# The Stationary State of the Granular Material Under the Action of Intense Vibration and Gravity

**Eugen Kremer**

**Abstract** A layer of granular material over an intensely vibrating plane in the field of gravity is considered as "granular gas". It simulates the vibrating fluidized bed widely used in various fields of technology. The processes of transport of kinetic energy and momentum are described with considering a non-Maxwell particle velocity distribution, which was discovered earlier in both numerical and full-scale experiments. Equations for the spatial variation of the particle concentration and their kinetic energy (granular temperature) are obtained and a general analytical solution of these equations is found, as well as the solution of the boundary value problem for a layer with a free surface from above and with a given motion of the plane from below. The influence of the coefficient of restitution, particle size, and vibration parameters on the spatial distributions of density and granular temperature as well as on the consumed power is analyzed. On the basis of the obtained equations, an analytical description of the effect of the instability of the symmetric state of a granular gas in two identical chambers separated by a baffle with a window ("Maxwell's demon" experiment) is given. It is shown that the instability arises from a certain height position of the window and has a maximum at a certain value of this position.

**Keywords** Granular material · Granular gas · Vibrating fluidized bed · "Maxwell's demon" · Granular temperature

## 1 Introduction

The dynamics of a granular medium under the influence of vibrations is not only of great practical importance for many fields of technology but also of theoretical interest as a source of various non-trivial and unexpected effects, such as instability

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of clusterization [1] and the "Maxwell's demon" experiments (the migration of material from a chamber with a smaller number of particles into a chamber with a larger number of them) [2, 3], the unexpected asymmetry of the circulation flows [4], the effect of the appearance of an air pressure drop across the vibrating layer [5, 6], the "Brazil nut effect" (floating of the larger particles) [4, 7, 8], the granular Leidenfrost effect [9], and others. Some of these effects are explained and described by reasonably convincing models. Concerning others, there are various hypotheses, and discussions continue. The complexity of the object and the variety of its behaviors under different conditions have given rise to a justifiable variety of models, approaches, and methods of investigation [10, 11]. The entire spectrum of applied models extends from very complex kinetic equations in which an attempt is made to take into account the maximum possible number of factors and the particle velocity distribution is considered in detail to comparatively simple semi-phenomenological models describing the average transfer of mass, energy, and momentum. There are specialized models related to the "solid", "liquid", and "gaseous" states of the granular medium [12]. The latter has attracted considerable interest in recent years within the framework of the concept of granular gas [1–3, 9–43]. Granular gas has a number of features that distinguish it from ordinary molecular gas. These features are determined primarily by the following circumstances.

Firstly, in collisions, energy dissipation takes place. In a minimal model, this can be described by assuming that the particles have a spherical shape, and the change in velocity during impacts is described by the coefficient of restitution $R_G = u_+/u_-$. Here, $u_+$ and $u_-$ are the relative velocities of two bodies in the direction of the normal at the point of their contact after and before the collision. There are known works that take into account the presence of particle rotation and the dependence of the coefficient on velocity [19], but for many practical applications, such complication of the model is not required. In particular, the highly rough particles do not have mutual penetration at the point of contact. The dissipation of energy as a result of a single collision is caused only by normal velocities and its average value can be calculated from the simple formula

$$\overline{\Delta \mathrm{E}} = \left(1 - R_G^2\right) m\overline{v^2} \tag{1}$$

Here, m is the mass of the particles, and the bar denotes averaging over the probability space (or in time). The average kinetic energy $\theta = m\overline{v^2}/2$ is usually called the granular temperature by analogy with ordinary gas.

Secondly, for a granular gas there is a deviation from the Maxwellian law of particle velocity distribution. For the probability distribution density of velocities in numerical and full-scale experiments, the following law is found [25, 30, 42, 43]:

$$f(v) = \eta e^{-\left(\frac{|v|}{v_s}\right)^{\kappa}} \tag{2}$$

Here, $v$ is the velocity, $v_s$ is some characteristic scale of velocity, κ is some constant, and $\eta$ is the factor determined from the normalization condition $\int_{-\infty}^{\infty} f(v)dv$, which is equal to $\eta = \kappa/(2\Gamma(1/\kappa)v_s)$. The constant κ is in the interval from 1 to 2 and is equal to 2 for the Maxwell distribution. According to [25, 30, 42, 43], on the basis of numerical and full-scale experiments, the characteristic value of κ can be taken as equal to 1.5 in many cases.

Third, for a granular gas, large gradients of both density and granular temperature are typical, which is rare in the molecular gases. This requires some modification of the transport equations in comparison with their usual approximate form, which is used for ordinary gases with a slightly varying density.

In the present paper, it is supposed that the features mentioned above are taken into account consistently when compiling transport equations and boundary conditions for a granular medium excited by a vibrating plane in a gravitational field. In this case, the analysis is performed for an arbitrary parameter κ in the distribution $f(v)$ [Formula (2)] and for arbitrary (not necessarily close to unity) the coefficient of restitution for particle–particle and particle–plane collisions. On the basis of these equations, we will consider the simplest but practically important stationary problem of a layer with a free surface from above and with a given motion of the plane from below. The aim is to obtain an analytical solution to this problem and with its help to trace the influence of the coefficient of restitution, particle size, and vibration parameters on the spatial distribution of density and granular temperature in the layer of granular material and also on the consumed power. As a verification of the model, we will consider the "Maxwell's demon" experiment [2, 3].

## 2 Formulation of the Problem

A layer of granular material located above the vibrating plane is considered. The oscillations of the plane have a period τ and are performed with the velocity $\boldsymbol{v}_p(\boldsymbol{t})$. It is assumed that these oscillations are sufficiently intense for the state of the medium to be characterized as a granular gas in at least some region adjacent to the plane. This means that the medium particles are on average at a sufficiently large distance from each other and experience chaotic motion with zero mean velocity. It is also assumed that the velocity distribution follows the law (2) with a constant parameter κ and with the parameter $\boldsymbol{v}_s$ depending only on the coordinate $x$. Thus, we consider an on average one-dimensional and stationary state of a granular gas.

The above requirement of a sufficiently large distance between particles can be specified by considering the free path of particles in the medium ξ. This quantity is random with the Poisson distribution, that is, with the probability density

$$f_P(\xi) = \frac{e^{-\xi/\lambda}}{\lambda} \tag{3}$$

where λ is the mean free path. The latter is calculated as known from the formula $\lambda = 1/\left(\pi d^2 n\sqrt{2}\right)$ [8], where $d$ is the effective diameter of the particles and $n$ is their average concentration. The condition of a sufficiently large distance between the particles can be specified as the requirement $\lambda \gg d$. At the same time, we will consider the change in the main characteristics of a granular gas—its concentration $n$ and the granular temperature $\theta$—at scales of length larger than the mean free path. The purpose of further consideration is to derive the averaged equations and boundary conditions for these quantities and find the solutions of the corresponding boundary-value problem.

