Springer Proceedings in Mathematics & Statistics

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Lie Theory and Its Applications in Physics

Varna, Bulgaria, June 2019

Springer

Springer Proceedings in Mathematics & Statistics

Volume 335

Springer Proceedings in Mathematics & Statistics

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Varna, Bulgaria, June 2019

Springer

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ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-981-15-7774-1 ISBN 978-981-15-7775-8 (eBook)
https://doi.org/10.1007/978-981-15-7775-8

Mathematics Subject Classification: 11G55, 11R42, 11S40, 14A22, 16G30, 16T25, 17A70, 17B10, 17B25, 17B37, 17B35, 17B65, 17B80, 17C40, 19F27, 20C33, 20C35, 20G42, 22E46, 22E70, 20G41, 20G42, 22E65, 33D80, 37K30, 58B34, 68T07, 70H06, 81R10, 81R50, 81R60, 81T13, 81T30, 81T35, 81T40, 81P40, 83C65, 83E50, 83F05, 91B80

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Preface

The workshop series ‘Lie Theory and Its Applications in Physics’ is designed to serve the community of theoretical physicists, mathematical physicists and mathematicians working on mathematical models for physical systems based on geometrical methods and in the field of Lie theory.

The series reflects the trend toward a geometrization of the mathematical description of physical systems and objects. A geometric approach to a system yields in general some notion of symmetry which is very helpful in understanding its structure. Geometrization and symmetries are meant in their widest sense, i.e., representation theory, algebraic geometry, number theory infinite-dimensional Lie algebras and groups, superalgebras and supergroups, groups and quantum groups, noncommutative geometry, symmetries of linear and nonlinear PDE, special functions, functional analysis. This is a big interdisciplinary and interrelated field.

The first three workshops were organized in Clausthal (1995, 1997, 1999), the 4th was part of the 2nd Symposium ‘Quantum Theory and Symmetries’ in Cracow (2001), the 5th, 7th, 8th, 9th and 11th were organized in Varna (2003, 2007, 2009, 2011, 2013, 2015), the 6th and the 12th were part of the 4th, resp. 10th, Symposium ‘Quantum Theory and Symmetries’ in Varna (2005, 2017).

The 13th Workshop of the series (LT-13) was organized by the Organizing Committee from the Institute of Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences (BAS) in June 2019 (17–23), at the Guest House of BAS near Varna on the Bulgarian Black Sea Coast.

The overall number of participants was 78 and they came from 24 countries. The number of talks was 64.

The scientific level was very high as can be judged by the *plenary speakers*: Burkhard Eden (Berlin), Malte Henkel (Nancy), Alexey Isaev (Dubna), Evgeny Ivanov (Dubna), Toshiyuki Kobayashi (Tokyo), Ivan Kostov (Saclay), Philip Phillips (Urbana), Gordon Semenoff (Vancouver), Andrei Smilga (Nantes), Birgit Speh (Cornell U.), Ivan Todorov (Sofia), Joris Van der Jeugt (Ghent), Kentaroh Yoshida (Kyoto), George Zoupanos (Athens).

The topics covered the most modern trends in the field of the workshop: symmetries in string theories and gravity theories, conformal field theory, integrable systems, representation theory, supersymmetry, quantum groups, deformations, quantum computing and deep learning, applications to quantum theory.

The International Steering Committee was: C. Burdik (Prague) V. K. Dobrev (Sofia, Chair), H. D. Doebner (Clausthal), B. Dragovich (Belgrade), G. S. Pogosyan (Yerevan & Guadalajara & Dubna).

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Acknowledgments

We express our gratitude to the

- Institute of Nuclear Research and Nuclear Energy
- Abdus Salam International Center for Theoretical Physics

for financial help. We thank the Bulgarian Academy of Sciences for providing its Guest House which contributed very much to the stimulating and pleasant atmosphere during the workshop. We thank the Publisher, Springer Japan, represented by Mr. Masayuki Nakamura (Editor, Mathematics), for assistance in the publication. Last, but not least, I thank the members of the Local Organizing Committee who, through their efforts, made the workshop run smoothly and efficiently.

May 2020 Vladimir Dobrev

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Topics on Global Analysis of Manifolds and Representation Theory of Reductive Groups

Toshiyuki Kobayashi

Abstract Geometric symmetry induces symmetries of function spaces, and the latter yields a clue to global analysis via representation theory. In this note we summarize recent developments on the general theory about how geometric conditions affect representation theoretic properties on function spaces, with focus on multiplicities and spectrum.

Mathematics Subject Classification (2020): Primary 22E46 · Secondary 43A85, 22F30

1 "Grip Strength" of Representations on Global Analysis—Geometric Criterion for Finiteness of Multiplicities

To which extent, does representation theory provide useful information for global analysis on manifolds?

As a guiding principle, we begin with the following perspective [14].

Basic Problem 1 ("Grip strength" of representations). Suppose that a Lie group G acts on X. Can the space of functions on X be "sufficiently controlled" by the representation theory of G?

The vague words, "sufficiently controlled", or conversely, "uncontrollable", need to be formulated as mathematics. Let us observe what may happen in the general setting of infinite-dimensional representations of Lie groups G.

Observation 2. For an infinite-dimensional G-module V, there may exist infinitely many different irreducible subrepresentations. Also, the multiplicity of each irreducible representation can range from finite to infinite.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_1

When confronting such a general situation, one focuses on the principle:

- even though there are infinitely many (sometimes uncountably many) different irreducible representations, the group action **can** distinguish the different parts;
- the group action **cannot** distinguish the parts where the same irreducible representations occur with multiplicities.

This observation suggests us to think of the multiplicity of irreducible representations as an obstruction of "grip strength of a group". For each irreducible representation Π of a group G, we define the multiplicity of Π in the regular representation $C^\infty(X)$ by

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\Pi, C^\infty(X)) \in \mathbb{N} \cup \{\infty\}. \tag{1}$$

The case $\dim_{\mathbb{C}} \mathrm{Hom}_G(\Pi, C^\infty(X)) = 1$ (multiplicity-one) provides a strong "grip strength" of representation theory on global analysis, which may be illuminated by the following example:

Example 1. Let X be a manifold, $D_1, \cdots, D_k$ differential operators on X, and G the group of diffeomorphisms T of X such that $T \circ D_j = D_j \circ T$ for all $j = 1, \cdots, k$. Then the space of solutions f to the differential equations on X:

$$D_j f = \lambda_j f \qquad \text{for } 1 \le j \le k \tag{2}$$

forms a G-module (possibly, zero) for any $\lambda_1, \cdots, \lambda_k \in \mathbb{C}$. The group G becomes a Lie group if $\{D_1, \cdots, D_k\}$ contains the Laplacian when X is a Riemannian manifold (or more generally a pseudo-Riemannian manifold). Assume now that the multiplicity of an irreducible representation Π of G in $C^\infty(X)$ is one. Then any function belonging to the image of a G-homomorphism from Π to $C^\infty(X)$ satisfies a system of differential equations (2) for some $\lambda_1, \cdots, \lambda_k \in \mathbb{C}$ by Schur's lemma.

We formalise Basic Problem 1 as follows.

Problem 1 (Grip strength of representations on global analysis). Let X be a manifold on which a Lie group G acts. Consider the regular representation of G on $C^\infty(X)$ by

$$C^\infty(X) \ni f(x) \mapsto f(g^{-1} \cdot x) \in C^\infty(X) \quad \text{for } g \in G.$$

(1) Find a necessary and sufficient on the pair (G, X) for which the multiplicity (1) of every irreducible representation Π of G in the regular representation $C^\infty(X)$ is **finite**.
(2) Determine a condition on the pair (G, X) for which the multiplicity is **uniformly bounded** with respect to all irreducible representations Π.

A solution to Problem 1 will single out a nice setting of (G, H) in which we could expect a detailed study of global analysis on the homogeneous manifold $X = G/H$ by using representation theory of G. The multiplicity may depend on the irreducible representations Π in (1), and thus we may think that the group G has "stronger grip power" in (2) than in (1). We may also consider a **multiplicity-free case**:

(3) Determine a condition on the pair (G, X) for which the multiplicity (1) is either 0 or 1 for any irreducible representation Π of G.

Clearly, (3) is stronger than (2), however, we do not discuss (3) here.

Problem 1 is settled in Kobayashi–Oshima [17] for homogeneous spaces X of reductive Lie groups G. To state the necessary and sufficient condition, we recall some notions from the theory of transformation groups. The following terminology was introduced in [11].

Definition 3 (Real sphericity). *Suppose that a reductive Lie group G acts continuously on a connected real manifold X. We say X is a* **real spherical** *if a minimal parabolic subgroup of G has an open orbit in X.*

As is seen in Example 2 below, the classical notion of spherical varieties is a special case of real sphericity because a minimal parabolic subgroup of a complex reductive group $G_{\mathbb{C}}$ is nothing but a Borel subgroup.

Example 2 (Spherical variety). Suppose that $X_{\mathbb{C}}$ is a connected complex manifold and that a complex reductive Lie group $G_{\mathbb{C}}$ acts biholomorphically on $X_{\mathbb{C}}$. Then $X_{\mathbb{C}}$ is called a **spherical variety** of $G_{\mathbb{C}}$, if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. Spherical varieties have been extensively studied in algebraic geometry, geometric representation theory, and number theory.

Here are some further examples.

Example 3. Let X be a homogeneous space of a reductive Lie group G and $X_{\mathbb{C}}$ its complexification.

(1) The following basic implications hold (Aomoto, Wolf, and Kobayashi–Oshima).

$$\begin{array}{l} X \text{ is a symmetric space} \\ \quad\Downarrow \text{ Aomoto, Wolf} \\ X_{\mathbb{C}} \text{ is a spherical variety} \\ \quad\Downarrow \text{ Kobayashi–Oshima [17, Prop. 4.3]} \\ X \text{ is a real spherical variety} \\ \quad\Uparrow \text{ obvious} \\ G \text{ is compact.} \end{array}$$

(2) When X admits a G-invariant Riemannian structure, the following are equivalent (see Vinberg [24], Wolf [25]):

$$\begin{array}{l} \quad X_{\mathbb{C}} \text{ is spherical} \\ \Longleftrightarrow X \text{ is weakly symmetric in the sense of Selberg} \\ \Longleftrightarrow X \text{ is a commutative space.} \end{array}$$

(3) The classification of irreducible symmetric spaces was accomplished by Berger [3] at the level of Lie algebras.
(4) The classification theory of spherical varieties $X_{\mathbb{C}}$ has been developed by Krämer, Brion, Mikityuk, and Yakimova.
(5) The triple space $(G \times G \times G)/\mathrm{diag}G$ is not a symmetric space. It is real spherical if and only if G is locally a direct product of compact Lie groups and $SO(n, 1)$, see [11]. This geometric result implies a finiteness criterion of multiplicities for the tensor product of two infinite-dimensional irreducible representations ([11], [12, Cor. 4.2]). The triple space is considered as a special case of the homogeneous space $(\widetilde{G} \times G)/\mathrm{diag}G$ for a pair of groups $\widetilde{G} \supset G$. More generally, the classification of real spherical manifolds $(\widetilde{G} \times G)/\mathrm{diag}G$ was accomplished in [16] when $(\widetilde{G}, G)$ are irreducible symmetric pairs in connection to the branching problem for $\widetilde{G} \downarrow G$, see [12].
(6) Let N be a maximal unipotent subgroup of a real reductive Lie group G. Then G/N is real spherical, as is easily seen from the Bruhat decomposition. Moreover, the following equivalence holds:

$$G_{\mathbb{C}}/N_{\mathbb{C}} \text{ is spherical } \iff G \text{ is quasi split.}$$

This is related to the fact that the theory of Whittaker models (*e.g.* Kostant–Lynch, H. Matumoto) yields stronger consequences when G is assumed to be quasi split, see Remark 4 below.

We denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible admissible smooth representations of G. We do not assume unitarity for here. The solutions of Problem 1, which is a reformalisation of Basic Problem 1, are given by the following two theorems.

Theorem 4 (Criterion for finiteness of multiplicity [17]**).** *Let G be a reductive Lie group and H a reductive algebraic subgroup of G, and set $X = G/H$. Then the following two conditions on the pair (G, H) are equivalent.*

(i) (representation theory) $\dim_{\mathbb{C}} \mathrm{Hom}_G(\Pi, C^\infty(X)) < \infty$ $({}^\forall \Pi \in \mathrm{Irr}(G))$.
(ii) (geometry) *X is a real spherical variety.*

In [17], the proof of the implication (ii) ⇒ (i) uses (hyperfunction-valued) boundary maps for a system of partial differential equations with regular singularities, whereas that of the implication (i) ⇒ (ii) is based on a generalization of the Poisson transform. This proof gives not only the equivalence of (i) and (ii) in Theorem 4 but also some estimates of the multiplicity from above and below. In turn, these estimates bring us to the following geometric criterion of the uniform boundedness of multiplicity.

Theorem 5 (Criterion for uniform boundedness of multiplicity [17]**).** *Let G be a reductive Lie group and H a reductive algebraic subgroup of G, and set $X = G/H$. Then the following three conditions on the pair (G, H) are equivalent.*

(i) (representation theory) *There exists a constant C such that*

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\Pi, C^\infty(X)) \leq C \quad ({}^\forall \Pi \in \operatorname{Irr}(G)).$$

(ii) (complex geometry) *The complexification $X_{\mathbb{C}}$ of X is a spherical variety of $G_{\mathbb{C}}$.*

(iii) (ring theory) *The ring of G-invariant differential operators on X is commutative.*

Remark 1. The equivalence (ii) $\Leftrightarrow$ (iii) in Theorem 5 is classical, see *e.g.,* [24], and the main part here is to characterize the representation theoretic property (i) by means of conditions in other disciplines.

Remark 2. In general, the constant C in (i) cannot be taken to be 1 when H is noncompact.

Remark 3. Theorem 5 includes the discovery that the property of "uniform boundedness of multiplicity" is determined only by the complexification $(G_{\mathbb{C}}, X_{\mathbb{C}})$ and is independent of a real form (G, X). It is expected that this kind of statements could be generalized for reductive algebraic groups over non-archimedean local fields. Recently, Sakellaridis–Venkatesh [20] has obtained some affirmative results in this direction.

Remark 4. Theorems 4 and 5 give solutions to Problem 1 (1) and (2), respectively. More generally, these theorems hold not only for the space $C^\infty(X)$ of functions but also for the space of distributions and the space of sections of an equivariant vector bundle. Furthermore, a generalization dropping the assumption that the subgroup H is reductive also holds, see [17, Thm. A, Thm. B] for precise formulation. For instance, the theory of the Whittaker model considers the case where H is a maximal unipotent subgroup, see also Example 3 (5). Even for such a case a generalization of Theorems 4 and 5 can be applied.

Remark 5. We may also consider parabolic subgroups Q instead of a minimal parabolic subgroup. In this case, we can also consider "generalized Poisson transform", and extend the implication (i) $\Rightarrow$ (ii) in Theorem 4, see [12, Cor. 6.8] for a precise formulation. On the other hand, an opposite implication (ii) $\Rightarrow$ (i) for parabolic subgroups Q is not always true, see Tauchi [22].

Theorems 4 and 5 suggest nice settings of global analysis in which the "grip strength" of representation theory is "strong". The well-studied cases such as the Whittaker model and the analysis on semisimple symmetric spaces may be thought of in this framework of "strong grip" as was seen in (6) and (1), respectively, of Example 3. As yet another set of promising directions, let us discuss briefly the restriction of representations to subgroups (**branching problems**).

In the spirit of "grip strength" (Basic Problem 1), we may ask "grip strength of a subgroup" on an irreducible representation of a larger group as follows:

Basic Problem 6 (Grip strength in branching problem). Let Π be an irreducible representation of a group G. We regard Π as a representation of a subgroup G' by restriction, and consider how many times another irreducible representation π of G' occurs in the restriction $\Pi|_{G'}$:

(1) When is the multiplicity of every irreducible representation π of G' occurring in the restriction $\Pi|_{G'}$ finite?
(2) When is the multiplicity of irreducible representation π of G' occurring in the restriction $\Pi|_{G'}$ uniformly bounded?

To be precise, we need to clarify what "occur" means, *e.g.,* as a submodule, as a quotient, or as a support of the direct integral (1) of the unitary representation, *etc*. Furthermore, since our concern is with infinite-dimensional irreducible representations, the definition of "multiplicity" depends also on the topology of the representation spaces of Π of G and π of G'. Typical definitions of multiplicities include:

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi^{\infty}|_{G'}, \pi^{\infty}), \tag{3}$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi), \tag{4}$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi, \Pi|_{G'}), \tag{5}$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}', K'}(\pi_{K'}, \Pi_K). \tag{6}$$

Here Π^{∞}, π^{∞} stand for smooth representations, whereas $\pi_{K'}$ and Π_K stand for the underlying $(\mathfrak{g}', K')$-modules and $(\mathfrak{g}, K)$-modules. If Π and π are both unitary representations, then the quantities (4) and (5) coincide. If (6) $\neq 0$ in addition, then all the quantities (3)–(6) coincide. In general the multiplicity (6) often vanishes, and its criterion is given in [9, 10].

Concerning the multiplicity (3), see [12, 21] and references therein for the general theory, in particular, for a geometric necessary and sufficient condition on the pair (G, G') such that (1) (or more strongly (2)) of Basic Problem 6 is always fulfilled. When the triple (Π, G, G') satisfies finiteness (or more strongly, uniform boundedness) of the multiplicity in Basic Problem 6, we could expect a detailed study of the restriction $\Pi|_{G'}$, see [13], for further "programs" of branching problems of reductive groups, such as the construction of "symmetry breaking operators" and "holographic operators" in concrete settings [15, 18, 19].

2 Spectrum of the Regular Representation $L^2(X)$—A Geometric Criterion for Temperedness

In the previous section, we focused on "multiplicity" from the perspective of "grip strength" of a group on a function space and proposed (real) spherical varieties as "a nice framework for **detailed** study of global analysis". On the other hand, even in a case in which the "grip strength" of representation theory is "weak", we may still expect to analyze the space of functions on X from representation theory in a "**coarse** standpoint". In this section, including *non-spherical cases*, let us focus on the support of the Plancherel measure and consider the following problem.

Suppose that a Lie group G acts on a manifold X with a Radon measure μ and that G leaves the measure invariant so that G acts naturally on the Hilbert space $L^2(X) \equiv L^2(X, d\mu)$ as a unitary representation.

Basic Problem 7 (Tempered space [1]). Find a necessary and sufficient condition on a pair (G, X) for which the regular representation $L^2(X)$ of G is a tempered representation.

We recall the general definition of tempered representations.

Definition 8 (Tempered representation). *A unitary representation π of a locally compact group G is called tempered if π is weakly contained in $L^2(G)$, namely, if any matrix coefficient $G \ni g \mapsto (\pi(g)u, v) \in \mathbb{C}$ can be approximated by a sequence of linear combinations of matrix coefficients of the regular representation $L^2(G)$ on every compact set of G.*

The classification of *irreducible* tempered representations of real reductive linear Lie groups G was accomplished by Knapp–Zuckerman [5]. In contrast to the long-standing problem of the classification of the unitary dual $\widehat{G}$, irreducible tempered representations form a subset of $\widehat{G}$ that is fairly well-understood. Loosely speaking, from the orbit philosophy due to Kirillov–Kostant–Duflo, irreducible tempered representations are supposed to be obtained as a "geometric quantization" of regular semisimple coadjoint orbits, see *e.g.,* [4, 8].

Tempered representations are unitary representations by definition, however, the classification theory of Knapp–Zuckerman played also a crucial role in the Langlands classification of irreducible admissible representations (without asking if they are unitarizable or not) of real reductive Lie groups.

The general theory of Mautner–Teleman tells that any unitary representation Π of a locally compact group G can be decomposed into the direct integral of irreducible unitary representations:

$$\Pi \simeq \int^{\oplus} \pi_\lambda d\mu(\lambda). \tag{1}$$

Then the following equivalence (i) $\Leftrightarrow$ (ii) holds [1, Rem. 2.6]:

(i) Π is tempered;
(ii) irreducible representation π_λ is tempered for a.e. λ with respect to the measure μ.

The irreducible decomposition of the regular representation of G on $L^2(X)$ is called the Plancherel-type theorem for X. Thus, if the Plancherel formula is "known", then we should be able to answer Basic Problem 7 in principle. However, things are not so easy:

Observation 9. The Plancherel-type theorem for semisimple symmetric spaces G/H was proved by T. Oshima, P. Delorme, E. van den Ban, and H. Schlichtkrull (up to nonvanishing condition of discrete series representation with singular parameters). However, it seems that a necessary and sufficient condition on a symmetric pair (G, H) for which $L^2(G/H)$ is tempered had not been found until the general theory [1] is established by a completely different approach. In fact, it is possible to show that temperedness of $L^2(G/H)$ implies a simple geometric condition that $(G/H)_{\mathrm{Am}}$ is dense in G/H (see the second statement of Theorem 14) from the Plancherel-type formula in the case where G/H is a symmetric space, whereas there is a counterexample to the converse statement, as was found in [1]. If one employs the Plancherel-type formula in order to derive the right answer to Problem 7 for symmetric spaces G/H, one will need a precise (non-)vanishing condition on certain cohomologies (Zuckerman derived functor modules) with singular parameters, and such a condition is combinatorially complicated in many cases [7, 23].

Observation 10. More generally, when $X_{\mathbb{C}}$ is not necessarily a spherical variety of $G_{\mathbb{C}}$, as shown in Theorem 5, the ring $\mathbb{D}_G(X)$ of G-invariant differential operators on X is not commutative and so we cannot use effectively the existing method on non-commutative harmonic analysis based on an expansion of functions on X into joint eigenfunctions with respect to the commutative ring $\mathbb{D}_G(X)$, *cf.* Example 1.

As observed above, to tackle Basic Problem 7, one needs to develop a new method itself. As a new approach, Benoist and I utilised an idea of dynamical system rather than differential equations. We begin with some basic notion:

Definition 11 (Proper action). *Suppose that a locally compact group G acts continuously on a locally compact space X. This action is called* **proper** *if the map*

$$G \times X \to X \times X, \qquad (g, x) \mapsto (x, g \cdot x)$$

is proper, namely, if

$$G_S := \{g \in G : gS \cap S \neq \emptyset\}$$

is compact for any compact subset S of X.

If G acts properly on X, then the stabilizer of any point $x \in X$ in G is compact. See [6] for a criterion of proper actions. On the other hand, if H is a compact subgroup of G, then $L^2(G/H) \subset L^2(G)$ holds, hence the regular representation on $L^2(G/H)$ is tempered. The following can be readily drawn from this.

Example 4. If the action of a group G on X is proper (Definition 11), then the regular representation in $L^2(X)$ is tempered.

Therefore, in the study of Basic Problem 7, we focus on the nontrivial case that the action of G on X is not proper. Properness of the action is **qualitative property**, namely, there exists a compact subset S of X such that the set $G_S = \{g \in G : gS \cap S \neq \emptyset\}$ is noncompact. In order to shift it **quantitatively**, we consider the volume $\mathrm{vol}(gS \cap S)$. Viewed as a function on G,

$$G \ni g \mapsto \mathrm{vol}(gS \cap S) \in \mathbb{R} \tag{2}$$

is a continuous function of $g \in G$. Definition 11 tells that the G-action on X is not proper if and only if the support of the function (2) is noncompact for some compact subset S of X. Hence the "decay" of the function (2) at infinity may be considered as capturing quantitatively a "degree" of non-properness of the action. By pursuing this idea, Basic Problem 7 is settled in Benoist–Kobayashi [1, 2] when X is an algebraic G-variety for a reductive group G. To describe the solution, let us introduce a piecewise linear function associated to a finite-dimensional representation of a Lie algebra.

Definition 12. *For a representation $\sigma : \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$ of a Lie algebra $\mathfrak{h}$ on a finite-dimensional real vector space V, we define a function ρ_V on $\mathfrak{h}$ by*

$$\rho_V : \mathfrak{h} \to \mathbb{R}, \quad Y \mapsto \text{the sum of the absolute values of the real parts of the eigenvalues of } \sigma(Y) \text{ on } V \otimes_{\mathbb{R}} \mathbb{C}.$$

The function ρ_V is uniquely determined by the restriction to a maximal abelian split subalgebra $\mathfrak{a}$ of $\mathfrak{h}$. Further, the restriction $\rho_V|_{\mathfrak{a}}$ is a piecewise linear function on $\mathfrak{a}$, namely, there exist finitely many convex polyhedral cones which cover $\mathfrak{a}$ and on which ρ_V is linear.

Example 5. When (σ, V) is the adjoint representation $(\mathrm{ad}, \mathfrak{h})$, the restriction $\rho_{\mathfrak{h}}|_{\mathfrak{a}}$ can be computed by using a root system. It coincides with twice the usual "ρ" in the dominant Weyl chamber.

With this notation, one can describe a necessary and sufficient condition for Basic Problem 7.

Theorem 13 (Criterion for temperedness of $L^2(X)$, [2]). *Let G be a reductive Lie group and H a connected closed subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Then the following two conditions on a pair (G, H) are equivalent.*

(i) (global analysis) *The regular representation $L^2(G/H)$ is tempered.*
(ii) (combinatorial geometry) $\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}/\mathfrak{h}}$.

Remark 6. If G is an algebraic group acting on an algebraic variety X, then, even when X is not a homogeneous space of G, one can give an answer to Basic Problem 7 by applying Theorem 13 to generic G-orbits [2].

Theorem 13 was proved in [1] in the special case where both G and H are real algebraic reductive groups. In this case, the following theorem also holds:

Theorem 14 **([1]).** *Let $G \supset H$ be a pair of real algebraic reductive Lie groups. We set*

$$(G/H)_{\mathrm{Am}} := \{x \in G/H : \text{the stabilizer of } x \text{ in } H \text{ is amenable}\}$$
$$(G/H)_{\mathrm{Ab}} := \{x \in G/H : \text{the stabilizer of } x \text{ in } H \text{ is abelian}\}.$$

Then the following implications hold.

$$\begin{array}{ll} \textit{geometry} & (G/H)_{\mathrm{Ab}} \text{ is dense in } G/H \\ & \qquad \Downarrow \\ \textit{representation} & L^2(G/H) \text{ is tempered} \\ & \qquad \Downarrow \\ \textit{geometry} & (G/H)_{\mathrm{Am}} \text{ is dense in } G/H. \end{array}$$

Since a complex Lie group is amenable if and only if it is abelian, Theorem 14 implies the following:

Corollary 1. *The following conditions on a pair of complex reductive Lie groups (G, H) are equivalent:*

(i) $L^2(G/H)$ *is tempered.*
(ii) $(G/H)_{\mathrm{Ab}}$ *is dense in* G/H.

Acknowledgements The author was partially supported by Grant-in-Aid for Scientific Research (A) (18H03669), Japan Society for the Promotion of Science. He also would like to thank Professor Vladimir Dobrev for his warm hospitality.

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A Hidden Symmetry of a Branching Law

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Abstract We consider branching laws for the restriction of some irreducible unitary representations Π of $G = O(p, q)$ to its subgroup $H = O(p-1, q)$. In Kobayashi (arXiv:1907.07994, [14]), the irreducible subrepresentations of $O(p-1, q)$ in the restriction of the unitary $\Pi|_{O(p-1,q)}$ are determined. By considering the restriction of packets of irreducible representations we obtain another very simple branching law, which was conjectured in Ørsted–Speh (arXiv:1907.07544, [17]).

Mathematics Subject Classication (2020): Primary 22E46; Secondary 22E30, 22E45, 22E50

1 Introduction

The restriction of a finite-dimensional irreducible representation Π^G of a connected compact Lie group G to a connected Lie subgroup H is a classical problem. For example, the restriction of irreducible representations of $SO(n+1)$ to the subgroup $SO(n)$ can be expressed as a combinatorial pattern satisfied by the highest weights of the irreducible representation Π^G of the large group and of the irreducible representations appearing in the restriction of π^H [20]. For the pair $(G, H) = (SO(n+1), SO(n))$, the branching law is always multiplicity-free, *i.e.,*

$$\dim \operatorname{Hom}_H(\pi^H, \Pi^G|_H) \leq 1.$$

In this article we consider a family of infinite-dimensional irreducible representations $\Pi^{p,q}_{\delta,\lambda}$ with parameters $\lambda \in \mathbb{Z} + \frac{1}{2}(p+q)$, and $\delta \in \{+, -\}$ of noncompact orthogonal groups $G = O(p, q)$ with $p \geq 3$ and $q \geq 2$, which have the

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_2

same infinitesimal character as a finite-dimensional representation and which are subrepresentations of $L^2(O(p,q)/O(p-1,q))$ for $\delta = +$, respectively of $L^2(O(p,q)/O(p,q-1))$ for $\delta = -$. We shall assume a *regularity condition* of the parameter λ (Definition 1). Similarly we consider a family of infinite-dimensional irreducible unitary representations $\pi_{\varepsilon,\mu}^{p-1,q}$, $\varepsilon \in \{+,-\}$ of noncompact orthogonal groups $H = O(p-1,q)$.

Reviewing the results of [14] we see in Section IV that the restriction of these representations to the subgroup $H = O(p-1,q)$ is either of "finite type" (Convention 8) if $\delta = +$ or of "discretely decomposable type" (Convention 5) if $\delta = -$. If the infinitesimal characters of $\Pi_{\delta,\lambda}^{p,q}$ and of a direct summand of $(\Pi_{\delta,\lambda}^{p,q})|_H$ satisfy an interlacing condition (12) similar to that of the finite-dimensional representations of $(SO(n+1), SO(n))$, then $\delta = +$ and the restriction of a representations $\Pi_{\delta,\lambda}^{p,q}$ is of finite type. On the other hand, if the infinitesimal characters $\Pi_{\delta,\lambda}^{p,q}$ and of a direct summand of $(\Pi_{\delta,\lambda}^{p,q})|_H$ satisfy another interlacing condition (9) similar to those of the holomorphic discrete series representations of $(SO(p,2), SO(p-1,2))$, then $\delta = -$ and the restriction of a representations $\Pi_{\delta,\lambda}^{p,q}$ is of discretely decomposable type.

For each λ we define a packet $\{\Pi_{+,\lambda}^{p,q}, \Pi_{-,\lambda}^{p,q}\}$ of representations with the same infinitesimal character. For simplicity, we assume $p \geq 3$ and $q \geq 2$. Using the branching laws for the individual representations we show in Section V:

Theorem 1. *Let $(G,H) = (O(p,q), O(p-1,q))$. Suppose that λ and μ are regular parameters.*

(1) Let Π_λ be a representation in the packet $\{\Pi_{+,\lambda}, \Pi_{-,\lambda}\}$. There exists exactly one representations π_μ in the packet $\{\pi_{+,\mu}, \pi_{-,\mu}\}$ so that

$$\dim \mathrm{Hom}_H(\Pi_\lambda|_H, \pi_\mu) = 1.$$

(2) Let π_μ be in the packet $\{\pi_{+,\mu}, \pi_{-,\mu}\}$. There exists exactly one representation Π_λ in the packet $\{\Pi_{+,\lambda}, \Pi_{-,\lambda}\}$ so that

$$\dim \mathrm{Hom}_H(\Pi_\lambda|_H, \pi_\mu) = 1.$$

Equivalently we may formulate the result as follows:

Theorem 2 (Version 2). *Suppose that λ and μ are regular parameters. Then*

$$\dim \mathrm{Hom}_H((\Pi_{+,\lambda} \oplus \Pi_{-,\lambda})|_H, (\pi_{+,\mu} \oplus \pi_{-,\mu})) = 1.$$

Another version of this theorem using interlacing properties of infinitesimal characters is stated in Section V.

Notation: $\mathbb{N} = \{0, 1, 2, \ldots, \}$ and $\mathbb{N}_+ = \{1, 2, \ldots, \}$.

2 Generalities

We will use in this article the notation and conventions of [14] which we recall now. These conventions differ from those used in [17].

Consider the standard quadratic form on $\mathbb{R}^{p+q}$

$$Q(X,X) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \tag{1}$$

of signature (p,q) in a basis $e_1,\ldots,e_p,e_{p+1},\ldots e_{p+q}$. We define $G = O(p,q)$ to be the indefinite special orthogonal group that preserves the quadratic form Q. Let H be the stabilizer of the vector e_1. Then H is isomorphic to $O(p-1,q)$.

Consider another quadratic form on $\mathbb{R}^{p+q}$

$$Q_-(X,X) = x_1^2 + \cdots + x_q^2 - x_{q+1}^2 - \cdots - x_{p+q}^2 \tag{2}$$

of signature (q,p) with respect to a basis $e_{-,1},\ldots,e_{-,p},e_{-,p+1},\ldots e_{-,p+q}$. The orthogonal group $G_- = O(q,p)$ that preserves the quadratic form Q_- is conjugate to $O(p,q)$ in $GL(p+q,\mathbb{R})$. Thus we may consider representations of $G_- = O(q,p)$ as representations of $G = O(p,q)$.

Since G and G_- are conjugate, the subgroup H of G is also conjugate to a subgroup H_- of G_- which is isomorphic to $O(q,p-1)$. This group isomorphism induces an isomorphism of homogeneous spaces $G/H = O(p,q)/O(p-1,q)$ and $G_-/H_- = O(q,p)/O(q,p-1)$. On the other hand $O(p,q)/O(p-1,q)$ and $O(q,p)/O(q-1,p)$ are not even homeomorphic to each other if $p \neq q$. In the rest of the article we will assume that the subgroup H_- preserves the vector $e_{-,p+q}$.

The maximal compact subgroups of G, G_- and H, H_- are denoted by K, K_- respectively K_H , K_{H_-}. The Lie algebras of the groups are denoted by the corresponding lowercase Gothic letters.

To avoid considering special cases we make in this article the following:

Assumption $\mathcal{O}$:

$$p \geq 3 \text{ and } q \geq 2.$$

3 Representations

We consider in this article a family of irreducible unitary representations introduced in [14]. Using the notation in [14] we recall their parametrization and some important properties in this section. The main reference is [14, Sect. 2].

The irreducible unitary subrepresentations of $L^2(O(p,q)/O(p-1,q))$ were considered by many authors after the pioneering work by I. M. Gelfand et. al. [6], T. Shintani, V. Molchanov, J. Faraut [4], and R. Strichartz [18]. For $p \geq 2$ and $q \geq 1$, they are parametrized by $\lambda \in \mathbb{Z} + \frac{1}{2}(p+q)$ with $\lambda > 0$. Following the notation of [14] we denote them by

$$\Pi^{p,q}_{+,\lambda}.$$

They have infinitesimal character

$$(\lambda, \frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \dots, \frac{p+q}{2} - [\frac{p+q}{2}]),$$

in the Harish-Chandra parametrization (see (8) below), and the minimal K-type

$$\begin{cases} \mathcal{H}^{b(\lambda)}(\mathbb{R}^p) \boxtimes \mathbf{1} & \text{if } b(\lambda) \geq 0, \\ \mathbf{1} \boxtimes \mathbf{1} & \text{if } b(\lambda) \leq 0, \end{cases} \tag{3}$$

where $b(\lambda) := \lambda - \frac{1}{2}(p-q-2)$ ($\in \mathbb{Z}$) and $\mathcal{H}^b(\mathbb{R}^p)$ stands for the space of spherical harmonics of degree b. We note that $\Pi^{p,q}_{+,\lambda}$ are so called Flensted-Jensen representations discussed in [5] if $b(\lambda) \geq 0$, namely, if $\lambda \geq \frac{1}{2}(p-q-2)$. This is the case if λ is regular (Definition 1). The underlying $(\mathfrak{g}, K)$-module of $\Pi^{p,q}_{+,\lambda}$ is given by a Zuckerman derived functor module. See [9, Thm. 3] or [14, Sect. 2.2].

Remark 1. When $p = 1$ and $q \geq 1$, there are *no* irreducible subrepresentations in $L^2(O(p,q)/O(p-1,q))$, and we regard $\pi^{p,q}_{+,\lambda}$ as zero in this case.

Remark 2. (1) For any $p \geq 2$, $q \geq 1$ and $\mathbb{Z} + \frac{1}{2}(p+q) \ni \lambda > 0$, the representation $\Pi^{p,q}_{+,\lambda}$ of $G = O(p,q)$ stays irreducible when restricted to $SO(p,q)$, see also Remark 5.
(2) If $p = 2$ and $\lambda \geq \frac{1}{2}(p+q-2)$, then the representation $\Pi^{p,q}_{+,\lambda}$ is a direct sum of a holomorphic discrete series representation and an anti-holomorphic discrete series representation when restricted to the identity component $G_0 = SO_0(p,q)$ of G.

Similarly there exist a family of irreducible unitary subrepresentations

$$\Pi^{q,p}_{+,\lambda} \qquad (\lambda \in \mathbb{Z} + \frac{1}{2}(p+q),\ \lambda > 0)$$

of $G_- = O(q,p)$ in $L^2(G_-/H_-) = L^2(O(q,p)/O(q-1,p))$ when $p \geq 1$ and $q \geq 2$, with the same infinitesimal character and the same properties. Via the isomorphism between (G_-, H_-) and (G, H), we may consider them as representations of $G = O(p,q)$ and irreducible subrepresentations of $L^2(G/H) = L^2(O(p,q)/O(p,q-1))$.

If no confusion is possible we use the simplified notation

$$\Pi_{+,\lambda} = \Pi^{p,q}_{+,\lambda}$$

and

$$\varPi_{-,\lambda} \simeq \varPi^{q,p}_{+,\lambda} \qquad (\text{via } G_- \simeq G),$$

to denote representations of $G = O(p, q)$.

Remark 3. The irreducible representation $\varPi_{+,\lambda}$ are nontempered if $p \geq 3$, and $\varPi_{-,\lambda}$ are nontempered if $q \geq 3$.

Lemma 1. *Assume that $\lambda \geq \frac{1}{2}(p + q - 2)$. The representations $\varPi_{+,\lambda}$, $\varPi_{-,\lambda}$ are inequivalent, but have the same infinitesimal character.*

Proof. The representation $\varPi_{+,\lambda}$ and $\varPi_{-,\lambda}$ are irreducible representations of $G = O(p, q)$ with respective minimal K-types

$$\begin{aligned} \mathcal{H}^b(\mathbb{R}^p) \boxtimes \mathbf{1}, \qquad & b := \lambda - \frac{1}{2}(p - q - 2), \\ \mathbf{1} \boxtimes \mathcal{H}^{b'}(\mathbb{R}^q), \qquad & b' := \lambda - \frac{1}{2}(q - p - 2), \end{aligned}$$

because the assumption $\lambda \geq \frac{1}{2}(p + q - 2)$ implies both $b \geq 0$ and $b' \geq 0$ by (3).

Remark 4. Lemma 1 holds in the more general setting where $\lambda \geq 0$, see [9, Thm. 3 (4)] for the proof.

Remark 5. For p and q positive and even, the restriction of the representations $\varPi_{+,\lambda}$, $\varPi_{-,\lambda}$ to $SO(p, q)$ are in an Arthur packet as discussed in [3, 16]. Global versions of Arthur packets were introduced by J. Arthur in the theory of automorphic representations and are inspired by the trace formula [1, 2]. Our considerations of Arthur packets of representations of the orthogonal groups which are discrete series representations for symmetric spaces are inspired by Arthur's considerations as well as by the conjectures of B. Gross and D. Prasad. In this article we will refer to $\{\varPi_{+,\lambda}, \varPi_{-,\lambda}\}$ as a **packet** of irreducible representations.

Similarly we have $\mu \in \mathbb{Z} + \frac{1}{2}(p + q - 1)$ satisfying $\mu \geq \frac{1}{2}(p + q - 3)$ a packet $\{\pi_{+,\mu}, \pi_{-,\mu}\}$ of unitary irreducible representations of $G' = O(p - 1, q)$.

Definition 1. We say $\lambda \in \mathbb{Z} + \frac{1}{2}(p + q)$ respectively $\mu \in \mathbb{Z} + \frac{1}{2}(p + q - 1)$ are **regular** if $\lambda \geq \frac{1}{2}(p + q - 2)$ respectively $\mu \geq \frac{1}{2}(p + q - 3)$.

Remark 6. The irreducible representation $\varPi_{+,\lambda}$ (or $\varPi_{-,\lambda}$) has the same infinitesimal character as a finite-dimensional irreducible representation of $G = O(p, q)$ if and only if $\lambda \geq \frac{1}{2}(p + q - 2)$, namely, λ is regular. Similarly, $\pi_{+,\mu}$ (or $\pi_{-,\mu}$) has the same infinitesimal character with a finite-dimensional representation of $G' = O(p - 1, q)$ if and only if $\mu \geq \frac{1}{2}(p + q - 3)$, namely, μ is regular.

For later use we define for regular λ and μ the reducible representations

$$U(\lambda) = \Pi_{+,\lambda} \oplus \Pi_{-,\lambda} \tag{4}$$

and

$$V(\mu) = \pi_{+,\mu} \oplus \pi_{-,\mu}. \tag{5}$$

of $G = O(p,q)$ respective of $H = O(p-1,q)$.

4 Branching Laws

In this section we summarize the results of [14]. For simplicity, we suppose that the assumption $\mathcal{O}$ is satisfied, namely, we assume $p \geq 3$ and $q \geq 2$. We note that the results in Sect. 4.2 hold in the same form for $p \geq 2$ and $q \geq 2$, and those in Sect. 4.3 hold for $p \geq 3$ and $q \geq 1$.

4.1 Quick Introduction to Branching Laws

Consider the restriction of a unitary representation Π of G to a subgroup G'. We say that an irreducible unitary representation π of H is in the discrete spectrum of the restriction $\Pi|_H$ if there exists an isometric H-homomorphism $\pi \to \Pi|_H$, or equivalently, if

$$\mathrm{Hom}_H(\pi, \Pi|_H) \neq \{0\}$$

where $\mathrm{Hom}_H(\ ,\ )$ denotes the space of continuous H-homomorphisms. We define the multiplicity for the unitary representations by

$$m(\Pi, \pi) := \dim \mathrm{Hom}_H(\pi, \Pi|_H) = \dim \mathrm{Hom}_H(\Pi|_H, \pi).$$

Remark 7. As in [7, 15, 19], we also may consider the multiplicity $m(\Pi^\infty, \pi^\infty)$ for smooth admissible representations Π^∞ of G and π^∞ of G' by

$$m(\Pi^\infty, \pi^\infty) := \dim \mathrm{Hom}_H(\Pi^\infty, \pi^\infty).$$

In general, one has

$$m(\Pi^\infty, \pi^\infty) \geq m(\Pi, \pi).$$

Besides the discrete spectrum there may be also continuous spectrum. Here are two interesting cases:

1. There is no continuous spectrum and the representation Π is a direct sum of irreducible representations of H, *i.e.*, the underlying Harish-Chandra module is

a direct sum of countably many Harish-Chandra modules of $(\mathfrak{h}, K_H)$. We say that the restriction $\Pi|_H$ is *discretely decomposable*.
2. There is continuous spectrum and there are only finitely many representations in the discrete spectrum in the irreducible decomposition of the restriction $\Pi|_H$.

We refer to the necessary and sufficient conditions of the parameters of the irreducible representations Π, π so that $m(\Pi, \pi) \neq 0$ (or $m(\Pi^\infty, \pi^\infty) \neq 0$) as a *branching law*. In the examples below, $m(\Pi^\infty, \pi^\infty), m(\Pi, \pi) \in \{0, 1\}$ for all Π and π.
Examples of branching laws:

1. Finite-dimensional representations of semisimple Lie groups are parametrized by highest weights. The classical branching law of the restriction of finite-dimensional representations of $SO(n)$ to $SO(n-1)$ is phrased as an interlacing pattern of highest weights, see Weyl [20].
2. The Gross–Prasad conjectures for the restriction of discrete series representations of $SO(2m, 2n)$ to $SO(2m-1, 2n)$ are expressed as interlacing properties of their parameters, see [7].
3. The branching laws for the restriction of irreducible self-dual representations Π^∞ of $SO(n+1, 1)$ to $SO(n, 1)$ are expressed by using *signatures*, *heights* and interlacing properties of weights, see [15].

If $\Pi \in \{\Pi_{+,\lambda}, \Pi_{-,\lambda}\}$, and

$$\mathrm{Hom}_H(\pi_H, \Pi|_H) \neq \{0\}$$

then for a character χ of O(1)

$$\mathrm{Hom}_{H\times O(1)}(\pi_H \boxtimes \chi, \Pi|_{H\times O(1)}) \neq \{0\}.$$

Moreover, by [14, Thm. 1.1] there exists a regular μ so that $\pi_H \in \{\pi_{+,\mu}, \pi_{-,\mu}\}$.

If Π is in the packet $\{\Pi_{+,\lambda}, \Pi_{-,\lambda}\}$ and π in the packet $\{\pi_{+,\mu}, \pi_{-,\mu}\}$ the branching laws discussed in the next part will involve the parameters $\lambda, \mu, \varepsilon, \delta$.

4.2 Branching Laws for the Restriction of $\Pi_{-,\lambda}$ to $H = O(p-1, q)$—discretely decomposable type

This section treats the restriction $\Pi_{-,\lambda}|_H$, which is discretely decomposable. We use the explicit branching law given in [14, Example 1.2 (1)]. The results were also obtained in [10] by using different techniques, see [12, 13] for details.

We begin with the pair $(G_-, H_-) = (O(q, p), O(q, p-1))$. The restriction of the representation $\Pi^{q,p}_{+,\lambda}$ of G_- to the subgroup $H_- \times O(1) = O(q, p-1) \times O(0, 1)$ is a direct sum of irreducible representations, and is isomorphic to the Hilbert direct sum of countably many Hilbert spaces:

$$\bigoplus_{n\in\mathbb{N}} \pi^{q,p-1}_{+,\lambda+n+\frac{1}{2}} \boxtimes (\mathrm{sgn})^n$$

where sgn stands for the nontrivial character of $O(1) = O(0,1)$. Then via the identification $(G_-, H_-) \simeq (G, H) = (O(p,q), O(p-1,q))$ and $\Pi^{q,p}_{+,\lambda} \simeq \Pi_{-,\lambda}$ as a representation of $G_- \simeq G$, we see the restriction of $\Pi_{-,\lambda}$ to $H \times O(1) = O(p-1,q) \times O(1,0)$ is discretely decomposable, and we have an isomorphism

$$\Pi_{-,\lambda}|_H \simeq \bigoplus_{n\in\mathbb{N}} \pi_{-,\lambda+n+\frac{1}{2}} \boxtimes (\mathrm{sgn})^n.$$

Hence

Proposition 3 (Version 1). *The restriction of $\Pi_{-,\lambda}$ to $H = O(p-1,q)$ is a Hilbert direct sum*

$$\bigoplus_{n\in\mathbb{N}} \pi_{-,\lambda+n+\frac{1}{2}}$$

and each representation has multiplicity one.

Remark 8. If λ is regular, then μ is regular whenever $\mathrm{Hom}_H(\pi_{-,\mu}, \Pi_{-,\lambda}|_H) \neq \{0\}$. In contrast, an analogous statement fails for the restriction $\Pi_{+,\lambda}|_H$, see Remark 12 below.

Remark 9. If $G = SO_0(p,2)$ the representation $\Pi_{-,\lambda}$ with λ regular is a holomorphic discrete series representation. In this case, this result follows from the work of H. Plesner-Jakobsen and M. Vergne [8, Cor. 3.1] or as a special case of the general formula proved in [11, Thm. 8.3].

We define $\kappa\colon \mathbb{N} \to \{0, \frac{1}{2}\}$ by

$$\kappa(n) = 0 \qquad \text{for } n \text{ even}; = \frac{1}{2} \qquad \text{for } n \text{ odd}.$$

Then the infinitesimal character of the representation $\Pi_{-,\lambda}$ of G is

$$(\lambda, \frac{p+q-4}{2}, \ldots, \kappa(p+q)), \tag{6}$$

and the infinitesimal character of the representations in $\pi_{-+,\mu}$ of H is

$$(\mu, \frac{p+q-5}{2}, \ldots, \kappa(p+q-1)). \tag{7}$$

Here we note that the groups G and H are not of Harish-Chandra class, but the infinitesimal characters of the centers $\mathfrak{Z}_G(\mathfrak{g}) := U(\mathfrak{g})^G$ and $\mathfrak{Z}_H(\mathfrak{h}) := U(\mathfrak{h})^H$ of the enveloping algebras can be still described by elements of $\mathbb{C}^M$ with $M := [\frac{1}{2}(p+q)]$

and $\mathbb{C}^N$ with $N := [\frac{1}{2}(p+q-1)]$ modulo finite groups via the Harish-Chandra isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}_G(\mathfrak{g}), \mathbb{C}) &\simeq \mathbb{C}^M/\mathfrak{S}_M \ltimes (\mathbb{Z}/2\mathbb{Z})^M, \\ \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}_H(\mathfrak{h}), \mathbb{C}) &\simeq \mathbb{C}^N/\mathfrak{S}_N \ltimes (\mathbb{Z}/2\mathbb{Z})^N. \end{aligned} \tag{8}$$

In our normalization, the infinitesimal character of the trivial one-dimensional representation of $G = O(p,q)$ is given by

$$(\frac{p+q-2}{2}, \frac{p+q-4}{2}, \cdots, \kappa(p+q)).$$

Hence we may also reformulate the branching laws in Proposition 3 as follows.

Proposition 4 (Version 2). *Suppose λ is a regular parameter (Definition 1). Then an irreducible representation π of $H = O(p-1,q)$ in the discrete spectrum of the restriction of $\Pi^{p,q}_{-,\lambda}$ must be isomorphic to $\pi_{-,\mu}$ for some regular parameter μ, and the infinitesimal characters have the interlacing property*

$$\mu > \lambda > \frac{p+q-4}{2} > \cdots > \frac{1}{2} > 0. \tag{9}$$

Conversely, $\pi = \pi_{-,\mu}$ occurs in the discrete spectrum of the restriction $\Pi^{p,q}_{-,\lambda}|_H$ if the interlacing property (9) is satisfied.

Convention 5. *We say that the restriction of the representation $\Pi_{-,\lambda}$ of G to $H = O(p-1,q)$ is of discretely decomposable type.*

4.3 Branching Laws for the Restriction of $\Pi_{+,\lambda}$ to $H = O(p-1,q)$—finite type

This section treats the restriction $\Pi_{+,\lambda}|_H$ which is *not* discretely decomposable. We use [14, Example 1.2 (2)] which determines the whole discrete spectrum in the restriction $\Pi_{+,\lambda}|_H$. A large part of discrete summands are also obtained in [17] using different techniques.

The restriction $\Pi_{+,\lambda}|_H$ contains at most finitely many irreducible summands. We recall from [14, Thm. 1.1] (or [14, Ex. 1.2 (2)]), an irreducible representation π of $H \times O(1,0) = O(p-1,q) \times O(1)$ occurs in the discrete spectrum of the restriction of $\Pi_{+,\lambda}$ if and only it is of the form

$$\pi^{p-1,q}_{+,\lambda-n-\frac{1}{2}} \boxtimes (\mathrm{sgn})^n \quad \text{for some } 0 \le n < \lambda - \frac{1}{2},$$

where sgn stands for the nontrivial character of $O(1)$.

Proposition 6 (Version 1). *An irreducible representation π of $H = O(p-1, q)$ occurs in the discrete spectrum of the restriction of $\Pi_{+,\lambda}$ of $G = O(p, q)$ when restricted to H if and only if it is of the form*

$$\pi^{p-1,q}_{+,\lambda-\frac{1}{2}-n} \text{ where } 0 \le n < \lambda - \frac{1}{2}.$$

Remark 10. There does not exist discrete spectrum in the restriction $\Pi_{+,\lambda}|_H$ if $p = 2$. In fact $\pi^{1,q}_{+,\mu}$ is zero for all μ if $q \geq 1$, see Remark 1.

Remark 11. The representation $\pi^{p-1,q}_{+,\lambda-\frac{1}{2}-n}$ has a regular parameter, or equivalently, has the same infinitesimal character as a finite-dimensional representation iff

$$\lambda - \frac{1}{2} - n > \frac{p+q-5}{2}.$$

Remark 12. In contrast to the discretely decomposable case (Remark 8), Proposition 6 tells that the implication

$$\lambda \text{ regular} \Rightarrow \mu \text{ regular}$$

does not necessarily hold when $\mathrm{Hom}_H(\pi_{+,\mu}, \Pi_{+,\lambda}|_H) \neq \{0\}$, see Remark 11 above.

We observe that for these representations the condition in the proposition depends only on $p + q$ and thus the proposition for these representations does not depend on the inner form $SO(r, s)$ of $SO(p + q, \mathbb{C})$ when $r + s = p + q$ with $r \geq 3$.

Recall that the infinitesimal character of the representation $\Pi_{+,\lambda}$ is

$$(\lambda, \frac{p+q-4}{2}, \dots, \kappa(p+q)) \tag{10}$$

and the infinitesimal character of the representations in $\pi_{+,\mu}$

$$(\mu, \frac{p+q-5}{2}, \dots, \kappa(p+q-1)) \tag{11}$$

as in (6) and (7).

Proposition 7 (Version 2). *Suppose π is an irreducible unitary representation of $H = O(p-1, q)$. If π occurs in the discrete spectrum of the restriction of $\Pi_{+,\lambda}$ to H, then π must be isomorphic to $\pi_{+,\mu}$ for some $\mu > 0$ with $\mu \in \mathbb{Z} + \frac{1}{2}(p+q-1)$. Assume further that λ and μ are regular. Then $\pi_{+,\mu}$ occurs in the discrete spectrum* *of the restriction $\Pi_{+,\lambda}|_H$ if and only if the two infinitesimal characters* (10) *and* (11) *have the interlacing property*

$$\lambda > \mu > \frac{p+q-4}{2} > \cdots > \frac{1}{2} > 0. \tag{12}$$

Remark 13. Consider the example: $q = 0$ and so G is compact. The representation $\Pi_{-,\lambda}$ is finite-dimensional and has highest weight

$$(\lambda - \frac{p}{2}, 0, \ldots, 0)$$

for an integer λ. A representation $\pi_{-,\mu}$ is a summand of the restriction to $H = SO(p-1)$ if it has highest weight

$$(\mu - \frac{p-1}{2}, 0, \ldots, 0)$$

for $\mu \in \mathbb{N} + \frac{1}{2}$ with $\mu \geq \frac{p-1}{2}$ and $\lambda - \frac{p}{2} \geq \mu - \frac{p-1}{2} \geq 0$, *i.e.*, if there exists and integer $n \in \mathbb{N}$ so that $\mu = \lambda - \frac{1}{2} - n \geq \frac{1}{2}(p-1)$.

This motivates the following:

Convention 8. *We say that the restriction of the representation $\Pi_{-,\lambda}$ to $H = SO(p-1, q)$ is of finite type.*

5 The Main Theorems

We retain Assumption $\mathcal{O}$, namely, $p \geq 3$ and $q \geq 2$. Combing the branching laws in the previous section proves the conjectures in [17, Sect. V] and suggests a generalization of a conjecture by B. Gross and D. Prasad [7], which was formulated for tempered representations.

5.1 Results for Pairs $(O(p, q), O(p-1, q))$

Theorem 9 (Version 1). *Suppose that λ and μ are regular parameters (Definition 1).*

1. *Let Π_λ be a representations in the packet $\{\Pi_{+,\lambda}, \Pi_{-,\lambda}\}$. There exists exactly one representations π_μ in the packet $\{\pi_{+,\mu}, \pi_{-,\mu}\}$ so that*

$$\dim \mathrm{Hom}_H(\Pi_\lambda|_H, \pi_\mu) = 1.$$

2. *Let π_μ be in the packet $\{\pi_{+,\mu}, \pi_{-,\mu}\}$. There exists exactly one representation Π_λ in the packet $\{\Pi_{+,\lambda}, \Pi_{-,\lambda}\}$ so that*

$$\dim \mathrm{Hom}_H(\Pi_\lambda|_H, \pi_\mu) = 1.$$

Equivalently we may formulate the results in terms of reducible representations $U(\lambda)$ and $V(\mu)$ defined in (4) and (5) as follows:

Theorem 10 (Version 2). *Suppose that λ and μ are regular parameters. Then*

$$\dim \operatorname{Hom}_H(U(\lambda)|_H, V(\mu)) = 1.$$

We may formulate the results in interlacing properties of parameter the infinitesimal characters similar to the results in [7].

Recall that the infinitesimal character of the representations of G in the packet $\{\Pi_{+,\lambda}, \Pi_{-,\lambda}\}$ is

$$(\lambda, \frac{p+q-4}{2}, \ldots, \kappa(p+q))$$

and the infinitesimal character of the representations of the subgroup H in the packet $\{\pi_{+,\mu}, \pi_{-,\mu}\}$ is

$$(\mu, \frac{p+q-5}{2}, \ldots, \kappa(p+q-1)),$$

where we recall $(\kappa(p+q), \kappa(p+q-1)) = (0, \frac{1}{2})$ if $p+q$ is even, $= (\frac{1}{2}, 0)$ if $p+q$ is odd.

Theorem 11 (Version 3). *Suppose that λ and μ are regular parameters.*

1. *If the two infinitesimal characters satisfy the following interlacing property:*

$$\mu > \lambda > \frac{p+q-4}{2} > \cdots > \frac{1}{2} > 0$$

then

$$\dim \operatorname{Hom}_H(\Pi_{-,\lambda}|_H, \pi_{-,\mu}) = 1.$$

2. *If the two infinitesimal characters satisfy the following interlacing property:*

$$\lambda > \mu > \frac{p+q-4}{2} > \cdots > \frac{1}{2} > 0$$

then

$$\dim \operatorname{Hom}_H(\Pi_{+,\lambda}|_H, \pi_{+,\mu}) = 1.$$

Remark 14. The trivial representation $\mathbf{1}$ of $H = O(p-1, q)$ is in the dual of the smooth representation $\Pi^{\infty}_{+,\lambda}$ but not in the dual of $\Pi^{\infty}_{-,\lambda}$. There is no other representation in the "packet" of the trivial representation of H and so we deduce

$$\dim \operatorname{Hom}_H(U(\lambda)^{\infty}|_H, \mathbf{1}) = 1,$$

or equivalently there is exactly one representation Π_λ in the set $\{\Pi^\infty_{+,\lambda}, \Pi^\infty_{-,\lambda}\}$ so that

$$\dim \mathrm{Hom}_H(\Pi^\infty_\lambda|_H, \mathbf{1}) = 1.$$

Acknowledgments The authors would like to acknowledge support by the MFO during research in pairs stay during which part of this work was accomplished.

The first author was partially supported by Grant-in-Aid for Scientific Research (A) (18H03669), Japan Society for the Promotion of Science.

The second author was partially supported by Simons Foundation collaboration grant, 633703.

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Exceptional Quantum Algebra for the Standard Model of Particle Physics

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Abstract The exceptional euclidean Jordan algebra J_3^8 of 3×3 hermitian octonionic matrices, appears to be tailor made for the internal space of the three generations of quarks and leptons. The maximal rank subgroup of the automorphism group F_4 of J_3^8 that respects the lepton-quark splitting is $(SU(3)_c \times SU(3)_{ew})/\mathbb{Z}_3$. Its restriction to the special Jordan subalgebra $J_2^8 \subset J_3^8$, associated with a single generation of fundamental fermions, is precisely the symmetry group $S(U(3) \times U(2))$ of the Standard Model. The Euclidean extension $\mathcal{H}_{16}(\mathbb{C}) \otimes \mathcal{H}_{16}(\mathbb{C})$ of J_2^8, the subalgebra of hermitian matrices of the complexification of the associative envelope of J_2^8, involves 32 primitive idempotents giving the states of the first generation fermions. The triality relating left and right $Spin(8)$ spinors to 8-vectors corresponds to the Yukawa coupling of the Higgs boson to quarks and leptons.

The present study of J_3^8 originated in the paper arXiv:1604.01247v2 by Michel Dubois-Violette. It reviews and develops ongoing work with him and with Svetla Drenska: 1806.09450; 1805.06739v2; 1808.08110.

1 Motivation. Alternative Approaches

The gauge group of the Standard Model (SM),

$$G_{SM} = \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6} = S(U(3) \times U(2)) \tag{1.1}$$

and its (highly reducible) representation for the first generation of 16 basic fermions (and as many antifermions),

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_3

$$\begin{pmatrix} \nu \\ e^- \end{pmatrix}_L \leftrightarrow (\mathbf{1}, \mathbf{2})_{-1}, \qquad \begin{pmatrix} u \\ d \end{pmatrix}_L \leftrightarrow (\mathbf{3}, \mathbf{2})_{\frac{1}{3}} \tag{1.2}$$
$$(\nu_R \leftrightarrow (\mathbf{1}, \mathbf{1})_0?), \qquad e_R^- \leftrightarrow (\mathbf{1}, \mathbf{1})_{-2}, \quad u_R \leftrightarrow (\mathbf{3}, \mathbf{1})_{\frac{4}{3}}, \quad d_R \leftrightarrow (\mathbf{3}, \mathbf{1})_{-\frac{2}{3}}$$

(the subscript standing for the value of the weak hypercharge Y), look rather baroque for a fundamental symmetry. Unsatisfied, the founding fathers proposed Grand Unified Theories (GUTs) with (semi)simple symmetry groups: (for a review see [4]):
$SU(5)$ H. Georgi - S.L. Glashow (1974);
$Spin(10)$ H. Georgi (1975), H. Fritzsch - P. Minkowski (1975);
$G_{PS} = Spin(6) \times Spin(4) = \frac{SU(4) \times SU(2) \times SU(2)}{\mathbb{Z}_2}$ J.C. Pati - A. Salam (1973).

The first two GUTs, based on simple groups, gained popularity in the beginning, since they naturally accommodated the fundamental fermions:

$$SU(5) : \mathbf{32} = \Lambda\mathbb{C}^5 = \bigoplus_{\nu=0}^{5} \Lambda^\nu, \ \Lambda^1 = \begin{pmatrix} \nu \\ e^- \end{pmatrix}_{-1} \oplus \bar{d}_{\frac{2}{3}} = \bar{\mathbf{5}},$$
$$\Lambda^3 = \begin{pmatrix} u \\ d \end{pmatrix}_{\frac{1}{3}} \oplus \overline{u}_{-\frac{4}{3}} \oplus e_2^+ = \mathbf{10}, \ \Lambda^5 = \bar{\nu}_L(?) = \bar{\mathbf{1}};$$
$$Spin(10) : \mathbf{32} = \mathbf{16}_L \oplus \mathbf{16}_R, \ \mathbf{16}_L = \Lambda^1 \oplus \Lambda^3 \oplus \Lambda^5. \tag{1.3}$$

(The question marks on the sterile (anti)neutrino indicate that their existence is only inferred indirectly - from the neutrino oscillations.) The Pati-Salam GUT is the only one to exploit the quark lepton symmetry: the group $SU(4) \subset G_{PS}$ combines the three colours with the lepton number. The left and right fermion octets are formed by $SU(2)_L$ and $SU(2)_R$ doublets, respectively (and conversely for the antifermions):

$$\mathbf{8}_L = (\mathbf{4}, \mathbf{2}, \mathbf{1}), \ \mathbf{8}_R = (\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2}) \ (\bar{\mathbf{8}}_R = (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}), \ \bar{\mathbf{8}}_L = (\mathbf{4}, \mathbf{1}, \mathbf{2})). \tag{1.4}$$

The quark-lepton symmetry plays a pivotal role in our approach, too, and the Lie subalgebra $su(4) \oplus su(2)$ of $\mathfrak{g}_{PS}$ will appear in Sect. 4.1.

If the fermions fit nicely in all GUTs, the gauge bosons start posing problems. The adjoint representations **24** (of $SU(5)$) and **45** (of $Spin(10)$) carry, besides the expected eight gluons and four electroweak gauge bosons, unwanted leptoquarks; for instance,

$$\mathbf{24} = (\mathbf{8}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{3}, \mathbf{2})_{\frac{5}{3}} \oplus (\bar{\mathbf{3}}, \mathbf{2})_{-\frac{5}{3}}. \tag{1.5}$$

Moreover, the presence of twelve gauge leptoquarks in (1.5) yields a proton decay rate that contradicts current experimental bounds [48]. It is noteworthy that the Pati-Salam GUT is the only one which does not predict a gauge triggered proton decay (albeit it allows model dependent interactions with scalar fields that would permit such a decay). Accordingly, the Pati-Salam group appears in a preferred reduction of the $Spin(10)$ GUT. Intriguingly, a version of this symmetry is also encountered in the noncommutative geometry approach to the SM, [13]. Concerning the most popular nowadays *supersymmetric GUTs* advocated authoritatively in [58], the lack of experimental evidence for any superpartner makes us share the misgivings expressed forcefully in [59] (see also the recent popular account [35]).

The noncommutative geometry approach, was started in 1988 (according to the dates of submission of [15, 25]), see [17, 24], "at the height of the string revolution" (to cite [14]) and pursued vigorously by Alain Connes and collaborators (work that can be traced back from [11–13, 16]) and by followers [8, 41] (for a pedagogical exposition see [53]).

The algebraic approach to quantum theory has, in fact, been initiated back in the 1930's by Pascual Jordan (1902–1980),[1] [46], who axiomatized the concept of *observable algebra*, the prime example of which is the algebra of complex hermitian matrices (or self-adjoint operators in a Hilbert space) equipped with the symmetrized product

$$A \circ B = \frac{1}{2}(AB + BA) \; (= B \circ A). \tag{1.6}$$

Such a (finite dimensional) Jordan algebra should appear as an "internal" counterpart of the algebra of smooth functions of classical fields. In the case of a *special Jordan algebra* (i.e., a Jordan subalgebra of an associative algebra equipped with the product (1.6)) one can of course work with its associative envelope, - i.e., with the corresponding matrix algebra. In the noncommutative geometry approach to the SM, based on a real spectral triple [11], one arrives at the finite algebra [12] (Proposition 3 of [14]):

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{C}[3] \tag{1.7}$$

($\mathbb{A}[n]$ standing for the algebra of $n \times n$ matrices with entries in the coordinate ring $\mathbb{A}$). The only hermitian elements of the quaternion algebra $\mathbb{H}$, however, are the real numbers, so $\mathcal{A}_F$ does not appear as the associative envelope of an interesting observable algebra. We shall, by contrast, base our treatment on an appropriate finite dimensional Jordan algebra[2] suited for a quantum theory - permitting, in particular, a spectral decomposition of observables.

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[1]The only one of the "Boys' Club", [36], that did not get a Nobel Prize. The work on *Jordan algebras* (called so by A.A. Albert, 1946), of 1932–1934, that culminated in [37], was preceded by the analysis (by Dirac, Jordan and von Neumann) of quantum transformation theory reviewed insightfully in [27]. There are but a few papers on the applications of Jordan algebras to quantum theory, [3, 5, 6, 21, 33, 43, 45, 57].

[2]Recently, a new paper, [9], was posted where an alternative approach, closer to Connes' real spectral triple, involving a different Jordan algebra, is being developed.

2 Euclidean Jordan Algebras

An euclidean Jordan algebra is a real vector space J equipped with a commutative product $X \circ Y$ with a unit 1 satisfying the *formal reality condition*

$$X_1^2 + ... + X_n^2 = 0 \Rightarrow X_1 = ... = X_n = 0 \ (X_i^2 := X_i \circ X_i) \tag{2.1}$$

and power associativity. Jordan has found a simple necessary and sufficient condition for power associativity. Introducing the operator $L(X)$ of multiplication by $X : L(X)Y = X \circ Y$, it can be written in the form:

$$[L(X), L(X^2)] = 0 \Leftrightarrow X \circ (Y \circ X^2) = X^2 \circ (Y \circ X), \ X, Y \in J. \tag{2.2}$$

(In general, non-associativity of the Jordan product is encoded in the noncommutativity of the maps $L(X)$.) A prototype example of a Jordan algebra is the space of $n \times n$ hermitian matrices with anticommutator product (1.6), $X \circ Y = \frac{1}{2}(XY + YX)$ where XY stands for the (associative) matrix multiplication. More generally, a Jordan algebra is called *special* if it is a Jordan subalgebra of an associative algebra with Jordan product defined by (1.6). If $\mathcal{A}$ is an associative involutive (star) algebra the symmetrized product (1.6) is not the only one which preserves hermiticity. The quadratic (in X) operator $U(X)Y = XYX$, $X, Y \in \mathcal{A}$ also maps a pair X, Y of hermitian elements into a hermitian element. For a general (not necessarily special) Jordan algebra the map $U(X)$ (whose role is emphasized in [44]) and its polarized form $U(X, Y) := U(X + Y) - U(X) - U(Y)$ can be defined in terms of $L(X)$:

$$U(X) = 2L^2(X) - L(X^2), U(X, Y) = 2(L(X)L(Y) + L(Y)L(X) - L(X \circ Y)). \tag{2.3}$$

The conditions (2.1) and (2.2) are necessary and sufficient to have spectral decomposition of any element of J and thus treat it as an observable.

2.1 Spectral Decomposition, Characteristic Polynomial

To begin with, power associativity means that the subalgebra (including the unit) generated by an arbitrary element X of J is associative. In particular, any power of X is defined unambiguously. In order to introduce spectral decomposition we need the algebraic counterpart of a projector. An element $e \in J$ satisfying $e^2 = e(\neq 0)$ is called a (non zero) *idempotent*. Two idempotents e and f are *orthogonal* if $e \circ f = 0$; then multiplication by e and f commute and $e + f$ is another idempotent. The formal reality condition (2.1) allows to define *partial order* in J saying that X is smaller than Y, $X < Y$, if $Y - X$ can be written as a sum of squares. Noting that $f = f^2$ we conclude that $e < e + f$. A non-zero idempotent is called *minimal* or *primitive* if it cannot be decomposed into a sum of (nontrivial) orthogonal idempotents. A *Jordan frame* is a set of orthogonal primitive idempotents $e_1, ..., e_r$ satisfying

$$e_1 + ... + e_r = 1 \;\; (e_i \circ e_j = \delta_{ij} e_i). \tag{2.4}$$

Each such frame gives rise to a complete set of commuting observables. The number of elements r in a Jordan frame is independent of its choice and is called the *rank of* J. Each $X \in J$ has a *spectral decomposition* of the form

$$X = \sum_{i=1}^{r} \lambda_i e_i, \;\; \lambda_i \in \mathbb{R}, \;\; \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_r. \tag{2.5}$$

For an X for which all λ_i in (2.5) are different the spectral decomposition is unique. Such *regular* X form a dense open set in J. The rank of J coincides with the degree of the characteristic polynomial (defined for any $X \in J$):

$$\begin{aligned} F_r(t, X) = t^r - a_1(X)t^{r-1} + ... + (-1)^r a_r(X), \\ a_k(X) \in \mathbb{R}, \; a_k(\alpha X) = \alpha^k a_k(X) \;\; (\alpha > 0). \end{aligned} \tag{2.6}$$

The roots of F_r are $(t =)\lambda_1, ..., \lambda_r$ (some of which may coincide). Given a regular X the idempotents e_i can be expressed as polynomials in X of degree $r - 1$, determined from the system of equations

$$\begin{aligned} e_1 + ... + e_r &= 1, \\ \lambda_1 e_1 + ... + \lambda_r e_r &= X, \\ &...... \\ \lambda_1^{r-1} e_1 + ... + \lambda_r^{r-1} e_r &= X^{r-1}, \end{aligned} \tag{2.7}$$

whose *Vandermonde determinant* is non zero for distinct λ_i.

We are now ready to define a trace and an inner product in J. The *trace*, $tr(X)$, is a linear functional on J taking value 1 on primitive idempotents:

$$tr(X) = \sum_i \lambda_i (= a_1(X)), \;\; tr(1) = r, \tag{2.8}$$

for X given by (2.5) (and $a_1(X)$ of (2.6)). The *inner product*, defined as the trace of the Jordan product, is positive definite:

$$(X, Y) := tr(X \circ Y) \Rightarrow (X, X) > 0 \quad for\; X \neq 0. \tag{2.9}$$

This justifies the name *euclidean* for a formally real Jordan algebra. The last coefficient, a_r, of (2.6) is the *determinant of* X:

$$a_r(X) = det(X) = \lambda_1 ... \lambda_r. \quad (2.10)$$

If $det(X) \neq 0$ then X is *invertible* and its inverse is given by

$$X^{-1} := \frac{(-1)^r}{det(X)}(X^{r-1} - a_1(X)X^{r-1} + ... + (-1)^{r-1}a_{r-1}(X)1). \quad (2.11)$$

The theory of euclidean Jordan algebras is simplified by the fact that any such algebra can be written as a direct sum of *simple* ones (which cannot be further decomposed into nontrivial direct sums).

2.2 Simple Jordan Algebras. Euclidean Extensions

The finite dimensional simple euclidean Jordan algebras were classified at the dawn of the theory, in 1934, by Jordan et al.[3] The argument is based on the *Peirce decomposition* in a Jordan algebra which we are going to sketch.

