Interface Wave Diffraction by a Permeable Thin Barrier

Uma Basu, Sandip Paul, and Soumen De

Abstract Wave interaction with a thin porous barrier submerged in the lower layer of an infinite depth fluid are analyzed based on plane wave approximations. By employing Greens integral theorem, we formulate a hypersingular integral equation in terms of unknown potential difference across the porous barrier. A collocation method using a finite series of Chebyshev polynomials of the second kind has been introduced to get the unknown difference potential numerically. The reflection and transmission coefficients and hydrodynamic forces on a porous barrier are analyzed for various physical parameters associated with the problem. The present study will be of significant importance in the design of various types of coastal structures used in the marine environment for the reflection and dissipation of wave energy.

Keywords Infinite depth fluid · Interface wave · Hypersingular integral equations · Reflection and transmission coefficients · Porous barrier

1 Introductions

Theoretical study on small-amplitude water wave scattering in a fluid by an obstacle is widely investigated by many researchers. A submerged breakwater allows the free exchange of water mass through the structures so that the water in the sheltered region can be kept circulating and, therefore, prevented from pollution. Scattering of surface waves by a thin vertical barrier submerged in a single layer fluid is considered by [3, 14] and others for deep water, wherein the barrier is either fully submerged and extended infinitely downwards. Moreover, the bodies which are immersed in a two-layer fluid have considerable applications. One of these is the suggestion by [4] that an underwater pipe bridge might be built across one of the Norwegian fjords,

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E. J. Sapountzakis et al. (eds.), *Proceedings of the 14th International Conference on Vibration Problems*, Lecture Notes in Mechanical Engineering,
https://doi.org/10.1007/978-981-15-8049-9_4

which typically consists of a layer of freshwater of finite depth on top of a very deep body of saltwater. Works on the study of water wave propagation in multilayered fluids with an interface (see, [1, 5, 6, 9, 13]). References [5, 6] studied the problems of an internal wave propagating at the surface separating two fluids by considering different types of singularities the finite depth fluid layers neglecting surface tension at the interface.

The bodies which are made of some porous materials have the ability of energy dissipation, so that transmission of energy through the barrier is reduced, which is of great interest in many ocean engineering applications. Studies related to water wave scattering by porous barriers have been carried out by a list of researchers. Water wave diffraction by porous breakwaters has been studied by [16]. Reference [2] gave a general review of the interaction between porous media and wave motion. Their studies focused on the effect of a porous structure on incoming wave trains and the movement of waves past a plate with regular gaps in it. Using the method of eigenfunction expansion in conjunction with least square approximations, [7] studied water wave scattering and radiation problems. They have considered different configurations of the porous barrier. Reference [8] analyzed the scattering of oblique water wave by a fully extended porous barrier in a two-layer fluid having a free surface and an interface. They used a modified orthogonal relation to solve the problem using the eigenfunction expansion method and derived the explicit forms of the reflection and transmission coefficients.

The study of interface wave diffraction by a completely submerged thin vertical barrier in the lower fluid of two superposed infinite fluids given by [10]. In this work, the barrier is extended infinitely downwards into the lower fluid. The present study is an extension of the work of [10] and concerned with the interface wave scattering by a permeable thin vertical barrier submerged infinitely in the lower layer of two superposed infinite-depth fluids. By suitable application of Green's integral theorem in two fluid regions, the problem is formulated as a hypersingular integral equation for the difference of potentials across the barrier. A collocation method involving Chebyshev polynomials of the second kind is applied to solve the hypersingular integral equation (cf. [11, 12]). This reduces the equation to a system of linear equations which can be solved by standard methods. The solutions are used to get the reflection and transmission coefficients, which are analyzed numerically.

2 Formulation of the Problem

A two-dimensional model of the problem under the linearized potential theory is being considered. The region occupied by two homogeneous, incompressible, and inviscid fluids of uniform densities is denoted by two layers of the fluid. The origin of the coordinate system is taken at the interface of the layers, where the X-axis is along the undisturbed interface and the Y-axis is taken vertically downwards. The upper and lower layers are given by $y \leq 0$ and $y \geq 0$, respectively. The densities of the fluids in the upper and lower layers are denoted by ρ_1 and ρ_2, respectively.