## 3 Momentum Equation

Let us consider an element of a granular layer enclosed between planes with coordinates $x$ and $x + \mathrm{d}x$ and a vertical cylindrical surface of a unit cross section. The flux of momentum entering this volume from below amounts to

$$p(x) = nm \int_0^\infty v^2 f(v) dv = \frac{nm\overline{v^2}}{2} = n(x)\theta(x), \tag{4}$$

where we use the above notation for the mean square of the velocity $\overline{v^2}$ and for the granular temperature $\theta$. Here, the concentration $n$ is a function of the coordinate $x$, and the dependence on $x$ of the values $f(v)$ and $\theta$ is due to the dependence of the parameter $v_s$ on $x$. The relationship between the parameter $v_s$ and the granular temperature $\theta$ can be found with allowance for (2):

$$\theta = \frac{mv_s^2}{2} \frac{\Gamma\left(\frac{3}{\kappa}\right)}{\Gamma\left(\frac{1}{\kappa}\right)} \tag{5}$$

The balance of momentum in the element of the granular layer leads to the well-known equation

$$\frac{dp}{dx} = -mgn \tag{6}$$

in which g is the free-fall acceleration and p can be interpreted as a granular pressure. If the temperature were independent of the coordinate, the well-known Boltzmann distribution would follow from Eq. (6). However, in most cases, the height dependence of temperature should not be neglected. In order to take this dependence into account, it is necessary to consider the transfer of kinetic energy.

## 4 The Kinetic Energy Transport Equation

Let us consider the flow of kinetic energy QT through some control surface. Taking into account the distribution along the mean free paths, this quantity can be calculated as

$$Q_T = m\int_0^\infty dv \int_0^\infty v^3(-f(x+\xi, v)n(x+\xi) + f(x-\xi, v)n(x-\xi))\frac{e^{-\xi/\lambda}}{\lambda}d\xi \quad (7)$$

Here, the total flux through the reference surface includes particles initially located at a distance ξ from the reference surface with the coordinate x and having a mean free path equal to this distance. Indeed, only such particles should be considered: particles with a shorter mean free path do not reach the reference surface, and particles with a larger length do not participate in the balance, since they have the same kinetic energy when approaching the control volume as when they leave it. Linearizing the functions in (7) with respect to ξ under the assumption that the mean free path is less than the characteristic scale of the change in the basic variables, we have, after integration,

$$Q_T = -\frac{m}{2}\frac{\Gamma\left(\frac{4}{\kappa}\right)}{\Gamma\left(\frac{1}{\kappa}\right)}\lambda\frac{d}{dx}\left(nv_s^3\right) \quad (8)$$

Taking into account the relation (5) and the formula for the mean free path [see the explanations Formula (3)], this expression can be transformed into the form

$$Q_T = -\sigma\left(\frac{2}{3n}\frac{dn}{dx}\theta^{3/2} + \sqrt{\theta}\frac{d\theta}{dx}\right) \quad (9)$$

Here we have introduced the notation

$$\sigma = \frac{3}{2}\frac{\Gamma\left(\frac{4}{\kappa}\right)\sqrt{\Gamma\left(\frac{1}{\kappa}\right)}}{\Gamma\left(\frac{3}{\kappa}\right)^{3/2}\pi\sqrt{m}d^2} \quad (10)$$

The balance of kinetic energy per unit volume can be described by the equation

$$\frac{dQ_T}{dx} = -q \quad (11)$$

Here, the value of q on the right-hand side of the equation represents the total energy dissipation in collisions. It is calculated as $q = \frac{1}{2}zn\overline{\Delta E}$, where $z = \sqrt{\overline{v^2}}/\lambda$ is the number of collisions experienced by one particle per unit time, and the dissipation for a single collision $\overline{\Delta E}$ is calculated by the formula (1). The factor 1/2 is included in order to avoid double-counting of the number of collisions. Thus, Eq. (11) can be rewritten after some transformations in the form

$$\sigma \frac{d}{dx}\left(\frac{2}{3n}\frac{dn}{dx}\theta^{3/2} + \sqrt{\theta}\frac{d\theta}{dx}\right) = \tilde{\gamma} n^2 \theta^{3/2}, \tag{12}$$

with

$$\tilde{\gamma} = \frac{\pi d^2}{\sqrt{m}}\left(1 - R_G^2\right) \tag{13}$$

## 5 General Analytical Solution of the Transport Equations

The stationary Eqs. (6) and (12) for the transfer of energy and momentum in a granular medium can be solved analytically in a general form. To find this solution, we take into account that these equations do not explicitly contain the coordinate *x,* and therefore their order can be reduced from the third to the second. We introduce the granular pressure *p* as a new independent variable and express the derivatives of the granular temperature with respect to *x* through its derivatives with respect to *p*, taking into account Eq. (6) and the equation of state (4):

$$\frac{d\theta}{dx} = -\mathrm{mg}\frac{p}{\theta}\frac{d\theta}{dp} \tag{14}$$

Substituting $n = p/\theta$ and $\frac{d\theta}{dx}$ into (12), we obtain, after some transformations, the following homogeneous linear differential equation for the variable $W = \sqrt{\theta}$:

$$p\frac{d^2W}{dp^2} + \frac{dW}{dp} - \gamma p W = 0, \tag{15}$$

with

$$\gamma = \frac{3\tilde{\gamma}}{2\sigma(mg)^2} = \frac{\left(1 - R_G^2\right)}{(\rho g d)^2} C_\gamma \tag{16}$$

Here ρ is the density of the particle material, and $C_\gamma = \frac{36\Gamma\left(\frac{3}{\kappa}\right)^{3/2}}{\Gamma\left(\frac{4}{\kappa}\right)\sqrt{\Gamma\left(\frac{1}{\kappa}\right)}}$ is the constant depending only on the distribution parameter κ. The value of $C_\gamma = 22.56$ is valid for the Msaxwell distribution (κ = 2), and for the distribution with κ = 1.5 we have $C_\gamma = 20.56$. Equation (15) has the general solution

$$W = \frac{C_1 sinh\left(\sqrt{\gamma}p\right)}{p} + \frac{C_2 cosh\left(\sqrt{\gamma}p\right)}{p} \tag{17}$$

where $C_1$ and $C_2$ are some constants determined from the boundary conditions. Then the granular temperature $\theta$, particle concentration $n$, and the coordinate $x$ corresponding to them are determined as functions of the granular pressure $p$ by the formulas

$$\theta = W^2, n = \frac{p}{W^2}, x = \frac{1}{mg} \int_{p}^{p_0} \frac{W^2 dp}{p} \tag{18}$$

The first of these formulas follows directly from the definition of $\theta$, the second from the equation of state (4), and the latter from Eq. (6). The parameter $p_0$ in the last expression is the pressure at $x = 0$, that is, on the vibrating plane.

Let us turn now to the formulation of the boundary conditions.

## 6 Energy Boundary Conditions on a Vibrating and on a Fixed Plane

Let us consider the flux of kinetic energy Q averaged over the period τ of oscillations of the plane from the plane to the depth of the layer. This flux is formed by particles striking the plane and changing their kinetic energy. It can be calculated with the following formula:

$$Q = n_G \frac{1}{\tau} \int_0^{\tau} dt \int_{-\infty}^{v_p(t)} \left(v_p(t) - v\right) f(v) \Delta T dv \tag{19}$$

In the expression (19), the quantity $n\left(v_p(t) - v\right) f(v) dv$ is the flux of the particles having a velocity less than the velocity $v_p(t)$ of the plane, that is, of those particles which are kinematically able to collide with the plane. The concentration in the immediate vicinity of the plane is denoted by $n_G$ and the change in the kinetic energy of a particle is denoted by $\Delta T$.

