the reproduction number ranged from 1.0 to 5.7, it dropped below 1 within 2–6 weeks [4, 23]. Out of 2449 total reported cases of MERS there has been 845 deaths [23] which is around 34% of the total cases.

Here the dynamics of the MERS virus for South Korea and Riyadh (Saudi Arabia) for a period of 50 days in 2015 have been analysed where a 1-week period is considered as a unit time step [4].

The number of cases y_t at time t, has been recorded from [4]. If the initial cases be y_0, then let the fuzzy initial value be $\tilde{y}_0 = (0.3y_0/y_0/1.7y_0)$.

Initially, $R_0(0)$ has been used as given in (see [4]) and its given value range corresponds to the limits of the considered fuzzy $\tilde{R}_{0(0)}$. For any $t \geq 1$, the crisp $R_{0(t)}$ values have been calculated from (30).

(1) **South Korea** from May 11, 2015–July 22, 2015

Let the initial number of cases be $y_0 = 1$. Also, $\tilde{R}_{0(0)} = (3.0/5.7/9.0)$ and $\forall t \geq 1$, $\tilde{R}_{0(t)} = (0.52R_{0(t)}/R_{0(t)}/1.6R_{0(t)})$ has been considered.
Case-I: Let $\tilde{y}_0 = (0.3/1/1.7)$ and $R_{0(t)}$ be crisp with $R_{0(0)} = 5.7$
Thus $\tilde{y}_0[\alpha] = [0.3 + 0.7\alpha, 1.7 - 0.7\alpha]$ and $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (31). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 8.

Here,

$$\tilde{y}_0 < \tilde{y}_1 < \tilde{y}_7 < \tilde{y}_6 < \tilde{y}_2 < \tilde{y}_3 < \tilde{y}_5 < \tilde{y}_4.$$

Thus the fuzzy phase points are $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_7$. The corresponding FCA will have cells $\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_7$ and the ON states of these cells will be $\tilde{A}_0(0)$, $\tilde{A}_1(1)$, $\tilde{A}_2(7)$, $\tilde{A}_3(6)$, $\tilde{A}_4(2)$, $\tilde{A}_5(3)$, $\tilde{A}_6(5)$, $\tilde{A}_7(4)$ as shown in (Fig. 6a).

Case-II: Let $y_0 = 1$ be crisp and $\tilde{R}_{0(t)} = (0.52R_{(t)}/R_{0(t)}/1.6R_{0(t)})$ be triangular fuzzy numbers with $\tilde{R}_{0(0)} = (3.0/5.7/9.0)$. Thus $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (32). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 9. Since the pattern

Table 9 $y_0 = 1$; $\tilde{R}_{0(t)} = (0.52R_{0(t)}/R_{0(t)}/1.6R_{0(t)})$

Time t	$\tilde{R}_{0(t)}$	$\tilde{R}_{0(t)}[0.25].\tilde{y}_t[0.25]$	$\tilde{R}_{0(t)}[0.50].\tilde{y}_t[0.50]$	$\tilde{R}_{0(t)}[0.75].\tilde{y}_t[0.75]$
0	(3.0/5.7/9.0)	[3.675, 8.175] × 1	[4.35, 7.35]× 1	[5.025, 6.525]× 1
1	(1.8/3.5/5.6)	[2.225, 5.075] [3.7, 8.2]	[2.65, 4.55] [4.3, 7.3]	[3.075, 4.025] [5.0, 6.5]
2	(0.8/1.5/2.4)	[0.975, 2.175] [8.2, 41.6]	[1.15, 1.95] [11.4, 33.2]	[1.325, 1.725] [15.4, 26.2]
3	(1.3/2.5/4.0)	[1.6,3.625] [8.0, 90.5]	[1.9, 3.25] [13.1, 64.7]	[2.2, 2.875] [20.4, 45.2]
4	(0.3/0.5/0.8)	[0.35, 0.725] [12.8, 328.1]	[0.4, 0.65] [24.9, 210.3]	[0.45, 0.575] [44.9, 130.0]
5	(0.2/0.3/0.5)	[0.225, 0.45] [4.5, 237.8]	[0.25, 0.4] [10.0, 136.7]	[0.275, 0.35] [20.2, 74.7]
6	(0.3/0.55/0.9)	[0.362, 0.812] [1.0, 107.0]	[0.425, 0.725] [2.5, 54.7]	[0.487, 0.637] [5.6, 26.1]
Day 50	–	$\tilde{y}_7[0.25] =$ [0.4, 86.9]	$\tilde{y}_7[0.50] =$ [1.1, 39.7]	$\tilde{y}_7[0.75] =$ [2.7, 16.6]

of the α-cut values of the fuzzy phase points from Tables 8 and 9 are similar, the corresponding FCA will be similar to as shown in (Fig. 6a).

