Meta-conformal Invariance and Their Covariant Correlation Functions

Malte Henkel and Stoimen Stoimenov

Abstract Meta-conformal invariance is a novel class of dynamical symmetries, with dynamical exponent $z = 1$, and distinct from the standard ortho-conformal invariance. The meta-conformal Ward identities can be directly read off from the Lie algebra generators, but this procedure implicitly assumes that the co-variant correlators should depend holomorphically on time- and space coordinates. Furthermore, making this assumptions leads to un-physical singularities in the co-variant correlators. We show how to carefully reformulate the meta-conformal Ward identities in order to obtain regular, but non holomorphic expressions for the co-variant two-point functions, both in $d = 1$ and $d = 2$ spatial dimensions.

1 Introduction

Many brilliant applications of conformal invariance are known, ranging from string theory and high-energy physics [36], or to two-dimensional phase transitions [9, 16, 19] or the quantum Hall effect [11, 17]. These applications are based on a geometric definition of conformal transformations, considered as local coordinate transformations $\boldsymbol{r} \mapsto \boldsymbol{r}' = \boldsymbol{f}(\boldsymbol{r})$, of spatial coordinates $\boldsymbol{r} \in \mathbb{R}^2$ such that angles are kept unchanged. The associated Lie algebra is called the '*conformal Lie algebra*'.

In Table 1, examples of infinite-dimensional Lie groups of time-space transformations are shown. They represent attempts to answer the question "Is it possible to adapt conformal invariance to dynamical problems ?" A minimal requirement

M. Henkel (✉)
Laboratoire de Physique et Chimie Théoriques (CNRS UMR 7019), Université de Lorraine Nancy, B.P. 70239, 54506 Vandœuvre-lès-Nancy Cedex, France
e-mail: malte.henkel@univ-lorraine.fr

Centro de Física Teórica e Computacional, Universidade de Lisboa, 1749-016 Lisboa, Portugal

MPIPKS Dresden, Nöthnitzer Straße 38, 01187 Dresden, Germany

S. Stoimenov
Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, Blvd., 1784 Sofia, Bulgaria
e-mail: stoimenn@yahoo.fr

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_5

is to distinguish time and space variables through their global rescaling, according to $t \mapsto t' = bt$ and $\boldsymbol{r} \mapsto \boldsymbol{r}' = b^{1/z}\boldsymbol{r}$ which defines the *dynamical exponent* z. In what follows, we shall consider infinitesimal transformations where $b = 1 + \varepsilon$, with $|\varepsilon| \ll 1$. Then a rescaling transformation is described by an infinitesimal generator, which for global dilatations on time- and space-coordinates takes the form $X_0 = -t\partial_t - \frac{1}{z}\boldsymbol{r}\cdot\partial_{\boldsymbol{r}} - \delta$. The parameter δ is the scaling dimension of the scaling operator $\varphi = \varphi(t, \boldsymbol{r})$ on which the generator X_0 is thought to act. Practical use of this is made for the computation of n-point correlation functions $C^{[n]} = C^{[n]}(t_1, \ldots, t_n; \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n) := \langle \varphi_1(t_1, \boldsymbol{r}_1) \cdots \varphi_n(t_n, \boldsymbol{r}_n)\rangle$. The dilatation-invariance of such a correlator is expressed via a *Ward identity*, which for the global dilatations described by X_0 takes the form

$$\sum_{j=1}^{n} \left(-t_j \frac{\partial}{\partial t_j} - \frac{1}{z}\boldsymbol{r}_j \cdot \frac{\partial}{\partial \boldsymbol{r}_j} - \delta_j \right) C^{[n]}(t_1, \ldots, t_n; \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n) = 0 \qquad (1)$$

and it becomes explicit how the dynamical exponent z distinguishes between temporal and spatial coordinates. Different symmetries will lead to different Ward identities which describe together constraints on the form of the n-point correlator $C^{[n]}$. Explicit examples will be given in later sections. These differential equation constraints are only consistent if the generators, such as X_0, belong to a well-defined algebraic structure, e.g. a Lie algebra.

It follows from time-space rotation-invariance that conformal invariance must have $z = 1$. In general, z has a non-trivial value [44]. In $1 + 1$ time-space dimensions, there exists an infinite hierarchy of models with dynamical exponent $1 < z \leq 2$ [37]. Lower bounds on z are derived from hydrodynamic projections of many-body dynamics [13]. Attempts of identifying dynamical conformal invariance goes back at least to critical dynamics of a two-dimensional statistical system [12]. In Table 1, we distinguish the well-studied *'ortho-conformal'* transformations [9], which in the two-dimensional space made from time-space points $(t, r) \in \mathbb{R}^2$ are angle-preserving, from recently constructed groups of *'meta-conformal'* transformations [20, 25, 28, 42], which in general are not angle-preserving but which share certain algebraic properties with ortho-conformal transformations in Table 1.

The most simple prediction of ortho-conformal invariance concerns the form of the co-variant two-point function $C = C(z_1, \bar{z}_1, z_2, \bar{z}_2) = \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle$ built from so-called 'quasi-primary' scaling operators ϕ_j, with 'conformal weights' Δ_j and $\overline{\Delta}_j$ [9]. In complex light-cone coordinates $z = t + \mathrm{i}\mu r$, $\bar{z} = t - \mathrm{i}\mu r$, one has

$$C_{\rm ortho}(z_1, \bar{z}_1, z_2, \bar{z}_2) = \delta_{\Delta_1,\Delta_2}\delta_{\overline{\Delta}_1,\overline{\Delta}_2}\left(z_1 - z_2\right)^{-2\Delta_1}\left(\bar{z}_1 - \bar{z}_2\right)^{-2\overline{\Delta}_1} \qquad (2)$$

up to normalisation. Herein, $1/\mu$ has the dimensions of a velocity. In deriving this kind of result, auxiliary assumptions are made. Analogously with Eq. (1), the requirement of ortho-conformal co-variance leads to a set of linear partial first-order differential equations for C, the so-called *global ortho-conformal Ward identities*. Their joint

Table 1 Several infinite-dimensional groups of time-space transformations, defined by the corresponding coordinate changes. Unspecified (vector) functions are assumed (complex) differentiable and $\mathscr{R}(t) \in SO(d)$ is a smoothly time-dependent rotation matrix. The physical time- and space-coordinates, the associated dynamical exponent z of this standard representation and the physical nature of the co-variant n-point functions is also indicated.

Group	Coordinate changes		Phys. coordinates	z	Co-variance
Ortho-conformal $(1+1)D$	$z' = f(z)$	$\bar{z}' = \bar{z}$	$z = t + \mathrm{i}\mu r$	1	Correlator
	$z' = z$	$\bar{z}' = \bar{f}(\bar{z})$	$\bar{z} = t - \mathrm{i}\mu r$		
Meta-conformal $1D$	$u = f(u)$	$\bar{u}' = \bar{u}$	$u = t$	1	Correlator
	$u' = u$	$\bar{u}' = \bar{f}(\bar{u})$	$\bar{u} = t + \mu r$		
Meta-conformal $2D$	$\tau' = b(\tau)$	$w' = w$ $\quad \bar{w}' = \bar{w}$	$\tau = t$		
	$\tau' = \tau$	$w' = f(w)$ $\quad \bar{w}' = \bar{w}$	$w = t + \mu(r_{\parallel} + \mathrm{i}r_{\perp})$	1	Correlator
	$\tau' = \tau$	$w' = w$ $\quad \bar{w}' = \bar{f}(\bar{w})$	$\bar{w} = t + \mu(r_{\parallel} - \mathrm{i}r_{\perp})$		
Conformal galilean	$t' = b(t)$	$\boldsymbol{r}' = (\mathrm{d}b(t)/\mathrm{d}t)\,\boldsymbol{r}$			
	$t' = t$	$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a}(t)$		1	Correlator
	$t' = t$	$\boldsymbol{r}' = \mathscr{R}(t)\boldsymbol{r}$			
Schrödinger-Virasoro	$t' = b(t)$	$\boldsymbol{r}' = (\mathrm{d}b(t)/\mathrm{d}t)^{1/2}\,\boldsymbol{r}$			
	$t' = t$	$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a}(t)$		2	Response
	$t' = t$	$\boldsymbol{r}' = \mathscr{R}(t)\boldsymbol{r}$			

solutions Eq. (2) are necessarily *holomorphic* (or anti-holomorphic) functions in the variables $z_j, \bar{z}_j$ [29].