The resulting motion in the fluids are described by the time-harmonic velocity potentials $\Re\left(\phi\,(x, y)\, e^{-\mathrm{i}\omega t}\right)$ and $\Re\left(\psi\,(x, y)\, e^{-\mathrm{i}\omega t}\right)$ in the lower and upper fluids respectively, with $\phi\,(x, y)$ and $\psi\,(x, y)$ satisfying the following Laplace's equations:

$$\nabla^2\phi = 0, \text{ in } 0 \le y < \infty \tag{1}$$

$$\nabla^2\psi = 0, \text{ in } -\infty < y \le 0. \tag{2}$$

The boundary conditions at the interface of the fluid are given by

$$K\phi + \phi_y = \rho\left(K\psi + \psi_y\right), \text{ on } y = 0 \tag{3}$$

$$\phi_y = \psi_y, \text{ on } y = 0, \tag{4}$$

where $\rho = \rho_1/\rho_2$, $K = \omega^2/g$, g being the gravitational acceleration and ω is the circular frequency.

The bottom and the free surface conditions are given by

$$\nabla\phi \to 0, \text{ as } y \to \infty \tag{5}$$

$$\nabla\psi \to 0, \text{ as } y \to -\infty. \tag{6}$$

A completely submerged thin porous barrier is situated in the lower fluid occupying the position $x = 0$, $a < y < \infty$. Based on Darcy's law, the boundary condition on the porous barrier is given by

$$\phi_x = -\mathrm{i}MG\left[\phi\,(0+, y) - \phi\,(0-, y)\right], \tag{7}$$

where $M = K/\sigma$, $\sigma = (1-\rho)/(1+\rho)$, $G \equiv G_r + \mathrm{i}G_i = \frac{\gamma(f^* + iS)}{Md(f^{*2} + S^2)}$ is a complex porous-effect parameter, d the thickness of the barrier, f^* the linearized resistance coefficient, S the coefficient of inertial force acting on the porous barrier, and γ denotes the porosity of the barrier (c.f.,[15]). Boundary condition (7) shows that the normal fluid velocity through the porous barrier is continuous and proportional to the pressure jump across the barrier.

When a train of waves propagating from negative infinity, the incident wave potentials takes the form

$$\begin{aligned} \phi_0\,(x, y) &= e^{\mathrm{i}Mx - My}, \text{ in the lower fluid;} \\ \psi_0\,(x, y) &= -e^{\mathrm{i}Mx + My}, \text{ in the upper fluid.} \end{aligned} \tag{8}$$

When a train of small-amplitude progressive wave is incident on the thin porous barrier, some part of the incident energy is transmitted above or through the pores of the plate and rest of it is reflected back. Let us denote the reflection and transmission coefficients by $|R|$ and $|T|$, respectively. Then the conditions at $x = \pm\infty$ are given by

$$\begin{bmatrix} \phi(x, y) \\ \psi(x, y) \end{bmatrix} \rightarrow \begin{cases} \begin{bmatrix} e^{\mathrm{i}Mx-My} \\ -e^{\mathrm{i}Mx+My} \end{bmatrix} + R \begin{bmatrix} e^{-\mathrm{i}Mx-My} \\ -e^{-\mathrm{i}Mx+My} \end{bmatrix}, & x \longrightarrow -\infty \\ T \begin{bmatrix} e^{\mathrm{i}Mx-My} \\ -e^{\mathrm{i}Mx+My} \end{bmatrix}, & x \longrightarrow \infty. \end{cases} \tag{9}$$