(2) **Riyadh** from July 13, 2015–August 31, 2015

Let the initial number of cases be $y_0 = 2$. Also, $\tilde{R}_{0(0)} = (2.0/2.9/5.0)$ and $\forall t \geq 1$, $\tilde{R}_{0(t)} = (0.7R_{0(t)}/R_{0(t)}/1.7R_{0(t)})$ has been considered.
Case-I: Let $\tilde{y}_0 = (0.6/2/3.4)$ and $R_{0(t)}$ be crisp with $R_{0(0)} = 2.9$
Thus $\tilde{y}_0[\alpha] = [0.6 + 1.4\alpha, 3.4 - 1.4\alpha]$ and $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (31). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 10.

Here,

$$\tilde{y}_0 < \tilde{y}_7 < \tilde{y}_1 < \tilde{y}_2 < \tilde{y}_6 < \tilde{y}_3 < \tilde{y}_4 \approx \tilde{y}_5$$

Thus the fuzzy phase points are $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_6$. The corresponding FCA will have cells $\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_6$ and the ON states of these cells will be $\tilde{A}_0(0)$, $\tilde{A}_1(7)$, $\tilde{A}_2(1)$, $\tilde{A}_3(2)$, $\tilde{A}_4(6)$, $\tilde{A}_5(3)$, $\tilde{A}_6(4)$, $\tilde{A}_6(5)$ as shown in (Fig. 6b).
Case-II: Let $y_0 = 2$ be crisp and $\tilde{R}_{0(t)} = (0.7R_{0(t)}/R_{0(t)}/1.7R_{0(t)})$ be triangular fuzzy numbers with $\tilde{R}_{0(0)} = (2.0/2.9/5.0)$. Thus $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (32). The pattern of values of asymslightly, moderately and highly infected active cases during the stipulated time period has been given in Table 11. Since the pattern of the α-cut values of the fuzzy phase points from Tables 10 and 11 are similar, the corresponding FCA will be similar to as shown in (Fig. 6b).

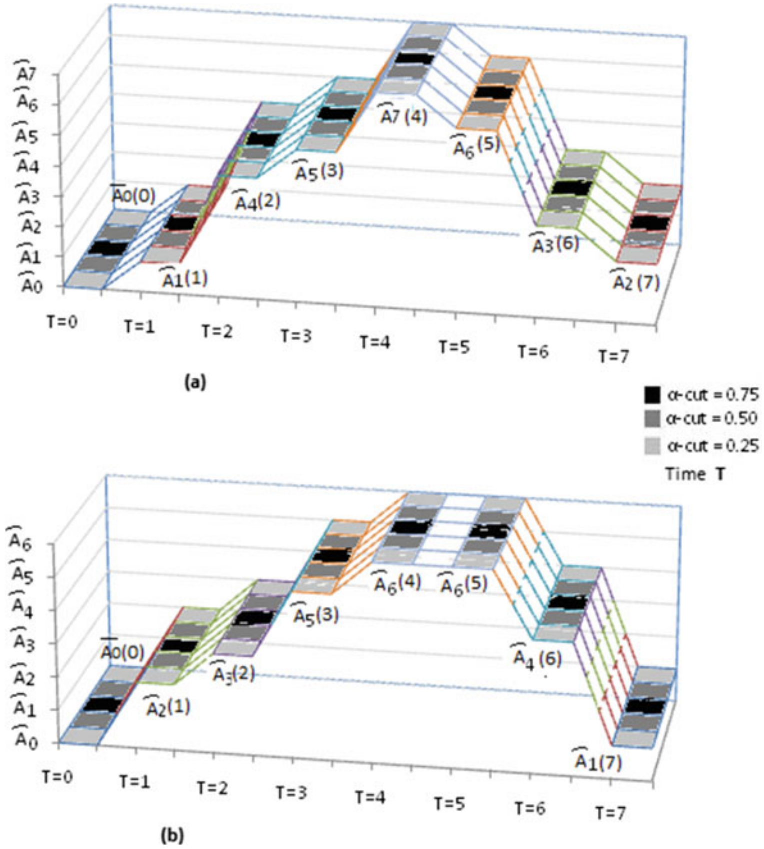

Fig. 6 FCA for spread of MERS during 2015 in **a** South Korea, **b** Riyadh (Saudi Arabia)

Table 10 $\tilde{y}_0 = (0.6/2/3.4)$; $R_{0(t)} = \frac{y_{t+1}}{y_t}$ be crisp with $R_{0(0)} = 2.9$