To begin with, by repeated manipulation of the Jordan identity (2.2) one obtains the basic third degree formula (see Proposition II.1.1 (iii) of [28]):

$$L(X^2 \circ Y) - L(X^2)L(Y) = 2(L(X \circ Y) - L(X)L(Y))L(X), \quad (2.12)$$

that is equivalent to

$$L(X^3) - 3L(X^2)L(X) + 2L^3(X) = 0,$$
$$[[L(X), L(Y)], L(Z)] + L([X, Z, Y]) = 0, \quad (2.13)$$

$[X, Z, Y] := (X \circ Z) \circ Y - X \circ (Z \circ Y)$ is the *associator*. For an idempotent, $X = e (= e^2)$, the first equation (2.13) takes the form:

$$2L^3(e) - 3L^2(e) + L(e) = L(e)(2L(e) - 1)(L(e) - 1) = 0, \quad (2.14)$$

thus restricting the eigenvalues of $L(e)$ to three possibilities $(0, 1/2, 1)$.

Let $e \in J$ be a nontrivial idempotent: $0 < e(= e^2) < 1$. The eigensubspace $J_1(e) \subset J$ of $L(e)$ corresponding to eigenvalue 1 coincides with $U(e)[J]$, the subspace of elements Y of the form $Y = U(e)X$, $X \in J$ where U is the quadratic map (2.3). If the idempotent e is minimal then $J_1(e)$ is one-dimensional: it us spanned by real multiples of e. Similarly, the subspace $J_0(e)$ annihilated by $L(e)$ can be written

[3] A streamlined pedagogical version of the original argument, [37], that follows [39], can be found in Chapters II-V of the book [28].

as $J_0(e) = U(e^{'})[J]$ for. $e^{'} = 1 - e$. For J simple (of rank $r > 1$) the subspace $J_{\frac{1}{2}}(e)$ has to be nontrivial as well:

$$J_{\frac{1}{2}}(e) = U(e, e^{'})[J] \Rightarrow (L(e) - \frac{1}{2})J_{\frac{1}{2}}(e) = (L(e^{'}) - \frac{1}{2})J_{\frac{1}{2}}(e) = 0. \quad (2.15)$$

Given a frame of primitive idempotents $(e_1, ..., e_r)$ in a rank r Jordan algebra J_r, we can introduce a set of $\binom{r+1}{2}$ orthogonal subspaces, a counterpart of Weyl's matrix units:

$$E_{ii} = U(e_i)[J_r],\ E_{ij} = U(e_i, e_j)[J_r](= E_{ji}),\ i, j = 1, ..., r,\ i \neq j. \quad (2.16)$$

They are eigenspaces of the set $\{L(e_k), k = 1, ..., r\}$ of commuting operators:

$$L(e_k)E_{ij} = \frac{1}{2}(\delta_{ik} + \delta_{jk})E_{ij}. \quad (2.17)$$

The subalgebras E_{ii} are one-dimensional while $E_{ij},\ i \neq j$ (for a given simple algebra J_r) have the same dimension, called the degree, $d > 0$. It turns out that the two positive numbers, the rank r and the degree d, determine all finite dimensional simple euclidean Jordan algebras, to be, hence, denoted J_r^d. (The single rank one Jordan algebra is the field $\mathbb{R}$ of real numbers - no room for off diagonal elements and no need for a degree in this case.) For $r = 2$ the degree d can take any positive integer value. For $r = 3$ the allowed values of d are the dimensions 1, 2, 4, 8 of the (normed) division rings. For $r \geq 4$ only the dimensions 1, 2, 4 of the associative division rings are permitted. The resulting simple Jordan algebras split into four infinite series and one exceptional algebra (proven to have no associative envelope by A. Albert also in 1934 and often called the *Albert algebra*):

$$\begin{aligned} J_r^1 &= \mathcal{H}_r(\mathbb{R}),\ r \geq 1; \quad J_r^2 = \mathcal{H}_r(\mathbb{C}),\ r \geq 2; \\ J_r^4 &= \mathcal{H}_r(\mathbb{H}),\ r \geq 2; \quad J_2^d = JSpin(d+1); \\ J_3^8 &= \mathcal{H}_3(\mathbb{O}), \quad dim(J_r^d) = \binom{r}{2}d + r \end{aligned} \quad (2.18)$$

$(dim(\mathcal{H}_r(\mathbb{R})) = \binom{r+1}{2},\ dim(\mathcal{H}_r(\mathbb{C})) = r^2,\ dim(J_2^d) = d + 2,\ dim(J_3^8) = 27)$. The first three algebras in the above list consist of familiar hermitian matrices (with entries in associative division rings). We stress once more that all items in (2.18) (including $\mathcal{H}_r(\mathbb{C})$ and $\mathcal{H}_r(\mathbb{H})$ which involve matrices with complex and quaternionic entries) are regarded as algebras over the reals. The *spin factor* $J_2^d \subset C\ell_{d+1}$ can be thought of as the set of 2×2 matrices of the form

$$\begin{aligned} X &= \xi\mathbf{1} + \hat{x},\ \xi \in \mathbb{R},\ tr\hat{x} = 0,\ X^2 = 2\xi X - detX, \\ detX &= \xi^2 - N(x),\ \hat{x}^2 = N(x)\mathbf{1},\ N(x) = \sum_{\mu=0}^{d} x_\mu^2 \end{aligned} \quad (2.19)$$

(cf. Remark 3.1 of [56]). The fact that the algebras $JSpin(n)(= J_2^{n-1})$ are special requires an argument (while it is obvious for the first three series of matrix algebras (2.18)). In fact they admit interesting *euclidean extensions*.

The algebra $JSpin(n)$ is isomorphic to the Jordan subalgebra of the real Clifford algebra $C\ell_n$ spanned by the unit element and an orthonormal basis of gamma matrices with Jordan product

$$\Gamma_i \circ \Gamma_j = \frac{1}{2}[\Gamma_i, \Gamma_j]_+ = \delta_{ij}\mathbf{1}. \tag{2.20}$$

To define an appropriate euclidean extension we use the classification of real (later also of complex) Clifford algebras (see e.g. [42, 54]):

$$C\ell_2 = \mathbb{R}[2],\ C\ell_3 = \mathbb{C}[2],\ C\ell_4 = \mathbb{H}[2],\ C\ell_5 = \mathbb{H}[2] \oplus \mathbb{H}[2],\ C\ell_6 = \mathbb{H}[4],$$
$$C\ell_7 = \mathbb{C}[8], C\ell_8 = \mathbb{R}[16], C\ell_9 = \mathbb{R}[16] \oplus \mathbb{R}[16];\ C\ell_{n+8} = C\ell_n \otimes \mathbb{R}[16]. \tag{2.21}$$

It seems natural to define the euclidean extensions of the spin factors $JSpin(n)$ as the corresponding subalgebras of hermitian (in the real case - symmetric) matrices: $\mathcal{H}_2(\mathbb{R})$, $\mathcal{H}_2(\mathbb{C})$, $\mathcal{H}_2(\mathbb{H})$, ..., $\mathcal{H}_{16}(\mathbb{R}) \oplus \mathcal{H}_{16}(\mathbb{R})$. In the case of real symmetric matrices (including the euclidean Jordan subalgebras of $C\ell_2, C\ell_8$ and $C\ell_9$), however, such a definition would exclude the most important observables: the hermitian counterparts of the symmetry generators. Indeed the derivations $\Gamma_{ab} = [\Gamma_a, \Gamma_b], a, b = 1, ..., n$ of $C\ell_n$ are antihermitian matrices; the hermitian observables $i\Gamma_{ab}$ only belong to the corresponding matrix Jordan algebra if we are dealing with complex hermitian (rather than real symmetric) matrices. More generally, we shall complexify from the outset the assoiciative envelope of the spin factors as postulated in [26]:

$$J_2^d \in C\ell_{d+1} \in C\ell_{d+1}(\mathbb{C}),\ C\ell_{2m}(\mathbb{C}) \cong \mathbb{C}[2^m],$$
$$C\ell_{2m+1}(\mathbb{C}) \cong \mathbb{C}[2^m] \oplus \mathbb{C}[2^m]. \tag{2.22}$$

(See the insightful discussion in (Sect. 3 of [3].) We will identify the optimal extension $\tilde{J}_2^d$ of J_2^d with the corresponding subalgebra of hermitian matrices. We shall exploit, in particular,

$$\tilde{J}_2^8 := J_{16}^2 \oplus J_{16}^2 = \mathcal{H}_{16}(\mathbb{C}) \oplus \mathcal{H}_{16}(\mathbb{C}). \tag{2.23}$$

Coming back to the list (2.18) we observe that it involves three obvious repetitions: the spin factors J_2^d for $d = 1, 2, 4$ coincide with the first items in the three families of matrix algebras in the above list. We could also write

$$J_2^8 = \mathcal{H}_2(\mathbb{O})(\subset C\ell_9); \tag{2.24}$$

here (as in J_3^8) $\mathbb{O}$ stands for the nonassociative division ring of *octonions* (see the review [2]). The (10-dimensional) spin factor J_2^8 (unlike J_3^8) is *special* - as a Jordan subalgebra of the (2^9-dimensional) associative algebra $C\ell_9$.

2.3 Symmetric Cone, States, Structure Group

Remarkably, an euclidean Jordan algebra gives room not only to the observables of a quantum theory, it also contains its states: these are, roughly speaking, the positive observables. We proceed to more precise definitions.

Each euclidean Jordan algebra J contains a convex, *open cone* $\mathcal{C}$ consisting of all positive elements of J (i.e., all invertible elements that can be written as sums of squares, so that all their eigenvalues are positive). Jordan frames belong to the closure $\bar{\mathcal{C}}$ (in fact, to the boundary) of the open cone, not to $\mathcal{C}$ itself, as primitive idempotents (for $r > 1$) are not invertible.

The *states* are (normalized) positive linear functionals on the space of observables, so they belong to the closure of the dual cone

$$\mathcal{C}^* = \{\rho \in J; (\rho, X) > 0 \ \forall X \in \bar{\mathcal{C}}\}. \tag{2.25}$$

In fact, the positive cone is *self-dual*, $\mathcal{C} = \mathcal{C}^*$. An element $\rho \in \bar{\mathcal{C}} \subset J$ of trace one defines a *state* assigning to any observable $X \in J$ an *expectation value*

$$<X> = (\rho, X) = tr(\rho \circ X), \ \rho \in \bar{\mathcal{C}}, \ tr\rho\,(=<1>) = 1. \tag{2.26}$$

The primitive idempotents define *pure states*; they are extreme points in the convex set of normalized states. All positive states (in the open cone $\mathcal{C}$) are (mixed) *density matrices*. There is a distinguished mixed state in J_r^d, the normalized unit matrix, called by Baez the *state of maximal ignorance*:

$$<X>_0 = \frac{1}{r} tr(X) \ (r = tr(1)). \tag{2.27}$$

Any other state can be obtained by multiplying it by a (suitably normalized) observable - thus displaying a *state observable correspondence* [3].

The cone $\mathcal{C}$ is *homogeneous*: it has a transitively acting symmetry group that defines the *structure group* of the Jordan algebra, $Aut(\mathcal{C}) =: Str(J)$, the product of a central subgroup $\mathbb{R}_+$ of uniform dilation with a (semi)simple Lie group $Str_0(J)$, the group that preserves the determinant of each element of J. Here is a list of the corresponding simple Lie algebras $str_0(J_r^d)$:

$$\begin{gathered} str_0(J_r^1) = sl(r, \mathbb{R}), \ str_0(J_r^2) = sl(r, \mathbb{C}), \ str_0(J_r^4) = su^*(2r), \\ str_0(J_2^d) = so(d+1, 1)(= spin(d+1, 1)), \ str_0(J_3^8) = \mathfrak{e}_{6(-26)}. \end{gathered} \tag{2.28}$$

The stabilizer of the point 1 of the cone is the maximal compact subgroup of $Aut(\mathcal{C})$ whose Lie algebra coincides with the derivation algebra[4] of J:

$$\mathfrak{Der}(J_r^1) = so(r),\ \mathfrak{Der}(J_r^2) = su(r),\ \mathfrak{Der}(J_r^4) = usp(2r),$$
$$\mathfrak{Der}(J_2^d) = so(d+1)(= spin(d+1)),\ \mathfrak{Der}(J_3^8) = \mathfrak{f}_4. \tag{2.29}$$

The structure Lie algebra acts by automorphisms on J. For the simple Jordan algebras J_r^d, $d = 1, 2, 4$ of hermitian matrices over an associative division ring an element u of $str(J_r^d)$ transforms hermitian matrices into hermitian by the formula:

$$u : (J_r^d \ni) X \to uX + Xu^*,\ d = 1, 2, 4, \tag{2.30}$$

where u^* is the hermitian conjugate of u. If u belongs to the derivation subalgebra $\mathfrak{Der}(J_r^d) \subset str(J_r^d)$ then $u^* = -u$ and (2.30) becomes a commutator (thus annihilating the Jordan unit). In general, (2.30) can be viewed as a Z_2 graded commutator (regarding the hermitian matrices as odd elements).

We shall argue that the exceptional Jordan algebra J_3^8 should belong to the observable algebra of the SM. It has three (special) Jordan subalgebras J_2^8 whose euclidean extensions match each one family of basic fermions.

3 Octonions, Quark-Lepton Symmetry, J_3^8

3.1 Why Octonions?

The *octonions* $\mathbb{O}$, the non-associative 8-dimensional composition algebra (reviewed in [2, 18], in Chaps. 19, 23 of [42, 50], and in [56], Sect. 1), were originally introduced as pairs of quaternions (the "Cayley-Dickson construction"). But it was the decomposition of $\mathbb{O}$ into complex spaces,

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3,\ x = z + \mathbf{Z},\ z = x^0 + x^7 e_7,\ \mathbf{Z} = Z^1 e_1 + Z^2 e_2 + Z^4 e_4,$$
$$Z^j = x^j + x^{3j(mod7)} e_7,\ j = 1, 2, 4;$$
$$e_i e_{i+1} = e_{i+3(mod7)},\ e_i e_k + e_k e_i = -2\delta_{ik},\ i, k = 1, ...7, \tag{3.1}$$

that led Feza Gürsey (and his student Murat Günaydin) back in 1973, [32, 34], to apply it to the quarks (then the newly proposed constituents of hadrons). They figured out that the subgroup $SU(3)$ of the automorphism group G_2 of the octonions, that fixes the first $\mathbb{C}$ in (3.1), can be identified with the quark colour group. Gürsey tried to relate the non-associativity of the octonions to the quark confinement - the unobservability of free quarks. Only hesitantly did he propose (in [34], 1974, Sect.

[4]The corresponding automorphism group need not be connected. For instance, $Aut(J_r^2)$ is the semi-direct product of $SU(r)/\mathbb{Z}_r$ with a $\mathbb{Z}_2$ generated by complex conjugation.

VII) "as another speculation" that the first $\mathbb{C}$ in (3.1) "could be related to leptons". Interpreting (3.1) as a manifestation of the quark-lepton symmetry was only taken seriously in 1987 by A. Govorkov [31] in Dubna. The subject has been later pursued by G.M. Dixon, - see e.g. [19, 20] and in [29] among others. M. Dubois-Violette pointed out [22] that, conversely, the unimodularity of the quark's colour symmetry yields - through an associated invariant volume form - an essentially unique octonion product with a multiplicative norm. The octonions (just like the quaternions) do not represent an observable algebra. They take part, however, in J_2^8 and in the exceptional Jordan algebra J_3^8 whose elements obey the following Jordan product rules:

$$X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix}$$
$$= \sum_{i=1}^{3} (\xi_i E_i + F_i(x_i)), \;\; E_i \circ E_j = \delta_{ij} E_i, \;\; E_i \circ F_j = \frac{1-\delta_{ij}}{2} F_j,$$
$$F_i(x) \circ F_i(y) = (x, y)(E_{i+1} + E_{i+2}), \;\; F_i(x) \circ F_{i+1}(y) = \frac{1}{2} F_{i+2}(y^* x^*) \qquad (3.2)$$

(indices being counted mod 3). The (order three) characteristic equation for $X(\xi, x)$ has the form:

$$X^3 - tr(X)X^2 + S(X)X - det(X) = 0; \;\; tr(X) = \xi_1 + \xi_2 + \xi_3,$$
$$S(X) = \xi_1\xi_2 - x_3 x_3^* + \xi_2\xi_3 - x_1 x_1^* + \xi_1\xi_3 - x_2^* x_2,$$
$$det(X) = \xi_1\xi_2\xi_3 + 2Re(x_1 x_2 x_3) - \sum_{i=1}^{3} \xi_i x_i x_i^*. \qquad (3.3)$$

The exceptional algebra J_3^8 incorporates *triality* that will be related - developing an idea of [26] - to the three generations of basic fermions.

3.2 Quark-Lepton Splitting of J_3^8 and Its Symmetry

The automorphism group of J_3^8 is the compact exceptional Lie group F_4 of rank 4 and dimension 52 whose Lie algebra is spanned by the (maximal rank) subalgebra $so(9)$ and its spinorial representation **16**, and can be expressed in terms of $so(8)$ and its three (inequivalent) 8-dimensional representations:

$$\mathfrak{Der}(J_3^8) = \mathfrak{f}_4 \cong so(9) + \mathbf{16} \cong so(8) \oplus \mathbf{8}_V \oplus \mathbf{8}_L \oplus \mathbf{8}_R; \qquad (3.4)$$

here $\mathbf{8}_V$ stands for the 8-vector, $\mathbf{8}_L$ and $\mathbf{8}_R$ for the left and right chiral $so(8)$ spinors. The group F_4 leaves the unit element $1 = E_1 + E_2 + E_3$ invariant and transforms the traceless part of J_3^8 into itself (under its lowest dimensional fundamental representation **26**).

The lepton-quark splitting (3.1) yields the following decomposition of J_3^8:

$$X(\xi, x) = X(\xi, z) + Z,\ X(\xi, z) \in J_3^2 = \mathcal{H}_3(\mathbb{C}),$$
$$Z = (Z_r^j,\ j = 1, 2, 4,\ r = 1, 2, 3) \in \mathbb{C}[3]. \quad (3.5)$$

The subgroup of $Aut(J_3^8)$ which respects this decomposition is the commutant $F_4^\omega \subset F_4$ of the automorphism $\omega \in G_2 \subset Spin(8) \subset F_4$ (of order three):

$$\omega X(\xi, x) = \sum_{i=1}^{3}(\xi_i E_i + F_i(\omega_7 x_i)),\ \omega_7 = \frac{-1+\sqrt{3}e_7}{2}\ (\omega^3 = 1 = \omega_7^3). \quad (3.6)$$

It consists of two $SU(3)$ factors (with their common centre acting trivially):

$$F_4^\omega = \frac{SU(3)_c \times SU(3)_{ew}}{\mathbb{Z}_3} \ni (U, V) :\ X(\xi, x) \to VX(\xi, z)V^* + UZV^*. \quad (3.7)$$

We see that the factor, U acts on each quark's colour index $j(= 1, 2, 4)$, so it corresponds to the exact $SU(3)_c$ colour symmetry while V acts on the first term in (3.5) and on the flavour index $r(= 1, 2, 3)$ and is identified with (an extension of) the broken electroweak symmetry as suggested by its restriction to the first generation of fermions (Sect. 4).

4 The First Generation Algebra

4.1 The G_{SM} Subgroup of $Spin(9)$ and Observables in $\tilde{J}_2^8$

The Jordan subalgebra $J_2^8 \subset J_3^8$, orthogonal, say, to the projector E_1,

$$J_2^8(1) = (1 - E_1)J_3^8(1 - E_1), \quad (4.1)$$

is special, its associative envelope being $C\ell_9$. Its automorphism group[5] is $Spin(9) \subset F_4$, whose intersection with F_4^ω, that respects the quark-lepton splitting, coincides with - and *explains* - *the gauge group (1.1) of the SM*:

$$G_{SM} = F_4^\omega \cap Spin(9) = Spin(9)^\omega = S(U(3) \times U(2)). \quad (4.2)$$

[5] Another instance of a (closed connected) maximal rank subgroup of a compact Lie group that have been studied by mathematicians [7] back in 1949 (see also [55]). The group $Spin(9)$ had another appearance in the *algebraic dreams* of Ramond [51].

As argued in Sect. 2.2 the optimal euclidean extension $\tilde{J}_2^8$ (2.23) of J_2^8 is obtained by replacing the real symmetric by complex hermitian matrices in the maximal euclidean Jordan subalgebra of its associative envelope:

$$J_2^8 \subset C\ell_9 = \mathbb{R}[16] \oplus \mathbb{R}[16] \to J_{16}^1 \oplus J_{16}^1 (\subset C\ell_9),\ J_{16}^1 = \mathcal{H}_{16}(\mathbb{R})$$
$$J_2^8 \to \tilde{J}_2^8 = J_{16}^2 \oplus J_{16}^2 \subset C\ell_9(\mathbb{C}),\ \ J_{16}^2 = \mathcal{H}_{16}(\mathbb{C}). \qquad (4.3)$$

The resulting (reducible) Jordan algebra of rank 32 gives room precisely to the state space (of internal quantum numbers) of fundamental fermions of one generation - including the right handed "sterile" neutrino. In fact, it is acted upon by the simple structure group of J_2^8 whose generators belong to the even part of the Clifford algebra $C\ell(9,1)$ isomorphic to $C\ell_9$; its Dirac spinor representation splits into two chiral (Majorana-)Weyl spinors:

$$Str_0(J_2^8) = Spin(9,1) \subset C\ell^0(9,1)(\cong C\ell_9) \Rightarrow \mathbf{32} = \mathbf{16}_L \oplus \mathbf{16}_R. \qquad (4.4)$$

Here is an explicit realization of the above embedding/extension. Let e_ν, $\nu = 0, 1, ..., 7$, be the octonionic units satisfying (3.1). The anticommutation relations of the imaginary octonion units e_j can be realized by the real skew-symmetric 8×8 matrices P_j generating $C\ell_{-6}$ and their product:

$$[P_j, P_k]_+ := P_j P_k + P_k P_j = -2\delta_{jk},\ P_7 := P_1 ... P_6 \Rightarrow [P_7, P_j]_+ = 0,\ P_7^2 = -1. \qquad (4.5)$$

The nine two-by-two hermitian traceless octonionic matrices $\hat{e}_a$ that generate J_2^8 (cf. (2.19)) are represented by similar real symmetric 16×16 matrices $\hat{P}_a$:

$$\hat{e}_0 = \sigma_1 e_0\ (e_0 = 1), \hat{e}_j = c e_j,\ j = 1, ..., 7,\ c = i\sigma_2,\ c^* = -c = c^3,$$
$$\hat{e}_a \hat{e}_b + \hat{e}_b \hat{e}_a = 2\delta_{ab} \Rightarrow \pm \hat{e}_8 = \hat{e}_0 ... \hat{e}_7 = \sigma_1 c^7(-1) = -\sigma_3;$$
$$\hat{P}_j = c \otimes P_j,\ j = 1, ..., 7,\ \hat{P}_0 = \sigma_1 \otimes P_0,\ P_0 = \mathbf{1}_8,\ \hat{P}_8 = \sigma_3 \otimes P_0. \qquad (4.6)$$

The nine matrices $\hat{P}_a$, $a = 0, 1, ..., 8$ generate an irreducible component of the Clifford algebra $C\ell_9$. Then the ten real 32×32 matrices

$$\Gamma_a = \sigma_1 \otimes \hat{P}_a,\ a = 0, 1, ..., 8,\ \Gamma_{-1} = c \otimes \mathbf{1}_{16} \Leftrightarrow$$
$$\Gamma_{-1} = \gamma_0 \otimes P_0, \Gamma_0 = \gamma_1 \otimes P_0, \Gamma_j = i\gamma_2 \otimes P_j,\ j = 1, ..., 7,\ \Gamma_8 = \gamma_3 \otimes P_0$$
$$(\gamma_0 = c^* \otimes \mathbf{1}_2, \gamma_j = \sigma_1 \otimes \sigma_j,\ j = 1, 2, 3,\ \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_3 \otimes \mathbf{1}_2) \quad (4.7)$$

generate the Clifford algebra $C\ell(9,1)$. We make correspond to the generators $\hat{e}_a$ (4.6) of J_2^8 the hermitian elements $\Gamma_{-1}\Gamma_a$ of the even subalgebra $C\ell^0(9,1) \simeq C\ell_9$. The generators of the symmetry algebra $so(9)$ of the spin factor J_2^8 are given by the antihermitian matrices

$$\Gamma_{ab} := [\Gamma_{-1}\Gamma_a, \Gamma_{-1}\Gamma_b] = [\Gamma_a, \Gamma_b] (= -\Gamma_{ab}^* = . = -\Gamma_{ba}). \qquad (4.8)$$

The corresponding observables $i\Gamma_{ab}$ only belong to the subalgebra $\tilde{J}_2^8$ (2.23) of pairs of complex hermitian 16×16 matrices.

In order to identify a complete commuting set of observables we introduce a maximal abelian subalgebra $\mathcal{A}$ of hermitian elements of the universal enveloping algebra $U(so(9,1))$ and the commutant $\mathcal{A}_8$ of Γ_8 in $\mathcal{A}$:

$$\mathcal{A} = \mathbb{R}[\Gamma_{-1,8}, i\Gamma_{07}, i\Gamma_{13}, i\Gamma_{26}, i\Gamma_{45}] \subset U(so(9,1)),$$
$$\mathcal{A}_8 = \mathbb{R}[i\Gamma_{07}, i\Gamma_{13}, i\Gamma_{26}, i\Gamma_{45}] \subset U(so(8)). \tag{4.9}$$

(The multilinear functions of $\mathcal{A}_8$ span a 16-dimensional vector subspace of the 32-dimensional real vector space $\mathcal{A}$.) To reveal the physical meaning of the generators of $\mathcal{A}_8 \subset \mathcal{A}$ we identify them with the Cartan elements of the maximal rank semisimple Lie subalgebra $so(6) \oplus so(3)$, the intersection

$$\mathfrak{g}_4 := so(6) \oplus so(3) \cong su(4) \oplus su(2) = \mathfrak{g}_{PS} \cap so(9) \tag{4.10}$$

of the Pati-Salam algebra $\mathfrak{g}_{PS} = su(4) \oplus su(2)_L \oplus su(2)_R$ with $so(9)$, both viewed as Lie subalgebras of $so(10)$. Here $su(2)$ is embedded diagonally[6] into $su(2)_L \oplus su(2)_R$. It is easily verified that the Lie algebra $\mathfrak{g}_4$ acting on the vector representation **9** of $Spin(9)$ preserves the quark lepton splitting in J_2^8. It does not preserve this splitting in the full Albert algebra J_3^8 which also involves the spinor representation **16** of $Spin(9)$. In accord with our statement in the beginning of this section only its subgroup G_{SM} (4.2) does respect the required symmetry for both nontrivial IRs of $Spin(9)$ contained in the fundamental representation **26** of F_4.

All elementary fermions can be labeled by the eigenvalues of two commuting operators in the centralizer of $su(3)_c$ in $\mathfrak{g}_4(\subset so(9))$:

$$2I_3 = -i\Gamma_{07}(= 2I_3^L + 2I_3^R),\ (2I_3)^2 = 1,\ \ B - L = \frac{i}{3} \sum_{j=1,2,4} \Gamma_{j,3j(mod 7)} \tag{4.11}$$

and of the *chirality* given by the Coxeter element of $C\ell(9,1)$:

$$\gamma := \omega_{9,1} = \Gamma_{-1}\Gamma_0\Gamma_1 \dots \Gamma_8 = \gamma_5 \otimes \mathbf{1}_8 \in C\ell^0(9,1)\ (= \gamma^*,\ \ \gamma^2 = 1). \tag{4.12}$$

Here $B - L$ (the difference between the baryon and the lepton number) is given by the commutant of $su(3)_c$ in $so(6)(= Span\{\Gamma_{ik}, i, k = 1, \dots 6\}) \subset \mathfrak{g}_4$. The electric charge is a linear function of $B - L$ and I_3:

$$Q = \frac{1}{2}(B - L) + I_3. \tag{4.13}$$

[6]We recall that while $I_3 \in su(2)$ is a convenient global label for fundamental fermions, it is $su(2)_L \oplus u(1)$ which carries the local gauge symmetry of weakly interacting bosons.

The right and left isospins $I_3^{R/L}$ and the hypercharge Y involve $\varGamma_{-1,8} \notin \mathcal{A}_8$ and their expressions in terms of I_3, $B-L$ and γ are more complicated:

$$4I_3^R = \varGamma_{-1,8} - i\varGamma_{07} = (1+\gamma(B-L))(3\gamma(B-L)-1)(3\gamma(B-L)-2)I_3,$$
$$4I_3^L = -\varGamma_{-1,8} - i\varGamma_{07} = 4(I_3 - I_3^R),\ Y = B-L+2I_3^R, \quad (4.14)$$

but we won't need them. Remarkably, all quarks and leptons are labeled by a single quantum number $B-L(=\pm 1, \pm 1/3)$ and two signs ($2I_3 = \pm 1, \gamma = \pm$).

As the quark colour is not observable we only have to distinguish $SU(3)_c$ representations as labels: **3** for a quark triplet, $\bar{\mathbf{3}}$ for an antiquark, and **1** for an $SU(3)_c$ singlet. These are encoded in the value of $B-L$:

$$B-L = \frac{1}{3} \leftrightarrow \mathbf{3},\ B-L = -\frac{1}{3} \leftrightarrow \bar{\mathbf{3}},\ B-L = \pm 1 \leftrightarrow \mathbf{1}.$$

We have eight primitive idempotents corresponding to the left and right (anti)leptons and eight (non primitive) chiral (anti)quark idempotents (colour singlets of trace three); for instance,

$$(\nu_L) := |\nu_L><\nu_L| \leftrightarrow (2I_3 = 1, B-L = -1, \gamma = 1),\ (e_R^+) \leftrightarrow (1, 1, -1);$$
$$(\bar{u}_L) := \sum_{j=1,2,4} |\bar{u}_L^j><\bar{u}_L^j| \leftrightarrow (-1, -\frac{1}{3}, 1),\ (d_L) \leftrightarrow (-1, \frac{1}{3}, 1). \quad (4.15)$$

4.2 Observables. Odd Chirality Operators

The 16-dimensional vector spaces S_R and S_L can be identified with the subspaces of J_3^8 spanned by $F_2(x_2)$ and $F_3(x_3)$ of Eq. (3.2), respectively. (As we shall recall in Sect. 5.1 below, they transform as expected under the $Spin(8)$ subgroup of the automorphism group F_4 of J_3^8.) The (extended) observables belong by definition to the complexification of the even part, $C\ell^0(9,1)$, of $C\ell(9,1)(\cong \mathbb{R}[32])$, thus commute with chirality and preserve individually the spaces S_R and S_L. The elements of the odd subspace, $C\ell^1(9,1)$, in particular the generators $\varGamma_\mu$ of $C\ell(9,1)$, by contrast, anticommute with chirality and interchange the left and right spinors. They can serve to define the internal space part of the Dirac operator.

We shall now present an explicit realization of both $\varGamma_\mu$ and the basic observables in terms of *fermionic oscillators* (anticommuting creation and annihilation operators - updating Sect. 5 of [56]), inspired by [1, 30]. To begin with, we note that the complexification of $C\ell(9,1)$ contains a five dimensional isotropic subspace spanned by the anticommuting operators

$$a_0 = \frac{1}{2}(\Gamma_0 + i\Gamma_7), a_1 = \frac{1}{2}(\Gamma_1 + i\Gamma_3), a_2 = \frac{1}{2}(\Gamma_2 + i\Gamma_6), a_4 = \frac{1}{2}(\Gamma_4 + i\Gamma_5)$$
$$(i.e. \, a_j = \frac{1}{2}(\Gamma_j + i\Gamma_{3j mod 7}), \; j = 1, 2, 4, \; i^2 = -1), \; a_8 = \frac{1}{2}(\Gamma_8 + \Gamma_{-1}). \quad (4.16)$$

The a_μ and their conjugate a_μ^* obey the canonical anticommutation relations,

$$[a_\mu, a_\nu]_+ = 0, \; [a_\mu, a_\nu^*]_+ = \delta_{\mu\nu}, \; \mu, \nu = 0, 1, 2, 4, 8, \quad (4.17)$$

equivalent to the defining Clifford algebra relations for $\Gamma_\alpha, \; \alpha = -1, 0, 1, ..., 8$.

Remark. The 32×32 matrices $\Gamma_{-1}, \Gamma_\mu(\mu = 0, 1, 2, 4, 8), i\Gamma_7, i\Gamma_3, i\Gamma_6, i\Gamma_5$ generate the split real form $C\ell(5,5)(\cong C\ell(9,1))$ of $C\ell_{10}(\mathbb{C})$. The split forms $C\ell(n,n)$ (which allow to treat spinors as differential forms) have been used by K. Krasnov [40] (for $n = 7$) in his attempt to make the SM natural.

We now proceed to translate the identification of basic observables of Sect. 4.1 in terms of the five products $a_\mu^* a_\mu, \; \mu = 0, 1, 2, 4, 8$. The set of 15 quadratic combinations of a_i^*, a_j (nine $a_i^* a_j$, three independent products $a_i a_j = -a_j a_i$, and as many $a_i^* a_j^*$) generate the Pati-Salam Lie algebra $su(4)$. The centralizer $B - L$ (4.11) of $su(3)_c$ in $su(4)$ now assumes the form:

$$B - L = \frac{1}{3}\sum_j [a_j^*, a_j] \Rightarrow [B - L, a_j^*] = \frac{1}{3}a_j^*, \; [B - L, a_1 a_2 a_4] = -a_1 a_2 a_4 \quad (4.18)$$

$(j = 1, 2, 4)$. The indices 0, 8 correspond to the left and right isospins:

$$I_+^L = a_8^* a_0, \; I_-^L = a_0^* a_8, \; 2I_3^L = [I_+^L, I_-^L] = a_8^* a_8 - a_0^* a_0;$$
$$I_-^R = a_0^* a_8^*, \; I_+^R = a_8 a_0, \; 2I_3^R = [I_+^R, I_-^R] = a_8 a_8^* - a_0^* a_0; \quad (4.19)$$
$$I_+ := I_+^L + I_+^R = \Gamma_8 a_0, \; I_- := I_-^L + I_-^R = a_0^* \Gamma_8, \; 2I_3 = [I_+, I_-] = [a_0, a_0^*].$$

The hypercharge and the chirality also involve $\Gamma_{-1}\Gamma_8 = [a_8, a_8^*]$:

$$Y = \frac{2}{3}\sum_j a_j^* a_j - a_0^* a_0 - a_8^* a_8, \; \gamma = [a_0^*, a_0][a_1^*, a_1][a_2^*, a_2][a_4^*, a_4][a_8^*, a_8]. \quad (4.20)$$

All fermion states and the commuting observables are expressed in terms of five basic idempotents π_μ and their complements:

$$\pi_\mu = a_\mu a_\mu^*, \; \bar{\pi}_\mu = 1 - \pi_\mu = a_\mu^* a_\mu, \pi_\mu^2 = \pi_\mu, \; \bar{\pi}_\mu^2 = \bar{\pi}_\mu, \pi_\mu \bar{\pi}_\mu = 0, \; [\pi_\mu, \pi_\nu] = 0,$$
$$\pi_\mu + \bar{\pi}_\mu = 1, \; tr 1 = 32 \Rightarrow tr\pi_\mu = tr\bar{\pi}_\mu = 16, \; \mu = 0, 1, 2, 4, 8. \quad (4.21)$$

As π_μ commute among themselves, products of π_μ are again idempotents. No product of less than five factors is primitive; for instance $\pi_0\pi_1\pi_2\pi_4 = \pi_0\pi_1\pi_2\pi_4(\pi_8 + \bar{\pi}_8)$. Any of the 2^5 products of five $\pi_\mu, \bar{\pi}_\nu$ of different indices is primitive. We shall take as “vacuum” vector the product of all π_μ which carry the quantum numbers of the right chiral (“sterile”) neutrino:

$$\Omega := \pi_0\pi_1\pi_2\pi_4\pi_8 = (\nu_R) \leftrightarrow (2I_3 = 1, B - L = -1, \gamma = -1),\ a_\nu\Omega = 0. \tag{4.22}$$

The left chiral states involve products of an even number of π_ν (an odd number for the right chiral states). In particular, $(\bar{\nu}_L)$, the antiparticle of (ν_R), only involves $\bar{\pi}_\nu$ factors:

$$(\bar{\nu}_L) = \bar{\Omega} := \bar{\pi}_0\bar{\pi}_1\bar{\pi}_2\bar{\pi}_4\bar{\pi}_8 \leftrightarrow (-1, 1, 1),\ a_\nu^*\bar{\Omega} = 0. \tag{4.23}$$

Each primitive idempotent can be obtained from either Ω or $\bar{\Omega}$ by consecutive action of the involutive chirality changing bilinear maps (cf. (2.3)):

$$U(a_\mu^*, a_\mu)X = a_\mu^* X a_\mu + a_\mu X a_\mu^*,\ X \in \mathcal{A} = \mathbb{R}[\pi],\ \mu = 0, 1, 2, 4, 8; \tag{4.24}$$

here $\pi = \{\pi_0, \pi_1, \pi_2, \pi_4, \pi_8\}$ is a basis of idempotents of the abelian multilinear algebra $\mathcal{A}$ (4.9). More economically, we obtain all eight lepton states by acting with the above operators for $\mu = 0, 8$ on both Ω and $\bar{\Omega}$ (or only on Ω but also using the “colourless” operator $U(a_1^*a_2^*a_4^*, a_4a_2a_1)$):

$$\begin{aligned}
&(\nu_R) = \Omega,\ (e_L^-) = U(a_0^*, a_0)\Omega = a_0^*\Omega a_0,\ (\nu_L) = a_8^*\Omega a_8\ (a_0\Omega a_0^* = 0 = a_8\Omega a_8^*);\\
&(e_R^-) = a_0^*a_8^*\Omega a_8 a_0,\ (e_L^+) = a_8 a_0\bar{\Omega}a_0^*a_8^* = a_1^*a_2^*a_4^*\Omega a_4a_2a_1;\\
&(\bar{\nu}_R) = a_8\bar{\Omega}a_8^* = a_0^*(e_L^+)a_0,\ (e_R^+) = a_0\bar{\Omega}a_0^* = a_8^*(e_L^+)a_8,\ (\nu_L) = \bar{\Omega} = a_0^*(e_R^+)a_0.
\end{aligned} \tag{4.25}$$

The $SU(3)_c$ invariant (trivalent) quark states are obtained by acting with the operator

$$U_q := \sum_{j=1,2,4} U(a_j^*, a_j) \tag{4.26}$$

on the corresponding lepton states:

$$\begin{aligned}
&U_q(\nu_R) = (\bar{d}_L),\ U_q(e_R^-) = (\bar{u}_L);\ U_q(\nu_L) = (\bar{d}_R),\ U_q(e_L^-) = (\bar{u}_R);\\
&U_q(e_L^+) = (u_R),\ U_q(\bar{\nu}_L) = (d_R);\ U_q(e_R^+) = (u_L),\ U_q(\bar{\nu}_R) = (d_L).
\end{aligned} \tag{4.27}$$

We note that while (ν_R) is the lowest weight vector of $so(9, 1)$ with five simple roots corresponding to the commutators $[a_\mu^*, a_\mu]$, the state that minimizes our choice of observables, $2I_3$, $B - L$, γ, is the right electron singlet (e_R^-) - the lowest weight vector of $\mathfrak{g}_4 \times \gamma$. (The corresponding highest weight vectors are given by the respective antiparticle states.)

Remark. Each of the rank 16 extensions of J_2^8 appears as a subrepresentation of the defining module **26** of the automorphism group F_4 of J_3^8, which splits into three irreducible components of $Spin(9)$:

$$\mathbf{26} = \mathbf{16}\ +\ \mathbf{9}\ +\mathbf{1}. \tag{4.28}$$

The vector representation **9** of $SO(9)$ is spanned by the generators Γ_a of $C\ell_9$. Their splitting into the defining representations of the two factors of its maximal rank subgroup $SO(6) \times SO(3)$ displays the quantum numbers of a pair of conjugate leptoquark and a triplet of weak interaction bosons:

$$\mathbf{9} = (\mathbf{6}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}); \;\; \mathbf{6} = (a_j, a_j^*, \; j = 1, 2, 4) = (D, \bar{D});$$

$$[B - L, a_j] = -\frac{2}{3} a_j, \; [B - L, a_j^*] = \frac{2}{3} a_j^* \Rightarrow D \leftrightarrow (0, -\frac{1}{3}, -\frac{2}{3}), \; \bar{D} \leftrightarrow (0, \frac{1}{3}, \frac{2}{3});$$

$$\mathbf{3} = (a_0^*, \Gamma_8, a_0) =: (W^+, W^0, W^-) \leftrightarrow Y = B - L = 0, \; Q = (1, 0, -1). \quad (4.29)$$

Combining the **3** with the singlet **1** one can find the mixtures that define the (neutral) Z-boson and the photon. The leptoquarks D, $\bar{D}$ also appear in the **27** of the E_6 GUT (see, e.g. [52] where they are treated as superheavy).

As pointed out in the beginning of this subsection we shall view instead the $\mathbf{9} \oplus \mathbf{1}$ as the gamma matrices which anticommute with chirality and (are not observables but) should enter the definition of the Dirac operator.

5 Triality and Yukawa Coupling

5.1 Associative Trilinear Form. The Principle of Triality

The trace of an octonion $x = \sum_\mu x^\mu e_\mu$ is a real valued linear form on $\mathbb{O}$:

$$tr(x) = x + x^* = 2x^0 = 2Re(x) \;\; (e_0 \equiv 1). \quad (5.1)$$

It allows to define an associative and symmetric under cyclic permutations *normed triality* form $t(x, y, z) = Re(xyz)$ satisfying:

$$2t(x, y, z) = tr((xy)z) = tr(x(yz)) =: tr(xyz) = tr(zxy) = tr(yzx). \quad (5.2)$$

The normalization factor 2 is chosen to have:

$$|t(x, y, z)|^2 \leq N(x)N(y)N(z), \;\; N(x) = xx^* (\in \mathbb{R}). \quad (5.3)$$

While the norm $N(x)$ and the corresponding scalar product are $SO(8)$-invariant, the trilinear form t corresponds to the invariant product of the three inequivalent 8-dimensional fundamental representations of $Spin(8)$, the 8-vector and the two chiral spinors, say $S^{\pm}$ (denoted by $\mathbf{8}_L$ and $\mathbf{8}_R$ in (3.4)). We proceed to formulating the more subtle trilinear invariance law.[7]

Theorem 5.1 (*Principle of triality* - see [60], Theorem 1.14.2). For any $g \in SO(8)$ there exists a pair (g^+, g^-) of elements of $SO(8)$, such that

$$g(xy) = (g^+x)(g^-y), \;\; x, y \in \mathbb{O}. \tag{5.4}$$

If the pair (g^+, g^-) satisfies (5.4) then the only other pair which obeys the principle of triality is $(-g^+, -g^-)$.

Corollary. If the triple g, g^+, g^- obeys (5.4) then the form t (5.2) satisfies the invariance condition

$$t(g^+x, g^-y, g^{-1}z) = t(x, y, z). \tag{5.5}$$

Proposition 5.2 (see [60] Theorem 1.16.2). The set of triples $(g, g^+, g^-) \in SO(8) \times SO(8) \times SO(8)$ satisfying the principle of triality form a group isomorphic to the double cover $Spin(8)$ of $SO(8)$.

An example of a triple (g^+, g^-, g^{-1}) satisfying (5.5) is provided by left-, right- and bi-multiplication by a unit octonion:

$$t(L_u x, R_u y, B_{u^*} z) = t(x, y, z), \; L_u x = ux, \, R_u y = yu, \, B_v z = vzv, \;\; uu^* = 1. \tag{5.6}$$

The permutations among g, g^+, g^- belong to the group of outer automorphisms of the Lie algebra $so(8)$ which coincides with the symmetric group $\mathcal{S}_3$ that permutes the nodes of the Dynkin diagram for $so(8)$. In particular, the map that permutes cyclicly L_u, R_u, B_{u^*} belongs to the subgroup $\mathbb{Z}_3 \subset \mathcal{S}_3$:

$$\nu : L_u \to R_u \to B_{u^*} \Rightarrow \nu^3 = 1. \tag{5.7}$$

Remark. The associativity law expressed in terms of left (or right) multiplication reads

$$L_x L_y = L_{xy}, \; R_x R_y = R_{yx}. \tag{5.8}$$

It is valid for complex numbers and for quaternions; for octonions Eq. (5.8) only takes place for real multiples of powers of a single element. Left and right multiplications by unit quaternions generate different $SO(3)$ subgroups of the full isometry group $SO(4)$ of quaternions. By contrast, products of up to 7 left multiplications of unit octonions (and similarly of up to 7 R_u or B_u) generate the entire $SO(8)$ (see Sect. 8.4 of [18]).

[7] For systematic expositions of the $Spin(8)$ triality see, in order of appearance, [50] (Chap. 24), [42] (Chap. 23), [2] (Sect. 2.4), [18] (Sect. 8.3), [60] (Sects. 1.14–1.16).

Proposition 5.3 (see [60] Theorem 2.7.1). The subgroup $Spin(8)$ of F_4 leaves the diagonal projectors E_i in the generic element $X(\xi, x)$ (3.2) of J_3^8 invariant and transforms the off diagonal elements as follows:

$$F_1(x_1) + F_2(x_2) + F_3(x_3) \to F_1(gx_1) + F_2(g^+x_2) + F_3(g^-x_3). \tag{5.9}$$

Thus if we regard x_1 as a $Spin(8)$ vector, then x_2 and x_3 should transform as S^+ and S^- spinors, respectively.

5.2 *Speculations About Yukawa Couplings*

It would be attractive to interpret the invariant trilinear form $t(x_1, x_2, x_3)$ as the internal symmetry counterpart of the Yukawa coupling between a vector and two (conjugate) spinors. Viewing the 8-vector x_1 as the finite geometry image of the Higgs boson, the associated Yukawa coupling would be responsible for the appearance of (the first generation) fermion masses. There are, in fact, three possible choices for the $SO(8)$ vector, one for each generation i corresponding to the Jordan subalgebra

$$J_2^8(i) = (1 - E_i)J_3^8(1 - E_i), \ i = 1, 2, 3. \tag{5.10}$$

According to Jacobson [38] any finite (unital) module over J_3^8 has the form $J_3^8 \otimes E$ for some finite dimensional real vector space E. The above consideration suggests that $dim(E)$ should be a multiple of three so that there would be room for an octonion counterpart of a vector current for each generation.

As demonstrated in Sect. 4 the natural euclidean extension of J_2^8 gives rise to a Jordan frame fitting nicely one generation of fermions. We also observed that the generators Γ_a of $C\ell_9$ anticommute with chirality and thus do not belong to the observable algebra but may serve to define (the internal part of) the Dirac operator. Unfortunately, according to Albert's theorem, J_3^8 admits no associative envelope and hence no such an euclidean extension either. To search for a possible substitute it would be instructive to see what exactly would go wrong if we try to imitate the construction of Sect. 4.1. The first step, the map (cf. (4.5))

$$\mathbb{O} \ni x \to P(x) = \sum_{\alpha=0}^{7} x^\alpha P_\alpha \in C\ell_{-6} \cong \mathbb{R}[8],$$
$$P(xx^*) = P(x)P(x)^* = xx^*P_0, \ P(x)^* = \sum P_\alpha^* x^\alpha, \ P_0 = \mathbf{1}_8, \tag{5.11}$$

for each of the arguments of $F_i, i = 1, 2, 3$, respects all binary relations. We have, for instance (in the notation of (3.2)),

$$F_1(P(x_1)) \circ F_2(P(x_2)) = \frac{1}{2}F_3(P(x_2^*x_1^*)), \ P(x^*y^*) = P(x)^*P(y)^* . \tag{5.12}$$

The map (5.11) fails, however, to preserve the trilinear form (5.2) which appears in $det(X)$ (3.3) and in triple products like

$$(F(x_1) \circ F(x_2)) \circ F(x_3) = \frac{1}{2} t(x_1, x_2, x_3)(E_1 + E_2); \tag{5.13}$$

setting $P(x_i) = X_i$ we find instead of t more general 8×8 matrices:

$$(F_1(X_1) \circ F_2(X_2)) \circ F_3(X_3) = \\ \frac{1}{4}((X_2^* X_1^* X_3^* + X_3 X_1 X_2)E_1 + (X_1 X_2 X_3 + X_3^* X_1^* X_2^*)E_2). \tag{5.14}$$

Moreover, while $t(e_1, e_2, e_3) = 0$, $t(e_1, e_2, e_4) = -1$, the corresponding (symmetric, traceless, mutually orthogonal) matrices $P_1 P_2 P_3$, $P_1 P_2 P_4$ have both square 1 and can be interchanged by an inner automorphism.

In order to apply properly the full exceptional Jordan algebra to particle dynamics we need further study of differential calculus and connection forms on Jordan modules, as pursued in [10, 23], on one side, and connect with current phenomenological understanding of the SM, on the other. In particular, the Cabibbo-Kobayashi-Maskawa (CKM) quark- and the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton-mixing matrices [47, 49] should be reflected in a corresponding mixing of their finite geometry counterpart. Its concrete realization should help us fit together the euclidean extensions of the three subalgebras $J_2^8(i)$ and the Yukawa couplings expressed in terms of the invariant trilinear form in J_3^8.

6 Outlook

The idea of a finite quantum geometry appeared in the late 1980's in an attempt to make the Standard Model natural, avoiding at the same time the excessive number of new unobserved states accompanying GUTs and higher dimensional theories. It was developed and attained maturity during the past thirty years in the framework of noncommutative geometry and the real spectral triple in work of Alain Connes and others [8, 14, 16]. Here we survey the progress in an alternative attempt, put forward by Michel Dubois-Violette, [22] (one of the originators of the noncommutative geometry approach, too) based on a finite dimensional counterpart of the *algebra of observables*, hence, a (commutative) euclidean Jordan algebra. As the deep ideas of Pascual Jordan seem to have found a more receptive audience among mathematicians than among physicists, we used the opportunity to emphasize (in Sect. 2) how well suited a Jordan algebra J is for describing both the observables and the states of a quantum system.

Recalling (Sect. 2.2) the classification of finite dimensional simple euclidean observable algebras (of [37], 1934) and the quark-lepton symmetry (Sect. 3.1) we argue that it is a multiple of the exceptional Jordan (or *Albert*) algebra J_3^8 that describes the three generations of fundamental fermions. We postulate that the symmetry group of the SM is the subgroup F_4^ω of $Aut(J_3^8) = F_4$ that respects the quark-lepton splitting. Remarkably, the intersection of F_4^ω with the automorphism group $Spin(9)$ of the subalgebra $J_2^8 \subset J_3^8$ of a single generation is just the gauge group of the SM.

The next big problem we should face is to fix an appropriate J_3^8 module (following our discussion in Sect. 5.2) and to write down the Lagrangian in terms of fields taking values in this module.

Acknowledgements This paper can be viewed as a progress report on an ongoing project pursued in [22, 26, 55, 56]. It is an extended version of talks at the: 13-th International Workshop "Lie Theory and Its Applications in Physics" (LT13), Varna, Bulgaria, June 2019; Workshop "Geometry and mathematical physics" (to the memory of Vasil Tsanov), Sofia, July 2019; Simplicity III Workshop at Perimeter Institute, Canada, September 9–12, 2019; Humboldt Kolleg "Frontiers in Physics, from Electroweak to Planck scales", Corfu, September 15–19, 2019; Conference "Noncommutative Geometry and the Standard Model", Jagiellonian University, Krakow, 8–9 November 2019. I thank my coauthors in these papers and Cohl Furey for discussions, Latham Boyle and Svetla Drenska for a careful reading of the manuscript, and both her and Michail Stoilov for their help in preparing these notes. I am grateful to Vladimir Dobrev, Latham Boyle, George Zoupanos and Andrzej Sitarz for invitation and hospitality in Varna, at the Perimeter Institute, in Corfu and in Kraków, respectively.

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Polylogarithms from the Bound State S-matrix

M. de Leeuw, B. Eden, D. le Plat, and T. Meier

Abstract Higher-point functions of gauge invariant composite operators in $\mathcal{N} = 4$ super Yang-Mills theory can be computed via triangulation. The elementary tile in this process is the hexagon introduced for the evaluation of structure constants. A gluing procedure welding the tiles back together is needed to return to the original object. In this note we present work in progress on n-point functions of BPS operators. In this case, quantum corrections are entirely carried by the gluing procedure. The lowest non-elementary process is the gluing of three adjacent tiles by the exchange of two single magnons. This problem has been analysed before. With a view to resolving some conceptional questions and to generalising to higher processes we are trying to develop an algorithmic approach using the representation of hypergeometric sums as integrals over Euler kernels.

1 Introductory Remarks

The spectrum problem of the AdS/CFT correspondence in the original form—so connecting $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions to IIB string theory on $AdS_5 \times S^5$—has been successfully described by an integrable system [7, 8, 10]. The effects of higher-loop corrections in the field theory can be incorporated into the corresponding Bethe equations using the Zhukowski variables $x(u)$ defined by

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_4

$$x + \frac{g^2}{2x} = u \tag{1}$$

where u is a Bethe rapidity. In particular, since the Bethe ansatz involves the quantities $u^{\pm} = u \pm \frac{i}{2}$ one introduces $x^{\pm}(u) = x(u^{\pm})$.

A double Wick rotation from the original model to a *mirror theory* enables one to use the thermodynamic Bethe ansatz (TBA) for the discussion of finite size effects in the AdS/CFT integrable model [1, 2, 11, 15, 18]. W.r.t. the Bethe rapidities, the mirror transformation is (here the scaling is adapted to the string side).

$$\gamma \quad : \quad x^+ \to \frac{1}{x^+} \tag{2}$$

while x^- stays inert. More generally we may define [5]

$$\begin{aligned} 2\gamma &: x^{\pm} \to \frac{1}{x^{\pm}}, \\ 3\gamma &: x^+ \to x^+,\ x^- \to \frac{1}{x^-}, \\ 4\gamma &: x^{\pm} \to x^{\pm}. \end{aligned} \tag{3}$$

The 2γ transformation has the interpretation of *crossing* from particles to antiparticles. Other $n\gamma$ transformations are the same modulo 4 on expressions only depending on the square root functions $x^{\pm}$. However, another element of the integrable system is the *dressing phase* [9] obeying a crossing equation [16] implying that it does not return to itself at 4γ. Developing the TBA has necessitated understanding the scattering of bound states of the model [3, 4].

For a long time, higher-point functions remained hard to address using these methods. The introduction of the *hexagon operator* meant a break-through w.r.t. the three-point problem [5]. In a nutshell, to evaluate the hexagon one can move all excitations to one spin chain by appropriate $n\gamma$ transformations and then scatter by the $psu(2|2)$ invariant S-matrix [6], or its bound state variant [4]. Every scattering is accompanied by a certain scalar factor h containing the (inverse of the) dressing phase and some factor of $x^{\pm}$ type.

Finally, it was noticed in [12, 13] that higher-point functions can likely be evaluated by tilings with hexagon patches, for a four-point function see Fig. 1. The circular openings in the figure are spin chains equivalent to the gauge theory operator. The faces of the figure yield four hexagons. The cut in the left panel is not promising because it introduces a sum over a complete set of physical states (the OPE), already at tree level. The second cut is suggested by tree-level Feynman diagrams and is much more useful because there is no sum over intermediate states.

To get back to the original uncut figure the tiles are *glued* by the procedure defined in [5] for the three-point case. This can be thought of as the insertion of a complete set of states. In fact, relevant are the bound states of the TBA analysis. This is a complicated but—as we shall see—hopefully manageable sum.

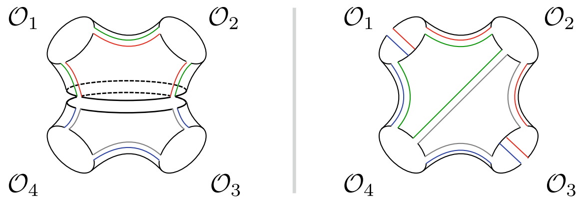

Fig. 1 OPE and non-OPE cuts of a four-point function

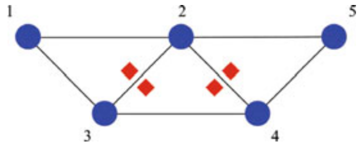

Fig. 2 Gluing three tiles by two virtual exchanges; the bound states are marked as red squares

The gluing of three adjacent tiles has been evaluated in [14] by matching a truncated residue calculation on an ansatz. The motivation for our study is to expand on this work: how can we integrate/sum in closed form? Second, in [14] it became apparent that extra braiding factors $e^{ip/2}$ (here p denotes the momentum of the bound state particles) have to be introduced. We would eventually like to answer whether the choice of braiding adopted there is the only possible one.

Now, according to [5] we have to choose $sl(2)$ sector bound states, c.f. Eq. (8). The scattering matrix available in the literature [4] is originally written for the opposite case: $su(2)$ bound states obtained from (8) by exchanging the rôle of bosonic and fermionic constituents. We argue below that—at least in the situation at hand—the formulae of [4] apply directly. We find a very clean integration scheme, although we cannot yet answer whether the result matches complete correlation functions.

2 Elements of the Calculation

Let us first consider gluing two neighbouring hexagons by a single mirror magnon. Let the first hexagon, as in the three-point problem in [5], connect operators at the positions

$$x_1 = \{0, 1, 0, 0\}, \quad x_2 = \{0, 0, 0, 0\}, \quad x_3 = \{0, \infty, 0, 0\}. \tag{4}$$

The second hexagon shares the edge between $0, \infty$ but its third point is parametrised by the variables $z, \bar{z}$, so it depends on

$$x_2 = \{0, 0, 0, 0\}, \quad x_3 = \{0, \infty, 0, 0\}, \quad x_4 = \{0, -\Im(z), \Re(z), 0\} \tag{5}$$

and the gluing is over the common edge 23. In a four-point problem this is not a restriction on the kinematics. One finds the frequently used parametrisation

$$\frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} = z\bar{z}\,, \qquad \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} = (1-z)(1-\bar{z}) \tag{6}$$

for the two independent cross-ratios. In [13] it is suggested to obtain the non-standard coordinates $z, \bar{z}$ from the usual situation $0, 1, \infty$ by the operator

$$W(z, \bar{z}) = e^{-D\,\log|z|} \sqrt{\frac{z}{\bar{z}}}^{L}\,, \qquad L = L_1^1 - L_2^2 \tag{7}$$

where D is the dilatation generator. This leaves $0, \infty$ invariant but maps $1 \mapsto (z, \bar{z})$. In calculations one will not transform coordinates but rather act on the states scattering over the second hexagon. Since the operator above is diagonal on these one can finally evaluate both hexagons as in the three-point problem.

Gluing means inserting $sl(2)$ bound states [5]. This *antisymmetric representation* at level (or length) a has the parts

$$(\psi^1)^{a-k-1}(\psi^2)^k \phi^i\,, \qquad (\psi^1)^{a-k}(\psi^2)^k\,, \qquad (\psi^1)^{a-k-1}(\psi^2)^{k-1}\phi^1\phi^2\,. \tag{8}$$

Customarily, in the first case one separately considers $i = 1, 2$. In a four-point calculation one can [13] act on the *states* on the second hexagon by the tilting transformation (7). To this end we rewrite

$$\frac{1}{2}(D - J) = E = i\tilde{p} = i\,u + \ldots \tag{9}$$

where the dots indicate $O(g^2)$ corrections in a weak coupling expansion. The generator J acts on the scalars in the $sl(2)$ bound states, of which there are only one or two. Our purpose in this note is to re-sum the infinite series in $z, \bar{z}$ that the weight factor creates; for now we turn a blind eye on all transformations required to rotate the second hexagon in the internal space. We can also send $\psi^{a-k-1} \to \psi^{a-k}$ etc. since these are constant shifts, while the summation ranges must, of course, be respected to obtain sensible results.

Then,

$$W(z, \bar{z})\,(\psi^1)^{a-k}(\psi^2)^k = (z\bar{z})^{-i\,u} \left(\frac{z}{\bar{z}}\right)^{\frac{a}{2}-k} (\psi^1)^{a-k}(\psi^2)^k \tag{10}$$

because L (the Cartan generator of the Lorentz transformation) attributes weight $1, -1$ to ψ^1, ψ^2, respectively.

In the five-point process in Fig. 2, the central tile is glued to two neighbouring hexagons. Full fledged five-point kinematics cannot be parametrised using only the coordinates of the 1, 2 plane, so that the Cartan generators used above are not enough to recover it. Nonetheless, to get started we follow [14] and use restricted kinematics. We then obtain a weight factor for either gluing, so $W(z_1, \bar{z}_1)\, W(z_2, \bar{z}_2)$. Clearly, the fifth cross ratio is lost.

On the left and the right hexagon there is only one bound state and thus no scattering. Yet, the contraction rule for the outer hexagons enforces the scattering on the middle tile to be diagonal. Further, let us choose $3\gamma,\ 1\gamma$ kinematics on the middle hexagon in which case the scalar factor becomes

$$h(u^{3\gamma}, v^{\gamma}) \;=\; \Sigma(u^{\gamma}, v^{\gamma}) \tag{11}$$

with the *improved BES dressing phase* [3, 9] in mirror/mirror kinematics

$$\Sigma^{ab} \;=\; \frac{\Gamma[1+\frac{a}{2}+i\,u]}{\Gamma[1+\frac{a}{2}-i\,u]}\,\frac{\Gamma[1+\frac{b}{2}-i\,v]}{\Gamma[1+\frac{b}{2}+i\,v]}\,\frac{\Gamma[1+\frac{a+b}{2}-i(u-v)]}{\Gamma[1+\frac{a+b}{2}+i(u-v)]}+O(g)\,. \tag{12}$$

A comprehensive discussion of the bound state S-matrix is given in [4]. By way of example, we consider the scattering of two states of the first type in (8), both with $i=1$ or both with $i=2$. The relevant scattering matrix is called $X_n^{kl}(a,u,b,v)$ in [4], where we associate the bound state counter a and the rapidity u as well as $x^{\pm}(u)$ with the first particle and $b,\ v,\ y^{\pm}(v)$ with the second.

In the symmetric representation, the rôle of bosons and fermions is exchanged, in particular in (8). As a consequence, at bound state length 1 (so for fundamental particles) the X element describes the scattering of two equal fermions. Hence in [4] it is called D in agreement with the nomenclature of [6]. The entire S-matrix can be changed by an overall factor, and indeed this D is equal to the A-element in [6].[1]

We repeated the steps of [4] to re-derive the S-matrix in the antisymmetric representation. Flipping the statistics means exchanging Poincaré and conformal supersymmetry, and also Lorentz and internal symmetry generators. Sticking to the same algebra conventions one obtains a sign flip on the rapidity parameters, so in particular $x^{\pm} \leftrightarrow x^{\mp}$. The X element at bound state length 1 now describes the scattering of two equal bosons. We observe that what was called D before now becomes A^{-1}. Hence for the antisymmetric representation the construction yields S^{-1} without any rescaling.

Next, by observation—at least in $3\gamma,\ 1\gamma$ kinematics and at leading order in g—the diagonal elements of our S^{-1} in the antisymmetric representation are related to those of [4] by flipping the sign of the rapidities, which has the interpretation of a complex conjugation or of taking a second inverse. It follows that we can use the S-matrix of [4] for our purposes, without any changes!

[1]It follows from here that the bound state length 1 part of the S-matrix of [4] is in fact the inverse of that derived in [6].

Verbatim,

$$X_n^{k,l} = D \frac{\prod_{j=1}^{n}(a-j)\,\prod_{j=1}^{k+l-n}(b-j)}{\prod_{j=1}^{k}(a-j)\,\prod_{j=1}^{l}(b-j)\,\prod_{j=1}^{k+l}(-i\,\delta u+\frac{a+b}{2}-j)} *$$
$$\sum_{m=0}^{k}\binom{k}{k-m}\binom{l}{n-m}\prod_{j=1}^{m}c_j^+\prod_{j=1-m}^{l-n}c_j^-\prod_{j=1}^{k-m}d_{k-j+2}\prod_{j=1}^{n-m}\tilde{d}_{k+l-m-j+2}\,,$$

$$c_j^{\pm} = -i\,\delta u \pm \frac{a-b}{2} - j + 1\,, \quad d_j = \frac{a+1-j}{2}\,, \quad \tilde{d}_j = \frac{b+1-j}{2}\,, \tag{13}$$

where $\delta u = u - v$ and

$$D = \frac{x^- - y^+}{x^+ - y^-}\sqrt{\frac{x^+}{x^-}}\sqrt{\frac{y^-}{y^+}} \tag{14}$$

simply applying the 3γ, 1γ transformation

$$x^- \to 1/x^-\,, \qquad y^+ \to 1/y^+\,. \tag{15}$$

At lowest order in g,

$$D = -\frac{u^- - v^+}{u^+ - v^-}\frac{\sqrt{u^+u^-}\sqrt{v^+v^-}}{u^-v^+}\,. \tag{16}$$

Including the *mirror measure* [5] for the propagation of either particle over an edge of width zero—so when no propagators run along the common edges of the hexagons—we obtain the expression

$$I(X) = \sum_{a,b=1}^{\infty}\sum_{k,l=0}^{a-1,b-1}\int\frac{du\,dv\,a\,b\,g^4}{4\pi^2(u^2+\frac{a^2}{4})^2(v^2+\frac{b^2}{4})^2}\,W_1\,W_2\,\Sigma^{ab}\,X_k^{k,l}\,.$$

Naively, this is not a one-loop contribution, because there is a factor g^4 from the measure for the two bound states. Yet, we expect the scattering of the scalar constituents to introduce braiding factors like

$$e^{i\,\frac{p_1}{2}}\,e^{-i\,\frac{p_2}{2}} \overset{3\gamma\,1\gamma}{\to} \frac{\sqrt{u^+u^-}\sqrt{v^+v^-}}{g^2} \tag{17}$$

where we have scaled back to the field theory convention of (1) to meet the weak-coupling expansion. Importantly, this factor does not only adjust the power to g^2, but it also removes the square-root branch cuts that would render inefficient the residue theorem as a means of evaluating the integrals over the rapidities u, v. In [14] an *averaging prescription* for such additional braiding factors is suggested. Building on the work here presented we want to study whether this prescription is the only possible one.

Despite of the appearance, the X matrix has singularities in δu only in the lower half-plane. Poles in X can therefore be avoided simply by closing the integration contour over the upper half-plane for u and the lower half-plane for v. Doing so, the poles $u^-,\ v^+$ from the measure can contribute. Likewise, in the numerator of the phase, $\Gamma[1+\frac{a}{2}+i\,u]$ and $\Gamma[1+\frac{b}{2}-i\,v]$ develop singularities. Note however, that we cannot localise both rapidities by poles from the phase:

$$u = i\left(m+\frac{a}{2}\right),\ v = -i\left(n+\frac{b}{2}\right) \quad \Rightarrow \quad \Gamma[1+\frac{a+b}{2}+i(u-v)] \\ = \Gamma[1-m-n] \tag{18}$$

for $m, n \in \mathbb{N}$ so that this denominator Γ-function creates a zero in these cases. Thus at least one pole, perhaps a higher one, must come from the measure. Then, e.g. with $u = i\frac{a}{2}$,

$$\Sigma^{ab} = \frac{\Gamma[1]\,\Gamma[1+\frac{b}{2}-i\,v]\,\Gamma[1+a+\frac{b}{2}+i\,v]}{\Gamma[1+a]\,\Gamma[1+\frac{b}{2}+i\,v]\,\Gamma[1+\frac{b}{2}-i\,v]} \tag{19}$$

and therefore the term in the phase that could create a pole at $v = -i(n+\frac{b}{2})$ actually drops. In conclusion, only the poles from the measure are relevant.

3 Integrating/Summing into Polylogarithms

Substituting $u = i\frac{a}{2},\ v = -i\frac{b}{2}$ the phase reduces to $\Sigma^{ab} = \Gamma[1+a+b]/(\Gamma[1+a]\Gamma[1+b])$ and we find the cross ratio dependence

$$z_1^{a-k}\,\bar{z}_1^{k}\,\bar{z}_2^{-b+l}\,z_2^{-l} \tag{20}$$

in accordance with the domain of convergence of the integrals over u, v. In order to unclutter the notation and to be able to straightforwardly Taylor-expand results in small quantities we will relabel the variables as

$$\left\{z_1,\ \bar{z}_1,\ \frac{1}{\bar{z}_2},\ \frac{1}{z_2}\right\} \quad \mapsto \quad \{z_1,\ b_1,\ y_2,\ a_2\}\,. \tag{21}$$

Due to the numerator of the A factor in X, the integrand of $I(x)$ has the pole structure $1/((u^-)^2v^+) - 1/(u^-(v^+)^2)$. In fact, the polylog level is set by the power of these poles. In the case at hand we obtain homogeneous transcendentality 2. The residue at either double pole can create a single $\log(b_1\,z_1)$ or $\log(a_2\,y_2)$, respectively, as is expected from one-loop Feynman graphs, while the remaining terms are regular when all four variables become small. Derivatives from the double poles can fall onto u^+, v^- creating an extra a or b in the denominator or onto δu in the m-sum in X, c.f. (13).

Due to explicit definition of X we can analytically evaluate all contributions to $I(X)$ by the methods developed below on a simpler example.

In (8) we list the four types of bound states forming the complete multiplet. The S-matrix will then have 16 diagonal elements. There is a second instance of X for the scattering of two bound states of the first type given in (8), but with $i = 2$. This matrix is algebraically equal. Second, scattering of the first type of bound state over the other two is called Y in [4]—there are four diagonal elements for both cases, $i = 1, 2$ and again, the two Y-matrices are equal. Last we have six diagonal elements Z_{jj} for the scattering of the last two types of bound states. Note that to some extent an averaging over braiding factors is automatic, if the dressing by momentum factors is tied to the i-index in Y; the set of Z-elements is symmetric in this respect.

The Y, Z cases are linear combinations of several instances of X with shifted k, l, n-indices, with coefficients depending on $x^\pm$, $y^\pm$ and the counters. All unphysical poles from the coefficient matrices cancel, but this property is not manifest in the formulae spelled out in [4]. Ultimately, all the diagonal Y, Z cases—at least to leading order in g in the given kinematics—share the property of X to have poles only in δu in the lower half-plane. Our reasoning about the locus of poles therefore directly carries over.

In the 3γ, 1γ kinematics, the Y elements are of order $1/g$. Nicely, to eliminate a single square root branch cut we also need one additional braiding factor $e^{\pm i p/2}$. Similarly, the Z elements start to come in at $1/g^2$ and $Z_{11} \ldots Z_{44}$ require no braiding, while Z_{55}, Z_{66} could be dressed by both, $e^{\pm i(p_1+p_2)/2}$. Here the averaging of [14] means to put in both possibilities with coefficient 1/2.

Remarkably, at the point $u = i\frac{a}{2}$, $v = -i\frac{b}{2}$, the matrix elements Y_{11}, Y_{22} and Z_{11}, Z_{22}, Z_{33}, Z_{55}, Z_{66} factor into simple products of Γ-functions. Below we describe how this enables us to calculate the part of the one-loop contribution where a derivative from the residue at $1/(u^-)^2$ falls upon $(z_1\bar{z}_1)^{-iu}$, or upon $1/u^+$, creating an extra factor $1/a$ (equivalently for the second particle with z_2, v, b). Since there is no fully explicit writing for Y, Z we cannot yet compute the contribution with a derivative on the scattering matrix itself in the way described in the following, because this destroys the factorisation properties. Yet, integrating all available pieces we could pin down the space of functions and use the symbol to fit the remaining parts, also for the non-factoring cases. This could be done separately for the individual contributions.