In this work, we shall examine the analogous question for meta-conformal invariance. Known physical examples of confirmed meta-conformal invariance are of two types. First, there exist spatially *non-local* representations, which arise as a dynamical symmetry of certain non-local equations of motion which occur for example in diffusion-limited erosion [34], the kink-terrace-step model for vicinal surfaces [39] or the associated quantum chain [31] which is a conformal field-theory with central charge $c = 1$ [38]. Some predictions of meta-conformal invariance for response functions have been confirmed in these models [26, 27]. Second, a different type of meta-conformal invariance, with spatially *local* representations, has been identified recently in the kinetics of biased spin systems, see Fig. 1, such as the kinetic $1D$ Glauber-Ising model with a bias, sufficiently long-ranged initial conditions and quenched to zero temperature [28, 43]. The influence of transverse dimensions on the representations of meta-conformal transformations is currently under investigation. However, the focus of this work rather is on the formal study of meta-conformal representations as time-space transformations and the boundedness of the resulting two-point correlators.

In order to do so, we begin by analysing the consequences of writing analogous global Ward identities for meta-conformal invariance [20, 28, 42]. As we shall see in Sect. 2, the straightforward implementation of the global meta-conformal Ward identities leads to un-physical singularities in the time-space behaviour of such correlators. These singularities arise since the meta-conformally co-variant correlators are no longer holomorphic functions of their arguments. Therefore, a more careful approach is required, which we shall explicitly describe in Sects. 3 and 4, respectively, for $d = 1$ and $d = 2$ spatial dimensions. Our main result is the explicit form of a meta-conformally co-variant two-point function which remains bounded everywhere, as stated in Eqs. (33, 34) in Sect. 5. An appendix contains mathematical background on Hardy spaces in restricted geometries, for both $d = 1$ and $d = 2$.

2 Global Meta-conformal Ward Identities

Meta-conformal invariance arises as a dynamical symmetry of the simple equation $\mathscr{S}\varphi(t,\boldsymbol{r}) = \left(-\mu\partial_t + \partial_{r_\parallel}\right)\varphi(t,\boldsymbol{r}) = 0$, which distinguishes a single preferred direction [41], with coordinate $r_\parallel$, from the transverse direction(s), with coordinate $\boldsymbol{r}_\perp$. This is sketched in Fig. 1. Throughout, we shall admit rotation-invariance in the transverse directions, if applicable. Therefore, in more than three spatial dimensions, the consideration of the two-point function can be reduced to the case of a single transverse direction, $r_\perp$. Therefore, it is enough to discuss explicitly either (i) the case of one spatial dimension, referred from now one as the $1D$ *case* (then there is no transverse direction), or else (ii) the case of two spatial dimensions, called the $2D$ *case* (with a single transverse direction).

The Lie algebra generators of meta-conformal invariance read off from Table 1 as follows. In the $1D$ case, in terms of time- and space-coordinates [20] (with $n \in \mathbb{Z}$)

$$
\begin{aligned}
\ell_n &= -t^{n+1}\left(\partial_t - \frac{1}{\mu}\partial_r\right) - (n+1)\left(\delta - \frac{\gamma}{\mu}\right)t^n \\
\bar{\ell}_n &= -\frac{1}{\mu}\left(t+\mu r\right)\partial_r - (n+1)\frac{\gamma}{\mu}\left(t+\mu r\right)^n
\end{aligned}
\tag{3}
$$

and in the $2D$ case [28]

$$
\begin{aligned}
A_n &= -t^{n+1}\left(\partial_t - \frac{1}{\mu}\partial_\parallel\right) - (n+1)\left(\delta - \frac{2\gamma_\parallel}{\mu}\right)t^n \\
B_n^{\pm} &= -\frac{1}{2\mu}\left(t+\mu(r_\parallel \pm \mathrm{i}r_\perp)\right)^{n+1}\left(\partial_\parallel \mp \mathrm{i}\partial_\perp\right) - (n+1)\frac{\gamma_\parallel \mp \mathrm{i}\gamma_\perp}{\mu}\left(t+\mu(r_\parallel \pm \mathrm{i}r_\perp)\right)^n
\end{aligned}
\tag{4}
$$

with the short-hands $\partial_\parallel = \frac{\partial}{\partial r_\parallel}$ and $\partial_\perp = \frac{\partial}{\partial r_\perp}$. The constants δ and γ (respectively $\gamma_{\parallel,\perp}$) are the scaling dimension and the rapidity of the scaling operators on which these generators act and μ^{-1} is a constant with the dimension of a velocity. Each

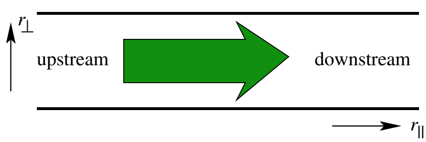

Fig. 1 Schematic illustration of ballistic transport in a channel, with the spatial coordinates $r_\parallel$, $r_\perp$

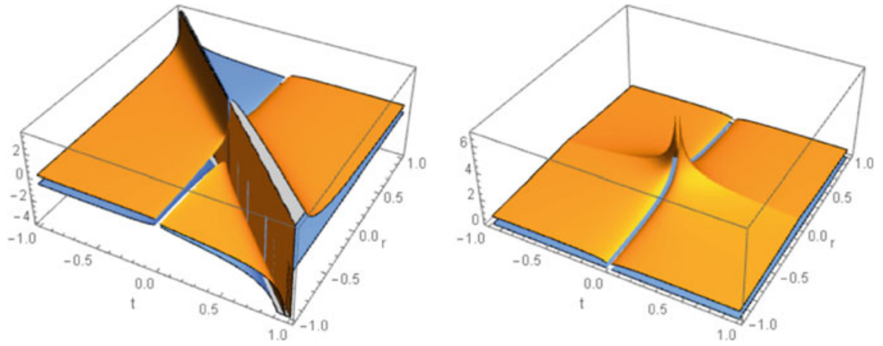

Fig. 2 Real part (orange) and imaginary part (blue) of the $1D$ meta-conformally co-variant two-point function $C(t, r)$, with $\delta_1 = 0.22$, $\gamma_1 = 0.33$ and $\mu = 1$. **Left panel:** Spurious singularities arise in (5). **Right panel:** Regularised form after correction of the spurious singular behaviour.

of the infinite families of generators in (3, 4) produces a Virasoro algebra (with zero central charge). Therefore, the $1D$ meta-conformal algebra is isomorphic to a direct sum of two Virasoro algebras. In the $2D$ case, there is an isomorphism with the direct sum of three Virasoro algebras. Their maximal finite-dimensional Lie sub-algebras (isomorphic to a direct sum of two or three $\mathfrak{sl}(2,\mathbb{R})$ algebras) fix the form of two-point correlators $C(t, \boldsymbol{r}) = \langle\varphi_1(t, \boldsymbol{r})\varphi_2(0, \boldsymbol{0})\rangle$ built from quasi-primary scaling operators. Since the generators (3, 4) already contain the terms which describe how the scaling operators $\varphi = \varphi(t, \boldsymbol{r})$ transform under their action, the global meta-conformal Ward identities can simply be written down. The requirement of meta-conformal co-variance leads to

$$C_{\rm meta}(t, \boldsymbol{r}) = \begin{cases} t^{-2\delta_1}\left(1+\mu\frac{r}{t}\right)^{-2\gamma_1/\mu} & ;\ \text{if } d=1 \\ t^{-2\delta_1}\left(1+\mu\frac{r_\parallel + {\rm i}r_\perp}{t}\right)^{-2\gamma_1/\mu}\left(1+\mu\frac{r_\parallel - {\rm i}r_\perp}{t}\right)^{-2\bar{\gamma}_1/\mu} & ;\ \text{if } d=2 \end{cases} \tag{5}$$

and where $\boldsymbol{r} = r \in \mathbb{R}$ for $d = 1$ and $\boldsymbol{r} = (r_\parallel, r_\perp) \in \mathbb{R}^2$ for $d = 2$ where we also write $\gamma := \gamma_\parallel - {\rm i}\gamma_\perp$ and $\bar{\gamma} := \gamma_\parallel + {\rm i}\gamma_\perp$. In addition, the constraints $\delta_1 = \delta_2$ and $\gamma_1 = \gamma_2$ in $1D$ or $\gamma_{\parallel,1} = \gamma_{\parallel,2}$ and $\gamma_{\perp,1} = \gamma_{\perp,2}$ in $2D$ are implied.