Time t	y_t	$R_{0(t)}$	$\tilde{y}_t[0.25]$	$\tilde{y}_t[0.50]$	$\tilde{y}_t[0.75]$
0	2	2.9	[0.95, 3.05]	[1.3, 2.7]	[1.65, 2.35]
1	5.8	1.4	[2.78, 8.84]	[3.77, 7.83]	[4.78, 6.81]
2	8	2.5	[3.89, 12.38]	[5.29, 10.96]	[6.69, 9.53]
3	20	1.4	[9.72, 30.95]	[13.22, 27.4]	[16.72, 23.82]
4	28	1.0	[13.61, 43.33]	[18.51, 38.36]	[23.41, 33.35]
5	28	0.54	[13.61, 43.33]	[18.51, 38.36]	[23.41, 33.35]
6	15	0.2	[7.35, 23.40]	[10.0, 20.71]	[12.64, 18.01]
Day 50	3	–	[1.47, 4.68]	[2.0, 4.14]	[2.53, 3.6]

Table 11 $y_0 = 2$; $\tilde{R}_{0(t)} = (0.7R_{0(t)}/R_{0(t)}/1.7R_{0(t)})$

Time t	$\tilde{R}_{0(t)}$	$\tilde{R}_{0(t)}[0.25].\tilde{y}_t[0.25]$	$\tilde{R}_{0(t)}[0.50].\tilde{y}_t[0.50]$	$\tilde{R}_{0(t)}[0.75].\tilde{y}_t[0.75]$
0	(2.0/2.9/5.0)	[2.225, 4.475] × 2	[2.43, 3.95]× 2	[2.675, 3.425]× 2
1	(1.0/1.4/2.4)	[1.1, 2.15] [4.4, 8.9]	[1.2, 1.9] [4.9, 7.9]	[1.3, 1.65] [5.3, 6.8]
2	(1.8/2.5/4.3)	[1.975, 3.85] [4.8, 19.1]	[2.15, 3.4] [5.9, 15.0]	[2.325, 2.95] [6.9, 11.2]
3	(1.0/1.4/2.4)	[1.1, 2.15] [9.4, 73.5]	[1.2, 1.9] [12.7, 51.0]	[1.3, 1.65] [16.0, 33.0]
4	(0.7/1.0/1.7)	[0.775, 1.525] [10.3, 158.0]	[0.85, 1.35] [15.2, 96.9]	[0.925, 1.175] [20.8, 54.4]
5	(0.4/0.54/0.9)	[0.435, 0.81] [8.0, 241.0]	[0.47, 0.72] [12.9, 130.8]	[0.505, 0.63] [19.2, 63.9]
6	(0.1/0.2/0.3)	[0.125, 0.275] [3.4, 195.2]	[0.15, 0.25] [6.1, 94.2]	[0.175, 0.225] [9.7, 40.3]
Day 50	–	$\tilde{y}_7[0.25] =$ [0.4, 53.7]	$\tilde{y}_7[0.50] =$ [0.9, 23.5]	$\tilde{y}_7[0.75] =$ [1.7, 9.1]

5.2.2 FCA Model Representing Spread of COVID-19 Virus

COVID-19 which emerged at China's Wuhan towards the end of 2019, became a pandemic around March 2020. Though more that 5.6 million people have been affected till recently, the global reproduction number which was around 1.66 in March has dropped below 1.1 by the beginning of May.

Here the dynamics (according to [24]) of the active cases of COVID-19 virus for India, USA and Germany for an entire period of 49 days(referred to as large period) from March 22, 2020 to May 09, 2020, has been analysed where a 4-day gap(referred to as short period) is considered as a unit time step. The number of active cases y_t at time t has been recorded from [24] and the $R_{0(t)}$ value at each time step t has been calculated from (30).

Again, from (29), we get, $R_{0(t)} \approx \rho^4$.

(1) India on 22/03/2020, had 365 active cases and 31 closed cases out of total reported 396 cases. On 09/05/2020 there were 41,406 active and 21,402 closed out of 62,808 cases[24]. The average percentage of closed cases during this period is found to be nearly 20%. Considering the scale of 1000:1, the initial cases are $y_0 \approx 0.36$. Here, $\tilde{y}_0 = (0.2y_0/y_0/1.8y_0)$ and $\tilde{R}_{0(t)} = (R_{0(t)} - 0.02/R_{0(t)}/R_{0(t)} + 0.02)$ have been considered.

Case-I: Let $\tilde{y}_0 = (0.07/0.36/0.65)$ be fuzzy and $R_{0(t)}$ be crisp with $R_{0(0)} = 1.83$. Thus $\tilde{y}_0[\alpha] = [0.07 + 0.29\alpha, 0.65 - 0.29\alpha]$ and $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (31). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 12. Here,

Table 12 $\tilde{y}_0 = (0.07/0.36/0.65)$; $R_{0(t)} \frac{y_{t+1}}{y_t}$ is crisp with $R_{0(0)} = 1.83$