Let us illustrate the idea on the example of Y_{11}, in particular the contribution in which a derivative acts on $(z_1 b_1)^{-iu}$:

$$\frac{I_{\log}(Y_{11})}{\log(z_1 b_1)} = \sum_{a,b,k,l} z_1^{a-k}\, b_1^k\, y_2^{l-b}\, a_2^l\, \frac{\Gamma[a-k+b-l] \quad \Gamma[1+k+l]}{4\,a\,\Gamma[a-k]\,\Gamma[1+b-l]\,\Gamma[1+k]\,\Gamma[1+l]} \tag{22}$$

where $a, b = 1 \ldots \infty$, $k, l = 0 \ldots a-1, b-1$. Define

$$r^2 = z_1 b_1\,, \quad p^2 = \frac{z_1}{b_1} \qquad \Rightarrow \qquad r\,\frac{\partial}{\partial r}\, z_1^{a-k} b_1^k = a\; z_1^{a-k} b_1^k\,. \tag{23}$$

The inverse operation is $\int dr/r$. Comparing to the original series the constant part of the indefinite integral must be subtracted. We can thus eliminate a from the denominator of (22) with no loss of information. Next we swap the sums over a, k and b, l respectively and shift the variables by $a \mapsto a+k,\ b \mapsto b+l$ in order to decouple the summations. The sums are of geometric type and yield

$$\frac{I_{\log}(Y_{11})}{\log(z_1 b_1)} = \int \frac{dr}{r} \frac{z_1\,(a_2 + y_2 - a_2 y_2 - b_1 y_2 - a_2 z_1)}{4\,(1-b_1)(1-a_2-b_1)(1-z_1)(1-y_2-z_1)}\,. \tag{24}$$

Hence the root of the procedure is a rational function and we add polylogarithm levels by the integration in the modulus r:

$$\begin{aligned}\frac{I_{\log}(Y_{11})}{\log(z_1 b_1)} &= \frac{z_1\,(\log[1-b_1] - \log[1-z_1])}{4\,(b_1 - z_1)} \\ &+ \frac{z_1\,(\log[1-a_2] - \log[1-a_2-b_1] - \log[1-y_2] + \log[1-y_2-z_1])}{4\,(b_1 - z_1 - b_1 y_2 + a_2 z_1)}\end{aligned} \tag{25}$$

upon subtraction of the constant part in r. To obtain the contribution from the derivative falling onto $1/u^+$ we can use the operation $\int dr/r$ a second time. As in [14] we assumed that factors like $|u|, |v|$ do not arise from the $x(u)$ functions or the expansion of Σ^{ab}, and that $(-1)^{ab}$ is unphysical and has to be undone by the contraction prescription on the central hexagon.

For the other factoring matrix elements we proceed similarly. For X one can again decouple the (five) sums by swapping the order of summation and shifting the counters. Since everything is expressed in terms of Γ-functions we can even address the contributions in which a single derivative falls on δu. The sums are of the type ${}_2F_1$ or ${}_3F_2$ and can be rewritten in terms of parametric integrations over Euler kernels by the standard formulae. Putting aside the integrations as long as possible we can find a path through the computation that always closes on the same type of summand/integrand. The final parametric integrations yield polylogarithms much as the integration in the modulus in the simple case above. Note that the Gauss hypergeometric function also appeared in the context of re-summing the POPE at one loop, see [17] and references therein.

In our problem we find eight different denominators:

$$\begin{aligned}&a-y,\ b-z,\ b\,y-a\,z,\ a-y+b\,y-a\,z,\ b-b\,y-z+a\,z,\\ &a\,b-a\,b\,y-a\,b\,z-y\,z+a\,y\,z+b\,y\,z,\\ &a\,b-a\,b\,y-y\,z+a\,y\,z,\ a\,b-a\,b\,z-y\,z+b\,y\,z\end{aligned} \tag{26}$$

Confusion with the bound state counters cannot arise anymore so that we dropped the 1, 2 subscript on the variables. The complete amplitudes are written as sums of weight two logarithmic functions over these denominators, with single terms of each denominator as a coefficient. Formula (25) illustrates what we mean here.

The symbol letters are

$$1-a,\ a,\ 1-b,\ 1-a-b,\ b,\ 1-y,\ a-y,\ y,\ 1-z,\ b-z,\ 1-y-z,\ z \tag{27}$$

and

$$a-y+b\,y-a\,z,\ b-b\,y-z+a\,z,\ a\,b-a\,b\,y-a\,b\,z-y\,z+a\,y\,z+b\,y\,z\,. \tag{28}$$

All the denominators are point permutations of the denominator $z-b$ of the Bloch-Wigner dilogarithm. Three of them also occur in the symbols. One can generate all the symbols of this type from

$$\mathrm{Li}_2\left(1-\frac{z}{b}\right)-\frac{1}{2}\log^2(b) \tag{29}$$

by permutations. At one loop, $\mathcal{N}=4$ field theory results contain only Bloch-Wigner dilogarithms. Since there will be several double gluing processes in complete correlators our results can correctly reproduce one-loop field theory if (29) cancels. A difficulty is that each of the three incarnations of the function occurs with various denominators making it hard to unambiguously associate terms to Bloch-Wigner dilogs or the part that has to drop.

Finally, as done in [14], one might choose to bring one of the particles around the central hexagon, so instead of scattering $X(u^\gamma)Y(v^{-\gamma})$ one studies $-\bar{Y}(v^{5\gamma})X(u^\gamma)$, where in this context X, Y are some bound states. Due to the odd number of crossing transformations the sign of the rapidities in the phase and the S-matrix in the antisymmetric representation is not aligned in this version of the computation. What is more, the scalar factor h contains the pole $1/(u^+-v^+)$ now, which we would have dubbed unphysical above. Picking the residue $u=i\frac{a}{2}$ from the measure this becomes a pole at $v=-i(\frac{b}{2}-a)$, which is (on the border of) the lower half-plane if $b\geq 2a$. Preliminary studies suggest that both effects introduce polylogarithms with root arguments,[2] and that uniform transcendentality may not be manifest. Yet, the end result must agree. In scattering processes with more virtual particles it will be hard to avoid this situation so that we should try to develop methods appropriate also in this kinematics. At the new residue all the factorisation properties are spoiled so that we could not proceed as before without substantial progress on simplifying Y, Z.

4 Conclusions

In the evaluation of n-point functions in $\mathcal{N}=4$ super Yang-Mills theory by hexagon tesselations, the first complicated process is the gluing of three adjacent tiles by two single mirror magnons. On the central tile this necessitates the evaluation of diagonal scattering of two so-called $sl(2)$ bound states.

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[2] We thank C. Duhr for some test calculations.

We find a beautiful and efficient integration scheme for this two-magnon problem, although we cannot yet ascertain that the outcome is the physical result. To answer this question must be one aim of future work.

Remarkably, the problem yields a multilinear alphabet of letters in the symbol of the relevant generalised polylogarithms, suggesting that the two-magnon problem can be integrated in closed form also beyond the leading order in the coupling constant.[3]

Last, another direction of future research will be to simplify the bound state scattering matrix in the various kinematical regimes in order to be able to address higher processes, too.

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[3] We thank O. Schnetz for a discussion on this point.

Meta-conformal Invariance and Their Covariant Correlation Functions

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Abstract Meta-conformal invariance is a novel class of dynamical symmetries, with dynamical exponent $z = 1$, and distinct from the standard ortho-conformal invariance. The meta-conformal Ward identities can be directly read off from the Lie algebra generators, but this procedure implicitly assumes that the co-variant correlators should depend holomorphically on time- and space coordinates. Furthermore, making this assumptions leads to un-physical singularities in the co-variant correlators. We show how to carefully reformulate the meta-conformal Ward identities in order to obtain regular, but non holomorphic expressions for the co-variant two-point functions, both in $d = 1$ and $d = 2$ spatial dimensions.

1 Introduction

Many brilliant applications of conformal invariance are known, ranging from string theory and high-energy physics [36], or to two-dimensional phase transitions [9, 16, 19] or the quantum Hall effect [11, 17]. These applications are based on a geometric definition of conformal transformations, considered as local coordinate transformations $\boldsymbol{r} \mapsto \boldsymbol{r}' = \boldsymbol{f}(\boldsymbol{r})$, of spatial coordinates $\boldsymbol{r} \in \mathbb{R}^2$ such that angles are kept unchanged. The associated Lie algebra is called the '*conformal Lie algebra*'.

In Table 1, examples of infinite-dimensional Lie groups of time-space transformations are shown. They represent attempts to answer the question "Is it possible to adapt conformal invariance to dynamical problems ?" A minimal requirement

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_5

is to distinguish time and space variables through their global rescaling, according to $t \mapsto t' = bt$ and $\boldsymbol{r} \mapsto \boldsymbol{r}' = b^{1/z}\boldsymbol{r}$ which defines the *dynamical exponent z*. In what follows, we shall consider infinitesimal transformations where $b = 1 + \varepsilon$, with $|\varepsilon| \ll 1$. Then a rescaling transformation is described by an infinitesimal generator, which for global dilatations on time- and space-coordinates takes the form $X_0 = -t\partial_t - \frac{1}{z}\boldsymbol{r} \cdot \partial_{\boldsymbol{r}} - \delta$. The parameter δ is the scaling dimension of the scaling operator $\varphi = \varphi(t, \boldsymbol{r})$ on which the generator X_0 is thought to act. Practical use of this is made for the computation of n-point correlation functions $C^{[n]} = C^{[n]}(t_1, \ldots, t_n; \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n) := \langle \varphi_1(t_1, \boldsymbol{r}_1) \cdots \varphi_n(t_n, \boldsymbol{r}_n) \rangle$. The dilatation-invariance of such a correlator is expressed via a *Ward identity*, which for the global dilatations described by X_0 takes the form

$$\sum_{j=1}^{n} \left(-t_j \frac{\partial}{\partial t_j} - \frac{1}{z} \boldsymbol{r}_j \cdot \frac{\partial}{\partial \boldsymbol{r}_j} - \delta_j \right) C^{[n]}(t_1, \ldots, t_n; \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n) = 0 \qquad (1)$$

and it becomes explicit how the dynamical exponent z distinguishes between temporal and spatial coordinates. Different symmetries will lead to different Ward identities which describe together constraints on the form of the n-point correlator $C^{[n]}$. Explicit examples will be given in later sections. These differential equation constraints are only consistent if the generators, such as X_0, belong to a well-defined algebraic structure, e.g. a Lie algebra.

It follows from time-space rotation-invariance that conformal invariance must have $z = 1$. In general, z has a non-trivial value [44]. In $1 + 1$ time-space dimensions, there exists an infinite hierarchy of models with dynamical exponent $1 < z \leq 2$ [37]. Lower bounds on z are derived from hydrodynamic projections of many-body dynamics [13]. Attempts of identifying dynamical conformal invariance goes back at least to critical dynamics of a two-dimensional statistical system [12]. In Table 1, we distinguish the well-studied *'ortho-conformal'* transformations [9], which in the two-dimensional space made from time-space points $(t, r) \in \mathbb{R}^2$ are angle-preserving, from recently constructed groups of *'meta-conformal'* transformations [20, 25, 28, 42], which in general are not angle-preserving but which share certain algebraic properties with ortho-conformal transformations in Table 1.

The most simple prediction of ortho-conformal invariance concerns the form of the co-variant two-point function $C = C(z_1, \bar{z}_1, z_2, \bar{z}_2) = \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2) \rangle$ built from so-called 'quasi-primary' scaling operators ϕ_j, with 'conformal weights' Δ_j and $\overline{\Delta}_j$ [9]. In complex light-cone coordinates $z = t + \mathrm{i}\mu r$, $\bar{z} = t - \mathrm{i}\mu r$, one has

$$C_{\text{ortho}}(z_1, \bar{z}_1, z_2, \bar{z}_2) = \delta_{\Delta_1, \Delta_2} \delta_{\overline{\Delta}_1, \overline{\Delta}_2} \left(z_1 - z_2\right)^{-2\Delta_1} \left(\bar{z}_1 - \bar{z}_2\right)^{-2\overline{\Delta}_1} \qquad (2)$$

up to normalisation. Herein, $1/\mu$ has the dimensions of a velocity. In deriving this kind of result, auxiliary assumptions are made. Analogously with Eq. (1), the requirement of ortho-conformal co-variance leads to a set of linear partial first-order differential equations for C, the so-called *global ortho-conformal Ward identities*. Their joint

Table 1 Several infinite-dimensional groups of time-space transformations, defined by the corresponding coordinate changes. Unspecified (vector) functions are assumed (complex) differentiable and $\mathscr{R}(t) \in SO(d)$ is a smoothly time-dependent rotation matrix. The physical time- and space-coordinates, the associated dynamical exponent z of this standard representation and the physical nature of the co-variant n-point functions is also indicated.

Group	Coordinate changes			Phys. coordinates	z	Co-variance
Ortho-conformal $(1+1)D$	$z' = f(z)$	$\bar{z}' = \bar{z}$		$z = t + \mathrm{i}\mu r$	1	Correlator
	$z' = z$	$\bar{z}' = \bar{f}(\bar{z})$		$\bar{z} = t - \mathrm{i}\mu r$		
Meta-conformal $1D$	$u = f(u)$	$\bar{u}' = \bar{u}$		$u = t$	1	Correlator
	$u' = u$	$\bar{u}' = \bar{f}(\bar{u})$		$\bar{u} = t + \mu r$		
Meta-conformal $2D$	$\tau' = b(\tau)$	$w' = w$	$\bar{w}' = \bar{w}$	$\tau = t$		
	$\tau' = \tau$	$w' = f(w)$	$\bar{w}' = \bar{w}$	$w = t + \mu(r_{\parallel} + \mathrm{i}r_{\perp})$	1	Correlator
	$\tau' = \tau$	$w' = w$	$\bar{w}' = \bar{f}(\bar{w})$	$\bar{w} = t + \mu(r_{\parallel} - \mathrm{i}r_{\perp})$		
Conformal galilean	$t' = b(t)$	$\boldsymbol{r}' = (\mathrm{d}b(t)/\mathrm{d}t)\,\boldsymbol{r}$				
	$t' = t$	$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a}(t)$			1	Correlator
	$t' = t$	$\boldsymbol{r}' = \mathscr{R}(t)\boldsymbol{r}$				
Schrödinger-Virasoro	$t' = b(t)$	$\boldsymbol{r}' = (\mathrm{d}b(t)/\mathrm{d}t)^{1/2}\,\boldsymbol{r}$				
	$t' = t$	$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{a}(t)$			2	Response
	$t' = t$	$\boldsymbol{r}' = \mathscr{R}(t)\boldsymbol{r}$				

solutions Eq. (2) are necessarily *holomorphic* (or anti-holomorphic) functions in the variables $z_j, \bar{z}_j$ [29].

In this work, we shall examine the analogous question for meta-conformal invariance. Known physical examples of confirmed meta-conformal invariance are of two types. First, there exist spatially *non-local* representations, which arise as a dynamical symmetry of certain non-local equations of motion which occur for example in diffusion-limited erosion [34], the kink-terrace-step model for vicinal surfaces [39] or the associated quantum chain [31] which is a conformal field-theory with central charge $c = 1$ [38]. Some predictions of meta-conformal invariance for response functions have been confirmed in these models [26, 27]. Second, a different type of meta-conformal invariance, with spatially *local* representations, has been identified recently in the kinetics of biased spin systems, see Fig. 1, such as the kinetic $1D$ Glauber-Ising model with a bias, sufficiently long-ranged initial conditions and quenched to zero temperature [28, 43]. The influence of transverse dimensions on the representations of meta-conformal transformations is currently under investigation. However, the focus of this work rather is on the formal study of meta-conformal representations as time-space transformations and the boundedness of the resulting two-point correlators.

In order to do so, we begin by analysing the consequences of writing analogous global Ward identities for meta-conformal invariance [20, 28, 42]. As we shall see in Sect. 2, the straightforward implementation of the global meta-conformal Ward identities leads to un-physical singularities in the time-space behaviour of such correlators. These singularities arise since the meta-conformally co-variant correlators are no longer holomorphic functions of their arguments. Therefore, a more careful approach is required, which we shall explicitly describe in Sects. 3 and 4, respectively, for $d = 1$ and $d = 2$ spatial dimensions. Our main result is the explicit form of a meta-conformally co-variant two-point function which remains bounded everywhere, as stated in Eqs. (33, 34) in Sect. 5. An appendix contains mathematical background on Hardy spaces in restricted geometries, for both $d = 1$ and $d = 2$.

2 Global Meta-conformal Ward Identities

Meta-conformal invariance arises as a dynamical symmetry of the simple equation $\mathscr{S}\varphi(t,\boldsymbol{r}) = \left(-\mu\partial_t + \partial_{r_\parallel}\right)\varphi(t,\boldsymbol{r}) = 0$, which distinguishes a single preferred direction [41], with coordinate $r_\parallel$, from the transverse direction(s), with coordinate $\boldsymbol{r}_\perp$. This is sketched in Fig. 1. Throughout, we shall admit rotation-invariance in the transverse directions, if applicable. Therefore, in more than three spatial dimensions, the consideration of the two-point function can be reduced to the case of a single transverse direction, $r_\perp$. Therefore, it is enough to discuss explicitly either (i) the case of one spatial dimension, referred from now one as the $1D$ *case* (then there is no transverse direction), or else (ii) the case of two spatial dimensions, called the $2D$ *case* (with a single transverse direction).

The Lie algebra generators of meta-conformal invariance read off from Table 1 as follows. In the $1D$ case, in terms of time- and space-coordinates [20] (with $n \in \mathbb{Z}$)

$$\begin{aligned} \ell_n &= -t^{n+1}\left(\partial_t - \frac{1}{\mu}\partial_r\right) - (n+1)\left(\delta - \frac{\gamma}{\mu}\right)t^n \\ \bar{\ell}_n &= -\frac{1}{\mu}\big(t+\mu r\big)\partial_r - (n+1)\frac{\gamma}{\mu}\big(t+\mu r\big)^n \end{aligned} \tag{3}$$

and in the $2D$ case [28]

$$\begin{aligned} A_n &= -t^{n+1}\left(\partial_t - \frac{1}{\mu}\partial_\parallel\right) - (n+1)\left(\delta - \frac{2\gamma_\parallel}{\mu}\right)t^n \\ B_n^\pm &= -\frac{1}{2\mu}\big(t+\mu(r_\parallel \pm \mathrm{i}r_\perp)\big)^{n+1}\big(\partial_\parallel \mp \mathrm{i}\partial_\perp\big) - (n+1)\frac{\gamma_\parallel \mp \mathrm{i}\gamma_\perp}{\mu}\big(t+\mu(r_\parallel \pm \mathrm{i}r_\perp)\big)^n \end{aligned} \tag{4}$$

with the short-hands $\partial_\parallel = \frac{\partial}{\partial r_\parallel}$ and $\partial_\perp = \frac{\partial}{\partial r_\perp}$. The constants δ and γ (respectively $\gamma_{\parallel,\perp}$) are the scaling dimension and the rapidity of the scaling operators on which these generators act and μ^{-1} is a constant with the dimension of a velocity. Each

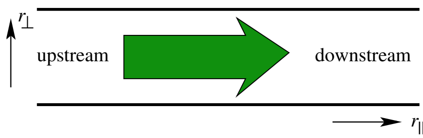

Fig. 1 Schematic illustration of ballistic transport in a channel, with the spatial coordinates $r_\parallel$, $r_\perp$

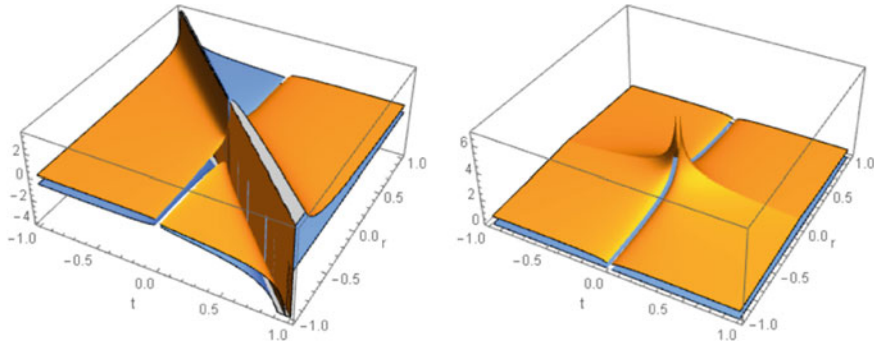

Fig. 2 Real part (orange) and imaginary part (blue) of the $1D$ meta-conformally co-variant two-point function $C(t, r)$, with $\delta_1 = 0.22$, $\gamma_1 = 0.33$ and $\mu = 1$. **Left panel:** Spurious singularities arise in (5). **Right panel:** Regularised form after correction of the spurious singular behaviour.

of the infinite families of generators in (3, 4) produces a Virasoro algebra (with zero central charge). Therefore, the $1D$ meta-conformal algebra is isomorphic to a direct sum of two Virasoro algebras. In the $2D$ case, there is an isomorphism with the direct sum of three Virasoro algebras. Their maximal finite-dimensional Lie sub-algebras (isomorphic to a direct sum of two or three $\mathfrak{sl}(2, \mathbb{R})$ algebras) fix the form of two-point correlators $C(t, \boldsymbol{r}) = \langle \varphi_1(t, \boldsymbol{r})\varphi_2(0, \mathbf{0})\rangle$ built from quasi-primary scaling operators. Since the generators (3, 4) already contain the terms which describe how the scaling operators $\varphi = \varphi(t, \boldsymbol{r})$ transform under their action, the global meta-conformal Ward identities can simply be written down. The requirement of meta-conformal co-variance leads to

$$C_{\text{meta}}(t, \boldsymbol{r}) = \begin{cases} t^{-2\delta_1}\left(1 + \mu\frac{r}{t}\right)^{-2\gamma_1/\mu} & ; \ \text{if } d = 1 \\ t^{-2\delta_1}\left(1 + \mu\frac{r_\parallel + \mathrm{i}r_\perp}{t}\right)^{-2\gamma_1/\mu}\left(1 + \mu\frac{r_\parallel - \mathrm{i}r_\perp}{t}\right)^{-2\bar{\gamma}_1/\mu} & ; \ \text{if } d = 2 \end{cases} \tag{5}$$

and where $\boldsymbol{r} = r \in \mathbb{R}$ for $d = 1$ and $\boldsymbol{r} = (r_\parallel, r_\perp) \in \mathbb{R}^2$ for $d = 2$ where we also write $\gamma := \gamma_\parallel - \mathrm{i}\gamma_\perp$ and $\bar{\gamma} := \gamma_\parallel + \mathrm{i}\gamma_\perp$. In addition, the constraints $\delta_1 = \delta_2$ and $\gamma_1 = \gamma_2$ in $1D$ or $\gamma_{\parallel,1} = \gamma_{\parallel,2}$ and $\gamma_{\perp,1} = \gamma_{\perp,2}$ in $2D$ are implied.

Formally, the procedure to derive (5) is completely analogous to the used above for the derivation of (2) from ortho-conformal co-variance. The explicit forms (5) make it apparent that $C_{\text{meta}}(t, \boldsymbol{r})$ is not necessarily bounded for all t or $\boldsymbol{r}$. In Fig. 2, we illustrate this for the $1D$ case—a spurious singularity appears whenever $\mu r = -t$.

In the limit $\mu \to 0$, the meta-conformal algebras contract into the galilean conformal algebras [18]. Carrying out the limit on the correlator (4), one obtains, as has been stated countless times in the literature, see e.g. [3–6, 35]

$$C_{cga}(t, \boldsymbol{r}) = \begin{cases} t^{-2\delta_1}\exp\left(-2\frac{\gamma_1 r}{t}\right) & ; \ \text{if } d = 1 \\ t^{-2\delta_1}\exp\left(-4\frac{\boldsymbol{\gamma}_1 \cdot \boldsymbol{r}}{t}\right) & ; \ \text{if } d = 2 \end{cases} \tag{6}$$

with the definition $\boldsymbol{\gamma} = (\gamma_{\|}, \gamma_{\perp})$. While this correlator decays in one spatial direction (where $\gamma_1 r > 0$ or $\boldsymbol{\gamma}_1 \cdot \boldsymbol{r} > 0$ and assuming $t > 0$), it diverges in the opposite direction. In view of the large interest devoted to conformal galilean field-theory, see [1, 6–8, 10, 14, 15, 30, 33, 35] and refs. therein, it appears important to be able to formulate well-defined correlators which remain bounded everywhere in time-space. We mention in passing that the $1D$ form of (6) can also be obtained from $2D$ ortho-conformal invariance: it is enough to consider complex conformal weights $\Delta = \frac{1}{2}(\delta - \mathrm{i}\gamma/\mu)$ and $\overline{\Delta} = \frac{1}{2}(\delta + \mathrm{i}\gamma/\mu)$. Then (2) can be rewritten as

$$C_{\rm ortho}(t,r) = t^{-2\delta}\left[1+\left(\frac{\mu r}{t}\right)^2\right]^{-\delta} \exp\left[-\frac{2\gamma}{\mu}\arctan\frac{\mu r}{t}\right] \stackrel{\mu\to 0}{\longrightarrow} t^{-2\delta}e^{-2\gamma r/t} \tag{7}$$

In what follows, we shall describe how to find correlators bounded everywhere. Since the implicit assumption of holomorphicity in the coordinates gave the unbounded results (5, 6), we shall explore how to derive non-holomorphic correlators. Our treatment follows [25], to be generalised to the case $d = 2$ where necessary.

3 Regularised Meta-conformal Correlator: The 1D Case

Non-holomorphic correlators can only be found by going beyond the local differential operators derived from the meta-conformal Ward identities. We shall do so in a few simple steps [25], restricting for the moment to the $1D$ case. First, we consider the 'rapidity' γ as a new variable. Second, it is dualised [22–24] through a Fourier transformation, which gives the quasi-primary scaling operator

$$\widehat{\varphi}(\zeta,t,r) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}\gamma\; e^{\mathrm{i}\gamma\zeta}\,\varphi_\gamma(t,r) \tag{8}$$

This leads to the following representation of the dualised meta-conformal algebra

$$\begin{aligned} X_n &= \mathrm{i}(n+1)\left[(t+\mu r)^n - t^n\right]\partial_\zeta - t^{n+1}\partial_t - \left[(t+\mu r)^{n+1} - t^{n+1}\right]\partial_r - (n+1)\delta t^n \\ Y_n &= \frac{\mathrm{i}(n+1)}{\mu}(t+\mu r)^n\,\partial_\zeta - \frac{1}{\mu}(t+\mu r)^{n+1}\,\partial_r \end{aligned} \tag{9}$$

such that meta-conformal Lie algebra is given by

$$[X_n, X_m] = (n-m)X_{n+m},\quad [X_n, Y_m] = (n-m)Y_{n+m},\quad [Y_n, Y_m] = (n-m)Y_{n+m} \tag{10}$$

This form will be more convenient for us than the one used in [25], since the parameter μ does no longer appear in the Lie algebra commutators (10). Third, it was suggested [22, 25] to look for a further generator N in the Cartan sub-algebra $\mathfrak{h}$, viz. $\mathrm{ad}_N \mathscr{X} = \alpha_{\mathscr{X}}\,\mathscr{X}$ for any meta-conformal generator $\mathscr{X}$. It can be shown that

$$N = -\zeta\partial_\zeta - r\partial_r + \mu\partial_\mu + \mathrm{i}\kappa(\mu)\partial_\zeta - \nu(\mu) \tag{11}$$

is the only possibility [25], where the functions $\kappa(\mu)$ and $\nu(\mu)$ remain undetermined. Since in this generator, the parameter μ is treated as a further variable, we see the usefulness of the chosen normalisation of the generators in (9). On the other hand, the generator of spatial translations now reads $Y_{-1} = -\mu^{-1}\partial_r$, with immediate consequences for the form of the two-point correlator. In dual space, the two-point correlator is defined as

$$\widehat{F} = \langle\widehat{\varphi}_1(\zeta_1, t_1, r_1, \mu_1)\widehat{\varphi}_2(\zeta_2, t_2, r_2, \mu_2)\rangle = \widehat{F}(\zeta_1, \zeta_2, t_1, t_2, r_1, r_2, \mu_1, \mu_2) \tag{12}$$

Lifting the generators from the representation (9) to two-body operators, the global meta-conformal Ward identities (derived from the maximal finite dimensional sub-algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$) become a set of linear partial differential equations of first order for the function $\widehat{F}$. While the solution will certainly be holomorphic in its variables, the back-transformation according to (8) can introduce non- holomorphic behaviour but will also lead to a correlator bounded everywhere.

The function $\widehat{F}$ is obtained as follows. First, co-variance under X_{-1} and Y_{-1} gives

$$\widehat{F} = \widehat{F}(\zeta_1, \zeta_2, t, \xi, \mu_1, \mu_2); \quad t = t_1 - t_2, \quad \xi = \mu_1 r_1 - \mu_2 r_2 \tag{13}$$

The action of the generators Y_0 and Y_1 on $\widehat{F}$ is best described by introducing the new variables $\eta := \mu_1\zeta_1 + \mu_2\zeta_2$ and $\zeta := \mu_1\zeta_1 - \mu_2\zeta_2$. Then the corresponding Ward identities become

$$\left(2\mathrm{i}\partial_\eta - (t+\xi)\partial_\xi\right)\widehat{F} = 0, \quad \partial_\zeta\widehat{F} = 0 \tag{14}$$

Finally, the Ward identities coming from the generators X_0 and X_1 become

$$\left(-t\partial_t - \xi\partial_\xi - \delta_1 - \delta_2\right)\widehat{F} = 0, \quad t\,(\delta_1 - \delta_2)\,\widehat{F} = 0 \tag{15}$$

The second of these gives the constraint $\delta_1 = \delta_2$. The two remaining equations have the general solution

$$\widehat{F} = (t_1 - t_2)^{-2\delta_1}\widehat{\mathscr{F}}\left(\frac{1}{2}(\mu_1\zeta_1 + \mu_2\zeta_2) + \mathrm{i}\ln\left(1 + \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right); \mu_1, \mu_2\right) \tag{16}$$

with an undetermined function $\widehat{\mathscr{F}}$. Spatial translation-invariance only holds in a more weak form, which could become useful for the description of physical situations where the propagation speed of each scaling operator can be different.

In [25], we tried to use co-variance under the further generator N in order to fix the function $\widehat{\mathscr{F}}$. However, therein a choice of basis in the meta-conformal Lie algebra was used where the parameter μ appears in the structure constants, but it became possible to fix $\widehat{\mathscr{F}}$ and furthermore to show that $\widehat{F}$ with respect to the variable η is in the Hardy space H_2^+, see the appendix for the mathematical details. If we want to consider μ as a further variable, as it is necessary because of the explicit form of N, objects such as "μY_{n+m}" are not part of the meta-conformal Lie algebra. Therefore, it is necessary, to use the normalisation (9) which leads to the Lie algebra (10) which is independent of μ. In order to illustrate the generic consequences, let $\nu = \nu(\mu)$ and $\sigma = -\mu\kappa(\mu)$ be constants. The co-variance condition $N\widehat{F} = 0$ gives

$$\widehat{\mathscr{F}}(w : \mu_1, \mu_2) = (\mu_1\mu_2)^\nu\, \widehat{\mathcal{F}}\left(w + \mathrm{i}\sigma\frac{\mu_1+\mu_2}{2}, \frac{\mu_1}{\mu_2}\right) \tag{17}$$

where the function $\widehat{\mathcal{F}}$ remains undetermined. In contrast to our earlier treatment, we can no longer show that $\widehat{F}$ had to be in the Hardy space H_2^+. On the other hand, this mathematical property had turned out to be very useful for the derivation of bounded correlators. This motivates the following.

First, we re-write the result (16) as follows (with the constraint $\delta_1 = \delta_2$)

$$\widehat{F} = (t_1 - t_2)^{-2\delta_1}\widehat{\mathscr{F}}(\zeta_+ + \mathrm{i}\lambda) \;\;,\;\; \zeta_+ := \frac{\mu_1\zeta_1 + \mu_2\zeta_2}{2} \;\;,\;\; \lambda := \ln\left(1 + \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right) \tag{18}$$

and we also denote $\widehat{\mathscr{F}}_\lambda(\zeta_+) := \widehat{\mathscr{F}}(\zeta_+ + \mathrm{i}\lambda)$. Then, we require:

Postulate. *If* $\lambda > 0$, *then* $\widehat{\mathscr{F}}_\lambda \in H_2^+$ *and if* $\lambda < 0$, *then* $\widehat{\mathscr{F}}_\lambda \in H_2^-$.

The Hardy spaces $H_2^\pm$ on the upper and lower complex half-planes $\mathbb{H}_\pm$ are defined in the appendix. There, it is also shown that, under mild conditions, that if $\lambda > 0$ and if there exist finite positive constants $\widehat{\mathscr{F}}^{(0)}$, ε such that $|\widehat{\mathscr{F}}(\zeta_+ + \mathrm{i}\lambda)| < \widehat{\mathscr{F}}^{(0)}e^{-\varepsilon\lambda}$, then $\widehat{\mathscr{F}}_\lambda$ is indeed in the Hardy space H_2^+. Physically, this amounts to a requirement of an algebraic decay with respect to the scaling variable.

The utility of our postulate is easily verified, following [25]. From Theorem 1 of the appendix, especially (A.3), we can write

$$\widehat{\mathscr{F}}_\lambda(\zeta_+) = \Theta(\lambda)\int_0^\infty \mathrm{d}\gamma_+ e^{\mathrm{i}(\zeta_+ + \mathrm{i}\lambda)\gamma_+}\widehat{\mathscr{F}}_+(\gamma_+) + \Theta(-\lambda)\int_0^\infty \mathrm{d}\gamma_- e^{-\mathrm{i}(\zeta_+ + \mathrm{i}\lambda)\gamma_-}\widehat{\mathscr{F}}_-(\gamma_-) \tag{19}$$

where the Heaviside functions $\Theta(\pm\lambda)$ select the two cases. For $\lambda > 0$, we find

$$\begin{aligned}
F &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\mathrm{d}\zeta_1\mathrm{d}\zeta_2\, e^{-\mathrm{i}\gamma_1\zeta_1 - \mathrm{i}\gamma_2\zeta_2}\widehat{F}\\
&= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^2}\mathrm{d}\zeta_1\mathrm{d}\zeta_2\, t^{-2\delta_1}\int_0^\infty \mathrm{d}\gamma_+\, e^{-\mathrm{i}\gamma_1\zeta_1 - \mathrm{i}\gamma_2\zeta_2}e^{\mathrm{i}(\mu_1\zeta_1 + \mu_2\zeta_2 + 2\mathrm{i}\lambda)\gamma_+/2}\widehat{\mathscr{F}}_+(\gamma_+)\\
&= \frac{\sqrt{32\pi}}{\mu_1\mu_2}t^{-2\delta_1}\int_0^\infty \mathrm{d}\gamma_+\, e^{-\lambda\gamma_+}\delta\left(\gamma_+ - \frac{2\gamma_1}{\mu_1}\right)\delta\left(\gamma_+ - \frac{2\gamma_2}{\mu_2}\right)\widehat{\mathscr{F}}_+(\gamma_+)\\
&= \frac{\sqrt{32\pi}}{\mu_1\mu_2}t^{-2\delta_1}\delta_{\gamma_1/\mu_1, \gamma_2/\mu_2}\int_0^\infty \mathrm{d}\gamma_+\, e^{-\lambda\gamma_+}\delta\left(\gamma_+ - \frac{2\gamma_1}{\mu_1}\right)\widehat{\mathscr{F}}_+(\gamma_+)\\
&= \text{cste.}\, \delta_{\gamma_1/\mu_1, \gamma_2/\mu_2}(t_1 - t_2)^{-2\delta_1}\left(1 + \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right)^{-2\gamma_1/\mu_1}\Theta\left(\frac{\gamma_1}{\mu_1}\right)
\end{aligned} \tag{20}$$

where the definitions (18) were used. Similarly, for $\lambda < 0$ we obtain

$$
\begin{aligned}
F &= \frac{1}{2\pi}\int_{\mathbb{R}^2} d\zeta_1 d\zeta_2\, e^{-i\gamma_1\zeta_1 - i\gamma_2\zeta_2}\,\widehat{F} \\
&= \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^2} d\zeta_1 d\zeta_2\, t^{-2\delta_1}\int_0^\infty d\gamma_-\, e^{-i\gamma_1\zeta_1 - i\gamma_2\zeta_2} e^{-i(\mu_1\zeta_1+\mu_2\zeta_2+2i\lambda)\gamma_-/2}\,\widehat{\mathscr{F}}_-(\gamma_-) \\
&= \frac{\sqrt{32\pi}}{\mu_1\mu_2} t^{-2\delta_1}\int_0^\infty d\gamma_-\, e^{\lambda\gamma_-}\delta\left(\gamma_- + \frac{2\gamma_1}{\mu_1}\right)\delta\left(\gamma_- + \frac{2\gamma_2}{\mu_2}\right)\widehat{\mathscr{F}}_-(\gamma_-) \\
&= \frac{\sqrt{32\pi}}{\mu_1\mu_2} t^{-2\delta_1}\delta_{\gamma_1/\mu_1,\gamma_2/\mu_2}\int_0^\infty d\gamma_-\, e^{\lambda\gamma_-}\delta\left(\gamma_- - \left|\frac{2\gamma_1}{\mu_1}\right|\right)\widehat{\mathscr{F}}_-(\gamma_-) \\
&= \text{cste.}\, \delta_{\gamma_1/\mu_1,\gamma_2/\mu_2}(t_1-t_2)^{-2\delta_1}\left(1 - \frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right)^{-2|\gamma_1/\mu_1|}\Theta\left(-\frac{\gamma_1}{\mu_1}\right)
\end{aligned} \tag{21}
$$

Combining these two forms gives our final $1D$ two-point correlator

$$
F = \delta_{\delta_1,\delta_2}\delta_{\gamma_1/\mu_1,\gamma_2/\mu_2}\left(1 + \left|\frac{\mu_1 r_1 - \mu_2 r_2}{t_1 - t_2}\right|\right)^{-2|\gamma_1/\mu_1|} \tag{22}
$$

up to normalisation. As shown in Fig. 2, this is real-valued and bounded in the entire time-space, although not a holomorphic function of the time-space coordinates.

Finally, it appears that our original motivation for allowing the μ_j to become free variables, is not very strong. We might have fixed the μ_j from the outset, had not included a factor $1/\mu$ into the generators Y_n (such that the spatial translations are generated by $Y_{-1} = -\partial_r$ and continue immediately with our Postulate. Since a consideration of the meta-conformal three-point function shows that $\mu_1 = \mu_2 = \mu_3$ [21, chap. 5], we can then consider μ^{-1} as an universal velocity.[1]

4 Regularised Meta-conformal Correlator: The $2D$ Case

The derivation of the $2D$ meta-conformal correlator starts essentially along the same lines as in the $1D$ case, but is based now on the generators (3). The dualisation is now carried out with respect to the chiral rapidities $\gamma = \gamma_\parallel - i\gamma_\perp$ and $\bar{\gamma} = \gamma_\parallel + i\gamma_\perp$ and we also use the light-cone coordinates $z = r_\parallel + ir_\perp$ and $\bar{z} = r_\parallel - ir_\perp$. Taking the translation generators A_{-1}, $B_{-1}^{\pm}$ into account, we consider the dual correlator

$$
\widehat{F} = \widehat{F}(\zeta_1, \zeta_2, \bar{\zeta}_1, \bar{\zeta}_2, t, \xi, \bar{\xi}, \mu_1, \mu_2) \tag{23}
$$

[1] In the conformal galilean limit $\mu \to 0$, recover the bounded result $F \sim \exp(-2|\gamma_1 r|/t)$ [25].

where we defined the variables

$$t = t_1 - t_2, \quad \xi = \mu_1 z_1 - \mu_2 z_2, \quad \bar{\xi} = \mu_1 \bar{z}_1 - \mu_2 \bar{z}_2 \tag{24}$$

In complete analogy with the $1D$ case, we further define the variables

$$\eta = \mu_1 \zeta_1 + \mu_2 \zeta_2, \quad \bar{\eta} = \mu_1 \bar{\zeta}_1 + \mu_2 \bar{\zeta}_2 \tag{25}$$

such that the correlator $\widehat{F} = \widehat{F}(\eta, \bar{\eta}, t, \xi, \bar{\xi}, \mu_1, \mu_2)$ obeys the equations

$$\begin{aligned} &\left(2\mathrm{i}\partial_\eta - (t+\xi)\partial_\xi\right)\widehat{F} = 0, \quad \left(2\mathrm{i}\partial_{\bar{\eta}} - (t+\bar{\xi})\partial_{\bar{\xi}}\right)\widehat{F} = 0, \\ &\left(t\partial_t + \xi\partial_\xi + \bar{\xi}\partial_{\bar{\xi}} + 2\delta_1\right)\widehat{F} = 0 \end{aligned} \tag{26}$$

along with the constraint $\delta_1 = \delta_2$. The most general solution of this system is

$$\widehat{F} = t^{-2\delta_1}\widehat{\mathscr{F}}\left(\frac{\eta}{2} + \mathrm{i}\ln(1+\xi/t), \frac{\bar{\eta}}{2} + \mathrm{i}\ln(1+\bar{\xi}/t)\right) = t^{-2\delta_1}\widehat{\mathscr{F}}\left(u + \mathrm{i}\lambda, \bar{u} + \mathrm{i}\lambda\right) \tag{27}$$

with the abbreviations ($\bar{u}$ is obtained from u by replacing $\zeta_j \mapsto \bar{\zeta}_j$)

$$u := \frac{\mu}{2}(\zeta_1 + \zeta_2) + \underbrace{\arctan\frac{\mu r_\perp/t}{1+\mu r_\parallel/t}}_{=:\,a}, \quad \lambda := \frac{1}{2}\ln\left[\left(1 + \frac{\mu r_\parallel}{t}\right)^2 + \left(\frac{\mu r_\perp}{t}\right)^2\right] \tag{28}$$

and we simplified the notation by letting $\mu_1 = \mu_2 = \mu$ and assumed translation-invariance in time and space. As before, we expect that a Hardy space will permit to derive the boundedness, see the appendix for details. We define $\widehat{\mathscr{F}}_\lambda(u, \bar{u}) := \widehat{\mathscr{F}}(u + \mathrm{i}\lambda, \bar{u} + \mathrm{i}\lambda)$ and require:

Postulate. *If* $\lambda > 0$, *then* $\widehat{\mathscr{F}}_\lambda \in H_2^{++}$ *and if* $\lambda < 0$, *then* $\widehat{\mathscr{F}}_\lambda \in H_2^{--}$.

Theorem 2 in the appendix, especially (A.11), then states that

$$\begin{aligned} \widehat{\mathscr{F}}_\lambda = \Theta(\lambda)\int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\bar{\tau}\; e^{\mathrm{i}(u+\mathrm{i}\lambda)\tau + \mathrm{i}(\bar{u}+\mathrm{i}\lambda)\bar{\tau}}\,\widehat{\mathscr{F}}_+(\tau, \bar{\tau}) \\ +\Theta(-\lambda)\int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\bar{\tau}\; e^{-\mathrm{i}(u+\mathrm{i}\lambda)\tau - \mathrm{i}(\bar{u}+\mathrm{i}\lambda)\bar{\tau}}\,\widehat{\mathscr{F}}_-(\tau, \bar{\tau}) \end{aligned} \tag{29}$$

Then, we can write the two-point function in the case $\lambda > 0$, with the short-hand $\mathscr{D}\zeta := \mathrm{d}\zeta_1 \mathrm{d}\bar{\zeta}_1 \mathrm{d}\zeta_2 \mathrm{d}\bar{\zeta}_2$ and the abbreviations from (28)

$$
\begin{aligned}
F &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4} \mathscr{D}\zeta\ e^{-\mathrm{i}\gamma_1\zeta_1-\mathrm{i}\bar{\gamma}_1\bar{\zeta}_1-\mathrm{i}\gamma_2\zeta_2-\mathrm{i}\bar{\gamma}_2\bar{\zeta}_2}\,\widehat{F} \\
&= \frac{t^{-2\delta_1}}{(2\pi)^3}\int_{\mathbb{R}^4} \mathscr{D}\zeta\ e^{-\mathrm{i}\gamma_1\zeta_1-\mathrm{i}\bar{\gamma}_1\bar{\zeta}_1-\mathrm{i}\gamma_2\zeta_2-\mathrm{i}\bar{\gamma}_2\bar{\zeta}_2}\ \times \\
&\quad\times \int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\bar{\tau}\ e^{\mathrm{i}(\mu(\zeta_1+\zeta_2)+2a)\tau/2-\lambda\tau}\, e^{\mathrm{i}(\mu(\bar{\zeta}_1+\bar{\zeta}_2)+2a)\bar{\tau}/2-\lambda\bar{\tau}}\,\widehat{\mathscr{F}}_+(\tau,\bar{\tau}) \\
&= \frac{t^{-2\delta_1}}{(2\pi)^3}\int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\bar{\tau}\ \widehat{\mathscr{F}}_+(\tau,\bar{\tau})\, e^{\mathrm{i}a(\tau-\bar{\tau})-\lambda(\tau+\bar{\tau})}\ \times \\
&\quad\times \int_{\mathbb{R}^4} \mathscr{D}\zeta\ e^{\mathrm{i}(-\gamma_1-\gamma_2+\mu\tau)\zeta_+ + \mathrm{i}(-\gamma_1+\gamma_2)\zeta_-}\, e^{\mathrm{i}(-\bar{\gamma}_1-\bar{\gamma}_2+\mu\bar{\tau})\bar{\zeta}_+ + \mathrm{i}(-\bar{\gamma}_1+\bar{\gamma}_2)\bar{\zeta}_-} \\
&= \mathrm{cste.}\ t^{-2\delta_1}\delta_{\gamma_1,\gamma_2}\delta_{\bar{\gamma}_1,\bar{\gamma}_2}\, e^{\mathrm{i}2a(\gamma_1-\bar{\gamma}_1)/\mu}\, e^{-\lambda 2(\gamma_1+\bar{\gamma}_1)/\mu}\,\Theta\left(\frac{\gamma_1}{\mu}\right)\Theta\left(\frac{\bar{\gamma}_1}{\mu}\right)
\end{aligned} \tag{30}
$$

Herein, variables were changed according to $\zeta_1 = \zeta_+ + \zeta_-$ and $\zeta_2 = \zeta_+ - \zeta_-$ and similarly for the $\bar{\zeta}_j$. The case $\lambda < 0$ is treated in the same manner

$$
\begin{aligned}
F &= \frac{t^{-2\delta_1}}{(2\pi)^3}\int_{\mathbb{R}^4} \mathscr{D}\zeta\ e^{-\mathrm{i}\gamma_1\zeta_1-\mathrm{i}\bar{\gamma}_1\bar{\zeta}_1-\mathrm{i}\gamma_2\zeta_2-\mathrm{i}\bar{\gamma}_2\bar{\zeta}_2}\ \times \\
&\quad\times \int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\bar{\tau}\ e^{-\mathrm{i}(\mu(\zeta_1+\zeta_2)+2a)\tau/2+\lambda\tau}\, e^{-\mathrm{i}(\mu(\bar{\zeta}_1+\bar{\zeta}_2)+2a)\bar{\tau}/2+\lambda\bar{\tau}}\,\widehat{\mathscr{F}}_+(\tau,\bar{\tau}) \\
&= \mathrm{cste.}\ t^{-2\delta_1}\delta_{\gamma_1,\gamma_2}\delta_{\bar{\gamma}_1,\bar{\gamma}_2}\, e^{\mathrm{i}2a(|\gamma_1/\mu|-|\bar{\gamma}_1/\mu|)}\, e^{-|\lambda|2(|\gamma_1/\mu|+|\bar{\gamma}_1/\mu|)}\,\Theta\left(-\frac{\gamma_1}{\mu}\right)\Theta\left(-\frac{\bar{\gamma}_1}{\mu}\right)
\end{aligned} \tag{31}
$$

In order to understand the meaning of these expression, we return to the physical interpretation of the conditions $\lambda > 0$ and $\lambda < 0$. From (28), the most restrictive case occurs for $r_\perp = 0$. Then $\lambda > 0$ is equivalent to $r_\parallel/t > 0$. On the other hand, since γ_1/μ will have a definite sign, it is *a fortiori* also real. Hence $\gamma_{1,\perp} = 0$ and we can conclude that

$$
F = \delta_{\delta_1,\delta_2}\delta_{\gamma_1,\gamma_2}\delta_{\bar{\gamma}_1,\bar{\gamma}_2}\, t^{-2\delta_1}\left[\left(1+\left|\frac{\mu r_\parallel}{t}\right|\right)^2 + \left(\frac{\mu r_\perp}{t}\right)^2\right]^{-2\gamma_{1,\parallel}/\mu} \tag{32}
$$

up to normalisation, is the final form for the $2D$ meta-conformally co-variant correlator which is bounded in the entire time-space.

5 Conclusions

It has been shown that via a dualisation procedure of the rapidities in the meta-conformal generators, a refined form of the global Ward identities can be found which leads to expressions of the quasi-primary two-point functions which remain bounded in the entire time-space. Herein, we postulate that the dualised two-point functions, whose dual variables are naturally seen to occur in a tube of the first (or the forth) quadrant, belong to a Hardy space. In this way, we can formulate a sufficient condition for the construction of bounded two-point functions, namely

$$F(t_1,t_2,r_1,r_2) = \delta_{\delta_1,\delta_2}\delta_{\gamma_1/\mu_1,\gamma_2/\mu_2}\,(t_1-t_2)^{-2\delta_1}\left(1+\left|\frac{\mu_1 r_1-\mu_2 r_2}{t_1-t_2}\right|\right)^{-2|\gamma_1/\mu_1|} \tag{33}$$

(up to normalisation) in $d=1$ spatial dimensions and

$$F(t_1,t_2,r_{\parallel,1},r_{\parallel,2},\boldsymbol{r}_{\perp,1},\boldsymbol{r}_{\perp,2}) = \delta_{\delta_1,\delta_2}\delta_{\gamma_{1,\parallel},\gamma_{2,\parallel}}\,(t_1-t_2)^{-2\delta_1}\times \\ \times\left[\left(1+\left|\frac{\mu_1 r_{\parallel,1}-\mu_2 r_{\parallel,2}}{t_1-t_2}\right|\right)^2+\left(\frac{\mu_1\boldsymbol{r}_{\perp,1}-\mu_2\boldsymbol{r}_{\perp,2}}{t_1-t_2}\right)^2\right]^{-2\gamma_{1,\parallel}/\mu} \tag{34}$$

in $d\geq 2$ spatial dimensions, where rotation-invariance in the $d-1$ transverse directions is assumed (provided $\boldsymbol{\gamma}_\perp=\boldsymbol{0}$).

Acknowledgements Most of this work was done during the visits of S.S. at Université de Lorraine Nancy and of M.H. at the workshop *"Lie theories and its applications in physics LT13"*. These visits were supported by PHC Rila. M.H. thanks the MPIPKS Dresden for warm hospitality, where other parts of this work were done.

Appendix. Background on Hardy Spaces

In the main text, we need precise statements on the Fourier transform on semi-infinite spaces. These can be conveniently formulated in terms of Hardy spaces. Here, we restrict to the special case H_2. Our brief summary is based on [2, 40].

We begin with the case of functions of a single complex variable z, defined in the upper half-plane $\mathbb{H}_+ := \{z\in\mathbb{C}\,|\,z=x+\mathrm{i}y,\ y\geq 0\}$.

Definition 1: *A function* $f:\mathbb{H}_+\to\mathbb{C}$ *belongs to the* Hardy space H_2^+ *if it is holomorphic on* $\mathbb{H}_+$ *and if*

$$M^2 := \sup_{y>0}\int_{-\infty}^{\infty}\mathrm{d}x\ |f(x+\mathrm{i}y)|^2 < \infty \tag{A.1}$$

The main results of interest to us can be summarised as follows.

Theorem 1 [2]**:** *Let* $f:\mathbb{H}_+\to\mathbb{C}$ *be a holomorphic function. Then the following statements are equivalent:*

1. $f\in H_2^+$
2. *there exists a function* $\mathcal{f}:\mathbb{R}\to\mathbb{C}$*, which is square-integrable* $\mathcal{f}\in L^2(\mathbb{R})$*, such that* $\lim_{y\to 0^+} f(x+\mathrm{i}y)=\mathcal{f}(x)$ *and*

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\xi\, \frac{f(\xi)}{\xi - z}, \quad 0 = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\xi\, \frac{f(\xi)}{\xi - z^*} \tag{A.2}$$

where $z^* = x - \mathrm{i}y$ denotes the complex conjugate of z. For notational simplicity, one often writes $f(x) = \lim_{y \to 0^+} f(x + \mathrm{i}y)$, with $x \in \mathbb{R}$.

3. *there exists a function $\widehat{f} : \mathbb{R}_+ \to \mathbb{C}$, $\widehat{f} \in L^2(\mathbb{R}_+)$, such that for all $y > 0$*

$$f(z) = f(x + \mathrm{i}y) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \mathrm{d}\zeta\, e^{\mathrm{i}(x+\mathrm{i}y)\zeta}\, \widehat{f}(\zeta) \tag{A.3}$$

The property (A.3) is of major interest to us in the main text.

If $f \in H_2^+$, one has the following bounds [2]

$$\lim_{y \to 0} f(x + \mathrm{i}y) = 0 \;\; ; \;\; \text{uniformly for all}\, x \in \mathbb{R} \tag{A.4a}$$

$$\lim_{x \to \pm\infty} f(x + \mathrm{i}y) = 0 \;\; ; \;\; \text{uniformly with respect to}\, y \geq y_0 > 0 \tag{A.4b}$$

Equation (A.4a) follows from the bound (in turn obtained from (A.3)), see also [32]

$$|f(x + \mathrm{i}y)| \leq f_\infty\, y^{-1/2} \tag{A.4c}$$

which holds for all $x \in \mathbb{R}$ and where the constant $f_\infty > 0$ depends on the function f. There is a simple sufficient criterion which can be used to establish that a given function f is in the Hardy space H_2^+:

Lemma: *If the complex function $f(z) = f(x + \mathrm{i}y)$ is holomorphic for all $y \geq 0$, obeys the bound $|f(z)| < f_0 e^{-\delta y}$, with the constants $f_0 > 0$ and $\delta > 0$ and if $\int_{-\infty}^{\infty} \mathrm{d}x\, |f(x)|^2 < \infty$, then $f \in H_2^+$.*

Proof: Since $f(z)$ is holomorphic on the closure $\overline{\mathbb{H}_+}$ (which includes the real axis), one has the Cauchy formula

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}} \mathrm{d}w\, \frac{f(w)}{w - z} = \frac{1}{2\pi \mathrm{i}} \int_{-R}^{R} \mathrm{d}w\, \frac{f(w)}{w - z} + \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{C}_{\mathrm{sup}}} \mathrm{d}w\, \frac{f(w)}{w - z} =: F_1(z) + F_2(z)$$

where the integration contour $\mathscr{C}$ consists of the segment $[-R, R]$ on the real axis and the superior semi-circle $\mathscr{C}_{\mathrm{sup}}$. One may write $w = u + \mathrm{i}v = Re^{\mathrm{i}\theta} \in \mathscr{C}_{\mathrm{sup}}$. It follows that on the superior semi-circle $|f(w)| < f_0 e^{-\delta v} = f_0 e^{-\delta R \sin\theta}$. Now, for R large enough, one has $|w - z| = |w(1 - z/w)| \geq R\frac{1}{2}$, for $z \in \overline{\mathbb{H}_+}$ fixed and $w \in \mathscr{C}_{\mathrm{sup}}$. We can then estimate the contribution $F_2(z)$ of the superior semi-circle

$$\begin{aligned} |F_2(z)| &\leq \frac{1}{2\pi} \int_{\mathscr{C}_{\mathrm{sup}}} |\mathrm{d}w| \frac{|F(w)|}{|w(1 - z/w)|} \quad \leq \frac{1}{2\pi} \int_0^{\pi} \mathrm{d}\theta\, \frac{f_0 e^{-\delta R \sin\theta} R}{R\, \frac{1}{2}} \\ &\leq \frac{2 f_0}{\pi} \int_0^{\pi/2} \mathrm{d}\theta\, \exp\left(-\frac{2\delta}{\pi} R\theta \right) \leq \frac{f_0}{\delta} \frac{1}{R} \to 0 \;\; , \;\; \text{for}\, R \to \infty \end{aligned}$$

Hence, the integral representation $f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{R}} \mathrm{d}w \; f(w)(w-z)^{-1}$ holds for $R \to \infty$. Since $f \in L^2(\mathbb{R})$, the assertion follows from Eq. (A.2) of Theorem 1. q.e.d.

Clearly, one may also define a Hardy space H_2^- for functions holomorphic on the lower complex half-plane $\mathbb{H}_-$, by adapting the above definition. All results transpose in an evident way.

Further conceptual preparations are necessary for the generalisation of these results to higher dimensions. Here, we shall merely treat the $2D$ case, which is enough for our purposes (and generalisations to $n > 2$ will be obvious). We denote $\boldsymbol{z} = (z_1, z_2) \in \mathbb{C}^2$ and write the scalar product $\boldsymbol{z} \cdot \boldsymbol{w} = z_1 w_1 + z_2 w_2$ for $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{C}^2$. Following [40], H_2-spaces can be defined as follows.

Definition 2: *If $B \subset \mathbb{R}^2$ is an open set, the* tube T_B with base B *is*

$$T_B := \left\{ \boldsymbol{z} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y} \in \mathbb{C}^2 \,\middle|\, \boldsymbol{y} \in B, \boldsymbol{x} \in \mathbb{R}^2 \right\} \tag{A.5}$$

A function $f : T_B \to \mathbb{C}$ which is holomorphic on T_B is in the Hardy space $H_2(T_B)$ *if*

$$M^2 := \sup_{\boldsymbol{y} \in B} \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{x} \; |f(\boldsymbol{x} + \mathrm{i}\boldsymbol{y})|^2 < \infty \tag{A.6}$$

However, it turns out that this definition is too general. More interesting results are obtained if one uses cônes as a base of the tubes.

Definition 3: *(i) An* open cône $\Gamma \subset \mathbb{R}^n$ *satisfies the properties $0 \notin \Gamma$ and if $\boldsymbol{x}, \boldsymbol{y} \in \Gamma$ and $\alpha, \beta > 0$, then $\alpha\boldsymbol{x} + \beta\boldsymbol{y} \in \Gamma$. A* closed cône *is the closure $\overline{\Gamma}$ of an open cône Γ. (ii) If Γ is a cône, and if the set*

$$\Gamma^* := \left\{ \boldsymbol{x} \in \mathbb{R}^n \,|\boldsymbol{x} \cdot \boldsymbol{t} \geq 0 \text{ with } \boldsymbol{t} \in \Gamma \right\} \tag{A.7}$$

has a non-vanishing interior, then Γ^* is the dual cone *with respect to Γ. The cône Γ is called* self-dual*, if $\Gamma^* = \overline{\Gamma}$.*

For illustration, note that in one dimension ($n = 1$) the only cône is $\Gamma = \{x \in \mathbb{R} \,|x > 0\} = \mathbb{R}_+$. It is self-dual, since $\Gamma^* = \overline{\Gamma} = \mathbb{R}_{0,+}$. In two dimensions ($n = 2$), consider the cône $\Gamma^{++} := \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,|\boldsymbol{x} = (x_1, x_2) \text{ with } x_1 > 0, x_2 > 0 \right\}$ which is the *first quadrant* in the $2D$ plane. Since

$$\Gamma^{++\,*} = \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,\middle|\boldsymbol{x} \cdot \boldsymbol{t} \geq 0, \;\text{ for all } \boldsymbol{t} \in \Gamma^{++} \right\} = \mathbb{R}_{0,+} \oplus \mathbb{R}_{0,+} = \overline{\Gamma^{++}} \tag{A.8}$$

the set Γ^{++} is a self-dual cône.

Hardy spaces defined on the tube $T_{\Gamma^{++}}$ of the first quadrant provide the structure required here.

Definition 4 [40]**:** *If Γ^{++} denotes the first quadrant of the plane $\mathbb{R}^2$, a function $f : T_{\Gamma^{++}} \to \mathbb{C}$ holomorphic on $T_{\Gamma^{++}}$ is in the* Hardy space $H_2^{++} := H_2(T_{\Gamma^{++}})$ *if*

$$M^2 := \sup_{\mathbf{y}\in\Gamma^{++}} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\, |f(\mathbf{x}+\mathrm{i}\mathbf{y})|^2 < \infty \tag{A.9}$$

Theorem 2 [40]**:** *Let the function $f : T_{\Gamma^{++}} \to \mathbb{C}$ be holomorphic. Then the following statements are equivalent:*

1. $f \in H_2^{++}$
2. *there exists a function $f : \mathbb{R} \to \mathbb{C}$, which is square-integrable $f \in L^2(\mathbb{R})$, such that* $\lim_{\mathbf{y}\to\mathbf{0}^+} f(\mathbf{x}+\mathrm{i}\mathbf{y}) = f(\mathbf{x})$ *and*

$$f(\mathbf{z}) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{w}\, \frac{f(\mathbf{w})}{\mathbf{w}-\mathbf{z}}, \quad 0 = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{R}^2} \mathrm{d}\mathbf{w}\, \frac{f(\mathbf{w})}{\mathbf{w}-\mathbf{z}^*} \tag{A.10}$$

where $(\mathbf{w}-\mathbf{z})^{-1} := (w_1 - z_1)^{-1}(w_2 - z_2)^{-1}$ and $\mathbf{z}^* = \mathbf{x} - \mathrm{i}\mathbf{y}$ denotes the complex conjugate of $\mathbf{z}$. For notational simplicity, one often writes $f(\mathbf{x}) = \lim_{\mathbf{y}\to\mathbf{0}^+} f(\mathbf{x}+\mathrm{i}\mathbf{y})$, with $\mathbf{x} \in \mathbb{R}^2$.

3. *there exists a function $\widehat{f} : \mathbb{R}_+ \oplus \mathbb{R}_+ \to \mathbb{C}$, with $\widehat{f} \in L^2(\mathbb{R}_+ \oplus \mathbb{R}_+)$ and $z_i \in \mathbb{H}_+$*

$$f(\mathbf{z}) = \frac{1}{2\pi} \int_{\Gamma^{++}} \mathrm{d}\mathbf{t}\, e^{\mathrm{i}\mathbf{z}\cdot\mathbf{t}} \widehat{f}(\mathbf{t}) = \frac{1}{2\pi} \int_0^\infty \mathrm{d}t_1 \int_0^\infty \mathrm{d}t_2\, e^{\mathrm{i}(z_1 t_1 + z_2 t_2)} \widehat{f}(\mathbf{t}) \tag{A.11}$$

The property (A.11) is of major interest to us in the main text. Summarising, the restriction to the first quadrant Γ^{++} allows to carry over the known results from the $1D$ case, separately for each component.

Of course, one may also define a Hardy space $H_2^{--} := H_2(T_{\Gamma^{--}})$ on the forth quadrant, in complete analogy.

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Infinite Spin Particles and Superparticles

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Abstract We give a new twistorial field formulation of a massless infinite spin particle. We quantize the world-line infinite spin particle model and construct a twistorial infinite spin field. The helicity decomposition of this field is derived. Making use of the field twistor transform, we construct the space-time infinite (continuous) spin field, which depends on the coordinate four-vector and additional commuting Weyl spinor. We show that the infinite integer-spin field and infinite half-integer-spin field form $\mathcal{N} = 1$ infinite spin supermultiplet. We prove that the supersymmetry transformations are closed on-shell and form the $\mathcal{N} = 1$ superalgebra.

1 Introduction

In our recent papers [1, 2] we constructed a new model of an infinite (continuous) spin particles. The states of these particles defines the space of massless unitary irreducible representation of the Poincaré group $ISO^{\uparrow}(1, 3)$ (or its covering $ISL(2, \mathbf{C})$).

Classification of the $ISO^{\uparrow}(1, 3)$ unitary irreducible representations was given in [3–5]. To characterize these irreducible representations we need to consider the corresponding irreducible representations of the Lie algebra $iso(1, 3)$ with the momentum $\hat{P}_n$ and the angular momentum $\hat{M}^{mk}$ Hermitian generators and defining relations

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_6

$$[\hat{P}_n,\ \hat{P}_m] = 0, \qquad [\hat{P}_n,\ \hat{M}_{mk}] = i\,(\eta_{kn}\hat{P}_m - \eta_{mn}\hat{P}_k),$$
$$[\hat{M}_{nm},\ \hat{M}_{k\ell}] = i\,(\eta_{nk}\hat{M}_{m\ell} - \eta_{mk}\hat{M}_{n\ell} + \eta_{m\ell}\hat{M}_{nk} - \eta_{n\ell}\hat{M}_{mk}), \tag{1}$$

where metric tensor is $||\eta_{mk}|| = \mathrm{diag}(+1, -1, -1, -1)$. The algebra $iso(1,3)$ has two Casimir operators

$$\hat{P}^n\hat{P}_n \quad \text{and} \quad \hat{W}^n\hat{W}_n, \tag{2}$$

where

$$\hat{W}_n = \frac{1}{2}\,\varepsilon_{nmkr}\,\hat{M}^{mk}\,\hat{P}^r \tag{3}$$

are components of the Pauli-Lubanski pseudovector which satisfy

$$\hat{W}_n\hat{P}^n = 0\,, \qquad [\hat{W}_k,\ \hat{P}_n] = 0\,, \qquad [\hat{W}_m,\ \hat{W}_n] = i\,\varepsilon_{mnkr}\hat{W}^k\,\hat{P}^r\,.$$

On the space of states of massless irreducible representation of infinite (continuous) spin the Casimir operators of $iso(1,3)$ take the values

$$\hat{P}^n\hat{P}_n = \mathsf{m}^2 = 0, \qquad \hat{W}^2 = \hat{W}^n\hat{W}_n = -\mu^2, \tag{4}$$

where m is the particle mass and μ is real mass-dimensional parameter.

To describe the massless irreducible representation of infinite (continuous) spin we have to introduce "canonically conjugate" to $\hat{P}_k$, $\hat{W}_n$ variables[1] x_k, y_n:

$$x = (x_0, x_1, x_2, x_3) \in \mathbf{R^{1,3}}, \qquad \mathbf{y} = (\mathbf{y_0}, \mathbf{y_1}, \mathbf{y_2}, \mathbf{y_3}) \in \mathbf{R^{1,3}}.$$

Then, as it was shown in [3–5], the massless infinite integer-spin irreducible representations of the Poincaré group are realized in the space of the fields $\Phi(x,y)$ which satisfy the conditions

$$\frac{\partial}{\partial x^m}\frac{\partial}{\partial x_m}\,\Phi = 0, \qquad \frac{\partial}{\partial x^m}\frac{\partial}{\partial y_m}\,\Phi = 0,$$
$$\frac{\partial}{\partial y^m}\frac{\partial}{\partial y_m}\,\Phi = \mu^2\,\Phi, \qquad -i\,y^m\,\frac{\partial}{\partial x^m}\,\Phi = \Phi. \tag{5}$$

The massless infinite half-integer-spin irreducible representations of the Poincaré group are realized in the space of wave functions (WF) $\Phi_A(x,y)$ with external Dirac index $A = 1, 2, 3, 4$ which satisfy the conditions [3–5]

[1] As we will see below in Sect. 2, variables y_n are not canonically conjugate to components $\hat{W}_n$ in the standard sense of phase space variables.

$$\frac{\partial}{\partial x^m}\left(\gamma^m\right)_A{}^B\,\Phi_B=0,\qquad \frac{\partial}{\partial x^m}\frac{\partial}{\partial y_m}\,\Phi_A=0,$$
$$\frac{\partial}{\partial y^m}\frac{\partial}{\partial y_m}\,\Phi_A=\mu^2\,\Phi_A\,,\qquad -i\,y^m\frac{\partial}{\partial x^m}\,\Phi_A=\Phi_A. \tag{6}$$

This report is devoted to some problems of theory of massless infinite (or continues) spin unitary irreducible representations of the $ISL(2,\mathbf{C})$ group. Various aspects of this theory were considered in a wide range of works (see, e.g., [6–18]). Motivation of the investigations of the infinite spin representations is caused by an identical spectrum of states of the infinite spin theory [6] and the higher-spin theory (see, e.g., [19]) and by its potential relation to the string theory (see [20], recent paper [21], and references therein). Here we present a generalization of the twistor formulation of standard (with fixed helicity) massless particle [22] to massless infinite spin representations. Making use of the field twistor transform, we obtain the space-time–spinorial presentation for infinite spin fields which describe all integer or all half-integer helicities. These fields form $\mathcal{N}=1$ supermultiplet of infinite spins [7, 14]. The present report is based on the results obtained in [1, 2].

2 Twistorial and Space-Time Formulations of Infinite Spin Particles

The twistorial formulation of the infinite (continuous) spin particle is described (see [1, 2]) in terms of even Weyl spinors[2]

$$\pi_\alpha,\quad \bar\pi_{\dot\alpha}:=(\pi_\alpha)^*,\qquad \rho_\alpha,\quad \bar\rho_{\dot\alpha}:=(\rho_\alpha)^*, \tag{7}$$

and their canonically conjugated spinors

$$\omega^\alpha,\quad \bar\omega^{\dot\alpha}:=(\omega^\alpha)^*,\qquad \eta^\alpha,\quad \bar\eta^{\dot\alpha}:=(\eta^\alpha)^*, \tag{8}$$

with Poisson brackets $\left\{\omega^\alpha,\pi_\beta\right\}=\left\{\eta^\alpha,\rho_\beta\right\}=\delta^\alpha_\beta$ and $\left\{\bar\omega^{\dot\alpha},\bar\pi_{\dot\beta}\right\}=\left\{\bar\eta^{\dot\alpha},\bar\rho_{\dot\beta}\right\}=\delta^{\dot\alpha}_{\dot\beta}$ (other Poisson brackets are equal to zero). Twistorial Lagrangian of the infinite (continuous) spin particle is written in the form [1, 2]:

$$\mathcal{L}_{twistor}=\pi_\alpha\dot\omega^\alpha+\bar\pi_{\dot\alpha}\dot{\bar\omega}^{\dot\alpha}+\rho_\alpha\dot\eta^\alpha+\bar\rho_{\dot\alpha}\dot{\bar\eta}^{\dot\alpha}+l\,\mathcal{M}+k\,\mathcal{U}+\ell\,\mathcal{F}+\bar\ell\,\bar{\mathcal{F}}, \tag{9}$$

[2] We will use the following two-spinor conventions. The totally antisymmetric tensor ϵ^{mnkl} has the component $\epsilon^{0123}=1$. We use the set of σ-matrices: $\sigma^n=(\sigma^0\equiv I_2,\,\sigma^1,\,\sigma^2,\,\sigma^3)$ and the set of dual σ-matrices: $\tilde\sigma^n=(\sigma^0,-\sigma^1,-\sigma^2,-\sigma^3)$, where σ^i are usual Pauli matrices. We also use standard van der Waerden spinor notation with dotted and undotted spinor indices and raise and lower them by means of metrics: $\epsilon_{\alpha\beta}$, $\epsilon_{\dot\alpha\dot\beta}$ and their inverse $\epsilon^{\alpha\beta}$, $\epsilon^{\dot\alpha\dot\beta}$ with components $\epsilon_{12}=-\epsilon_{21}=1$. In particular $(\tilde\sigma_m)^{\dot\alpha\beta}=\epsilon^{\dot\alpha\dot\delta}\epsilon^{\beta\gamma}(\sigma_m)_{\gamma\dot\delta}$. The links between the Minkowski four-vectors and spinorial quantities are $A_{\alpha\dot\beta}=\frac{1}{\sqrt2}A_m(\sigma^m)_{\alpha\dot\beta}$, $A_m=\frac{1}{\sqrt2}A_{\alpha\dot\beta}(\tilde\sigma_m)^{\dot\beta\alpha}$, so that $A^mB_m=A_{\alpha\dot\beta}B^{\dot\beta\alpha}$.

where $\dot{\omega}(\tau) := \partial_\tau \omega(\tau)$ and τ is an evolution parameter. Functions $l(\tau)$, $k(\tau)$, $\ell(\tau)$, $\bar{\ell}(\tau)$ are Lagrange multipliers for the constraints

$$\mathcal{M} := \pi^\alpha \rho_\alpha \, \bar{\rho}_{\dot{\alpha}} \bar{\pi}^{\dot{\alpha}} - \mu^2/2 \approx 0, \tag{10}$$

$$\mathcal{F} := \eta^\alpha \pi_\alpha - 1 \approx 0, \qquad \bar{\mathcal{F}} := \bar{\pi}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}} - 1 \approx 0, \tag{11}$$

$$\mathcal{U} := i \, (\omega^\alpha \pi_\alpha - \bar{\pi}_{\dot{\alpha}} \bar{\omega}^{\dot{\alpha}} + \eta^\alpha \rho_\alpha - \bar{\rho}_{\dot{\alpha}} \bar{\eta}^{\dot{\alpha}}) \approx 0. \tag{12}$$

The first-class constraints (10), (11), (12) generate abelian Lie group which acts in the phase space of spinors (7), (8) as follows:

$$\begin{pmatrix} \pi_1 & \rho_1 \\ \pi_2 & \rho_2 \end{pmatrix} \to \begin{pmatrix} \pi_1 & \rho_1 \\ \pi_2 & \rho_2 \end{pmatrix} \begin{pmatrix} e^{i\beta} & \alpha e^{i\beta} \\ 0 & e^{i\beta} \end{pmatrix}, \tag{13}$$

$$\begin{pmatrix} \eta_1 & \omega_1 \\ \eta_2 & \omega_2 \end{pmatrix} \to \begin{pmatrix} \eta_1 & \omega_1 \\ \eta_2 & \omega_2 \end{pmatrix} \begin{pmatrix} e^{-i\beta} & -\alpha e^{-i\beta} \\ 0 & e^{-i\beta} \end{pmatrix} + \frac{2}{\mu^2} (\bar{\rho}_{\dot{\alpha}} \bar{\pi}^{\dot{\alpha}}) \begin{pmatrix} \pi_1 & \rho_1 \\ \pi_2 & \rho_2 \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \tag{14}$$

where $\beta(\tau), \gamma(\tau) \in \mathbf{R}$ and $\alpha(\tau) \in \mathbf{C}\backslash\mathbf{0}$ are the parameters of the gauge group which is generated by constraints (10), (11), (12).