Formally, the procedure to derive (5) is completely analogous to the used above for the derivation of (2) from ortho-conformal co-variance. The explicit forms (5) make it apparent that $C_{\rm meta}(t, \boldsymbol{r})$ is not necessarily bounded for all t or $\boldsymbol{r}$. In Fig. 2, we illustrate this for the $1D$ case—a spurious singularity appears whenever $\mu r = -t$.

In the limit $\mu \to 0$, the meta-conformal algebras contract into the galilean conformal algebras [18]. Carrying out the limit on the correlator (4), one obtains, as has been stated countless times in the literature, see e.g. [3–6, 35]

$$C_{cga}(t, \boldsymbol{r}) = \begin{cases} t^{-2\delta_1}\exp\left(-2\frac{\gamma_1 r}{t}\right) & ;\ \text{if } d=1 \\ t^{-2\delta_1}\exp\left(-4\frac{\boldsymbol{\gamma}_1\cdot\boldsymbol{r}}{t}\right) & ;\ \text{if } d=2 \end{cases} \tag{6}$$

with the definition $\boldsymbol{\gamma} = (\gamma_\parallel, \gamma_\perp)$. While this correlator decays in one spatial direction (where $\gamma_1 r > 0$ or $\boldsymbol{\gamma}_1 \cdot \boldsymbol{r} > 0$ and assuming $t > 0$), it diverges in the opposite direction. In view of the large interest devoted to conformal galilean field-theory, see [1, 6–8, 10, 14, 15, 30, 33, 35] and refs. therein, it appears important to be able to formulate well-defined correlators which remain bounded everywhere in time-space. We mention in passing that the $1D$ form of (6) can also be obtained from $2D$ ortho-conformal invariance: it is enough to consider complex conformal weights $\Delta = \frac{1}{2}(\delta - \mathrm{i}\gamma/\mu)$ and $\overline{\Delta} = \frac{1}{2}(\delta + \mathrm{i}\gamma/\mu)$. Then (2) can be rewritten as

$$C_{\rm ortho}(t,r) = t^{-2\delta}\left[1+\left(\frac{\mu r}{t}\right)^2\right]^{-\delta} \exp\left[-\frac{2\gamma}{\mu}\arctan\frac{\mu r}{t}\right] \stackrel{\mu\to 0}{\longrightarrow} t^{-2\delta}e^{-2\gamma r/t} \tag{7}$$

In what follows, we shall describe how to find correlators bounded everywhere. Since the implicit assumption of holomorphicity in the coordinates gave the unbounded results (5, 6), we shall explore how to derive non-holomorphic correlators. Our treatment follows [25], to be generalised to the case $d = 2$ where necessary.

3 Regularised Meta-conformal Correlator: The $1D$ Case

Non-holomorphic correlators can only be found by going beyond the local differential operators derived from the meta-conformal Ward identities. We shall do so in a few simple steps [25], restricting for the moment to the $1D$ case. First, we consider the 'rapidity' γ as a new variable. Second, it is dualised [22–24] through a Fourier transformation, which gives the quasi-primary scaling operator

$$\widehat{\varphi}(\zeta,t,r) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}\gamma\; e^{\mathrm{i}\gamma\zeta}\,\varphi_\gamma(t,r) \tag{8}$$

This leads to the following representation of the dualised meta-conformal algebra

$$\begin{aligned} X_n &= \mathrm{i}(n+1)\left[(t+\mu r)^n - t^n\right]\partial_\zeta - t^{n+1}\partial_t - \left[(t+\mu r)^{n+1} - t^{n+1}\right]\partial_r - (n+1)\delta t^n \\ Y_n &= \frac{\mathrm{i}(n+1)}{\mu}(t+\mu r)^n\,\partial_\zeta - \frac{1}{\mu}(t+\mu r)^{n+1}\,\partial_r \end{aligned} \tag{9}$$

such that meta-conformal Lie algebra is given by

$$[X_n, X_m] = (n-m)X_{n+m},\quad [X_n, Y_m] = (n-m)Y_{n+m},\quad [Y_n, Y_m] = (n-m)Y_{n+m} \tag{10}$$

This form will be more convenient for us than the one used in [25], since the parameter μ does no longer appear in the Lie algebra commutators (10). Third, it was suggested [22, 25] to look for a further generator N in the Cartan sub-algebra $\mathfrak{h}$, viz. $\mathrm{ad}_N \mathscr{X} = \alpha_{\mathscr{X}}\,\mathscr{X}$ for any meta-conformal generator $\mathscr{X}$. It can be shown that

$$N = -\zeta\partial_\zeta - r\partial_r + \mu\partial_\mu + \mathrm{i}\kappa(\mu)\partial_\zeta - \nu(\mu) \tag{11}$$

is the only possibility [25], where the functions $\kappa(\mu)$ and $\nu(\mu)$ remain undetermined. Since in this generator, the parameter μ is treated as a further variable, we see the usefulness of the chosen normalisation of the generators in (9). On the other hand, the generator of spatial translations now reads $Y_{-1} = -\mu^{-1}\partial_r$, with immediate consequences for the form of the two-point correlator. In dual space, the two-point correlator is defined as

$$\widehat{F} = \langle\widehat{\varphi}_1(\zeta_1, t_1, r_1, \mu_1)\widehat{\varphi}_2(\zeta_2, t_2, r_2, \mu_2)\rangle = \widehat{F}(\zeta_1, \zeta_2, t_1, t_2, r_1, r_2, \mu_1, \mu_2) \tag{12}$$

Lifting the generators from the representation (9) to two-body operators, the global meta-conformal Ward identities (derived from the maximal finite dimensional sub-algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$) become a set of linear partial differential equations of first order for the function $\widehat{F}$. While the solution will certainly be holomorphic in its variables, the back-transformation according to (8) can introduce non- holomorphic behaviour but will also lead to a correlator bounded everywhere.

The function $\widehat{F}$ is obtained as follows. First, co-variance under X_{-1} and Y_{-1} gives

$$\widehat{F} = \widehat{F}(\zeta_1, \zeta_2, t, \xi, \mu_1, \mu_2);\quad t = t_1 - t_2,\quad \xi = \mu_1 r_1 - \mu_2 r_2 \tag{13}$$

The action of the generators Y_0 and Y_1 on $\widehat{F}$ is best described by introducing the new variables $\eta := \mu_1\zeta_1 + \mu_2\zeta_2$ and $\zeta := \mu_1\zeta_1 - \mu_2\zeta_2$. Then the corresponding Ward identities become

$$\left(2\mathrm{i}\partial_\eta - (t+\xi)\partial_\xi\right)\widehat{F} = 0,\quad \partial_\zeta\widehat{F} = 0 \tag{14}$$

Finally, the Ward identities coming from the generators X_0 and X_1 become

$$\left(-t\partial_t - \xi\partial_\xi - \delta_1 - \delta_2\right)\widehat{F} = 0,\quad t\,(\delta_1 - \delta_2)\,\widehat{F} = 0 \tag{15}$$

The second of these gives the constraint $\delta_1 = \delta_2$. The two remaining equations have the general solution

$$\widehat{F} = (t_1 - t_2)^{-2\delta_1}\widehat{\mathscr{F}}\left(\frac{1}{2}(\mu_1\zeta_1 + \mu_2\zeta_2) + \mathrm{i}\ln\left(1 + \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right); \mu_1, \mu_2\right) \tag{16}$$

with an undetermined function $\widehat{\mathscr{F}}$. Spatial translation-invariance only holds in a more weak form, which could become useful for the description of physical situations where the propagation speed of each scaling operator can be different.