Time t	y_t	$R_{0(t)}$	$\tilde{y}_t[0.25]$	$\tilde{y}_t[0.50]$	$\tilde{y}_t[0.75]$
0(22/03)	0.36	1.83	[0.142, 0.577]	[0.215, 0.505]	[0.287, 0.432]
1(26/03)	0.66	1.69	[0.260, 1.056]	[0.393, 0.924]	[0.525, 0.791]
2(30/03)	1.12	2.48	[0.439, 1.785]	[0.664, 1.562]	[0.887, 1.337]
3(03/04)	2.78	1.70	[1.089, 4.427]	[1.647, 3.874]	[2.20, 3.316]
4(07/04)	4.72	1.52	[1.851, 7.526]	[2.80, 6.586]	[3.74, 5.637]
5(11/04)	7.19	1.45	[2.814, 11.44]	[4.256, 10.011]	[5.685, 8.568]
6(15/04)	10.44	1.36	[4.08, 16.588]	[6.171, 14.516]	[8.243, 12.424]
7(19/04)	14.20	1.22	[5.549, 22.56]	[8.393, 19.742]	[11.210, 16.897]
8(23/04)	17.31	1.23	[6.77, 27.523]	[10.239, 24.085]	[13.676, 20.614]
9(27/04)	21.37	1.22	[8.327, 33.853]	[12.594, 29.625]	[16.821, 25.355]
10(01/05)	26.03	1.29	[10.159, 41.301]	[15.365, 36.142]	[20.522, 30.933]
11(05/05)	33.57	1.23	[13.105, 53.278]	[19.821, 46.623]	[26.473, 39.904]
12(09/05)	41.41	–	[16.120, 65.532]	[24.38, 57.346]	[32.562, 49.082]

$$\tilde{y}_0 < \tilde{y}_1 < \cdots < \tilde{y}_{12}$$

Thus the fuzzy phase points are $\tilde{x}_0$, $\tilde{x}_1$, …, $\tilde{x}_{12}$. The corresponding FCA will have cells $\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_{12}$ and the ON states of these cells will be $\tilde{A}_0(0), \tilde{A}_1(1), \ldots, \tilde{A}_{12}(12)$ as shown in (Fig. 7a).

Case-II: Let $y_0 = 0.36$ be crisp and $\tilde{R}_{0(t)} = (R_{0(t)} - 0.02/R_{0(t)}/R_{0(t)} + 0.02)$ be fuzzy triangular numbers with $\tilde{R}_{0(0)} = (1.81/1.83/1.85)$. Thus $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (32). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 13. Since the pattern of the α-cut values of the fuzzy phase points from Tables 12 and 13 are similar, the corresponding FCA will be similar to as shown in (Fig. 7a).

(2) USA on 22/03/2020, had 33,150 active cases and 690 closed cases out of total reported 33,840 cases. On 09/05/2020 there were 1,015,164 active and 332,145 closed out of 1,347,309 cases [24]. The average percentage of closed cases during this period is found to be nearly 13%. Considering the scale of 1000:1, the initial cases are $y_0 \approx 33.15$. Here, $\tilde{y}_0 = (0.13y_0/y_0/0.87y_0)$ and $\tilde{R}_{0(t)} = (R_{0(t)} - 0.04/R_{0(t)}/R_{0(t)} + 0.04)$ have been considered.

Table 13 $y_0 = 0.36$; $\tilde{R}_{0(t)} = (R_{0(t)} - 0.02/R_{0(t)}/R_{0(t)} + 0.02)$

Time t	$\tilde{R}_{0(t)}$	$\tilde{R}_{0(t)}[0.25].\tilde{y}_t[0.25]$	$\tilde{R}_{0(t)}[0.50].\tilde{y}_t[0.50]$	$\tilde{R}_{0(t)}[0.75].\tilde{y}_t[0.75]$
0(22/03)	(1.81/1.83/1.85)	[1.815, 1.845]× 0.36	[1.82, 1.84]× 0.36	[1.825, 1.835]×0.36
1(26/03)	(1.67/1.69/1.71)	[1.675, 1.705] [0.653, 0.664]	[1.68, 1.70] [0.655, 0.662]	[1.685, 1.695] [0.657, 0.661]
2(30/03)	(2.46/2.48/2.50)	[2.465, 2.495] [1.094, 1.132]	[2.47, 2.49] [1.10, 1.125]	[2.475, 2.485] [1.107, 1.120]
3(03/04)	(1.68/1.70/1.72)	[1.685, 1.715] [2.698, 2.826]	[1.690, 1.710] [2.717, 2.801]	[1.695, 1.705] [2.740, 2.783]
4(07/04)	(1.50/1.52/1.54)	[1.505, 1.535] [4.546, 4.846]	[1.510, 1.530] [4.592, 4.79]	[1.515, 1.525] [4.644, 4.745]
5(11/04)	(1.43/1.45/1.47)	[1.435, 1.465] [6.841, 7.438]	[1.44, 1.460] [6.934, 7.329]	[1.445, 1.455] [7.036, 7.236]
6(15/04)	(1.34/1.36/1.38)	[1.345, 1.375] [9.817, 10.897]	[1.350, 1.37] [9.985, 10.70]	[1.355, 1.365] [10.167, 10.528]
7(19/04)	(1.20/1.22/1.24)	[1.205, 1.235] [13.204, 14.984]	[1.210, 1.230] [13.480, 14.659]	[1.215, 1.225] [13.776, 14.371]
8(23/04)	(1.21/1.23/1.25)	[1.215, 1.245] [15.911, 18.505]	[1.22, 1.24] [16.311, 18.031]	[1.225, 1 .235] [16.738, 17.604]
9(27/04)	(1.20/1.22/1.24)	[1.205, 1.235] [19.332, 23.039]	[1.210, 1.23] [19.899, 22.358]	[1.215, 1.225] [20.504, 21.741]
10(01/05)	(1.27/1.29/1.31)	[1.275, 1.305] [23.295, 28.453]	[1.28, 1.30] [24.078, 27.50]	[1.285, 1.295] [24.912, 26.633]
11(05/05)	(1.21/1.23/1.25)	[1.215, 1.245] [29.701, 37.131]	[1.22, 1.24] [30.820, 35.750]	[1.225, 1.235] [32.012, 34.490]
12(09/05)	–	$\tilde{y}_{12}[0.25] =$ [36.087,46.228]	$\tilde{y}_{12}[0.50] =$ [37.60,44.33]	$\tilde{y}_{12}[0.75] =$ [39.215,42.595]