3 Solution of the Problem

Consider the scattered potentials due to the presence of porous barrier be the functions $\phi_1(x, y)$ and $\psi_1(x, y)$ given by

$$\begin{aligned} \phi_1(x, y) &= \phi(x, y) - \phi_0(x, y), \\ \psi_1(x, y) &= \psi(x, y) - \psi_0(x, y). \end{aligned} \tag{10}$$

which satisfies the following boundary value problem:

$$\nabla^2 \phi_1 = 0 \text{ in } y \geq 0, \tag{11}$$
$$\nabla^2 \psi_1 = 0 \text{ in } y \leq 0, \tag{12}$$
$$K\phi_1 + \phi_{1y} = \rho \left(K\psi_1 + \psi_{1y} \right) \text{ on } y = 0, \tag{13}$$
$$\phi_{1y} = \psi_{1y} \text{ on } y = 0, \tag{14}$$
$$\nabla\phi_1 \rightarrow 0 \text{ as } y \rightarrow \infty, \tag{15}$$
$$\nabla\psi_1 \rightarrow 0, \text{ as } y \rightarrow -\infty, \tag{16}$$

and

$$\phi_{1x} = -\mathrm{i}M \left(e^{-My} + Gf(y) \right), \tag{17}$$

where $f(y) = \phi_1(+0, y) - \phi_1(-0, y) = \phi(+0, y) - \phi(-0, y)$.

The far-field conditions are takes the form

$$\begin{bmatrix} \phi_1(x, y) \\ \psi_1(x, y) \end{bmatrix} \rightarrow \begin{cases} R \begin{bmatrix} e^{-\mathrm{i}Mx-My} \\ -e^{-\mathrm{i}Mx+My} \end{bmatrix}, & x \longrightarrow -\infty \\ (T-1) \begin{bmatrix} e^{\mathrm{i}Mx-My} \\ -e^{\mathrm{i}Mx+My} \end{bmatrix}, & x \longrightarrow \infty. \end{cases} \tag{18}$$

To solve the boundary value problem described by Eqs. (11)–(17), we require two-dimensional source potentials due to a line source submerged in the lower fluid. Let $\mathcal{G}(x, y; \xi, \eta)$ and $\mathcal{H}(x, y; \xi, \eta)$ be the source potentials in the lower and upper fluids, respectively, due to a line source submerged in the upper fluid at $(\xi, \eta)(\eta > 0)$. Now apply Green's integral theorem to the functions $\phi_1(x, y)$ and $\mathcal{G}(x, y; \xi, \eta)$ in the region bounded externally by the lines $x = \pm X, 0 \leq y < Y$; $y = 0, Y, -X \leq x \leq X$; $x = \pm 0, a \leq y \leq Y$ and internally by a circle of small radius ϵ with center at (ξ, η) and ultimately we make $X, Y \rightarrow \infty$ and $\epsilon \rightarrow 0$. Again we apply Green's integral theorem to the functions $\psi_1(x, y)$ and $\mathcal{H}(x, y; \xi, \eta)$ in the region bounded externally by the lines $x = \pm X, -Y < y \leq 0$; $y = -Y, 0, -X \leq x \leq X$ and ultimately we make $X, Y \to \infty$. After few simplifications the above boundary value problem reduces to the integral equation given by

$$\phi_1(\xi, \eta) = -\frac{1}{2\pi} \int_a^\infty f(y)\mathcal{G}_x(0, y; \xi, \eta)dy \tag{19}$$

The discontinuity in pressure across the porous barrier tends to zero as we approach to the edge at $y = a$, i.e.,

$$f(a) = 0. \tag{20}$$

Form the condition given by Eq. (17), we have

$$\phi_{1\xi}(0, \eta) = -\mathrm{i}M(e^{-K\eta} + Gf(\eta)), \quad \eta \in (a, \infty). \tag{21}$$

Using Eq. (21) in Eq. (19), we get the following integro-differential equation

$$\frac{\partial}{\partial \xi} \int_a^\infty f(y)\mathcal{G}_x(0, y; 0, \eta)dy - 2\pi \mathrm{i}MGf(\eta) = 2\pi \mathrm{i}Me^{-M\eta}, \quad \eta \in (a, \infty). \tag{22}$$