The Noether charges of the Poincaré transformations have the following form

$$M_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}} M_{\alpha\beta} + \epsilon_{\alpha\beta} \bar{M}_{\dot{\alpha}\dot{\beta}}, \qquad P_{\alpha\dot{\alpha}} = \pi_\alpha \bar{\pi}_{\dot{\beta}}, \tag{15}$$

where (anti)self-dual spin-tensors are

$$M_{\alpha\beta} = \pi_{(\alpha} \omega_{\beta)} + \rho_{(\alpha} \eta_{\beta)}, \qquad \bar{M}_{\dot{\alpha}\dot{\beta}} = \bar{\pi}_{(\dot{\alpha}} \bar{\omega}_{\dot{\beta})} + \bar{\rho}_{(\dot{\alpha}} \bar{\eta}_{\dot{\beta})}. \tag{16}$$

In the Weyl-spinor notation the Pauli-Lubanski vector (3) has the form

$$W_{\alpha\dot{\alpha}} = -i \left(M_{\alpha\beta} P^{\beta}_{\dot{\alpha}} - \bar{M}_{\dot{\alpha}\dot{\beta}} P^{\dot{\beta}}_{\alpha} \right) \tag{17}$$

and for considered twistorial realization (15), (16) we obtain

$$\begin{aligned} W_{\alpha\dot{\alpha}} = \Lambda P_{\alpha\dot{\alpha}} &- \frac{i}{2} \left[(\bar{\pi}_{\dot{\beta}} \bar{\eta}^{\dot{\beta}}) \pi_\alpha \bar{\rho}_{\dot{\alpha}} - (\pi_\beta \eta^\beta) \rho_\alpha \bar{\pi}_{\dot{\alpha}} \right] \\ &+ \frac{i}{2} \left[(\bar{\pi}^{\dot{\beta}} \bar{\rho}_{\dot{\beta}}) \pi_\alpha \bar{\eta}_{\dot{\alpha}} - (\pi^\beta \rho_\beta) \eta_\alpha \bar{\pi}_{\dot{\alpha}} \right], \end{aligned} \tag{18}$$

where

$$\Lambda := \frac{i}{2} \left(\pi_\beta \omega^\beta - \bar{\pi}_{\dot{\beta}} \bar{\omega}^{\dot{\beta}} \right). \tag{19}$$

The square of the vector (18) gives Casimir operator

$$W^2 = W^{\alpha\dot{\alpha}} W_{\alpha\dot{\alpha}} = -2 \, |\pi^\alpha \rho_\alpha|^2 \, |\pi_\beta \eta^\beta|^2 \tag{20}$$

and, due to the constraints (10), (11), we reproduce (4): $W^2 = -\mu^2$. So, the twistorial model (9) indeed describes massless particles of infinite spin.

The pairs of spinors π_α, $\bar{\omega}^{\dot\alpha}$ and ρ_α, $\bar{\eta}^{\dot\alpha}$ form two Penrose twistors [22]

$$Z_A := \left(\pi_\alpha, \bar{\omega}^{\dot\alpha}\right), \qquad Y_A := \left(\rho_\alpha, \bar{\eta}^{\dot\alpha}\right). \tag{21}$$

Conjugated spinors $\bar{\pi}_{\dot\alpha}$, ω^α and $\bar{\rho}_{\dot\alpha}$, η^α constitute the dual twistors

$$\bar{Z}^A := \begin{pmatrix} \omega^\alpha \\ -\bar{\pi}_{\dot\alpha} \end{pmatrix}, \qquad \bar{Y}^A := \begin{pmatrix} \eta^\alpha \\ -\bar{\rho}_{\dot\alpha} \end{pmatrix}. \tag{22}$$

It means that our description of infinite spin particles uses a couple of twistors as opposed to the one-twistor description of the massless particle with fixed helicity.

Following [22], we choose the norms of twistors (21), (22) as

$$\bar{Z}^A Z_A \;=\; \omega^\alpha \pi_\alpha - \bar{\pi}_{\dot\alpha}\bar{\omega}^{\dot\alpha}, \qquad \bar{Y}^A Y_A \;=\; \eta^\alpha \rho_\alpha - \bar{\rho}_{\dot\alpha}\bar{\eta}^{\dot\alpha}, \tag{23}$$

and write the constraint (12) in concise form

$$\mathcal{U} \;=\; i\,(\bar{Z}^A Z_A + \bar{Y}^A Y_A) \;\approx\; 0. \tag{24}$$

The norm $\bar{Z}^A Z_A$ of the twistor Z commutes with constraints (10), (11), (12) and therefore is independent of τ. For a massless particle with fixed helicity the norm $\bar{Z}^A Z_A$ defines the helicity operator

$$\Lambda = \frac{i}{2}\,\bar{Z}^A Z_A. \tag{25}$$

So in the considered model of the infinite (continuous) spin particle, in view of the constraint (24), the particle helicity is not fixed since it is proportional to $-\bar{Y}^A Y_A$.

Now we consider Wigner-Bargmann space-time formulation [3–5] of the irreducible infinite integer-spin massless representation. This formulation can be realized by means of quantization of the particle model with the following Lagrangian

$$\begin{aligned} \mathcal{L}_{sp.-time} = {} & p_m \dot{x}^m \;+\; q_m \dot{y}^m \;+\; e\, p_m p^m \\ & + e_1\, p_m q^m \;+\; e_2\, \left(q_m q^m + \mu^2\right) \;+\; e_3\, \left(p_m y^m - 1\right), \end{aligned} \tag{26}$$

where $p_n(\tau)$ and $q_n(\tau)$ are momenta canonically conjugated to coordinates $x_n(\tau)$ and $y_n(\tau)$, respectively. The Lagrangian (26) yields the canonical Poisson brackets

$$\left\{x^m, p_n\right\} = \delta^m_n, \qquad \left\{y^m, q_n\right\} = \delta^m_n$$

and first-class constraints

$$T := p_m p^m \;\approx\; 0, \qquad T_1 := p_m q^m \;\approx\; 0,$$
$$T_2 := q_m q^m + \mu^2 \;\approx\; 0, \qquad T_3 := p_m y^m - 1 \;\approx\; 0, \tag{27}$$

which correspond to the Wigner-Bargmann equations (5). The functions $e(\tau)$, $e_1(\tau)$, $e_2(\tau)$ and $e_3(\tau)$ are Lagrange multipliers for the constraints (27). Nonvanishing Poisson brackets of the constraints (27) are $\{T_1, T_3\} = -T$ and $\{T_2, T_3\} = -2T_1$, i.e. the algebra of the constraints is nonabelian.

The action $S_{sp.-time} = \int d\tau \mathcal{L}_{sp.-time}$ is invariant under the transformations which are generated by quantities

$$P_m = p_m, \qquad M_{mn} = (x_m p_n - x_n p_m + y_m q_n - y_n q_m).$$

These charges form the classical analog of the Poincaré algebra (1) with respect to Poisson brackets. We see that additional coordinates y^m in the arguments of these fields play the role of spin variables.

Now by making use of constraints $T \approx 0$, $T_1 \approx 0$, $T_2 \approx 0$, $T_3 \approx 0$ we obtain relations

$$P_m P^m \approx 0, \qquad W_m W^m = \frac{1}{2} M_{nk} M^{nk} P_m P^m - M_{mk} M^{nl} P^k P_l \approx -\mu^2.$$

where $W_m = \frac{1}{2}\,\varepsilon_{mnkl} P^n M^{kl}$ are the components of the Pauli-Lubanski pseudovector. Therefore, the model with Lagrangian $\mathcal{L}_{sp.-time}$ indeed describes the massless particle with continuous spin. We note that vectors q_m and $W_m = \varepsilon_{mnkl} p^n y^l q^l$ do not coincide to each other and components W_m strictly speaking are not canonically conjugated to y_m.

After canonical quantization the constraints (27) yield the Wigner-Bargmann equations (5) for the continuous spin fields $\Phi(x, y)$.

Proposition 1. *The Wigner-Bargmann space-time (26) and twistorial (9) formulations of the infinite (continuous) spin particle are equivalent on the classical level by means of the generalized Cartan-Penrose relations [22]*

$$p_{\alpha\dot\beta} = \pi_\alpha \bar\pi_{\dot\beta}, \qquad q_{\alpha\dot\beta} = \pi_\alpha \bar\rho_{\dot\beta} + \rho_\alpha \bar\pi_{\dot\beta}, \tag{28}$$

and by the following generalized incidence relations [22]:

$$\omega^\alpha = \bar\pi_{\dot\alpha} x^{\dot\alpha\alpha} + \bar\rho_{\dot\alpha} y^{\dot\alpha\alpha}, \qquad \bar\omega^{\dot\alpha} = x^{\dot\alpha\alpha}\pi_\alpha + y^{\dot\alpha\alpha}\rho_\alpha, \tag{29}$$

$$\eta^\alpha = \bar\pi_{\dot\alpha} y^{\dot\alpha\alpha}, \qquad \bar\eta^{\dot\alpha} = y^{\dot\alpha\alpha}\pi_\alpha. \tag{30}$$

The proof of this ***Proposition*** is straightforward and is given in [1, 2].

3 Quantization of the Twistorial Model and Twistor Field of the Infinite Spin Particle

Quantization of the model is vastly simplified if we introduce new spinorial variables by means of Bogolyubov canonical transformations (cf. gauge transformations (13), (14)):

$$\begin{pmatrix} \pi_1 & \rho_1 \\ \pi_2 & \rho_2 \end{pmatrix} = \sqrt{M} \begin{pmatrix} p^{(z)}_1 & 0 \\ p^{(z)}_2 & p^{(s)}/p^{(z)}_1 \end{pmatrix} \begin{pmatrix} 1 & p^{(t)} \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \eta_1 & \omega_1 \\ \eta_2 & \omega_2 \end{pmatrix} = \begin{pmatrix} 0 & z_1/\sqrt{M} \\ -t/\pi_1! & z_2/\sqrt{M} \end{pmatrix} \begin{pmatrix} 1 & -p^{(t)} \\ 0 & 1 \end{pmatrix} + \frac{s}{M} \begin{pmatrix} \pi_1 & \rho_1 \\ \pi_2 & \rho_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $M := \mu/\sqrt{2}$ and new variables are defined by relations

$$\begin{gathered} p^{(z)}_\alpha = \pi_\alpha/\sqrt{M}, \qquad p^{(s)} = \pi^\alpha \rho_\alpha/M, \qquad p^{(t)} = \rho_1/\pi_1, \\ \omega^\alpha = \frac{1}{\sqrt{M}}\, z^\alpha - \frac{1}{M}\, s\, \rho^\alpha - \frac{\delta^{\alpha 1}}{\pi_1}\, t\, p^{(t)}, \qquad \eta^\alpha = \frac{1}{M}\, s\, \pi^\alpha + \frac{\delta^{\alpha 1}}{\pi_1}\, t. \end{gathered} \tag{31}$$

The nonzero canonical Poisson brackets of the new variables and their complex conjugated variables $\bar z^{\dot\alpha}$, $\bar s$, $\bar t$, $\bar p^{(z)}_{\dot\alpha}$, $\bar p^{(s)}$, $\bar p^{(t)}$ are

$$\left\{ z^\alpha, p^{(z)}_\beta \right\} = \delta^\alpha_\beta, \quad \left\{ \bar z^{\dot\alpha}, \bar p^{(z)}_{\dot\beta} \right\} = \delta^{\dot\alpha}_{\dot\beta},$$

$$\left\{ s, p^{(s)} \right\} = \left\{ \bar s, \bar p^{(s)} \right\} = 1, \quad \left\{ t, p^{(t)} \right\} = \left\{ \bar t, \bar p^{(t)} \right\} = 1.$$

In terms of new variables (31) the constraints (10), (11), (12) of spinorial model (9) take very simple form

$$\mathcal{M}' := p^{(s)} \bar p^{(s)} - 1 \approx 0, \tag{32}$$

$$\mathcal{F}' := t - 1 \approx 0, \qquad \bar{\mathcal{F}}' := \bar t - 1 \approx 0, \tag{33}$$

$$\mathcal{U}' := \frac{i}{2} \left(z^\alpha p^{(z)}_\alpha - \bar z^{\dot\alpha} \bar p^{(z)}_{\dot\alpha} \right) + i \left(s p^{(s)} - \bar s \bar p^{(s)} \right) \approx 0. \tag{34}$$

After canonical quantization $[.,.] = i\,\{.,.\}$ these constraints turn into equations of motion

$$\left(p^{(s)} \bar p^{(s)} - 1 \right) \Psi^{(c)} = 0, \tag{35}$$

$$\frac{\partial}{\partial p^{(t)}}\, \Psi^{(c)} = \frac{\partial}{\partial \bar p^{(t)}}\, \Psi^{(c)} = -i\, \Psi^{(c)}, \tag{36}$$

$$\left[\frac{1}{2} \left(p^{(z)}_\alpha \frac{\partial}{\partial p^{(z)}_\alpha} - \bar p^{(z)}_{\dot\alpha} \frac{\partial}{\partial \bar p^{(z)}_{\dot\alpha}} \right) + p^{(s)} \frac{\partial}{\partial p^{(s)}} - \bar p^{(s)} \frac{\partial}{\partial \bar p^{(s)}} \right] \Psi^{(c)} = c\, \Psi^{(c)}, \tag{37}$$

where differential operators in their left hand sides are quantum counterparts of the constraints (32), (33), (34). In equations (35), (36), (37) wave function (or spinorial field)

$$\Psi^{(c)}\big(p^{(z)}_{\alpha},\ \bar{p}^{(z)}_{\dot{\alpha}};\ p^{(s)},\ \bar{p}^{(s)};\ p^{(t)},\ \bar{p}^{(t)}\big),$$

is taken in "momentum representation" and describes physical states, which form the space of irreducible representation of Poincaré group with continues spin. The constant c is related to the ambiguity of operator ordering in Eq. (37). In other words, constant c is an analog of the vacuum energy in the quantum oscillator model.

Equations of motion (35), (36) can be solved explicitly in the form

$$\Psi^{(c)} \;=\; \delta\left(p^{(s)}\cdot\bar{p}^{(s)}-1\right)\; e^{-i(p^{(t)}+\bar{p}^{(t)})}\sum_{k=-\infty}^{\infty} e^{-ik\varphi}\,\tilde{\psi}^{(c+k)}\big(p^{(z)},\bar{p}^{(z)}\big), \tag{38}$$

where $e^{i\varphi} := (p^{(s)}/\bar{p}^{(s)})^{1/2}$. Due to the constraint (37) the coefficient functions $\tilde{\psi}^{(c+k)}(p_z,\bar{p}_z)$ satisfy the equations

$$\frac{1}{2}\left(p^{(z)}_{\alpha}\frac{\partial}{\partial p^{(z)}_{\alpha}} \;-\; \bar{p}^{(z)}_{\dot{\alpha}}\frac{\partial}{\partial \bar{p}^{(z)}_{\dot{\alpha}}}\right)\tilde{\psi}^{(c+k)} = \big(c+k\big)\,\tilde{\psi}^{(c+k)}. \tag{39}$$

Now we can restore the dependence of the wave function (38) on the twistor variables. As result we obtain the following statement.

Proposition 2. *The twistor wave function which is general solution of the equations of motion (35), (36), (37) is represented in the form*

$$\Psi^{(c)}(\pi,\bar{\pi};\rho,\bar{\rho}) \;=\; \delta\left((\pi\rho)(\bar{\rho}\bar{\pi})-M^2\right)\, e^{-i\left(\frac{\rho_1}{\pi_1}+\frac{\bar{\rho}_1}{\bar{\pi}_1}\right)}\,\hat{\Psi}^{(c)}(\pi,\bar{\pi};\rho,\bar{\rho}), \tag{40}$$

where we make use the shorthand notation $(\pi\rho) := \pi^{\beta}\rho_{\beta}$, $(\bar{\rho}\bar{\pi}) := \bar{\rho}_{\dot{\beta}}\bar{\pi}^{\dot{\beta}}$ *and*

$$\hat{\Psi}^{(c)}(\pi,\bar{\pi};\rho,\bar{\rho}) = \psi^{(c)}(\pi,\bar{\pi}) + \sum_{k=1}^{\infty}(\bar{\rho}\bar{\pi})^k\,\psi^{(c+k)} + \sum_{k=1}^{\infty}(\pi\rho)^k\,\psi^{(c-k)}. \tag{41}$$

The coefficient functions $\psi^{(c\pm k)}\big(\pi,\bar{\pi}\big)$ *obey the condition*

$$\Lambda\cdot\psi^{(c\pm k)}(\pi,\bar{\pi}) \;=\; -\big(c\pm k\big)\,\psi^{(c\pm k)}(\pi,\bar{\pi}), \tag{42}$$

where $\Lambda = -\frac{1}{2}\left(\pi_{\alpha}\frac{\partial}{\partial\pi_{\alpha}} \;-\; \bar{\pi}_{\dot{\alpha}}\frac{\partial}{\partial\bar{\pi}_{\dot{\alpha}}}\right)$ *is the helicity operator.*

Equations of motion (35), (36), (37) are written in terms of twistor variables as following

$$i\,\pi_\alpha \frac{\partial}{\partial\rho_\alpha}\,\Psi^{(c)} = \Psi^{(c)}, \qquad i\,\bar\pi_{\dot\alpha}\frac{\partial}{\partial\bar\rho_{\dot\alpha}}\,\Psi^{(c)} = \Psi^{(c)}, \tag{43}$$

$$\left(\pi_\alpha\frac{\partial}{\partial\pi_\alpha} \;-\; \bar\pi_{\dot\alpha}\frac{\partial}{\partial\bar\pi_{\dot\alpha}} \;+\; \rho_\alpha\frac{\partial}{\partial\rho_\alpha} \;-\; \bar\rho_{\dot\alpha}\frac{\partial}{\partial\bar\rho_{\dot\alpha}}\right)\Psi^{(c)} \;=\; 2c\,\Psi^{(c)}. \tag{44}$$

To describe the bosonic infinite spin representation related to all integer helicities, we put (in view of condition (42)) $c = 0$ and therefore consider the twistorial field $\Psi^{(0)}(\pi,\bar\pi;\rho,\bar\rho)$. Similarly, to describe the infinite spin representation related to half-integer helicities we take $c = -\frac{1}{2}$. According to the condition (42), the corresponding wave function $\Psi^{(-1/2)}(\pi,\bar\pi;\rho,\bar\rho)$ contains in its expansion only half-integer helicities. Note that the complex conjugate field $\bar\Psi^{(0)}$ also has zero charge $c = 0$, but the complex conjugate field $\bar\Psi^{(+1/2)}(\pi,\bar\pi;\rho,\bar\rho)$ has the opposite charge $c = +1/2$.

Proposition 3. *The twistor wave function $\Psi^{(c)}(\pi,\bar\pi;\rho,\bar\rho)$, defined in Proposition 2, describes the massless particle of the infinite (continuous) spin:*

$$W^{\alpha\dot\gamma}W_{\alpha\dot\gamma}\cdot\Psi^{(c)} \;=\; -\mu^2\,\Psi^{(c)}, \tag{45}$$

where $W_{\alpha\dot\gamma} = \frac{1}{\sqrt{2}}W_m(\sigma^m)_{\alpha\dot\gamma}$ is the Pauli-Lubański operator

$$\begin{aligned} W_{\alpha\dot\gamma} &= \pi_\alpha\bar\pi_{\dot\gamma}\,\Lambda + \frac{1}{2}\left[\pi_\alpha\bar\rho_{\dot\gamma}\Big(\bar\pi_{\dot\beta}\,\frac{\partial}{\partial\bar\rho_{\dot\beta}}\Big) - \rho_\alpha\bar\pi_{\dot\gamma}\Big(\pi_\beta\,\frac{\partial}{\partial\rho_\beta}\Big)\right] \\ &\quad + \frac{1}{2}\left[(\bar\rho\bar\pi)\,\pi_\alpha\,\frac{\partial}{\partial\bar\rho^{\dot\gamma}} - (\pi\rho)\,\bar\pi_{\dot\gamma}\,\frac{\partial}{\partial\rho^\alpha}\right]. \end{aligned} \tag{46}$$

Proof. See [1] and [2].

In conclusion of this Section we stress once again that the twistorial wave function $\Psi^{(c)}$ is complex and therefore all component fields $\psi^{(c\pm k)}(\pi,\bar\pi)$ in its expansion are also complex. In view of this we must consider together with the field $\Psi^{(c)}$ its complex conjugated field $(\Psi^{(c)})^* := \bar\Psi^{(-c)}$ which has the opposite charge $c \to -c$.

4 Twistor Transform for Infinite Spin Fields

Here we establish a correspondence between twistor fields in momentum representation and fields defined in the four-dimensional Minkowski space-time.

For further convenience we introduce the dimensionless spinor

$$\xi_\alpha := M^{-1/2}\rho_\alpha, \qquad \bar\xi_{\dot\alpha} := M^{-1/2}\bar\rho_{\dot\alpha}.$$

Then, the twistor wave function $\Psi^{(c)}$ of infinite integer-spin particle (40) for $c = 0$ can be represented in the form [2]

$$\Psi^{(0)}(\pi,\bar\pi;\xi,\bar\xi) = \delta\left((\pi\xi)(\bar\xi\bar\pi) - M\right) e^{-iq_0/p_0}\,\hat\Psi^{(0)}(\pi,\bar\pi;\xi,\bar\xi)\,, \tag{47}$$
$$\hat\Psi^{(0)} = \psi^{(0)}(\pi,\bar\pi) + \sum_{k=1}^{\infty}(\bar\xi\bar\pi)^k\,\psi^{(k)}(\pi,\bar\pi) + \sum_{k=1}^{\infty}(\pi\xi)^k\,\psi^{(-k)}(\pi,\bar\pi).$$

In the expansion of $\hat\Psi^{(0)}$, all components $\psi^{(k)}(\pi,\bar\pi)$ ($k \in \mathbf{Z}$) in general are complex functions (fields). Moreover, the quantity p_0/q_0 is expressed by means of the generalized Cartan-Penrose representations (28) in spinorial form as

$$\frac{q_0}{p_0} = \frac{\sqrt{M}\sum\limits_{\alpha=\dot\alpha}(\pi_\alpha\bar\xi_{\dot\alpha} + \xi_\alpha\bar\pi_{\dot\alpha})}{\sum\limits_{\beta=\dot\beta}\pi_\beta\bar\pi_{\dot\beta}}.$$

In the case $c = -1/2$, the wave function of the infinite half-integer spin particle is

$$\Psi^{(-\frac12)}(\pi,\bar\pi;\xi,\bar\xi) = \delta\left((\pi\xi)(\bar\xi\bar\pi) - M\right) e^{-iq_0/p_0}\,\hat\Psi^{(-\frac12)}(\pi,\bar\pi;\xi,\bar\xi), \tag{48}$$
$$\hat\Psi^{(-\frac12)} = \psi^{(-\frac12)}(\pi,\bar\pi) + \sum_{k=1}^{\infty}(\bar\xi\bar\pi)^k\,\psi^{(-\frac12+k)}(\pi,\bar\pi) + \sum_{k=1}^{\infty}(\pi\xi)^k\,\psi^{(-\frac12-k)}(\pi,\bar\pi).$$

The expansion of the complex conjugated wave function $\bar\Psi^{(+\frac12)}$ has the form

$$\bar\Psi^{(+\frac12)}(\pi,\bar\pi;\xi,\bar\xi) = \delta\left((\pi\xi)(\bar\xi\bar\pi) - M\right) e^{iq_0/p_0}\,\hat{\bar\Psi}^{(+\frac12)}(\pi,\bar\pi;\xi,\bar\xi), \tag{49}$$
$$\hat{\bar\Psi}^{(+\frac12)} = \bar\psi^{(\frac12)}(\pi,\bar\pi) + \sum_{k=1}^{\infty}(\bar\xi\bar\pi)^k\,\bar\psi^{(\frac12+k)}(\pi,\bar\pi) + \sum_{k=1}^{\infty}(\pi\xi)^k\,\bar\psi^{(\frac12-k)}(\pi,\bar\pi),$$

where the component fields $\bar\psi^{(r)}(\pi,\bar\pi)$ are complex conjugation of the component fields $\psi^{(-r)}(\pi,\bar\pi)$:

$$\left(\psi^{(-\frac12+k)}\right)^* = \bar\psi^{(\frac12-k)}, \qquad k \in \mathbf{Z}.$$

In the case of integer spins the U(1)-charge is zero, $c = 0$, and the space-time wave function is determined by means of the integral Fourier transformation of twistor field $\Psi^{(0)}(\pi,\bar\pi;\xi,\bar\xi)$:

$$\Phi(x;\xi,\bar\xi) = \int d^4\pi\, e^{\,i\pi_\alpha\bar\pi_{\dot\alpha}x^{\dot\alpha\alpha}}\,\Psi^{(0)}(\pi,\bar\pi;\xi,\bar\xi), \tag{50}$$

where we have used the representation $p_{\alpha\dot\alpha} = \pi_\alpha \bar\pi_{\dot\alpha}$ and perform integration over the measure $d^4\pi := \frac{1}{2}\, d\pi_1 \wedge d\pi_2 \wedge d\bar\pi_{\dot 1} \wedge d\bar\pi_{\dot 2} = d\phi\; d^4p\; \delta(p^2)$ (here ϕ is common phase in π_α which is not presented in $p_{\alpha\dot\alpha} = \pi_\alpha\bar\pi_{\dot\alpha}$).

Proposition 4. *The field $\Phi(x;\xi,\bar\xi)$ defined by the integral transformation (50) in coordinate representation satisfies four equations*

$$\begin{aligned} &\partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\, \Phi(x;\xi,\bar\xi) = 0, \qquad \left(i\frac{\partial}{\partial\xi_\alpha}\partial_{\alpha\dot\alpha}\frac{\partial}{\partial\bar\xi_{\dot\alpha}} - M\right)\Phi(x;\xi,\bar\xi) = 0, \\ &\left(i\xi^\alpha\partial_{\alpha\dot\alpha}\bar\xi^{\dot\alpha} + M\right)\Phi(x;\xi,\bar\xi) = 0, \qquad \left(\xi_\alpha\frac{\partial}{\partial\xi_\alpha} - \bar\xi_{\dot\alpha}\frac{\partial}{\partial\bar\xi_{\dot\alpha}}\right)\Phi(x;\xi,\bar\xi) = 0. \end{aligned} \tag{51}$$

Proof. Make use the integral transformation (50) and equations of motion (43), (44) for $c = 0$.

In the case of half-integer spins the U(1)-charge equals $c = -1/2$. Then we use the standard prescription of the twistorial definition of space-time fields with nonvanishing helicities. Namely, we have to insert the twistorial spinor π_α in the integrand in the Fourier transformation:

$$\Phi_\alpha(x;\xi,\bar\xi) = \int d^4\pi\; e^{\,i\pi_\beta\bar\pi_{\dot\beta}x^{\dot\beta\beta}}\, \pi_\alpha\, \Psi^{(-1/2)}(\pi,\bar\pi;\xi,\bar\xi), \tag{52}$$

and obtain the external spinor index α. Then the complex conjugate twistorial field with charge $c = +1/2$ is defined analogously

$$\bar\Phi_{\dot\alpha}(x;\xi,\bar\xi) \;=\; \int d^4\pi\; e^{\,-i\pi_\beta\bar\pi_{\dot\beta}x^{\dot\beta\beta}}\, \bar\pi_{\dot\alpha}\, \bar\Psi^{(+1/2)}(\pi,\bar\pi;\xi,\bar\xi). \tag{53}$$

Proposition 5. *The space-time fields $\Phi_\alpha(x;\xi,\bar\xi)$ and $\bar\Phi_{\dot\alpha}(x;\xi,\bar\xi)$, which correspond to the states with half-integer helicities, satisfy massless Dirac-Weyl equations*

$$\partial^{\dot\alpha\alpha}\,\Phi_\alpha(x;\xi,\bar\xi) \;=\; 0, \qquad \partial^{\dot\alpha\alpha}\,\bar\Phi_{\dot\alpha}(x;\xi,\bar\xi) \;=\; 0, \tag{54}$$

and integer spin equations:

$$\begin{aligned} &\left(i\xi^\beta\partial_{\beta\dot\beta}\bar\xi^{\dot\beta} + M\right)\Phi_\alpha(x;\xi,\bar\xi) = 0, \qquad \left(i\xi^\beta\partial_{\beta\dot\beta}\bar\xi^{\dot\beta} - M\right)\bar\Phi_{\dot\alpha}(x;\xi,\bar\xi) = 0, \\ &\left(i\frac{\partial}{\partial\xi_\beta}\partial_{\beta\dot\beta}\frac{\partial}{\partial\bar\xi_{\dot\beta}} - M\right)\Phi_\alpha(x;\xi,\bar\xi) = 0, \quad \left(i\frac{\partial}{\partial\xi_\beta}\partial_{\beta\dot\beta}\frac{\partial}{\partial\bar\xi_{\dot\beta}} + M\right)\bar\Phi_{\dot\alpha}(x;\xi,\bar\xi) = 0, \\ &\left(\xi_\beta\frac{\partial}{\partial\xi_\beta} - \bar\xi_{\dot\beta}\frac{\partial}{\partial\bar\xi_{\dot\beta}}\right)\Phi_\alpha(x;\xi,\bar\xi) \;=\; 0\,, \quad \left(\xi_\beta\frac{\partial}{\partial\xi_\beta} - \bar\xi_{\dot\beta}\frac{\partial}{\partial\bar\xi_{\dot\beta}}\right)\bar\Phi_{\dot\alpha}(x;\xi,\bar\xi) \;=\; 0. \end{aligned} \tag{55}$$

Proof. Make use the integral transformations (52), (53) and equations of motion (43), (44) for $c = \pm 1/2$.

We stress that although the fields $\Psi^{(-1/2)}(\pi,\bar\pi;\xi,\bar\xi)$ and $\bar\Psi^{(+1/2)}(\pi,\bar\pi;\xi,\bar\xi)$ have nonvanishing charges $c = \mp 1/2$, their integral transformed versions $\Phi_{\dot\alpha}(x;\xi,\bar\xi)$ and $\bar\Phi_{\dot\alpha}(x;\xi,\bar\xi)$ have zero U(1)-charge. This fact is crucial for forming infinite spin supermultiplets, as we will see below.

5 Infinite Spin Supermultiplet

We unify fields $\Phi(x;\xi,\bar\xi)$ and $\Phi_\alpha(x;\xi,\bar\xi)$ with integer and half-integer helicities into one supermultiplet. The fields $\Phi(x;\xi,\bar\xi)$ and $\Phi_\alpha(x;\xi,\bar\xi)$ contain the bosonic $\psi^{(k)}(\pi,\bar\pi)$ and fermionic $\psi^{(k-1/2)}(\pi,\bar\pi)$ component fields ($k \in \mathbf{Z}$) with all integer and half-integer spins, respectively.

As in the case of the Wess-Zumino supermultiplet (see, e.g., [23, 24]), we write supersymmetry transformations of the fields Φ and Φ_α in the form

$$\delta\,\Phi \;=\; \varepsilon^\alpha \Phi_\alpha\,, \qquad \delta\,\Phi_\alpha \;=\; 2i\bar\varepsilon^{\dot\beta}\partial_{\alpha\dot\beta}\Phi\,, \tag{56}$$

where $\varepsilon_\alpha, \bar\varepsilon_{\dot\alpha}$ are the constant odd Weyl spinors. Applying these transformations twice we obtain

$$\begin{aligned}(\delta_1\delta_2-\delta_2\delta_1)\,\Phi \;&=-2ia^{\beta\dot\beta}\partial_{\beta\dot\beta}\Phi,\\ (\delta_1\delta_2-\delta_2\delta_1)\,\Phi_\alpha &= -2ia^{\beta\dot\beta}\partial_{\beta\dot\beta}\Phi_\alpha + 2ia_{\alpha\dot\beta}\partial^{\dot\beta\beta}\Phi_\beta,\end{aligned} \tag{57}$$

where $a_{\alpha\dot\beta} \;:=\; \varepsilon_{1\alpha}\bar\varepsilon_{2\dot\beta} - \varepsilon_{2\alpha}\bar\varepsilon_{1\dot\beta}$. According to the Dirac-Weyl equations of motion (54), the commutators of variations in the left hand side of (57) are closed on-shell since in the right hand side of (57) we obtain generator of translations

$$P_{\beta\dot\beta} = -i\partial_{\beta\dot\beta}.$$

Moreover, one can show that the whole system of equations of motion (51), (54), (55) is invariant with respect to supersymmetry transformations (57).

Using the inverse integral Fourier transformations, we rewrite (57) as supersymmetry transformations for the fields $\Psi^{(0)}(\pi,\bar\pi;\xi,\bar\xi)$, $\Psi^{(-1/2)}(\pi,\bar\pi;\xi,\bar\xi)$ in the momentum representation:

$$\delta\,\Psi^{(0)} \;=\; \varepsilon^\alpha\pi_\alpha\Psi^{(-1/2)}, \qquad \delta\,\Psi^{(-1/2)} \;=\; -2\,\bar\varepsilon^{\dot\alpha}\pi_{\dot\alpha}\Psi^{(0)}, \tag{58}$$

or in terms of bosonic $\psi^{(k)}(\pi,\bar\pi)$ and fermionic $\psi^{(-\frac12+k)}(\pi,\bar\pi)$ twistorial components we have

$$\delta\,\psi^{(k)} \;=\; \varepsilon^\alpha\pi_\alpha\,\psi^{(-\frac12+k)}\,, \qquad \delta\,\psi^{(-\frac12+k)} \;=\; -2\,\bar\varepsilon^{\dot\alpha}\pi_{\dot\alpha}\,\psi^{(k)}, \qquad \forall k\in\mathbf{Z}. \tag{59}$$

Recall that bosonic field $\psi^{(k)}$ and fermionic field $\psi^{(-\frac{1}{2}+k)}$ at fixed $k \in \mathbf{Z}$ describe massless states with helicities $(-k)$ and $(\frac{1}{2}-k)$, respectively. Thus, the infinite-component supermultiplet of the infinite spin splits into an infinite number of $\mathcal{N}=1$ supermultiplets of the component fields $\psi^{(k)}$, $\psi^{(-\frac{1}{2}+k)}$ with fixed $k \in \mathbf{Z}$. However we stress that boosts of the Poincare group mix the fields with different values of k.

We point out that superfield description of infinite spin supermultiplet was considered in recent paper [25].

6 Conclusion

Let us summarize obtained results and discuss some open problems.

- We have presented the new twistorial formulation of the massless infinite spin particles and fields.
- We gave the helicity decomposition of twistorial infinite spin fields and constructed the field twistor transform to define the space-time infinite (continuous) spin fields $\Phi(x; \xi, \bar{\xi})$ and $\Phi_\alpha(x; \xi, \bar{\xi})$.
- As opposed to the Wigner-Bargmann space-time formulation [3–5] the space-time infinite spin fields $\Phi(x; \xi, \bar{\xi})$ and $\Phi_\alpha(x; \xi, \bar{\xi})$ depend on the Weyl spinor variables ξ_α, $\bar{\xi}_{\dot{\alpha}}$.
- The use of auxiliary spinor variables ξ_α, $\bar{\xi}_{\dot{\alpha}}$ instead of vector variables allowed us to describe massless irreducible representations of infinite half-integer spin without introducing additional Grassmann variables.
- We found the equations of motion for $\Phi(x; \xi, \bar{\xi})$ and $\Phi_\alpha(x; \xi, \bar{\xi})$ and showed that these fields form the $\mathcal{N}=1$ infinite spin supermultiplet.
- A natural question arises about status of such fields in Lagrangian field theory and also about possibility to construct self-consistent interaction of such fields. One of the commonly used methods for this purpose is the BRST approach, which was used in the case of continuous spin particles in [10, 16, 17]. In a recent paper [18] the covariant Lagrangian formulation of the infinite integer-spin field was constructed by using the methods developed in [26]. A natural continuation of our research is the construction of the BRST Lagrangian formulation of infinite half-integer spin representation in the framework of the approach developed here.

Acknowledgements We acknowledge the partial support of the Ministry of Science and High Education of Russian Federation, project No. 3.1386.2017. I.L.B. acknowledges the support of the Russian Foundation for Basic Research, project No. 18-02-00153. A.P.I. acknowledges the support of the Russian Foundation for Basic Research, project No. 19-01-00726-a.

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Supersymmetric Calogero-Type Models via Gauging in Superspace

Evgeny Ivanov

Abstract A new kind of $\mathscr{N}=2$ and $\mathscr{N}=4$ supersymmetric extensions of both the rational and hyperbolic Calogero models is derived by gauging $U(n)$ symmetry of the appropriate superfield matrix models. The common feature of these systems is the non-standard number $\mathscr{N}n^2$ of fermionic variables. An essential ingredient of the construction of $\mathscr{N}=4$ models is the semi-dynamical spin variables described by $d=1$ Wess-Zumino terms. The bosonic cores of $\mathscr{N}=4$ models are $U(2)$ spin Calogero and Calogero-Sutherland models. In the hyperbolic case there exist two non-equivalent $\mathscr{N}=4$ extensions, with and without the interacting center-of-mass coordinate in the bosonic sector.

1 Motivations and Contents

Calogero-type models (CM) [1] (see also [2]) are notorious text-book examples of integrable $d=1$ systems. Most known is the rational n-particle Calogero model

$$S^c = \int dt \Big[\sum_a \dot{x}_a \dot{x}_a - \sum_{a\neq b} \frac{c^2}{4(x_a - x_b)^2} \Big]. \tag{1}$$

The integrable Calogero-Moser system corresponds to adding the oscillator-type term $\sim \sum_{a\neq b}(x_a - x_b)^2$ to (1). The rational CM models are conformal: the action (1) is invariant under $d=1$ conformal group $SO(1,2)$

$$\delta t = \alpha, \;\; \delta x_a = \frac{1}{2}\dot{\alpha} x_a, \;\;\; \partial_t^3 \alpha = 0.$$

Conformal CM models can be closely related to black holes and M-theory [3]. Besides conformal models, there exist other integrable CM-type models, e.g., the trigonometric and hyperbolic Calogero-Sutherland systems, the latter being presented by the $d=1$ action

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_7

$$S^{cs} = \int dt \Big[\sum_a \dot{x}_a \dot{x}_a - \sum_{a \neq b} \frac{c^2}{4 \sinh^2 \frac{x_a - x_b}{2}} \Big]. \tag{2}$$

The first version of $\mathscr{N} = 2$ superextended CM model was constructed in [4]: each bosonic coordinate x_a was enlarged to the supermultiplet $(\mathbf{1}, \mathbf{2}, \mathbf{1})$, i.e. n bosonic particles were completed by $2n$ fermionic $d = 1$ fields. The appropriate $\mathscr{N} = 2, d = 1$ superfield action can be shown to yield the rational CM model in the limit of vanishing fermions. Analogously, $\mathscr{N} = 2$ extension of the Calogero-Sutherland models can be constructed. An important role in producing the correct pairwise potential terms in the bosonic limit is played by auxiliary fields of the supermultiplets $(\mathbf{1}, \mathbf{2}, \mathbf{1})$.

Higher $\mathscr{N}$ extensions meet some problems [5–7]. In $\mathscr{N} = 4$ case, x_a should be enlarged to the supermultiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$, i.e. there are present n bosonic and $4n$ fermionic fields of physical dimension. It is very difficult to construct the appropriate superfield action yielding the n-particle Calogero potential in the bosonic sector. There appear a few functions of x_a related by the complicated WDVV [8, 9] equations the explicit solutions of which are known only for a few lowest values of n.

No universal convenient method was suggested so far for constructing the "standard" $\mathscr{N} n$-extended supersymmetric CM systems with $\mathscr{N} > 2$.

Fortunately, exists another type of supersymmetrization, such that the above-mentioned problems do not arise. The models constructed in this way are "non-minimal": they contain $\mathscr{N} n^2$ fermions for n bosonic coordinates [10–12]. The method for constructing such models generalizes the gauge approach to bosonic CM models developed earlier in [13–15]. One takes some simple free matrix model as the departure point and gauge the appropriate linear isometries by non-propagating $d = 1$ gauge fields. Eliminating gauge fields by their algebraic equations of motion leaves us with one or another CM model.

Generalization to the supersymmetry case is rather straightforward: one gauges the isometry of some free superfield matrix model by a gauge superfield. After passing to components and fixing a gauge, some supersymmetric CM model is recovered.

I will explain this approach on the examples of $\mathscr{N} = 2$ and $\mathscr{N} = 4$ superfield matrix models and show that some new versions of the supersymmetric CM models can be discovered in this way.

To a large extent, the talk is based on the papers [10–12].

2 Conformal Mechanics via Gauging

The well-known conformal mechanics model [16] is described by the action:

$$S_0 = \int dt \, \left(\dot{x}^2 - \gamma^2 x^{-2} \right) \equiv \int dt \, \mathscr{L}_0.$$

Let us demonstrate that this action can be recovered by gauging procedure applied to some another action. Consider the Lagrangian of free complex field $z(t)$

$$L_z = \dot{z}\,\dot{\bar{z}} + im\left(\dot{z}\bar{z} - z\dot{\bar{z}}\right).$$

It is obviously invariant under rigid phase transformations

$$z' = e^{-i\lambda}z, \quad \bar{z}' = e^{i\lambda}\bar{z}.$$

Then we gauge this abelian symmetry, $\lambda \to \lambda(t)$, and introduce $d = 1$ gauge field $A(t)$, so that $\dot{z} \to \dot{z} + iAz$:

$$L_z \to L_z^g = (\dot{z} + iAz)\,(\dot{\bar{z}} - iA\bar{z}) + im\left(\dot{z}\bar{z} - z\dot{\bar{z}} + 2iAz\bar{z}\right) + 2\gamma\, A,$$
$$A' = A + \dot{\lambda}.$$

Here, a "Fayet-Iliopoulos term" $\propto \gamma$ has been added. It is gauge invariant (up to a total derivative) by itself.

The next step is to impose the appropriate gauge:

$$z = \bar{z} \equiv q(t).$$

Plugging it back into L^g, we obtain:

$$L_z^g = (\dot{q} + iAq)\,(\dot{q} - iAq) + 2im\,Aq^2 + 2\gamma\,A = (\dot{q})^2 + A^2q^2 - 2mAq^2 + 2\gamma A.$$

The field $A(t)$ is a typical auxiliary field: it can be eliminated by its algebraic equation of motion:

$$\delta A: \qquad A = m - \gamma\, q^{-2}.$$

The final form of the gauge-fixed Lagrangian is

$$L^g \Rightarrow (\dot{q})^2 - \left(mq - \gamma q^{-1}\right)^2 = (\dot{q})^2 - m^2q^2 - \gamma^2 q^{-2} + 2\gamma m.$$

Up to an additive constant, this Lagrangian coincides with the mass-modified AFF. At $m = 0$, the standard conformal mechanics is recovered:

$$L^g_{(m=0)} = (\dot{q})^2 - \gamma^2\, q^{-2}.$$

The initial action $S_z = \int dt L_z$ at $m = 0$ is invariant under the conformal $SO(1,2)$ transformations $\delta t = f(t), \delta z = \frac{1}{2}\dot{f}\, z,\ (\partial_t)^3 f = 0$. The conformal invariance is preserved by the gauging procedure, if the gauge field $A(t)$ is assumed to transform as ∂_t, *i.e.* $\delta A(t) = -\dot{f}\; A(t)$.

3 Calogero and Calogero-Sutherland by Gauging

The generalization of the above setting to the CM case is accomplished as follows.

One starts from the $U(n)$ invariant free action of the system of $n \times n$ hermitian matrix field X_a^b and complex U(n)-spinor field $Z_a(t)$, $\bar{Z}^a = (\overline{Z_a})$, $a, b = 1, \ldots, n$. The $U(n)$ symmetry is gauged by n^2 hermitian gauge fields A_a^b. The resulting gauge invariant action reads

$$S_C = \int dt \left[\mathrm{tr}\,(\nabla X \nabla X) + \tfrac{i}{2}\,(\bar{Z}\nabla Z - \nabla\bar{Z} Z) + c\,\mathrm{tr} A \right], \tag{3}$$
$$\nabla X = \dot{X} + i[A, X], \qquad \nabla Z = \dot{Z} + iAZ, \qquad \nabla\bar{Z} = \dot{\bar{Z}} - i\bar{Z}A.$$

The last term (Fayet-Iliopoulos term) contains only $U(1)$ gauge field, c being a real constant. As the next step, one fixes $U(n)$ gauge in such a way that all non-diagonal components of X_a^b are gauged away

$$X_a{}^b = x_a \delta_a{}^b, \Rightarrow [X, A]_a{}^b = (x_a - x_b) A_a{}^b.$$

Just $n^2 - n$ gauge parameters have been used, but there still remains the residual abelian gauge subgroup $[U(1)]^n$, with local parameters $\varphi_a(t)$:

$$Z_a \to e^{i\varphi_a} Z_a, \;\; \bar{Z}^a \to e^{-i\varphi_a} \bar{Z}^a, \;\; A_a{}^a \to A_a{}^a - \dot{\varphi}_a \quad \text{(no sum with respect to } a).$$

The next gauge-fixing is as follows:

$$\bar{Z}^a = Z_a.$$

It leads to the gauge-fixed action in the form

$$S_C = \int dt \sum_{a,b} \left[\dot{x}_a \dot{x}_a + (x_a - x_b)^2 A_a{}^b A_b{}^a - Z_a Z_b A_a{}^b + c\, A_a{}^a \right]$$

Varying it with respect to the non-propagating gauge fields, we obtain

$$A_a{}^b = \frac{Z_a Z_b}{2(x_a - x_b)^2} \qquad \text{for } a \neq b,$$
$$Z_a Z_a = c \qquad \forall a \quad \text{(no sum with respect to a)}.$$

The diagonal entries A_a^a drop out from the action and, after substituting the explicit expressions for the remaining A_a^b, one obtains

$$S_C = \tfrac{1}{2} \int dt \left[\sum_a \dot{x}_a \dot{x}_a - \sum_{a \neq b} \frac{c^2}{(x_a - x_b)^2} \right],$$

which is the rational Calogero model action. The original action is conformal, so is the final Calogero action.

The Calogero-Sutherland model can be re-derived by the same techniques, the only difference is that the initial action should be of the nonlinear sigma-model type:

$$S_C = \int dt \left[\operatorname{tr}\left(X^{-1}\nabla X X^{-1}\nabla X\right) + \tfrac{i}{2}\left(\bar{Z}\nabla Z - \nabla\bar{Z} Z\right) + c \operatorname{tr} A \right]. \tag{4}$$

Passing through the same steps as in the rational case yields the gauge-fixed action in the form

$$S_C = \tfrac{1}{2} \int dt \Big[\sum_a \frac{\dot{x}_a \dot{x}_a}{(x_a)^2} - \sum_{a\neq b} \frac{x_a x_b c^2}{(x_a - x_b)^2} \Big].$$

Introducing the new variables as $x_a = e^{q_a}$ brings this action to the standard Calogero-Sutherland form

$$S^{cs} = \int dt \Big[\sum_a \dot{q}_a \dot{q}_a - \sum_{a\neq b} \frac{c^2}{4\sinh^2 \frac{q_a - q_b}{2}} \Big].$$

This action is not conformal, since the initial action lacks this symmetry.

4 $\mathcal{N} = 2$ Calogero and Calogero-Sutherland

Once again, both cases follow the same strategy, are defined on the same set of $d = 1$ superfields and differ only in the choice of the initial $\mathcal{N} = 2$ matrix model action to be gauged.

The starting point in the first case is the free $\mathcal{N} = 2, d = 1$ action of the $n \times n$ matrix hermitian superfield $\mathscr{X}_a{}^b(t, \theta, \bar{\theta}),\ a, b = 1, \ldots, n$, with each entry carrying $(\mathbf{1}, \mathbf{2}, \mathbf{1})$ multiplet, and of chiral $U(n)$-spinor superfield $\mathscr{Z}_a(t_R, \bar{\theta}),\ \bar{\mathscr{Z}}^a(t_L, \theta)$, $D\mathscr{Z}_a = 0$, $\bar{D}\bar{\mathscr{Z}}^a = 0$,

$$S_{sC}^{(N=2)} = \int dt d\theta d\bar{\theta} \left[\operatorname{tr}\left(\bar{D}\mathscr{X}\, D\mathscr{X}\right) + \tfrac{1}{2}\, \bar{\mathscr{Z}}\, \mathscr{Z} \right].$$

This action is evidently invariant under rigid $U(n)$ transformations acting as rotations of the fundamental and co-fundamental indices a, b.

In order to preserve the chiralities of $\mathscr{Z}_a$, $\bar{\mathscr{Z}}^a$, we gauge this global symmetry by chiral and anti-chiral superfield parameters λ and $\bar{\lambda}$,

$$\mathscr{Z}' = e^{i\lambda}\mathscr{Z}, \quad \bar{\mathscr{Z}}' = \bar{\mathscr{Z}}\, e^{-i\bar{\lambda}}, \quad \mathscr{X}' = e^{i\lambda}\, \mathscr{X}\, e^{-i\bar{\lambda}}.$$

In order to construct the gauge invariant action, one introduces the hermitian gauge superfield V,

$$e^{2V'} = e^{i\bar{\lambda}}\, e^{2V}\, e^{-i\lambda}.$$

Then, the gauge-covariantized action reads

$$\tilde{S}^{(N=2)}_{sC} = \int dt d^2\theta \left[\mathrm{tr}\left(\bar{\mathscr{D}}\mathscr{X}\, e^{2V}\, \mathscr{D}\mathscr{X}\, e^{2V}\right) + \tfrac{1}{2}\, \bar{\mathscr{Z}}\, e^{2V} \mathscr{Z} - c\, \mathrm{tr} V \right], \qquad (5)$$

where

$$\mathscr{D}\mathscr{X} = D\mathscr{X} + e^{-2V}(De^{2V})\, \mathscr{X}, \qquad \bar{\mathscr{D}}\mathscr{X} = \bar{D}\mathscr{X} - \mathscr{X}\, e^{2V}(\bar{D}e^{-2V}).$$

It can be shown that the original matrix action, as well as the final gauge-covariantized one, are invariant under the $\mathscr{N} = 2$ superconformal symmetry $SU(1, 1|1)$.

In the component expansions,

$$\mathscr{X} = X + \theta\Psi - \bar{\theta}\bar{\Psi} + \theta\bar{\theta}Y,\ \mathscr{Z} = Z + 2i\theta\Upsilon + i\theta\bar{\theta}\dot{Z},\ \bar{\mathscr{Z}} = \bar{Z} + 2i\bar{\theta}\bar{\Upsilon} - i\theta\bar{\theta}\dot{\bar{Z}},$$

all the fields, except for $2n^2$ fermionic ones,

$$\Psi_a^b, \quad \bar{\Psi}_b^a,$$

are auxiliary and can be eliminated by their equations of motion.

After choosing the standard Wess-Zumino gauge $V = \theta\bar{\theta}A(t)$, the component action takes the form

$$S^{WZ}_{sC} = \int dt \left[\mathrm{tr}\, \nabla X\, \nabla X + \tfrac{i}{2}\,(\bar{Z}\nabla Z - \nabla\bar{Z} Z) - c\, \mathrm{tr} A\ +\ i\, \mathrm{tr}(\bar{\Psi}\nabla\Psi - \nabla\bar{\Psi}\Psi) \right],$$

where

$$\nabla\Psi = \dot{\Psi} + i[\Psi, A], \quad \nabla\bar{\Psi} = \dot{\bar{\Psi}} + i[\bar{\Psi}, A]$$

and ∇Z, ∇X are given by the purely bosonic expressions presented earlier. In the limit of zero fermions, the standard gauge-invariant "pre-action" of the rational Calogero is recovered.

So we have gained a new $\mathscr{N} = 2$ extension of the n-particle Calogero model with n bosons and $2n^2$ fermions, as opposed to the standard $\mathscr{N} = 2$ super Calogero system of Freedman and Mende with only $2n$ fermions.

In terms of the physical variables, the component action reads

$$S_{sC}^{(N=2)} = \int dt \, \Big[\sum_a \dot{x}_a \dot{x}_a + i \, (\bar{\Psi}_a{}^b \dot{\Psi}_b{}^a - \dot{\bar{\Psi}}_a{}^b \Psi_b{}^a) - V \Big], \tag{6}$$

$$V = \sum_{a \neq b} \frac{4}{(x_a - x_b)^2} \Big(Z_a \bar{Z}^a Z_b \bar{Z}^b + 2\bar{Z}^a \{\Psi, \bar{\Psi}\}_a{}^b Z_b + \{\Psi, \bar{\Psi}\}_a{}^b \{\Psi, \bar{\Psi}\}_b{}^a \Big).$$

The constraint on Z (after fixing the gauge $Z_a = \bar{Z}^a$) also essentially involves fermions:

$$(Z_a)^2 = c - R_a, \quad R_a \equiv \{\Psi, \bar{\Psi}\}_a{}^a = \sum_b (\Psi_a{}^b \bar{\Psi}_b{}^a + \bar{\Psi}_a{}^b \Psi_b{}^a),$$
$$(R_a)^{2n-1} \equiv 0 \text{ (nilpotency)}.$$

It is as yet unclear how to treat this huge amount of fermionic fields, and whether this number could be reduced by imposing some extra (perhaps, fermionic) gauge invariance. More detailed study of the fermionic sectors of such models was recently performed in [17, 18].

Like in the bosonic case, $\mathcal{N} = 2$ CS system is obtained by proceeding from a sigma-model type gauged action

$$S_{cs}^{N=2} = \frac{1}{2} \int \mathrm{d}t \mathrm{d}^2\theta \left[\mathrm{tr}\big(\mathscr{X}^{-1} \bar{\mathscr{D}} \mathscr{X} \, \mathscr{X}^{-1} \mathscr{D} \mathscr{X} \big) - \bar{\mathscr{Z}} \, e^{2V} \, \mathscr{Z} + 2c \, \mathrm{tr} V \right]. \tag{7}$$

After passing through the same steps as before, we obtain the component action

$$S^{N=2} = \int \mathrm{d}t \, L^{N=2},$$
$$L^{N=2} = \frac{1}{2} \mathrm{tr}\big(X^{-1} \nabla X \, X^{-1} \nabla X \big) + \frac{i}{2} \big(\bar{Z} \nabla Z - \nabla \bar{Z} Z \big) + c \, \mathrm{tr} A$$
$$+ \frac{i}{2} \mathrm{tr}\big(X^{-1} \bar{\Psi} X^{-1} \nabla \Psi - X^{-1} \nabla \bar{\Psi} X^{-1} \Psi \big)$$
$$- \frac{1}{4} \mathrm{tr}\big(X^{-1} \bar{\Psi} X^{-1} \bar{\Psi} X^{-1} \Psi X^{-1} \Psi \big).$$

In the bosonic limit it yields the gauge-invariant CS action. A new interesting feature of the hyperbolic case as compared to $\mathcal{N} = 2$ rational Calogero, is the appearance of the quartic fermionic term in the gauge-covariantized action.

5 $\mathcal{N}=4$ Calogero system

The $\mathcal{N}=4$ model is of special interest because the same gauge procedure applied to it results in the $U(2)$ spin Calogero systems. The reason is that the additional multiplets $Z, \bar{Z}$ now cannot be entirely eliminated by using the gauge freedom and/or the constraints, as earlier. What remains is just a sort of the target space $U(2)$ harmonics.

The universal superfield approach to $\mathcal{N}=4$ mechanics, both one-particle and multi-particle, is $\mathcal{N}=4, d=1$ harmonic superspace (HSS) [19], which is the $d=1$ version of $\mathcal{N}=2, d=4$ HSS [20].

The $d=1$ HSS is an extension of the standard $\mathcal{N}=4, d=1$ superspace $(t, \theta_i, \bar{\theta}^k)$ by the harmonic coordinates $u_i^{\pm}$:

$$(t, \theta^{\pm}, \bar{\theta}^{\pm}, u_i^{\pm}), \quad \theta^{\pm}=\theta^i u_i^{\pm},\ \bar{\theta}^{\pm}=\bar{\theta}^i u_i^{\pm},\ u^{+i}u_i^{-}=1.$$

The commuting $SU(2)$ variables $u_i^{\pm}$ parametrize the 2-sphere $S^2 \sim SU(2)_R/U(1)_R$, where $SU(2)_R$ is the automorphism group of $\mathcal{N}=4, d=1$ "Poincaré" superalgebra. The most important property of HSS is the presence of the invariant harmonic analytic superspace in it, involving half of the original Grassmann coordinates:

$$(\zeta, u)=(t_A, \theta^{+}, \bar{\theta}^{+}, u_i^{\pm}), \quad t_A=t+i(\theta^{+}\bar{\theta}^{-}+\theta^{-}\bar{\theta}^{+}).$$

Most off-shell $\mathcal{N}=4, d=1$ multiplets are described by superfields "living" on this subspace, the analytic superfields.

The direct analog of the $\mathcal{N}=2$ multiplet (**1**, **2**, **1**) is the multiplet (**1**, **4**, **3**) represented by a general superfield $\mathscr{X}(t, \theta^{\pm}, \bar{\theta}^{\pm}, u)$ subjected to the constraints

$$D^{++}\mathscr{X}=0, \quad D^{++}=u^{+i}\frac{\partial}{\partial u^{-i}}+2i\theta^{+}\bar{\theta}^{+}\partial_{t_A},$$
$$D^{+}D^{-}\mathscr{X}=0,\ \bar{D}^{+}\bar{D}^{-}\mathscr{X}=0,\ (D^{+}\bar{D}^{-}+\bar{D}^{+}D^{-})\mathscr{X}=0.$$

These constraints are solved for by the analytic prepotential $\mathscr{V}$,

$$\mathscr{X}(t, \theta_i, \bar{\theta}^i)=\int du\, \mathscr{V}(t_A, \theta^{+}, \bar{\theta}^{+}, u)\Big|_{\theta^{\pm}=\theta^i u_i^{\pm},\ \bar{\theta}^{\pm}=\bar{\theta}^i u_i^{\pm}}.$$

The needed field content is ensured by the invariance under gauge analytic transformations

$$\delta\mathscr{V}=D^{++}\lambda^{--}, \quad \lambda^{--}=\lambda^{--}(\zeta, u)$$

and is recovered in the appropriate WZ gauge for $\mathscr{V}$.

The $\mathcal{N}=4$ analogs of the chiral $\mathcal{N}=2$ multiplet $\mathscr{Z}_a, \bar{\mathscr{Z}}^a$ are the complex analytic superfields $\mathscr{Z}^{+}, \bar{\mathscr{Z}}^{+}$, subjected to the additional constraints

$$D^{++}\mathscr{Z}^{+} = D^{++}\bar{\mathscr{Z}}^{+} = 0,$$

which imply the off-shell content $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ for these superfields:

$$\mathscr{Z}^{+} = z^i u_i^+ + \theta^+\varphi + \bar{\theta}^+\phi - 2i\,\theta^+\bar{\theta}^+\partial_{t_A} z^i u_i^- .$$

Finally, the gauge field $A(t)$ is a component of the analytic unconstrained gauge prepotential V^{++},

$$V^{++\,\prime} = e^{i\lambda}\,V^{++}\,e^{-i\lambda} - i\,e^{i\lambda}(D^{++}e^{-i\lambda}),$$

where $\lambda_a^b(\zeta, u^\pm) \in u(n)$ is the hermitian analytic matrix parameter. Using this gauge freedom we can choose the WZ gauge

$$V^{++} = 2i\,\theta^+\bar{\theta}^+ A(t_A).$$

Now we have all the objects needed for constructing $\mathscr{N}=4$ Calogero and Calogero-Sutherland models. We will firstly discuss the first class of systems.

Our guiding principle will be invariance under the most general $\mathscr{N}=4, d=1$ conformal supergroup $D(2,1;\alpha)$. The appropriate matrix superfield action reads

$$S = -\tfrac{1}{4(1+\alpha)}\int \mu_H \left(\mathrm{tr}\mathscr{X}^2\right)^{-1/2\alpha} - \tfrac{1}{2}\int \mu_A^{(-2)}\mathscr{V}_0\,\widetilde{\mathscr{Z}}^{a\,+}\mathscr{Z}_a^{+} - \tfrac{i}{2}\,c\int \mu_A^{(-2)}\,\mathrm{tr}\,V^{++}. \qquad (8)$$

Here, all the superfields defined above are involved, with all derivatives properly covariantized with respect to local $U(n)$ group, which acts as

$$\mathscr{X}^{\,\prime} = e^{i\lambda}\,\mathscr{X}\,e^{-i\lambda}, \quad \mathscr{Z}^{+\prime} = e^{i\lambda}\,\mathscr{Z}^{+}, \;\; \bar{\mathscr{Z}}^{+\prime} = \bar{\mathscr{Z}}^{+}e^{-i\lambda}.$$

E.g., $D^{++}\mathscr{Z}^{+} \to \mathscr{D}^{++}\mathscr{Z}^{+} = D^{++}\mathscr{Z}^{+} + i\,V^{++}\mathscr{Z}^{+}$. The object $\mathscr{V}_0$ is a real analytic gauge prepotential for the U(n) singlet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ superfield $\mathscr{X}_0 \equiv \mathrm{tr}\,(\mathscr{X})$. It is defined by the integral transform

$$\mathscr{X}_0(t, \theta_i, \bar{\theta}^i) = \int du\,\mathscr{V}_0\left(t_A, \theta^+, \bar{\theta}^+, u^\pm\right)\Big|_{\theta^\pm = \theta^i u_i^\pm,\;\; \bar{\theta}^\pm = \bar{\theta}^i u_i^\pm}.$$

In what follows, we will be interested in the choice $\alpha = -1/2$, which corresponds to the free superfield Lagrangian $\sim \mathrm{tr}\mathscr{X}^2$ for the multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$, and with which $D(2,1;\alpha) \sim osp(4|2)$.

In WZ gauge, and with auxiliary fields eliminated, we end up with the action:

$$
\begin{aligned}
S_4 &= S_b + S_f, \\
S_b &= \int dt \big[\mathrm{tr}\big(\nabla X \nabla X + c\,A\big) + \tfrac{n}{8}(\bar{Z}^{(i} Z^{k)})(\bar{Z}_i Z_k) + \tfrac{i}{2}\, X_0 \big(\bar{Z}_k \nabla Z^k - \nabla \bar{Z}_k\, Z^k\big)\big], \\
S_f &= i\,\mathrm{tr} \int dt \big(\bar{\Psi}_k \nabla \Psi^k - \nabla \bar{\Psi}_k \Psi^k\big) - \int dt\, \frac{\Psi_0^{(i} \bar{\Psi}_0^{k)} (\bar{Z}_i Z_k)}{X_0}, \\
X_0 &:= \mathrm{tr}(X), \quad \Psi_0^i := \mathrm{tr}(\Psi^i), \quad \bar{\Psi}_0^i := \mathrm{tr}(\bar{\Psi}^i).
\end{aligned}
$$

After fixing gauges with respect to the residual gauge group, elimination of A_a^b, $a \neq b$, and some further redefinitions, the bosonic core of this action proves to be as follows

$$
S_b = \int dt \Big\{ \sum_a \dot{x}_a \dot{x}_a + \tfrac{i}{2} \sum_a (\bar{Z}_k^a \dot{Z}_a^k - \dot{\bar{Z}}_k^a Z_a^k) + \sum_{a \neq b} \frac{\mathrm{tr}(S_a S_b)}{4(x_a - x_b)^2} - \frac{n\,\mathrm{tr}(\hat{S}\hat{S})}{2(X_0)^2} \Big\}. \tag{9}
$$

Here, the fields Z_a^k are subject to the constraints

$$
\bar{Z}_i^a Z_a^i = c \qquad \forall\, a,
$$

and

$$
(S_a)_i{}^j := \bar{Z}_i^a Z_a^j, \qquad (\hat{S})_i{}^j := \sum_a \Big[(S_a)_i{}^j - \tfrac{1}{2} \delta_i^j (S_a)_k{}^k \Big].
$$

To clarify the meaning of these composite objects, we note that, in the Hamiltonian approach, the kinetic WZ term for Z gives rise to the following Dirac brackets:

$$
[\bar{Z}_i^a, Z_b^j]_D = i \delta_b^a \delta_i^j,
$$

that implies

$$
[(S_a)_i{}^j, (S_b)_k{}^l]_D = i \delta_{ab} \Big\{ \delta_i^l (S_a)_k{}^j - \delta_k^j (S_a)_i{}^l \Big\}.
$$

So, for each index a the quantities S_a form mutually commuting $u(2)$ algebras. The object $\hat{S}$ is just the conserved Noether $SU(2)$ current of the total system. Thus, the new feature of the $\mathcal{N}=4$ case is that not all out of the bosonic variables Z_a^i are eliminated by fixing gauges and solving the constraint; there survives a non-vanishing WZ term for them. After quantization these variables become purely internal ($U(2)$-spin) degrees of freedom. Since $tr\,\hat{S}\hat{S}$ is a constant of motion, the conformal inverse-square potential appears even in the sector of the center-of-mass coordinate X_0. This is an essential difference of the $\mathcal{N}=4$ case from the $\mathcal{N}=1, 2$ cases where this coordinate decouples. Modulo this extra conformal potential, the bosonic limit of the $\mathcal{N}=4$ system constructed is none other than the integrable $U(2)$-spin Calogero model as it was formulated in [2, 22].

6 $\mathscr{N} = 4$ Calogero-Sutherland systems

Like in the previous case, the input superfield action in the hyperbolic case is the sum of three parts

$$S^{N=4} = S_{\mathscr{X}} + S_{WZ} + S_{FI}. \tag{10}$$

The basic distinguishing feature of this system is the choice of $\mathscr{X}$ action

$$S_{\mathscr{X}} = \frac{1}{2}\int \mu_H \,\mathrm{tr}\Big(\ln \mathscr{X}\Big), \tag{11}$$

while the structure of the remaining two pieces is the same, as well as the form of the superfield constraints.

The full structure of the component action is restored by passing through the same steps as in the rational Calogero case, *i.e.*, imposing various gauges, elimination of the auxiliary fields, etc. It is rather involved, especially in the fermionic sector. In particular, it contains a few terms quartic in fermions. The number of physical fermions is the same, just $4n^2$. The actions, both the superfield and the component ones, lack superconformal symmetry, only "flat" $\mathscr{N} = 4$, $d = 1$ supersymmetry and $SU(2)$ R-symmetry are present.

The bosonic sector is described by the action

$$S^{N=4}_{bose} = \frac{1}{2}\int \mathrm{d}t\,\Big[\mathrm{tr}\big(X^{-1}\nabla X X^{-1}\nabla X + 2c\,A\big) + i\big(\bar{Z}_k\nabla Z^k - \nabla\bar{Z}_k\,Z^k\big) + \frac{(\bar{Z}^{(i}Z^{k)})(\bar{Z}_i Z_k)\,\mathrm{tr}\big(X^2\big)}{2(X_0)^2}\Big].$$

Upon fixing $U(n)$ gauge as ${X_a}^b = 0, a \neq b$, eliminating fields A^c_b, using the constraint $\bar{Z}^a_i Z^i_a = c\,\forall\, a$, and passing to $x_a = \mathrm{e}^{q_a}$, the action becomes

$$S^{N=4}_{bose} = \frac{1}{2}\int \mathrm{d}t\Big\{\sum_a\big[\dot{q}_a\dot{q}_a + i(\bar{Z}^a_k\dot{Z}^k_a - \dot{\bar{Z}}{}^a_k Z^k_a)\big] - \sum_{a\neq b}\frac{{(S_a)_i}^k{(S_b)_k}^i}{4\sinh^2\frac{q_a-q_b}{2}} + \sum_{a,b}\frac{(S_a)^{(ik)}\,(S_b)_{(ik)}\,\mathrm{tr}\big(X^2\big)}{2(X_0)^2}\Big\}, \tag{12}$$

$$\mathrm{Tr}\,\big(X^2\big) = \sum_c \mathrm{e}^{2q_c},\quad X_0 = \sum_c \mathrm{e}^{q_c},\quad {(S_a)_i}^k := \bar{Z}^a_i Z^k_a.$$

The quantities ${(S_a)_i}^k$ generate n copies of $U(2)$ algebra. As a result, the above action, up to the last term, describes the hyperbolic $U(2)$-spin Calogero-Sutherland system [2, 14].

The choice of the $\mathscr{Z}^+$ action in the $\mathscr{N} = 4$ rational Calogero model was mainly caused by the requirement of superconformal invariance. In the hyperbolic case, no such a symmetry is present from the very beginning. In particular, the $\mathscr{X}$ part of the action already lacks such an invariance. So, there are no intrinsic reasons to require it for other pieces.

Then it is natural to choose the simplest action for the (**4**, **4**, **0**) multiplets, just $-1/2 \int \mu_A^{(-2)} \bar{\mathscr{Z}}^{+a} \mathscr{Z}_a^+$. Under this choice, all steps are radically simplified, in particular, all fermionic auxiliary fields of (**4**, **4**, **0**) multiplets become zero on shell. The bosonic sector of the component action reads

$$\tilde{S}_{bose}^{N=4} = \frac{1}{2} \int dt \Big\{ \sum_a \big[\dot{q}_a \dot{q}_a + i (\bar{Z}_k^a \dot{Z}_a^k - \dot{\bar{Z}}_k^a Z_a^k) \big] - \sum_{a \neq b} \frac{(S_a)_i{}^k (S_b)_k{}^i}{4 \sinh^2 \frac{q_a - q_b}{2}} \Big\}. \quad (13)$$

The previous bosonic action involved $\mathrm{tr}\big(X^2\big)$ and X_0. The latter coordinate (the center-of-mass coordinate) decouples only for the trivial cases $n = 1, 2$. In contrast, the new action in the bosonic sector yields the pure hyperbolic $U(2)$-spin Calogero-Sutherland system for any n, without any additional interaction. The center-of-mass coordinate is fully detached and it is described by the free action in this sector.

While for $n = 1$ the $\mathscr{X}$ sector in the system obtained fully decouples from the $\mathscr{Z}$ sector and describes a free dynamics, at $n = 2$ one is left with a non-trivial system. The relative coordinate $\phi := \frac{1}{2}(q_1 - q_2)$ involves a non-trivial interaction with the spin variables Z_1^k and Z_2^k

$$\sim \quad \frac{\mathrm{tr}\,(S_1 S_2)}{4 \sinh^2 \phi}. \quad (14)$$

So, in the bosonic sector we find an extension of the standard hyperbolic Pöschl-Teller mechanics [23] by the spin variables. This new $\mathscr{N} = 4$ superextended Pöschl-Teller system certainly deserves a careful analysis. In the $\mathscr{N} = 1, 2$ cases analogous Pöschl-Teller potential without any additional variables is recovered (the known versions of supersymmetric Pöschl-Teller mechanics were given, e.g., in [24]).

7 Summary and Outlook

- We have presented a universal method of constructing supersymmetric extensions of the Calogero-type models through the superfield gauging procedure, which directly generalizes the similar one for the n-particle bosonic Calogero systems. This method yields a non-standard supersymmetrization, with $\mathscr{N} n^2$ physical fermionic fields, in contrast to $\mathscr{N} n$ such fields within the standard supersymmetrization.
- In this way, we have explicitly constructed new $\mathscr{N} = 2$ and $\mathscr{N} = 4$ superfield systems containing the rational Calogero and hyperbolic Calogero-Sutherland models

as the bosonic cores. In the hyperbolic models, new superxtensions of the Pöschl-Teller mechanics for the relative coordinate arise at $n = 2$, with the extra spin variables in the $\mathcal{N} = 4$ case.

- Recently, it was shown, in the on-shell Hamiltonian approach, that similar systems, at least in the rational case, can be formulated for arbitrary $\mathcal{N}$ [17, 25]. It is unclear, whether such systems can be re-obtained within the superfield gauging since the *off-shell* superfield matrix models (the starting point of the gauging procedure) are known only until $\mathcal{N} = 8$.