In [25], we tried to use co-variance under the further generator N in order to fix the function $\widehat{\mathscr{F}}$. However, therein a choice of basis in the meta-conformal Lie algebra was used where the parameter μ appears in the structure constants, but it became possible to fix $\widehat{\mathscr{F}}$ and furthermore to show that $\widehat{F}$ with respect to the variable η is in the Hardy space H_2^+, see the appendix for the mathematical details. If we want to consider μ as a further variable, as it is necessary because of the explicit form of N, objects such as "μY_{n+m}" are not part of the meta-conformal Lie algebra. Therefore, it is necessary, to use the normalisation (9) which leads to the Lie algebra (10) which is independent of μ. In order to illustrate the generic consequences, let $\nu = \nu(\mu)$ and $\sigma = -\mu\kappa(\mu)$ be constants. The co-variance condition $N\widehat{F} = 0$ gives

$$\widehat{\mathscr{F}}(w : \mu_1, \mu_2) = (\mu_1\mu_2)^{\nu}\, \widehat{\mathcal{F}}\left(w + \mathrm{i}\sigma\frac{\mu_1+\mu_2}{2}, \frac{\mu_1}{\mu_2}\right) \tag{17}$$

where the function $\widehat{\mathcal{F}}$ remains undetermined. In contrast to our earlier treatment, we can no longer show that $\widehat{F}$ had to be in the Hardy space H_2^+. On the other hand, this mathematical property had turned out to be very useful for the derivation of bounded correlators. This motivates the following.

First, we re-write the result (16) as follows (with the constraint $\delta_1 = \delta_2$)

$$\widehat{F} = (t_1 - t_2)^{-2\delta_1}\widehat{\mathscr{F}}(\zeta_+ + \mathrm{i}\lambda) \;\;,\;\; \zeta_+ := \frac{\mu_1\zeta_1 + \mu_2\zeta_2}{2} \;\;,\;\; \lambda := \ln\left(1 + \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right) \tag{18}$$

and we also denote $\widehat{\mathscr{F}}_{\lambda}(\zeta_+) := \widehat{\mathscr{F}}(\zeta_+ + \mathrm{i}\lambda)$. Then, we require:

Postulate. *If* $\lambda > 0$*, then* $\widehat{\mathscr{F}}_{\lambda} \in H_2^+$ *and if* $\lambda < 0$*, then* $\widehat{\mathscr{F}}_{\lambda} \in H_2^-$.

The Hardy spaces $H_2^{\pm}$ on the upper and lower complex half-planes $\mathbb{H}_{\pm}$ are defined in the appendix. There, it is also shown that, under mild conditions, that if $\lambda > 0$ and if there exist finite positive constants $\widehat{\mathscr{F}}^{(0)}$, ε such that $|\widehat{\mathscr{F}}(\zeta_+ + \mathrm{i}\lambda)| < \widehat{\mathscr{F}}^{(0)}e^{-\varepsilon\lambda}$, then $\widehat{\mathscr{F}}_{\lambda}$ is indeed in the Hardy space H_2^+. Physically, this amounts to a requirement of an algebraic decay with respect to the scaling variable.

The utility of our postulate is easily verified, following [25]. From Theorem 1 of the appendix, especially (A.3), we can write

$$\widehat{\mathscr{F}}_{\lambda}(\zeta_+) = \Theta(\lambda)\int_0^{\infty}\mathrm{d}\gamma_+ e^{\mathrm{i}(\zeta_+ + \mathrm{i}\lambda)\gamma_+}\widehat{\mathscr{F}}_+(\gamma_+) + \Theta(-\lambda)\int_0^{\infty}\mathrm{d}\gamma_- e^{-\mathrm{i}(\zeta_+ + \mathrm{i}\lambda)\gamma_-}\widehat{\mathscr{F}}_-(\gamma_-) \tag{19}$$

where the Heaviside functions $\Theta(\pm\lambda)$ select the two cases. For $\lambda > 0$, we find

$$\begin{aligned}
F &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}\zeta_1\mathrm{d}\zeta_2\, e^{-\mathrm{i}\gamma_1\zeta_1 - \mathrm{i}\gamma_2\zeta_2}\widehat{F} \\
&= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^2}\mathrm{d}\zeta_1\mathrm{d}\zeta_2\, t^{-2\delta_1}\int_0^{\infty}\mathrm{d}\gamma_+\, e^{-\mathrm{i}\gamma_1\zeta_1 - \mathrm{i}\gamma_2\zeta_2}e^{\mathrm{i}(\mu_1\zeta_1 + \mu_2\zeta_2 + 2\mathrm{i}\lambda)\gamma_+/2}\widehat{\mathscr{F}}_+(\gamma_+) \\
&= \frac{\sqrt{32\pi}}{\mu_1\mu_2}t^{-2\delta_1}\int_0^{\infty}\mathrm{d}\gamma_+\, e^{-\lambda\gamma_+}\delta\left(\gamma_+ - \frac{2\gamma_1}{\mu_1}\right)\delta\left(\gamma_+ - \frac{2\gamma_2}{\mu_2}\right)\widehat{\mathscr{F}}_+(\gamma_+) \\
&= \frac{\sqrt{32\pi}}{\mu_1\mu_2}t^{-2\delta_1}\delta_{\gamma_1/\mu_1,\gamma_2/\mu_2}\int_0^{\infty}\mathrm{d}\gamma_+\, e^{-\lambda\gamma_+}\delta\left(\gamma_+ - \frac{2\gamma_1}{\mu_1}\right)\widehat{\mathscr{F}}_+(\gamma_+) \\
&= \mathrm{cste.}\, \delta_{\gamma_1/\mu_1,\gamma_2/\mu_2}(t_1 - t_2)^{-2\delta_1}\left(1 + \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right)^{-2\gamma_1/\mu_1}\Theta\left(\frac{\gamma_1}{\mu_1}\right)
\end{aligned} \tag{20}$$

where the definitions (18) were used. Similarly, for $\lambda < 0$ we obtain

$$
\begin{aligned}
F &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2\, e^{-\mathrm{i}\gamma_1\zeta_1 - \mathrm{i}\gamma_2\zeta_2} \widehat{F} \\
&= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2\, t^{-2\delta_1} \int_0^\infty \mathrm{d}\gamma_-\, e^{-\mathrm{i}\gamma_1\zeta_1 - \mathrm{i}\gamma_2\zeta_2} e^{-\mathrm{i}(\mu_1\zeta_1+\mu_2\zeta_2+2\mathrm{i}\lambda)\gamma_-/2} \widehat{\mathscr{F}}_-(\gamma_-) \\
&= \frac{\sqrt{32\pi}}{\mu_1\mu_2} t^{-2\delta_1} \int_0^\infty \mathrm{d}\gamma_-\, e^{\lambda\gamma_-} \delta\left(\gamma_- + \frac{2\gamma_1}{\mu_1}\right) \delta\left(\gamma_- + \frac{2\gamma_2}{\mu_2}\right) \widehat{\mathscr{F}}_-(\gamma_-) \\
&= \frac{\sqrt{32\pi}}{\mu_1\mu_2} t^{-2\delta_1} \delta_{\gamma_1/\mu_1, \gamma_2/\mu_2} \int_0^\infty \mathrm{d}\gamma_-\, e^{\lambda\gamma_-} \delta\left(\gamma_- - \left|\frac{2\gamma_1}{\mu_1}\right|\right) \widehat{\mathscr{F}}_-(\gamma_-) \\
&= \text{cste.}\, \delta_{\gamma_1/\mu_1, \gamma_2/\mu_2} (t_1 - t_2)^{-2\delta_1} \left(1 - \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right)^{-2|\gamma_1/\mu_1|} \Theta\left(-\frac{\gamma_1}{\mu_1}\right)
\end{aligned} \tag{21}
$$

Combining these two forms gives our final $1D$ two-point correlator

$$
F = \delta_{\delta_1,\delta_2} \delta_{\gamma_1/\mu_1, \gamma_2/\mu_2} \left(1 + \left|\frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right|\right)^{-2|\gamma_1/\mu_1|} \tag{22}
$$

up to normalisation. As shown in Fig. 2, this is real-valued and bounded in the entire time-space, although not a holomorphic function of the time-space coordinates.