The order of integration and differentiation in Eq. (22) can be interchanged provided the integral is interpreted as a finite part integral. This leads to the following hypersingular integral equation:

$$⨍_a^\infty f(y)\mathcal{G}_{x\xi}(0, y; 0, \eta)dy - 2\pi \mathrm{i}MGf(\eta) = 2\pi \mathrm{i}Me^{-M\eta}, \quad \eta \in (a, \infty). \tag{23}$$

The reflection and transmission coefficients are obtained as

$$R = -\frac{M}{1+\rho} \int_a^\infty f(y)e^{-My}dy, \tag{24}$$

$$T = 1 + \frac{M}{1+\rho} \int_a^\infty f(y)e^{-My}dy. \tag{25}$$

The difference potential $f(y)$ involved in Eqs. (24) and (25) can be computed by solving the hypersingular integral equation given in Eq. (23) numerically.

4 Hypersingular Integral Equation

As (x, ξ) approaches $(0, 0)$, the source potential takes the form

$$\mathcal{G}_{x\xi}(0, y; 0, \eta) = -\frac{1}{(y-\eta)^2} - L(y, \eta), \tag{26}$$

where

$$L(y,\eta)=\frac{\sigma}{(y+\eta)^2}+\frac{2M}{(1+\rho)\,(y+\eta)}+\frac{2M^2}{(1+\rho)}\zeta\,(y,\eta)\,, \tag{27}$$

$$\zeta\,(y,\eta)=\fint_0^\infty \frac{e^{-k(y+\eta)}}{k-M}dk. \tag{28}$$

Using Eq. (26) in Eq. (23), we get

$$\fint_a^\infty f(y)\left[\frac{1}{(y-\eta)^2}+L(y,\eta)\right]dy+2\pi\mathrm{i}MGf(\eta)=-2\pi\mathrm{i}Me^{-M\eta},\, a<\eta<\infty. \tag{29}$$

On substituting $y=\frac{2a}{1+p}, \eta=\frac{2a}{1+q}$ in Eq. (29) and considering $M_1=Ma$ the non-dimensionalized form of the hypersingular integral is

$$\fint_{-1}^1 f_1(p)\left[\frac{1}{(p-q)^2}+L_1(p,q)\right]dp+4\pi\mathrm{i}\frac{M_1G}{(1+q)^2}f_1(q)=H(q) \tag{30}$$

with

$$L_1(p,q)=\frac{\sigma}{(p+q+2)^2}+\frac{4M_1}{(1+\rho)\,(p+1)\,(q+1)\,(p+q+2)} \\ \frac{8M_1^2}{(1+\rho)\,(p+1)^2\,(q+1)^2}\zeta_1\,(p,q)\,, \tag{31}$$

$$\zeta_1\,(p,q)=-e^{-\mu M_1}\left[\ln\,(\mu M_1)+\nu-\pi\mathrm{i}+\sum_{r=0}^{\infty}\frac{(\mu M_1)^r}{r.r!}\right] \tag{32}$$

$$H(q)=-4\pi\mathrm{i}\frac{M_1}{(1+q)^2}e^{-\frac{2M_1}{1+q}},\, -1<q<1, \tag{33}$$

where $\mu=\frac{2(p+q+2)}{(p+1)(q+1)}$ and $\nu=0.5772$ being the Eulers constant.

The edge condition given in Eq. (20) suggests us to consider

$$f_1(p)=\sqrt{1-p^2}\sum_{n=0}^{N}a_nU_n(p), \tag{34}$$

where $U_n(p)$ is a Chebyshev's polynomial of second kind given by

$$U_n(\cos\theta)=\frac{\sin\,(n+1)\theta}{\sin\theta} \tag{35}$$

and the value of the unknowns a_n are to be determined.