Case-I: Let $\tilde{y}_0 = (4.3/33.15/62.0)$ be fuzzy and $R_{0(t)}$ be crisp with $R_{0(0)} = 2.5$. Thus $\tilde{y}_0[\alpha] = [4.3 + 28.85\alpha, 62.0 - 28.85\alpha]$ and $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (31). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 14. Here,

$$\tilde{y}_0 < \tilde{y}_1 < \cdots < \tilde{y}_{12}$$

Thus the fuzzy phase points are $\tilde{x}_0$, $\tilde{x}_1$, …, $\tilde{x}_{12}$. The corresponding FCA will have cells $\tilde{A}_0$, $\tilde{A}_1$, …, $\tilde{A}_{12}$ and the ON states of these cells will be $\tilde{A}_0(0)$, $\tilde{A}_1(1)$,…,$\tilde{A}_{12}(12)$ as shown in (Fig. 7a).

Case-II: Let $y_0 = 33.15$ be crisp and $\tilde{R}_{0(t)} = (R_{0(t)} - 0.04/R_{0(t)}/R_{0(t)} + 0.04)$ be triangular fuzzy numbers with $\tilde{R}_{0(0)} = (2.46/2.5/2.54)$. Thus $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from *(32)*. The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 15. Since the pattern of the α-cut values of the fuzzy phase points of Tables 14 and 15 are similar,

Table 14 $\tilde{y}_0 = (4.3/33.15/62.0)$; $R_{0(t)} = \frac{y_{t+1}}{y_t}$ is crisp with $R_{0(0)} = 2.5$

Time t	y_t	$R_{0(t)}$	$\tilde{y}_t[0.25]$	$\tilde{y}_t[0.50]$	$\tilde{y}_t[0.75]$
0(22/03)	33.15	2.5	[11.512, 54.787]	[18.725, 47.575]	[25.937, 40.362]
1(26/03)	82.87	1.91	[28.78,136.967]	[46.812, 118.937]	[64.842, 100.905]
2(30/03)	158.56	1.65	[54.97, 261.607]	[89.411, 227.170]	[123.848, 192.729]
3(03/04)	262.26	1.42	[90.7, 431.652]	[147.528, 374.830]	[204.349, 318.003]
4(07/04)	371.82	1.31	[128.794, 612.946]	[209.49, 532.259]	[290.176, 451.564]
5(11/04)	485.43	1.18	[168.720, 802.959]	[274.432, 697.259]	[380.131, 591.549]
6(15/04)	571.06	1.15	[199.09, 947.492]	[323.83, 822.766]	[448.555, 698.028]
7(19/04)	656.75	1.13	[228.953, 1089.616]	[372.404, 946.181]	[515.838, 802.732]
8(23/04)	745.15	1.09	[258.717, 1231.266]	[420.817, 1069.185]	[582.897, 907.087]
9(27/04)	808.52	1.11	[282.0, 1342.08]	[458.691, 1165.412]	[635.358, 988.725]
10(01/05)	895.11	1.07	[313.02, 1489.709]	[509.147, 1293.607]	[705.247, 1097.485]
11(05/05)	953.56	1.06	[334.931, 1593.989]	[544.787, 1384.159]	[754.614, 1174.309]
12(09/05)	1015.16	–	[355.027, 1689.628]	[577.474, 1467.209]	[799.891, 1244.768]

the corresponding FCA will be similar to as shown in (Fig. 7a).