Finally, it appears that our original motivation for allowing the μ_j to become free variables, is not very strong. We might have fixed the μ_j from the outset, had not included a factor $1/\mu$ into the generators Y_n (such that the spatial translations are generated by $Y_{-1} = -\partial_r$ and continue immediately with our Postulate. Since a consideration of the meta-conformal three-point function shows that $\mu_1 = \mu_2 = \mu_3$ [21, chap. 5], we can then consider μ^{-1} as an universal velocity.[1]

4 Regularised Meta-conformal Correlator: The $2D$ Case

The derivation of the $2D$ meta-conformal correlator starts essentially along the same lines as in the $1D$ case, but is based now on the generators (3). The dualisation is now carried out with respect to the chiral rapidities $\gamma = \gamma_\parallel - \mathrm{i}\gamma_\perp$ and $\bar{\gamma} = \gamma_\parallel + \mathrm{i}\gamma_\perp$ and we also use the light-cone coordinates $z = r_\parallel + \mathrm{i}r_\perp$ and $\bar{z} = r_\parallel - \mathrm{i}r_\perp$. Taking the translation generators A_{-1}, $B_{-1}^{\pm}$ into account, we consider the dual correlator

$$
\widehat{F} = \widehat{F}(\zeta_1, \zeta_2, \bar{\zeta}_1, \bar{\zeta}_2, t, \xi, \bar{\xi}, \mu_1, \mu_2) \tag{23}
$$

[1] In the conformal galilean limit $\mu \to 0$, recover the bounded result $F \sim \exp\left(-2|\gamma_1 r|/t\right)$ [25].

where we defined the variables

$$t = t_1 - t_2, \quad \xi = \mu_1 z_1 - \mu_2 z_2, \quad \bar{\xi} = \mu_1 \bar{z}_1 - \mu_2 \bar{z}_2 \tag{24}$$

In complete analogy with the $1D$ case, we further define the variables

$$\eta = \mu_1 \zeta_1 + \mu_2 \zeta_2, \quad \bar{\eta} = \mu_1 \bar{\zeta}_1 + \mu_2 \bar{\zeta}_2 \tag{25}$$

such that the correlator $\widehat{F} = \widehat{F}(\eta, \bar{\eta}, t, \xi, \bar{\xi}, \mu_1, \mu_2)$ obeys the equations

$$\begin{aligned} &\left(2\mathrm{i}\partial_\eta - (t+\xi)\partial_\xi\right)\widehat{F} = 0, \quad \left(2\mathrm{i}\partial_{\bar{\eta}} - (t+\bar{\xi})\partial_{\bar{\xi}}\right)\widehat{F} = 0, \\ &\left(t\partial_t + \xi\partial_\xi + \bar{\xi}\partial_{\bar{\xi}} + 2\delta_1\right)\widehat{F} = 0 \end{aligned} \tag{26}$$

along with the constraint $\delta_1 = \delta_2$. The most general solution of this system is

$$\widehat{F} = t^{-2\delta_1}\widehat{\mathscr{F}}\left(\frac{\eta}{2} + \mathrm{i}\ln(1+\xi/t), \frac{\bar{\eta}}{2} + \mathrm{i}\ln(1+\bar{\xi}/t)\right) = t^{-2\delta_1}\widehat{\mathscr{F}}(u + \mathrm{i}\lambda, \bar{u} + \mathrm{i}\lambda) \tag{27}$$

with the abbreviations ($\bar{u}$ is obtained from u by replacing $\zeta_j \mapsto \bar{\zeta}_j$)

$$u := \frac{\mu}{2}(\zeta_1 + \zeta_2) + \underbrace{\arctan\frac{\mu r_\perp/t}{1+\mu r_\parallel/t}}_{=:\,a}, \quad \lambda := \frac{1}{2}\ln\left[\left(1+\frac{\mu r_\parallel}{t}\right)^2 + \left(\frac{\mu r_\perp}{t}\right)^2\right] \tag{28}$$

and we simplified the notation by letting $\mu_1 = \mu_2 = \mu$ and assumed translation-invariance in time and space. As before, we expect that a Hardy space will permit to derive the boundedness, see the appendix for details. We define $\widehat{\mathscr{F}}_\lambda(u, \bar{u}) := \widehat{\mathscr{F}}(u + \mathrm{i}\lambda, \bar{u} + \mathrm{i}\lambda)$ and require:

Postulate. *If* $\lambda > 0$, *then* $\widehat{\mathscr{F}}_\lambda \in H_2^{++}$ *and if* $\lambda < 0$, *then* $\widehat{\mathscr{F}}_\lambda \in H_2^{--}$.

Theorem 2 in the appendix, especially (A.11), then states that

$$\begin{aligned} \widehat{\mathscr{F}}_\lambda = \Theta(\lambda)\int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\bar{\tau}\; e^{\mathrm{i}(u+\mathrm{i}\lambda)\tau + \mathrm{i}(\bar{u}+\mathrm{i}\lambda)\bar{\tau}}\,\widehat{\mathscr{F}}_+(\tau, \bar{\tau}) \\ +\Theta(-\lambda)\int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\bar{\tau}\; e^{-\mathrm{i}(u+\mathrm{i}\lambda)\tau - \mathrm{i}(\bar{u}+\mathrm{i}\lambda)\bar{\tau}}\,\widehat{\mathscr{F}}_-(\tau, \bar{\tau}) \end{aligned} \tag{29}$$

Then, we can write the two-point function in the case $\lambda > 0$, with the short-hand $\mathscr{D}\zeta := \mathrm{d}\zeta_1\mathrm{d}\bar{\zeta}_1\mathrm{d}\zeta_2\mathrm{d}\bar{\zeta}_2$ and the abbreviations from (28)

$$
\begin{aligned}
F &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \mathscr{D}\zeta \; e^{-\mathrm{i}\gamma_1\zeta_1-\mathrm{i}\bar{\gamma}_1\bar{\zeta}_1-\mathrm{i}\gamma_2\zeta_2-\mathrm{i}\bar{\gamma}_2\bar{\zeta}_2}\,\widehat{F} \\
&= \frac{t^{-2\delta_1}}{(2\pi)^3}\int_{\mathbb{R}^4} \mathscr{D}\zeta \; e^{-\mathrm{i}\gamma_1\zeta_1-\mathrm{i}\bar{\gamma}_1\bar{\zeta}_1-\mathrm{i}\gamma_2\zeta_2-\mathrm{i}\bar{\gamma}_2\bar{\zeta}_2} \times \\
&\quad\times \int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\bar{\tau}\; e^{\mathrm{i}(\mu(\zeta_1+\zeta_2)+2a)\tau/2-\lambda\tau} e^{\mathrm{i}(\mu(\bar{\zeta}_1+\bar{\zeta}_2)+2a)\bar{\tau}/2-\lambda\bar{\tau}}\,\widehat{\mathscr{F}}_+(\tau,\bar{\tau}) \\
&= \frac{t^{-2\delta_1}}{(2\pi)^3}\int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\bar{\tau}\; \widehat{\mathscr{F}}_+(\tau,\bar{\tau})\, e^{\mathrm{i}a(\tau-\bar{\tau})-\lambda(\tau+\bar{\tau})} \times \\
&\quad\times \int_{\mathbb{R}^4} \mathscr{D}\zeta \; e^{\mathrm{i}(-\gamma_1-\gamma_2+\mu\tau)\zeta_+ +\mathrm{i}(-\gamma_1+\gamma_2)\zeta_-} e^{\mathrm{i}(-\bar{\gamma}_1-\bar{\gamma}_2+\mu\bar{\tau})\bar{\zeta}_+ +\mathrm{i}(-\bar{\gamma}_1+\bar{\gamma}_2)\bar{\zeta}_-} \\
&= \mathrm{cste.}\, t^{-2\delta_1}\delta_{\gamma_1,\gamma_2}\delta_{\bar{\gamma}_1,\bar{\gamma}_2}\, e^{\mathrm{i}2a(\gamma_1-\bar{\gamma}_1)/\mu}\, e^{-\lambda 2(\gamma_1+\bar{\gamma}_1)/\mu}\,\Theta\left(\frac{\gamma_1}{\mu}\right)\Theta\left(\frac{\bar{\gamma}_1}{\mu}\right)
\end{aligned}
\tag{30}
$$