At this stage, Eq. (30) produces a system of $N+1$ linear equations with $N+1$ unknowns $a_n, n = 0, 1, 2, \ldots, N$ of the form

$$\sum_{n=0}^{N} a_n A_n(q) = H(q), -1 < q < 1, \tag{36}$$

where

$$A_n(q) = -\pi \left(n + 1 - 4\mathrm{i} \frac{M_1 G \sqrt{1-q^2}}{(1+q)^2} \right) U_n(q) + \int_{-1}^{1} \sqrt{1-p^2} U_n(p) L_1(p,q) dp. \tag{37}$$

Forms of the reflection and transmission coefficients are therefore obtained as

$$R = -\frac{2M_1}{1+\rho} \sum_{n=0}^{N} a_n \int_{-1}^{1} \frac{\sqrt{1-p^2}}{(1+p)^2} e^{-\frac{2M_1}{(1+p)}} U_n(p) dp. \tag{38}$$

$$T = 1 + \frac{2M_1}{1+\rho} \sum_{n=0}^{N} a_n \int_{-1}^{1} \frac{\sqrt{1-p^2}}{(1+p)^2} e^{-\frac{2M_1}{(1+p)}} U_n(p) dp. \tag{39}$$

We consider a collocation scheme with collocation points $q_j, j = 0, 1, 2, \ldots N$ (c.f. [11]) and solve the $(N+1)$ linear equations

$$\sum_{n=0}^{N} a_n A_n(q_j) = H(q_j), \quad j = 0, 1, 2, \ldots N, \tag{40}$$

where the collocation points, $q_j, j = 0, 1, 2, \ldots N$ are considered as

$$q_j = \cos\left[\frac{(2j+1)\pi}{2N+2} \right], \quad j = 0, 1, 2, \ldots N. \tag{41}$$

5 Analysis and Discussion

This section provides a numerical study in deciding the effectiveness of the porous barrier. The reflection and transmission coefficients are analyzed for different parameters such as the density ratio ρ and the porosity parameter G as a function of the wavenumber Ka. The results are computed by taking $N = 10$.

In Table 1, the values of reflection coefficients are computed by the present method for the porosity parameter $G = 0$ with $\rho = 0.01$ and compared with the results obtained in [10]. The data reveal an excellent agreement of [10] results with those obtained by the current work. Thus, we have been successful to provide a

Table 1 Comparison of the results with Mandal et al. [10] when $\rho = 0.01$

Ka	Mandal et al. [10]	Present result
0.001	0.912800	0.912800
0.005	0.864877	0.864876
0.1	0.610188	0.610188
0.5	0.266129	0.266127
1.0	0.105712	0.105711

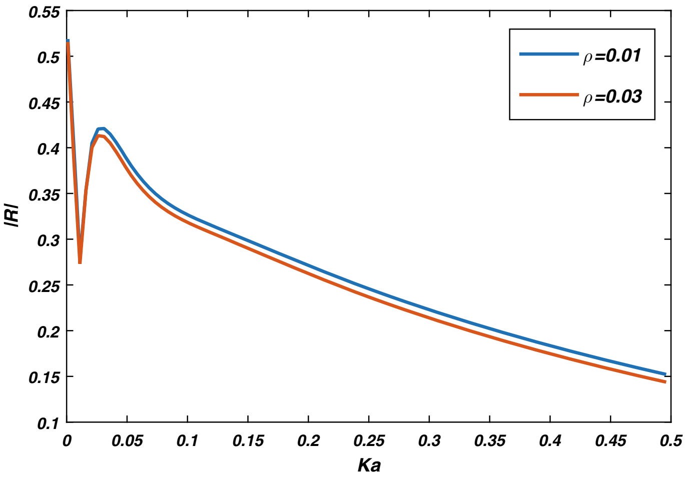

Fig. 1 Reflection coefficients versus Ka for different values of $\rho(= 0.01, 0.03)$ and $G = 0.5$

straightforward and efficient method to study the effects of a fixed vertical porous barrier on water waves propagating along the interface of a two-layered fluid.