7.1 Further Lines of Study

In conclusion, we indicate some further possible lines of study:

(a) An interesting question is as to whether the new super Calogero models preserve the remarkable classical and quantum integrability of the bosonic models.
(b) It would be also interesting to learn whether some other $\mathcal{N} = 4$, $d = 1$ multiplets (e.g., the multiplets $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ or $(\mathbf{3}, \mathbf{4}, \mathbf{1})$) can be used to represent spin variables in various $\mathcal{N} = 4$ Calogero systems.
(c) To generalize new $\mathcal{N}$ super Calogero to the case of “weak” $\mathcal{N} = 4$ ($SU(2|1)$) supersymmetry [26–28], as well as to analogous deformed versions of $\mathcal{N} = 8$ supersymmetry [29, 30]. Such generalizations involve an intrinsic mass-dimension parameter m which deforms the Calogero models to a kind of Calogero-Moser systems, with extra oscillator-type terms.
(d) Quantizing all these models. In fact, the quantization of the deformed $SU(2|1)$ Calogero-Moser systems was recently done in [31]. The quantization of the $\mathcal{N} = 2$ models described here was undertaken in a recent paper [32].
(e) At last, other integrable Calogero-like multiparticle models are known [33, 34], e.g., the trigonometric Calogero-Sutherland models, the elliptic models, etc. All of them still wait their supersymmetrization. It is likely that the trigonometric Calogero-Sutherland models, at least up to $\mathcal{N} = 2$, can be constructed by starting from the matrix models with the *unitary* $U(n)$ matrix as the basic $d = 1$ field, in contrast to the *hermitian* such matrix in the hyperbolic case.

Acknowledgements The author thanks the organizers of the XIII International Workshop “Lie Theory and Its Applications in Physics” for the kind hospitality in Varna. This contribution is based on joint papers with S. Fedoruk, O. Lechtenfeld and S. Sidorov. The author is indebted to them for a useful collaboration. A partial support from the RFBR grant 19-02-01046 is acknowledged.

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The Octagon Form Factor in $\mathcal{N} = 4$ SYM and Free Fermions

Ivan Kostov

Abstract The computation of a certain class of polarised four-point functions of heavily charged BPS in $\mathcal{N} = 4$ SYM operators boils down to the computation of a special form factor - the octagon. Here I review the representation of the octagon in terms of free fermions and the determinant formulas that follow. The presentation is based mainly on a common work with Valentina Petkova and Didina Serban [1, 2], but I also mention some recent developments obtained by other authors.

1 Introduction: The Octagon

Starting with the pioneer paper by Minahan and Zarembo [3], the integrability techniques developed for two-dimensional models were conditioned to solve higher-dimensional field theories. A notorious example is the planar maximally supersymmetric Yang-Mills theory, or shortly $\mathcal{N} = 4$ SYM. This field of activity is now referred to as Integrability in Gauge and String Theories (IGST). IGST appeared as the result of interbreeding of ideas and conjectures about gauge-string duality, in particular the AdS/CFT correspondence, and the powerful technology developed for solving two-dimensional integrable models. The interbreeding was successful because the four-dimensional $\mathcal{N} = 4$ SYM is secretly a two-dimensional integrable theory defined by a factorised scattering matrix. The world-sheet description of $\mathcal{N} = 4$ SYM involves the particle excitations of this two-dimensional theory.

The spectrum of the single-trace operators in $\mathcal{N} = 4$ SYM is determined by the two-point function, which is described by a world sheet with the geometry of an asymptotically long cylinder, with the two operators associated with the two extremities. The two-point function can be computes in principle with methods of the Thermodynamical Bethe Ansatz (TBA), which tells us how to sum over the virtual particles wrapping the cylinder.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_8

For $n > 2$, the n-point correlation function of single-trace operators are given by a semi-classical n-closed-string amplitude, the world sheet of which is topologically a sphere with n punctures. It happens that this world-sheet picture applies for any coupling, after being properly formulated.

The vicinity of each puncture can be described by an asymptotic Hilbert space. Hence one can formulate the problem in terms of factorised scattering of these asymptotic states. It is important that particles from *different* Hilbert spaces are allowed to scatter. In this way one can construct a "worldsheet" which is the arena for the scattering of ingoing and outcoming particles for the asymptotic spaces associated with the different operators whose correlation function is computed. Besides the physical particles representing asymptotic states in the physical Hilbert spaces, one should admit that there are also virtual particles circling around the main cycles or connecting pairs of singularities on the world sheet which are other than the physical operators.

There is a single metric compatible with the factorised scattering. In this metric the proper time on the world sheet is defined globally. In a worldsheet with n punctures and no handles there should be $2(n-2)$ local curvature defects, each housing a negative curvature $-\pi$. In a local QFT such defects are known as twist operators [4]. In exceptional cases two twist operators can merge into a branch point. In this case on speaks of extremal correlators. Importantly, the world-sheet distance between each pair of operators is infinite after removing the regularisation.

A intuitively appealing non-perturbative Ansatz for the computation of the n-point correlation functions, known as "hexagonisation" [5–10], prescribes to construct the punctured sphere from elementary blocks called hexagon form factors or shortly hexagons. The hexagon is a special form factor of the above mentioned generalised twist operator. It has three physical edges (time slices) and three mirror edges (space slices). The lengths of the physical edges are determined by the charges of the n operators while the length of the mirror edges is asymptotically large. The hexagons are glued together along their mirror edges by inserting complete sets of virtual particles. In other words, the hexagonalisation can be viewed as a higher-genus generalisation of the thermodynamical Bethe Ansatz techniques.

The result is a sum over all possible physical and virtual particle excitations, with quantisation conditions for the momenta determined by the moduli of the world sheet. The computation of this sum in full generality is a formidable challenge, and has been performed only in the lowest wrapping order [11].

Remarkably, a class of four-point functions of half-BPS operators with large R-charges and specially tuned polarisations, discovered in [12, 13], can be evaluated exactly. In this case all virtual particles are suppressed except those associated with two pairs of hexagons. Such a correlation function factorises into a product or a sum of products of two simpler objects called *octagons*. An octagon is represents two hexagons glued together by summing over a complete set of mirror particles associates with their common edge. Very recently it was discovered that similar factorisation occurs in any given order of the $1/N_c$ expansion [14].

1.1 The Simplest 4-Point Correlation Function

The simplest one-trace operators in $\mathcal{N} = 4$ SYM are the half-BPS ones. Such an operator is characterised by its position x in the Minkowski space, a null vector y giving its $\mathfrak{so}(6)$ polarisation, and its scaling dimension K,

$$\mathcal{O}(x) = \mathrm{Tr}[(y \cdot \Phi(x))^K]. \tag{1.1}$$

The correlation function for four such operators depends on the 't Hooft coupling g, the coordinates x_1, ..., x_4 and the polarisations y_1, ..., y_4 of the four operators. Thanks to the conformal symmetry the dependence on x_i, y_i is only through the cross ratios in the coordinate and in the flavour spaces

$$\begin{aligned} z\bar{z} &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u, \qquad & (1-z)(1-\bar{z}) &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = v, \\ \alpha\bar{\alpha} &= \frac{(y_1 \cdot y_2)(y_3 \cdot y_4)}{(y_1 \cdot y_3)(y_2 \cdot y_4)}, \qquad & (1-\alpha)(1-\bar{\alpha}) &= \frac{(y_1 \cdot y_4)(y_2 \cdot y_3)}{(y_1 \cdot y_3)(y_2 \cdot y_4)}. \end{aligned} \tag{1.2}$$

The hexagonalisation prescription gives an expression of the four-point function as a sum over virtual particles associated with the six mirror channels associated with pair of operators $\mathcal{O}_i$ and $\mathcal{O}_j$. Each mirror channel is characterised by a "bridge" of length $\ell_{ij} = \#$[tree-level Wick contractions between the operators $\mathcal{O}_i$ and $\mathcal{O}_j$]. The six bridge lengths obey four constraints of the type $K_1 = \ell_{12} + \ell_{13} + \ell_{14}$. The propagation of virtual particles across a bridge is exponentially suppressed for large bridge length.

The sum over virtual particles simplifies for heavy fields (large K) and particular choices of the polarisations because some of the channels for propagation are suppressed [12]. In this case the four-point function factorises into a product or into a sum of products of two simpler objects, octagons. The octagon $\mathbb{O}_\ell$ is composed of two hexagons glued together by inserting a complete set of virtual states The two hexagons may be separated by a bridge of ℓ. The octagon depends only on the 't Hooft coupling g, the cross ratios (1.2) and the bridge length ℓ,

$$\mathbb{O}_\ell = \mathbb{O}_\ell(z, \bar{z}, \alpha, \bar{\alpha}). \tag{1.3}$$

The simplest four-point function that leads to such a factorisation, named in [12] the *simplest*, is characterised by $(y_1 \cdot y_4) = (y_2 \cdot y_3) = 0$. For example (the dots stand for the sum over permutations)

$$\begin{aligned} \mathcal{O}_1(0) &= \mathrm{tr}(Z^K \bar{X}^K) + ..., \qquad & \mathcal{O}_2(z, \bar{z}) &= \mathrm{tr}(X^{2K}), \\ \mathcal{O}_4(\infty) &= \mathrm{tr}(Z^K \bar{X}^K) + ..., \qquad & \mathcal{O}_3(1) &= \mathrm{tr}(\bar{Z}^{2K}). \end{aligned} \tag{1.4}$$

At tree level, the four-point function is given by a single Feynman graph, Fig. 1, left.

In the limit $K \to \infty$ the *simplest* four-point correlator factorises to a product of two identical octagons with $\ell = 0$, and $\alpha = \bar{\alpha} = 1$, sketched in Fig. 1, right,

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle \underset{K\to\infty}{=} \frac{[\mathbb{O}_0(z, \bar{z}, 1, 1)]^2}{(x_{12}^2 x_{34}^2 x_{13}^2 x_{24}^2)^K}. \tag{1.5}$$

There is another class of four-point functions considered in [12], which are expressed in terms of octagons $\mathbb{O}_\ell$ with $\ell > 0$. In the recent paper [14] the non-planar corrections to the four-point correlators are expressed as polynomials of $\mathbb{O}_\ell^2$.

1.2 The Octagon Form Factor

The *octagon* has four physical and four mirror edges with the corresponding BMN vacuum at each physical edge, as shown schematically in Fig. 3. Because of the choice of the four operators there are no physical excitations. There are only multi-particle mirror excitations associated with the common edge of the two hexagons. The octagon is given by a sum over a complete set of such mirror states. Symbolically

$$\begin{aligned}\mathbb{O}_\ell(z, \bar{z}, \alpha, \bar{\alpha}) &= \sum_\psi \mu(\psi)\, \langle \mathcal{H}_2 | \psi \rangle\, e^{-E_\psi \ell}\, \langle \psi | \mathcal{H}_1 \rangle \\ &= \sum_\psi \mu(\psi)\, \langle g^{-1} \mathcal{H} | \psi \rangle\, e^{-E_\psi \ell} \langle \psi | \mathbf{g} | \psi \rangle \langle \psi | \mathcal{H} \rangle.\end{aligned} \tag{1.6}$$

In the second line the two hexagons are rotated as shown in Fig. 4 to the canonical hexagon $\mathcal{H}$ which is defined for collinear operators. The dependence of the cross ratios is through the rotation $\mathbf{g} \in PSU(2, 2|4)$,

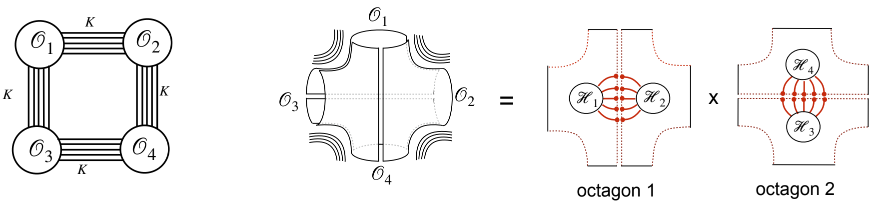


Fig. 1 Left: The simplest 4p function is given at tree level by a single Feynman graph. Right: Factorisation of the 4p function into a product of two octagons

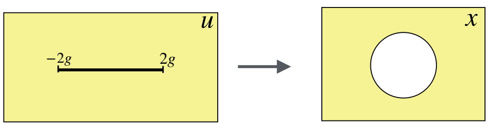


Fig. 2 The Zhukovskymap. The physical sheet of the rapidity Riemann surface is parametrised by the exteriour of the unit circle in the x-plane

$$\langle\psi|\mathbf{g}|\psi\rangle = e^{2i\tilde{p}_\psi\xi}\, e^{iL_\psi\phi}\, e^{iR_\psi\theta}\, e^{J_\psi\varphi} \tag{1.7}$$

where L_ψ is the angular momentum, R_ψ and J_ψ are the R-charges and it is used that $\frac{D-J}{2} = E = i\tilde{p}$ [6]. The parameters $\phi, \xi, \theta, \varphi$ are related to the cross ratios in the Minkowski and in the flavour spaces, Eq. (1.2), as

$$\begin{aligned} z &= e^{-\xi+i\phi}, & \bar{z} &= e^{-\xi-i\phi}, \\ \alpha &= e^{\varphi-\xi+i\theta}, & \bar{\alpha} &= e^{\varphi-\xi-i\theta}. \end{aligned} \tag{1.8}$$

For an n-particle virtual state ψ the contribution of the chemical potentials generated by the rotation $\mathbf{g}$ together with the matrix part of the hexagon weights factorise into a product of one-particle weights. An n-particle mirror state is characterised by the rapidities u_i and the bound-state numbers a_i of its particles. The octagon (1.6) can be thus expanded as a series of multiple integrals with integrand given by a product of local and bi-local weights [12]

$$\mathbb{O}_\ell = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{a_1,\ldots,a_n\geq 1} \int \prod_{j=1}^{n} \frac{du_j}{2\pi i}\, W_{a_j}(u_j) \prod_{j<k}^{n} W_{a_j,a_k}(u_j,u_k). \tag{1.9}$$

• *Bi-local weights.*

The bi-local weights are defined in terms of the function

$$W(u,v) = \frac{x(u)-x(v)}{x(u)x(v)-1} \tag{1.10}$$

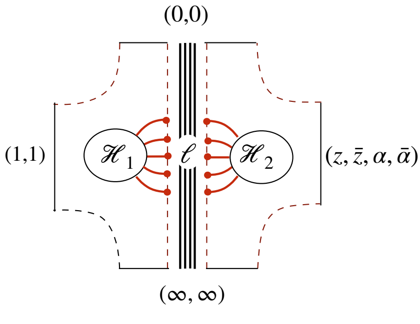


Fig. 3 Symbolic representation of the octagon obtained by gluing the hexagons $\mathcal{H}_1$ and $\mathcal{H}_2$ along the common edge $(0,0)$–(∞,∞) by inserting a complete set of virtual states ψ with energies $\tilde{E}_\psi$. Each state consists of particles transforming in the skew-symmetric representations of $\mathfrak{psu}(2|2)\times\mathfrak{psu}(2|2)$

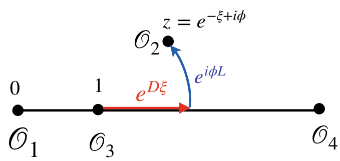


Fig. 4 Rotation from canonical hexagons to a general position. The rotation angles ξ and ϕ correspond respectively to a dilatation and rotation

where the function $x(u)$ is defined by the Zhukovskymap $\frac{u}{g} = x + \frac{1}{x}$ transforming the physical sheet in the rapidity plane into the exterior of the unit circle (Fig. 2). Namely

$$W_{a,b}(u, v) = \prod_{\pm,\pm} W(u \pm \tfrac{i}{2}a, v \pm \tfrac{i}{2}b). \tag{1.11}$$

• *Local weights.*

Assume for simplicity that $\alpha = \bar{\alpha} = 0$, that is $\theta = 0$ and $\varphi = \xi$. Then the one-particle factor is given by the product

$$W_a(u) = \frac{1}{g}\Omega_\ell(u + \tfrac{i}{2}a)\Omega_\ell(u - \tfrac{i}{2}a)\,(-1)^a \chi_a(\phi, \xi) \quad \times W(u + \tfrac{i}{2}a, u - \tfrac{i}{2}a) \tag{1.12}$$

where

$$\Omega_\ell(u) = \frac{1}{x^\ell}\frac{e^{ig\xi\,[x-1/x]}}{x - 1/x} = g\,\frac{e^{ig\xi\,[x-1/x]}}{x^\ell}\,\frac{d\log x}{du}\,, \tag{1.13}$$

χ_a is the character of the a-th antisymmetric representation of $\mathfrak{psu}(2|2)$

$$\chi_a(\phi, \xi) = \operatorname{tr}_a[(-1)^F\, e^{\xi \mathbf{J} + i\phi \tilde{L}}] = (-1)^a\,\frac{\sin(a\phi)}{\sin\phi}\,[2\cos\phi - 2\cosh\xi]\,, \tag{1.14}$$

and the last factor completes the product od the bi-local weights, as we will see later.

The function $\Omega(u)$ reflects the form of the momentum and the energy of the mirror magnons as functions of the rapidity u,

$$\begin{aligned} \tilde{p}_a(u) &= \tfrac{1}{2}g(x - \tfrac{1}{x})_{u+ia/2} + (x - \tfrac{1}{x})_{u-ia/2}, \\ \tilde{E}_a(u) &= \log x|_{u+ia/2} + \log x|_{u-ia/2}, \end{aligned} \tag{1.15}$$

For later use let us remind the generating function of the $\mathfrak{psu}(2|2)$ characters

$$\begin{aligned} \mathcal{W}(k) &= \sum_{a=1}^{\infty}(-1)^a \chi_a\, e^{-ak} = \frac{(1 - e^{\xi} e^{-k})(1 - e^{-\xi}\, e^{-k})}{(1 - e^{i\phi}\, e^{-k})(1 - e^{-i\phi}\, e^{-k})} - 1 \\ &= \frac{\cos\phi - \cosh\xi}{\cosh k - \cos\phi}. \end{aligned} \tag{1.16}$$

2 The Octagon in Terms of Free Fermions

The expansion (1.9) of the octagon can be nicely written in terms of free fermions. The fermionic representation allows to sum up the series with no pain and also opens the possibility to use the intuition and the techniques coming from CFT. Before proceeding with the operator formulation we will remind some standard formulas about free fermions.

2.1 Free Fermions: Conventions

We mostly follow the conventions in the review by Jimbo and Miwa [15]. The fermionic amplitudes satisfy canonical anti-commutation relations

$$[\psi_m, \psi_n^*]_+ = \delta_{m,n}, \qquad m, n \in \mathbb{Z}, \tag{2.1}$$

and the left and right vacua represent the Fermi sea filled up to the level ℓ,

$$\langle \ell | = \langle \text{vac} | \prod_{n<0} \psi_n^*, \quad |\ell\rangle = \prod_{n<0} \psi_n |\text{vac}\rangle. \tag{2.2}$$

Here $\langle\text{vac}|$ is the absolute vacuum annihilated by all ψ_n and its dual $|\text{vac}\rangle$ is annihilated by all ψ_n^*. The two vacua satisfy

$$\begin{aligned} &\psi_n|\ell\rangle = 0, \ \ \langle \ell|\psi_n^* = 0 \qquad (n < \ell), \\ &\langle \ell|\psi_n = 0, \ \ \psi_n^*|\ell\rangle = 0 \qquad (n \geq \ell). \end{aligned} \tag{2.3}$$

The non-vanishing correlators are

$$\langle \psi_m \psi_n^* \rangle = \delta_{m,n} \quad (m < \ell), \qquad \langle \psi_m^* \psi_n \rangle = \delta_{m,n} \quad (m \geq \ell), \tag{2.4}$$

where $\langle ... \rangle \equiv \langle \ell | ... | \ell \rangle$.

A pair of analytic fermionic fields $\psi(x)$, $\psi^*(x)$ is defined by the mode expansions

$$\psi(x) = \sum_{n\in\mathbb{Z}} \psi_n x^{-n}, \quad \psi^*(u) = \sum_{n\in\mathbb{Z}} \psi_n^* \, x^n. \tag{2.5}$$

The two-point correlator of these fields is

$$G(x, y) = \langle \ell | \psi^*(x) \ \psi(y) | \ell \rangle = \left(\frac{x}{y}\right)^{-\ell} \frac{1}{1 - x/y}, \qquad x \neq y. \tag{2.6}$$

The correlation function of a product of fermions is given by the determinant of the two-point correlators

$$\langle \prod_{j=1}^{n} \psi(x_j) \prod_{j=1}^{n} \psi^*(x_j) \rangle = \det_{n\times n} G(x_j, x_k). \tag{2.7}$$

2.2 The Octagon as a Fock Space Expectation Value

Our goal is to construct real fermions defined on the physical sheet of the rapidity Riemann surface parametrised by the exteriour of the unit disk, $|x| > 1$, and having the two-point function

$$\langle\psi(x)\psi(y)\rangle = \frac{1}{x^\ell y^\ell}\,\frac{x-y}{xy-1} \qquad (|x|>1,\, |y|>1). \tag{2.8}$$

generating the bi-local weights in the expansion (1.9). The idea of the fermionic construction was given in [2].

With the bare Fock vacua the two-point correlator of the field ϕ vanishes. In order to obtain the correlator (2.12) we replace the left vacuum by a coherent state[1]

$$\langle C| = \langle\ell|C, \quad C = \exp\Big(\tfrac{1}{2}\sum_{m,n\in\mathbb{Z}} \psi^*_m C_{mn}\psi^*_n\Big) = \exp\oint_{|x|=1}\frac{dx}{2\pi i}\,\psi^*(1/x)\psi^*(x). \tag{2.9}$$

$$C_{mn} = \delta_{m+1,n} - \delta_{m-1,n}. \tag{2.10}$$

The state $\langle B|$ can be seen as a boundary state associated with the unit circle $|x| = 1$. The chiral fermion $\phi(x)$ lives in the exterior of the unit disk which is the image of the physical sheet in the rapidity plane.

The non-vanishing correlations of the ψ oscillators

$$\langle C|\psi_m\psi_n|\ell\rangle = C_{mn} \qquad (m\ge\ell,\, n\ge\ell). \tag{2.11}$$

are obtained by commuting the fermions with the operator B and applying the rules (2.3) which also imply $\langle\ell|B|\ell\rangle = 1$.

The two-point function in the coordinate representation has the desired form

$$\begin{aligned}\langle\ell|B\ \psi(x)\psi(y)|\ell\rangle &= \sum_{m,n\ge 0} C_{mn}x^{-m-\ell}y^{-n-\ell}\\ &= \frac{1}{x^\ell y^\ell}\,\frac{x-y}{xy-1}, \qquad (|x|>1,\,|y|>1).\end{aligned} \tag{2.12}$$

The $2n$-point correlator is the pfaffian of the two-point correlators:

$$\begin{aligned}\langle C|\psi(x_1)\ldots\psi(x_{2n})|\ell\rangle &= \mathrm{Pf}\left(\left[\frac{x_j-x_k}{x_jx_k-1}\right]_{i,j=1}^{2n}\right)\\ &= \prod_{j=1}^{2n}\frac{1}{x_j^\ell}\prod_{j<k}^{2n}\frac{x_j-x_k}{x_jx_k-1}.\end{aligned} \tag{2.13}$$

---

[1] I thank Y. Matsuo for suggesting to construct the Fock space in this way.

Now we can readily sum up the expansion (1.9) of Sect. 1. Nicely, all the bilinear factors are produced by the correlation functions of the fermions,

$$\mathbb{O}_\ell = \sum_{n=0}^{\infty} \frac{g^n}{n!} \sum_{a_1,\dots,a_n \geq 1} \int \prod_{j=1}^{n} \frac{du_j}{2\pi i} \, (-1)^{a_j} \chi_{a_j}(\phi,\xi) \, [\Omega_0]_{u_j+ia_j/2} [\Omega_0]_{u_j-ia_j/2} \\ \times \langle C| \prod_{j=1}^{n} \psi(u_j + \tfrac{1}{2} i a_j) \psi(u_j - \tfrac{1}{2} i a_j) |\ell\rangle. \tag{2.14}$$

Since the length of the bridge ℓ became the charge of the left and right vacuum states, ω_ℓ is replaced by $\omega_{\ell=0}$. The series (2.14) exponentiates,

$$\mathbb{O}_\ell = \langle \ell | C \ \exp(\tfrac{1}{2} \, \psi K \psi) | \ell \rangle \tag{2.15}$$

with the quadratic form defined by the infinite sum

$$\begin{aligned} \psi K \psi &= g \int_{\mathbb{R}} \frac{du}{2\pi i} \sum_{a\geq 1} (-1)^a \chi_a(\phi,\xi) \ [\Omega_0 \psi]_{u+ia/2} \ [\Omega_0 \psi]_{u-ia/2} \ . \\ &= g \int_{\mathbb{R}} \frac{du}{2\pi i} \sum_{a\geq 1} (-1)^a \chi_a(\phi,\xi) \ [\Omega_0 \psi]_{u+i0} \ [\Omega_0 \psi]_{u-ia} \\ &= g \int_{\mathbb{R}} \frac{du}{2\pi i} [\Omega_0 \psi]_{u+i0} \sum_{a\geq 1} (-1)^a \chi_a(\phi,\xi) \mathbb{D}_u^{-2a} \ [\Omega_0 \psi(x)]_{u-i0} \ . \end{aligned} \tag{2.16}$$

In the second line of (2.16) the integration variable in the a-th term of the series is shifted as $u \to u - ia/2 - i0$ which is justified by the integrand being analytic in the strip $|\Im u| < a/2$. After the shift of the argument, the integrands in all terms have the same cut $[-2g, 2g]$ on the real axis. In the last line we wrote a more compact representation of the sum in terms of the shift operator $\mathbb{D}_u = e^{i\partial_u/2}$,

$$\mathbb{D}_u : \quad f[x(u)] \to f[x(u + i/2)]. \tag{2.17}$$

Because of the cut, the action of the shift operator should be considered as analytic continuation from the interval $|u| > 2g$.

The sum in the last line of (2.16) is essentially the generating function for the $\mathfrak{sl}(2|2)$ characters, Eq. (1.16) of Sect. 2, with operator spectral parameter $k = i\partial/\partial u$. Introducing the operator

$$\begin{aligned} \mathbb{W} &= \sum_{a=1}^{\infty} (-1)^a e^{-\xi \tilde{p}_a(u)} \chi_a(\phi,\xi) \, \mathbb{D}_u^{-2a} \\ &= -e^{ig\xi\,[x-1/x]} \, \frac{\cosh\xi - \cos\phi}{\cos\partial_u - \cos\phi} \, e^{ig\xi\,[x-1/x]} \,, \end{aligned} \tag{2.18}$$

which can be regarded as the *quantum spectral curve* for the octagon, we can write the quadratic form as

$$\psi \mathrm{K} \psi = g \int_{\mathbb{R}} \frac{du}{2\pi i} [\partial_u \log x \ \psi(x)]_{u+i0} \, \mathbb{W} \, [\partial_u \log x \ \psi(x)]_{u-i0}. \tag{2.19}$$

By Fourier transformation the action of the operator $\mathbb{W}$ can be represented by an integration kernel

$$\psi \mathrm{K} \psi = \int_{\mathbb{R}} \frac{du}{2\pi i} \oint_{\mathbb{R}} \frac{dv}{2\pi i} \, [\partial_u \log x \ \psi(x)]_{u+i0} \, \check{K}(u,v) \, [\partial_v \log x \ \psi(x)]_{v-i0} \tag{2.20}$$

with

$$\begin{aligned} \check{K}(u,v) &= -g \, 2\pi i \ e^{ig\xi\,[x(u)-1/x(u)]} \ e^{ig\xi\,[x(v)-1/x(v)]} \\ &\quad \times \int_{\mathbb{R}} \frac{dk}{2\pi} i \sin k(u-v) \ \frac{\cosh\xi - \cos\phi}{\cosh k - \cos\phi}. \end{aligned} \tag{2.21}$$

When this kernel is substituted in (2.19), an important simplification occurs. Namely the integrals in the rapidities u and v can be transformed into a contour integrals along the init circle in the x-space. Indeed, whenever $k - |\xi| > 0$, the integral over u which goes below the real axis can be transformed into an integral around the Zhukovskycut, otherwise the integral vanishes. The same holds for the integration in v. Therefore one can write the quadratic form as a double contour integral

$$\psi \mathrm{K} \psi = \int_{|x|=1} \frac{d \log x}{2\pi i} \int_{|y|=1} \frac{d \log x}{2\pi i} \ K(x,y), \tag{2.22}$$

with

$$\begin{aligned} K(x,y) &= g \ e^{ig\xi\,[x-1/x+y-1/y]} \\ &\quad \times \int_{\xi}^{\infty} dk \sin[gk(x+1/x+y+1/y)] \ \frac{\cosh\xi - \cos\phi}{\cosh k - \cos\phi} \\ &= -\frac{g}{2i} \int_{\xi}^{\infty} dk \ e^{ig[(k+\xi)x + \frac{k-\xi}{x}]} e^{-ig[(k-\xi)y + \frac{k+\xi}{y}]} \ \frac{\cosh\xi - \cos\phi}{\cosh k - \cos\phi} \\ &\quad - \{x \leftrightarrow y\}. \end{aligned} \tag{2.23}$$

In conclusion, we have seen that the expansion of the octagon as a sum over mirror magnons can be formulated as a Fock expectation value of a free chiral fermion with correlator (2.12) ,

$$\mathbb{O}_\ell = \langle \ell | e^{\frac{1}{2} \oint_{|x|=1} \frac{dx}{2\pi i} \ \psi^*(x)\psi^*(\frac{1}{x})} e^{\frac{1}{2} \oint_{|x|=1} \frac{dx/x}{2\pi i} \oint_{|y|=1} \frac{dy/y}{2\pi i} \ \psi(x)K(x,y)\psi(y)} | \ell \rangle, \tag{2.24}$$

with the kernel $\mathrm{K}(x,y)$ defined by (2.23).

2.3 Discrete Basis

When expressed in terms of the fermionic amplitudes ψ_n, the quadratic form is given by the semi-infinite matrix

$$\psi K \psi = \sum_{m,n\in\mathbb{Z}} K_{mn}\, \psi_m\, \psi_n, \tag{2.25}$$

where K_{mn} are the coefficients in the Laurent expansion of the analytic kernel (2.23)

$$K(x, y) = \sum_{m,n\in\mathbb{Z}} K_{mn}\, x^m y^n \tag{2.26}$$

valid in a ring containing the unit circle. After acting on the right vacuum, only the modes with $m, n \geq \ell$ survive in the double sum.

Assume that the relevant fermionic oscillators ψ_n^*, ψ_n are truncated to $0 < n - \ell \leq N$ and define the $N \times N$ matrices

$$\boldsymbol{K} = \{K_{mn}\}_{\ell\leq m,n\leq N+\ell}, \quad \boldsymbol{\mathcal{C}} = \{C_{mn}\}_{\ell\leq m,n\leq N+\ell}, \tag{2.27}$$

with $C_{m,n} = \delta_{m+1,n} - \delta_{m,n+1}$. In terms of the discrete modes the operator in the fermionic representation (2.24) has the form of the Balian-Brézin decomposition for a Bogolyubov transformation [16]

$$\mathcal{O}_\ell = \lim_{N\to\infty} \mathcal{O}_\ell^{(N)},$$

$$\begin{aligned} \mathcal{O}_\ell^{(N)} &= \langle \ell | e^{\frac{1}{2}\psi^* \boldsymbol{\mathcal{C}} \psi^*} e^{\frac{1}{2}\psi \boldsymbol{K} \psi} | \ell \rangle \\ &= \int \prod_{m=\ell}^{N+\ell} [d\theta_m^* d\theta_m]\ e^{\frac{1}{2}\theta^* \boldsymbol{\mathcal{C}} \theta^*} e^{\theta^* \theta}\ e^{\frac{1}{2}\theta_m \boldsymbol{K}_\theta}. \end{aligned} \tag{2.28}$$

In the second line the expectation value is represented as a Grassmannian integral, where we ignored the sign factor in the measure. The expectation value is evaluated as the pfaffian of the $2N \times 2N$ skew-symmetric matrix

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{\mathcal{C}} & \boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{K} \end{pmatrix}. \tag{2.29}$$

The octagon is equal to the pfaffian of this matrix,

$$\mathcal{O}_\ell = \mathrm{Pf}\, \boldsymbol{M} = \sqrt{\mathrm{Det}\,[1 - \boldsymbol{\mathcal{C}} \boldsymbol{K}]}. \tag{2.30}$$

The matrix elements can be evaluated as double contour integrals [2]

$$\begin{aligned} K_{mn} &= \int_{|x|=1} \frac{dx/x}{2\pi i} \int_{|y|=1} \frac{dy/y}{2\pi i}\, x^m y^n\, K(x,y) \\ &= -\frac{g}{i} \int_{\xi}^{\infty} dk \left[\left(i\sqrt{\frac{k-\xi}{k+\xi}} \right)^{m-n} - \left(i\sqrt{\frac{k-\xi}{k+\xi}} \right)^{n-m} \right] \\ &\quad \times J_m(2g\sqrt{k^2-\xi^2})\, J_n(2g\sqrt{k^2-\xi^2})\, \frac{\cosh\xi - \cos\phi}{\cosh k - \cos\phi}, \end{aligned} \tag{2.31}$$

or, after changing the integration variable to $t = \sqrt{k^2 - \xi^2}$,

$$K_{mn} = -2g \int_0^\infty dt\, \mathrm{X}(t)\, \Pi_{m-n}(\xi/t)\, J_m(2gt)\, J_n(2gt) \tag{2.32}$$

with

$$\mathrm{X}(t) = \frac{\cosh\xi - \cos\phi}{\cosh\sqrt{t^2+\xi^2} - \cos\phi} \tag{2.33}$$

and the polynoms $\Pi_n(z)$ are the Chebyshev polynomials of second kind with imaginary argument,

$$\Pi_n(z) = U_{n-1}(iz) = \frac{i^n \left(\sqrt{z^2+1}+z\right)^n - i^{-n}\left(\sqrt{z^2+1}-z\right)^n}{2i\sqrt{z^2+1}} \tag{2.34}$$

and satisfy the recurrence relation

$$\Pi_{n-1}(z) + \Pi_{n+1}(z) = 2iz\, \Pi_n(z)\,. \tag{2.35}$$

3 Species Doubling Phenomenon: The Pfaffian as a Determinant

3.1 Doubling of the Eigenvalues and Similarity Transformation

The matrix elements of the matrix $\boldsymbol{CK}$ decrease rapidly with m and n, which allows to truncate it with reasonable precision to a finite-dimensional matrix. The truncation works better for small coupling g.

The numerical diagonalisation reveals that the eigenvalues of the matrix $\boldsymbol{CK}$ are real, negative and doubly degenerate. This is a non-trivial fact because the matrix K_{mn} is anti-symmetric but not real, its even-odd elements are real while the even-even and odd-odd elements are pure imaginary. It is therefore plausible that the antisymmetric matrix $\boldsymbol{M}$ can be transformed by a similarity transformation into a real skew-symmetric matrix, which is diagonalised by orthogonal transformation as

$$\boldsymbol{M} = O\boldsymbol{\Sigma}O^{\mathrm{T}}, \qquad \boldsymbol{\Sigma} = \mathrm{diag}\{\begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix}\}. \tag{3.1}$$

In [2], it was conjectured that there exist a similarity transformation from $\boldsymbol{K}$ to a real anti-symmetric matrix $\boldsymbol{K}^{\circ}$ with non-zero matrix elements only if $m - n$ is odd. A perturbative series for the matrix $\boldsymbol{K}^{\circ}$ was proposed and it was checked that the matrices $\mathbf{K}$ and $\mathbf{K}^{\circ}$ give identical results for the first several orders of the perturbative expansion of the octagon.

Later Belitsky and Korchemsky [17] proposed an elegant integral expression for the matrix elements of $\mathbf{K}^{\circ}$ which reproduces the perturbative series found in [2] and presumably hold for any value of the gauge coupling. The integral formula is obtained by replacing in the integrand in (2.31) of the previous section the polynomials $\Pi_n(\xi/t)$ by their constant terms s_n:

$$K^{\circ}_{mn} = -2g\, s_{m-n}\, L_{m,n}, \qquad L_{m,n} \equiv \int_0^{\infty} dt\; \mathrm{X}(t)\; J_m(2gt)\; J_n(2gt). \tag{3.2}$$

It is easy to see that the matrix elements of $\boldsymbol{K}$ are linear combinations of the matrix elements of $\boldsymbol{K}^{\circ}$. This can be demonstrated using the fact that $\Pi_n(\xi/t)$ are polynomials in $1/t$ and the recurrence relations for the Bessel functions

$$\frac{\xi}{t} J_n[2gt] = \xi g\, \frac{J_{n+1}[2gt] + J_{n-1}[2gt]}{n}. \tag{3.3}$$

Since the form of the matrix elements K_{mn} is linked to the parity of $m - n$, it is natural to reorder the lines and the rows of the $2N \times 2N$ matrix $\boldsymbol{M}$ in order to reveal an additional 2×2 block structure.[2] Assuming that N is even (otherwise the pfaffian vanishes), we have

$$\boldsymbol{M} = \begin{pmatrix} 0 & \mathbf{C} & \mathbf{I} & 0 \\ -\mathbf{C} & 0 & 0 & \mathbf{I} \\ -\mathbf{I} & 0 & 0 & \mathbf{K} \\ 0 & -\mathbf{I} & -\mathbf{K} & 0 \end{pmatrix}. \tag{3.4}$$

The matrices in the blocks are defined as follows,

$$\mathbf{K} = \{\mathrm{K}_{ij}\}_{i,j=0,\dots,\frac{N}{2}-1}, \qquad \mathrm{K}_{ij} = K^{\circ}_{2i,2j+1} = -g\,(-1)^{i-j} L_{2i+\ell,2j+1+\ell}, \tag{3.5}$$

$$\mathbf{C} = \{\mathrm{C}_{ij}\}_{i,j=0,\dots,\frac{N}{2}-1}, \qquad \mathrm{C}_{ij} = C_{2i,2j+1} = \delta_{i,j} - \delta_{i,j+1}. \tag{3.6}$$

The pfaffian of the matrix (3.4) becomes a determinant

[2] Such a appears also in other problems in $\mathcal{N} = 4$ SYM as pointed out by Basso and Dixon [18].

$$\mathbb{O}_\ell = \mathrm{Pf}[\boldsymbol{M}] = \pm \det[1 + \mathbf{R}^{(\ell)}], \qquad \mathbf{R}^{(\ell)} = -\mathbf{CK}. \tag{3.7}$$

The matrix elements of $\mathbf{R}(\ell) = \mathbf{CK}$ are expressed in terms of the integrals L_{mn}, Eq. (3.2), as

$$\mathrm{R}^{(\ell)}_{ij} = 2g(-1)^{i-j}(L_{2i+\ell,2j+1+\ell} + L_{2i+2+\ell,2j+\ell}), \quad i, j \geq 0, \tag{3.8}$$

Applying the recurrence relation (3.3), one can write the matrix elements of $\mathbf{R}_\ell$ as

$$\mathrm{R}^{(\ell)}_{mn} = -2(2m+1+\ell)\int_0^\infty \frac{dt}{t} J_{2m+1+\ell}(2gt)\, J_{2n+1+\ell}(2gt)\, \mathrm{X}(t), \tag{3.9}$$

$$\mathrm{X}(t) = \frac{\cosh\xi - \cos\phi}{\cosh\sqrt{t^2+\xi^2} - \cos\phi}. \tag{3.10}$$

Here we ignored the sign factor $(-1)^{m-n}$ as it cancels when computing the moments.

4 Weak Coupling Expansion

4.1 Expansion of the Kernel in Polylogs

The coefficients of the weak coupling expansion of the matrix elements (3.9) can be expressed in terms of a set of functions $f_n(z,\bar{z})$ known as ladder integrals. The (conveniently normalised) ladder integrals evaluate the so-called ladder Feynman graphs represented in Fig. 5.

The ladder integrals can be expanded in polylogarithms, but we will need here only their integral representation which can be extracted from the results reported in [19]

$$\begin{aligned} f_k(z,\bar{z}) &= \int_0^\infty \frac{e^\xi\, t^{2k-1}}{\cosh\sqrt{t^2+\xi^2} - \cos\phi}\, dt \\ &= \sum_{j=k}^{2k} \frac{(k-1)!\, j!}{(j-k)!(2k-j)!}\,(-\log z\bar{z})^{2k-j}\,\frac{\mathrm{Li}_j(z) - \mathrm{Li}_j(\bar{z})}{z-\bar{z}}. \end{aligned} \tag{4.1}$$

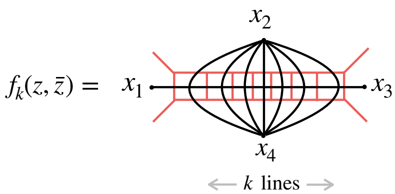


Fig. 5 A ladder Feynman graph and its dual. In the x-space the graph has k vertices and k lines meeting at the points x_2 and x_4. The dual graph in the momentum space has the form of a lader with $k+1$ rungs

The integrands in the expressions for the matrix elements (2.32) or (3.9) are linear combinations of the integrand of (4.1). Hence both the original matrix $\boldsymbol{K}$, Eq. (2.31), and the "improved" matrix $\boldsymbol{K}^{\circ}$, Eq. (3.2), can be expanded in ladder integrals.

Here we will give the expression for the improved kernel only. For that it is sufficient to expand the numerator of the integrand in (3.9), which is done according to the formula

$$\begin{aligned} J_k(2gt)J_l(2gt) &= \sum_{p=0}^{\infty} C_{kl}^p \,(gt)^{k+l+2p}, \\ C_{kl}^p &= \binom{2p+k+l}{p}\frac{(-1)^p}{(k+p)!(l+p)!}. \end{aligned} \tag{4.2}$$

We thus find $\mathbb{O}_\ell(z,\bar{z},g^2) = \mathrm{Det}\left[\mathbf{I}+\mathbf{R}^{(\ell)}\right]$ with

$$\begin{aligned} \mathbf{R}^{(\ell)} &= \{\mathrm{R}_{i,j}^{(\ell)}\}_{i,j=0}^{\infty}, \\ \mathrm{R}_{i,j}^{(\ell)} &= (1-z)(1-\bar{z})\,(2i+1)\sum_{p=0}^{\infty} f_{i+j+p+1}\, g^{2i+2j+2p+2}\, C_{2i+1,2j+1}^p. \end{aligned} \tag{4.3}$$

The determinant can be evaluated by expanding the logarithm of the octagon in the moments of the matrix $\boldsymbol{R}^{(\ell)}$,

$$\log\mathbb{O}_\ell = \sum_{i=0}^{\infty}\mathrm{R}_{i,i}^{(\ell)} - \frac{1}{2}\sum_{i,j=0}^{\infty}\mathrm{R}_{i,j}^{(\ell)}\mathrm{R}_{j,i}^{(\ell)} + \ldots \tag{4.4}$$

Alternatively, and more efficiently from the computational point of view, one can truncate the semi-infinite matrix $\mathbf{R}^{(\ell)}$ to a $N\times N$ matrix $\mathbf{R}_{N\times N}^{(\ell)} = \{\mathrm{R}_{k,j}^{(\ell)}\}_{0\le k,j\le N-1}$. Such a truncation reproduces the perturbative expansion of the octagon to loop order $2N+\ell-1$,

$$\mathbb{O}_\ell = \det(\mathbf{I}+\mathbf{R}_{N\times N}^{(\ell)}) + o(g^{4N+2\ell}). \tag{4.5}$$

For example, the truncation to a 3×3 matrix gives the perturbative expansion up to $o(g^{12})$,

$$\begin{aligned} \mathbb{O}_{\ell=0} &= \det\left(1+\mathbf{R}^{(\ell)}\right)_{3\times 3} + o(g^{12}) \\ &= 1 + (1-z)(1-\bar{z})\,\left(f_1 g^2 - f_2 g^4 + \tfrac{1}{2} f_3 g^6 - \tfrac{5}{36} f_4 g^8 + \tfrac{7}{288} f_5 g^{10}\right) \\ &\quad + [(1-z)(1-\bar{z})\,]^2\,\left(\tfrac{1}{12}(f_1 f_3 - f_2^2)g^8 - \tfrac{1}{24}(f_1 f_4 - f_2 f_3)g^{10}\right) + o(g^{12}). \end{aligned}$$

The nine loop result presented in [12] is reproduced by truncating to a 5×5 matrix. To compare with [12] one should take $f_n = n!(n-1)!F_n$.

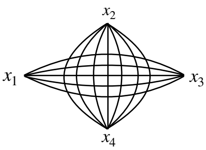
4.2 Ladders and Fishnets

There is an intriguing relation with the fishnet graphs studied originally by A. Zamolodchikov [20] and recently rediscovered in certain limits of the $\mathcal{N} = 4$ SYM theory [21–25]. Namely, in [13] it was conjectured that the octagon was expanded in a basis made of the minors of the semi-infinite matrix

$$\mathbf{f} = \begin{pmatrix} f_1 & f_2 & f_3 & \cdot \\ f_2 & f_3 & f_4 & \cdot \\ f_3 & f_4 & f_5 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{4.6}$$

Some of these minors evaluate rectangular fishnet graphs. The diagonal minor of the determinant (4.6) with rows and lines labeled by $\ell, \ell+1, ..., \ell+n$ equals the $n \times (n+\ell)$ rectangular fishnet graph [23]

$$[\text{fish}]_{n,n+\ell} =$$



$$= \det\left[\frac{f_{i+j+1+\ell}}{(2i+\ell)!(2j+1+\ell)!}\right]_{i,j=0,\dots,N-1} \tag{4.7}$$

The $n \times (n+\ell)$ fishnet Feynman diagram gives the lowest order of the n-magnon contribution to the octagon,

$$\mathbb{O}_\ell = \sum_{n=0}^{\infty} [(1-z)(1-\bar{z})]^n \; g^{2n(n+\ell)} \; \left([\text{fish}]_{n,n+\ell} + o(g^2)\right) , \tag{4.8}$$

This property of the octagon is obvious from the representation (4.5), which can be written as a sum over minors of the matrix $\mathbf{K}$, Eq. (4.3),

$$\begin{aligned} \mathbb{O}_\ell &= \sum_{n=0}^{\infty} \sum_{\substack{0 \le i_1 < \dots < i_n \\ 0 \le j_1 < \dots < j_n}} \det\left(\left[\mathrm{R}^{(\ell)}_{i_\alpha j_\beta}\right]_{\alpha,\beta=1,\dots,n}\right) \\ &= \sum_{n=0}^{\infty} \det \mathbf{R}^{(\ell)}_{n\times n} \; (1 + O(g^2)). \end{aligned} \tag{4.9}$$

Indeed, to the lowest order the determinant of the matrix $\boldsymbol{R}^{[\ell]}_{N\times N}$ is given by the fishnet integral normalised as in (4.7),

$$\det \boldsymbol{R}^{(\ell)}_{n\times n} = [(1-z)(1-\bar{z})]^n \; g^{2n(n+\ell)} \; [\text{fish}]_{n,n+\ell} \; + o(g^{2n(n+\ell)+2}). \tag{4.10}$$

5 Strong Coupling Limit

The strong coupling limit studied in [26] can be viewed as the semiclassical limit of the fermionic system where the free energy is given by an integral over the Fermi sea. First let us note that the pole of the fermionic correlator $C(x, y)$ is at $xy = 1$. It is more natural to replace the correlator by

$$C(x, y) = \frac{1}{x^\ell y^\ell} \frac{x - y}{xy - 1} \quad \to C(x, 1/y) = \frac{y^\ell}{x^\ell} \frac{xy - 1}{x - y} \tag{5.1}$$

and simultaneously replace the kernel by $K(1/x, y)$. The semiclassical kernel is evaluates by using the strong coupling approximation of the mirror momentum $2\tilde{p}_a(u) = a\tilde{p}(u)$, with $\tilde{p}(u) = ig\partial_u(x - 1/x)$,

$$e^{-E_{\rm cl}(u,k)} = \sum_{a=1}^{\infty} (-1)^a \chi_a\, e^{-ak} e^{2i\xi\, \tilde{p}_a(u)} e^{ia\ell/gx} = -\frac{\cosh\xi - \cos\phi}{\cosh\tilde{k} - \cos\phi}, \tag{5.2}$$

with

$$\tilde{k}(k, u, \ell) = k - iP, \quad iP = i\xi\,\tilde{p}(u) - i\ell/x = \xi \frac{x + \frac{1}{x}}{x - \frac{1}{x}} + i\frac{\ell}{gx}. \tag{5.3}$$

Here k is coupled to the fast variable $u_1 - u_2$ and the mirror momentum depends on the slow variable $u = (u_1 + u_2)/2$. The function $k_{\pm}^{\rm Fermi}(k, u) = k - iP(u)$ gives the profile of the Fermi sea. The dependence on ℓ shows up only in the subleading order except if $\ell \sim g$.

As is well-known, the grand potential in the semi-classical limit is given by the integral over the phase space,

$$\begin{aligned}\ln \mathbb{O}_\ell &\simeq \int \frac{du}{2\pi} \int_0^\infty dk\, \ln\left(1 + e^{-E_{\rm cl}(u,k)}\right) \\ &= \int \frac{du}{2\pi} \int_0^\infty dk\, \log \frac{(1 - e^{-k+iP+\xi})(1 - e^{-k+iP-\xi})}{(1 - e^{-k+iP+i\phi})(1 - e^{-k+iP-i\phi})}.\end{aligned} \tag{5.4}$$

The integral in k gives a sum of dilogarithms and the result matches with the expression obtained in [26].

$$\log \mathbb{O} = \int_{-2g}^{2g} \frac{du}{2\pi} \left(-{\rm Li}_2(e^{iP+\xi}) - {\rm Li}_2(e^{iP-\xi}) + {\rm Li}_2(e^{iP+i\phi}) + {\rm Li}_2(e^{iP-i\tilde{\phi}})\right), \tag{5.5}$$

$$iP = \xi \frac{x + 1/x}{x - 1/x} + i\frac{\ell/g}{x}.$$

6 The Octagon as a Fredholm Determinant with Modified Bessel Kernel

6.1 A Generalised Bessel Kernel

Here we give the representation of the octagon as a Fredholm determinant of a modified Bessel kernel which was thoroughly studied in [17] for $\ell = 0$. Using the summation formula

$$\sum_{m=\ell}^{\infty}(2m+1)\frac{J_{2m+1}(2w)J_{2m+1}(2z)}{w\,z} = \frac{w\,J_{\ell+1}(2w)J_\ell(2z) - z\,J_{\ell+1}(2z)J_\ell(2w)}{w^2 - z^2} \tag{6.1}$$

the determinant can be written [17] as a Fredholm determinant of the type

$$D_\ell(z,\bar{z}) = \mathrm{Det}\,[1 - K_\ell]\,, \quad [K_\ell f](t) = \int_0^\infty dt_1 K_\ell(t,t_1) f(t_1). \tag{6.2}$$

The integration kernel $K_\ell(t,t')$ is given by

$$\begin{aligned} K_\ell(t_1,t_2] &= K_\ell^0(t_1,t_2)\,\mathrm{X}(t_2), \\ K_\ell^0(t_1,t_2) &= \frac{t_1 t_2}{\sqrt{t_1 t_2}} \frac{t_1 J_{\ell+1}(gt_1)J_\ell(2gt_2) - t_2 J_{\ell+1}(2gt_2)J_\ell(2gt_1)}{t_1^2 - t_2^2}. \end{aligned} \tag{6.3}$$

All dependence on ϕ and ξ is carried by the factor $\Omega(t)$.

It is convenient, following [27], to change the variable so that also the dependence on the 't Hooft coupling g gets absorbed into the local factor. In terms of the new variable x defined by

$$2gt = \sqrt{x} \tag{6.4}$$

the kernel K_0 becomes the Bessel kernel which describes the statistics of the spacing of the eigenvalues of orthogonal ($\ell = -1/2$) or symplectic ($\ell = 1/2$) matrices of large order [28],

$$\begin{aligned} K_B(x,y) &= \frac{\phi(x)\psi(y) - \psi(x)\phi(y)}{2(x-y)} \\ &= \frac{\phi(x)y\partial_y\phi(y) - \phi(y)x\partial_x\phi(x)}{x-y} = \frac{1}{4}\int_0^1 d\alpha\phi(\alpha x)\phi(\alpha y), \\ \phi(x) &= J_\ell(\sqrt{x}),\;\; \psi(x) = -\sqrt{x}J_{\ell+1}(\sqrt{x}), \\ [K_B f](x) &= \int_0^\infty K_B(x,y)f(y)dy. \end{aligned} \tag{6.5}$$

The representations in the second line follow from the form of the derivatives

$$2x\partial_x\phi(x) = \ell\phi(x) + \psi(x), \quad -2x\partial_x\psi(x) = \ell\psi(x) + x\phi(x), \tag{6.6}$$

which allows one to replace in the definition of the Bessel kernel in the first line

$$\psi(x) \to 2x\partial_x\phi(x). \tag{6.7}$$

In our case the Bessel kernel is modified by a factor $\chi(x)$ obtained by changing the variable of X,

$$\begin{aligned} K_\chi(x, y) &= K_B(x, y)\chi(y), \\ \chi(x) = \mathrm{X}(\sqrt{x}/4g) &= \frac{\cosh\xi - \cos\phi}{\cosh\left(\xi\sqrt{1 + x/(4g\xi)^2}\right) - \cos\phi}. \end{aligned} \tag{6.8}$$

6.2 Differential Equations

The analysis by Belitsky and Korchemsky [17, 27] works for any function χ provided it satisfies the homogeneity property

$$\Delta\chi = 0, \qquad \Delta \equiv 2x\frac{\partial}{\partial x} + g\frac{\partial}{\partial g}. \tag{6.9}$$

They derived a system of integro-differential equations for the functions

$$U = -2g\partial_g \log\mathbb{O}, \qquad Q(x) = \langle x|\frac{1}{1 - K_\chi}|\phi\rangle, \tag{6.10}$$

assuming that $\ell = 0$. Here we formulate the equations for general bridge length ℓ,

$$\partial_\alpha U = \int_0^\infty dx\, Q^2(x)\partial_\alpha\chi(x), \qquad \alpha = g, \phi, \xi, \tag{6.11}$$

$$\left(\Delta^2 - g(\partial_g U) + U\right) Q = (\ell^2 - x)Q. \tag{6.12}$$

The differential equations are supplemented with the boundary condition at weak coupling

$$Q(x) = J_\ell(\sqrt{x}) + O(g^2). \tag{6.13}$$

Assuming that the solution is found, the octagon is given by the integral

$$\mathbb{O} = \int_0^g \frac{dg'}{g'} U(\phi, \xi, g'). \tag{6.14}$$

Once the function Q is found, the derivatives of the octagon with respect of $\alpha = \{\xi, g, \phi\}$ are computed as

$$\partial_\alpha \log \mathbb{O} = \tfrac{1}{2} \int_0^\infty dx (\partial_\alpha \chi)\ Q^2 \partial_x \Delta \log Q. \tag{6.15}$$

Taking into account the particular form of χ the three differential equations take the form

$$\partial_\phi U = \int_0^\infty dx\ Q^2(x) \partial_\phi \chi(x), \tag{6.16}$$

$$g\partial_g U = -2 \int_0^\infty dx\ Q^2(x) x \partial_x \chi(x), \tag{6.17}$$

$$\partial_\xi U = 8g^2 \xi \int_0^\infty dx\ Q^2(x) \partial_x \chi(x) + \frac{\sinh(\xi)}{\cosh \xi - \cos \phi} \int_0^\infty dx\ Q^2(x) \chi(x) \tag{6.18}$$

together with the Eq. (6.12) for Q. In [17] perturbative methods of solution were developed for the weak and strong coupling limits.

7 Conclusions

Here we reviewed the methods for computing the octagon form factor, which appears as a building block of a class of polarised four-point functions of heavy half-BPS operators. The octagon is made of two hexagons are glued together which do not interact with the rest and appears whenever a skeleton graph contains a square delimited by four large bridges.

We reviewed the determinantal representation of the octagon obtained in [1, 2] and gave a simplified derivation based on the Fock-space representation in terms of free fermions. The fermionic representation was possible because the dressing factors of the scattering matrices do not appear in the expansion.

The pure octagon appears in the simplest case where the hexagons are forced to stay in couples. It is worth exploring the possibility of higher order form factors representing chains of hexagons in interaction (decagon, dodecagon) by several species of free fermions. Such form factors occur the higher-point polarised correlation functions of heavy half-BPS operators or if one of the charges in the four-point function is kept finite.

Very recently, Belitsky and Korchemsky developed a powerful approach to the octagon based on a system of integro-differential equations [17, 27]. Their approach allows to construct systematically the strong coupling expansion, which is notoriously difficult problem in general.

Concerning weak coupling, the appearance of the fishnet graphs is somewhat mysterious, although not completely unexpected. It would be interesting to find an interpretation of the higher order graphs as fishnet graphs with defects.

Finally, it seems that the integrability structure of the octagon is not completely unveiled. The operator fermionic representation of the octagon resembles the tau-function of a Toda chain hierarchy and it is thus possible that there exist Hirota-like equations relating octagons with bridges ℓ and $\ell \pm 1$. Once the integrable structure of the octagon is completely understood, one can try to generalise it to non-BPS operators.

Acknowledgements I am grateful to Gregory Korchemsky and Valentina Petkova for useful discussions during the preparation of this text.

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Nöther's Second Theorem as an Obstruction to Charge Quantization

Philip Phillips and Gabriele La Nave

Abstract While it is a standard result in field theory that the scaling dimension of conserved currents and their associated gauge fields are determined strictly by dimensional analysis and hence cannot change under any amount of renormalization, it is also the case that the standard conservation laws for currents, $dJ = 0$, remain unchanged in form if any differential operator that commutes with the total exterior derivative, $[d, \hat{Y}] = 0$, multiplies the current. Such an operator, effectively changing the dimension of the current, increases the allowable gauge transformations in electromagnetism and is at the heart of Nöther's second theorem. We review here our recent work on one particular instance of this theorem, namely fractional electromagnetism and highlight the holographic dilaton models that exhibit such behavior and the physical consequences this theory has for charge quantization. Namely, the standard electromagnetic gauge and the fractional counterpart cannot both yield integer values of Planck's constant when they are integrated around a closed loop, thereby leading to a breakdown of charge quantization.

1 Preliminaries

Although Nöther [12] has two theorems, the second is little known but ultimately more important, as we will see. The first theorem which sits at the foundation of gauge theories asserts that applying the gauge-invariant condition of electromagnetism $A_\mu \to A_\mu + \partial_\mu \Lambda$ to the Maxwell action

$$S = -\frac{1}{4}\int d^d x (F^2 + J_\mu A^\mu) \tag{1}$$

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_9

results in the conservation law

$$\partial_\mu J_\mu = 0, \tag{2}$$

with $F = dA$. Because Λ is a dimensionless function, $[A] = 1$ and the current has fixed dimension $[J] = d - 1$. Had we retained the dimensionful charge, we would have that $[qA] = 1$. Note the covariant derivative, heuristically written as $D - iqA$, only fixes the dimension of the product $[qA] = 1$. Hence, it is entirely possible to construct theories [4] in which q and A have arbitrary dimensions without changing how the gauge group acts. In what remains, we have set $q = 1$ but our remarks apply to the dimensionful case as well. The well known ambiguity (or "improvement transformations" [15]) of the current, namely that the conservation laws remain fixed under shifting the current by a total derivative of the form, $J_0 \to J_0 + \partial_\mu X^\mu$ and $J_\mu \to J_\mu + \partial^0 X_\mu$, have no effect on the conserved charge nor the dimension of the current. In fact as Gross [6, 13] pointed out, because it is the action of the $U(1)$ group that ultimately fixes the dimension of the current through

$$\delta(x_0 - y_0)[J_0(x), \phi(y)] = \delta\phi(y)\delta^d(x - y), \tag{3}$$

the dimension of the current, $[J_\mu] = d - 1$, is sacrosanct unless one changes how the $U(1)$ group acts.

This is the context [12] for NST. Nöther [12] noticed that the form of the conservation law for the current is determined by the order of the derivative retained in the degeneracy condition for A_μ. In fact, there is no unique way of specifying this as can be seen from the following argument. Consider the Maxwell action,

$$\begin{aligned} S &= \frac{1}{2}\int \frac{d^d k}{2\pi^d} A_\mu(k)[k^2\eta^{\mu\nu} - k^\mu k^\nu]A_\nu(k) \\ &= \frac{1}{2}\int d^d k A_\mu(k) M^{\mu\nu} A_\nu(k). \end{aligned} \tag{4}$$

All gauge transformations arise as zero-eigenvalues of M. For example,

$$M^{\mu\nu} k_\nu = 0, \tag{5}$$

which yields the standard gauge-invariant condition in electromagnetism because ik_ν is just the Fourier transform of ∂_μ. The ambiguity that leads to NST comes from noticing that if k_ν is an eigenvector, then so is $f k_\nu$, where f is a scalar. Whence, there are a whole family of eigenvectors,

$$M_{\mu\nu} f k^\nu = 0, \tag{6}$$

that satisfy the zero eigenvalue condition, each characterizing a perfectly valid electromagnetism. It is for this reason that Nöther [12] devoted the second half of her paper to the consequences of retaining all possible integer derivatives,

$$A_\mu \to A_\mu + \partial_\mu \Lambda + \partial_\mu \partial_\nu G^\nu + \cdots, \tag{7}$$

in the gauge-invariant condition for A_μ on the conservation laws for the current. Stated succinctly, the second theorem finds that the full family of generators of $U(1)$ invariance determines the dimension of the current not just the linear derivative term $A_\mu \to A_\mu + \partial_\mu \Lambda$. In general, the second theorem applies anytime there are either a collection of infinitesimal symmetries or one symmetry parametrized by an arbitrary number of functions as in Eq. (7). What Nöther [12] found is that the higher-order derivatives in the gauge-invariant condition add further constraints on the current. They can even change the order of the current. However, as long as only integer derivatives [1] are retained, the constraint equations yield no new content. It is for this reason that Nöther's second theorem has garnered little interest.

However, there is a generalization of Eq. (7) that does yield non-trivial results. Consider the fact that the current conservation equation remains unchanged if a differential operator $\hat{Y}$ exists such that $[d, \hat{Y}] = 0$. If such an operator exists then the conservation law becomes

$$\partial_\mu \hat{Y} J_\mu = \partial_\mu \tilde{J}_\mu = 0, \tag{8}$$

which the redefines the current to be $\tilde{J} = \hat{Y} J$. This is an ambiguity **distinct** from the "improvement transformations" of the first theorem because $\hat{Y}$ is linked to the gauge symmetry. We can construct $\hat{Y}$ directly from Eq. (6). Since $f k_\nu$ is the generator of the gauge symmetry, there are some constraints on f. (1) f must be rotationally invariant. (2) f cannot change the fact that Λ is dimensionless; equivalently it cannot change the fact that A is a 1-form. (3) f must commute with the total exterior derivative; that is, $[f, k_\mu] = 0$ just as $[d, \hat{Y}] = 0$. Hence, finding f is equivalent to fixing $\hat{Y}$. A form of f that satisfies all of these constraints is $f \equiv f(k^2)$. In momentum space, k^2 is simply the Fourier transform of the Laplacian, $-\Delta$. As a result, the general form of $f(k^2)$ in real space is just the Laplacian raised to an arbitrary power, and the generalization in Eq. (6) implies that there are a multitude of possible electromagnetisms (in vacuum) that are invariant under the transformation,

$$A_\mu \to A_\mu + f(k^2) i k_\mu \Lambda, \tag{9}$$

or in real space,

$$A_\mu \to A_\mu + (-\Delta)^\gamma \partial_\mu \Lambda, \tag{10}$$

resulting in $[A_\mu] = 1 + 2[f] = \gamma$. The definition of the fractional Laplacian we adopt here is due to Reisz:

$$(-\Delta_x)^\gamma f(x) = C_{n,\gamma} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{\mid x - \xi \mid^{n+2\gamma}} \, d\xi \tag{11}$$

for some constant $C_{n,\gamma}$. Note rather than just depending on the information of $f(x)$ at a point, the fractional Laplacian requires information everywhere in $\mathbb{R}^n$. The standard Maxwell theory is just a special case in which $\gamma = 1$. In general, the theories that result for $\gamma \neq 1$ allow for the current to have an arbitrary dimension not necessarily $d - 1$. Identifying $\hat{Y}$ with the fractional Laplacian yields the conservation law

$$\partial^\mu (-\Delta)^{(\gamma-1)/2} J_\mu = 0. \tag{12}$$

Conservation laws such as the one in Eq. (12) are in some sense more fundamental, as one can infer the standard ones from them but more importantly they can occur earlier [9, 11] in the hierarchy of conservation laws that stem from Nöther's first theorem. This is the same conclusion reached from the degeneracy of the eigenvalue of Eq. (6). This consilience is not surprising because the degeneracy of the eigenvalue is another way of stating Nöther's second theorem. That is, the current is not unique in gauge theories. It is the lack of the uniqueness of the current that yields a breakdown of charge quantization. As expected, this ambiguity shows up at the level of the Ward identities. The current-current correlator for the photon

$$C^{ij}(k) \propto (k^2)^\gamma \left(\eta^{ij} - \frac{k^i k^j}{k^2} \right) \tag{13}$$

does not just satisfy $k_\mu C^{\mu\nu} = 0$ but also $k^{\gamma-1} k_\mu C^{\mu\nu} = 0$. This translates into either $\partial_\mu C^{\mu\nu} = 0$, the standard Ward identity, or

$$\partial_\mu (-\Delta)^{\frac{\gamma-1}{2}} C^{\mu\nu} = 0 \tag{14}$$

which illustrates beautifully the fact that the current conservation equation only specifies the current up to any operator that commutes with the total differential. As we mentioned previously, this appears to be the first time this ambiguity has been linked to Nöther's Second Theorem.

Because the fractional Laplacian is a non-local operator, the corresponding gauge theories are all non-local and offer a much broader formulation of electricity magnetism than previously thought possible. All such anomalies can be understood as particular instances of Nöther's Second Theorem. We will show how such theories arise from holographic bulk dilaton models [5, 8] and show that Eq. (8) leads to a breakdown of charge quantization as can be seen from the fractional version of the Aharonov-Bohm effect [10, 11].

2 Charge Quantization

Changing the dimension of the vector potential has profound consequences for the quantization of charge. This can be seen immediately because the integration of the gauge field around a closed loop

$$q \oint \mathbf{A} \cdot d\ell = h\mathbb{Z} \tag{15}$$

must be an integer multiple of Planck's constant, h. This condition amounts to the integrability condition for the cohomology class of qA to be an *integral class*. Consequently, charge quantization is equivalent to the geometric requirement that the form $F_A = dA$ be indeed the curvature of a connection $D = d - qA$ on a $U(1)$ principal bundle $\mathcal{P}$. It is on this fact that the Byers-Yang [2] theorem is based. Clearly then when $[A] \neq 1$, the integral above is no longer dimensionless, leading to an inapplicability of the Byers-Yang theorem. What is required in such cases is the construction of a new fictitious gauge field that does have the requisite dimension. While the new gauge will preserve Eq. (15), the original one will not [8, 10]. Consequently, if it is the fractional gauge field that describes the material in question, strictly speaking, charge is not quantized. That is, both gauges cannot preserve Eq. (15) simultaneously. Maxwell's equations amount to setting $f = 1$ or $\gamma = 1$. As $f \neq 1$ is a perfectly valid electromagnetism, charge quantization is essentially a choice. This is a physical consequence of Nöther's second theorem.

3 Holographic Models with Fractional Gauge Transformations

The preliminaries lay play that within a model with local interactions and with $U(1)$ symmetry in tact, there is no way around Gross's [6] argument that the dimension of the gauge field and the current are fixed to $[A] = 1$ and $[J] = d - 1$, respectively. However, Nöther's second theorem suggests that other possibilities exist. Interestingly [17, 19], superconductivity provides a simple counter example, in which the current,

$$J_i = -\int K_{ij}(\boldsymbol{x}, \boldsymbol{x}') \left(\boldsymbol{A}_j(\boldsymbol{x}') - \nabla'_j \phi(\boldsymbol{x}')\right) d^3x' \tag{16}$$

has dimension $d - d_K - 1$ and hence is a non-local function of the gauge field, $A(\boldsymbol{x}')$ as a result of the kernel K_{ij} which arises from expanding the free energy around the minimum $\nabla\phi - A = 0$ with ϕ the $U(1)$ phase. The Pippard kernel [14], relevant to explaining the disorder dependence of the Meissner effect, amounts to a particular choice for K_{ij}. Holographic constructions offer a possibility as a result of the extra dimension which allows for the boundary (either at the UV or the IR) to have properties quite distinct from the bulk. A distinct claim of dilatonic models of the form

$$S = \int \mathrm{d}^{d+1}x dy \sqrt{-g}\left[\mathcal{R} - \frac{\partial\phi^2}{2} - \frac{Z(\phi)}{4}F^2 + V(\phi)\right], \tag{17}$$

is that the boundary gauge field acquires an anomalous dimension that is determined solely by the asymptotic form of the action

$$S_{\mathrm{Max}} = \int dV_d dy (y^a F^2 + \cdots), \tag{18}$$

where y is the radial coordinate in the anti-de Sitter spacetime. That such models change the gauge structure at the boundary can be seen by interpreting the dilaton term y^a as a running charge coupling $g(a)$ which depending on the exponent a can yield a relevant interaction at either the UV or at the IR horizon. In the standard holographic set-up [7, 18], the boundary lacks a global $U(1)$ structure only the bulk does where the gauge field acts as source for the boundary current. That is, the conformal boundary, which we denote by the zero of the radial coordinate, $y = 0$, is not imbued by a local gauge structure in which $A(y = 0, x) = A_{\parallel} + d\Lambda$. More explicitly, once the boundary condition is set, $A(y = 0, x) = A_{\parallel}$, the gauge degree of freedom is lost. Of course, the gauge structure can be reinstated simply by changing the boundary conditions from Dirichlet to von Neumann. Alternatively, the theory can have a non-trivial structure at the IR or at the horizon. Theories valid at either the UV or the IR boundary can be constructed using the membrane paradigm [16]. In this case, this approach is particularly apropos as either the IR or UV limits are relevant depending on the value of a as can be seen from the equations of motion,

$$\nabla^{\mu}(\rho^a F_{\mu\nu}) = 0, \tag{19}$$

where we have introduced the radius $\rho = r - r_h$ which measures the distance from the horizon. As depicted in Fig. 1 it is the IR limit which is relevant if $a > 0$ and the UV in the opposite limit.

To construct solve this boundary value problem, we appeal to a well known theorem in analysis. In 2007, Caffarelli and Silvestre (CS) [3] proved that standard second-order elliptic differential equations in the upper half-plane in $\mathbb{R}^{n+1}_{+}$ reduce to one with the fractional Laplacian, $(-\Delta)^{\gamma}$, when one of the dimensions is eliminated to achieve $\mathbb{R}^n$. For $\gamma = 1/2$, the equation is non-degenerate and the well known reduction of the elliptic problem to that of Laplace's obtains. The precise statement of this highly influential theorem is as follows. Let $f(x)$ be a smooth *bounded* function in $\mathbb{R}^n$ that we use to solve the extension problem,

$$\begin{aligned} g(x, y = 0) &= f(x) \\ \triangle_x g + \frac{a}{y} g_y + g_{yy} &= 0, \end{aligned} \tag{20}$$

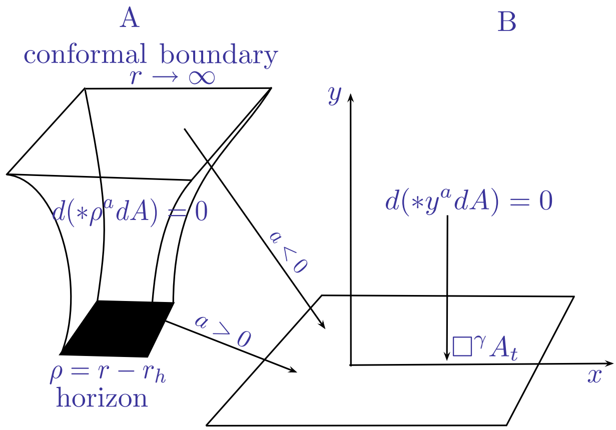

Fig. 1 **A.)** A depiction on an AdS spacetime with a conformal boundary at $r = \infty$ and a black hole horizon at $r = r_h$. The Maxwell-dilaton action in the bulk has equations of motion of the form $d(*\rho^a dA) = 0$. **B.)** p-form generalization of the Caffarelli-Silvestre [3]-extension theorem. A_t are the boundary (tangential) components of the bulk gauge field, A. For a dilaton action in $\mathcal{R}^n$ with the equations of motion $d(*y^a dA) = 0$, the restriction of these equations of motion to the boundary yields the fractional Box operator where the exponent is given by $\gamma = (1 - a)/2$. Depending on the sign of a, the bulk dilaton action either yields fractional Maxwell equations of motion at the conformal UV ($a < 0$) boundary or at the IR limit ($a > 0$) demarcated by the horizon radius, r_h

to yield a smooth *bounded* function, $g(x, y)$ in $\mathbb{R}^{n+1}_+$. $f(x)$ functions as the Dirichlet boundary condition of $g(x, y)$ at $y = 0$. These equations can be recast in degenerate elliptic form,

$$\mathrm{div}(y^a \nabla g) = 0 \quad \in \mathbb{R}^{n+1}_+, \tag{21}$$

which CS proved has the property that

$$\lim_{y \to 0^+} y^a \frac{\partial g}{\partial y} = C_{n,\gamma} \, (-\triangle)^\gamma f \tag{22}$$

for some (explicit) constant $C_{n,\gamma}$ only depending on d and $\gamma = \frac{1-a}{2}$ with $(-\triangle)^\gamma$, the Reisz fractional Laplacian defined earlier. That is, the fractional Laplacian serves as a Dirichlet to Neumann map for elliptic differential equations when the number of dimensions is reduced by one. Consider a simple solution in which, $g(x, 0) = b$, a constant, but also $g_x = 0$. This implies that $g(y) = b + y^{1-a} h$ with $(1 - a) > 0$. Imposing that the solution be bounded as $y \to \infty$ requires that $h = 0$ leading to a vanishing of the LHS of Eq. (22). The RHS also vanishes because $(-\Delta_x)^\gamma b = 0$. As a final note on the theorem, from the definition of the fractional Laplacian, it is clear that it is a non-local operator in the sense that it requires knowledge of the function everywhere in space for it to be computed at a single point. In fact, it is explicitly an anti-local operator. Anti locality of an operator $\hat{T}$ in a space $V(x)$

means that for any function $f(x)$, the only solution to $f(x) = 0$ (for some $x \in V$) and $\hat{T} f(x) = 0$ is $f(x) = 0$ everywhere. Fractional Laplacians naturally satisfy this property of anti-locality as can seen from their Fourier transform of Eq. (11).