Herein, variables were changed according to $\zeta_1=\zeta_+ + \zeta_-$ and $\zeta_2=\zeta_+ - \zeta_-$ and similarly for the $\bar{\zeta}_j$. The case $\lambda<0$ is treated in the same manner

$$
\begin{aligned}
F &= \frac{t^{-2\delta_1}}{(2\pi)^3}\int_{\mathbb{R}^4} \mathscr{D}\zeta \; e^{-\mathrm{i}\gamma_1\zeta_1-\mathrm{i}\bar{\gamma}_1\bar{\zeta}_1-\mathrm{i}\gamma_2\zeta_2-\mathrm{i}\bar{\gamma}_2\bar{\zeta}_2} \times \\
&\quad\times \int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\bar{\tau}\; e^{-\mathrm{i}(\mu(\zeta_1+\zeta_2)+2a)\tau/2+\lambda\tau} e^{-\mathrm{i}(\mu(\bar{\zeta}_1+\bar{\zeta}_2)+2a)\bar{\tau}/2+\lambda\bar{\tau}}\,\widehat{\mathscr{F}}_+(\tau,\bar{\tau}) \qquad (31)\\
&= \mathrm{cste.}\, t^{-2\delta_1}\delta_{\gamma_1,\gamma_2}\delta_{\bar{\gamma}_1,\bar{\gamma}_2}\, e^{\mathrm{i}2a(|\gamma_1/\mu|-|\bar{\gamma}_1/\mu|)}\, e^{-|\lambda|2(|\gamma_1/\mu|+|\bar{\gamma}_1/\mu|)}\,\Theta\left(-\frac{\gamma_1}{\mu}\right)\Theta\left(-\frac{\bar{\gamma}_1}{\mu}\right)
\end{aligned}
$$

In order to understand the meaning of these expression, we return to the physical interpretation of the conditions $\lambda>0$ and $\lambda<0$. From (28), the most restrictive case occurs for $r_\perp=0$. Then $\lambda>0$ is equivalent to $r_\parallel/t>0$. On the other hand, since γ_1/μ will have a definite sign, it is *a fortiori* also real. Hence $\gamma_{1,\perp}=0$ and we can conclude that

$$
F = \delta_{\delta_1,\delta_2}\delta_{\gamma_1,\gamma_2}\delta_{\bar{\gamma}_1,\bar{\gamma}_2}\, t^{-2\delta_1}\left[\left(1+\left|\frac{\mu r_\parallel}{t}\right|\right)^2+\left(\frac{\mu r_\perp}{t}\right)^2\right]^{-2\gamma_{1,\parallel}/\mu} \tag{32}
$$

up to normalisation, is the final form for the $2D$ meta-conformally co-variant correlator which is bounded in the entire time-space.

5 Conclusions

It has been shown that via a dualisation procedure of the rapidities in the meta-conformal generators, a refined form of the global Ward identities can be found which leads to expressions of the quasi-primary two-point functions which remain bounded in the entire time-space. Herein, we postulate that the dualised two-point functions, whose dual variables are naturally seen to occur in a tube of the first (or the forth) quadrant, belong to a Hardy space. In this way, we can formulate a sufficient condition for the construction of bounded two-point functions, namely

$$F(t_1, t_2, r_1, r_2) = \delta_{\delta_1,\delta_2}\delta_{\gamma_1/\mu_1,\gamma_2/\mu_2}\,(t_1 - t_2)^{-2\delta_1}\left(1 + \left|\frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right|\right)^{-2|\gamma_1/\mu_1|} \tag{33}$$

(up to normalisation) in $d = 1$ spatial dimensions and

$$\begin{aligned} F(t_1, t_2, r_{\|,1}, r_{\|,2}, \boldsymbol{r}_{\perp,1}, \boldsymbol{r}_{\perp,2}) &= \delta_{\delta_1,\delta_2}\delta_{\gamma_{1,\|},\gamma_{2,\|}}\,(t_1 - t_2)^{-2\delta_1} \times \\ &\times \left[\left(1 + \left|\frac{\mu_1 r_{\|,1} - \mu_2 r_{\|,2}}{t_1 - t_2}\right|\right)^2 + \left(\frac{\mu_1 \boldsymbol{r}_{\perp,1} - \mu_2 \boldsymbol{r}_{\perp,2}}{t_1 - t_2}\right)^2\right]^{-2\gamma_{1,\|}/\mu} \end{aligned} \tag{34}$$

in $d \geq 2$ spatial dimensions, where rotation-invariance in the $d - 1$ transverse directions is assumed (provided $\boldsymbol{\gamma}_\perp = \boldsymbol{0}$).

Acknowledgements Most of this work was done during the visits of S.S. at Université de Lorraine Nancy and of M.H. at the workshop *"Lie theories and its applications in physics LT13"*. These visits were supported by PHC Rila. M.H. thanks the MPIPKS Dresden for warm hospitality, where other parts of this work were done.

Appendix. Background on Hardy Spaces

In the main text, we need precise statements on the Fourier transform on semi-infinite spaces. These can be conveniently formulated in terms of Hardy spaces. Here, we restrict to the special case H_2. Our brief summary is based on [2, 40].

We begin with the case of functions of a single complex variable z, defined in the upper half-plane $\mathbb{H}_+ := \{z \in \mathbb{C}\,|\,z = x + \mathrm{i}y,\, y \geq 0\}$.

Definition 1: *A function $f : \mathbb{H}_+ \to \mathbb{C}$ belongs to the* Hardy space H_2^+ *if it is holomorphic on $\mathbb{H}_+$ and if*

$$M^2 := \sup_{y>0} \int_{-\infty}^{\infty} \mathrm{d}x\;|f(x + \mathrm{i}y)|^2 < \infty \tag{A.1}$$

The main results of interest to us can be summarised as follows.

Theorem 1 [2]**:** *Let $f : \mathbb{H}_+ \to \mathbb{C}$ be a holomorphic function. Then the following statements are equivalent:*

1. $f \in H_2^+$
2. *there exists a function $\mathfrak{f} : \mathbb{R} \to \mathbb{C}$, which is square-integrable $\mathfrak{f} \in L^2(\mathbb{R})$, such that* $\lim_{y\to 0^+} f(x + \mathrm{i}y) = \mathfrak{f}(x)$ *and*

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\xi \, \frac{f(\xi)}{\xi - z}, \quad 0 = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\xi \, \frac{f(\xi)}{\xi - z^*} \tag{A.2}$$

where $z^* = x - \mathrm{i}y$ denotes the complex conjugate of z. For notational simplicity, one often writes $f(x) = \lim_{y\to 0^+} f(x + \mathrm{i}y)$, with $x \in \mathbb{R}$.

3. *there exists a function $\widehat{f} : \mathbb{R}_+ \to \mathbb{C}$, $\widehat{f} \in L^2(\mathbb{R}_+)$, such that for all $y > 0$*

$$f(z) = f(x + \mathrm{i}y) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \mathrm{d}\zeta \; e^{\mathrm{i}(x+\mathrm{i}y)\zeta} \, \widehat{f}(\zeta) \tag{A.3}$$

The property (A.3) is of major interest to us in the main text.

If $f \in H_2^+$, one has the following bounds [2]

$$\lim_{y\to 0} f(x + \mathrm{i}y) = 0 \;\; ; \;\; \text{uniformly for all } x \in \mathbb{R} \tag{A.4a}$$

$$\lim_{x\to\pm\infty} f(x + \mathrm{i}y) = 0 \;\; ; \;\; \text{uniformly with respect to } y \geq y_0 > 0 \tag{A.4b}$$

Equation (A.4a) follows from the bound (in turn obtained from (A.3)), see also [32]

$$|f(x + \mathrm{i}y)| \leq f_\infty \, y^{-1/2} \tag{A.4c}$$

which holds for all $x \in \mathbb{R}$ and where the constant $f_\infty > 0$ depends on the function f. There is a simple sufficient criterion which can be used to establish that a given function f is in the Hardy space H_2^+:

Lemma: *If the complex function $f(z) = f(x + \mathrm{i}y)$ is holomorphic for all $y \geq 0$, obeys the bound $|f(z)| < f_0 e^{-\delta y}$, with the constants $f_0 > 0$ and $\delta > 0$ and if $\int_{-\infty}^{\infty} \mathrm{d}x \, |f(x)|^2 < \infty$, then $f \in H_2^+$.*

Proof: Since $f(z)$ is holomorphic on the closure $\overline{\mathbb{H}_+}$ (which includes the real axis), one has the Cauchy formula