(3) Germany on 22/03/2020, had 24,513 active cases and 360 closed cases out of total reported 24,873 cases. On 09/05/2020 there were 20,475 active and 150,849 closed out of 171,324 cases [24]. The average percentage of closed cases during this period is found to be nearly 44%. Considering the scale of 1000:1, the initial cases $y_0 \approx 24.51$. Here, $\tilde{y}_0 = (0.44y_0/y_0/1.56y_0)$ and $\tilde{R}_{0(t)} = (R_{0(t)} - 0.02/R_{0(t)}/R_{0(t)} + 0.02)$ have been considered.

Case-I: Let $\tilde{y}_0 = (10.8/24.51/38.2)$ be fuzzy and $R_{0(t)}$ be crisp with $R_{0(0)} = 1.55$. Thus $\tilde{y}_0[\alpha] = [10.8 + 13.71\alpha, 38.2 - 13.69\alpha]$ and $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (31). The pattern of values of slightly, moderately and highly infected active cases during the stipulated time period has been given in Table 16. Here, $\tilde{y}_0 < \cdots < \tilde{y}_4$ and $\tilde{y}_5 > \cdots > \tilde{y}_{12}$ giving

$$\tilde{y}_{12} < \tilde{y}_0 \approx \tilde{y}_{11} < \tilde{y}_{10} < \tilde{y}_1 \approx \tilde{y}_9 < \tilde{y}_8 < \tilde{y}_2 \approx \tilde{y}_7 < \tilde{y}_6 < \tilde{y}_3 \approx \tilde{y}_5 < \tilde{y}_4$$

Table 15 $y_0 = 33.15$; $\tilde{R}_{0(t)} = (R_{0(t)} - 0.04/R_{0(t)}/R_{0(t)} + 0.04)$

Time t	$\tilde{R}_{0(t)}$	$\tilde{R}_{0(t)}[0.25].\tilde{y}_t[0.25]$	$\tilde{R}_{0(t)}[0.50].\tilde{y}_t[0.50]$	$\tilde{R}_{0(t)}[0.75].\tilde{y}_t[0.75]$
0(22/03)	(2.46/2.5/2.54)	[2.47, 2.53] × 33.15	[2.48, 2.52] × 33.15	[2.49, 2.51]× 33.15
1(26/03)	(1.87/1.91/1.95)	[1.88, 1.94] [81.880, 83.869]	[1.89, 1.93] [82.212, 83.538]	[1.90, 1.92] [82.543, 83.206]
2(30/03)	(1.61/1.65/1.69)	[1.62, 1.68] [153.934, 162.706]	[1.63, 1.67] [155.381, 161.228]	[1.64, 1.66] [156.832, 159.756]
3(03/04)	(1.38/1.42/1.46)	[1.39, 1.45] [249.373, 273.346]	[1.40, 1.44] [253.271, 269.251]	[1.41, 1.43] [257.204, 265.195]
4(07/04)	(1.27/1.31/1.35)	[1.28, 1.34] [346.628, 396.352]	[1.29, 1.33] [354.579, 387.721]	[1.30, 1.32] [362.658, 379.229]
5(11/04)	(1.14/1.18/1.22)	[1.15, 1.21] [443.684, 531.112]	[1.16, 1.2] [457.407, 515.669]	[1.17, 1.19] [471.455, 500.582]
6(15/04)	(1.11/1.15/1.19)	[1.12, 1.18] [510.237, 642.646]	[1.13, 1.17] [530.592, 618.803]	[1.14, 1.16] [551.602, 595.693]
7(19/04)	(1.09/1.13/1.17)	[1.10, 1.16] [571.465, 758.322]	[1.11, 1.15] [599.569, 724.0]	[1.12, 1.14] [628.826, 691.004]
8(23/04)	(1.05/1.09/1.13)	[1.06, 1.12] [628.612, 879.654]	[1.07, 1.11] [666.522, 832.6]	[1.08, 1.10] [704.285, 787.745]
9(27/04)	(1.07/1.11/1.15)	[1.08, 1.14] [666.329, 985.212]	[1.09, 1.13] [712.109, 924.186]	[1.10, 1.12] [760.628, 866.52]
10(01/05)	(1.03/1.07/1.11)	[1.04, 1.10] [719.635, 1123.142]	[1.05, 1.09] [776.199, 1044.33]	[1.06, 1.08] [836.691, 970.502]
11(05/05)	(1.02/1.06/1.10)	[1.03, 1.09] [748.420, 1235.456]	[1.04, 1.08] [815.009, 1138.320]	[1.05, 1.07] [886.892, 1048.142]
12(09/05)	–	$\tilde{y}_{12}[0.25] =$ [770.823, 1346.647]	$\tilde{y}_{12}[0.50] =$ [847.609, 1229.386]	$\tilde{y}_{12}[0.75] =$ [931.237, 1121.512]