Figures 1 and 2 illustrate the effect of density ratio on the scattering of interface wave. The reflection coefficients $|R|$ and the transmission coefficients $|T|$ are plotted for two different density ratios of the fluid, viz., $\rho = 0.01$ and $\rho = 0.3$ versus the dimensionless wavenumber Ka. For increasing values of ρ, it is observed that the amplitude of reflection coefficient decreases. The opposite pattern of the graphs can be seen in the case of transmission coefficients. Therefore, the density ratio of the fluid has a significant role in the scattering behavior of the interface wave.

In Figs. 3 and 4, the reflection and transmission coefficients, $|R|$ and $|T|$, respectively, are depicted for various values of the porosity parameter $G(= 0.25 + 0.25i, 0.5 + 0.5i)$ in a fluid with density ratio $\rho = 0.1$ as a function of the wavenumber Ka. It is observed from Fig. 3 that $|R|$ decreases with increasing Kb as the absolute value of the porosity parameter of the barrier increases. Perhaps, this is due to the ability of the porous barrier that some of the incident energy may be dissipated by the

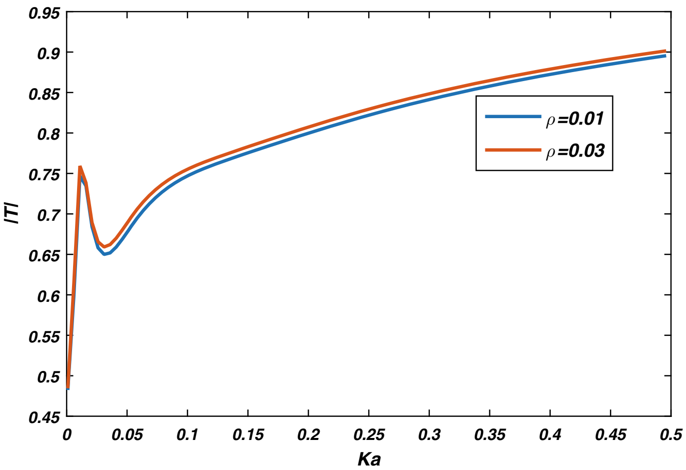

Fig. 2 Transmission coefficients versus Ka for different values of $\rho(= 0.01, 0.03)$ and $G = 0.5$

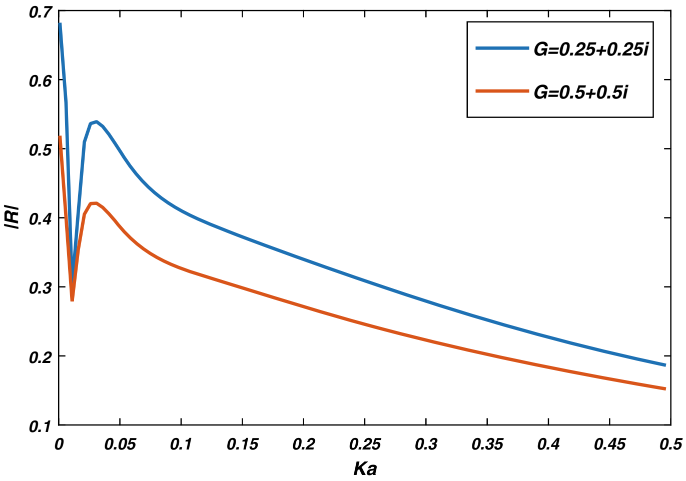

Fig. 3 Reflection coefficients versus Ka for different values of $G(= 0.25 + 0.25i, 0.5 + 0.5i)$ and $\rho = 0.01$

barrier. The transmission coefficients in Fig. 4 are decreasing functions of Ka due to the energy dissipation by the barrier. Indeed, in the presence of porous barriers, the value of $|R|^2 + |T|^2$ is always less than 1.