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}} \mathrm{d}w \, \frac{f(w)}{w - z} = \frac{1}{2\pi \mathrm{i}} \int_{-R}^{R} \mathrm{d}w \, \frac{f(w)}{w - z} + \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}_{\text{sup}}} \mathrm{d}w \, \frac{f(w)}{w - z} =: F_1(z) + F_2(z)$$

where the integration contour $\mathscr{C}$ consists of the segment $[-R, R]$ on the real axis and the superior semi-circle $\mathscr{C}_{\text{sup}}$. One may write $w = u + \mathrm{i}v = Re^{\mathrm{i}\theta} \in \mathscr{C}_{\text{sup}}$. It follows that on the superior semi-circle $|f(w)| < f_0 e^{-\delta v} = f_0 e^{-\delta R \sin\theta}$. Now, for R large enough, one has $|w - z| = |w(1 - z/w)| \geq R\frac{1}{2}$, for $z \in \overline{\mathbb{H}_+}$ fixed and $w \in \mathscr{C}_{\text{sup}}$. We can then estimate the contribution $F_2(z)$ of the superior semi-circle

$$\begin{aligned}
|F_2(z)| &\leq \frac{1}{2\pi} \int_{\mathscr{C}_{\text{sup}}} |\mathrm{d}w| \frac{|F(w)|}{|w(1 - z/w)|} \quad \leq \frac{1}{2\pi} \int_0^{\pi} \mathrm{d}\theta \; \frac{f_0 e^{-\delta R \sin\theta} R}{R \; \frac{1}{2}} \\
&\leq \frac{2 f_0}{\pi} \int_0^{\pi/2} \mathrm{d}\theta \; \exp\left(-\frac{2\delta}{\pi} R\theta\right) \leq \frac{f_0}{\delta} \frac{1}{R} \to 0 \;\; , \;\; \text{for} R \to \infty
\end{aligned}$$

Hence, the integral representation $f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \mathrm{d}w\, f(w)(w-z)^{-1}$ holds for $R \to \infty$. Since $f \in L^2(\mathbb{R})$, the assertion follows from Eq. (A.2) of Theorem 1. q.e.d.

Clearly, one may also define a Hardy space H_2^- for functions holomorphic on the lower complex half-plane $\mathbb{H}_-$, by adapting the above definition. All results transpose in an evident way.

Further conceptual preparations are necessary for the generalisation of these results to higher dimensions. Here, we shall merely treat the $2D$ case, which is enough for our purposes (and generalisations to $n > 2$ will be obvious). We denote $\boldsymbol{z} = (z_1, z_2) \in \mathbb{C}^2$ and write the scalar product $\boldsymbol{z} \cdot \boldsymbol{w} = z_1 w_1 + z_2 w_2$ for $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{C}^2$. Following [40], H_2-spaces can be defined as follows.

Definition 2: *If $B \subset \mathbb{R}^2$ is an open set, the* tube T_B with base B *is*

$$T_B := \left\{ \boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y} \in \mathbb{C}^2 \,\middle|\, \boldsymbol{y} \in B, \boldsymbol{x} \in \mathbb{R}^2 \right\} \tag{A.5}$$

A function $f : T_B \to \mathbb{C}$ which is holomorphic on T_B is in the Hardy space $H_2(T_B)$ *if*

$$M^2 := \sup_{\boldsymbol{y} \in B} \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\, |f(\boldsymbol{x} + \mathrm{i}\boldsymbol{y})|^2 < \infty \tag{A.6}$$

However, it turns out that this definition is too general. More interesting results are obtained if one uses cônes as a base of the tubes.

Definition 3: *(i) An* open cône $\Gamma \subset \mathbb{R}^n$ *satisfies the properties* $0 \notin \Gamma$ *and if* $\boldsymbol{x}, \boldsymbol{y} \in \Gamma$ *and* $\alpha, \beta > 0$*, then* $\alpha\boldsymbol{x} + \beta\boldsymbol{y} \in \Gamma$. *A* closed cône *is the closure* $\overline{\Gamma}$ *of an open cône* Γ. *(ii) If Γ is a cône, and if the set*

$$\Gamma^* := \left\{ \boldsymbol{x} \in \mathbb{R}^n \,|\boldsymbol{x} \cdot \boldsymbol{t} \geq 0 \text{ with } \boldsymbol{t} \in \Gamma \right\} \tag{A.7}$$

has a non-vanishing interior, then Γ^* is the dual cone *with respect to Γ. The cône Γ is called* self-dual*, if $\Gamma^* = \overline{\Gamma}$.*

For illustration, note that in one dimension ($n = 1$) the only cône is $\Gamma = \{x \in \mathbb{R}\,|x > 0\} = \mathbb{R}_+$. It is self-dual, since $\Gamma^* = \overline{\Gamma} = \mathbb{R}_{0,+}$. In two dimensions ($n = 2$), consider the cône $\Gamma^{++} := \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,|\boldsymbol{x} = (x_1, x_2) \text{ with } x_1 > 0, x_2 > 0 \right\}$ which is the *first quadrant* in the $2D$ plane. Since

$$\Gamma^{++\,*} = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,\middle|\, \boldsymbol{x} \cdot \boldsymbol{t} \geq 0,\ \text{for all } \boldsymbol{t} \in \Gamma^{++} \right\} = \mathbb{R}_{0,+} \oplus \mathbb{R}_{0,+} = \overline{\Gamma^{++}} \tag{A.8}$$

the set Γ^{++} is a self-dual cône.

Hardy spaces defined on the tube $T_{\Gamma^{++}}$ of the first quadrant provide the structure required here.

Definition 4 [40]**:** *If Γ^{++} denotes the first quadrant of the plane $\mathbb{R}^2$, a function $f : T_{\Gamma^{++}} \to \mathbb{C}$ holomorphic on $T_{\Gamma^{++}}$ is in the* Hardy space $H_2^{++} := H_2(T_{\Gamma^{++}})$ *if*

$$M^2 := \sup_{\boldsymbol{y}\in\Gamma^{++}} \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x}\, |f(\boldsymbol{x}+\mathrm{i}\boldsymbol{y})|^2 < \infty \tag{A.9}$$

Theorem 2 [40]**:** *Let the function $f : T_{\Gamma^{++}} \to \mathbb{C}$ be holomorphic. Then the following statements are equivalent:*

1. $f \in H_2^{++}$
2. *there exists a function $f : \mathbb{R} \to \mathbb{C}$, which is square-integrable $f \in L^2(\mathbb{R})$, such that* $\lim_{\boldsymbol{y}\to\boldsymbol{0}^+} f(\boldsymbol{x}+\mathrm{i}\boldsymbol{y}) = f(\boldsymbol{x})$ *and*

$$f(\boldsymbol{z}) = \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{w}\, \frac{f(\boldsymbol{w})}{\boldsymbol{w}-\boldsymbol{z}},\quad 0 = \frac{1}{(2\pi\mathrm{i})^2}\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{w}\, \frac{f(\boldsymbol{w})}{\boldsymbol{w}-\boldsymbol{z}^*} \tag{A.10}$$

   where $(\boldsymbol{w}-\boldsymbol{z})^{-1} := (w_1-z_1)^{-1}(w_2-z_2)^{-1}$ and $\boldsymbol{z}^* = \boldsymbol{x}-\mathrm{i}\boldsymbol{y}$ denotes the complex conjugate of $\boldsymbol{z}$. For notational simplicity, one often writes $f(\boldsymbol{x}) = \lim_{\boldsymbol{y}\to\boldsymbol{0}^+} f(\boldsymbol{x}+\mathrm{i}\boldsymbol{y})$, with $\boldsymbol{x}\in\mathbb{R}^2$.
3. *there exists a function $\widehat{f} : \mathbb{R}_+ \oplus \mathbb{R}_+ \to \mathbb{C}$, with $\widehat{f} \in L^2(\mathbb{R}_+ \oplus \mathbb{R}_+)$ and $z_i \in \mathbb{H}_+$*

$$f(\boldsymbol{z}) = \frac{1}{2\pi}\int_{\Gamma^{++}} \mathrm{d}\boldsymbol{t}\, e^{\mathrm{i}\boldsymbol{z}\cdot\boldsymbol{t}}\,\widehat{f}(\boldsymbol{t}) = \frac{1}{2\pi}\int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2\, e^{\mathrm{i}(z_1 t_1 + z_2 t_2)}\,\widehat{f}(\boldsymbol{t}) \tag{A.11}$$

The property (A.11) is of major interest to us in the main text. Summarising, the restriction to the first quadrant Γ^{++} allows to carry over the known results from the $1D$ case, separately for each component.

Of course, one may also define a Hardy space $H_2^{--} := H_2(T_{\Gamma^{--}})$ on the forth quadrant, in complete analogy.