Thus the different fuzzy phase points are $\tilde{x}_{-1}, \tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_7$. The corresponding FCA will have cells $\tilde{A}_{-1}, \tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_7$ and the ON states of these cells will be $\tilde{A}_{-1}(12)$, $\tilde{A}_0(0)$, $\tilde{A}_0(11)$, $\tilde{A}_1(10)$, $\tilde{A}_2(1)$, $\tilde{A}_2(9)$, $\tilde{A}_3(8)$, $\tilde{A}_4(2)$, $\tilde{A}_4(7)$, $\tilde{A}_5(6)$, $\tilde{A}_6(3)$, $\tilde{A}_6(5)$, $\tilde{A}_7(4)$ as shown in (Fig. 7b).

Case-II: Let $y_0 = 24.51$ be crisp and $\tilde{R}_{0(t)} = (R_{0(t)} - 0.02/R_{0(t)}/R_{0(t)} + 0.02)$ be triangular fuzzy numbers with $\tilde{R}_{0(0)} = (1.53/1.55/1.57)$. Thus $\forall t \geq 1$, $\tilde{y}_t[\alpha]$ has been calculated from (32). The pattern of values of slightly, moderately and highly

Table 16 $\tilde{y_0} = (10.8/24.51/38.2)$; $R_{0(t)} = \frac{y_{t+1}}{y_t}$ be crisp with $R_{0(0)} = 1.55$

Time t	y_t	$R_{0(t)}$	$\tilde{y}_t[0.25]$	$\tilde{y}_t[0.50]$	$\tilde{y}_t[0.75]$
0(22/03)	24.51	1.55	[14.227, 34.777]	[17.655, 31.355]	[21.082, 27.932]
1(26/03)	38.0	1.39	[22.052, 53.904]	[27.365, 48.60]	[32.677, 43.295]
2(30/03)	52.74	1.24	[30.652, 74.927]	[38.037, 67.554]	[45.421, 60.180]
3(03/04)	65.31	1.06	[38.008, 92.909]	[47.166, 83.767]	[56.322, 74.623]
4(07/04)	69.57	0.94	[40.288, 98.484]	[49.996, 88.793]	[59.701, 79.100]
5(11/04)	65.18	0.90	[37.871, 92.575]	[46.996, 83.465]	[56.119, 74.354]
6(15/04)	58.35	0.91	[34.084, 83.317]	[42.296, 75.118]	[50.507, 66.919]
7(19/04)	53.10	0.83	[31.016, 75.818]	[38.489, 68.357]	[45.961, 60.896]
8(23/04)	44.25	0.86	[25.743, 62.929]	[31.946, 56.736]	[38.148, 50.544]
9(27/04)	38.13	0.80	[22.139, 54.119]	[27.474, 48.793]	[32.807, 43.468]
10(01/05)	30.44	0.82	[17.711, 43.295]	[21.979, 39.034]	[26.246, 34.774]
11(05/05)	24.91	0.82	[14.523, 35.502]	[18.023, 32.008]	[21.522, 28.515]
12(09/05)	20.47	–	[11.901, 29.112]	[14.779, 26.246]	[17.648, 23.382]

infected active cases during the stipulated time period has been given in Table 17. Since the pattern of the α-cut values of the fuzzy phase points of Tables 16 and 17 are similar, hence the corresponding FCA will be similar to as shown in (Fig. 7b).

6 Conclusion

The basic reproduction number associated with any virus outbreak changes after a short period of time. We have designed an FCA which turned out to be temporally hybrid, representing the growth of the number of infected population from MERS virus outbreak in 2015, and for Covid-19 virus. However, a moderately large time period may be divided into short equal time intervals. In each of these equal time intervals there will be growth of the number of infected population which may be designed by an FCA. So alternate use of FCA and temporally hybrid FCA will

Table 17 $y_0 = 24.51$; $\tilde{R}_{0(t)} = (R_{0(t)} - 0.02/R_{0(t)}/R_{0(t)} + 0.02)$