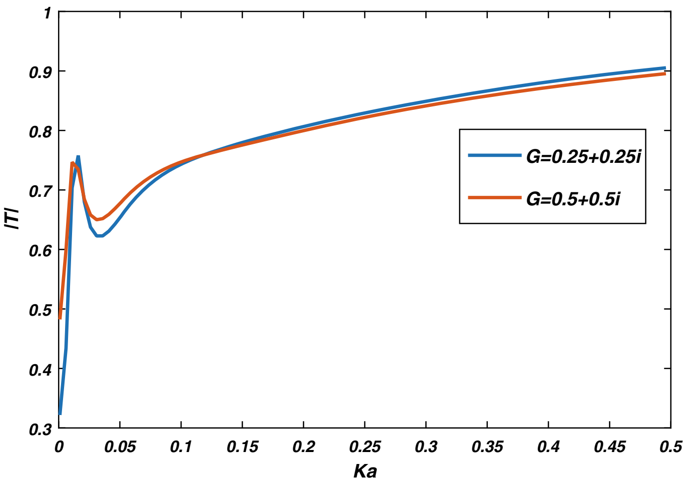

Fig. 4 Transmission coefficients versus Ka for different values of $G(0.25 + 0.25i, 0.5 + 0.5i)$ and $\rho = 0.01$

6 Conclusion

A model for the interface wave scattering by a thin porous barrier submerged completely in the lower layer of two superposed fluids is studied. Both the fluids are considered as of infinite depth and the porous barrier is placed at a depth a beneath the interface and extended to infinity. Green's integral theorem is applied to formulate the problem in terms of the hypersingular integral equation for the unknown difference potential across the barrier. These difference potentials are approximated by a finite series of Chebyshev polynomials and the coefficients of reflection and transmission coefficients are calculated in terms of integrals involving difference potentials. The numerical results show that the scattering behavior depends highly on the density ratio of the fluid and porosity parameter of the barrier.

References

1. Chakrabarti RN, Mandal BN (1983) Singularities in a two-fluid medium. Internat. J Math Math Sci 6(4):737–754
2. Chwang AT, Chan AT (1998) Interaction between porous media and wave motion. Annu Rev Fluid Mech 30:53–84
3. Dean WR (1945) On the reexion of surface waves by a submerged plane barrier. Proc Camb Phil Soc 41:231–238
4. Friis A, Grue J, Palm E (1991) Application of Fourier transform to the second order 2D wave diffraction problem. In: Tulin's MP, Miloh T (eds) Festschrqt: Mathematical approaches in hydrodynamics. SIAM, pp 209–227

5. Kassem SE (1982) Multipole expansion for two superposed fluids each of finite depth. Indian J pure appl Math 13:1109–1116
6. Kassem SE (1986) Wave source potentials for two superposed fluids each of finite depth. Math Proc Camb Phil Soc 100:595–599
7. Lee MM, Chwang AT (2000) Scattering and radiation of water waves by permeable barriers. Phys Fluids 12:54–65
8. Manam SR, Sahoo T (2005) Waves past porous structures in a two-layer fluid. J Eng Math 52:355–377
9. Mandal BN, Chakrabarti RN (1986) Two-dimensional source potentials in a two fluid medium for the modified Helmholtz's equation. Int J Math Math Sci 9:175–184
10. Mandal BN, Banerjea S, Dolai DP (1995) Interface wave diffraction by a thin vertical barrier submerged in the lower fluid. Appl Ocean Res 17:93–102
11. Parsons NF, Martin PA (1992) Scattering of water waves by submerged plates using hypersingular integral equations. Appl Ocean Res 14:313–321
12. Parsons NF, Martin PA (1994) Scattering of water waves by submerged curved plates and by surface-piercing flat plates. Appl Ocean Res 14:129–139
13. Rhodes-Robinson PF (1994) On wave motion a two-layered liquid of infinite depth in the presence of surface and interfacial tension. J Aust Math Soc, Series B 35:302–322
14. Ursell F (1947) The effect of fixed vertical barrier on surface waves in deep water. Proc Camb Phil Soc 43:374–382
15. Yu X, Chwang AT (1994) Wave induced oscillation in harbor with porous breakwaters. J Waterw Port Coast Ocean Eng 120:125–144
16. Yu X (1995) Diffraction of water waves by porous breakwaters. J Waterw Port Coast Ocean Eng 121:275–282