Equation (19) is highly reminiscent of Eq. (21) of the CS construction. The only difference is that g is a scalar in the CS-extension theorem and the gauge field is a 1-form. Hence, the p-form generalization [11] of the CS-extension theorem is precisely the tool we need to determine the gauge structure either at the conformal boundary or at the horizon. The key ingredient in this proof is the fractional differential. Because the Hodge Laplacian

$$\Delta = dd^* + d^*d : \Omega^p(M) \to \Omega^p(M), \tag{23}$$

does not change the order of a p-form, as it is a product of d and d^*, it can be used to define the fractional differential

$$d_\gamma = d\Delta^{\frac{\gamma-1}{2}} = \Delta^{\frac{\gamma-1}{2}}, \quad d_\gamma^* = d^*\Delta^{\frac{\gamma-1}{2}}. \tag{24}$$

Since $[d, \Delta^b] = 0$ for any power b, a key benefit of d_γ is that the composition

$$(d_\gamma d_\gamma^* + d_\gamma^* d_\gamma)\omega = \Delta^\gamma \omega \tag{25}$$

offers a way of computing the action of fractional Laplacian on the differential form ω.

These definitions allow an immediate construction of the p-form generalization of the CS-extension theorem for $\alpha \in \Omega^p$ and a bounded solution to the extension problem

$$\begin{gathered} d(y^a d^*\alpha) + d^*(y^a d\alpha) = 0 \in M \times \mathbb{R}_+ \\ \alpha \mid_{\partial M} = \omega \text{ and } d^*\alpha \mid_{\partial M} = d_x^*\omega, \end{gathered} \tag{26}$$

then

$$\lim_{y \to 0} y^a \mathrm{i}_\nu d\alpha = C_{n,a}(\Delta)^\gamma \omega, \tag{27}$$

with $2\gamma = 1 - a$ and where $\mathrm{i}_V\omega$ indicates the $(p-1)$-form determined by $\mathrm{i}_V\omega$ $(X_1, \cdots, X_{p-1}) = \omega(X_1, \cdots, X_{p-1}, V)$, $\nu = \frac{\partial}{\partial y}$, for some positive constant $C_{n,a}$. This is the p-form generalization of the Caffarelli/Silvestre extension theorem. It implies that the CS extension theorem on forms is the CS extension theorem on the components of the p-form. The method of proof was simply component-by-component. The succinct statement in terms of the components is easiest to formulate from the equations of motion

$$\begin{gathered} \mathrm{div}(y^a \nabla \alpha_{i_1 \cdots i_p}) = 0 \in M \times \mathbb{R}_+ \\ \left(\alpha_{i_1 \cdots i_p}\right) \mid_{\partial M} = \omega_{i_1 \cdots i_p} \text{ and } d^*\alpha \mid_{\partial M} = d_x^*\omega. \end{gathered} \tag{28}$$

Therefore, using the CS theorem, we have that

$$\lim_{y\to 0} \; y^a \frac{\partial \alpha_{i_1\cdots i_p}}{\partial y} = C_{n,a}(-\Delta)^\alpha \; \omega_{i_1\cdots i_p}, \tag{29}$$

which proves that

$$\lim_{y\to 0} y^a \mathrm{i}_\nu d\alpha = (\Delta)^a \omega, \tag{30}$$

since by (elliptic) regularity of solutions to Eq. (26)

$$\lim_{y\to 0} y^a \frac{\partial \alpha_{0\ell_1,\cdots \ell_{p-1}}}{\partial x^{j_k}} = 0. \tag{31}$$

Applying this result to the dilaton equations of motion, Eq. (19) results in the fractional Maxwell equations

$$\Delta^\gamma A_t = 0. \tag{32}$$

for the boundary components of the gauge field. Since the only restriction is that $2\gamma = 1 - a$, this proof applies equally, with the use of the membrane paradigm [16], at the conformal boundary and the horizon. Hence, even the dynamics in the IR (horizon) are governed by a fractional Maxwell action. The curvature that generates these boundary equations of motion is

$$F_\gamma = d_\gamma A = d\Delta^{\frac{\gamma-1}{2}} A, \tag{33}$$

with gauge-invariant condition,

$$A \to A + d_\gamma \Lambda, \tag{34}$$

where the fractional differential is as before in Eq. (24) which preserves the 1-form nature of the gauge field. This feature is guaranteed because by construction, the fractional Lagrangian cannot change the order of a form. As is evident, $[A_\mu] = \gamma$, rather than unity. This gauge transformation is precisely of the form permitted by the preliminary considerations on Nöther's second theorem presented at the outset of this article and also consistent with the zero eigenvalue of the matrix M in Eq. (4).

4 Nöther's Second Theorem Revisited

In order to determine how the fractional gauge acts, we first define the covariant derivative. To this end, we consider the ansatz,

$$D_{\alpha,\beta,A}\phi = (d + i\Box^\beta A)\Box^\alpha \phi, \tag{35}$$

with α and β to be determined. The reason behind this ansatz for the covariant derivative is that we require the existence of a non-local transformation of the field ϕ, the vector potential A and the infinitesimal gauge group generator Λ such that the covariant derivative transforms in the usual way $D_{\alpha,\beta,A}$ to the standard $D_{A'}\phi' = (d + iA')\phi'$ with the field redefinitions

$$\phi' = \Box^{\alpha}\phi \qquad A' = \Box^{\beta}A. \tag{36}$$

The Gauge action on ϕ' and A' is thus the classical one

$$\phi' \to e^{i\Lambda'}\phi' \qquad A' \to A' + d\Lambda' \tag{37}$$

and

$$D_{A'}\left(e^{i\Lambda'}\phi'\right) = e^{i\Lambda'}D_{A'+d\Lambda'}\phi'. \tag{38}$$

Following the non-local transformations of Eq. (36), it is natural to suppose a field redefinition for the infinitesimal generators of the Gauge group as

$$\Lambda' = \Box^{\delta}\Lambda. \tag{39}$$

Naturally, after such a change, there is only one way to define the Gauge group action,

$$e^{i\Lambda} \odot \phi = \Box^{-\alpha}\left(e^{i\Box^{\delta}\Lambda}\Box^{\alpha}\phi\right), \tag{40}$$

to make $D_{\alpha,\beta,A}$ equivariant. The equivariant condition is then

$$D_{\alpha,\beta,A}\left(e^{i\Lambda} \odot \phi\right) = e^{i\Box^{\delta}\Lambda}D_{\alpha,\beta,A+d\Box^{\delta}\Lambda}\phi. \tag{41}$$

We will define the curvature of $D_{\alpha,\beta,A}$ to be

$$F_{\alpha,\beta,A}\phi = (d + i\Box^{\beta}A)D_{\alpha,\beta,A}\phi. \tag{42}$$

This definition has the feature that it reduces to the curvature $F_{A'}$ after the transformations in Eq. (36). In fact, it also reduces to the curvature $F_{\Box^{\beta}A}$ after the mere change of fields $\phi \to \phi'$. At this point, we have not fixed the values of α, β and δ. There are three natural conditions which we impose that will determine uniquely their values (hence the covariant derivative) and the nature of the Gauge group action at the same time:

1. $D_{\alpha,\beta,A}\phi$ restricts to the fractional differential $d_{\gamma}\phi$ on functions when $A = 0$.
2. The Gauge group action on connection fields must be $A \to A + d_{\gamma}\Lambda$.
3. The curvature $F_{\alpha,\beta,A} = id_{\gamma}A$.

The restriction that the covariant derivative reduce to the fractional differential, d_γ, when the fields are functions (Condition 1) imposes that $\alpha = \frac{\gamma-1}{2}$. Next, we use Condition 2 to determine the value δ. A quick read of Eq. (41) will convince the reader that the Gauge transformation sends A to $A + d\Box^\delta \Lambda$. Therefore, in order for condition 2 to hold, we require that $\delta = \frac{\gamma-1}{2}$. Finally, in order to satisfy Condition 3, we make explicit the formula in Eq. (42).

$$\begin{aligned} F_{\alpha,\beta,A}\phi &= (d + i\Box^\beta A)(d + i\Box^\beta A)\Box^\alpha \phi \\ &= dd\Box^\alpha\phi + id\left(\Box^\beta A\Box^\alpha\phi\right) + i\Box^\beta A d\Box^\alpha\phi - \Box^\beta A \wedge \Box^\beta A\Box^\alpha\phi \\ &= id(\Box^\beta A)\phi - i\Box^\beta A d\Box^\alpha\phi + i\Box^\beta A d\Box^\alpha\phi = id(\Box^\beta A)\phi. \end{aligned} \tag{43}$$

Therefore for Condition 3 to hold, we require that $\beta = \frac{\gamma-1}{2}$. Summarizing, we have

$$\begin{aligned} D_{\gamma,A}\phi &= (d + i\Box^{\frac{\gamma-1}{2}} A)\Box^{\frac{\gamma-1}{2}}\phi \\ e^{i\Lambda} \odot \phi &= \Box^{\frac{1-\gamma}{2}}\left(e^{i\Box^{\frac{\gamma-1}{2}}\Lambda}\Box^{\frac{\gamma-1}{2}}\phi\right) \\ F_{\gamma,A} &= (d + i\Box^{\frac{\gamma-1}{2}} A)D_{\gamma,A}\phi, \end{aligned} \tag{44}$$

and the equivariance condition is

$$D_{\gamma,A}\left(e^{i\Lambda}\odot\phi\right) = e^{i\Box^{\frac{\gamma-1}{2}}\Lambda} D_{\alpha,\beta,A+d_\gamma\Lambda}\phi. \tag{45}$$

We can now put Nöther's second theorem in this context of the redefined fields A' and ϕ'. What we will show is that the standard version of Nöther's second theorem can be applied straightforwardly to a gauge action with A' and ϕ' which can then be translated back to its non-local counterpart in terms of A and ϕ. Given a schematic action for some field $\Phi(\mathbf{x})$,

$$S(\Phi) = \int dx L(x, \Phi(\mathbf{x})), \tag{46}$$

we consider the infinitesimal action of Lie algebras and infinitesimal generators represented by vector fields

$$X = (D_I Q^\alpha)\frac{\partial}{\partial \Phi^\alpha_{,I}}, \tag{47}$$

with $D_I = D_1^{i_1}\cdots D_p^{i_p}$ for any multi-index $I = (i_1, \cdots, i_p)$, $\Phi(x)^\alpha_{,I} = \frac{\partial}{\partial x^{i_1}}\cdots\frac{\partial}{\partial x^{i_p}}\Phi^\alpha$, and $D_m = \frac{\partial}{\partial x_m} + \sum_{\alpha,I}\frac{\partial\Phi^\alpha_{,I}}{\partial x_m}\frac{\partial}{\partial\Phi^\alpha_{,I}}$ by solving the (linearized) symmetry conditions

$$X(E_\alpha(L)) = 0, \tag{48}$$

where $E_\alpha(L)$ are the Euler-Lagrange operators

$$E_\alpha(L) = (-1)^{|I|} D_I L \tag{49}$$

and $|I| = \sum_m i_m$. The solution, $Q(\Phi) = (Q_1, \cdots Q_q)$ is called the characteristic of the symmetries generated by X. Nöther's second theorem can now be stated as follows.

Theorem 1. *The action S admits an infinite dimensional group of symmetries with characteristics $Q(\Phi, B)$ that depend on arbitrary functions B if and only if there exist differential operators P_i such that*

$$\sum_i P_i E_i(L) = 0. \tag{50}$$

In this language, the content of the second theorem is that P_i are determined by the vector field X and the statement is that there are infinitely many characteristics, that is, charges, if the sum preserves the total symmetry of the Euler-Lagrange equations.

We apply this discussion and Nöther's second theorem to the Lagrangian

$$L' = \frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} + (D'_A)_\mu \phi' (D'_A)^\mu \phi' + m^2 \phi' \phi'^* \tag{51}$$

of the redefined fields A', ϕ' defined in the previous section. The Euler-Lagrange equations have components

$$\begin{aligned} E_{\phi'}(L) &\equiv -\left((D'_A)_\mu (D'_A)^\mu \phi'\right)^* + m^2 \phi'^* = 0 \\ E_{\phi'^*}(L) &\equiv -(D'_A)_\mu (D'_A)^\mu \phi' + m^2 \phi' = 0 \\ E_{A_\mu}(L) &\equiv i\phi' \left((D'_A)_\mu \phi'\right)^* - i\phi'^* (D'_A)_\mu \phi' + \eta_{\mu\alpha} F^{\alpha\beta}_{,\beta} = 0. \end{aligned} \tag{52}$$

Amongst the variational symmetries, one finds the Gauge symmetries

$$\phi' \to e^{i\Lambda'} \phi'; \qquad A' \to A' + d\Lambda' \tag{53}$$

The generalized characteristics of the Gauge symmetries in components define an infinite set of charges

$$Q^{\phi'} = -i\phi' B \qquad Q^{\phi'^*} = i(\phi')^* B \qquad Q^\mu = \eta^{\mu\nu} B_{,\nu}, \tag{54}$$

for some arbitrary real function B. The differential identity, Eq. (50), in Nöther's theorem is now

$$-i\phi' E_{\phi'}(L) + i\phi'^* E_{\phi'^*}(L') - D_\alpha \left(\eta^{\mu\nu} E_\nu(L')\right) = 0, \tag{55}$$

where D_α is the total derivative. The key here is that the charges are arbitrary but yield nothing new [1] in the classical theory where only integer derivatives are present. Clearly this is an operator equation as one can see by carrying out the calculations for $\int \mathcal{D}\phi\mathcal{D}Ae^{-S(\phi,A)}$ with arbitrary insertions.

When going back to the fields A, ϕ and the infinitesimal Gauge parameter Λ, we find that there is now a new non-trivial relation which gives rise to the action $A \to A + d_\gamma \Lambda$ and a new charge $Q = \int j_\gamma$ which did not exist in the theory corresponding to the action S' (i.e., the classical Maxwell's equations). Effectively, one can see the fractional Maxwell equations as emergent from imposing the symmetry to be generated by the non-local action $e^{i\Box^\alpha \Lambda}$ for some α.

5 Aharonov-Bohm and Charge Quantization

The inherent problem the degree of freedom f (see Eq. (6)) introduces into electromagnetism is that the multiplicity of gauge fields that are related by the fractional Laplacian, A and A', each satisfy

$$\int_\Sigma d_\gamma A = \oint_{\partial\Sigma} A', \tag{56}$$

with $A' = \Delta^{\frac{(\gamma-1)}{2}} A$. As pointed out previously, although this equality follows from Stokes' theorem, the result does not seem to have the units to be a *quantizable* flux. That is, it is not simply an integer $\times hc/e$. The implication is then that the charge depends on the scale. In fact, because $[d, \Box^\gamma] = 0$, the equations of motion can be rewritten as

$$\Box^{\frac{\gamma-1}{2}} d(\star d \Box^{\frac{\gamma-1}{2}} A) = \star J. \tag{57}$$

The current that emerges when Eq. (57) is invertible has the equations of motion,

$$d(\star d\tilde{A}) = \star\Box^{\frac{1-\gamma}{2}} J \equiv \star j. \tag{58}$$

Similarly, the classical electromagnetic gauge $a \equiv \Box^{\frac{1-\gamma}{2}} A \equiv \Box^{1-\gamma} A'$, hence having unit dimension, obeys the equations of motion

$$d(\star da) = \Box^{1-\gamma} j = \Box^{\frac{3}{2}(1-\gamma)} J. \tag{59}$$

Each of these choices for the gauge field defining different currents are all equally valid descriptions of nature. The problem is that they are not all quantizable simultaneously. For example, we have shown [9] that

$$\mathrm{N}orm\left(\int_\ell A'\right) = \frac{\int_\ell A}{\Gamma(s+1)}, \tag{60}$$

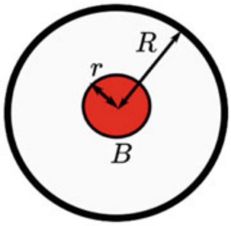

Fig. 2 Disk geometry for the Aharonov-Bohm phase [10]. The fractional magnetic field pierces the disk in a small region of radius, r

with $s = \frac{1-\gamma}{2}$, provided $\gamma < 1$, and

$$\int_\ell A' = 0 \tag{61}$$

if $\gamma > 1$. Hence, the line integral A or A' cannot both yield integer values, the basic requirement for quantization. Similarly,

$$\text{Norm}\left(\int_\ell a\right) = \frac{\int_\ell A}{\Gamma(s+1)} \tag{62}$$

with $s = \frac{1-\gamma}{2}$, when $\gamma > 1$ and

$$\int_\ell a = 0 \tag{63}$$

when $\gamma < 1$.

All of this is a consequence of Nöther's second theorem: ambiguity in the gauge transformation leads to a breakdown of the standard charge quantization rules. What is the convention then for choosing the value of γ? The answer is material dependent. If either A or $\tilde{A}'$ are the physical gauge fields then the corresponding electric and magnetic fields in the material are indeed fractional. That is, each has an anomalous dimension. Consequently, the flux enclosed in a disk of radius r is no longer $\pi r^2 B$ simply because $[B] \neq 2$ and hence a failure of the key ingredient of the Byers-Yang theorem [2]. The Aharonov-Bohm phase in this case for the disk geometry shown in Fig. 2 must be constructed by constructing using the fictitious gauge $a \equiv \Box^{\frac{1-\gamma}{2}} A \equiv \Box^{1-\gamma} A'$ so that the correct dimensions are engineered in the usual covariant derivative $d - iqa$. The result for the phase when a is integrated around a closed loop

$$\Delta\phi_{\mathrm{D}} = \frac{e}{\hbar}\pi {r^2}_\alpha B R^{2\alpha-2}\left(\frac{2^{2-2\alpha}\Gamma(2-\alpha)}{\Gamma(\alpha)}{}_2F_1(1-\alpha, 2-\alpha, 2; \frac{r^2}{R^2})\right). \tag{64}$$

involves the standard result, $\pi r^2 B$ multiplied by a quantity that depends on the total outer radius of the sample such that the total quantity is dimensionless. Here ${}_2F_1(a, b; c; z)$ is a hypergeometric function and the terms in the parenthesis reduce to unity in the limit $\alpha \rightarrow 1$. This is the key experimental prediction of the fractional formulation of electricity and magnetism: the flux depends on the outer radius. This stems from the non-local nature of the underlying theory and is the key signature that charge is no longer quantized in that is determined by a topological integral.

6 Concluding Remarks

In actuality, the ambiguity in defining the redundancy condition for the gauge field, Eq. (6), ultimately leads to a landscape problem for charge quantization. This is the physical import of Nöther's Second Theorem and the guiding mathematical idea behind our work on fractional electromagnetism [9–11]. There is no easy fix here. Each choice for γ defines a valid vacuum theory of electromagnetism. Ultimately it is a materials problem whether or not the fractional or standard gauge describe the interaction of matter with radiation. In this sense, charge is ultimately emergent.

Acknowledgements We thank K. Limtragool for a collaboration on the Aharonov-Bohm effect and E. Witten and S. Avery for insightful remarks and DMR19-19143 for partial support.

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Entanglement and the Infrared

Gordon W. Semenoff

Abstract We shall outline some results regarding the infrared catastrophes of quantum electrodynamics and perturbative quantum gravity and their implications for information loss in quantum processes involving electrically or gravitationally charged particles. We will argue that two common approaches to the solution of the infrared problem, using transition probabilities which are inclusive of copious soft photon and graviton production and using dressed states describe fundamentally different quantizations of electrodynamics and low energy gravity which are, in principle, distinguishable by experiments.

1 Prologue

Motivated by the idea that subtle infrared effects could be relevant to the black hole information paradox, interest in the infrared problems in quantum electrodynamics and in perturbative quantum gravity has recently seen a rebirth [1–22]. These happen to be the two known theories of nature which contain massless physical particles and which describe long-ranged interactions. There are two well developed ways of dealing with the infrared divergences in these theories.

The first of the two has been known since the early days of quantum electrodynamics [23–25], and was generalized to perturbative quantum gravity by Weinberg [26]. In this approach, the infrared divergences that occur in internal loops in Feynman diagrams, and which afflict the S-matrix that is computed in renormalized perturbation theory, are canceled by computing the probabilities of processes which also include the production of soft photons and soft gravitons. In this approach, the infrared divergences of the perturbative S-matrix cancel with those which occur in the integration of transition probabilities over the wave-vectors of the outgoing soft particles, leaving infrared finite inclusive transition probabilities. The precise order by order cancellation of the infrared divergences by this mechanism is due to unitarity and it can be seen as a consequence an optical theorem for the S-matrix.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_10

The second formalism considers dressed states where the quantum states of charged particles are dressed by adding soft on-shell photons and gravitons. The soft particle content of the dressed state is fine-tuned in such a way that transition amplitudes between dressed states are infrared finite [27–32]. Moreover, to an accuracy which is governed by the detector resolution, the transition probabilities which are computed in this second approach are identical to those of the first approach.

The replacement of charged particle states by dressed states can be implemented as a canonical transformation [19] which decouples the infrared, so that the copious production of arbitrarily soft particles, beyond those already included in the dressed states, no longer occurs in a scattering processes. In this approach, the S-matrix elements between dressed states is infrared finite. However, the canonical transformation which dresses the charged particles is an improper unitary transformation. All of the dressed states are orthogonal to all of the multi-particle Fock states. As a result, the first and second approaches are not equivalent, they have different, orthogonal, Hilbert spaces. They should be considered different, inequivalent theories of how to deal with infrared divergences.

Recently, it has been noted that the two approaches, dressed and un-dressed, have important and potentially physically observable differences in how quantum information is distributed by the interactions when a scattering process occurs [33–36]. It is known that even elastic scattering results in entanglement of the quantum states of the out-going particles [38–40]. In the first approach to the infrared, the copious production of a cloud of soft photons or soft gravitons, which then fly away, undetected, from a scattering event, results in a quantum state where the soft photon or soft graviton cloud and the hard particles that are left behind are highly entangled. The result of this entanglement and the inaccessibility of the soft photon cloud to measurements is decoherence which, although very small in any realistic experiment, could in principle be measured. If the particles are dressed, and the infrared is decoupled, so that pure states evolve to pure states, this fundamental decoherence must be absent.

We will mostly use the language of quantum electrodynamics in the following as we anticipate that it may be more familiar to the reader. Practically all of our considerations also apply to perturbative quantum gravity in the low energy regime and we will give some of the relevant formulae. Of course quantized gravity is not a consistent, renormalizable quantum field theory. Moreover, it is not clear that it can have an infrared cutoff which leaves it unitary. We will ignore these difficulties here.

2 Inclusive Approach to Infrared Singularity Cancellation

If we wanted to use quantum electrodynamics to compute the amplitude for Moller scattering, for example, we would begin with the Feynman diagram which is illustrated in Fig. 1. That diagram gives an estimate of the quantum amplitude that two incoming electrons will interact and then re-emerge as two electrons. The modulus square of this amplitude gives an answer for the probability that the process will happen which, because of the small value of the electromagnetic coupling constant $\frac{e^2}{4\pi} \sim \frac{1}{137}$ is already accurate to one percent.

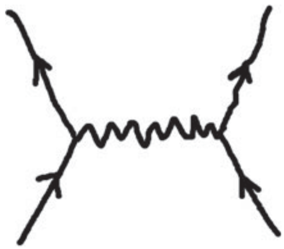

Fig. 1 The Feynman diagram which is used to compute the quantum amplitude for Moller scattering is depicted. The probability of the two-electron state, incoming from the bottom of the diagram, emerging as a two-electron state is gotten by taking the square of the modulus of this amplitude. Because quantum electrodynamics is a weakly coupled theory, the result is already accurate to the one percent level

If we want to improve the accuracy of the computation, we must include higher order corrections in the way of loop diagrams. The next correction occurs at one loop and it consists of several processes. One of them is illustrated in the second diagram in Fig. 2 where the electron emits a virtual photon, interacts with the other electron and then re-absorbs the virtual photon. This contribution will be infrared divergent. Unlike ultraviolet divergences, which are well understood, and are dealt with by using the usual renormalization procedure, the infrared divergence is physical and it must be dealt with by using physical reasoning.

The solution to this infrared problem is well known and it dates back to the early days of quantum electrodynamics [23–25], in fact it predates the understanding of ultraviolet renormalization by a few decades. The solution is to consider an additional process which is physically indistinguishable from the process that we have described up to now. That process considers the same Moller scattering, but with the additional production of a soft photon. The photon should be so soft that it eludes detection by the detection apparatus, and thus, it flies away undetected from our Moller scattering experiment. The idea is that we should add the possibility of this process to the one which where no soft photon is produced. That probability is the one represented by the last term in Fig. 2. If that last contribution is integrated over the wave-vectors of the soft photon, it is also infrared divergent. In fact, it is divergent in such a way as to cancel the infrared divergence in the same order (e^6) cross-term in the first contribution. This cancellation is exact. Its fine-tuning is a result of unitarity—the optical theorem—and this sort of argument can be seen to cancel the infrared divergences encountered in any amplitude which involves charged particle scattering and to all orders in perturbation theory.

An important consequence of the argument in the paragraphs above is the fact that, even though the lowest order Feynman diagram in Fig. 1 turns out to be the correct one to accurately analyze Moller scattering, the physics of what is happening is much more complicated. The amplitude for the process in Fig. 1 is zero. The processes which dominate are those where infinite numbers of soft photons are produced, as in Fig. 3.

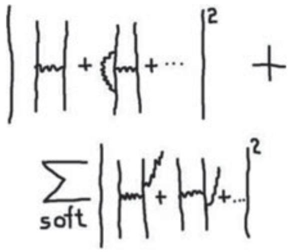

Fig. 2 The probability of a Moller scattering process is gotten by taking the squared modulus of the sum of the leading order and higher order Feynman diagrams which contribute to Moller scattering amplitude and then adding a similar squared modulus of the amplitude for Moller scattering plus the production of a soft photon. Here, only one example diagram of the several that contribute at the next-to-leading order are displayed. The infrared divergence from the internal loop is canceled by the integration over soft momenta of the extra emitted photon. The cancellation is between the last term and the cross term in the first bracket

The infinite numbers of photons which fly away undetected carry very little energy or momentum. To accuracy of the detector resolution, their influence on the kinematics of the experiment is not noticeable. However, even if they have very little energy, each photon has a polarization and a direction of motion. Specifying the details of their quantum state involves a significant amount of information. A question that one could then ask is, when this cloud of photons escapes detection, how much information is lost? What we mean here is information in the quantum sense, as we shall try to explain in the next section. This question has only been recently addressed [33–36] and as we will explain in the rest of this review, the results were somewhat surprising.

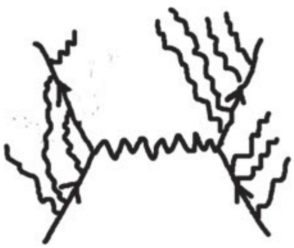

Fig. 3 The physical processes which contribute to Moller scattering and which have non-zero probability involve the copious production of soft photons

3 Information Loss Due to Quantum Entanglement

Let us try to explain precisely what we mean by information loss. Let us consider a model system of two qubits, qubit #1 and qubit #2. We could think of qubit #1 as the analog of the hard particles in our scattering experiment and quit #2 as the soft photons and gravitons that are produced. Bases for the Hilbert spaces of the quantum states of qubit #1 are the two vectors $|\uparrow>_1$ and $|\downarrow>_1$ and for the qubit #2 the states $|\uparrow>_2$ and $|\downarrow>_2$. Let us assume that the qubits are dynamically independent, that is, they do not interact with each other.

The question that we want to ask is, if qubit #2 becomes inaccessible to us, how much information about the quantum state of qubit #1 have we lost. In the classical world, if these were classical bits, rather then qubits, the answer would be easy—none! Everything that we could find out by classical measurements of qubit #1 before qubit #2 was misplaced could still be done afterward. As far as qubit #1 is concerned, we would have lost no information at all.

In the quantum world the answer will depend on the quantum state of the joint two-qubit system at the time when qubit #2 was lost. Let us consider two examples for that quantum state, an un-entangled state

$$|\psi> \; = [\cos\varphi|\uparrow>_1 + \sin\varphi|\downarrow>_2]\otimes|\uparrow>_2$$

and an entangled state

$$|\tilde{\psi}> \; = [\cos\varphi|\uparrow>_1 \otimes|\uparrow>_1 + \sin\varphi|\downarrow>_1\otimes|\downarrow>_2]$$

These two states have the same expectation values of the “spin” of qubit 1, that is, the expectation values of the operator $|\uparrow>_1<\uparrow|\otimes\mathcal{I}_2$, which is $\cos^2\varphi$ or $|\downarrow>_1<\downarrow|\otimes\mathcal{I}_2$ which is $\sin^2\varphi$. The difference between the two states is that the un-entangled state has a wave-function which is a direct product of the wave-functions of qubit #1 and qubit #2. The entangled state, on the other hand, is a superposition of direct products, which cannot itself be written as a single direct product of states of #1 and #2.

Now, let us assume that, in the quantum world, we have lost track of qubit #2. What is the implication for qubit #1. We get information from a quantum system by quantum measurements. Quantum measurements are represented mathematically by projection operators. If we have no access to qubit #2, all quantum measurements that we can do must act on qubit #2 like the unit operator on its factor in the Hilbert space. Therefore, for the sake of quantum measurements, we can once and for all contract the states of qubit #2 with the unit operator, that is, we can form the reduced density matrix which describes qubit #1, in our first example, by tracing over the states of qubit #2,

$$\rho = \mathrm{Tr}_2|\psi><\psi| = [\cos\varphi|\uparrow>_1 + \sin\varphi|\downarrow>_1]\,[_1<\uparrow|\cos\varphi + _1<\downarrow|\sin\varphi]$$

$$= \begin{bmatrix} \cos^2\varphi & \cos\varphi\sin\varphi \\ \cos\varphi\sin\varphi & \sin\varphi\sin\varphi \end{bmatrix}$$

or, in our second example,

$$\tilde{\rho} = \mathrm{Tr}_2|\tilde{\psi} >< \tilde{\psi}| = \left(\cos^2\varphi| \uparrow>_1<\uparrow | + \sin^2\varphi| \downarrow>_1<\downarrow |\right)$$

$$= \begin{bmatrix} \cos^2\varphi & 0 \\ 0 & \sin^2\varphi \end{bmatrix}$$

In the first, unentangled example, the reduced density matrix is still that of a pure state. Qubit #1 is sure to be in the quantum state $[\cos\varphi| \uparrow>_1 + \sin\varphi| \downarrow>_1]$. No information about its state has been lost. However, in the second case, the reduced density matrix is now that of a mixed state with classical probabilities $\cos^2\varphi$ of finding $| \uparrow>_1$ and $\sin^2\varphi$ of $| \downarrow>_1$. What is missing are the off-diagonal elements of the density matrix. These contain interference terms. We can see the difference if we ask what is the expectation value of the Hermitian operator

$$\mathcal{O} = \left[\alpha| \uparrow>_1 + \sqrt{1-|\alpha|^2}| \downarrow>_1\right]\left[{}_2 <\uparrow |\alpha^* + {}_2 <\downarrow |\sqrt{1-|\alpha|^2}\right] \otimes \mathcal{I}_2$$

In the first case, the expectation value is

$$\mathrm{Tr}\mathcal{O}\rho = \left|\cos\varphi\,\alpha + \sin\varphi\sqrt{1-|\alpha|^2}\right|^2$$

whereas in the second case it is

$$\mathrm{Tr}\mathcal{O}\rho = |\cos\varphi\,\alpha|^2 + \left|\sin\varphi\sqrt{1-|\alpha|^2}\right|^2$$

The difference is, in the second case the cross-terms, that is, the interference terms are missing. In the second case, we have lost the possibility of interference. This is called decoherence. In the entangled case, when qubit #2 was lost, the quantum probabilities of the two spin outcomes became classical probabilities. On the other hand, in the un-entangled case, no information was lost. The outcomes of all possible measurements of qubit #1 remain unchanged.

The property of the state $|\tilde{\psi} >$ which distinguishes it from state $|\psi >$ and which results in decoherence is quantum entanglement. A quantitative measure of entanglement is the entanglement entropy, defined as the Von Neumann entropy of the reduced density matrix,

$$S = -\mathrm{Tr}\rho\ln\rho$$

In the un-entangled case, $S = 0$, whereas in the entangled case, $\tilde{S} = -\cos^2\varphi\ln\cos^2\varphi - \sin^2\varphi\ln\sin^2\varphi$.

4 Entanglement of Soft and Hard

Let us return to quantum electrodynamics and consider a scattering event where an incoming state $|\alpha >$ evolves to an out-going state. The outgoing state is a superposition of incoming states. The coefficients in this super-position are the elements of the S-matrix,

$$|\alpha > \ \rightarrow \ \sum_{\beta,\gamma} |\beta, \gamma > S^{\dagger}_{\beta\gamma,\alpha}$$

Here, in $|\beta, \gamma >$, we are separating the soft photons, which we call γ, from the hard particles, which we denote by β.

In a perturbative computation, the S matrix turns out to be logarithmically infrared divergent and an infrared cutoff is needed in order to define it. We shall introduce such an infrared cutoff which we will denote by μ. A nice example of how this could be done is by assuming that the photon has a small mass, μ, so Maxwell theory coupled to charged matter becomes Proca theory of a massive vector field, coupled to the conserved charged currents of the charged matter. This is still a Lorentz invariant, renormalizable quantum field theory with a unitary S-matrix that we shall denote $S^{\mu}_{\alpha\beta}$ where the superscript μ reminds us that it is to be computed with the infrared cutoff μ in internal loops. The infrared cutoff S-matrix is unitary,

$$\sum_{\alpha} S^{\mu\dagger}_{\beta\gamma\alpha} S^{\mu}_{\alpha\beta'\gamma'} = \delta_{\beta\beta'}\delta_{\gamma\gamma'}$$

where the sum on the left-hand side is schematic for integrations and sums over the momenta and quantum numbers of the particles in the incoming state and the right-hand-side is schematic for an assembly of Dirac and Kronecker delta functions which identify momenta and discrete quantum numbers in the states $|\beta, \gamma >$ and $|\beta', \gamma' >$.

Generally, our incoming states can be either eigenstates of energy and momentum or they can be wave-packets. In order to address the most general consideration, we will consider an in-coming density matrix of the form

$$|\alpha >< \alpha'|$$

where α and α' are states where each of the incoming particles has a fixed energy and momentum. If these states contain photons, they are hard photons, with energies and wave-vectors much larger than the fundamental infrared cutoff μ and we will also need them to be much larger than another intermediate cutoff, which we shall call λ, the detector resolution.

We could make a wave-packet from this state as, for example

$$|f >< f| \ \equiv \ \sum_{\alpha\alpha'} f_{\rm in}(\alpha) f^{*}_{\rm in}(\alpha')|\alpha >< \alpha'|$$

with

$$\sum_{\alpha} |f_{\text{in}}(\alpha)|^2 = 1 \ , \quad < f|f >= 1$$

During the scattering process our in-state evolves to the out-state which is given by

$$|\alpha >< \alpha'| \ \rightarrow \ \left[\sum_{\beta\gamma} |\beta\gamma > S^{\mu\dagger}_{\beta\gamma\alpha} \right] \left[\sum_{\beta'\gamma'} S^{\mu}_{\alpha'\beta'\gamma'} < \beta'\gamma'| \right]$$

where we have separated the scattering products into hard particles, β, β', those whose momenta are above a the cutoff λ, the detector resolution, and soft particles γ, γ' whose frequencies and wave-numbers are greater than the fundamental cutoff, μ but smaller than the detector resolution, λ. Any state of free particles can be divided in this way.

We then reduce the density matrix of the final state by tracing over the soft degrees of freedom. This yields

$$\rho = \sum_{\tilde{\gamma}} <\tilde{\gamma}|\rho_{\text{out}}|\gamma'> \ = \sum_{\tilde{\gamma}} < \tilde{\gamma}| \left[\sum_{\beta\gamma} |\beta\gamma > S^{\mu\dagger}_{\beta\gamma\alpha} \right] \left[\sum_{\beta'\gamma'} S^{\mu}_{\alpha'\beta'\gamma'} < \beta'\gamma'| \right] |\tilde{\gamma} >$$

or, simplifying the notation,

$$<\beta|\rho|\beta'> \ = \sum_{\gamma} S^{\mu *}_{\alpha\beta\gamma} S^{\mu}_{\alpha'\beta'\gamma} \tag{1}$$

Now, we would like to use a soft photon theorem to simplify this expression, particularly the trace over soft photons. A nice derivation and discussion of the soft photon theorem can be found in Weinberg's quantum field theory book [43].

A soft photon theorem is valid only when we have a large hierarchy of scales. That means that we can apply it to our out-state only when the masses, energies and momenta of all of the particles in the states $|\beta >$ and $|\beta' >$ are much greater than the detector resolution, λ and also when λ is much greater than the fundamental cutoff, μ. This means that we cannot analyze every possible out-state, but only those which meet this requirement. We will not worry about this limitation here or in the following. We emphasize that we shall also need that $\lambda >> \mu$. In addition, we shall cut off the total energy of the photons that escape with a cutoff E. To be clear, E is the maximum value of the sum of all of the energies of the soft photons. The soft photon theorem then tells us that, when there is a hierarchy of scales, $\lambda >> \mu$, we can replace Eq. (1) by the expression

$$<\beta|\rho|\beta'> \ = S^{\mu\dagger}_{\beta\alpha} S^{\mu}_{\alpha'\beta'} \left(\frac{\lambda}{\mu} \right)^{A_{\alpha\beta,\alpha'\beta'}} f\left(\frac{\lambda}{E}, A_{\alpha\beta,\alpha'\beta'} \right) \tag{2}$$

where the exponent is a complicated function of the four-momenta of the hard particles in the initial and final states,

$$A_{X,Y} = -\sum_{n\in X}\sum_{n'\in Y}\frac{e_n e_{n'}\eta_n\eta_{n'}}{8\pi^2}\beta_{nn'}^{-1}\ln\frac{1+\beta_{nn'}}{1-\beta_{nn'}} \tag{3}$$

where e_n are the charges of particles, and $\eta_n = 1$ for an incoming particle and $\eta_n = -1$ for an outgoing particle.

$$\beta_{nn'} = \sqrt{1 - \frac{(mm')^2}{(p\cdot p')^2}}$$

are the relativistic relative velocities of particles m and n. The last factor comes from imposing the cutoff on the total energy and it contributes

$$f(x,A) = \frac{1}{\pi}\int_{-\infty}^{\infty} du\frac{\sin u}{u}\exp\left(A\int_0^x \frac{d\omega}{\omega}\left(e^{i\omega u}-1\right)\right) \tag{4}$$

$$f(1,A) = \frac{e^{-\gamma A}}{\Gamma[1+A]}\,,\ \gamma = .05772... \tag{5}$$

The factor $f(\lambda/E, A)$ is smooth, of order one and obeys $f(0, A) = 1$. We have included the result of an energy cutoff for completeness, however, it will play no role in the following, so we will put $E \to \infty$ where $f(\lambda/E, A) \to 1$.

The trace over soft photons produces energy and momentum-dependent factors multiplying the S-matrix for the hard particles alone. These factors, as well as the S-matrix, depend on the fundamental cutoff μ. Now that we have assumed a hierarchy of scales, we can also exchange the infrared cutoff μ for a larger cutoff Λ where it appears in the S-matrix. We can choose the new cutoff Λ and a further soft photon theorem tells us that

$$S^{\mu}_{\alpha\beta} = S^{\Lambda}_{\alpha\beta}\left(\frac{\mu}{\Lambda}\right)^{A_{\alpha\beta,\alpha\beta}/2}$$

The right-hand-side of this equation does not depend on Λ, at least over a range of Λ that respects the hierarchy of scales $\alpha\alpha'\beta\beta' >> E, \lambda, \Lambda >> \mu$, where, by $\alpha\alpha'\beta\beta'$, we mean the masses, energies and momenta of all of the particles in the states.

Using this equation, we find

$$<\beta|\rho|\beta'> = S^{\Lambda *}_{\alpha\beta}\, S^{\Lambda}_{\alpha'\beta'}\left(\frac{\mu}{\Lambda}\right)^{A_{\alpha\beta,\alpha\beta}/2}\left(\frac{\mu}{\Lambda}\right)^{A_{\alpha'\beta',\alpha'\beta'}/2}\left(\frac{\lambda}{\mu}\right)^{A_{\alpha\alpha',\beta\beta'}} \tag{6}$$

This is our expression for the reduced density matrix that we use to describe the quantum state of the out-going hard particles. We can see, by studying the exponents, A_{XY} that its diagonal matrix elements, for a fixed in-state $\alpha = \alpha'$, are

$$<\beta|\rho|\beta> \;=\; |S^{\Lambda}_{\alpha\beta}|^2 \left(\frac{\lambda}{\Lambda}\right)^{A_{\alpha\beta,\alpha\beta}} \tag{7}$$

which no longer depends on the fundamental cutoff. In fact it simply has the form of the square of the transition amplitude for $|\alpha>\to|\beta>$, computed with an infrared cutoff Λ for internal loops in Feynman diagrams, times the Sudakov-like factor $\left(\frac{\lambda}{\Lambda}\right)^{A_{\alpha\alpha\beta\beta}}$ with the ratio of detector resolution λ and Λ. This result is well known.

Now, what about the off-diagonal elements? They can be written as

$$<\beta|\rho|\beta'> \;=\; S^{\Lambda *}_{\alpha\beta}\, S^{\Lambda}_{\alpha'\beta'} \left(\frac{\lambda}{\Lambda}\right)^{A_{\alpha\alpha',\beta\beta'}} \;\; \left(\frac{\mu}{\Lambda}\right)^{\Delta A(\alpha\alpha'\beta\beta')} \tag{8}$$

Now the small μ behaviour is dependent on the exponent

$$\Delta A(\alpha\alpha', \beta\beta') = A_{\alpha\beta,\alpha'\beta'} - A_{\alpha\beta,\alpha\beta}/2 - A_{\alpha'\beta',\alpha'\beta'}/2 \geq 0$$

This exponent can be shown to be positive semi-definite [33, 35]. This means that, as we remove the fundamental cutoff, to make the photon truly massless, some off-diagonal elements of the density matrix are set to zero. Only those where $\Delta A(\alpha\alpha', \beta\beta') = 0$ survive. This turns out to be a surprisingly strict restriction on which elements survive. It turns out that, $\Delta A(\alpha\alpha', \beta\beta') = 0$ if and only if the four sets of ingoing and outgoing electric currents.

$$\left\{ \frac{e_i p_i^{\mu}}{\sqrt{\mathbf{p}_i^2 + m^2}} : e_i, p_i \in \alpha \right\} = \left\{ \frac{e_j p_j^{\mu}}{\sqrt{\mathbf{p}_j^2 + m^2}} : e_j, p_j \in \beta' \right\}$$

$$\left\{ \frac{e_k p_k^{\mu}}{\sqrt{\mathbf{p}_k^2 + m^2}} : e_k, p_k \in \alpha' \right\} = \left\{ \frac{e_\ell p_\ell^{\mu}}{\sqrt{\mathbf{p}_\ell^2 + m^2}} : e_\ell, p_\ell \in \beta \right\}$$

That is, the sets of electric currents are identical. In conclusion, the matrix element of the reduced density matrix survives if and only if the set of all electric currents contained in the states α, β' is identical (up to permutations) to the set of all electric currents in the states α', β. If these currents do not match, $\Delta A > 0$ and the matrix elements vanish in the limit where the photon is massless. Perturbative quantum gravity has a similar conclusion with the matching condition on the in-coming and outgoing energy-momentum currents

$$\left\{ \frac{p_i^\nu p_i^\mu}{\sqrt{\mathbf{p}_i^2 + m^2}} : e_i, p_i \in \alpha \right\} = \left\{ \frac{p_j^\nu p_j^\mu}{\sqrt{\mathbf{p}_j^2 + m^2}} : e_j, p_j \in \beta' \right\}$$

$$\left\{ \frac{p_k^\nu p_k^\mu}{\sqrt{\mathbf{p}_k^2 + m^2}} : e_k, p_k \in \alpha' \right\} = \left\{ \frac{p_\ell^\nu p_\ell^\mu}{\sqrt{\mathbf{p}_\ell^2 + m^2}} : e_\ell, p_\ell \in \beta \right\}$$

This is remarkably restrictive. If we assume that the incoming state is a pure state with in-coming plane waves, $\alpha = \alpha'$, we find that the off-diagonal elements of the density matrix vanish unless the electric and energy-momentum currents in the two states match exactly. For some simple processes, this can mean that the out-going density matrix is just diagonal. Of course this argument says nothing about their diagonal elements, they are as they have always been, the transition probabilities between plane-wave states.

The zeroing of off-diagonal elements of the density matrix is decoherence. One loses the quantum coherence that is necessary for quantum interference to occur. An even more dramatic effect occurs with incoming wave-packets, superpositions of plane-wave states. There, scattering seems to be suppressed in many cases. For example, if we look at even diagonal components of the final state density matrix for in-coming wave-packets,

$$<\beta|\rho|\beta> = \sum_{ij} f_i f_j^* S^{\Lambda\dagger}_{\beta\alpha_i} \; S^{\Lambda}_{\alpha_j\beta'} \left(\frac{\lambda}{\Lambda}\right)^{A_{\alpha_i\alpha_j,\beta\beta}} \quad \left(\frac{\mu}{\Lambda}\right)^{\Delta A(\alpha_i\alpha_j\beta\beta)}$$

and the massless limit of the photon still requires that we now put $\mu \to 0$. This, at least partially, concentrates the sum over i, j in the region $i \sim j$. However, this sum is actually an integral and the limit $\mu \to 0$ suppresses scattering. There are many processes for which only the unit matrix part of the S-matrix will contribute to the scattering of wave packets [35].

5 Dressed Quantum States

Now we turn to the second way of dealing with the infrared problem, that of dressing the incoming and out-going states of charged particles with soft on-shell photons and gravitons with the dressing fine tuned in a way that cancels the infrared singularities. For a given distribution of incoming currents, the dressed state is obtained by a canonical transformation which creates a coherent state of the photons which is tuned to the currents of the charged particles,

$$
\begin{aligned}
&|p_1, p_2, ... > \to |p_1, p_2, ... >_D \\
&\equiv \exp\left(-e \int_\mu^\lambda \frac{d^3k}{\sqrt{2|k|}} \sum_{j\in\alpha} \frac{p_j^\mu \epsilon_\mu^s(k)}{p_j^\nu k_\nu + i\epsilon} a_s(k) - \text{ h.c.} \right) |p_1, p_2, ... > \qquad (9) \\
&a_s(k) \to a_s(k) + \frac{e}{\sqrt{2|k|}} \sum_j \frac{p_j^\mu \epsilon_\mu^s(k)}{p_j^\nu k_\nu - i\epsilon}\ ,\ \mu < |\mathbf{k}| < \lambda
\end{aligned}
$$

where $\epsilon^s(k)$ is the physical polarization of the photon. If we take matrix elements of the S-matrix in these states, the infrared singularities which are contained in the S-matrix are canceled by additional ones coming from the interactions with the photons in the dressed states. These matrix elements are finite. The statement is that ${}_D\!<\alpha|S^\mu|\beta>_D$ are have a finite limit as $\mu \to 0$. This moreover, the probabilities of transitions agree with those which are computed in the inclusive approach,

$$
|{}_D\!<\alpha|S^\mu|\beta>_D|^2 = \sum_{\mu<\gamma<\lambda} | < \alpha|S^\mu|\beta, \gamma > |^2 = | < \alpha|S^\lambda|\beta >|^2
$$

and the result is as if one simply computed the usual perturbative S matrix for hard particles, but with the detector resolution λ as an infrared cutoff for the otherwise infrared divergent internal loops in Feynman diagrams.

Dressing is a canonical transformation. However, when the fundamental cutoff is removed, the canonical transformation in Eq. (9) is not a proper unitary transformation. Every undressed state in the undressed Hilbert space is orthogonal to every dressed state in the dressed Hilbert space. This means that, if the photon were truly massless, the dressed and undressed formalisms are inequivalent quantizations of quantum electrodynamics.

What is more, there is a fundamental difference between the two procedures. This difference appears on the off-diagonal elements of the density matrix. With dressed states, the production of soft photons is already included in the state and there is no further soft photon production when charged particles scatter. Pure states evolve to pure states and there is no decoherence. In the inclusive formalism, as we have argued, there should be some fundamental decoherence and even suppression of some scattering. These are, in principle, physical differences which could be measured by experiments. The conclusion is that there are two different quantizations of electrodynamics, with physically measurable differences, and only one of them can be the correct fundamental theory to describe nature.

When $\mu \to 0$, the dressed states have other peculiarities. For example the dressed states are never eigenstates of the total momentum. They are always mixtures of states with different momenta, the spread of momenta being governed by the detector resolution. They are also not Lorentz invariant. This is apparent in that the coherent photon field has a classical piece. If the dressing were obtained by a unitary transformation, we could Lorentz transform a state simply by undressing it, Lorentz transforming it, and then re-dressing it. When this is not possible, the Lorentz transformation itself is not a proper unitary transformation. It is not clear what the implications of this subtlety are. There has already been some discussion of it in the context of infrared divergences [44–47] and it would be interesting to understand that work in the present context.

6 Epilogue

We have argued that, in the limit where fundamental infrared cutoffs are removed, there are two fundamentally different interpretations of quantum electrodynamics and perturbative quantum gravity. What is more, the differences are measurable in principle, although perhaps very difficult in practice. For example, in a non-ideal scattering experiment, one which takes place over a finite time, a rough estimate of the decoherence effect would be to replace μ by the inverse time. For example, if we consider Compton scattering, where the in-state is an electron and a hard photon and the out-state is also an electron and hard photon, the off-diagonal elements of the density matrix have the suppression factor

$$\rho_{p,k;p',k'} \sim \left(\frac{\mu}{\lambda}\right)^{\frac{e^2}{4\pi^2}\left(\frac{1}{2\beta}\ln\frac{1+\beta}{1-\beta}-1\right)}$$

where $\beta^2 = 1 - \frac{m^4}{(p^\mu p'_\mu)^2}$ is the relativistic relative velocity of the out-going electrons, with momenta p, p', on the two legs of the reduced density matrix. If we take the detector resolution λ to be the electron mass and μ to be an inverse second, the value of this suppression factor is graphed as a function of β in Fig. 4. We see there that the suppression is significant only for very far off-diagonal elements where the relative velocity is close to that of light.

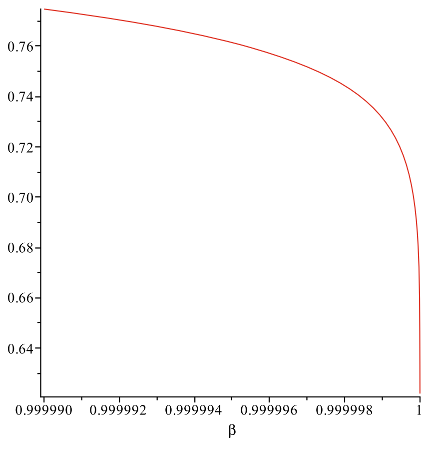

Fig. 4 The magnitude of the suppression factor for off-diagonal elements of the outgoing density matrix for Compton scattering when the time scale of the experiment is of the order of one second and the detector resolution is the electron mass is plotted on the vertical axis versus the relative velocity β of the outgoing electrons on the horizontal axis

For dressed states, one might worry about locality as the state is created by excitations which occupy far separated positions in space. The breakdown of Lorentz invariance and the fact that states are not eigenstates of the momentum are also consequences that deserve attention. This balances the alternative of fundamental decoherence of the inclusive approach. This fundamental decoherence is likely very small (and even smaller for perturbative quantum gravity) in any realistic interaction of charged particles. It would be interesting to find an experimental scenario where it would be detectable.

Acknowledgements The author acknowledges the collaboration of Dan Carney, Dominik Neuenfeld, Laurent Chaurette and Gianluca Grignani on most of the ideas and results that have been reported here. This work was supported by NSERC of Canada.

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Comments on the Newlander-Nirenberg Theorem

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Abstract The Newlander-Nirenberg theorem says that a necessary and sufficient condition for the complex coordinates associated with a given almost complex structure tensor $I_M{}^N$ to exist is the vanishing of the Nijenhuis tensor $\mathcal{N}_{MN}{}^K$. In the first part of the paper, we give a heuristic but very simple proof of this fact. In the second part, we discuss a supersymmetric interpretation of this theorem. *(i)* The condition $\mathcal{N}_{MN}{}^K = 0$ is necessary for certain $\mathcal{N} = 1$ supersymmetric mechanical sigma models to enjoy $\mathcal{N} = 2$ supersymmetry. *(ii)* The sufficiency of this condition for the existence of complex coordinates implies that the representation of the supersymmetry algebra realized by the superfields associated with all the real coordinates and their superpartners can be presented as a direct sum of d irreducible representations (d is the complex dimension of the manifold).

1 Introduction

Since 1982, we know that many well-known structures of differential geometry, such as the de Rham complex, allow for a supersymmetric interpretation [1]. For any manifold, one can define a certain supersymmetric quantum mechanical model. The dynamical time-dependent variables of this model include the coordinates and their Grassmann-valued superpartners.

Supersymmetric language is very useful. Besides giving a new unexpected interpretation of known mathematical facts, it allows one to derive many *new* nontrivial results, which are difficult to derive in a traditional way. I give here only one example. The so-called HKT manifolds were first discovered by supersymmetric methods [2] and only then they attracted the attention of pure mathematicians who gave their traditional description [3]. The full classification of HKT metrics was also recently constructed using supersymmetric tools [4, 5].

Supersymmetry is a standard method to study geometrical properties of the manifolds used by "physicists" (I've put here the quotation marks because we are talking in this case about the scholars who may have studied physics at university, but who

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_11

are now solving pure mathematical problems without much relationship to the physical world) in the papers published in the hep-th section of the *arXiv*. On the other hand, pure mathematicians are reluctant to use it, preferring traditional methods.

It is an unfortunate fact of our life that a large gap exists between the two communities. The languages in which the papers are written and the ways of thinking derived from these languages are often very different, to the extent that mathematicians and physicists do not often understand each other, even though the subject of their studies could be practically identical.

That is exactly the reason by which I've decided to write this methodical paper. Its second half is mainly addressed to mathematicians who might be curious to learn that a certain well-known mathematical fact admits an unexpected interpretation in the supersymmetry framework. And its first half is addressed to physicists who might have heard about the NN theorem, but probably do not know how it is proven. Indeed, its rigourous mathematical proof is not so trivial. So I give here a heuristic but simple reasoning, presenting the solution to the Eq. (5) as the perturbative series over a deviation of the complex structure tensor ${I_M}^N(x)$ from its flat form. This reasoning might be upgraded to a rigourous proof if the convergence of this series is proven.

2 Geometry

2.1 Preliminaries

Definition 1. A complex manifold is a manifold of even dimension $D = 2d$ which can be represented as a union of several overlapping charts such that:

1. Each chart is homeomorphic to $\mathbb{R}^D$.
2. In each chart, one can define complex coordinates z^n.
3. In a region where two charts overlap, the coordinates z^n in one chart and the coordinates w^n in another chart are related by *holomorphic* transition functions $z^n = f^n(w^m)$.

Definition 2. A Hermitian manifold is a complex manifold endowed by Hermitian metric

$$ds^2 \;=\; 2h_{n\bar{m}}\, dz^n d\bar{z}^{\bar{m}} \tag{1}$$

with $\overline{h_{n\bar{m}}} = h_{m\bar{n}}$.

The factor 2 was introduced here for further conveniences—to make contact with the standard normalization in (58) and (60). Mathematicians sometimes consider manifolds not endowed with the metric. In particular, the NN theorem can be formulated and proven without using the notion of metric. But we are interested in a supersymmetric interpretation of the NN theorem, and we can only give it if the Hermitian metric (1) is defined. Thus, its existence will be assumed.

An interesting and important fact is that one can describe complex manifolds without explicitly introducing complex charts, but working exclusively in the real terms.[1] To this end, we introduce first the notion of an *almost* complex manifold:

Definition 3. An *almost complex manifold* is a manifold of even dimension D endowed with a globally defined tensor field I_{MN} satisfying the properties *(i)* $I_{MN} = -I_{NM}$ and *(ii)* $I_M{}^N I_N{}^P = -\delta_M^P$. The tensor $I_M{}^N$ is called the *almost complex structure*.

To understand why a *real* tensor is called *complex* structure, consider first the simplest possible example—flat 2-dimensional Euclidean space. It can be parametrized by the real Cartesian coordinates x^1, x^2 or by the complex coordinate $z = (x^1 + ix^2)/\sqrt{2}$. An obvious relation $\partial z/\partial x^2 = i\partial z/\partial x^1$ holds, which can also be presented in the form

$$\frac{\partial z}{\partial x_A} - i\varepsilon_{AB}\frac{\partial z}{\partial x_B} \;=\; 0 \tag{2}$$

with

$$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

The tensor ε_{AB} satisfies both conditions in the definition above and *is* the complex structure in this case. Note that the property (2) holds not only for z, but for any holomorphic function $f(z)$. In the latter case, the real and imaginary parts of (2) are none other than the Cauchy-Riemann conditions.

If a 2-dimensional manifold is not flat, $I_M{}^N$ may have a little bit more complicated form, but its tangent space projection $I_{AB} = I_{MN}e_A^M e_B^N$ coincides with the matrix ϵ or probably with $-\epsilon$. Indeed, an antisymmetric 2×2 matrix whose square is $-\mathbb{1}$ coincides with (3) up to a sign. It describes rotations by $\pi/2$ or by $-\pi/2$.

In the general multidimensional case, one can prove a simple theorem:

Theorem 1. Take a tensor $I_M{}^N$ satisfying the conditions above. With a proper choice of the vielbeins e_A^M (with a proper choice of the orthonormal base in the tangent space), its tangent space projection can be brought to the canonical form

$$I_{AB} \;=\; \text{diag}\,(\varepsilon, \ldots, \varepsilon). \tag{4}$$

Proof. To construct an orthonormal base in the tangent space E where the complex structure acquires the form (4), we start with choosing in E an arbitrary unit vector e_1. It follows from $I = -I^T$ and $I^2 = -\mathbb{1}$ that the vector $e_2 = Ie_1$ has also unit length and is orthogonal to e_1. Obviously, $Ie_2 = I^2 e_1 = -e_1$. Consider the subspace $E^* \subset E$ that is orthogonal to e_1 and e_2. If it is not empty, choose there an arbitrary

[1]It is convenient—especially, for supersymmetric applications—but is not *necessary*. For example, the popular textbook [6] uses only complex but not real description.

unit vector f_1 and consider $f_2 = If_1$. One can easily see that f_2 also belongs to E^*. Now consider the subspace $E^{**} \subset E^* \subset E$ that is orthogonal to $e_{1,2}$, $f_{1,2}$ and, if E^{**} is not empty, repeat the procedure. We arrive at the matrix (4).

Now consider the equation system

$$\frac{\partial z^n}{\partial x^M} - i I_M{}^N \frac{\partial z^n}{\partial x^N} \;=\; 0 \tag{5}$$

If not only I_{AB}, but also $I_M{}^N$ has the form (4), solutions to (5) can be easily found. A simple set of d independent solutions is

$$z^1_{(0)} \;=\; \frac{x^1 + ix^2}{\sqrt{2}}\,, \quad z^2_{(0)} \;=\; \frac{x^3 + ix^4}{\sqrt{2}}\,, \ldots \tag{6}$$

or any set of d non-degenerate analytic functions of $z^n_{(0)}$.

In a generic case, the solutions to (5) are more complicated. Moreover, they do not always exist. The conditions under which they do, is the content of the NN theorem to be proven in the next section. For the time being, we will prove that

Theorem 2. If the equation system (5) has d independent solutions, the manifold is complex. Its metric is Hermitian.

Actually, as follows from Theorem 3 below, it is sufficient to require the existence of only one such solution.

Proof. We will show first that the metric has a Hermitian form (i.e. the components g^{nm} etc vanish) Let us trade x^M for $(z^n\,,\ \bar{z}^{\bar{n}})$ and write

$$g^{nm} \;=\; \frac{\partial z^n}{\partial x^M}\frac{\partial z^m}{\partial x^N}\, g^{MN} = iI_M{}^P \frac{\partial z^n}{\partial x^P}\frac{\partial z^m}{\partial x^N}\, g^{MN} = iI^{NP} \frac{\partial z^n}{\partial x^P}\frac{\partial z^m}{\partial x^N} = 0$$

by symmetry considerations. The vanishing of $g^{\bar{n}\bar{m}}$ follows from the same argument. The properties $g^{\bar{n}\bar{m}} = g^{nm} = 0$ imply also the vanishing of the components g_{nm} and $g_{\bar{n}\bar{m}}$ of the inverse tensor.

Next, we need to show that the transition functions between two overlapping charts with the coordinates $(z^n, \bar{z}^{\bar{n}})$ and $(w^m, \bar{w}^{\bar{m}})$ are holomorhic. To this end, we express, using (5), $I_M{}^N$ in the complex frame,

$$\begin{aligned} I_m{}^n = I_M{}^N \frac{\partial z^n}{\partial x^N}\frac{\partial x^M}{\partial z^m} &= -i\frac{\partial z^n}{\partial x^M}\frac{\partial x^M}{\partial z^m} \;=\; -i\delta^n_m\,, \\ I_{\bar{m}}{}^{\bar{n}} \;=\; i\delta^{\bar{n}}_{\bar{m}}\,, \quad I_m{}^{\bar{n}} &\;=\; I_{\bar{m}}{}^n \;=\; 0 \end{aligned} \tag{7}$$

and consider the transformation of the tensor (7) from one chart to another. Knowing that I keeps the form (7) after this transformation, one can derive that $\partial w^m / \partial \bar{z}^{\bar{n}} = 0$.

2.2 NN Theorem

Not wishing to plunge into not relevant for us details, we assume that the manifold and all its structures are real analytic (can be expanded in the Taylor series). The traditional proof of the NN theorem in [7] assumes the existence of $D = 2d$ derivatives. Hörmander proved that it is sufficient to require the existence of the first derivative [8].

Introduce the object

$$\mathcal{N}_{MN}{}^K = \partial_{[M} I_{N]}{}^K - I_M{}^P I_N{}^Q \partial_{[P} I_{Q]}{}^K. \tag{8}$$

It is a tensor, in spite of the presence of the ordinary rather than covariant derivatives. This is so because one can replace the ordinary derivatives by the covariant ones—the terms involving the Christoffel symbols cancel out in this case. Using a sloppy language, we will call the L.H.S. of Eq. (8) the *Nijenhuis tensor*.[2] We will do so because the object (8) has a more transparent structure, and it is this combination that will directly appear later in (12).

The NN theorem says that

Theorem 3 [7]**.** The complex coordinates satisfying the condition (5) can be introduced and the manifold is complex iff the condition

$$\mathcal{N}_{MN}{}^K \;=\; 0 \tag{10}$$

holds.

Proof.
Necessity. Represent the system (5) as $\mathcal{D}_M z^n = 0$ with

$$\mathcal{D}_M = \partial_M - i I_M{}^N \partial_N. \tag{11}$$

For self-consistency, the conditions $[\mathcal{D}_M, \mathcal{D}_N] z^n = 0$ should also hold. We derive

$$\begin{aligned} [\mathcal{D}_M, \mathcal{D}_N] z^n \;&=\; \left[-i\partial_{[M} I_{N]}{}^Q - I_{[M}{}^P (\partial_P I_{N]}{}^Q)\right] \partial_Q z^n \\ = \left[-i\partial_{[M} I_{N]}{}^K - i I_{[M}{}^P (\partial_P I_{N]}{}^Q) I_Q{}^K\right] &\partial_K z^n - I_{[M}{}^P (\partial_P I_{N]}{}^Q) \mathcal{D}_Q z^n. \end{aligned} \tag{12}$$

Bearing in mind that $\mathcal{D}_Q z^n = 0$, the last term in the R.H.S. vanishes. The middle term can be transformed by flipping the derivative, $(\partial_P I_N{}^Q) I_Q{}^K = -I_N{}^Q \partial_P I_Q{}^K$ (this holds due to $I^2 = -\mathbb{1}$), and we finally obtain

[2] A conventional definition of the Nijenhuis tensor is a little bit different:

$$\mathcal{N}_{MN}{}^{K\,(\text{conventional})} = I_M{}^P \mathcal{N}_{PN}{}^{K\,(\text{this paper})} \;=\; I_M{}^P \partial_{[P} I_{N]}{}^K + I_N{}^P \partial_{[M} I_{P]}{}^K \tag{9}$$

(the last equality holds due to $I^2 = -\mathbb{1}$).

$$[\mathcal{D}_M, \mathcal{D}_N] z^n \ = \ -i\mathcal{N}_{MN}{}^K \partial_K z^n. \tag{13}$$

For this to vanish, the tensor $\mathcal{N}_{MN}{}^K$ should also vanish (to see this, choose the real coordinates as the real and imaginary parts of z^n).

Sufficiency. This part of the theorem [the proof of existence of the solution to the system (5) under the condition (10)] is more difficult. Well, it might be not so difficult for the mathematicians in the case when the complex structures $I_M{}^N$ represent analytic functions of the coordinates. Then the sufficiency of the conditions $[\mathcal{D}_M, \mathcal{D}_N] = \mathcal{K}_{MN}{}^Q \mathcal{D}_Q$ for the equation system $\mathcal{D}_M z^n = 0$ to have a solution is a corollary of the classical Frobenius theorem [9]. We will give here instead a heuristic proof of the NN theorem using "physical" language. This proof will elucidate the meaning of the constraint (10). Its linearized version is similar in spirit to multidimensional Cauchy-Riemann conditions.

- Let the complex structure $I_M{}^N$ has a canonic form (4). Then the solutions to (5) exist, and one of the solution is given by (6).
  Suppose now that the complex structure does not coincide with $(I_0)_M{}^N$ = $\text{diag}(\varepsilon, \ldots, \varepsilon)$, but is close to it: $I = I_0 + \Delta,\ \Delta \ll 1$. As a first step in the proof, we will show that, after such an infinitesimal deformation, solutions to (5) still exist.
- Let us first do so in the simplest case $D = 2$. Then the condition (10) is fulfilled identically. The condition $I^2 = -\mathbb{1}$ means that $\{\Delta, I_0\} = 0$, which is so iff [3]

$$\Delta_1^1 = -\Delta_2^2, \quad \Delta_1^2 = \Delta_2^1. \tag{14}$$

  Look now at the system (5). We set $z = z_{(0)} + \delta z$. The equations acquire the form

$$\begin{aligned}
\frac{\partial}{\partial x^1}(\delta z) + i\frac{\partial}{\partial x^2}(\delta z) &= \frac{1}{\sqrt{2}}(i\Delta_1{}^1 - \Delta_1{}^2)\,, \\
\frac{\partial}{\partial x^2}(\delta z) - i\frac{\partial}{\partial x^1}(\delta z) &= \frac{1}{\sqrt{2}}(i\Delta_2{}^1 - \Delta_2{}^2).
\end{aligned} \tag{15}$$

  Bearing in mind (14), these two equations coincide. Introducing the notation $X^{1+i2} = X^1 + iX^2$, they can be expressed as

$$\frac{\partial(\delta z)}{\partial \bar{z}_{(0)}} \ = \ \frac{i}{2}\Delta_1{}^{1+i2}, \tag{16}$$

  which can be easily integrated on a disk. Indeed, the whole discussion applies to a particular topologically trivial chart in a set of which a manifold is subdivided.
- The simplest nontrivial case is $D = 4$. The condition $\{\Delta, I_0\} = 0$ implies

[3] In physical notation, $\Delta = \alpha\sigma^1 + \beta\sigma^3$, where $\sigma^{a=1,2,3}$ are the Pauli matrices.

$$
\begin{aligned}
\Delta_1{}^1 &= -\Delta_2{}^2, & \Delta_1{}^2 &= \Delta_2{}^1, \\
\Delta_1{}^3 &= -\Delta_2{}^4, & \Delta_1{}^4 &= \Delta_2{}^3, \\
\Delta_3{}^1 &= -\Delta_4{}^2, & \Delta_3{}^2 &= \Delta_4{}^1, \\
\Delta_3{}^3 &= -\Delta_4{}^4, & \Delta_3{}^4 &= \Delta_4{}^3.
\end{aligned} \tag{17}
$$

We pose $z^1 \to z$, $z^2 \to w$. A short calculation shows that, bearing the relations (17) in mind, the Eq. (5) are reduced to

$$
\begin{aligned}
\frac{\partial(\delta z)}{\partial \bar{z}_{(0)}} &= \frac{i}{2}\,\Delta_1{}^{1+i2}, \\
\frac{\partial(\delta z)}{\partial \bar{w}_{(0)}} &= \frac{i}{2}\,\Delta_3{}^{1+i2}, \\
\frac{\partial(\delta w)}{\partial \bar{z}_{(0)}} &= \frac{i}{2}\,\Delta_1{}^{3+i4}, \\
\frac{\partial(\delta w)}{\partial \bar{w}_{(0)}} &= \frac{i}{2}\,\Delta_3{}^{3+i4}.
\end{aligned} \tag{18}
$$

If $D > 2$, the conditions (10) provide nontrivial constraints. Their linearized version is

$$
\partial_P \Delta_N{}^M - \partial_N \Delta_P{}^M \;=\; (I_0)_P{}^Q (I_0)_N{}^S \left[\partial_Q \Delta_S{}^M - \partial_S \Delta_Q{}^M\right]. \tag{19}
$$

Again, bearing in mind (17), one can show that, for $D = 4$, out of 24 real conditions in (19), only 4 independent real or 2 independent complex constraints are left. The latter have a simple form

$$
\begin{aligned}
\frac{\partial}{\partial \bar{z}_{(0)}}\,\Delta_3{}^{1+i2} - \frac{\partial}{\partial \bar{w}_{(0)}}\,\Delta_1{}^{1+i2} &= 0, \\
\frac{\partial}{\partial \bar{z}_{(0)}}\,\Delta_3{}^{3+i4} - \frac{\partial}{\partial \bar{w}_{(0)}}\,\Delta_1{}^{3+i4} &= 0.
\end{aligned} \tag{20}
$$

The first equation in (20) is the integrability condition for the system of the first two equations in (18). It is necessary and also sufficient for the solution of this system to exist. Indeed, it implies that the (0,1)-form

$$
\omega \;=\; \Delta_1{}^{1+i2}\, d\bar{z}_{(0)} + \Delta_3{}^{1+i2}\, d\bar{w}_{(0)}
$$

is closed, $\bar{\partial}_0\,\omega = 0$. Bearing in mind the trivial topology of a chart of our complex manifold that we are discussing, ω is also exact (see e.g. Theorem 6.1 in [6]), which is tantamount to saying that the solution exists. The second relation in (20) is the necessary and sufficient integrability condition for the system of the third and fourth equations in (18).