The value ΔT can be obtained by taking into account that a particle having velocity v acquires, after a collision, a velocity equal to $v_+ = v_p(t)(1 + R) - Rv$ . Here, R is the coefficient of restitution when the particle strikes a plane. This coefficient is not necessarily equal to the coefficient of restitution for mutual collisions of the particles $R_G$. The change in the kinetic energy of the particle $\Delta T = \mathrm{m}\left(v_+^2 - v^2\right)/2$ can thus be calculated as

$$\Delta T = (\mathrm{m}/2)\left(v_p(t)^2 (1 + R)^2 - 2R(1 + R) v v_p(t) + \left(R^2 - 1\right) v^2\right) \tag{20}$$

Substituting $\Delta T$ from (20) and $f(v)$ from (2) into the formula (19) and performing the corresponding integration leads, after transformations, to the following expression for the energy flux Q:

$$Q = n_G(\mathrm{m}/2)v_s^3\left((1+R)^2 H_2\left(\frac{U}{v_s}\right) - 2R(1+R)H_1\left(\frac{U}{v_s}\right) + \left(R^2-1\right)H_0\left(\frac{U}{v_s}\right)\right), \tag{21}$$

Here, the functions $H_s$ of the ratio $U/v_s$ are introduced, where U is the amplitude of the plane velocity. They are expressed in terms of their argument $\zeta$ as follows:

$$H_0(\zeta) = \left(\frac{1}{2}\Gamma\left(\frac{4}{\kappa}\right) + \Gamma\left(\frac{3}{\kappa}\right)\varsigma^3 Z_1(\zeta) - \Gamma\left(\frac{4}{\kappa}\right)\varsigma^4 Z_0(\zeta)\right)/\Gamma\left(\frac{1}{\kappa}\right)$$
$$H_1(\zeta) = \zeta\left(-\frac{1}{4}\Gamma\left(\frac{2}{\kappa}\right) + \Gamma\left(\frac{2}{\kappa}\right)\varsigma^2 Z_2(\zeta) - \Gamma\left(\frac{3}{\kappa}\right)\varsigma^3 Z_1(\zeta)\right)/\Gamma\left(\frac{1}{\kappa}\right)$$
$$H_2(\zeta) = \zeta^2\left(\frac{1}{4}\Gamma\left(\frac{2}{\kappa}\right) + \Gamma\left(\frac{1}{\kappa}\right)\varsigma Z_3(\zeta) - \Gamma\left(\frac{2}{\kappa}\right)\varsigma^2 Z_2(\zeta)\right)/\Gamma\left(\frac{1}{\kappa}\right) \tag{22}$$

The auxiliary functions $Z_k$ entering these expressions are computed as a rapidly convergent power series with an infinite radius of convergence:

$$Z_k(\zeta) = \frac{1}{2\sqrt{\pi}\,\Gamma\left(\frac{4-k}{\kappa}\right)}\sum_{j=0}^{\infty}\frac{(-1)^j \zeta^{j\kappa}\Gamma\left(\frac{5+jk}{2}\right)}{\Gamma(j+1)\Gamma\left(\frac{6+jk}{2}\right)\left(\frac{4-k}{\kappa}+j\right)} \tag{23}$$

If we introduce the granular temperature of the vibrating plane as

$$\theta_p = \frac{m\overline{v_p^2}}{2} = \frac{mU^2}{4} \tag{24}$$

and use the relationship (5) between the parameter $v_s$ and the granular temperature $\theta_G$ in the immediate vicinity of the plane, we can rewrite the expression for the flow (21) in the following form:

$$Q = \sqrt{2}n_G\theta_G^{\frac{3}{2}}m^{-\frac{1}{2}}\left(\frac{\Gamma\left(\frac{1}{\kappa}\right)}{\Gamma\left(\frac{3}{\kappa}\right)}\right)^{\frac{3}{2}}((1+R)^2 H_2\left(\sqrt{\frac{2\theta_p}{\theta_G}}\right) - 2R(1+R)H_1\left(\sqrt{\frac{2\theta_p}{\theta_G}}\right)$$
$$+ \left(R^2-1\right)H_0\left(\sqrt{\frac{2\theta_p}{\theta_G}}\right)) \tag{25}$$

Note that the relation (25) for the kinetic energy flux from the plane is also valid for a fixed plane ($\theta_p = 0$). In this case, the relation (25) takes the form

$$Q = -n_G\theta_G^{\frac{3}{2}}m^{-\frac{1}{2}}\left(\frac{\Gamma\left(\frac{4}{\kappa}\right)^2\Gamma\left(\frac{1}{\kappa}\right)}{2\Gamma\left(\frac{3}{\kappa}\right)^3}\right)^{\frac{1}{2}}\left(1-R^2\right) \tag{26}$$

The minus sign indicates the obvious fact that in this case the resultant flux of kinetic energy is directed to the plane, not from the plane.

Taking into account (25), (26), and (9), we can formulate the boundary conditions expressing the energy balance for the vibrating and fixed planes. These conditions express the fact that in the stationary state the flux of the radiated energy Q must be equal to the flux of energy $Q_T$ transferred to the depth of the layer, calculated from the formula (9)

$$Q_T = Q \tag{27}$$

Before specifying the conditions (27) of the variables p and W, it is necessary to understand which variable determines the concentration of $n_G$ near the plane that enters the expression for Q and also to consider the problem of specifying the granular pressure near the boundaries. To do this, consider the force acting on the vibrating plane and on the fixed plane.

## 7 Power Boundary Conditions on a Vibrating and on a Fixed Plane and the Granular Leidenfrost Effect

Let us consider the resulting flux of momentum to the plane averaged over the period τ of oscillations of the plane. As in the case of the kinetic energy considered in the previous section, the flow to be calculated is formed by particles striking the plane and changing their momentum as a result of this impact. This flow is equal to the averaged force of the granular gas acting on the plane, that is, the pressure $p_G$, and can be found by the following formula:

$$p_G = m(1+R)n_G \frac{1}{\tau}\int_0^{\tau} dt \int_{-\infty}^{v_p(t)} \left(v_p(t) - v\right)^2 f(v)dv \tag{28}$$

The integration with allowance for (9) and the subsequent transformations analogous to those performed in the analysis of the formula (19) lead to the following expression for the pressure $p_G$:

$$p_G = (1+R)n_G\left(\frac{mU^2}{2} + \frac{mv_s^2}{2}\frac{\Gamma\left(\frac{3}{\kappa}\right)}{\Gamma\left(\frac{1}{\kappa}\right)}\right) \tag{29}$$

Thus, taking into account the relations (5) and (24), we have

$$p_G = n_G(1+R)\left(1 + \frac{\theta_p}{\theta_G}\right)\theta_G \tag{30}$$

Consider the lower layer of the medium. It is in equilibrium under the action of the pressure $p_G$ from below [calculated with (30)], and the pressure $p_0$ at $x = 0$ from above [calculated with formula (4)], that is,

$$p_0 = n_0\theta_0 \tag{31}$$

Here, $n_0$ and $\theta_0$ are the concentration and the granular temperature at some small distance from the plane. Here, we take into account the possibility that the granular temperature and the concentration of particles near the surface undergo a jump. The appearance of such a jump is known as the granular Leidenfrost effect [9]. This effect is especially pronounced for the "hot" plane, when $\theta_p/\theta \gg 1$. However, for $\theta_p = 0$, the jump also takes place if $R \neq 0$.