Time t	$\tilde{R}_{0(t)}$	$\tilde{R}_{0(t)}[0.25].\tilde{y}_t[0.25]$	$\tilde{R}_{0(t)}[0.50].\tilde{y}_t[0.50]$	$\tilde{R}_{0(t)}[0.75].\tilde{y}_t[0.75]$
0(22/03)	(1.53/1.55/1.57)	[1.535, 1.565] × 24.51	[1.54, 1.56] × 24.51	[1.545, 1.555] × 24.51
1(26/03)	(1.37/1.39/1.41)	[1.375, 1.405] [37.623, 38.358]	[1.38, 1.40] [37.745, 38.236]	[1.385, 1.395] [37.868, 38.113]
2(30/03)	(1.22/1.24/1.26)	[1.225, 1.255] [51.732, 53.893]	[1.23, 1.25] [52.088, 53.530]	[1.235, 1.245] [52.447, 53.168]
3(03/04)	(1.04/1.06/1.08)	[1.045, 1.075] [63.372, 67.636]	[1.05, 1.07] [64.068, 66.912]	[1.055, 1.065] [64.772, 66.194]
4(07/04)	(0.92/0.94/0.96)	[0.925, 0.955] [66.224, 72.709]	[0.93, 0.95] [67.271, 71.596]	[0.935, 0.945] [68.334, 70.497]
5(11/04)	(0.88/0.90/0.92)	[0.885, 0.915] [61.257, 69.437]	[0.89, 0.91] [62.562, 68.016]	[0.895, 0.905] [63.892, 66.620]
6(15/04)	(0.89/0.91/0.93)	[0.895, 0.925] [54.212, 63.535]	[0.90, 0.92] [55.680, 61.895]	[0.905, 0.915] [57.183, 60.291]
7(19/04)	(0.81/0.83/0.85)	[0.815, 0.845] [48.520, 58.77]	[0.82, 0.84] [50.112, 56.943]	[0.825, 0.835] [51.751, 55.166]
8(23/04)	(0.84/0.86/0.88)	[0.845, 0.875] [39.544, 49.661]	[0.85, 0.87] [41.092, 47.832]	[0.855, 0.865] [42.695, 46.064]
9(27/04)	(0.78/0.80/0.82)	[0.785, 0.815] [33.415, 43.453]	[0.79, 0.81] [34.928, 41.614]	[0.795, 0.805] [36.504, 39.845]
10(01/05)	(0.80/0.82/0.84)	[0.805, 0.835] [26.231, 35.414]	[0.81, 0.83] [27.593, 33.707]	[0.815, 0.825] [29.021, 32.075]
11(05/05)	(0.80/0.82/0.84)	[0.805, 0.835] [21.116, 29.571]	[0.81, 0.83] [22.350, 27.977]	[0.815, 0.825] [23.652, 26.462]
12(09/05)	–	$\tilde{y}_{12}[0.25] =$ [16.998, 24.692]	$\tilde{y}_{12}[0.50] =$ [18.104, 23.221]	$\tilde{y}_{12}[0.75] =$ [19.276, 21.831]

completely explain the model. Use of the concept of α-cut enabled us to introduce a gradation of the number of infected people. Changing the values of α will give a different gradation.

Study of the models obtained by replacing basic reproduction number with effective reproduction number may be an interesting exercise.

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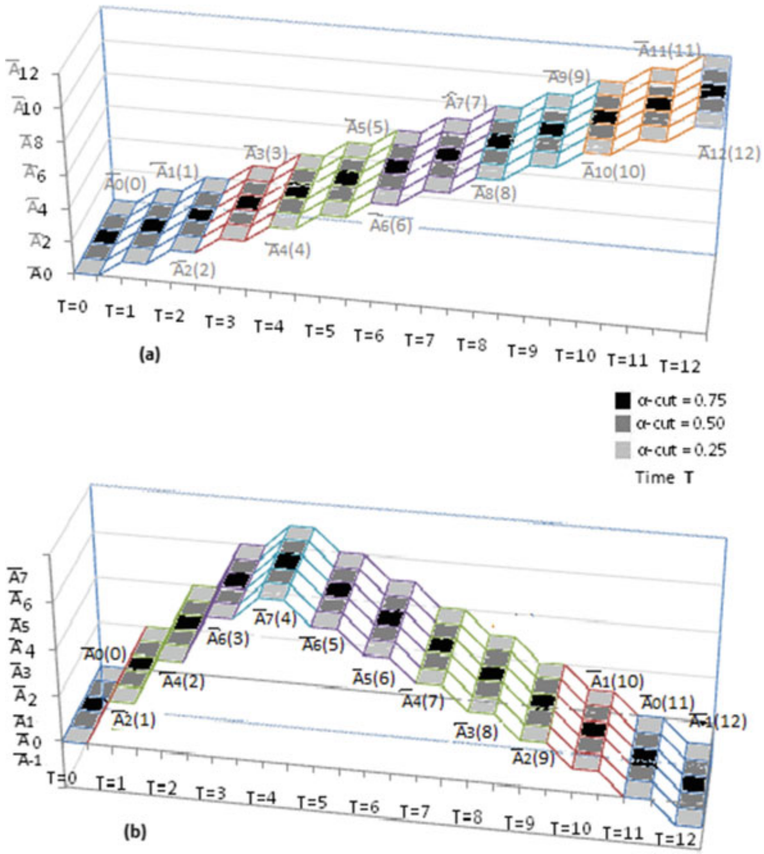

Fig. 7 FCA for spread Of COVID-19 from 22/03/2020 to 09/05/2020 in **a** India/USA, **b** Germany

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