- This reasoning can be translated to the case of higher dimensions. For an arbitrary $D = 2d$, the Eq. (5) are reduced, bearing in mind $I^2 = -\mathbb{1}$, to d^2 conditions similar to (18) but with differentiation over each antiholomorphic variable $\bar{z}^{\bar{n}}_{(0)}$ for each complex function δz^n. The conditions (10) lead to $d^2(d-1)/2$ complex constraints which represent integrability conditions of the type (20). They imply that the forms

$$\begin{aligned} \omega_1 &= \Delta_1{}^{1+i2}\, d\bar{z}^1_{(0)} + \Delta_3{}^{1+i2}\, d\bar{z}^2_{(0)} + \ldots, \\ \omega_2 &= \Delta_1{}^{3+i4}\, d\bar{z}^1_{(0)} + \Delta_3{}^{3+i4}\, d\bar{z}^2_{(0)} + \ldots, \end{aligned} \tag{21}$$

  etc. are all closed. Due to the trivial topology of the chart, it also means that they are exact.
- Once the complex coordinates $z^n = z^n_{(0)} + \delta z^n$ satisfying the Eq. (5) are found, the complex structure acquires in these new coordinates the canonical form (7) and (4). Thus, we have actually proven that a small deformation of $I_M{}^N$ can be brought to the form (4) by an infinitesimal diffeomorphism, provided the condition (10) is satisfied.
- Let now $I_M{}^N(x)$ be arbitrary, not necessarily close to I_0 of Eq. (4). Using analyticity, we expand it into a formal series in a small parameter α:

$$I(x) = I_0 + \alpha I_1(x) + \alpha^2 I_2(x) + \ldots \tag{22}$$

  Do the same for the solutions $z^n(x)$ that we are looking for:

$$z^n(x) = z^n_{(0)} + \alpha z^n_{(1)}(x) + \alpha^2 z^n_{(2)}(x) + \ldots \tag{23}$$

  The correction $\alpha z^n_{(1)}(x)$ was determined before. Let $\tilde{z}^n(x) = z^n_{(0)} + \alpha z^n_{(1)}(x)$. As was just mentioned, the complex structure in these new coordinates has the canonical form (7) up to the terms $\propto \alpha^2$. Introducing the real and imaginary parts of $\tilde{z}^n(x)$ and calling them $\tilde{x}^M$, we may bring it to the form (4).
- Taking also into account the term $\alpha^2 I_2(x)$ in (22), we may express the complex structure in the new coordinates $\tilde{x}$ as

$$I(\tilde{x}) = I_0 + \alpha^2 \tilde{I}_2(\tilde{x}) + \text{ higher-order terms.} \tag{24}$$

  Repeating the same procedure that we used to determine $z^n_{(1)}(x)$, we can now determine $\tilde{z}^n_{(2)}(\tilde{x})$, from that $z^n_{(2)}(x)$, and likewise all the terms in the series (23).
- With the only reservation that we did not address a difficult question of the convergence of the series (23), the theorem is proven.

3 Supersymmetry

3.1 Preliminaries

To begin with, we present some basic "superfacts", bearing in mind a reader who is an expert in differential geometry, but may not know much about supersymmetry. We give, however, only the minimal necessary information assuming that our reader knows the basics of Grassmann algebra and, which is not so much necessary but desirable, of classical and quantum mechanics of the systems involving Grassmann dynamical variables. More details can be found in the review [10]. See especially Chap. 8.1 there.

The simplest supersymmetry algebra reads

$$Q_1^2 = Q_2^2 \;=\; H, \qquad\qquad Q_1 Q_2 + Q_2 Q_1 \;=\; 0. \tag{25}$$

Here H is the Hamiltonian and $Q_{1,2}$ are two different Hermitian operators called *supercharges*. As follows from (25), they commute with H. If one introduces a complex supercharge $Q \;=\; (Q_1 + iQ_2)/2$, one can also present (25) in the form

$$Q^2 = (\bar{Q})^2 = 0, \qquad\qquad Q\bar{Q} + \bar{Q}Q \;=\; H. \tag{26}$$

The algebra (25) involves two supercharges and, correspondingly, is usually called the algebra of $\mathcal{N} = 2$ *supersymmetric quantum mechanics* (SQM). More complicated algebras may involve extra supercharges[4] or also the momentum operators P_j. The latter algebras are relevant for supersymmetric quantum field theories. But in this paper we are going to discuss only the algebra (25) and also still more simple $\mathcal{N} = 1$ supersymmetry algebra,

$$\mathcal{Q}^2 = H \tag{27}$$

with real $\mathcal{Q}$. Physically, the latter is too simple to be interesting. After diagonalisation, one can always extract a square root of the Hamiltonian whose spectrum is bounded from below. If some energies in the spectrum are negative, one just redefines H by adding an appropriate positive constant. However, we will use in what follows the algebra (27) and its representations as a *technical tool*.

The algebra (25) leads to a double degeneracy of the spectrum. It also follows from (25) that the eigenvalues of the Hamiltonian are positive or zero. The doublets involving two positive energy states $|B\rangle$ and $|F\rangle$ with the properties

$$\begin{aligned} H|B\rangle \;&=\; E|B\rangle, & H|F\rangle \;&=\; E|F\rangle\,, \\ Q|B\rangle \;&=\; \sqrt{E}|F\rangle\,, & Q|F\rangle \;&=\; 0\,, \\ \bar{Q}|B\rangle \;&=\; 0\,, & \bar{Q}|F\rangle \;&=\; \sqrt{E}|B\rangle \end{aligned} \tag{28}$$

[4]The SQM systems enjoying $\mathcal{N} = 4$ or $\mathcal{N} = 8$ supersymmetry are known.

represent a simple 2-dimensional irreducible representation of the algebra (26). There exist also finite-dimensional representations involving a larger even number of states, but it is easy to show that they are all reducible. In physical language, any set of $2n$ states providing a representation of (26) is split into n doublets.

The only irreducible finite-dimensional representations of the algebra (27) are the trivial singlets—the eigenstates of $\mathcal{Q}$ and H.

We will be interested, however, in more complicated infinite-dimensional representations of the $\mathcal{N}=1$ and $\mathcal{N}=2$ algebra where the supercharges and the Hamiltonian are realized as linear differential operators acting in *superspace*.[5]

The $\mathcal{N}=1$ superspace includes time t and a real Grassmann nilpotent variable θ: $\theta^2=0$. The supercharges and the Hamiltonian are realized as the differential operators.

$$\begin{aligned} \mathcal{Q} &= -i\left(\frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}\right), \\ H &= -i\frac{\partial}{\partial t} \end{aligned} \tag{29}$$

The Hamiltonian is the generator for the time shifts. The supercharge is the generator for somewhat more complicated transformations:

$$\begin{aligned} \theta &\to \theta + \eta\,, \\ t &\to t + i\eta\theta \end{aligned} \tag{30}$$

with a real Grassmann parameter η.

Consider now $\mathcal{N}=1$ *superfields* (or supervariables) representing functions of t and θ. Due to the nilpotency of θ, they can be presented as

$$\mathcal{X}(t,\theta) = x(t) + i\theta\psi(t). \tag{31}$$

The ordinary real function $x(t)$ and the Grassmann-odd real function $\psi(t)$ are called the *components* of the superfield (31). The shifts (30) induce the shift

$$\delta\mathcal{X} \;=\; \mathcal{X}(t+i\eta\theta, \theta+\eta) - \mathcal{X}(t,\theta) \;=\; i\eta\mathcal{Q}\mathcal{X} \tag{32}$$

of the superfield $\mathcal{X}$ implying the following shifts of its components:

$$\delta x(t) \;=\; i\eta\psi(t)\,, \qquad \delta\psi(t) \;=\; -\eta\dot{x}. \tag{33}$$

Note that the product of two superfields is also a superfield: $\delta(\mathcal{X}_1\mathcal{X}_2) = i\eta\mathcal{Q}(\mathcal{X}_1\mathcal{X}_2)$.

Now we introduce the *covariant supersymmetric derivative*

[5] Well, in supersymmetric mechanical problems, we are dealing not with “superspace”, but rather with “supertime”, because we do not have any space variables and spatial dependence. But we stick to the terms commonly used in the literature.

$$\mathcal{D} \;=\; \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}. \tag{34}$$

This operator is Hermitian, nilpotent and anticommutes with $\mathcal{Q}$. The property

$$\mathcal{D}^2 \;=\; -i\frac{\partial}{\partial t} \tag{35}$$

holds.

Theorem 4. If $\mathcal{X}$ is a superfield, the same is true for $\mathcal{D}\mathcal{X}$.

Proof. We have

$$\delta(\mathcal{D}\mathcal{X}) = \mathcal{D}\,\delta\mathcal{X} = i\mathcal{D}(\eta\mathcal{Q}\mathcal{X}) = i\eta\mathcal{Q}(\mathcal{D}\mathcal{X})$$

(do not forget that η anticommutes with $\mathcal{D}$).

We understand now why $\mathcal{D}$ is called the *covariant* derivative. In the same way as the covariant derivative in Riemannian geometry makes a tensor out of a tensor, the derivative (34) makes a superfield out of a superfield.

The superfield (31) with its transformation law (33) defines an infinite-dimensional representation of the algebra (27). But it is a *reducible* representation. Indeed, one can now impose the constraint of reality $\bar{\mathcal{X}} = \mathcal{X}$. A real superfield stays real under the variation (32).

$\mathcal{N} = 2$ superspace and the $\mathcal{N} = 2$ superfields are defined in a similar manner. The superspace now includes time t and a *complex* Grassmann anticommuting variable θ: $\theta^2 = \bar\theta^2 = \{\theta, \bar\theta\}_+ = 0$. The supertransformations are

$$\begin{aligned}
\theta &\to \theta + \epsilon\,, \\
\bar\theta &\to \bar\theta + \bar\epsilon\,, \\
t &\to t + i(\epsilon\bar\theta + \bar\epsilon\theta)
\end{aligned} \tag{36}$$

with complex Grassmann ϵ. These transformations are generated by a complex supercharge Q and its Hermitian conjugate:

$$\begin{aligned}
Q &= -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial\theta} + i\bar\theta\frac{\partial}{\partial t}\right), \\
\bar Q &= -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial\bar\theta} + i\theta\frac{\partial}{\partial t}\right)
\end{aligned} \tag{37}$$

[the factor $1/\sqrt{2}$ is added to ensure the validity of (26)]. A generic $\mathcal{N} = 2$ superfield reads

$$\Phi(t,\theta,\bar\theta) \;=\; z(t) + i\theta\chi(t) + i\bar\theta\lambda(t) + \theta\bar\theta F(t) \tag{38}$$

with Grassmann-even complex $z(t)$ and $F(t)$ and Grassmann-odd complex $\chi(t)$ and $\lambda(t)$. The supersymmetric variation of Φ reads

$$\delta\Phi \;=\; i\sqrt{2}(\epsilon Q + \bar{\epsilon}\bar{Q})\Phi. \tag{39}$$

The covariant supersymmetric derivatives which are nilpotent and anticommute with Q and $\bar{Q}$ are

$$\begin{aligned} D &= \frac{\partial}{\partial\theta} - i\bar{\theta}\frac{\partial}{\partial t}, \\ \bar{D} &= -\frac{\partial}{\partial\bar{\theta}} + i\theta\frac{\partial}{\partial t}. \end{aligned} \tag{40}$$

The operator $i\bar{D}$ is the Hermitian conjugate of iD. If Φ is a superfield, then $D\Phi$ and $\bar{D}\Phi$ are also superfields.

The superfield (38) defines an infinite-dimensional representation of the algebra (26). This representation is reducible. Two different irreducible representations are obtained after imposing the constraints:

- The *reality* constraint $\bar{\Phi} = \Phi$. If Φ is real, the variation $\delta\Phi$ is also real.
- The *chirality* constraints $D\Phi = 0$ or $\bar{D}\Phi = 0$. Again, if $D\Phi$ vanishes, so does $D\delta\Phi$, and the same for $\bar{D}$. Note that if $\bar{D}Z = 0$, then $D\bar{Z} = 0$. We will call Z a *left* chiral superfield and $\bar{Z}$ a *right* chiral superfield.[6]

In what follows, we will not be interested in the real $\mathcal{N} = 2$ superfields, but exclusively in the chiral ones.

For a chiral superfield, the component expansion (38) can be simplified if one introduces "left" and "right" times:

$$t_L \;=\; t - i\theta\bar{\theta}\,, \qquad\qquad t_R \;=\; t + i\theta\bar{\theta}.$$

The supersymmetric variation of t_L depends only on θ, $\delta t_L = 2i\bar{\epsilon}\theta$, and the supersymmetric variation of t_R depends only on $\bar{\theta}$.

The set of coordinates (t_L, θ) describes the *holomorphic chiral* $\mathcal{N} = 2$ *superspace* and the set $(t_R, \bar{\theta})$ describes the *antiholomorphic chiral* $\mathcal{N} = 2$ *superspace.*

Then, if $\bar{D}Z = 0$, we may write

$$\begin{aligned} Z &= Z(t_L, \theta) = z(t_L) + i\sqrt{2}\,\theta\,\chi(t_L)\,, \\ \bar{Z} &= \bar{Z}(t_R, \bar{\theta}) = \bar{z}(t_R) + i\sqrt{2}\,\bar{\theta}\,\bar{\chi}(t_R). \end{aligned} \tag{41}$$

The components of a left chiral superfield are transformed as

$$\delta z \;=\; i\sqrt{2}\epsilon\,\chi\,, \qquad\qquad \delta\chi \;=\; -\sqrt{2}\bar{\epsilon}\,\dot{z}. \tag{42}$$

[6]The terms "left" and "right" have a physical origin which is irrelevant for us here.

Let us pose now

$$z = \frac{x_1 + i x_2}{\sqrt{2}}, \qquad \chi = \frac{\psi_1 + i\psi_2}{\sqrt{2}}, \qquad \epsilon = \frac{\eta + i\tilde{\eta}}{\sqrt{2}}. \tag{43}$$

Suppose first that ϵ is real, $\tilde{\eta} = 0$. Then we derive

$$\begin{aligned} \delta x_1 &= i\eta\psi_1, & \delta\psi_1 &= -\eta\dot{x}_1, \\ \delta x_2 &= i\eta\psi_2, & \delta\psi_2 &= -\eta\dot{x}_2. \end{aligned} \tag{44}$$

We see that the components (x_1, ψ_1) are not mixed with the components (x_2, ψ_2); each set is transformed in the same way as the components of an $\mathcal{N} = 1$ superfield [see Eq. (33)]! In other words, the representation Z is an irreducible representation of the $\mathcal{N} = 2$ superalgebra, but it can also be thought of as a *reducible* representation of $\mathcal{N} = 1$ superalgebra realized by the transformations (42) with real ϵ. When going down from $\mathcal{N} = 2$ to $\mathcal{N} = 1$, the chiral superfield Z is split into two real superfields $\mathcal{X}_1$ and $\mathcal{X}_2$. To see it quite explicitly, substitute $\theta = (\theta_1 + i\theta_2)/\sqrt{2}$ in (41). Then $t_L = t + \theta_2\theta_1$. We derive

$$Z \;=\; \frac{1}{\sqrt{2}}\{\mathcal{X}_1(t,\theta_1) + i\mathcal{X}_2(t,\theta_1) + i\theta_2[\mathcal{D}\mathcal{X}_1(t,\theta_1) + i\mathcal{D}\mathcal{X}_2(t,\theta_1)]\}\,. \tag{45}$$

Look now at the transformations (42) when $\epsilon = i\tilde{\eta}/\sqrt{2}$ is imaginary. We obtain

$$\begin{aligned} \tilde{\delta} x_1 &= -i\tilde{\eta}\psi_2, & \tilde{\delta}\psi_1 &= -\tilde{\eta}\dot{x}_2, \\ \tilde{\delta} x_2 &= i\tilde{\eta}\psi_1, & \tilde{\delta}\psi_2 &= \tilde{\eta}\dot{x}_1 \end{aligned} \tag{46}$$

or in a compact form:

$$\tilde{\delta}\mathcal{X}_A \;=\; \tilde{\eta}\,\varepsilon_{AB}\,\mathcal{D}\mathcal{X}_B \tag{47}$$

[with ε defined as in (3)].

The generators of the transformations (44 and (46) obey the algebra (25). Indeed,

- It is rather evident that the transformations (44) and (46, 47) commute. Indeed, $\delta\mathcal{X}_A$ is a superfield, and hence $\delta(\tilde{\delta}\mathcal{X}_A)$ and $\tilde{\delta}(\delta\mathcal{X}_A)$ coincide, having both the form (32) with $\mathcal{X}$ replaced by $\tilde{\delta}\mathcal{X}_A$. A corollary of this is the vanishing of the anticommutator $\mathcal{Q}\tilde{\mathcal{Q}} + \tilde{\mathcal{Q}}\mathcal{Q}$ of the corresponding quantum supercharges.
- Bearing in mind (35), the Lie bracket of two different tilde-transformations reads

$$(\tilde{\delta}_1\tilde{\delta}_2 - \tilde{\delta}_2\tilde{\delta}_1)\mathcal{X}_A = -2i\tilde{\eta}_1\tilde{\eta}_2\dot{\mathcal{X}}_A, \tag{48}$$

which is tantamount to saying that $\tilde{\mathcal{Q}}^2$ coincides with the Hamiltonian (the generator of time shifts).

3.2 NN Theorem: Supersymmetric Interpretation

The tensor ε_{AB} entering (47) can be interpreted as a 2×2 block in the flat complex structure (4). The components x_A of the superfields $\mathcal{X}_A$ can be interpreted as the flat Cartesian coordinates. Suppose now that we have $2d$ $\mathcal{N} = 1$ superfields $\mathcal{X}^M$. One of the supersymmetries follows from the transformations of the superspace coordinates as in (44):

$$\delta x^M = i\eta\psi^M, \qquad\qquad \delta\psi^M = -\eta\dot{x}^M. \tag{49}$$

Looking for a generalization of (47), we anticipate the presence of the second supersymmetry,

$$\tilde{\delta}\mathcal{X}^M \;=\; \tilde{\eta}\, I_N{}^M(\mathcal{X}^P)\, \mathcal{D}\mathcal{X}^N, \tag{50}$$

where

$$I^2 = -\mathbb{1}, \tag{51}$$

and ask: *under what conditions is it possible?* Under what conditions do the generators of the transformations (49) and (50) obey the algebra (25)?

Theorem 5. The algebra (25) holds iff the Nijenhuis tensor (10) vanishes.

Proof. The Lie bracket $[\delta, \tilde{\delta}]$ vanishes by the same reason as in the flat case treated before: the transformation δ mixes the components of each multiplet, while the transformation $\tilde{\delta}$ mixes different superfields and does not bother much about their internal structure. Thus, we only need to explore the Lie bracket $(\tilde{\delta}_1\tilde{\delta}_2 - \tilde{\delta}_2\tilde{\delta}_1)\mathcal{X}^M$.

Note first that

$$\tilde{\delta}(\mathcal{D}\mathcal{X}^N) = \mathcal{D}(\tilde{\delta}\mathcal{X}^N) \;=\; -\tilde{\eta}\mathcal{D}(I_L{}^N\mathcal{D}\mathcal{X}^L) \;=\; -\tilde{\eta}(\partial_K I_L{}^N)\mathcal{D}\mathcal{X}^K\mathcal{D}\mathcal{X}^L + i\tilde{\eta}I_L{}^N\dot{\mathcal{X}}^L.$$

The commutator of two transformations (50) is then derived to be

$$\begin{aligned}\left(\tilde{\delta}_1\tilde{\delta}_2 - \tilde{\delta}_2\tilde{\delta}_1\right)\mathcal{X}^M &= 2i\tilde{\eta}_1\tilde{\eta}_2(I^2)_K{}^M\dot{\mathcal{X}}^K \\ &\quad -2\tilde{\eta}_1\tilde{\eta}_2\Big[I_K{}^L\left(\partial_L I_N{}^M\right) + \left(\partial_N I_K{}^L\right)I_L{}^M\Big]\mathcal{D}\mathcal{X}^K\mathcal{D}\mathcal{X}^N.\end{aligned} \tag{52}$$

If we want it to coincide with $-2i\,\tilde{\eta}_1\tilde{\eta}_2\,\partial_t\mathcal{X}^M$ [as is dictated by Eq.(25)] the conditions (51) as well as

$$\left(\partial_L I_{[N}{}^M\right) I_{K]}{}^L + \left(\partial_{[N} I_{K]}{}^L\right) I_L{}^M \;=\; 0 \tag{53}$$

follow. Using again (51) and flipping the derivative in the second term, the L.H.S. of Eq. (53) can be brought into the form (9). The condition (10) follows.

Thus, the condition $\mathcal{N}_{MN}{}^K = 0$ is necessary and sufficient for $\mathcal{N} = 2$ supersymmetry associated with the given complex structure to hold. But the NN theorem is formulated differently: it affirms that the condition (10) is necessary and sufficient for the existence of complex coordinates.

Well, as far as necessity is concerned, the equivalence of Theorems 3 and 5 is rather clear. Suppose that complex coordinates z^n exist. But then each such coordinate can be upgraded to a complex chiral superfield Z^n whose components are transformed under supersymmetry as in (42). Each superfield Z^n can be expressed via a pair of $\mathcal{N} = 1$ real superfields as in (45). The complex structure tensor $I_M{}^N$ has in this case the form (4) and does not depend on the coordinates. The tensor $\mathcal{N}_{MN}{}^K$ vanishes automatically.

Now, if the Nijenhuis tensor vanishes, we know from Theorem 5 that the algebra of $\mathcal{N} = 2$ supersymmetry holds. The set of $2d$ superfields $\mathcal{X}^M$ is an infinite-dimensional representation of this algebra. Then the sufficiency of (10) means that, for $d > 1$, this representation is *reducible* and can be decomposed in a direct sum of d irreducible representations realized by the components of the chiral complex superfields Z^n.

This latter statement looks very natural, it is widely used by physicists, but I am not aware of its independent proof. The only known proof of this fact is the proof of the sufficiency part of the NN theorem that we outlined in Sect. 2 and that does not resort to supersymmetric description.

3.2.1 Invariant Actions

Up to now, when talking about the supersymmetric aspects of the NN theorem, we stayed at the purely algebraic level, having discussed only the algebras (25), (27) and their representations. A reader-mathematician may stop reading this paper at this point.

But, when a physicist thinks of a symmetry, s/he is always interested in *dynamical systems* that enjoy these symmetries. An industrial method to find supersymmetric dynamical systems is based on the following theorem:

Theorem 6. Let $\mathcal{X}(t, \theta)$ be an $\mathcal{N} = 1$ superfield that vanishes at $t = \pm\infty$. Then the integral (associated with the physical action)

$$S = \int d\theta \int_{-\infty}^{\infty} dt\, \mathcal{X} \tag{54}$$

is invariant under transformations (30).

Here the symbol $\int d\theta$ is the Berezin integral,

$$\int d\theta \mathcal{X} \equiv \frac{\partial}{\partial \theta} \mathcal{X}. \tag{55}$$

Proof. We have

$$\delta S = \int d\theta \int_{-\infty}^{\infty} dt\, \delta \mathcal{X} = -\epsilon \int d\theta \int_{-\infty}^{\infty} dt \left(\frac{\partial}{\partial \theta} + i\theta \frac{\partial}{\partial t} \right) \mathcal{X}.$$

The first term vanishes due to the definition (55) and the Grassmannian nature of θ. The second term vanishes due to the condition $\mathcal{X}(\pm\infty, \theta) = 0$.

Obviously, the same property holds for the integral

$$S = \int d\bar{\theta} d\theta \int_{-\infty}^{\infty} dt\, \Phi \qquad (56)$$

of a $\mathcal{N} = 2$ superfield Φ.

The superfield $\mathcal{X}$ in Eq. (54) and the superfield Φ in Eq. (56) can be constructed out of certain basic superfields by multiplications, time differentiations and covariant differentiations with the operator $\mathcal{D}$ in the $\mathcal{N} = 1$ case and with the operators D and $\bar{D}$ in the $\mathcal{N} = 2$ case. In particular, one can write [11]

$$S = \frac{1}{4} \int d\bar{\theta} d\theta dt\, h_{m\bar{n}}(Z^k, \bar{Z}^{\bar{k}})\, \bar{D}\bar{Z}^{\bar{n}}(t_R) D Z^m(t_L), \qquad (57)$$

where $Z^{k=1,\ldots,d}$ are left chiral superfields and $h_{m\bar{n}}$ is Hermitian. Substituting there the expansions (41), not forgetting to expand over θ and $\bar{\theta}$ also $t_{L,R} = t \mp i\theta\bar{\theta}$ and performing the integral over $d\bar{\theta} d\theta dt$, one can derive the following expression for the Lagrangian:

$$L = h_{m\bar{n}}(z, \bar{z}) \dot{z}^m \dot{\bar{z}}^{\bar{n}} + \text{terms including superpartners } \chi^m(t) \qquad (58)$$

We can now interpret z^m and $\bar{z}^{\bar{m}}$ as the coordinates on a complex manifold with the metric $h_{m\bar{n}}(z, \bar{z})$. The displayed term of the Lagrangian can be interpreted as the kinetic energy of a particle with unit mass moving along the manifold. The dynamical system describing such a motion is called *sigma model*. And the whole Lagrangian [due to Theorem 6, the corresponding action is invariant under (42)] represents its supersymmetric version.

The *same* dynamical system can also be described in the $\mathcal{N} = 1$ superfield language. Consider the action [12]

$$S = \frac{i}{2} \int d\theta dt\, g_{MN}(\mathcal{X})\, \dot{\mathcal{X}}^M \mathcal{D} \mathcal{X}^N, \qquad (59)$$

This is not a most general form. The action (59) describes (under the condition that ${\mathcal{N}_{MN}}^K$ vanishes) only the Kähler manifolds; to describe generic complex manifolds, one should add an extra term. But we do not want to plunge into too much details here, addressing an interested reader to Sect. 4 of Ref. [5].

After integration over $d\theta dt$, we obtain the Lagrangian

$$L = \frac{1}{2} g_{MN} \dot{x}^M \dot{x}^N + \text{terms including superpartners } \psi^M(t), \tag{60}$$

i.e. g_{MN} has the meaning of the real metric.

By construction, the action (59) is invariant under $\mathcal{N} = 1$ transformations, but it is also invariant under the extra supersymmetry transformations (50) provided the conditions (51), (10) *and* the condition $I_{MN} = -I_{NM}$ hold.

Note that, to relate I_{MN} to $I_M{}^N$, we need the metric. The notion of metric was *not* used in the proof of Theorem 5 or Theorem 3, which thus hold also for non-metric manifolds. Indeed, the equation system (5) for the complex coordinates has solutions provided the condition (10) is fulfilled even when $I_{MN} \neq -I_{NM}$. But we need the metric for the physical applications. And then the condition of the antisymmetry of I_{MN} should be imposed.

The equations of motion that follow from the Lagrangian (60) describe classical supersymmetric dynamics. The Legendre transformation of (60) gives us the classical Hamiltonian from which the quantum Hamiltonian can be derived. The quantum system has the same symmetry as the classical one. If we are dealing with $\mathcal{N} = 2$ supersymmetry, a pair of Hermitially conjugate supercharges satisfying the algebra (26) exist. This guarantees the two-fold degeneracy of all positive energy states as in (28).

Acknowledgements I am indebted to G. Carron, G. Papadopoulos and A. Rosly for illuminating discussions.

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A Class of Representations of the Orthosymplectic Lie Superalgebras $\mathcal{B}(n, n)$ and $\mathcal{B}(\infty, \infty)$

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Abstract In 1982 Palev showed that the algebraic structure generated by the creation and annihilation operators of a system of m parafermions and n parabosons, satisfying the mutual parafermion relations, is the Lie superalgebra $\mathfrak{osp}(2m+1|2n)$. The "parastatistics Fock spaces" of order p of such systems are then certain lowest weight representations of $\mathfrak{osp}(2m+1|2n)$. We investigate now the situation when the number of parafermions and parabosons becomes infinite, which is of interest not only in a physics context but also from the mathematical point of view. In this contribution, we will discuss the various steps that are needed to understand the infinite-rank case. First, we will introduce appropriate bases and Dynkin diagrams for $\mathfrak{B}(n,n) = \mathfrak{osp}(2n+1|2n)$ that allow us to extend $n \to \infty$. Then we will develop a new matrix form for $\mathfrak{B}(n,n) = \mathfrak{osp}(2n+1|2n)$, because the standard one is not appropriate for taking this limit. Following this, we construct a new Gelfand-Zetlin basis of the parastatistics Fock spaces in the finite rank case (in correspondence with this new matrix form). The new structures, related to a non-distinguished simple root system, allow the extension to $n \to \infty$. This leads to the definition of the algebra $\mathfrak{B}(\infty,\infty)$ as a Lie superalgebra generated by an infinite number of creation and annihilation operators (subject to certain relations), or as an algebra of certain infinite-dimensional matrices. We study the parastatistics Fock spaces, as certain lowest weight representations of $\mathfrak{B}(\infty,\infty)$. In particular, we construct a basis consisting of well-described row-stable Gelfand-Zetlin patterns.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_12

1 Introduction

Throughout this paper we will use square brackets for a commutator, $[A, B] = AB - BA$; curly brackets for an anti-commutator: $\{A, B\} = AB + BA$; and double brackets if we are dealing with operators from a $\mathbb{Z}_2$ graded algebra: $[\![A, B]\!] = AB - (-1)^{\langle A\rangle\langle B\rangle} BA$, where $\langle A\rangle = \deg(A) \in \{0, 1\}$ is the degree of A.

In this contribution we will consider Fock spaces for bosons, fermions, parabosons, parafermions, and combined systems of parabosons and parafermions. The emphasis is on algebraic structures behind these systems, on identifying Fock spaces with a class of representations of these algebras, and on constructing a basis for these representations.

For a system described by n pairs of boson (creation and annihilation) operators $B_i^\pm$ $(i = 1, \ldots, n)$, satisfying

$$[B_i^-, B_j^+] = \delta_{ij} \tag{1}$$

and all other commutators zero, the Fock space with vacuum vector $|0\rangle$ characterized by $(B_i^\pm)^\dagger = B_i^\mp$ and $B_i^-|0\rangle = 0$ has a very simple (orthonormal) basis:

$$|k_1, \ldots, k_n\rangle = \frac{(B_1^+)^{k_1} \cdots (B_n^+)^{k_n}}{\sqrt{k_1! \cdots k_n!}} |0\rangle \tag{2}$$

with $k_i \in \{0, 1, 2, \ldots\}$. Similarly, a system described by m pairs of fermion operators $F_i^\pm$ $(i = 1, \ldots, m)$, with

$$\{F_i^-, F_j^+\} = \delta_{ij} \tag{3}$$

and all other anti-commutators zero, the Fock space is characterized by $(F_i^\pm)^\dagger = F_i^\mp$ and $F_i^-|0\rangle = 0$, and has a basis similar to (2) but with all $k_i \in \{0, 1\}$.

More interesting structures are provided by parabosons and parafermions, especially from the algebraic point of view. These were first introduced by Green [1] and their Fock spaces were first studied by Greenberg and Messiah [2].

A system of n pairs of parabosons $b_j^\pm$ $(j = 1, \ldots, n)$ is defined by means of triple relations:

$$[\{b_j^\xi, b_k^\eta\}, b_l^\epsilon] = (\epsilon - \xi)\delta_{jl} b_k^\eta + (\epsilon - \eta)\delta_{kl} b_j^\xi, \tag{4}$$

where $j, k, l \in \{1, 2, \ldots, n\}$ and $\eta, \epsilon, \xi \in \{+, -\}$ (to be interpreted as $+1$ and -1 in the algebraic expressions $\epsilon - \xi$ and $\epsilon - \eta$). In this case, there is not a unique Fock space, but for every positive integer p (referred to as the order of statistics) there is a Fock space $\mathcal{V}(p)$ characterized by $(b_j^\pm)^\dagger = b_j^\mp$, $b_j^-|0\rangle = 0$ and

$$\{b_j^-, b_k^+\}|0\rangle = p\, \delta_{jk}\, |0\rangle. \tag{5}$$

Similarly, a system of m pairs of parafermions $f_j^{\pm}$ ($j = 1, \ldots, m$) is defined by the triple relations

$$[[f_j^{\xi}, f_k^{\eta}], f_l^{\epsilon}] = |\epsilon - \eta|\delta_{kl} f_j^{\xi} - |\epsilon - \xi|\delta_{jl} f_k^{\eta}. \tag{6}$$

Their Fock spaces $\mathcal{W}(p)$, also labelled by a positive integer p, are characterized by $(f_j^{\pm})^{\dagger} = f_j^{\mp}$, $f_j^{-}|0\rangle = 0$ and

$$[f_j^{-}, f_k^{+}]|0\rangle = p\,\delta_{jk}\,|0\rangle. \tag{7}$$

These cubic or triple relations involve nested (anti-)commutators, just like the Jacobi identity of Lie (super)algebras. It was indeed shown later [3, 4] that the parafermionic algebra generated by $2m$ elements $f_i^{\pm}$ subject to (6) is the orthogonal Lie algebra $\mathfrak{so}(2m+1)$. The Fock space $\mathcal{W}(p)$ is the unitary irreducible representation of $\mathfrak{so}(2m+1)$ with lowest weight $(-\frac{p}{2}, -\frac{p}{2}, \ldots, -\frac{p}{2})$ in the standard basis.

Many years later, it was shown that the parabosonic algebra generated by $2n$ *odd* elements $b_i^{\pm}$ subject to (4) is the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [5]. In this case the Fock space $\mathcal{V}(p)$ is the unitary irreducible $\mathfrak{osp}(1|2n)$ representation with lowest weight $(\frac{p}{2}, \frac{p}{2}, \ldots, \frac{p}{2})$ in the standard basis.

For $p = 1$, $\mathcal{V}(p)$ becomes the ordinary boson Fock space and $\mathcal{W}(p)$ becomes the ordinary fermion Fock space.

Already in their first paper, Greenberg and Messiah [2] considered combined systems of parafermions and parabosons. In combined systems, it will be convenient to use negative indices for parafermions and positive indices for parabosons, and to use the common operator notation $c_i^{\pm}$:

$$c_j^{\pm} = f_j^{\pm} \quad (-m \le j \le -1); \qquad c_i^{\pm} = b_i^{\pm} \quad (1 \le i \le n). \tag{8}$$

Apart from two trivial combinations, there are two non-trivial relative commutation relations between parafermions and parabosons, also expressed by means of triple relations. The case considered here is the so-called "relative parafermion relation" and is determined by the parastatistics relations

$$[\![[\![c_j^{+}, c_k^{-}]\!], c_l^{+}]\!] = 2\delta_{kl} c_j^{+}, \qquad [\![[\![c_j^{+}, c_k^{+}]\!], c_l^{+}]\!] = 0, \tag{9}$$

$$[\![c_j^{-}, [\![c_k^{+}, c_l^{-}]\!]]\!] = 2\delta_{jk} c_l^{-}, \qquad [\![[\![c_j^{-}, c_k^{-}]\!], c_l^{-}]\!] = 0. \tag{10}$$

The complete set of relations can also be written in the somewhat complicated form

$$[\![[\![c_j^{\xi}, c_k^{\eta}]\!], c_l^{\epsilon}]\!] = -2\delta_{jl}\delta_{\epsilon,-\xi}\epsilon^{\langle l\rangle}(-1)^{\langle k\rangle\langle l\rangle} c_k^{\eta} + 2\epsilon^{\langle l\rangle}\delta_{kl}\delta_{\epsilon,-\eta} c_j^{\xi}, \tag{11}$$

where $\langle k\rangle$ refers to the grading of $c_k^{\pm}$, and thus is 0 for negative k and 1 for positive k, following (8).

It was shown by Palev [6] that the Lie superalgebra (LSA) generated by $2m$ even elements $f_j^{\pm}$ and $2n$ odd elements $b_j^{\pm}$ subject to the above relations (11) is

$\mathfrak{B}(m, n) = \mathfrak{osp}(2m+1|2n)$. The Fock spaces, denoted by $V(p)$ and labelled by a positive integer p, are characterized by $(c_j^{\pm})^\dagger = c_j^{\mp}$, $c_j^-|0\rangle = 0$ and $[\![c_j^-, c_k^+]\!]|0\rangle = p\,\delta_{jk}\,|0\rangle$. $V(p)$ is the unitary irreducible representation of $\mathfrak{osp}(2m+1|2n)$ with lowest weight $(-\frac{p}{2}, \ldots, -\frac{p}{2}|\frac{p}{2}, \ldots, \frac{p}{2})$ in the standard basis. These are referred to as the parastatistics Fock spaces.

Understanding the algebraic structure behind such systems of parabosons/parafermions is one step. But understanding the structure of the corresponding Fock spaces is another important step. A major contribution here is the so-called Green ansatz, where one considers the p-fold tensor product of an ordinary boson/fermion Fock space and extracts an irreducible component herein. This is far from trivial, and computing matrix elements for generators remains a difficult problem in this approach [7, 8]. For the case of parabosons ($\mathfrak{osp}(1|2n)$ representations $\mathcal{V}(p)$), a complete basis with all matrix elements was given for the first time in [9]. The same type of construction was given for parafermions ($\mathfrak{so}(2m+1)$ representations $\mathcal{W}(p)$) in [10]. Interesting character formulas for these representations were also given, and these could be extended to characters of the parastatistics representations $V(p)$ of $\mathfrak{osp}(2m+1|2n)$ [11]. An actual basis of the parastatistics Fock spaces was constructed in [12], where again all matrix elements of the generators could be computed.

All the above constructions of basis vectors rely on the development of an appropriate Gelfand-Zetlin (GZ) basis, which in turn depends on an appropriate chain of subalgebras under which the reduction of $V(p)$ is multiplicity free at every step of the chain. For the parastatistics case, this subalgebra chain is

$$\begin{aligned}&\mathfrak{osp}(2m+1|2n) \supset \mathfrak{gl}(m|n) \supset \mathfrak{gl}(m|n-1) \supset \mathfrak{gl}(m|n-2) \supset \cdots \\ &\supset \mathfrak{gl}(m|1) \supset \mathfrak{gl}(m) \supset \mathfrak{gl}(m-1) \supset \cdots \supset \mathfrak{gl}(2) \supset \mathfrak{gl}(1).\end{aligned} \tag{12}$$

Since it follows from the character formula [12] that the decomposition of $V(p)$ in the chain $\mathfrak{osp}(2m+1|2n) \supset \mathfrak{gl}(m|n)$ is easy and multiplicity free, the GZ-basis consists of a (triangular) pattern with $m+n$ rows, each row corresponding to a highest of a $\mathfrak{gl}$ algebra in the chain (12).

In the present contribution, we consider the case for which m and n become infinite. If one tries to extend the above mentioned GZ-patterns to infinite patterns, starting from the bottom row corresponding to $\mathfrak{gl}(1)$ and gradually increasing the rank of the algebra, it is obvious that one cannot let both m and n go to infinity.

In the next paragraph, we shall explain how the introduction of an "odd GZ-basis" can overcome this problem, however only in the case $m = n$. This will lead to a new basis for the Fock spaces of $\mathfrak{B}(n, n) = \mathfrak{osp}(2n+1|2n)$. This new basis was constructed in [13], to which we refer for further details. The current contribution summarizes some of the main results in [13] and it is inevitable to have some overlap with [13]. Here, we first give a justification for the necessity of a new GZ-basis. Then we will proceed to a new matrix realization of $\mathfrak{B}(n, n)$, and give the parastatistics generators in this new basis. The parastatistics Fock representations are then described in the new GZ-basis. We also include an example (given in the Appendix) to illustrate the various notions. Finally, it is shown how to extend this to the case when $n \to \infty$, where so-called row-stable GZ-patters are of importance. For some details and explicit formulas, the reader will be referred to [13].

2 Introducing an Odd GZ-Basis

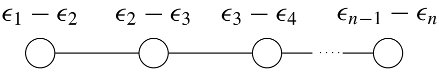
Looking back at the original idea of a Gelfand-Zetlin basis, for the case of the Lie algebra $\mathfrak{gl}(n)$, the construction of the basis is according to the chain of subalgebras

$$\mathfrak{gl}(n) \supset \mathfrak{gl}(n-1) \supset \cdots \supset \mathfrak{gl}(2) \supset \mathfrak{gl}(1). \tag{13}$$

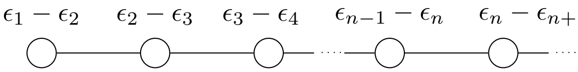
Every row of a GZ-basis vector consist of a highest weight of $\mathfrak{gl}(k)$, the top row ("row n") corresponding to $\mathfrak{gl}(n)$ and the bottom row ("row 1") to $\mathfrak{gl}(1)$. Such GZ-patterns can easily be extended to the infinite rank case by introducing infinitely large GZ-patterns according to

$$\mathfrak{gl}(1) \subset \mathfrak{gl}(2) \subset \cdots \subset \mathfrak{gl}(n-1) \subset \mathfrak{gl}(n) \subset \cdots . \tag{14}$$

In order to label basis vectors of an irreducible $\mathfrak{gl}(\infty)$ representation, with locally finite action of $\mathfrak{gl}(\infty)$ generators, one should require certain stability properties of the infinite GZ-patterns. The main idea is however that one can reverse the chain (13) to (14) allowing the limit $n \to \infty$. Also in terms of Dynkin diagrams, this process of letting n increase to infinity is somehow clear from the Dynkin diagram of $\mathfrak{gl}(n)$,



and its extension as n increases:



For the Lie superalgebra $\mathfrak{gl}(m|n)$, one can also construct (at least for a class of representations) a GZ-basis [14] according to the chain

$$\begin{aligned} &\mathfrak{gl}(m|n) \supset \mathfrak{gl}(m|n-1) \supset \mathfrak{gl}(m|n-2) \supset \cdots \\ &\supset \mathfrak{gl}(m|1) \supset \mathfrak{gl}(m) \supset \mathfrak{gl}(m-1) \supset \cdots \supset \mathfrak{gl}(2) \supset \mathfrak{gl}(1). \end{aligned} \tag{15}$$

In an attempt to let m and n increase to infinity, the GZ-patterns corresponding to the above chain are no longer appropriate. Indeed, if one reverses the chain (15) in which m grows to infinity,

$$\mathfrak{gl}(1) \subset \mathfrak{gl}(2) \subset \cdots \subset \mathfrak{gl}(m-1) \subset \mathfrak{gl}(m) \subset \mathfrak{gl}(m+1) \subset \cdots \tag{16}$$

one somehow never reaches the point where a Lie superalgebra can be included, and there is no way of having also $n \to \infty$.

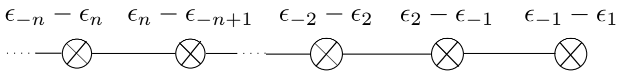
In a previous paper [15], this was solved by introducing the so-called odd GZ-basis for $\mathfrak{gl}(n|n)$ (m and n must be equal). This arises from the chain of superalgebras

$$\begin{aligned} &\mathfrak{gl}(n|n) \supset \mathfrak{gl}(n|n-1) \supset \mathfrak{gl}(n-1|n-1) \supset \cdots \\ &\cdots \supset \mathfrak{gl}(2|2) \supset \mathfrak{gl}(2|1) \supset \mathfrak{gl}(1|1) \supset \mathfrak{gl}(1). \end{aligned} \tag{17}$$

This chain can easily be reversed and continued to infinity,

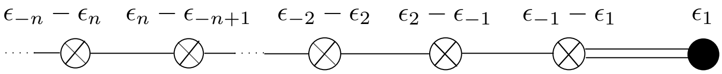
$$\begin{aligned} &\mathfrak{gl}(1) \subset \mathfrak{gl}(1|1) \subset \mathfrak{gl}(2|1) \subset \mathfrak{gl}(2|2) \subset \cdots \\ &\subset \mathfrak{gl}(n-1|n-1) \subset \mathfrak{gl}(n|n-1) \subset \mathfrak{gl}(n|n) \subset \cdots \end{aligned} \tag{18}$$

leading to an appropriate GZ-basis for $\mathfrak{gl}(\infty|\infty)$ representations [15], in which each row of the infinite GZ-pattern corresponds to a highest weight in the chain (18) (with certain stability requirements). Note that such a chain corresponds to a consecutive inclusion of Dynkin diagrams of Lie superalgebras of type $\mathfrak{gl}$ with odd simple roots only. In a convenient basis $(\ldots, \epsilon_{-3}, \epsilon_{-2}, \epsilon_{-1}; \epsilon_1, \epsilon_2, \epsilon_3, \ldots)$, the Dynkin diagram is



Hence, starting from the right and extending each time by one node to the left, one finds consecutively the Dynkin diagrams of $\mathfrak{gl}(1|1)$, $\mathfrak{gl}(1|2)$, $\mathfrak{gl}(2|2)$, etc. This process can continue to the left basically up to infinity.

It is in this context that the convenient GZ-basis and Dynkin diagrams for $\mathfrak{B}(n,n) = \mathfrak{osp}(2n+1|2n)$ are introduced. Adding the extra odd root ϵ_1 to the right, one finds by extending to the left consecutive Dynkin diagrams of $\mathfrak{B}(n,n)$ or $\mathfrak{B}(n,n+1)$.



3 New Matrix Realization of $\mathfrak{B}(n,n)$

Following the previous remarks, it is convenient to work in a new matrix realization of $\mathfrak{B}(n,n)$. Rows and columns, and indices of other objects, will be labelled by both negative and positive numbers. For non-negative integers m and n we will use the following notation for ordered sets:

$$\begin{aligned} &[-m,n] = \{-m,\ldots,-2,-1,0,1,2,\ldots,n\}, \\ &[-m,n]^* = \{-m,\ldots,-2,-1,1,2,\ldots,n\}. \end{aligned} \tag{19}$$

When more convenient, we write the minus sign of an index as an overlined number, e.g. $[\bar{2}, 3]^* = \{\bar{2}, \bar{1}, 1, 2, 3\}$. We will also use $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_+^* = \{1, 2, 3, \ldots\}$.

Let I and J be the (2×2)-matrices

$$I := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{20}$$

and let B be the $(4n+1) \times (4n+1)$-matrix, with indices in $[-2n, 2n]$, given by $B = I \oplus \cdots \oplus I \oplus 1 \oplus J \oplus \cdots \oplus J$, or, written in block form:

$$B := \left(\begin{array}{ccc|c|ccc} I & 0 & & & & & \\ 0 & \ddots & 0 & & & & \\ & 0 & I & & & & \\ \hline & & & 1 & & & \\ \hline & & & & J & 0 & \\ & & & & 0 & \ddots & 0 \\ & & & & & 0 & J \end{array}\right). \tag{21}$$

Herein, 0 stands for the zero (2×2)-matrix, the entry 1 is at position $(0, 0)$, and the empty parts of the matrix consist of zeros.

The matrices X of the Lie superalgebra $\mathfrak{B}(n, n)$ will have the following block form:

$$X := \left(\begin{array}{ccc|c|ccc} X_{\bar{n},\bar{n}} & \cdots & X_{\bar{n},\bar{1}} & X_{\bar{n},0} & X_{\bar{n},1} & \cdots & X_{\bar{n},n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{\bar{1},\bar{n}} & \cdots & X_{\bar{1},\bar{1}} & X_{\bar{1},0} & X_{\bar{1},1} & \cdots & X_{\bar{1},n} \\ \hline X_{0,\bar{n}} & \cdots & X_{0,\bar{1}} & 0 & X_{0,1} & \cdots & X_{0,n} \\ \hline X_{1,\bar{n}} & \cdots & X_{1,\bar{1}} & X_{1,0} & X_{1,1} & \cdots & X_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{n,\bar{n}} & \cdots & X_{n,\bar{1}} & X_{n,0} & X_{n,1} & \cdots & X_{n,n} \end{array}\right). \tag{22}$$

Herein, any matrix of the form X_{ij} with $i, j \in [\bar{n}, n]^*$ is a (2×2)-matrix, $X_{0,i}$ is a (1×2)-matrix and $X_{i,0}$ a (2×1)-matrix.

The Lie superalgebra $\mathfrak{B}(n, n) = \mathfrak{osp}(2n+1|2n)$ is $\mathbb{Z}_2$-graded and its homogeneous elements are referred to as even and odd elements, with the degree denoted by $\deg(X)$. The even matrices X will have zeros in the upper right and bottom left blocks, i.e. $X_{ij} = 0$ for all $(i, j) \in [\bar{n}, 0] \times [1, n]$ and $(i, j) \in [1, n] \times [\bar{n}, 0]$. The odd matrices X will have zeros in the upper left and bottom right blocks, i.e. $X_{ij} = 0$ for all $(i, j) \in [\bar{n}, 0] \times [\bar{n}, 0]$ and $(i, j) \in [1, n] \times [1, n]$.

The actual definition, derived from [16], is then as follows: $\mathfrak{B}(n, n)_0$ consists of all even matrices X of the form (22) such that

$$X^T B + BX = 0;$$

$\mathfrak{B}(n,n)_1$ consists of all odd matrices X of the form (22) such that

$$X^{ST}B - BX = 0.$$

Herein X^T is the ordinary transpose of X and X^{ST} is the supertranspose of X [13, 16]. For homogeneous elements of type (22), the Lie superalgebra bracket is

$$[\![X, Y]\!] = XY - (-1)^{\deg(X)\deg(Y)} YX,$$

with ordinary matrix multiplication in the right hand side.

Denote, as usual, by e_{ij} the matrix with zeros everywhere except a 1 on position (i, j), where the row and column indices run from $-2n$ to $2n$. A basis of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{B}(n,n)$ consists of the elements $h_i = e_{2i-1,2i-1} - e_{2i,2i}$ $(i \in [1,n])$ and $h_i = e_{2i,2i} - e_{2i+1,2i+1}$ $(i \in [\bar{n}, \bar{1}])$. The corresponding dual basis of $\mathfrak{h}^*$ will be denoted by ϵ_i $(i \in [\bar{n}, n]^*)$. The following elements are even root vectors with roots ϵ_{-i} and $-\epsilon_{-i}$ respectively $(i \in [1,n])$:

$$\begin{aligned} c_{-i}^{+} &\equiv f_{-i}^{+} = \sqrt{2}(e_{-2i,0} - e_{0,-2i+1}), \\ c_{-i}^{-} &\equiv f_{-i}^{-} = \sqrt{2}(e_{0,-2i} - e_{-2i+1,0}), \end{aligned} \tag{23}$$

and odd root vectors with roots ϵ_i and $-\epsilon_i$ respectively $(i \in [1,n])$ are given by:

$$\begin{aligned} c_{i}^{+} &\equiv b_{i}^{+} = \sqrt{2}(e_{0,2i} + e_{2i-1,0}), \\ c_{i}^{-} &\equiv b_{i}^{-} = \sqrt{2}(e_{0,2i-1} - e_{2i,0}). \end{aligned} \tag{24}$$

The remaining root vectors of $\mathfrak{B}(n,n)$ are given by elements of the form $[\![c_i^{\xi}, c_j^{\eta}]\!]$. The matrices (23)–(24) satisfy the triple relations (11), hence they realize the parastatistics operators.

In our development, it is also important to note that the $4n^2$ elements

$$[\![c_i^{+}, c_j^{-}]\!] \qquad (i, j \in [\bar{n}, n]^*) \tag{25}$$

are a basis of the subalgebra $\mathfrak{gl}(n|n)$. Observe also that

$$[c_i^{+}, c_i^{-}] = 2h_i \quad (i \in [\bar{n}, \bar{1}]), \qquad \{c_i^{+}, c_i^{-}\} = 2h_i \quad (i \in [1,n]). \tag{26}$$

Hence $\mathfrak{h} = \text{span}\{h_i,\ i \in [\bar{n}, n]^*\}$, the Cartan subalgebra of $\mathfrak{B}(n,n)$, is also the Cartan subalgebra of $\mathfrak{gl}(n|n)$.

4 The Fock Representations $V(p)$ of $\mathfrak{B}(n,n)$

The Fock representation $V(p)$ of $\mathfrak{B}(n,n)$ was already introduced in the first section. Note that the condition $[\![c_j^-, c_k^+]\!]|0\rangle = p\delta_{jk}\,|0\rangle$ implies that we are dealing with a lowest weight representation of $\mathfrak{B}(n,n)$, with lowest weight $(-\frac{p}{2},\ldots,-\frac{p}{2}|\frac{p}{2},\ldots,\frac{p}{2})$ in the basis $\{\epsilon_{-n},\ldots,\epsilon_{-1};\epsilon_1,\ldots,\epsilon_n\}$. These representations have been analyzed in [12]. The main result is the decomposition with respect to the subalgebra chain $\mathfrak{B}(n,n)\supset\mathfrak{gl}(n|n)$, because then the Gelfand-Zetlin basis of the $\mathfrak{gl}(n|n)$ representations can be used to label the vectors of $V(p)$. In the decomposition of $V(p)$ with respect to $\mathfrak{B}(n,n)\supset\mathfrak{gl}(n|n)$, all covariant representations of $\mathfrak{gl}(n|n)$ labelled by a partition $\lambda=(\lambda_1,\lambda_2,\ldots)$ appear with multiplicity 1, subject to $\lambda_1\le p$ and $\lambda_{n+1}\le n$. For each $\mathfrak{gl}(n|n)$ covariant representation labelled by λ, the highest weight can be determined [17], and is given by an array of $2n$ integers denoted by

$$[m]^{2n} = [m_{\bar{n},2n},\ldots,m_{\bar{2},2n},m_{\bar{1},2n};m_{1,2n},m_{2,2n},\ldots,m_{n,2n}] \tag{27}$$

satisfying certain conditions. Next, one can follow the chain (17), leading in each step to the highest weight of the subalgebra, and thus yielding a labelling with $2n$ rows for the corresponding vectors. This is the actual odd GZ-basis for the Fock representation $V(p)$ of $\mathfrak{B}(n,n)$.

Explicitly, it is described as follows. For any positive integer p, a basis of the Fock representation $V(p)$ of $\mathfrak{B}(n,n)$ is given by the set of vectors of the following form:

$$|p;m)^{2n} \equiv |m)^{2n} = \left| \begin{array}{l} [m]^{2n} \\ |m)^{2n-1} \end{array} \right) = \tag{28}$$

$$\left| \begin{array}{ccccc|cccccc}
m_{\bar{n},2n} & m_{\overline{n-1},2n} & \cdots & m_{\bar{2},2n} & m_{\bar{1},2n} & m_{1,2n} & m_{2,2n} & \cdots & m_{n-2,2n} & m_{n-1,2n} & m_{n,2n} \\
\uparrow & \uparrow & \cdots & \uparrow & \uparrow & & & & & & \\
m_{\bar{n},2n-1} & m_{\overline{n-1},2n-1} & \cdots & m_{\bar{2},2n-1} & m_{\bar{1},2n-1} & m_{1,2n-1} & m_{2,2n-1} & \cdots & m_{n-2,2n-1} & m_{n-1,2n-1} & \\
 & & & & & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \\
 & m_{\overline{n-1},2n-2} & \cdots & m_{\bar{2},2n-2} & m_{\bar{1},2n-2} & m_{1,2n-2} & m_{2,2n-2} & \cdots & m_{n-2,2n-2} & m_{n-1,2n-2} & \\
 & \uparrow & \cdots & \uparrow & \uparrow & & & & & & \\
 & m_{\overline{n-1},2n-3} & \cdots & m_{\bar{2},2n-3} & m_{\bar{1},2n-3} & m_{1,2n-3} & m_{2,2n-3} & \cdots & m_{n-2,2n-3} & & \\
 & & \ddots & \vdots & \vdots & \vdots & \vdots & .^{.^{.}} & & & \\
 & & & m_{\bar{2}4} & m_{\bar{1}4} & m_{14} & m_{24} & & & & \\
 & & & \uparrow & \uparrow & & & & & & \\
 & & & m_{\bar{2}3} & m_{\bar{1}3} & m_{13} & & & & & \\
 & & & & & \downarrow & & & & & \\
 & & & & m_{\bar{1}2} & m_{12} & & & & & \\
 & & & & \uparrow & & & & & & \\
 & & & & m_{\bar{1}1} & & & & & &
\end{array} \right)$$

where all $m_{ij}\in\mathbb{Z}_+$, satisfying $m_{\bar{n},2n}\le p$ and the GZ-conditions

$$
\begin{aligned}
&1.\ m_{j,2n} - m_{j+1,2n} \in \mathbb{Z}_+,\ \ j \in [\bar{n}, \bar{2}] \cup [1, n] \text{ and}\\
&\quad m_{-1,2n} \geq \#\{i : m_{i,2n} > 0,\ i \in [1, n]\};\\
&2.\ m_{-i,2s} - m_{-i,2s-1} \equiv \theta_{-i,2s-1} \in \{0, 1\},\quad 1 \leq i \leq s \leq n;\\
&3.\ m_{i,2s} - m_{i,2s+1} \equiv \theta_{i,2s} \in \{0, 1\},\quad 1 \leq i \leq s \leq n-1;\\
&4.\ m_{-1,2s} \geq \#\{i : m_{i,2s} > 0,\ i \in [1, s]\},\ s \in [1, n];\\
&5.\ m_{-1,2s-1} \geq \#\{i : m_{i,2s-1} > 0,\ i \in [1, s-1]\},\ s \in [2, n];\\
&6.\ m_{i,2s} - m_{i,2s-1} \in \mathbb{Z}_+ \text{ and } m_{i,2s-1} - m_{i+1,2s} \in \mathbb{Z}_+,\\
&\qquad 1 \leq i \leq s-1 \leq n-1;\\
&7.\ m_{-i-1,2s+1} - m_{-i,2s} \in \mathbb{Z}_+ \text{ and } m_{-i,2s} - m_{-i,2s+1} \in \mathbb{Z}_+,\\
&\qquad 1 \leq i \leq s \leq n-1.
\end{aligned}
\tag{29}
$$

Conditions 2 and 3 are referred to as "θ-conditions". Conditions 6 and 7 are often referred to as "betweenness conditions." Conditions 1, 4 and 5 assure that each row of (28) corresponds to the highest weight of a covariant representation of $\mathfrak{gl}(t|t)$ or $\mathfrak{gl}(t|t-1)$ in the chain (17). Note that the arrows in this pattern have no real function, and can be omitted. We find it useful to include them, just in order to visualize the θ-conditions. When there is an arrow $a \to b$ between labels a and b, it means that either $b = a$ or else $b = a + 1$ (a θ-condition). We will also refer to "rows" and "columns" of the GZ-pattern. Rows are counted from the bottom: row 1 is the bottom row in (28), and row $2n$ is the top row in (28). In an obvious way, columns 1, 2, 3, ... refer to the columns to the right of the dashed line in (28), and columns $-1, -2, -3, \ldots$ (or $\bar{1}, \bar{2}, \bar{3}, \ldots$) to the columns to the left of this dashed line. For two consecutive rows in the GZ-pattern (28), about half of the labels involve θ-conditions, and the other half involves betweenness conditions.

It should already be clear from this construction that the GZ-patterns of $\mathfrak{gl}(n|n)$ consist of those of $\mathfrak{gl}(n-1|n-1)$ to which two rows are added at the top. Hence it will be possible to gradually increase n, and we are in a setting for which the limit $n \to \infty$ can be examined.

One of the main computational results of [13] is the determination of the action of the parastatistics operators $c_i^{\pm}$ on the GZ basis vectors $|m)^{2n}$. For this, it is necessary to note that the $2n$ elements c_i^+ themselves form a standard $\mathfrak{gl}(n|n)$ tensor. Thus every element of $(c_n^+, c_{-n}^+, \cdots, c_2^+, c_{-2}^+, c_1^+, c_{-1}^+)$ corresponds, in this order, to a GZ-pattern of type (28) consisting of k top rows of the form $10\cdots0$ and $2n-k$ bottom rows of the form $0\cdots0$ for $k = 1, 2, \ldots, 2n$. It will be convenient to introduce a notation for the order in which these $2n$ elements appear:

$$
\rho(i) = \begin{cases} 2i \text{ for } i \in [1, n] \\ -2i - 1 \text{ for } i \in [\bar{n}, \bar{1}] \end{cases}. \tag{30}
$$

Then the pattern corresponding to c_i^+ has rows of the form $10\cdots0$ for each row index $j \in [\rho(i), 2n]$ and zero rows for each row index $j \in [1, \rho(i) - 1]$.

Following standard methods [9, 18], and knowing the tensor product rule in $\mathfrak{gl}(n|n)$ for covariant representations, the matrix elements of c_i^+ in $V(p)$ can be written as follows:

$$
{}^{2n}(m'|c_i^+|m)^{2n} = \left(\begin{matrix} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{matrix} \middle| c_i^+ \middle| \begin{matrix} [m]^{2n} \\ |m)^{2n-1} \end{matrix} \right)
$$

$$
= \left(\begin{matrix} 10\cdots 00 \\ 10\cdots 0 \\ \cdots \\ 0 \end{matrix} ; \begin{matrix} [m]^{2n} \\ |m)^{2n-1} \end{matrix} \middle| \begin{matrix} [m]^{2n}_{+(k)} \\ |m')^{2n-1} \end{matrix} \right) \times ([m]^{2n}_{+(k)}||c^+||[m]^{2n}). \tag{31}
$$

Herein, the GZ-pattern with 0's and 1's is the one corresponding to c_i^+, as described earlier, and $[m]^{2n}_{\pm(k)}$ is the pattern obtained from $[m]^{2n}$ by the replacement of $m_{k,2n}$ by $m_{k,2n} \pm 1$. The first factor in the right hand side of (31) is a $\mathfrak{gl}(n|n)$ Clebsch-Gordan coefficient (CGC), where all patterns are of the form (28). These CGC's have been determined in the Appendix of [13], and will not be repeated here. The second factor in (31) is a *reduced matrix element* for the standard $\mathfrak{gl}(n|n)$ tensor. The possible values of the patterns $|m')^{2n}$ are determined by the $\mathfrak{gl}(n|n)$ tensor product rule and the first line of $|m')^{2n}$ is of the form $[m]^{2n}_{+(k)}$. The reduced matrix elements themselves depend only upon the $\mathfrak{gl}(n|n)$ highest weights $[m]^{2n}$ and $[m]^{2n}_{+k}$ (and not on the type of GZ basis that is being used.) These reduced matrix elements have actually been determined in [12, Proposition 4].

Note furthermore that by the Hermiticity requirement one has

$$
{}^{2n}(m'|c_i^-|m)^{2n} = {}^{2n}(m|c_i^+|m')^{2n}. \tag{32}
$$

So in this way, one obtains a complete action of all parastatistics operators:

$$
c_i^+|m)^{2n} = \sum_{m'} C^+\left[i, |m)^{2n}, |m')^{2n}\right] \; |m')^{2n}, \tag{33}
$$

$$
c_i^-|m)^{2n} = \sum_{m'} C^-\left[i, |m)^{2n}, |m')^{2n}\right] \; |m')^{2n}, \tag{34}
$$

where $C^+\left[i, |m)^{2n}, |m')^{2n}\right]$ is just a shorthand notation for the element ${}^{2n}(m'|c_i^+|m)^{2n}$ computed in (31), and similarly for $C^-\left[i, |m)^{2n}, |m')^{2n}\right]$.

Examining the action of the creation operators c_i^+ in detail, one deduces the following property [13]: the action of c_i^+ on $|m)^{2n}$ yields vectors $|m')^{2n}$ such that rows $1, 2, \ldots, \rho(i) - 1$ of $|m')^{2n}$ are the same as those of $|m)^{2n}$. And in rows $\rho(i), \ldots, 2n$ there is a change by one unit for just one particular column index s: $[m']^j = [m]^j + [0, \ldots, 0, 1, 0, \ldots, 0]$ for $j \in [\rho(i), 2n]$. The increase can be in any possible column, as long as the remaining pattern is still valid, i.e. as long as (29) is satisfied.

An important observation is a certain stability property. For this, one introduces the following definition: the pattern, or equivalently the associated basis vector, $|m)^{2n}$

is *row-stable* with respect to row s if there exists a partition ν such that all rows $s, s+1, \ldots, 2n$ are of the form

$$[\nu_1, \nu_2, \ldots, 0; 0, 0, \ldots].$$

In that case, s is called a *stability index* of $|m)^{2n}$.

The following properties were proven in [13]:

- The action of a consecutive number of c_i^+'s on the vacuum vector produces row-stable patterns if n is sufficiently large. More precisely, if $k < n$, then all basis vectors appearing in

$$c_{i_k}^+ \cdots c_{i_2}^+ c_{i_1}^+ |0\rangle \qquad (\text{each } i_r \in [\bar{n}, n]^*) \tag{35}$$

  are row-stable with respect to some row index s.
- Row-stable patterns remain row-stable under the action of c_i^+'s (but the stability index might increase). Specifically, let $|m)^{2n}$ be row-stable with respect to row s, where $s < 2n-1$. Then the vectors $|m')^{2n}$ appearing in $c_i^+|m)^{2n}$ are row-stable with respect to row $\max\{s+2, \rho(i)+1\}$.
- Row-stable patterns remain row-stable under the action of c_i^-'s for the same stability index.

Also the matrix elements (33)–(34) satisfy a stability property. To specify this, one defines a map from GZ-patterns with $2n$ rows to GZ-patterns with $2n+2$ rows. For this, suppose that the top row of $|m)^{2n}$ has the zero partition as second part, i.e. it is of the form

$$[m]^{2n} = [\nu_1, \nu_2, \ldots; 0, \ldots, 0]$$

with ν a partition. Define the map $\phi_{2n,+2}$ from the set of GZ-patterns $|m)^{2n}$ with zero second part to the set of GZ-patterns $|m)^{2n+2}$ with stability index $2n$ by:

$$|m)^{2n+2} = \phi_{2n,+2}\left(|m)^{2n}\right), \text{ where} \tag{36}$$
$$[m]^{2n+1} = [\nu_1, \nu_2, \ldots, 0, 0; 0, \ldots, 0], \quad [m]^{2n+2} = [\nu_1, \nu_2, \ldots, 0, 0; 0, \ldots, 0, 0].$$

In other words, the top row of $|m)^{2n}$ is just repeated twice, with the extra addition of zeros in order to have sufficient entries for the pattern $|m)^{2n+2}$. Clearly, the action of $\phi_{2n,+2}$ can also be extended by linearity, on a linear combination of vectors $|m)^{2n}$ with zero second part.

The final important stability property can now be formulated: let $|m)^{2n}$ be row-stable with respect to row $2n$, and $|m)^{2n+2} = \phi_{2n,+2}\left(|m)^{2n}\right)$ Then for all i with $\rho(i) \leq 2n$ (or equivalently, $i \in [-n, n]^*$):

$$c_i^+|m)^{2n+2} = \phi_{2n,+2}\left(c_i^+|m)^{2n}\right).$$

5 The Fock Representations $V(p)$ of $\mathfrak{B}(\infty, \infty)$

Due to the stability properties just described, we can extend both the parastatistics algebra $\mathfrak{B}(n, n)$ and its Fock representations $V(p)$ to the infinite rank case $\mathfrak{B}(\infty, \infty)$.

The infinite rank Lie superalgebra $\mathfrak{B}(\infty, \infty)$ consists of infinite matrices X of the form (22) with $n \to \infty$ but with a finite number of non-zero elements, see [13] for a more precise definition. The indices of the matrices X now belong to $\mathbb{Z}$ instead of $[-n, n]$. The matrices e_{ij} consist of zeros everywhere except a 1 on position (i, j), where the row and column indices belong to $\mathbb{Z}$. A basis of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{B}(\infty, \infty)$ consists of the elements $h_i = e_{2i-1,2i-1} - e_{2i,2i}$ $(i \in \mathbb{Z}_+^*)$ and $h_i = e_{2i,2i} - e_{2i+1,2i+1}$ $(i \in \mathbb{Z}_-^*)$. The corresponding dual basis of $\mathfrak{h}^*$ is denoted by ϵ_i $(i \in \mathbb{Z}^*)$. As in the finite rank case, we can identify the following even root vectors with roots ϵ_{-i} and $-\epsilon_{-i}$ respectively $(i \in \mathbb{Z}_+^*)$:

$$\begin{aligned} c_{-i}^+ &\equiv f_{-i}^+ = \sqrt{2}(e_{-2i,0} - e_{0,-2i+1}), \\ c_{-i}^- &\equiv f_{-i}^- = \sqrt{2}(e_{0,-2i} - e_{-2i+1,0}), \end{aligned} \tag{37}$$

and odd root vectors with roots ϵ_i and $-\epsilon_i$ respectively $(i \in \mathbb{Z}_+^*)$:

$$\begin{aligned} c_i^+ &\equiv b_i^+ = \sqrt{2}(e_{0,2i} + e_{2i-1,0}), \\ c_i^- &\equiv b_i^- = \sqrt{2}(e_{0,2i-1} - e_{2i,0}). \end{aligned} \tag{38}$$

The operators c_i^+ can be chosen as positive root vectors, and the c_i^- as negative root vectors.

The operators introduced here satisfy the triple relations of parastatistics. But now we are dealing with an infinite number of parafermions and an infinite number of parabosons, satisfying the mutual relative parafermion relations. In other words, the triple relations (11) are satisfied, but now with $j, k, l \in \mathbb{Z}^*$. We also have: as a Lie superalgebra defined by generators and relations, $\mathfrak{B}(\infty, \infty)$ is generated by the elements $c_i^\pm$ $(i \in \mathbb{Z}^*)$ subject to the relations (11).

The parastatistics Fock space of order p, with p a positive integer, can be defined as before, and will correspond to a lowest weight representation $V(p)$ of the algebra $\mathfrak{B}(\infty, \infty)$. $V(p)$ is the Hilbert space generated by a vacuum vector $|0\rangle$ and the parastatistics creation and annihilation operators, i.e. subject to $\langle 0|0\rangle = 1, c_j^-|0\rangle = 0$, $(c_j^\pm)^\dagger = c_j^\mp$,

$$[\![c_j^-, c_k^+]\!]|0\rangle = p\delta_{jk}|0\rangle \quad (j, k \in \mathbb{Z}^*) \tag{39}$$

and which is irreducible under the action of the algebra $\mathfrak{B}(\infty, \infty)$. Clearly $|0\rangle$ is a lowest weight vector of $V(p)$ with weight $(\ldots, -\frac{p}{2}, -\frac{p}{2}|\frac{p}{2}, \frac{p}{2}, \ldots)$ in the basis $\{\ldots, \epsilon_{-2}, \epsilon_{-1}; \epsilon_1, \epsilon_2, \ldots\}$.

The basis vectors of $V(p)$ will consist of infinite GZ-patterns. Not all possible infinite GZ-patterns will appear, but only row-stable ones. Such row-stable infinite GZ-patterns consist of an infinite number of rows, of the type introduced in (28), but such that from a certain row index s all rows $s, s+1, s+2, \ldots$ are of the same form. As an example,

$$
|m)^{\infty} = \left| \begin{array}{cccc|cccc}
\ddots & \ddots & & & & & \iddots & \iddots \\
4 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\
4 & 3 & 1 & 0 & 0 & 0 & 0 & \\
 & 4 & 3 & 1 & 0 & 0 & 0 & \\
 & 4 & 3 & 1 & 0 & 0 & & \\
 & & 3 & 3 & 1 & 0 & & \\
 & & 3 & 2 & 1 & & & \\
 & & & 2 & 2 & & & \\
 & & & 1 & & & &
\end{array} \right) \tag{40}
$$

where the row $(4, 3, 1, 0, \ldots)$ is repeated up to infinity.

The basis of $V(p)$ is described as follows.