At the same time, continuity of the granular pressure p near the vibrating surface should be assumed. This requirement follows from the equilibrium condition of the lower layer of the medium under consideration. Thus, $p_G = p_0$, and from the relations (30) and (31) it follows that

$$n_0\theta_0 = n_G(1 + R)\left(1 + \frac{\theta_p}{\theta_G}\right)\theta_G \tag{32}$$

Another relation connecting the values of concentration and temperature before and after the jump follows from the continuity of the material flux through the jump:

$$\int\limits_0^\infty (vn_G f_G(v) - vn_0 f_0(v))dv = 0 \tag{33}$$

Here, as before, $v$, $n$, and $f(v)$ denote the velocity, concentration, and distribution of the particle flux through the jump, and the indexes G and 0 correspond to the states before and after the jump. As a result of integration, we have

$$n_G\sqrt{\theta_G} = n_0\sqrt{\theta_0} \tag{34}$$

Taking into account (32), we obtain the following relations connecting quantities with the indices G and 0:

$$\theta_0 = (1 + R)^2\left(1 + \frac{\theta_p}{\theta_G}\right)^2\theta_G, n_0 = \frac{n_G}{(1 + R)\left(1 + \frac{\theta_p}{\theta_G}\right)} \tag{35}$$

The expression for the kinetic energy flux from the vibrating plane (25) can be rewritten in the form

$$Q = p_0\sqrt{\theta_0}m^{-\frac{1}{2}}B\left(\frac{\theta_p}{\theta_0}\right) \tag{36}$$

where the function B (z) can be calculated with sufficient accuracy for moderate values of ζ (ζ ≤ O (1)) as a fourth-degree polynomial in $\zeta^{1/2}$:

$$B(\zeta) = b_0 + b_1\zeta^{1/2} + b_2\zeta + b_3\zeta^{3/2} + b_4\zeta^2 \tag{37}$$

with the coefficients

$$b_0 = -\frac{(1-R)}{(1+R)}\frac{\Gamma\left(\frac{4}{\kappa}\right)\Gamma\left(\frac{1}{\kappa}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{\kappa}\right)^{\frac{3}{2}}\sqrt{2}}, b_1 = R\frac{\Gamma\left(\frac{2}{\kappa}\right)\Gamma\left(\frac{1}{\kappa}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{\kappa}\right)^{\frac{3}{2}}},$$

$$b_2 = \sqrt{2}\frac{\Gamma\left(\frac{1}{\kappa}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{\kappa}\right)^{\frac{3}{2}}}(1+R)\left(\frac{1}{2}\Gamma\left(\frac{2}{\kappa}\right)(1+R) + \Gamma\left(\frac{4}{\kappa}\right)(1-R)\right),$$

$$b_3 = (R+1)^2\frac{\Gamma\left(\frac{1}{\kappa}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{\kappa}\right)^{\frac{3}{2}}}\left(R\Gamma\left(\frac{2}{\kappa}\right) + \frac{1}{4}\kappa(2+R)\right),$$

$$b_4 = -\frac{(R+1)^2}{\sqrt{2}}\frac{\kappa\Gamma\left(\frac{1}{\kappa}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{3}{\kappa}\right)^{\frac{3}{2}}}\left(3\left(1-R^2\right)\Gamma\left(\frac{4}{\kappa}\right) + \frac{1}{8}\kappa(3+R)\right) \tag{38}$$

depending on the coefficient of restitution R and on the distribution parameter κ.

Figure 1 shows the function B (ζ) for different coefficients of restitution ($R = 0.6$, 0.7, 0.8, and 0.9) and for the distribution parameter $\kappa = 1.5$.

Figure 2 shows the function $B\,(\zeta)$ for various distribution parameters κ ($\kappa = 1.5$, 1.6, 1.8, and 2) and with the coefficient of restitution $R = 0.8$.

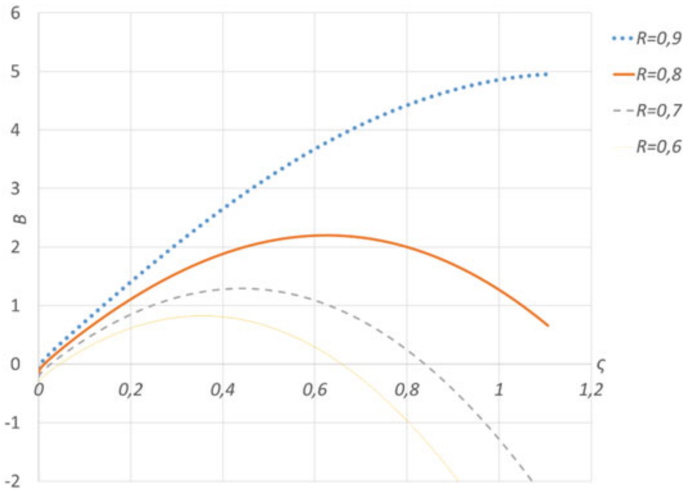



**Fig. 1** Function B (ζ) for different coefficients of restitution (R = 0.6, 0.7, 0.8, and 0.9) for the value of the distribution parameter κ = 1.5

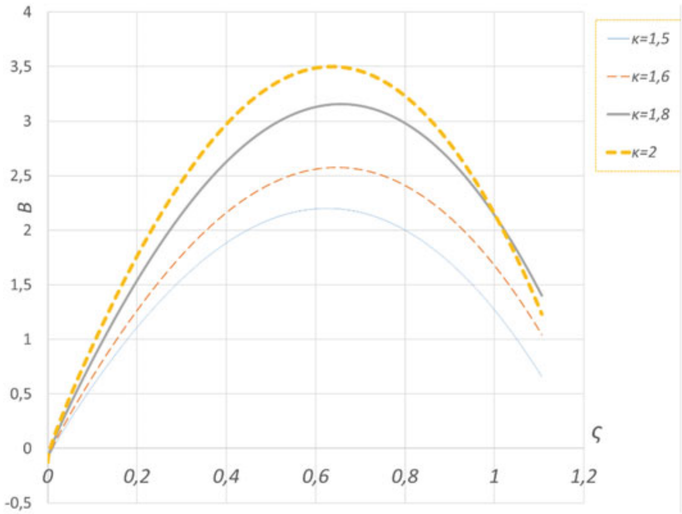



**Fig. 2** The function B (ζ) for various distribution parameters κ (κ = 1.5, 1.6, 1.8, and 2) at the value of the coefficient of restitution R = 0.8

The dependence B $\left(\theta_p/\theta\right)$, as might be expected, has points of intersection with the abscissa axis. This point corresponds to the equilibrium between the energy that the plane radiates (transmits to the layer) and absorbs due to the inelasticity of the impacts. There are two such points: to the right and to the left of the maximum. Let us consider the qualitative behavior of the system in the vicinity of the left point of these intersections. For a sufficiently small amplitude of velocity oscillations (the "temperature" of the plane) or for a sufficiently large granular temperature of the medium near the plane, the energy absorption prevails and the granular gas "cools" near the plane. Conversely, if the granular temperature is below a certain value, the granular gas "heats up". This circumstance indicates the possibility of a stable stationary state. Oppositely, the right point of intersection corresponds to an unstable equilibrium.

## 8 Statement of Boundary-Value Problems on a Stationary Vibro-Excited Granular Layer

Let us return to the boundary condition on the vibrating plane (27) and concretize it under the assumption that at this boundary the pressure $p_G$ is given and is equal to some value $p_0$. We rewrite condition (27) for the variables $p$ and $W$.

The expression (36) for the flow Q leads, after the transition to the variable $W = \sqrt{\theta_0}$, to the following representation for the kinetic energy flux from the plane:

$$Q = p_0 m^{-1/2} W B\left(W_p^2 / W^2\right) \tag{39}$$

Here we denote $W_p = \sqrt{\theta_p}$ and, for brevity, the index 0 for the quantity $W = W(p_0)$ is omitted.