References

1. Aizawa, N., Kuznetsova, Z., Toppan, F.: Invariant partial differential equations with two-dimensional exotic centrally extended conformal Galilei symmetry. J. Math. Phys. **57**, 041701 (2016). arXiv:1512.02290
2. Akhiezer, N.I.: Lectures on Integral Transforms. American Mathematical Society, Providence (1988)
3. Bagchi, A., Gopakumar, R.: Galilean conformal algebras and AdS/CFT. JHEP **0907**, 037 (2009). arXiv:0902.1385
4. Bagchi, A., Gopakumar, R., Mandal, I., Miwa, A.: CGA in 2D. JHEP **1008**, 004 (2010). arXiv:0912.1090
5. Bagchi, A., Detournay, S., Fareghbal, R., Simón, J.: Holographies of $3D$ flat cosmological horizons. Phys. Rev. Lett. **110**, 141302 (2013). arxiv:1208.4372
6. Bagchi, A., Gary, M., Zodinmawia: Bondi-Metzner-Sachs bootstrap. Phys. Rev. **D96**, 025007 (2017). arxiv:1612:01730
7. Barnich, G., Compère, G.: Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions. Class. Quant. Grav. **24** F15 (2007). Corrigendum **24**, 3139 (2007). [arxiv:gr-qc/0610130]

8. Barnich, G., Gomberoff, A., González, H.A.: Three-dimensional Bondi-Metzner-Sachs invariant two-dimensional field-theories as the flat limit of Liouville theory. Phys. Rev. D **87**, 124032 (2007). arxiv:1210.0731
9. Belavin, A.A., Polykaov, A.M., Zamolodchikov, A.B.: Infinite conformal symmetry in two-dimensional quantum field-theory. Nucl. Phys. B **241**, 333 (1984)
10. Bondi, H., van der Burg, M.G.J., Metzner, A.W.K.: Gravitational waves in general relativity. Proc. Roy. Soc. London A **269**, 21 (1962)
11. Cappelli, A., Dunne, G.V., Trugenberger, C.A., Zemba, G.R.: Conformal symmetry and universal properties of quantum Hall states. Nucl. Phys. B **398**, 531 (1993). [arXiv:hep-th/9211071]
12. Cardy, J.L.: Conformal invariance and critical dynamics. J. Phys. A **18**, 2271 (1985)
13. Doyon, B.: Diffusion and superdiffusion from hydrodynamic projection. arxiv:1912.01551
14. Duval, C., Gibbons, G.W., Horváthy, P.A.: Conformal Carroll groups. J. Phys. A: Math. Theor. **47**, 335204 (2014). arXiv:1403.4213
15. Duval, C., Gibbons, G.W., Horváthy, P.A.: Conformal Carroll groups and BMS symmetry. Class. Quantum Gravity **34**, 092001 (2014). arXiv:1402.5894
16. di Francesco, P., Mathieu, P., Sénéchal, D.: Conformal Field Theory. Springer, Heidelberg (1997)
17. Hansson, T.H., Hermanns, M., Simon, S.H., Viefers, S.F.: Quantum Hall physics: hierarchies and conformal field-theory techniques. Rev. Mod. Phys. **89**, 025005 (2017). arXiv:1601.01697
18. Havas, P., Plebanski, J.: Conformal extensions of the Galilei group and their relation to the Schrödinger group. J. Math. Phys. **19**, 482 (1978)
19. Henkel, M.: Conformal Invariance and Critical Phenomena. Springer, Heidelberg (1999)
20. Henkel, M.: Phenomenology of local scale-invariance: from conformal invariance to dynamical scaling. Nucl. Phys. B **641**, 405 (2002). [arxiv:hep-th/0205256]
21. Henkel, M., Pleimling, M.: Non-equilibrium Phase Transitions. Volume 2: Ageing and Dynamical Scaling far from Equilibrium. Springer, Heidelberg (2010)
22. Henkel, M.: Causality from dynamical symmetry: an example from local scale-invariance. In: Makhlouf, A., et al. (eds.) Algebra, Geometry and Mathematical Physics. Springer Proceedings in Mathematics & Statistics, vol. 85, p. 511 (2014). arxiv:1205.5901
23. Henkel, M.: Dynamical symmetries and causality in non-equilibrium phase transitions. Symmetry **7**, 2108 (2015). arxiv:1509.03669
24. Henkel, M., Stoimenov, S.: Physical ageing and Lie algebras of local scale-invariance. In: Dobrev, V. (ed.) Lie Theory and its Applications in Physics. Springer Proceedings in Mathematics & Statistics, vol. 111, p. 33 (2015). arxiv:1401.6086
25. Henkel, M., Stoimenov, S.: Meta-conformal invariance and the boundedness of two-point correlation functions. J. Phys A: Math. Theor. **49**, 47LT01 (2016). arxiv:1607.00685
26. Henkel, M.: Non-local meta-conformal invariance, diffusion-limited erosion and the XXZ chain. Symmetry **9**, 2 (2017). arxiv:1611.02975
27. Henkel, M., Stoimenov, S.: Infinite-dimensional meta-conformal symmetries: $1D$ diffusion-limited erosion and ballistic transport in (1+2) dimensions. In: Dobrev, V. (ed.) Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume 1. Springer Proceedings in Mathematics and Statistics, vol. 263, p. 113 (2018)
28. Henkel, M., Stoimenov, S.: Infinite-dimensional meta-conformal Lie algebras in one and two spatial dimensions. J. Stat. Mech. 084009 (2019). arxiv:1810.09855v3
29. Hille, E.: Ordinary Differential Equations in the Complex Domain. Wiley, New York (1976); Dover, New York (1997)
30. Hosseiny, A., Rouhani, S.: Affine extension of galilean conformal algebra in $2 + 1$ dimensions. J. Math. Phys. **51**, 052307 (2016). arXiv:0909.1203
31. Karevski, D., Schütz, G.M.: Conformal invariance in driven diffusive systems at high currents. Phys. Rev. Lett. **118**, 030601 (2017). arXiv:1606.04248
32. Koosis, P.: Introduction to H_p Spaces, 2nd edn. Cambridge University Press, Cambridge (1998)
33. Krivonos, S., Lechtenfeld, O., Sorin, A.: Minimal realization of ℓ-conformal Galilei algebra, Pais-Uhlenbeck oscillators and their deformation. JHEP **1610**, 073 (2016). arxiv:1607.03756
34. Krug, J., Meakin, P.: Kinetic roughening of laplacian fronts. Phys. Rev. Lett. **66**, 703 (1991)

35. Martelli, D., Tachikawa, Y.: Comments on Galilean conformal field theories and their geometric realization. JHEP **1005**, 091 (2010). arXiv:0903.5184
36. Polchinski, J.: String Theory, vol. 2. Cambridge University Press, Cambridge (2001)
37. Popkov, V., Schadschneider, A., Schmidt, J., Schütz, G.M.: Fibonacci family of dynamical universality classes. Proc. Nat. Acad. Sci. **112**, 12645 (2019). arxiv:1505.04461
38. Schütz, G.M.: Conformal invariance in conditioned stochastic particle systems. J. Phys. A **50**, 314002 (2017)
39. Spohn, H.: Bosonization, vicinal surfaces, and hydrodynamic fluctuation theory. Phys. Rev. **E60**, 6411 (1999). arxiv:cond-mat/9908381
40. Stein, E.M., Weiss, G.: Fourier Analysis in Euclidean Spaces. Princeton University Press, Princeton (1971)
41. Stoimenov, S., Henkel, M.: From conformal invariance towards dynamical symmetries of the collisionless Boltzmann equation. Symmetry **7**, 1595 (2015). arxiv:1509.00434
42. Stoimenov, S., Henkel, M.: Construction of meta-conformal algebras in d spatial dimensions. In: AIP Conference Proceedings, vol. 2075, p. 090026 (2019)
43. Stoimenov, S., Henkel, M.: Meta-conformal invariance in the directed Glauber-Ising chain. In: Dobrev, V. (ed.) Lie Theory and its Applications in Physics, vol. 335, pp. 463–471. Springer, Singapore (2020). https://doi.org/10.1007/978-981-15-7775-8_37
44. Täuber, U.C.: Critical Dynamics. Cambridge University Press, Cambridge (2014)