On the other hand, from (9) with (4) and (6), the following representation for $Q_T$ can be obtained:

$$Q_T = \frac{2}{3}\sigma g m\left(W + p\frac{dW}{dp}\right) \tag{40}$$

Thus, we have a boundary condition on the vibrating plane:

$$p_0 m^{-1/2} W B(W_p^2 / W^2) = \frac{2}{3}\sigma g m\left(W + p_0\frac{dW}{dp}\right), \mathrm{p} = p_0 \tag{41}$$

The determination of the constants $C_1$ and $C_2$ in the general solution (17) of the Eq. (15) for W (p) needs one more boundary condition: on the upper boundary of the layer. Different boundary conditions are of interest for different technical applications. So in the case of a fixed plane bounding a granular layer from above, one should set a condition similar to (41) which has the form

$$p_L m^{-1/2} W B(0) = \frac{2}{3}\sigma g m\left(W + p_L\frac{dW}{dp}\right), \mathrm{p} = P_L \tag{42}$$

Here, $p_L$ is the pressure at the upper, fixed boundary, that is, for x = L, where L is the height of the tank with the granular medium. Note that in this problem formulation the values of $p_0$ and $p_L$ are not known in advance and thus there are four unknown parameters: $p_0$, $p_L$, $C_1$, and $C_2$. To determine them, we can use the conditions (36) and (37) and the following relations:

$$p_0 - p_L = mgN, \tag{43}$$

$$L = \frac{1}{mg}\int_{p_L}^{p_0}\frac{W^2 dp}{p} \tag{44}$$

Here, $N$ is the total number of particles over the unit surface. The relation (43) follows from (6) as a result of integration over the total height of the layer L. The condition (44) corresponds to the last formula (18) for $x = L$.

We will not dwell on the solution of this problem in detail, but consider a simpler problem concerning a layer with a free surface, that is, a layer unbounded from above.

## 9 A Granular Layer with a Free Surface

For a layer unbounded from above, the concentration and therefore the pressure $p_L$ on its upper, free boundary are equal to 0. The pressure $p_0$ on the lower, vibrating surface is immediately determined from the formula (43)

$$p_0 = mgN, \tag{45}$$

Since the integral in the formula (44) is not convergent at $p = 0$, the height of the layer is formally infinitely large, although a decrease in the concentration to practically insignificant values allows us to speak of a certain height of the layer. In any case, we should take $C_2 = 0$ in the general solution of (17) for W to exclude a physically meaningless infinite increase of W, and hence also of the granular temperature $\theta = W^2$, to infinity at p → 0. Thus, we have

$$W = \frac{C_1 \sinh\left(\sqrt{\gamma}\,p\right)}{p} \tag{46}$$

In particular, it follows that when the coordinate x is unbounded, the granular temperature tends to the value $\theta_\infty = C_1^2\gamma$.

The constant $C_1$ is obtained from condition (25). It is easy to verify that, due to (17), the complex $W + p_0\frac{dW}{dp}$ entered in (36) is equal to $Wp_0\sqrt{\gamma}\,\mathrm{cth}\left(\sqrt{\gamma}\,p_0\right)$. Therefore, the quantities $W$ and $p_0$ in both parts of the condition (41) cancel out and it takes the form

$$F = B(\zeta) \tag{47}$$

with

$$F = \frac{2}{3}\sigma g m^{3/2}\sqrt{\gamma}\,\mathrm{cth}\left(\sqrt{\gamma}\,p_0\right), \tag{48}$$

The previously introduced value of $\zeta$ [see formula (37)] is related to the granular temperature in accordance with the formula (49), and (48) can also be expressed in terms of the formula

$$\frac{\theta(p_0)}{\theta_p} = \frac{1}{\zeta} \tag{49}$$

The dimensionless pressure on the vibrating plane can be expressed with the help of (47) and (48) as follows:

$$\sqrt{\gamma}\,p_0 = \frac{\Gamma\left(\frac{3}{\kappa}\right)^{\frac{3}{4}}}{\Gamma\left(\frac{4}{\kappa}\right)^{\frac{1}{2}}\Gamma\left(\frac{1}{\kappa}\right)^{\frac{1}{4}}}\pi d^2 N\sqrt{\left(1 - R_G^2\right)} \tag{50}$$

or through the dimensionless parameters ζ and

$$\varphi = \frac{2}{3}\sigma g m^{3/2}\sqrt{\gamma} = 6\Gamma\left(\frac{3}{\kappa}\right)\sqrt{\left(1 - R_G^2\right)} \tag{51}$$

as follows:

$$\sqrt{\gamma}p_0 = \frac{1}{2}\ln\left(\frac{\varphi + B(\zeta)}{\varphi - B(\zeta)}\right) \tag{52}$$

Together, the formulas (49) and (50) form a parametric dependence of the dimensionless granular temperature $\theta(p_0)/\theta_p$ on the dimensionless pressure $\sqrt{\gamma}p_0$ through the parameter ζ. Note that the parameter $\sqrt{\gamma}p_0$ can also be presented as $\varphi$, where ν is calculated by the formula

$$\nu = \frac{\pi d^2 N}{6\Gamma\left(\frac{3}{\kappa}\right)^{\frac{1}{4}}\Gamma\left(\frac{1}{\kappa}\right)^{\frac{1}{4}}\Gamma\left(\frac{4}{\kappa}\right)^{\frac{1}{2}}} \tag{53}$$

and characterizes the number of layers of particles. Using this parameter, formula (52) can be rewritten in the form

$$\nu = \frac{1}{2\varphi}\ln\left(\frac{\varphi + B(\zeta)}{\varphi - B(\zeta)}\right) \tag{54}$$

Together with (49), it gives the parametric dependence of $\frac{\theta(p_0)}{\theta_p}$ on the parameter $\nu$. Note that this dependence contains additionally only the distribution parameter κ and the two coefficients of restitution: R (through the function B) and $R_G$ (via the parameter $\varphi$). These parameters are specified for a given medium. The variable parameters that can be easily changed in the technological processing conditions—the particle size d, the particle number N, and the amplitude of the velocity U of the plane—are involved in the dimensionless parameter $\nu$ and in the scale factor $\theta_p$. Therefore, the transition to other technological process conditions (d, N, U) is carried out by simple scaling of the obtained universal dependence. Figure 3 shows this dependence for $\kappa = 1.5$ and $R = 0.8$.

The limit of the granular temperature at x → ∞ can be calculated as

$$\theta_\infty = \mathrm{C}_1^2\gamma = \theta(p_0)\frac{p_0^2\gamma}{\sinh^2\left(\sqrt{\gamma}p_0\right)} \tag{55}$$

The corresponding dependence is shown in Fig. 4.

After $\theta(p_0)$ is found, the granular temperature $\theta(p)$ at any pressure $p < p_0$ is calculated, taking into account (46) as

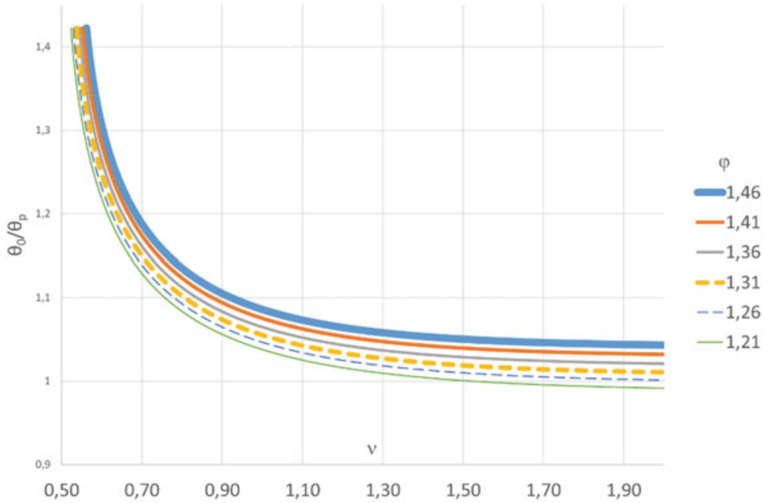



**Fig. 3** Dependence of the dimensionless temperature near the plane on the parameter $\nu$ for κ = 1.5, R = 0.8, and different values of φ

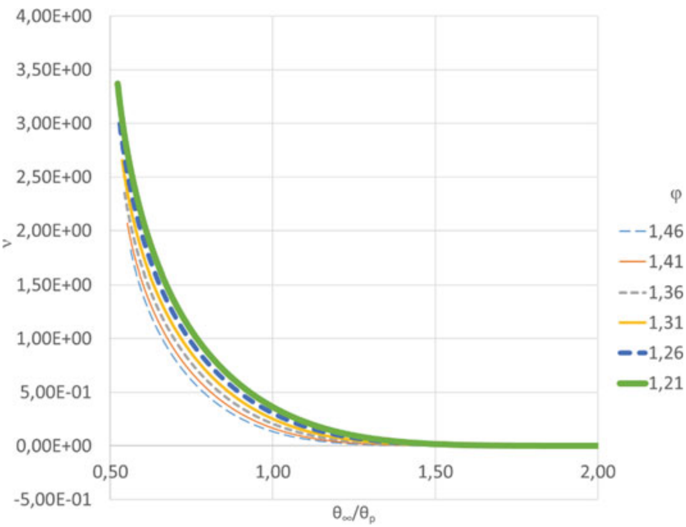



**Fig. 4** Connection between the temperature away from the plane and the parameter $\nu$

$$\theta(p) = \theta(p_0)\frac{p_0^2 \sinh^2\left(\sqrt{\gamma}\, p\right)}{p^2 \sinh^2\left(\sqrt{\gamma}\, p_0\right)} \tag{56}$$

Now, using the formulas (18), the concentration n can be found as a function of pressure and coordinate x, which corresponds to some pressure $p < p_0$:

$$n = \frac{p}{\theta(p)}, x = \frac{1}{mg}\int_{p}^{p_0} \frac{\theta(p)dp}{p} \tag{57}$$

It is also possible to find the Leidenfrost jumps in the concentration near the vibrating plane in accordance with Eq. (35) and the consumed power per unit transverse surface, which is equal to the kinetic energy flux from the plane

$$Q_T = \frac{2}{3}\sigma gm\left(W + p\frac{dW}{dp}\right) = \frac{2}{3}\sigma gm\sqrt{\theta(p_0)}p_0\sqrt{\gamma}\mathrm{cth}\left(\sqrt{\gamma}p_0\right) \tag{58}$$

## 10 Examples of Calculations

Table 1 shows the parameters and calculated values for some basic variants to be varied, giving an idea of the orders of magnitude under consideration.

Figure 5a, b shows the dependence of the granular temperature θ reduced to its value near the vibrating plane θ0 on the dimensionless distance to the vibrating plane

In accordance with intuitive imagination, the temperature decreases rather rapidly with $x$ and reaches its limit value $\theta_\infty$.

Figure 6a, b shows the corresponding dependence of the concentration $n$ scaled to its value near the vibrating plane $\theta_0$.

It can be seen from the figure that the concentration as a function of the vertical coordinate has a pronounced maximum. This circumstance is determined by the rapid decrease in the granular temperature with distance from the vibrating plane. In the figure, a Leidenfrost jump in the concentration near the vibrating plane is also seen.

In the case of a constant temperature, a monotonic exponential decrease in the concentration according to the Boltzmann law would be observed. A similar exponential drop of the pressure with x in the case of a constant temperature would take place. In fact, according to the calculation of the proposed model, the pressure as a function of the coordinate has a characteristic inflection. These dependencies are shown in Fig. 7a, b.

Formula (58) makes it possible, in particular, to calculate the power needed to maintain a stationary state. Figure 8a, b shows the dependencies of the power on the coefficient of restitution for mutual collisions of the particles $R_G$ at different values of the coefficient of restitution for particle impacts on the plane R and the distribution parameters κ.

Note that these dependencies decrease for each fixed value of R, as expected.

Only some examples of calculations are presented here. However, the method underlying them can be used, with further development, for energy optimization of a number of processes in the fields of chemical technology and the processing of mineral and technogenic raw materials [44–48].

**Table 1** Parameters and calculated values for a basic variant

| | | | |
|---|---|---|---|
| R | Coefficient of restitution for particle–plane collision | 0.8 | |
| $R_G$ | Coefficient of restitution for particle–particle collision | 0.9 | |
| D | Diameter of the particles | 0.005 | M |
| δL | Bulk density of the layer material | 2500 | $kg/m^2$ |
| Δ | Density of particles | 2700 | $kg/m^3$ |
| G | Acceleration of free fall | 9.8 | $m/s^2$ |
| l | Height of the layer in bulk state | 0.01 | M |
| U | Amplitude of the plane velocity | 1 | m/s |
| κ | The parameter in the velocity distribution law | 1.5 | |
| *Calculated parameters* | | | |
| m | Mass of the particle | 1.77E−04 | Kg |
| N | Number of particles over the unit area | 1.41E+05 | $1/m^2$ |
| $p_0$ | Pressure of the granular medium in the lower section | 2.45E+02 | Pa |
| γ~ | Dissipation parameter | 1.12E−03 | $m^2/kg^{1/2}$ |
| σ | Parameter σ | 2.52E+06 | $1/m^2/kg^{1/3}$ |
| γ | Modified dissipation parameter | 2.23E−04 | $Pa^{-1/2}$ |
| $p_0$*γ^0.5 | Dimensionless pressure of the granular medium in the lower section | 3.66E+00 | – |
| nmax | Maximum possible concentration | 1.41E+07 | $1/m^3$ |
| f | Parameter f | 5.78E−01 | – |
| $b_0$ | Coefficient in function B | −1.38E−01 | |
| $b_1$ | Coefficient in function B | 8.31E−01 | |
| $b_2$ | Coefficient in function B | 3.27E+00 | |
| $b_3$ | Coefficient in function B | 6.65E+00 | |
| $b_4$ | Coefficient in function B | −9.35E+00 | |
| ζ | Ratio of granular temperatures of the plane and particles in the lower section | 0.101 | |
| $\theta_p$ | Granular temperature of the plane | 4.42E−05 | J |
| $\theta_0$ | Granular temperature in the lower section | 4.37E−04 | J |
| $\theta_\infty$ | Limit of temperature at x–>∞ | 1.55E−05 | J |
| n | Concentration at some distance from the lower section | 5.60E+05 | Pa |
| $n_G$ | Concentration near the plane | 2.83E+05 | Pa |
| $Q_T$ | Power | 2.23E+02 | $Wt/m^2$ |

In the next section, the developed theory is applied to the description of a phenomenon observed in experiments: competitive clustering. This serves primarily to verify the proposed model. However, this consideration can also be of independent interest, since it can give a new impulse to the practical application of this phenomenon.

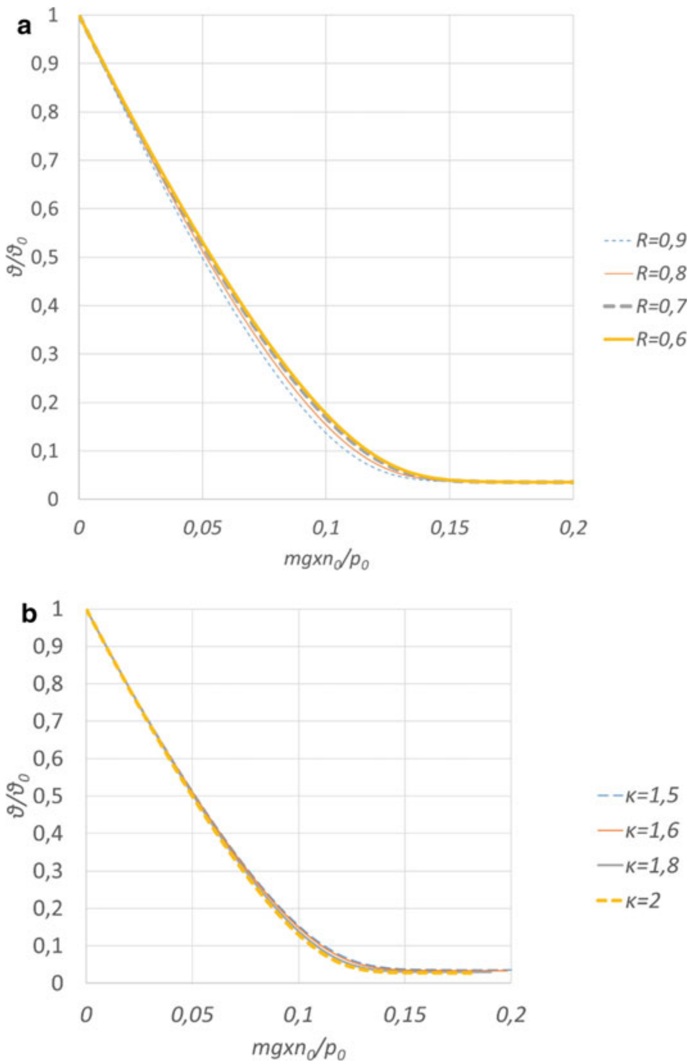



**Fig. 5** **a** Typical temperature dependence of the granular temperature on the distance to the vibrating plane (basic variant with a variation of R). **b** Typical temperature dependence of the granular temperature on the distance to the vibrating plane (basic variant with a variation of κ)

## 11 Application to the Experiment with "Maxwell's Demon" (Competitive Clustering)

In reference [2], one can find a demonstration of an enchanting experiment called competitive clustering. In article [3], this experiment is discussed under the name of the "Maxwell's demon" experiment. In this experiment, the particles are placed

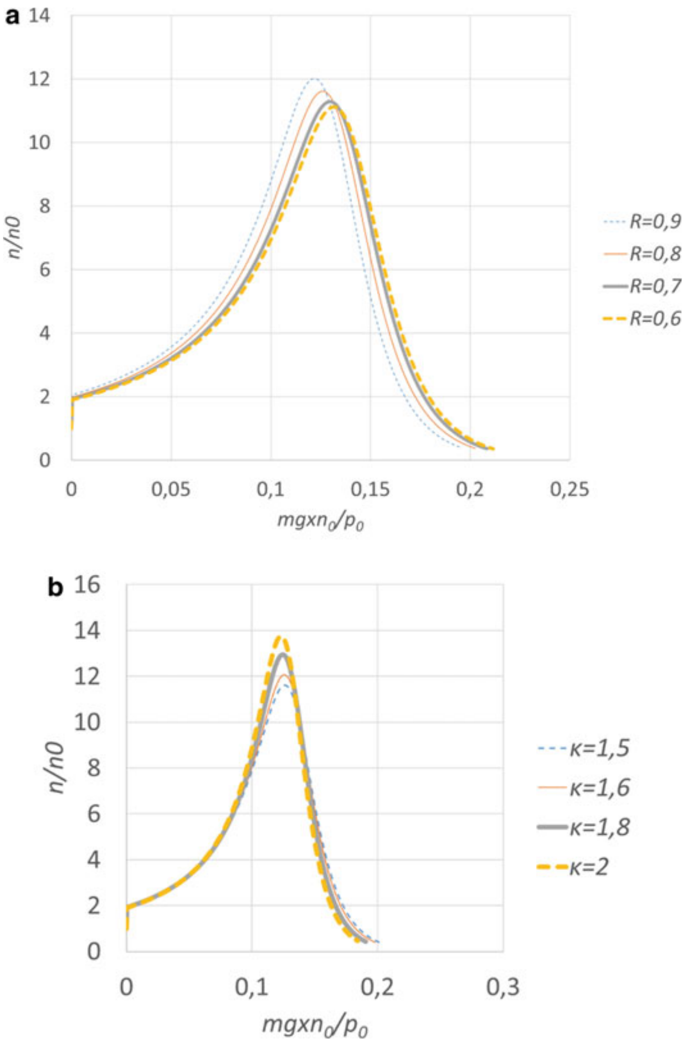



**Fig. 6** **a** Typical dependence of concentration on distance from the vibrating plane (basic variant with a variation of R). **b** Typical dependence of concentration on distance from the vibrating plane (basic variant with a variation of κ)

in two identical chambers, between which is a partition containing a window at a certain height. It turns out that the chamber that initially contains fewer particles is emptied fairly quickly—the particles pass from this chamber to the one that initially contains more particles (Fig. 9).

This result is at first glance paradoxical. It seems intuitively that the resultant particle flux is proportional to the mean concentration difference and is directed to the side where the mean concentration is lower. However, this is not the case in this

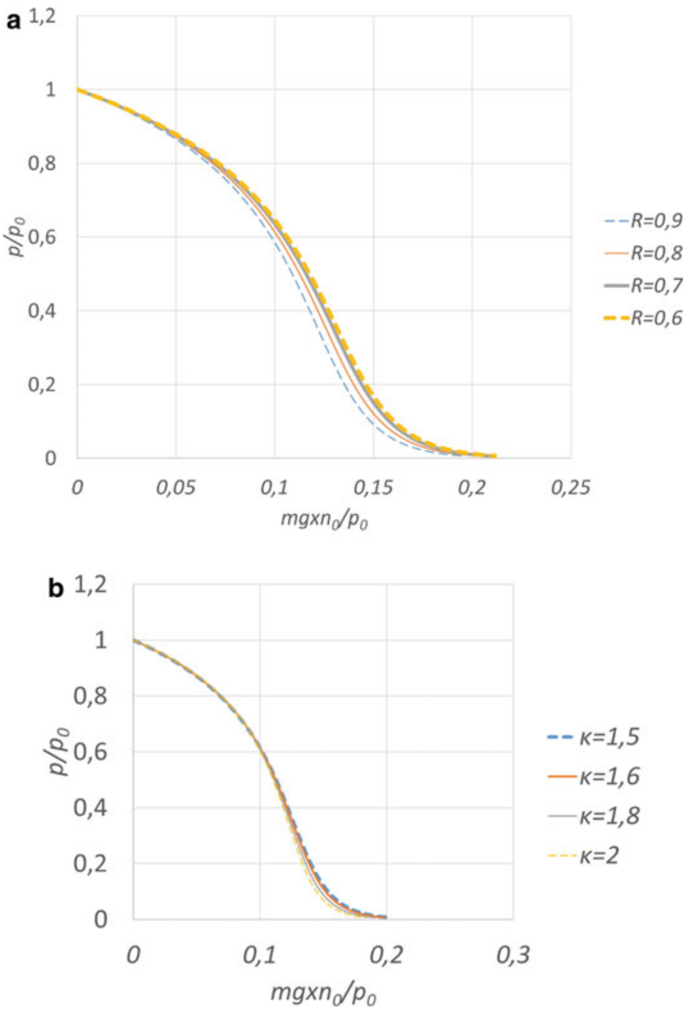
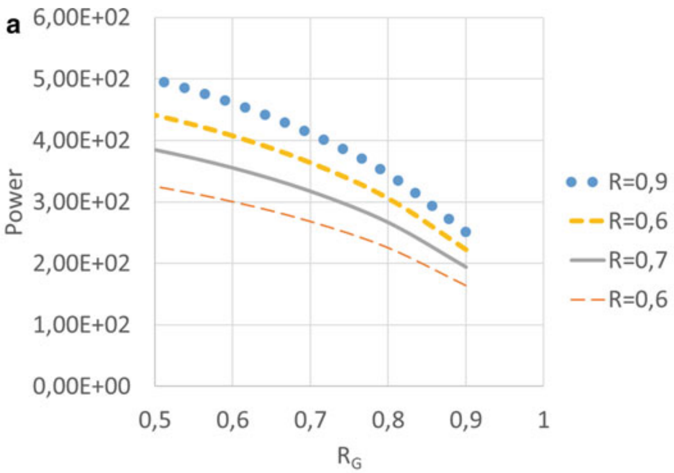



**Fig. 7** **a** Typical dependence of the concentration on the distance from the vibrating plane (basic variant with a variation of R). **b** Typical dependence of the concentration on the distance from the vibrating plane (basic variant with a variation of κ)

experiment. A qualitative explanation of the effect is that the rate of the transitions is determined not only by concentrations but by the particle velocities. An increase in the number of particles in one of the chambers (in Fig. 9 on the left) leads to a decrease in the granular temperature in it and hence in the speed at which the particles pass from it to the neighboring chamber. This reduction in speed under certain conditions is not compensated by the possible increase in concentration at the window level in a chamber with a large number of particles.



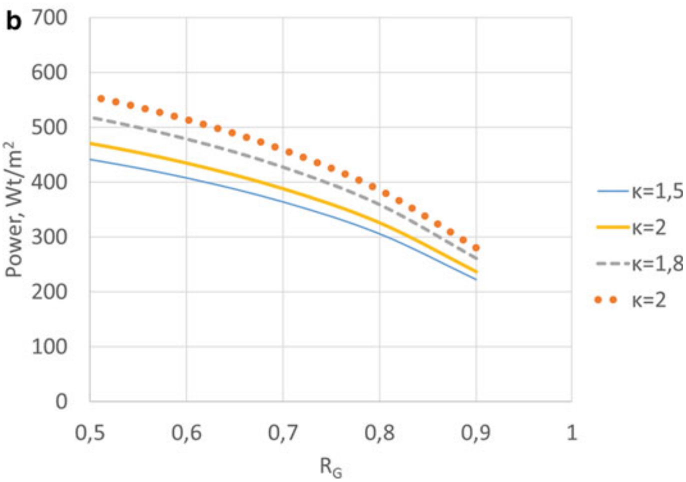




**Fig. 8** **a** Power [Wt/m$^2$] versus the coefficient of restitution for mutual collisions of $R_G$ at different values of the coefficient of restitution for particle impacts on the plane R. **b** Power [Wt/m$^2$] versus the coefficient of restitution for mutual collisions of $R_G$ at different values of the distribution parameters κ

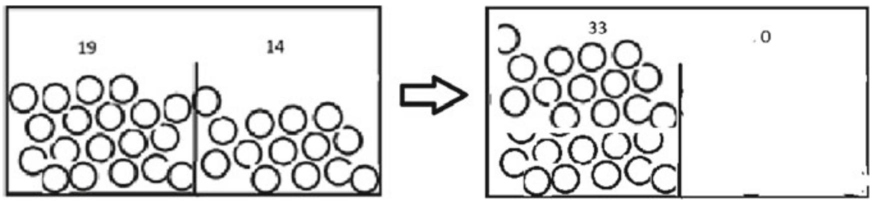



**Fig. 9** Scheme of the experiment

Based on the results described in the previous sections, a quantitative analysis of this experiment can be proposed.

We take into account that the resultant flux of particles through the window of area S from chamber 1 to chamber 2 is

$$q_{12} = S\int_{0}^{\infty}(vn_1 f_1(v) - vn_2 f_2(v))dv \tag{59}$$

Here, as before, $v$, $n$, and $f(v)$ denote the velocity, concentration, and velocity distribution according to (2) at the window level, and the indexes 1 and 2 correspond to the camera numbers. Integration leads to the formula

$$q_{12} = \chi\left(n_1\sqrt{\theta_1} - n_2\sqrt{\theta_2}\right) \tag{60}$$

or

$$q_{12} = \chi\left(\frac{p_1}{\sqrt{\theta_1}} - \frac{p_2}{\sqrt{\theta_2}}\right) \tag{61}$$

Here, is denoted

$$\chi = \frac{S\Gamma\left(\frac{2}{\kappa}\right)}{\sqrt{2m\Gamma\left(\frac{1}{\kappa}\right)\Gamma\left(\frac{3}{\kappa}\right)}} \tag{62}$$

Let the number of particles in chamber 1 decrease and accordingly let the number in chamber 2 increase. If this leads to an increase in $q_{12}$, then obviously there is a further increase in the flow of particles from chamber 1 to chamber 2 and the symmetric state is not stable. Otherwise, the concentration is levelled. Thus, the instability of a symmetric state takes place under the condition

$$D = \frac{d}{dN}\left(\frac{p}{\sqrt{\theta}}\right) < 0 \tag{63}$$

The derivative is most easily determined directly by giving a small increment in the total number of particles in the base case (Table 1) and computing the quantities $\frac{p}{\sqrt{\theta}}$ at the same values of the coordinate x interpreted as one of the possible positions of the window.

Figure 10 shows the dependence of D on the dimensionless coordinate characterizing the position of the window.

Figure 10 shows the following.

- The instability of the symmetric state, that is, the "Maxwell's demon" effect, arises only starting from a certain height of the window position (D < 0).

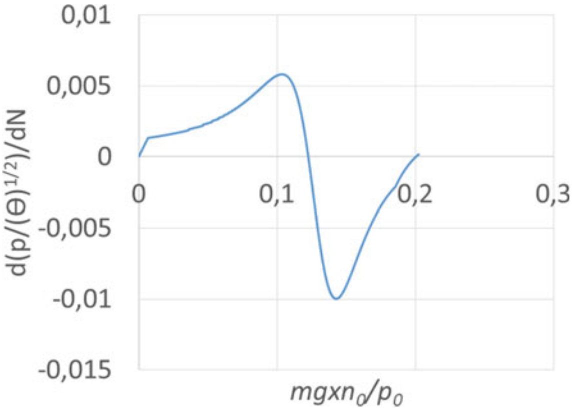



**Fig. 10** Dependence of the D value characterizing the stability of the symmetric state on the dimensionless coordinate characterizing the position of the window (D > 0 means stability)

- There is a certain height of the location of the window for which the effect is expressed most strongly (D < 0 and | D | is maximal).

## 12 Conclusion

The main results of this paper are as follows:

- Equations for the stationary state of a vibro-excited granular material (granular gas) are obtained, taking into account the non-Maxwellian law of velocity distribution and the presence of significant gradients of both the concentration and the granular temperature.
- Nonlinear boundary conditions on the vibrating and fixed planes are formulated.
- The problem of a layer with a free surface is posed and analytically solved.
- Simple formulas for calculating the granular temperature, concentration, and pressure near the vibrating plane and at any point along the height of the layer and also for calculating the Leidenfrost jump in the concentration near the exciting plane are obtained.
- A formula is derived for the power of the layer spent to maintain the steady state of the layer.
- A quantitative description of the phenomenon of "Maxwell's demon" (competitive clustering) is developed.

The developed theory can be used in further development and validation for energy optimization of the processes of chemical technology and processing of mineral and technogenic raw materials.

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