Proposition 1 *A basis of $V(p)$ is given by all infinite row-stable GZ-patterns $|m)^{\infty}$ of the form (28) with $n \to \infty$ where for each $|m)^{\infty}$ there should exist a row index s (depending on $|m)^{\infty}$) such that row s is of the form*

$$[m]^s = [\nu_1, \nu_2, \ldots, 0; 0, 0, \ldots]$$

with ν a partition, all rows above s are of the same form (up to extra zeros), and $\nu_1 \leq p$. Furthermore all $m_{ij} \in \mathbb{Z}_+$ and the usual GZ-conditions should be satisfied:

1. $m_{-i,2r} - m_{-i,2r-1} \equiv \theta_{-i,2r-1} \in \{0, 1\}, \quad 1 \leq i \leq r;$
2. $m_{i,2r} - m_{i,2r+1} \equiv \theta_{i,2r} \in \{0, 1\}, \quad 1 \leq i \leq r;$
3. $m_{-1,2r} \geq \#\{i : m_{i,2r} > 0,\ i \in [1, r]\},\ r \in \mathbb{Z}_+^*;$
4. $m_{-1,2r+1} \geq \#\{i : m_{i,2r+1} > 0,\ i \in [1, r]\},\ r \in \mathbb{Z}_+^*;$
5. $m_{i,2r+2} - m_{i,2r+1} \in \mathbb{Z}_+$ *and* $m_{i,2r+1} - m_{i+1,2r+2} \in \mathbb{Z}_+, \quad 1 \leq i \leq r;$
6. $m_{-i-1,2r+1} - m_{-i,2r} \in \mathbb{Z}_+$ *and* $m_{-i,2r} - m_{-i,2r+1} \in \mathbb{Z}_+, \quad 1 \leq i \leq r.$

The process of adding an infinite number of identical rows (up to additional zeros) at the top of a finite GZ-pattern can now be formalized by means of a map, just as we did by adding two identical rows in the previous section. Let $|m)^{2n}$ be a finite GZ-pattern of type (28) with $2n$ rows, such that row $2n$ is of the form $[\nu_1, \nu_2, \ldots; 0, 0, \ldots, 0]$. Then $\phi_{2n,\infty}\left(|m)^{2n}\right)$ is the infinite GZ-pattern consisting of the rows of $|m)^{2n}$ to which an infinite number of rows $[\nu_1, \nu_2, \ldots; 0, 0, \ldots, 0]$ are added at the top (all identical, up to additional zeros). Conversely, if an infinite GZ-pattern $|m)^{\infty}$ is given, which is stable with respect to row $2s$, then one can restrict the infinite pattern to a finite GZ-pattern, and

$$|m)^{2s} = \phi_{2s,\infty}^{-1}\left(|m)^{\infty}\right).$$

Both maps can be extended by linearity. Then one can define the action of $c_i^{\pm}$ on vectors $|m)^{\infty}$:

Definition 1 Given a vector $|m)^{\infty}$ of $V(p)$ with stability index $2s$, and a generator $c_i^{\pm}$. Let $2n$ be such that $2n > \max\{2s, \rho(i)\}$. Then

$$c_i^{\pm}|m)^{\infty} = \phi_{2n,\infty}\left(c_i^{\pm}|m)^{2n}\right), \text{ where } |m)^{2n} = \phi_{2n,\infty}^{-1}\left(|m)^{\infty}\right). \tag{41}$$

The main theorem, proved in [13] is then

Theorem 1 *The vector space $V(p)$, with basis vectors all infinite row-stable GZ-patterns for which $\nu_1 \le p$, on which the action of the $\mathfrak{B}(\infty, \infty)$ generators $c_i^{\pm}$ ($i \in \mathbb{Z}^*$) is defined by (41), is an irreducible unitary Fock representation of $\mathfrak{B}(\infty, \infty)$.*

To conclude, we have managed to give a description of parastatistics Fock spaces with an infinite number of parafermions and parabosons. Our developments in previous years had already led to such a description for m parafermions and n parabosons by means of representations of $\mathfrak{osp}(2m+1|2n)$. The GZ basis for these representations, determined in [12], is however not appropriate for the limit to an infinite number of parastatistics operators. We therefore constructed a new GZ basis for $\mathfrak{B}(n, n) = \mathfrak{osp}(2n+1|2n)$ representations. In this new basis, there is a natural limit for $n \to \infty$, and the corresponding infinite row-stable GZ-patterns label the basis vectors of the corresponding Fock space $V(p)$ of $\mathfrak{B}(\infty, \infty)$.

Acknowledgements N. I. Stoilova was supported by the Bulgarian National Science Fund, grant DN 18/1, and J. Van der Jeugt was partially supported by KP-06-N28/6 and by the EOS Research Project 30889451.

Appendix

Although a low-rank example is not very instructive for the case $n \to \infty$, it is still useful to the reader to visualize the basic structure of the basis vectors (28) and the action (31) with matrix elements (33). This is why we include the basis of $V(p)$ for $n = 1$, i.e. for $\mathfrak{B}(n, n)$. Let

$$|m) \equiv \left| \begin{array}{c:c} m_{\bar{1}2} & m_{12} \\ m_{\bar{1}1} & \end{array} \right)$$

where

$$\begin{array}{l} 1.\ m_{ij} \in \mathbb{Z}_+,\ \ m_{\bar{1}2} \le p; \\ 2.\ m_{\bar{1}2} \in \{0, 1, 2, \cdots\} \text{ if } m_{12} = 0;\ \ m_{\bar{1}2} \in \{1, 2, \cdots\} \text{ if } m_{12} \ne 0;. \\ 3.\ m_{\bar{1}1} \in \{m_{\bar{1}2}, m_{\bar{1}2} - 1\}. \end{array} \tag{42}$$

The action of the Cartan algebra elements is:

$$
\begin{aligned}
h_{\bar{1}}|m) &= \left(-\frac{p}{2} + m_{\bar{1}1}\right)|m),\\
h_{1}|m) &= \left(\frac{p}{2} + m_{\bar{1}2} + m_{12} - m_{\bar{1}1}\right)|m).
\end{aligned}
\tag{43}
$$

The action of the parastatistics creation operators reads

$$
\begin{aligned}
c_{\bar{1}}^{+}\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12} \\ m_{\bar{1}2} & \end{array}\right) &= G_{\bar{1}}(m_{\bar{1}2}, m_{12})\left|\begin{array}{l:l} m_{\bar{1}2}+1 & m_{12} \\ m_{\bar{1}2}+1 & \end{array}\right),\\
c_{\bar{1}}^{+}\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12} \\ m_{\bar{1}2}-1 & \end{array}\right) &= \sqrt{\frac{m_{\bar{1}2}+m_{12}}{m_{\bar{1}2}+m_{12}+1}}\,G_{\bar{1}}(m_{\bar{1}2}, m_{12})\left|\begin{array}{l:l} m_{\bar{1}2}+1 & m_{12} \\ m_{\bar{1}2} & \end{array}\right)\\
&\quad -\sqrt{\frac{1}{m_{\bar{1}2}+m_{12}+1}}\,G_{1}(m_{\bar{1}2}, m_{12})\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12}+1 \\ m_{\bar{1}2} & \end{array}\right),\\
c_{1}^{+}\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12} \\ m_{\bar{1}2} & \end{array}\right) &= \sqrt{\frac{1}{m_{\bar{1}2}+m_{12}+1}}\,G_{\bar{1}}(m_{\bar{1}2}, m_{12})\left|\begin{array}{l:l} m_{\bar{1}2}+1 & m_{12} \\ m_{\bar{1}2} & \end{array}\right)\\
&\quad +\sqrt{\frac{m_{\bar{1}2}+m_{12}}{m_{\bar{1}2}+m_{12}+1}}\,G_{1}(m_{\bar{1}2}, m_{12})\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12}+1 \\ m_{\bar{1}2} & \end{array}\right),\\
c_{1}^{+}\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12} \\ m_{\bar{1}2}-1 & \end{array}\right) &= -G_{1}(m_{\bar{1}2}, m_{12})\left|\begin{array}{l:l} m_{\bar{1}2} & m_{12}+1 \\ m_{\bar{1}2}-1 & \end{array}\right).
\end{aligned}
\tag{44}
$$

Herein, $G_{\bar{1}}$ and G_1 are shorthand notations for the reduced matrix elements in (31): $G_{\bar{1}}(m_{\bar{1}2}, m_{12}) = (m_{\bar{1}2}+1, m_{12}||c^{+}||m_{\bar{1}2}, m_{12})$ and $G_{1}(m_{\bar{1}2}, m_{12}) = (m_{\bar{1}2}, m_{12}+1\ ||c^{+}||m_{\bar{1}2}, m_{12})$, explicitly given by

$$
\begin{aligned}
G_{\bar{1}}(m_{\bar{1}2}, m_{12}) &= \sqrt{\frac{m_{\bar{1}2}(m_{\bar{1}2}+m_{12}+1)(p-m_{\bar{1}2})}{m_{\bar{1}2}+m_{12}}}, \quad \text{if } \ m_{12} \text{ is even},\\
G_{\bar{1}}(m_{\bar{1}2}, m_{12}) &= \sqrt{m_{\bar{1}2}(p-m_{\bar{1}2})}, \quad \text{if } \ m_{12} \text{ is odd},\\
G_{1}(m_{\bar{1}2}, m_{12}) &= \sqrt{m_{\bar{1}2}+m_{12}+1}, \quad \text{if } \ m_{12} \text{ is even},\\
G_{1}(m_{\bar{1}2}, m_{12}) &= \sqrt{\frac{(m_{12}+1)(p+m_{12}+1)}{m_{\bar{1}2}+m_{12}}}, \quad \text{if } \ m_{12} \text{ is odd}.
\end{aligned}
\tag{45}
$$

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Recent Progress on Yang-Baxter Deformation and Generalized Supergravity

Kentaroh Yoshida

Abstract In recent years, great progress has been made on a systematic method to perform an integrable deformation of a two-dimensional relativistic non-linear sigma model. The deformations are labeled by classical r-matrices satisfying the classical Yang-Baxter equation, and this method is called the Yang-Baxter deformation. It was generalized to type IIB superstring theory defined on the $AdS_5 \times S^5$ background and gave rise to a lot of integrable backgrounds including well-known backgrounds such as the Lunin-Maldacena background, a gravity dual for a non-commutative gauge theory, and a Schrödinger spacetime. In addition, the study of Yang-Baxter deformation led to the discovery of a generalized type IIB supergravity. In this proceeding, I will give a short summary of the recent progress on the Yang-Baxter deformation and the generalized supergravity.

1 Introduction

A conjectured duality between a string theory on a $(d+1)$-dimensional anti de Sitter (AdS) space and a conformal field theory (CFT) in d dimensions, which is called the AdS/CFT correspondence (or simply AdS/CFT) [1], is one of the fascinating topics in String Theory. A typical example of this correspondence is a duality between type IIB string theory defined on $AdS_5 \times S^5$ and the four-dimensional $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills (SYM) theory in the large N limit.

One of the great achievements is the discovery of the integrable structure that exists behind AdS/CFT (For a comprehensive review, see [2]). As a tip of the iceberg of this integrable structure, type IIB superstring theory on $AdS_5 \times S^5$ [3], which is often abbreviated as the $AdS_5 \times S^5$ superstring, is classically integrable [4]. In the following, we will be concerned with this classical integrability.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_13

An intriguing direction is to study an integrable deformation of the $AdS_5 \times S^5$ superstring. There are some possible ways in the context of integrable models, hence by employing one of them, one can perform an integrable deformation of the system as a two-dimensional (2D) non-linear sigma model. Accordingly, the target-space geometry is also deformed. The resulting background can be seen as a deformed $AdS_5 \times S^5$ geometry. Then, one may ask the following question:

Does the deformation give a solution to type IIB supergravity or not?
If not, is the deformed background a solution to some new theory?

The main issue of this proceeding is to answer these questions for a specific class of integrable deformation called the Yang-Baxter deformation [5, 6].

2 Yang-Baxter Deformation

The Yang-Baxter (YB) deformation is a systematic method to perform an integrable deformation of 2D non-linear sigma model. It was originally invented by Klimcik for 2D principal chiral model [5, 6]. Then it was generalized to the symmetric coset case [7, 8] and further to the $AdS_5 \times S^5$ superstring [9–11].

We will first introduce the YB deformation of 2D principal chiral model. Then we present YB deformation of the $AdS_5 \times S^5$ superstring and explain the scheme of the supercoset construction. Finally, some examples of classical r-matrices and the associated deformed backgrounds are presented.

2.1 YB Deformation of 2D Principal Chiral Model

We consider a 2D non-linear sigma model whose target space is a Lie group G, which is called 2D G-principal chiral model (PCM).

The classical action is given by

$$S = \int d^2x\, \eta^{\mu\nu} \mathrm{tr}(J_\mu J_\nu)\,. \tag{1}$$

Here $\eta_{\mu\nu} = \mathrm{diag}(-1, +1)$ is 2D Minkowski metric and J_μ is the left-invariant one-form defined as

$$J_\mu \equiv g^{-1}\partial_\mu g\,, \tag{2}$$

where g is a group element of G. It is well-known that 2D PCM is classically integrable in the sense of kinematical integrability.

Next, let us consider YB deformation of the action (1) by following [5, 6, 8]. The deformed action is given by

$$S^{(\eta)} = \int d^2x\, \eta^{\mu\nu} \mathrm{tr}\left(J_\mu \frac{1}{1-\eta R} J_\nu \right) . \tag{3}$$

The deformation is characterized by the insertion of the factor $1/(1-\eta R)$. Here η is a constant parameter which measures the deformation. When $\eta = 0$, the original action of 2D PCM (1) is reproduced. Then R is a linear map from $\mathfrak{g}$ to $\mathfrak{g}$ (where $\mathfrak{g}$ is the Lie algebra associated with G) and satisfies the (modified) classical Yang-Baxter equation ((m)CYBE),

$$[R(X), R(Y)] - R\left([R(X), Y] + [X, R(Y)]\right) = -c^2[X, Y] . \tag{4}$$

Here c is a constant parameter. The right-hand side of (4) is a modification to the homogeneous CYBE (i.e., the case with $c = 0$).

In summary, an integrable deformation is specified by a linear R-operator satisfying the (m)CYBE. Hence this deformation is called the YB deformation.

R-operator and Classical r-matrix

It is useful to see the relation between the linear R-operator and a classical r-matrix in the tensorial notation, $r \in \mathfrak{g} \otimes \mathfrak{g}$. Given a non-degenerate inner product $\langle\ ,\ \rangle$ for the Lie algebra generators, one can see the one-to-one correspondence between a linear R-operator and a skew-symmetric r-matrix by taking the inner product on the second site of the tensor product like

$$R(X) \equiv \langle r_{12}, 1 \otimes X\rangle = \sum_i \Big(a_i \langle b_i, X\rangle - b_i \langle a_i, X\rangle \Big) \quad \text{for} \quad X \in \mathfrak{g} , \tag{5}$$

where the skew-symmetric r-matrix is expressed as

$$r_{12} = \sum_i (a_i \otimes b_i - b_i \otimes a_i) \quad \text{with} \quad a_i,\ b_i \in \mathfrak{g} . \tag{6}$$

Thus YB deformation may also be labeled by a skew-symmetric classical r-matrix in the tensorial notation, in which the physical meaning of the Lie algebra generators is clear and often useful.

Example: G=$SU(2)$ case

Let us see an example of linear R-operator and the associated geometry.

The simplest case is $G = SU(2)$. The Lie algebra $\mathfrak{su}(2)$ is given by

$$[T^3, T^{\pm}] = \pm 2T^{\pm}, \qquad [T^+, T^-] = T^3 , \tag{7}$$

where the generators are represented by the following matrices:

$$T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

In the $SU(2)$ case, the unique (non-trivial) solution to mCYBE is given by a classical r-matrix of Drinfeld-Jimbo (DJ) type [12, 13],

$$r_{\mathrm{DJ}} = -i \left[T^+ \otimes T^- - T^- \otimes T^+ \right]. \tag{9}$$

Then, the relation (5) leads to the associated linear R-operator,

$$R(T^+) = -i\, T^+, \quad R(T^-) = +i\, T^-, \quad R(T^3) = 0. \tag{10}$$

Finally, by using the R-operator (10), the resulting action is given by

$$S = \frac{1}{1+\eta^2} \int \mathrm{d}^2x\, \gamma^{\alpha\beta} \left[\mathrm{Tr}(J_\alpha J_\beta) + \frac{\eta^2}{2} \mathrm{Tr}(T^3 J_\alpha) \mathrm{Tr}(T^3 J_\beta) \right]. \tag{11}$$

Just a single piece has been added and this describes a deformation from the round S^3 to a squashed S^3.

Derivation of (11)

It would be instructive to see the derivation of (11). In addition to J, let us introduce a new quantity, projected current A defined as

$$J \equiv g^{-1}\mathrm{d}g, \qquad A \equiv \frac{1}{1-\eta R} J. \tag{12}$$

Both J and A take values in the Lie algebra $\mathfrak{su}(2)$ and can be expanded as

$$J = J^+T^- + J^-T^+ + J^3T^3, \tag{13}$$
$$A = A^+T^- + A^-T^+ + A^3T^3. \tag{14}$$

By multiplying $1-\eta R$ to A and using (14) and (10), J can be expressed as

$$\begin{aligned} J &= (1-\eta R)A \\ &= (1-i\eta)A^+T^- + (1+i\eta)A^-T^+ + A^3T^3. \end{aligned} \tag{15}$$

Then, by comparing (15) with (13), the following relations are obtained,

$$A^+ = \frac{J^+}{1-i\eta}, \quad A^- = \frac{J^-}{1+i\eta}, \quad A^3 = J^3. \tag{16}$$

As a result, A is expressed in terms of J and η.

Finally, by putting (16) into (3), the action is evaluated as

$$
\begin{aligned}
S &= \frac{2}{1+\eta^2}\int d^2x\,\gamma^{\alpha\beta}\left[J^+_\alpha J^-_\beta + (1+\eta^2)J^3_\alpha J^3_\beta\right] \\
&= \frac{1}{1+\eta^2}\int d^2x\,\gamma^{\alpha\beta}\left[\mathrm{Tr}(J_\alpha J_\beta) + \frac{\eta^2}{2}\mathrm{Tr}(T^3 J_\alpha)\mathrm{Tr}(T^3 J_\beta)\right].
\end{aligned}
$$

This is nothing but (11).

It should be remarked that this is the simplest example but the essence in computation is common even for higher dimensional and supersymmetric cases, though the computation becomes messy and intricate technically.

Coset Construction of Metric

The deformed action (11) is written in terms of group element. Hence the target-space metric is not manifest. To see the metric explicitly, let us introduce a parametrization of group element g like

$$
g = \exp\left[-i\left(\frac{\phi(x)}{2}\right)T_1\right]\cdot\exp\left[i\left(\frac{\theta(x)}{2}\right)T_2\right]\cdot\exp\left[i\left(\frac{\psi(x)}{2}\right)T_3\right]. \qquad (17)
$$

Here $\phi(x)$, $\theta(x)$ and $\psi(x)$ are the angle variables for S^3.

Now J is expressed in terms of the angle variables. By expanding J like

$$
J = g^{-1}dg = J^1T_1 + J^2T_2 + J^3T_3\,,
$$

the target-space metric is given by

$$
\begin{aligned}
ds^2 &= -[(J^1)^2 + (J^2)^2 + (1+\eta^2)(J^3)^2] \\
&= \frac{1}{4}\left[\,d\theta^2 + \cos^2\theta\,d\phi^2 + (1+\eta^2)(d\psi + \sin\theta\,d\phi)^2\,\right]. \qquad (18)
\end{aligned}
$$

This is the metric of squashed S^3. When $\eta = 0$, it is reduced to the metric of the round S^3. This metric describes S^3 as a $U(1)$-fibration over S^2. In this metric, the left $SU(2)$ symmetry is manifest as expected from the construction based on the left-invariant one-form J.

2.2 YB Deformation of the $AdS_5 \times S^5$ Superstring

Let us introduce YB deformation of the $AdS_5 \times S^5$ superstring [9–11]. To be pedagogical, we start from the explanation about the classical integrability of the $AdS_5 \times S^5$ superstring [4]. Then we introduce YB deformed action and outline the supercoset construction. Finally, some examples are presented.

Classical Integrability of the $AdS_5 \times S^5$ Superstring

The classical action of the $AdS_5 \times S^5$ superstring is constructed based on the following supercoset [3]:

$$\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}\,. \tag{19}$$

The bosonic part of this coset describes the $AdS_5 \times S^5$ geometry,

$$\frac{SO(2,4)}{SO(5)}\times\frac{SO(6)}{SO(5)} = \mathrm{AdS}_5\times \mathrm{S}^5\,. \tag{20}$$

The fermionic part of (19) corresponds to the spacetime fermions, whose dynamics is described in the Green-Schwarz (GS) formulation of superstring [3].

The bosonic part is nothing but a symmetric coset, hence the classical integrability of the system is ensured automatically. This symmetric coset structure is equivalent to the $\mathbb{Z}_2$-grading property. Remarkably, the supercoset (19) exhibits the $\mathbb{Z}_4$-grading as a supersymmetric generalization of the symmetric coset. This grading property ensures the classical integrability for the supersymmetric case, as elucidated by Bena, Polchinski and Roiban [4].

YB Deformed Action and Supercoset Construction

The YB deformation can also be applied to the $AdS_5 \times S^5$ superstring [9–11]. The deformed action is given by

$$S = -\frac{1}{2}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\ P_-^{ab}\mathrm{Str}\left[J_a d\circ \frac{1}{1-\eta\,[R]_g\circ d}(J_b)\right]. \tag{21}$$

When $\eta = 0$, the original Metsaev-Tseytlin action [3] is reproduced. For the detail of this action, see [9–11].

Since the deformed action (21) is written in terms of the group element, the target-space geometry is not clear. In addition, since the spacetime fermions are included, the dilaton and Ramond–Ramond (R-R) field strengths also appear as well as the metric and the Neveu-Schwarz–Neveu-Schwarz (NS-NS) two-form. In order to see the deformed background explicitly, one needs to perform supercoset construction by taking a parametrization of the group element [14–16]. Then, by expanding the action in terms of the spacetime fermion θ, the second-order action can be compared with the following canonical form of the GS superstring on an arbitrary background [17],

$$\begin{aligned} S = &-\frac{\sqrt{\lambda_c}}{4}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma\ \left[\gamma^{ab}G_{MN}\partial_a X^M\partial_b X^N - \epsilon^{ab}B_{MN}\partial_a X^M\partial_b X^N\right] \\ &-\frac{\sqrt{\lambda_c}}{2}\, i\bar{\Theta}_I(\gamma^{ab}\delta^{IJ} - \epsilon^{ab}\sigma_3^{IJ})\, e_a^m \Gamma_m D_b^{JK}\Theta_K + \mathcal{O}(\theta^4)\,. \end{aligned} \tag{22}$$

This expression contains the metric G_{MN} and the NS–NS two-form B_{MN} manifestly. The covariant derivative D for the spacetime fermion θ is

$$
\begin{aligned}
D_a^{IJ} \equiv \delta^{IJ}\left(\partial_a - \frac{1}{4}\omega_a^{mn}\Gamma_{mn}\right) + \frac{1}{8}\sigma_3^{IJ} e_a^m H_{mnp}\Gamma^{np} \\
-\frac{1}{8}e^{\Phi}\left[\epsilon^{IJ}\Gamma^p F_p + \frac{1}{3!}\sigma_1^{IJ}\Gamma^{pqr}F_{pqr} + \frac{1}{2\cdot 5!}\epsilon^{IJ}\Gamma^{pqrst}F_{pqrst}\right] e_a^m \Gamma_m
\end{aligned}
$$

and it contains the dilaton Φ , and the R–R-field strengths F_p , F_{pqr} and F_{pqrst} . Thus, one can read off all of the (bosonic) components of type IIB supergravity from (22).

Here we should go back to the original questions made in Introduction. In principle, a new deformed background can be obtained by performing the supercoset construction with a classical r-matrix. Then the question can be rephrased as follows:

> Are the resulting backgrounds solutions of type IIB supergravity?

The answer depends on classical r-matrices utilized as the initial input. Now we know the significant condition for this issue, which is called the unimodularity condition [18].

The unimodularity condition [18] is given by

$$
r^{ij}[b_i, b_j] = 0 \qquad (r = r^{ij} b_i \wedge b_j \in \mathfrak{g} \otimes \mathfrak{g}) . \tag{23}
$$

When the classical r-matrix satisfies this condition, the resulting background is a solution to type IIB supergravity. If not, the background does not satisfy the on-shell condition of the supergravity and becomes a solution to a *generalized* supergravity. In the next section, we will explain what the generalized supergravity is. Before concluding this section, we will present some unimodular examples, which are well-known examples in different contexts (For short reviews, see [19, 20]).

Unimodular Examples

(1) Gamma-deformation of $\mathbf{S^5}$

A simple unimodular r-matrix is given by [21]

$$
r = \frac{1}{8}\left(\mu_3\, h_1 \wedge h_2 + \mu_1\, h_2 \wedge h_3 + \mu_2\, h_3 \wedge h_1\right) , \tag{24}
$$

where h_i $(i = 1, 2, 3)$ are the Cartan generators of $\mathfrak{su}(4)$ and μ_i $(i = 1, 2, 3)$ are constant parameters.

Then, the supercoset construction [16] leads to the following background:

$$
\begin{aligned}
ds^2 &= ds^2_{\mathrm{AdS}_5} + \sum_{i=1}^{3}(d\rho_i^2 + G\rho_i^2 d\phi_i^2) + \eta^2 G\rho_1^2\rho_2^2\rho_3^2 \left(\sum_{i=1}^{3}\mu_i\, d\phi_i\right)^2 , \\
B_2 &= \eta\, G \left(\mu_3\, \rho_1^2\rho_2^2\, d\phi_1 \wedge d\phi_2 + \mu_1\, \rho_2^2\rho_3^2\, d\phi_2 \wedge d\phi_3 + \mu_2\, \rho_3^2\rho_1^2\, d\phi_3 \wedge d\phi_1\right) , \\
F_5 &= 4\left[\omega_{\mathrm{AdS}_5} + G\,\omega_{\mathrm{S}^5}\right] , \qquad \Phi = \frac{1}{2}\log\, G , \\
F_3 &= -4\eta \sin^3\alpha\, \cos\alpha\, \sin\theta\, \cos\theta \left(\sum_{i=1}^{3}\mu_i\ d\phi_i\right) \wedge d\alpha \wedge d\theta . \qquad (25)
\end{aligned}
$$

Here the scalar function G is given by

$$
G^{-1} \equiv 1 + \eta^2(1 + \mu_3^2\rho_1^2\rho_2^2 + \mu_1^2\rho_2^2\rho_3^2 + \mu_2^2\rho_3^2\rho_1^2) , \qquad \sum_{i=1}^{3}\rho_i^2 = 1 , \qquad (26)
$$

where ρ_i's are parametrized as

$$
\rho_1 = \sin\,\alpha\, \cos\,\theta , \quad \rho_2 = \sin\,\alpha\, \sin\,\theta , \quad \rho_3 = \cos\,\alpha . \qquad (27)
$$

This is the gamma-deformation of S^5 presented in [22, 23]. Indeed, the classical r-matrix corresponds to three TsT transformations.

(2) Gravity dual for non-commutative gauge theory

Next, let us consider the following classical r-matrix [24],

$$
r = \frac{1}{2}\, p_2 \wedge p_3 . \qquad (28)
$$

Here the generators are represented by

$$
p_\mu \equiv \frac{1}{2}\gamma_\mu - m_{\mu 5} , \qquad m_{\mu 5} \equiv \frac{1}{4}[\gamma_\mu, \gamma_5] , \qquad (29)
$$

where γ_μ's are matrices of $\mathfrak{su}(2,2)$.

Then the supercoset construction [16] gives rise to the background:

$$
\begin{aligned}
ds^2 &= \frac{1}{z^2}(-dx_0^2 + dx_1^2) + \frac{z^2}{z^4+\eta^2}(dx_2^2 + dx_3^2) + \frac{dz^2}{z^2} + d\Omega_5^2 , \\
B_2 &= \frac{\eta}{z^4+\eta^2}\, dx^2 \wedge dx^3 , \qquad \Phi = \frac{1}{2}\log\left(\frac{z^4}{z^4+\eta^2}\right) , \\
F_3 &= \frac{4\eta}{z^5}\, dx^0 \wedge dx^1 \wedge dz , \qquad F_5 = 4\left[\mathrm{e}^{2\Phi}\omega_{AdS_5} + \omega_{\mathrm{S}^5}\right] . \qquad (30)
\end{aligned}
$$

This is nothing but a gravity dual of a noncommutative gauge theory[1] constructed in [26, 27]. The classical r-matrix (28) also correspond to a TsT transformation.

(3) Schrödinger spacetime

The last one is composed of the generators of both $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)$ [28]:

$$r = -\frac{i}{4}\, p_- \wedge (h_4 + h_5 + h_6)\,, \tag{31}$$

where the above generators have already appeared.

By the supercoset construction [16], the resulting background is given by

$$\begin{aligned} ds^2 &= \frac{-2dx^+dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \eta^2 \frac{(dx^+)^2}{z^4} + ds^2_{\mathrm{S}^5} \\ B_2 &= \frac{\eta}{z^2} dx^+ \wedge (d\chi + \omega)\,, \qquad \Phi = \text{const.}\,, \\ F_5 &= 4\left[\mathrm{e}^{2\Phi}\omega_{AdS_5} + \omega_{\mathrm{S}^5}\right]\,, \end{aligned} \tag{32}$$

where the S^5-coordinates are taken as

$$ds^2_{S^5} = (d\chi + \omega)^2 + ds^2_{\mathbb{C}\mathrm{P}^2}\,, \tag{33}$$

$$ds^2_{\mathbb{C}\mathrm{P}^2} = d\mu^2 + \sin^2\mu\,\left(\Sigma_1^2 + \Sigma_2^2 + \cos^2\mu\,\Sigma_3^2\right)\,. \tag{34}$$

This is the 5D Schrödinger spacetime embedded in type IIB supergravity [29–31]. The classical r-matrix (31) corresponds to a null Melvin twist.

In fact, the three examples presented so far belong to the class of abelian classical r-matrix. All of the Yang-Baxter deformations in this class can be expressed as TsT transformations [32]. For more general cases, see [33].

3 Generalized Supergravity

In this section, let us introduce an extension of the type IIB supergravity, called the generalized supergravity. The bosonic part was discovered originally by Arutunov, Frolov, Hoare, Roiban and Tseytlin [34] in the study of YB-deformation of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ superstring. After that, Tseytlin and Wulff succeeded in reproducing the generalized supergravity including the fermionic sector (dilatino and gravitino) by solving the kappa-symmetry constraints of the GS formulation of type IIB superstring on an arbitrary background [35].

The equations of motion in (the bosonic sector of) the generalized supergravity are given by

[1] This means a gauge theory defined on a noncommutative spacetime [25].

$$R_{MN} - \frac{1}{4}H_{MKL}H_N{}^{KL} - T_{MN} + D_M X_N + D_N X_M = 0\,, \tag{35}$$

$$\frac{1}{2}D^K H_{KMN} + \frac{1}{2}F^K F_{KMN} + \frac{1}{12}F_{MNKLP}F^{KLP} \tag{36}$$

$$= X^K H_{KMN} + D_M X_N - D_N X_M\,, \tag{37}$$

$$R - \frac{1}{12}H^2 + 4D_M X^M - 4X_M X^M = 0\,, \tag{38}$$

$$D^M \mathcal{F}_M - Z^M \mathcal{F}_M - \frac{1}{6}H^{MNK}\mathcal{F}_{MNK} = 0\,, \qquad I^M \mathcal{F}_M = 0\,, \tag{39}$$

$$D^K \mathcal{F}_{KMN} - Z^K \mathcal{F}_{KMN} - \frac{1}{6}H^{KPQ}\mathcal{F}_{KPQMN} - (I \wedge \mathcal{F}_1)_{MN} = 0\,, \tag{40}$$

$$D^K \mathcal{F}_{KMNPQ} - Z^K \mathcal{F}_{KMNPQ} + \frac{1}{36}\epsilon_{MNPQRSTUVW}H^{RST}\mathcal{F}^{UVW} - (I \wedge \mathcal{F}_3)_{MNPQ} = 0\,. \tag{41}$$

The energy-momentum tensor T_{MN} in (35) is given by

$$T_{MN} \equiv \frac{1}{2}\mathcal{F}_M\mathcal{F}_N + \frac{1}{4}\mathcal{F}_{MKL}\mathcal{F}_N{}^{KL} + \frac{1}{4\times 4!}\mathcal{F}_{MPQRS}\mathcal{F}_N{}^{PQRS} - \frac{1}{4}G_{MN}(\mathcal{F}_K\mathcal{F}^K + \frac{1}{6}\mathcal{F}_{PQR}\mathcal{F}^{PQR})\,. \tag{42}$$

The modified parts are three vector fields X_M, I_M and Z_M. But $X_M = I_M + Z_M$. So two of them, say I_M and Z_M are independent fields. Note that Z_M was originally the derivative of dilaton but now has undergone some modification.

Now the Bianchi identities are also modified as

$$\begin{aligned}
&(d\mathcal{F}_1 - Z \wedge \mathcal{F}_1)_{MN} - I^K \mathcal{F}_{MNK} = 0\,,\\
&(d\mathcal{F}_3 - Z \wedge \mathcal{F}_3 + H_3 \wedge \mathcal{F}_1)_{MNPQ} - I^K \mathcal{F}_{MNPQK} = 0\,,\\
&(d\mathcal{F}_5 - Z \wedge \mathcal{F}_5 + H_3 \wedge \mathcal{F}_3)_{MNPQRS} + \frac{1}{6}\epsilon_{MNPQRSTUVW}I^T\mathcal{F}^{UVW} = 0\,.
\end{aligned} \tag{43}$$

Furthermore, we need to explain more constraints,

$$D_M I_N + D_N I_M = 0\,, \tag{44}$$

$$D_M Z_N - D_N Z_M + I^K H_{KMN} = 0\,, \tag{45}$$

$$I^M Z_M = 0\,. \tag{46}$$

In particular, the first condition (44) is nothing but the Killing condition for I. Namely, I should be taken as a Killing vector. This condition may sound a bit stronger but this condition is necessary in solving the kappa-symmetry constraints [35]. Furthermore, this Killing condition is necessary to consider the embedding of the generalized supergravity into the Double Field Theory [36–38].

The Lie derivative of NS-NS two form along the Killing direction

$$(\mathcal{L}_I B)_{MN} = I^K \partial_K B_{MN} + B_{KN} \partial_M I^K - B_{KM} \partial_N I^K$$

should vanish. Then, by solving the second condition (45), one can obtain the following expression:

$$Z_M = \partial_M \Phi - B_{MN} I^N \,.$$

From this expression, one can understand that Z_M is a modification of the dilaton derivative with non-vanishing I and that Z is not independent of I. In this sense, only the Killing vector I characterizes the generalized supergravity. When $I = 0$, the original type IIB supergravity is reproduced.

Non-unimodular Example

As denoted previously, a non-unimodular classical r-matrix leads to a solution to the generalized supergravity with $I \neq 0$.

Let consider here the following non-unimodular example [39, 40]:

$$\begin{aligned} r &= E_{24} \wedge (c_1 E_{22} - c_2 E_{44}) \\ &= (p_0 - p_3) \wedge \left[a_1 \left(\frac{1}{2}\gamma_5 - n_{03} \right) - a_2 \left(n_{12} - \frac{i}{2} \mathbf{1}_4 \right) \right], \end{aligned} \tag{47}$$

where this is a two-parameter family and the deformation parameters (c_1, c_2) are related (a_1, a_2) each other through the relation

$$a_1 \equiv \frac{c_1 + c_2}{2} = \mathrm{Re}(c_1)\,, \qquad a_2 \equiv \frac{c_1 - c_2}{2\,i} = \mathrm{Im}(c_1)\,. \tag{48}$$

Then, by performing the supercoset construction [16], one can obtain the following background:

$$\begin{aligned} ds^2 &= \frac{-2dx^+dx^- + d\rho^2 + \rho^2 d\phi^2 + dz^2}{z^2} + ds^2_{S^5} \\ &\quad -4\eta^2 \left[(a_1^2 + a_2^2)\frac{\rho^2}{z^6} + \frac{a_1^2}{z^4} \right] (dx^+)^2 \,, \\ B_2 &= 8\eta \left[\frac{a_1 x^1 + a_2 x^2}{z^4} dx^+ \wedge dx^1 + \frac{a_1 x^2 - a_2 x^1}{z^4} dx^+ \wedge dx^2 \right. \\ &\quad \left. + a_1 \frac{1}{z^3} dx^+ \wedge dz \right], \\ F_3 &= 8\eta \left[\frac{a_2 x^1 - a_1 x^2}{z^5} dx^+ \wedge dx^1 \wedge dz + \frac{a_1 x^1 + a_2 x^2}{z^5} dx^+ \wedge dx^2 \wedge dz \right. \\ &\quad \left. + \frac{a_1}{z^4} dx^+ \wedge dx^1 \wedge dx^2 \right], \\ F_5 &= \text{undeformed}\,, \qquad \Phi = \text{const.} \end{aligned} \tag{49}$$

This background is not a solution to type IIB supergravity. It is easy to check this statement by taking an exterior derivative of F_3,

$$dF_3 = 16\eta\,\frac{a_1}{z^5}\,dx^+ \wedge dx^1 \wedge dx^2 \wedge dz \neq 0\,. \tag{50}$$

This does not vanish and the equation of motion for B_2 is also not satisfied.

However, by taking the extra vector field I as

$$I = -\frac{2\eta a_1}{z^2}\,dx^+\,, \quad Z = 0\,,$$

the background (49) becomes a solution to the generalized supergravity [16]. For other non-unimodular solutions, for example, see [41, 42].

Hoare-Tseytlin Conjecture

What of the generalized supergravity is so interesting? In the long history that String Theory has been studied, a number of so-called "pathological backgrounds," which are not solutions to supergravities, have been discovered. For example, it is well-known that non-abelian T-dualities generate such pathological backgrounds. It may be a good idea to check whether these backgrounds may be solutions to the generalized supergravity.

In fact, Hoare and Tseytlin advocated a interesting conjecture, the homogeneous YB deformations are equivalent to (a certain class of) non-abelian T-dualities [43]. Then this conjecture was proven by Borsato and Wulff [44]. The YB deformed backgrounds are solutions to the generalized supergravity, hence the accompanying non-abelian T-dualized backgrounds are also solutions as well.

Non-YB Solution

As we have seen so far, the YB deformation can be regarded as a solution generation technique in the generalized supergravity. However, as a matter of course, it does not give all of the solutions. That is, there exist a number of solutions which cannot be obtained as YB deformations.

Such an example is the Gasperini-Ricci-Veneziano background [45]:

$$\begin{aligned} ds^2 &= -dt^2 + \frac{(t^4+y^2)\,dx^2 - 2x\,y\,dx\,dy + (t^4+x^2)\,dy^2 + t^4 dz^2}{t^2(t^4+x^2+y^2)} + ds^2_{T^6}\,, \\ B_2 &= \frac{(x\,dx + y\,dy)\wedge dz}{t^4+x^2+y^2}\,, \qquad \Phi = \frac{1}{2}\ln\left[\frac{1}{t^2(t^4+x^2+y^2)}\right]. \end{aligned} \tag{51}$$

This is not a solution of the usual supergravity. However, by taking the vector field I like

$$I^z = -2\,,$$

the background (51) becomes a solution to the generalized supergravity [46]. Note here that this background can be obtained through a non-abelian T-duality, but cannot be expressed as a YB deformation. Hence this background is not included in the Hoare-Tseytlin conjecture. The background (51) is just an example, but further confirmation was made in [47], in which a number of similar solutions were listed.

4 Other Topics

Due to the page limit, a number of other issues could not be covered here. The list of them includes

- Open string picture, non-commutativity and Killing spinor formula [24, 48–56].
- Embedding of the generalized supergravity into Double Field Theory and the DFT perspective [36–38, 57–59].
- Non-geometric backgrounds obtained as YB deformations [46].
- Arguments on Weyl invariance of string theory on a generalized supergravity background [38, 60].
- Relation between Costello-Yamazaki [61] and YB deformation [62].

and more. I apologize for not being able to make a complete list, and hope that I have the opportunity to write a more comprehensive review.

Acknowledgements The work of K.Y. was supported by the Supporting Program for Interaction-based Initiative Team Studies (SPIRITS) from Kyoto University, a JSPS Grant-in-Aid for Scientific Research (B) No. 18H01214. This work is also supported in part by the JSPS Japan-Russia Research Cooperative Program.

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Gauge Theories on Fuzzy Spaces and Gravity

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Abstract We start by briefly reviewing the description of gravity theories as gauge theories in four dimensions. More specifically we recall the procedure leading to the results of General Relativity and Weyl Gravity in a gauge-theoretic manner. Then, after a brief reminder of the formulation of gauge theories on noncommutative spaces, we review our recent work, where gravity is constructed as a gauge theory on the fuzzy dS_4.

1 Introduction

One of the main research areas addressing the problem of the lack of knowledge of the spacetime quantum structure is based on the idea that at extremely small distances (Planck length) the coordinates exhibit a noncommutative structure. Then it is natural to wonder which are the implications for gravity of such an idea. On the other hand at more ordinary (say LHC) distances the Strong, Weak and Electromagnetic interactions are successfully formulated using gauge theories, while at much smaller distances the Grand Unified Gauge Theories provide a very attractive unification scheme of the three interactions. The gravitational interaction is not part of this picture, admitting a geometric formulation, the Theory of Relativity. However there exists a gauge-theoretic approach to gravity besides the geometric one [1–12]. This approach started with the pioneer work of Utiyama [1] and was refined by other authors [2–12] as a gauge theory of the de Sitter $SO(1,4)$ group, spontaneously broken by a scalar field to the Lorentz $SO(1,3)$ group. Similarly using the gauge-theoretic approach the

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_14

Weyl gravity has been constructed as a gauge theory of the 4-d conformal group [7, 8]. Returning to the noncommutative framework and taking into account the gauge-theoretic description of gravity, the well-established formulation of gauge theories on noncommutative spaces leads to the construction of models of noncommutative gravity [13–21]. In these treatments the authors use the constant noncommutativity (Moyal-Weyl), the formulation of the $\star$-product and the Seiberg-Witten map [22]. In addition to these treatments noncommutative gravitational models can be constructed using the noncommutative realization of matrix geometries [23–35], while it should also be noted that there exist alternative approaches [36–38] (see also [39]), which will not be considered here. It should also be noted that the formulation of noncommutative gravity implies, in general, noncommutative deformations which break the Lorentz invariance. However, "covariant noncommutative spaces" have been constructed too [41, 42] which preserve the Lorentz invariance. Consequently noncommutative deformations of field theories have been constructed [43–52] (see also [53–57]. The main point of this article is to present the various features of a 4-d gravity that we have constructed recently [50] as a gauge theory on a fuzzy dS_4. Motivated by Heckman-Verlinde [42], who were based on Yang's early work [41], we have considered a 4-d covariant fuzzy dS space which preserves Lorentz invariance. The requirement of covariance led us to an enlargement of the isometries of the fuzzy dS_4, specifically from $SO(1, 4)$ to $SO(1, 5)$. Then the construction of a gauge theory on this noncommutative space by gauging a subgroup of the full isometry, led us to an enlargement of the gauge group and in fixing its representation. In addition the covariance of the field strength tensor required the inclusion of a 2-form gauge field. Eventually we have proposed an action of Yang-Mills type, including the kinetic term of the 2-form.

2 Gravity as a Gauge Theory

In this section we recall the interpretation of the four-dimensional Einstein and Weyl gravities as gauge theories in order to be used later in the framework of noncommutative fuzzy spaces.

2.1 4-D Einstein's Gravity as a Gauge Theory

Gravitational interaction in four dimensions is described by General Relativity, a solid and successful theory which has been well-tested over decades since its early days. It is formulated geometrically in contrast to the rest of the interactions, which are described as gauge theories. Targeting to a unified description of gravity with the other interactions, a gauge-theoretic approach to gravity has been developed [1–6]. Lets us recall the main features of this approach to describe the 4-d Einstein's gravity. To achieve a gauge-theoretic approach of 4-d gravity, as a first step the vierbein formulation of General Relativity has to be employed. Then depending on the presence and sign of the cosmological constant gauge theories have been constructed on the Minkowski M^4, de Sitter dS_4 and anti-de-Sitter AdS_4 spacetimes based on the gauge groups Poincare, de Sitter and Anti-de Sitter, respectively. The choice of these groups as the symmetry gauge groups being that they are the isometry groups of the corresponding spacetimes. Let us start with the case in which there is no cosmological constant included, i.e., the case of the Poincaré group. In this case the generators of the corresponding algebra satisfy the following commutation relations:

$$[M_{ab}, M_{cd}] = 4\eta_{[a[c}M_{d]b]}, \quad [P_a, M_{bc}] = 2\eta_{a[b}P_{c]}, \quad [P_a, P_b] = 0, \tag{1}$$

where $\eta_{ab} = diag(-1, 1, 1, 1)$ is the metric tensor of the 4-d Minkowski spacetime, M_{ab} are the generators of the Lorentz group (the Lorentz transformations) and P_a are the generators of the local translations. Then according to the standard gauging procedure, the gauge potential, A_μ, is introduced and it is expressed as a decomposition on the generators of the Poincare algebra, as follows:

$$A_\mu(x) = e_\mu{}^a(x)P_a + \frac{1}{2}\omega_\mu{}^{ab}(x)M_{ab}. \tag{2}$$

The functions attached to the generators are the gauge fields of the theory and, in this case, they are identified as the vierbein, $e_\mu{}^a$, and the spin connection, $\omega_\mu{}^{ab}$, which correspond to the translations, P_a, and the Lorentz generators, M_{ab}, respectively. In this way, i.e. considering the vierbein as gauge field, it is achieved a mixing among the internal and spacetime symmetries and that is what makes this kind of construction special, as compared to the gauge theories describing other interactions. The gauge connection A_μ transforms according to the following rule:

$$\delta A_\mu = \partial_\mu\epsilon + [A_\mu, \epsilon], \tag{3}$$

where $\epsilon = \epsilon(x)$ is the gauge transformation parameter which is also expanded on the generators of the algebra:

$$\epsilon(x) = \xi^a(x)P_a + \frac{1}{2}\lambda^{ab}(x)M_{ab}. \tag{4}$$

Combining Eqs. (2) and (4) with (3) result to the following expressions of the transformations of the gauge fields:

$$\delta e_\mu{}^a = \partial_\mu\xi^a + \omega_\mu{}^{ab}\xi_b - \lambda^a{}_b e_\mu{}^b, \tag{5}$$

$$\delta\omega_\mu{}^{ab} = \partial_\mu\lambda^{ab} - 2\lambda^{[a}{}_c\omega_\mu{}^{cb]}. \tag{6}$$

According to the standard procedure followed in gauge theories, the corresponding field strength tensor of the gauge theory is defined as:

$$R_{\mu\nu}(A) = 2\partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu] \tag{7}$$

and since it is valued in the algebra of generators is also expanded on them as:

$$R_{\mu\nu}(A) = R_{\mu\nu}{}^{a}(e)P_a + \frac{1}{2}R_{\mu\nu}{}^{ab}(\omega)M_{ab}, \tag{8}$$

where $R_{\mu\nu}{}^{a}$ and $R_{\mu\nu}{}^{ab}$ are the curvatures associated to the component gauge fields, identified as the torsion and curvature, respectively. Replacing Eqs. (2) and (8) in the (7) results to the following explicit expressions:

$$R_{\mu\nu}{}^{a}(e) = 2\partial_{[\mu}e_{\nu]}{}^{a} - 2\omega_{[\mu}{}^{ab}e_{\nu]b}, \tag{9}$$

$$R_{\mu\nu}{}^{ab}(\omega) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b}. \tag{10}$$

Concerning the dynamics of the theory, the obvious choice is an action of Yang-Mills type, invariant under the gauge Poincaré group ISO(1,3). However, the aim is to result with the Einstein-Hilbert action, which is Lorentz invariant and, therefore, the gauge Poincaré group ISO(1,3) of the initial action has to be broken to the gauge Lorentz group SO(1,3). This can be achieved by gauging the SO(1,4) group, instead of the Poincaré group ISO(1,3), and employing its spontaneous symmetry breaking, induced by a scalar field that belongs to its fundamental representation [3, 5]. The choice of the 4-d de Sitter group is an alternative and preferred choice to that of the Poincare group, since all generators of the algebra can be considered on equal footing. The spontaneous symmetry breaking leads to the breaking of the translational generators, resulting to a constrained theory with vanishing torsion involving the Ricci scalar (and a topological Gauss-Bonnet term), respecting only the Lorentz symmetry, that is the Einstein-Hilbert action!

Concluding, Einstein's four-dimensional gravity can be formulated as a gauge theory of the Poincare group, as far as the kinematic part is concerned, i.e. the transformation of the fields and the expressions of the curvature tensors. Going to the dynamics though, instead of the Poincare group, it is the de Sitter symmetry which the initial Yang-Mills action has to respect. In turn, the inclusion of a scalar field and the addition of an appropriate kinetic term in the Lagrangian leads to a spontaneous symmetry breaking to the Lorentz gauge symmetry, i.e. to the Einstein-Hilbert action.

An alternative way to obtain an action with Lorentz symmetry, is to impose that the action is invariant only under the Lorentz symmetry and not under the total Poincare symmetry with which one starts. This means that the curvature tensor related to the translations has to vanish. In other words the torsionless condition is imposed in this way as a constraint that is necessary in order to result with an action respecting only the Lorentz symmetry. Solution of this constraint leads to a relation of the spin connection with the vielbein:

$$\begin{aligned}\omega_{\mu}^{ab} &= \frac{1}{2}e^{\nu a}(\partial_{\mu}e_{\nu}^{\ b} - \partial_{\nu}e_{\mu}^{\ b})\frac{1}{2}e^{\nu b}(\partial_{\mu}e_{\nu}^{\ a} - \partial_{\nu}e_{\mu}^{\ a}) \\ &\quad - \frac{1}{2}e^{\rho a}e^{\sigma b}(\partial_{\rho}e_{\sigma c} - \partial_{\sigma}e_{\rho c})e_{\mu}^{\ c}.\end{aligned} \tag{11}$$

However, straightforward consideration of an action of Yang-Mills type with Lorentz symmetry, would lead to an action involving the $R(M)^2$ term, which is not the correct one, since the aim is to obtain the Einstein-Hilbert action. Also, such an action would imply the wrong dimensionality (zero) of the coupling constant of gravity. In order to result with the Einstein-Hilbert action, which includes a dimensionful coupling constant, the action has to be considered in an alternative, non-straightforward way, that is the construction of Lorentz invariants out of the quantities (curvature tensor) of the theory. The one that is built by certain contractions of the curvature tensor is the correct one, ensuring the correct dimensionality of the coupling constant, and is identified as the Ricci scalar and the corresponding action is eventually the Einstein-Hilbert action.

2.2 4-D Weyl Gravity as a Gauge Theory

Besides Einstein's gravity, also Weyl's gravity has been successfully described as a gauge theory of the 4-d conformal group, SO(2,4). In this case, too, the transformations of the fields and the expressions of the curvature tensors are determined in a straightforward way. The initial action that is considered is an SO(2,4) gauge invariant action of Yang-Mills type which is broken by imposition of specific conditions (constraints) on the curvature tensors. After taking into account the constraints, the resulting action of the theory is the scale invariant Weyl action [7–9] (see also [10, 11]).

The generators of the conformal algebra of SO(2,4) are the local translations (P_a), the Lorentz transformations (M_{ab}), the conformal boosts (K_a) and the dilatations (D). Their algebra is determined by their commutation relations:

$$\begin{gathered}[M_{ab}, M^{cd}] = 4M_{[a}^{\ [d}\delta_{b]}^{c]}, \quad [M_{ab}, P_c] = 2P_{[a}\delta_{b]c}, \quad [M_{ab}, K_c] = 2K_{[a}\delta_{b]c} \\ [P_a, D] = P_a, \quad [K_a, D] = -K_a, \quad [P_a, K_b] = 2(\delta_{ab}D - M_{ab}),\end{gathered} \tag{12}$$

where $a, b, c, d = 1...4$. Then, according to the gauging procedure, the gauge potential, A_μ of the theory is in turn determined and is given as an expansion on the generators of the gauge group, i.e.:

$$A_\mu = e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab} M_{ab} + b_\mu D + f_\mu{}^a K_a, \tag{13}$$

where a gauge field has been associated with each generator. In this case, too, the vierbein and the spin connection are identified as gauge fields of the theory. The transformation rule of the gauge potential, (13), is given by:

$$\delta_\epsilon A_\mu = D_\mu \epsilon = \partial_\mu \epsilon + [A_\mu, \epsilon], \tag{14}$$

where ϵ is a gauge transformation parameter valued in the Lie algebra of the SO(2,4) group and therefore it can be written as:

$$\epsilon = \epsilon_P{}^a P_a + \frac{1}{2}\epsilon_M{}^{ab} M_{ab} + \epsilon_D D + \epsilon_K{}^a K_a. \tag{15}$$

Combining the Eqs. (14), (13) and (15) result to the following expressions of the transformations of the gauge fields of the theory:

$$\begin{aligned}
\delta e_\mu{}^a &= \partial_\mu \epsilon_P{}^a + 2ie_{\mu b}\epsilon_M{}^{ab} - i\omega_\mu{}^{ab}\epsilon_{Pb} - b_\mu \epsilon_K{}^a + f_\mu{}^a \epsilon_D, \\
\delta\omega_\mu{}^{ab} &= \frac{1}{2}\partial_\mu \epsilon_M{}^{ab} + 4ie_\mu{}^a \epsilon_P{}^b + \frac{i}{4}\omega_\mu{}^{ac}\epsilon_M{}^b{}_c + if_\mu{}^a \epsilon_K{}^b, \\
\delta b_\mu &= \partial_\mu \epsilon_D - e_\mu{}^a \epsilon_{Ka} + f_\mu{}^a \epsilon_{Pa}, \\
\delta f_\mu{}^a &= \partial_\mu \epsilon_K{}^a + 4ie_\mu{}^a \epsilon_D - i\omega_\mu{}^{ab}\epsilon_{Kb} - 4ib_\mu \epsilon_P{}^a + if_\mu{}^b \epsilon_M{}^a{}_b.
\end{aligned} \tag{16}$$

Accordingly the field strength tensor is defined by the relation:

$$R_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} - i[A_\mu, A_\nu] \tag{17}$$

and is expanded on the generators as:

$$R_{\mu\nu} = \tilde{R}_{\mu\nu}{}^a P_a + \frac{1}{2}R_{\mu\nu}{}^{ab} M_{ab} + R_{\mu\nu} + R_{\mu\nu}{}^a K_a. \tag{18}$$

Then combining the Eq. (17) and (18) result in the following expressions of the component curvature tensors:

$$\begin{aligned}
R_{\mu\nu}{}^a(P) &= 2\partial_{[\mu} e_{\nu]}{}^a + f_{[\mu}{}^a b_{\nu]} + e_{[\mu}{}^b \omega_{\nu]}{}^{ac}\delta_{bc}, \\
R_{\mu\nu}{}^{ab}(M) &= \partial_{[\mu}\omega_{\nu]}{}^{ab} + \omega_{[\mu}{}^{ca}\omega_{\nu]}{}^{db}\delta_{cd} + e_{[\mu}{}^a e_{\nu]}{}^b + f_{[\mu}{}^a f_{\nu]}{}^b, \\
R_{\mu\nu}(D) &= 2\partial_{[\mu} b_{\nu]} + f_{[\mu}{}^a e_{\nu]}{}^b \delta_{ab}, \\
R_{\mu\nu}{}^a(K) &= 2\partial_{[\mu} f_{\nu]}{}^a + e_{[\mu}{}^a b_{\nu]} + f_{[\mu}{}^b \omega_{\nu]}{}^{ac}\delta_{bc}.
\end{aligned} \tag{19}$$

Concerning the action, it is taken to be a gauge SO(2,4) invariant of Yang-Mills type. Then the initial SO(2,4) gauge symmetry can be broken by the imposition of certain constraints [7–9], namely the torsionless condition, $R(P) = 0$ and an additional constraint on $R(M)$. The two constraints admit an algebraic solution leading to expressions of the fields $\omega_\mu{}^{ab}$ and $f_\mu{}^a$ in terms of the independent fields $e_\mu{}^a$ and b_μ. In addition, b_μ can be gauged fixed to $b_\mu = 0$ and, imposing all the constraints in the initial action lead to the well-known Weyl action, which is diffeomorphism and scale invariant.

Besides the above breaking of the conformal symmetry which led to the Weyl action, another breaking pattern via constraints has been suggested [51], leading to an action with Lorentz symmetry, i.e. explicitly the Einstein-Hilbert action. From our perspective, the latter can be achieved through an alternative symmetry breaking mechanism, specifically with the inclusion of two scalar fields in the fundamental representation of the conformal group [52]. Then the spontaneous symmetry breaking could be triggered just as a generalization of the case of the breaking of the 4-d de Sitter group down to the Lorentz group by the inclusion of a scalar in the fundamental representation of SO(1,4), as discussed in Sect. 2.1. Calculations and details on this issue will be included in a future work.

Moreover, the argument used in the previous section in the 4-d Poincaré gravity case as an alternative way to break the initial symmetry to the Lorentz, can be generalized in the case of conformal gravity too. Since it is desired to result with the Lorentz symmetry starting from the initial gauge SO(2,4) symmetry, the vacuum of the theory is considered to be directly SO(4) invariant, which means that every other tensor, except for the R(M), has to vanish. Setting these tensors to zero will produce the constraints of the theory leading to expressions that relate the gauge fields. In particular, in [51], it is argued that if both tensors $R(P)$ and $R(K)$ are simultaneously set to zero, then from the constraints of the theory it is understood that the corresponding gauge fields, $f_\mu{}^a, e_\mu{}^a$ are equal—up to a rescaling factor—and $b_\mu = 0$.

3 Gauge Theories on Noncommutative Spaces

Let us now briefly recall the main concepts of the formulation of gauge theories on noncommutative spaces, in order to use them later in the construction of the noncommutative gravity models.

Gauge fields arise in noncommutative geometry and in particular on fuzzy spaces very naturally; they are linked to the notion of covariant coordinate [58]. Consider a field $\phi(X_a)$ on a fuzzy space described by the non-commuting coordinates X_a and transforming according to a gauge group G. An infinitesimal gauge transformation $\delta\phi$ of the field ϕ with gauge transformation parameter $\lambda(X_a)$ is defined by:

$$\delta\phi(X) = \lambda(X)\phi(X). \tag{20}$$

If $\lambda(X)$ is a function of the coordinates, X_a, then it is an infinitesimal Abelian transformation and $G = U(1)$, while if $\lambda(X)$ is valued in the Lie algebra of hermitian $P \times P$ matrices, then the transformation is non-Abelian and the gauge group is $G = U(P)$. The coordinates are invariant under an infinitesimal transformation of the gauge group, G, i.e. $\delta X_a = 0$. In turn the gauge transformation of the product of a coordinate and the field is not covariant:

$$\delta(X_a\phi) = X_a\lambda(X)\phi, \tag{21}$$

since, in general, it holds:

$$X_a\lambda(X)\phi \neq \lambda(X)X_a\phi. \tag{22}$$

Following the ideas of the construction of ordinary gauge theories, where a covariant derivative is defined, in the noncommutative case, the covariant coordinate, ϕ_a, is introduced by its transformation property:

$$\delta(\phi_a \phi) = \lambda \phi_a \phi, \tag{23}$$

which is satisfied if:

$$\delta(\phi_a) = [\lambda, \phi_a]. \tag{24}$$

Eventually, the covariant coordinate is defined as:

$$\phi_a \equiv X_a + A_a, \tag{25}$$

where A_a is identified as the gauge connection of the noncommutative gauge theory. Combining Eqs. (24), (25), the gauge transformation of the connection, A_a, is obtained:

$$\delta A_a = -[X_a, \lambda] + [\lambda, A_a]. \tag{26}$$

justifying the interpretation of A_a as a gauge field.[1] Correspondingly the field strength tensor, F_{ab}, is defined as:

$$F_{ab} \equiv [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - C_{ab}^c A_c = [\phi_a, \phi_b] - C_{ab}^c \phi_c, \tag{27}$$

which is covariant under a gauge transformation,

$$\delta F_{ab} = [\lambda, F_{ab}]. \tag{28}$$

In the following sections, the above methodology will be applied in the construction of gravity models as gauge theories on fuzzy spaces.

4 A 4-D Noncommutative Gravity Model

Let us now proceed with the presentation of a 4-d gravity model as a gauge theory on a fuzzy space. We start with the construction of an appropriate 4-d fuzzy space and then we build a gravity theory as a gauge theory on this noncommutative space.

4.1 Fuzzy de Sitter Space

Let us construct first the fuzzy 4-d de Sitter space, dS_4, which will be used as the background space on which we will define the gauge theory that we propose to describe gravity. The continuous dS_4 is defined as a submanifold of the 5-d Minkowski

[1] For more details see [39].

spacetime and can be viewed as the Lorentzian analogue of the definition of the four-sphere as an embedding in the 5-d Euclidean space. The defining embedding equation of dS_4 is:

$$\eta^{MN} x_M x_N = R^2, \tag{29}$$

$M, N = 0, \ldots, 4$ and η^{MN} is the metric tensor of the 5-d Minkowski spacetime, $\eta^{MN} = \text{diag}(-1, +1, +1, +1, +1)$. In order to obtain the fuzzy analogue of this space, one has to consider its coordinates, X_m, to be operators that do not commute with each other:

$$[X_m, X_n] = i\theta_{mn}, \tag{30}$$

where the spacetime indices are $m, n = 1, \ldots, 4$. In analogy to the fuzzy sphere case, where the corresponding coordinates are identified as the rescaled three generators of $SU(2)$ in a high N-dimensional representation, we expect that the right hand side in Eq. (30), should be identified with a generator of the underlying algebra, ensuring covariance, i.e $\theta_{mn} = C_{mn}{}^{r} X_r$, where C_{mnr} is a rescaled Levi-Civita symbol. Otherwise, if the right hand side in Eq. (30) is a fixed antisymmetric tensor the Lorentz invariance will be violated. However, in the present fuzzy de Sitter case, such an identification cannot be achieved, since the algebra is not closing [42].[2] To achieve covariance, the suggestion [41, 42] is to use a group with a larger symmetry, in which we will be able to incorporate all generators and the noncommutativity in it. The minimal extension of the symmetry leads us to adopt the $SO(1, 5)$ group. Therefore, a fuzzy dS_4 space, with its coordinates being operators represented by N-dimensional matrices, respecting covariance, too, is obtained after the enlargement of the symmetry to the $SO(1, 5)$ [50]. To facilitate the construction we make use of the Euclidean signature, therefore, instead of the $SO(1, 5)$, the resulting symmetry group is considered to be that of $SO(6)$.

In order to formulate explicitly the above 4-d fuzzy space, let us consider the $SO(6)$ generators, denoted as $\mathrm{J}_{AB} = -\mathrm{J}_{BA}$, with $A, B = 1, \ldots, 6$, satisfying the following commutation relation:

$$[J_{AB}, J_{CD}] = i(\delta_{AC} J_{BD} + \delta_{BD} J_{AC} - \delta_{BC} J_{AD} - \delta_{AD} J_{BC}). \tag{31}$$

These generators can be written as a decomposition in an $SO(4)$ notation, with the component generators identified as various operators, including the coordinates, i.e.:

$$J_{mn} = \frac{1}{\hbar}\Theta_{mn}, \quad J_{m5} = \frac{1}{\lambda} X_m, \quad J_{m6} = \frac{\lambda}{2\hbar} P_m, \quad J_{56} = \frac{1}{2}\mathrm{h}, \tag{32}$$

where $m, n = 1, \ldots, 4$. For dimensional reasons, an elementary length, λ, has been introduced in the above identifications, in which the coordinates, momenta and non-commutativity tensor are denoted as X_m, P_m and Θ_{mn}, respectively. Then the coordinate and momentum operators satisfy the following commutation relations:

[2] For more details on this issue, see [54, 55], where the same problem emerges in the construction of the fuzzy four-sphere.

$$[X_m, X_n] = i\frac{\lambda^2}{\hbar}\Theta_{mn}, \qquad [P_m, P_n] = 4i\tfrac{\hbar}{\lambda^2}\Theta_{mn}, \tag{33}$$

$$[X_m, P_n] = i\hbar\delta_{mn}\mathrm{h}, \qquad [X_m, \mathrm{h}] = i\tfrac{\lambda^2}{\hbar}P_m, \tag{34}$$

$$[P_m, \mathrm{h}] \qquad = 4i\tfrac{\hbar}{\lambda^2}X_m\,, \tag{35}$$

while the algebra of spacetime transformations is given by:

$$[X_m, \Theta_{np}] \qquad = i\hbar(\delta_{mp}X_n - \delta_{mn}X_p) \tag{36}$$

$$[P_m, \Theta_{np}] \qquad = i\hbar(\delta_{mp}P_n - \delta_{mn}P_p) \tag{37}$$

$$[\Theta_{mn}, \Theta_{pq}] = i\hbar(\delta_{mp}\Theta_{nq} + \delta_{nq}\Theta_{mp} - \delta_{np}\Theta_{mq} - \delta_{mq}\Theta_{np}) \tag{38}$$

$$[\mathrm{h}, \Theta_{mn}] \qquad = 0\,. \tag{39}$$

It is very interesting to note that the above algebra in contrast to the Heisenberg algebra (see [59]) admits finite-dimensional matrices to represent the operators X_m, P_m and Θ_{mn} and therefore the spacetime obtained above is a finite quantum system. Then clearly the above fuzzy dS_4 falls into the general class of the fuzzy covariant spaces [42, 56, 60].

4.2 Gravity as Gauge Theory on the Fuzzy dS_4

In the previous section, the fuzzy dS_4 space was constructed and the appropriate symmetry group to be used was found to be the $SO(6)$. Following the recipe of the construction of Einstein gravity as gauge theory in Sect. 2.1, in which the isometry group (the Poincaré group) was chosen to be gauged, in this case the gauge group would be given by the isometry group of the fuzzy dS_4 space, namely the $SO(5)$, viewed as a subgroup of the $SO(6)$ group.

However, it is known that in noncommutative gauge theories, the use of the anticommutators of the generators of the algebra is inevitable, as we have explained in detail in our previous works [43, 44] (see also [16]). Specifically, the anticommutation relations of the generators of the gauge group, $SO(5)$, produce operators that, in general, do not belong to the algebra. The indicated treatment is to fix the representation of the generators and all operators produced by the anticommutators of the generators to be included into the algebra, identifying them as generators, too. This procedure led us to an extension of the $SO(5)$ to $SO(6) \times U(1)$ ($\sim U(4)$) group[3] with the generators being represented by 4×4 matrices in the spinor representation of $SO(6)$ (or the fundamental of $SU(4)$), 4.

In order to obtain the specific expressions of the matrices representing the generators, the four Euclidean Γ-matrices are employed, satisfying the following anticommutation relation:

[3]Most probably the extension of the gauge group from $SO(5)$ to $SO(6)$ is not a coincidence, while the inclusion of a $U(1)$ is quite intrinsic property of noncommutative theories.

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\mathbb{1}, \tag{40}$$

where $a, b = 1, \ldots, 4$. Also the Γ_5 matrix is defined as $\Gamma_5 = \Gamma_1\Gamma_2\Gamma_3\Gamma_4$. Therefore, the generators of the SO(6)×U(1) gauge group are identified as:

(a) Six generators of the Lorentz transformations: $\mathrm{M}_{ab} = -\frac{i}{4}[\Gamma_a, \Gamma_b] = -\frac{i}{2}\Gamma_a \Gamma_b, a < b$,
(b) four generators of the conformal boosts: $\mathrm{K}_a = \frac{1}{2}\Gamma_a$,
(c) four generators of the local translations: $\mathrm{P}_a = -\frac{i}{2}\Gamma_a\Gamma_5$,
(d) one generator for special conformal transformations: $\mathrm{D} = -\frac{1}{2}\Gamma_5$ and
e) one U(1) generator: $\mathbb{1}$.

The Γ-matrices are determined as tensor products of the Pauli matrices, specifically:

$$\Gamma_1 = \sigma_1 \otimes \sigma_1, \quad \Gamma_2 = \sigma_1 \otimes \sigma_2, \quad \Gamma_3 = \sigma_1 \otimes \sigma_3$$

$$\Gamma_4 = \sigma_2 \otimes \mathbb{1}, \quad \Gamma_5 = \sigma_3 \otimes \mathbb{1}.$$

Therefore, the generators of the algebra are represented by the following 4×4 matrices:

$$M_{ij} = -\frac{i}{2}\Gamma_i\Gamma_j = \frac{1}{2}\mathbb{1} \otimes \sigma_k, \tag{41}$$

where $i, j, k = 1, 2, 3$ and:

$$M_{4k} = -\frac{i}{2}\Gamma_4\Gamma_k = -\frac{1}{2}\sigma_3 \otimes \sigma_k. \tag{42}$$

Straightforward calculations lead to the following commutation relations, which the operators satisfy:

$$\begin{aligned}
[K_a, K_b] &= iM_{ab}, \quad [P_a, P_b] = iM_{ab} \\
[X_a, P_b] &= i\delta_{ab}D, \quad [X_a, D] = iP_a \\
[P_a, D] &= iK_a, \quad [K_a, P_b] = i\delta_{ab}D, \quad [K_a, D] = -iP_a \\
[K_a, M_{bc}] &= i(\delta_{ac}K_b - \delta_{ab}K_c) \\
[P_a, M_{bc}] &= i(\delta_{ac}P_b - \delta_{ab}P_c) \\
[M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} + \delta_{bd}M_{ac} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc}) \\
[D, M_{ab}] &= 0\,.
\end{aligned} \tag{43}$$

Having determined the commutation relations of the generators of the algebra, the noncommutative gauging procedure can be done in a rather straightforward way. To start with, the covariant coordinate is defined as:

$$\hat{X}_m = X_m \otimes \mathbb{1} + A_m(X). \tag{44}$$

The coordinate $\hat{X}_m$ is covariant by construction and this property is expressed as:

$$\delta \hat{X}_m = i[\epsilon, \hat{X}_m], \tag{45}$$

where $\epsilon(X)$ is the gauge transformation parameter, which is a function of the coordinates ($N \times N$ matrices), X_m, but also is valued in the $SO(6) \times U(1)$ algebra. Therefore, it can be decomposed on the sixteen generators of the algebra:

$$\epsilon = \epsilon_0(X) \otimes \mathbb{1} + \xi^a(X) \otimes K_a + \tilde{\epsilon}_0(X) \otimes D + \lambda_{ab}(X) \otimes \Sigma^{ab} + \tilde{\xi}^a(X) \otimes P_a. \tag{46}$$

Taking into account that a gauge transformation acts trivially on the coordinate X_m, namely $\delta X_m = 0$, the transformation property of the A_m is obtained by combining the Eqs. (44), (45) and (46). According to the corresponding procedure in the commutative case, the A_m transforms in such a way that admits the interpretation of the connection of the gauge theory. Also similarly to the case of the gauge transformation parameter, ϵ, the A_m, is a function of the coordinates X_m of the fuzzy space dS_4, but also takes values in the $SO(6) \times U(1)$ algebra, which means that it can be expanded on its sixteen generators as follows:

$$\begin{aligned} A_m(X) = e_m^{\ a}(X) \otimes P_a + \omega_m^{\ ab}(X) \otimes \Sigma_{ab}(X) + b_m^{\ a}(X) \otimes K_a(X) \\ + \tilde{a}_m(X) \otimes D + a_m(X) \otimes \mathbb{1}, \end{aligned} \tag{47}$$

where it is clear that the various gauge fields have been corresponded to the generators of the $SO(6) \times U(1)$. The component gauge fields are functions of the coordinates of the space, X_m, therefore they have the form of $N \times N$ matrices, where N is the dimension of the representation in which the coordinates are accommodated. Thus, instead of the ordinary product, between the gauge fields and their corresponding generators, the tensor product is used, since the factors are matrices of different dimensions, given that the generators are represented by 4×4 matrices. Then, each term in the expression of the gauge connection is a $4N \times 4N$ matrix.

After the introduction of the gauge fields, the covariant coordinate is written as:

$$\hat{X}_m = X_m \otimes \mathbb{1} + e_m^{\ a}(X) \otimes P_a + \omega_m^{\ ab}(X) \otimes \Sigma_{ab} + b_m^{\ a} \otimes K_a + \tilde{a}_m \otimes D + a_m \otimes \mathbb{1}. \tag{48}$$

Then the next step in the theory that we are developing is to calculate its field strength tensor. We found that for the fuzzy de Sitter space, the field strength tensor has to be defined as:

$$\mathcal{R}_{mn} = [\hat{X}_m, \hat{X}_n] - \frac{i\lambda^2}{\hbar}\hat{\Theta}_{mn}, \tag{49}$$

where $\hat{\Theta}_{mn} = \Theta_{mn} \otimes \mathbb{1} + \mathcal{B}_{mn}$. The $\mathcal{B}_{mn}$ is a 2-form gauge field, which takes values in the SO(6)×U(1) algebra. The $\mathcal{B}_{mn}$ field was introduced in order to make the field strength tensor covariant, since in its absence it does not transform covariantly.[4]

[4] Details on this generic issue on such spaces are given in Appendix A of the first paper of [50].

The $\mathcal{B}_{mn}$ field will contribute in the total action of the theory with a kinetic term of the following form:

$$\mathcal{S}_{\mathcal{B}} = \mathrm{Tr}\,\mathrm{tr}\,\hat{\mathcal{H}}_{mnp}\hat{\mathcal{H}}^{mnp}. \tag{50}$$

The $\hat{\mathcal{H}}_{mnp}$ field strength tensor transforms covariantly under a gauge transformation, therefore the above action is gauge invariant.

The field strength tensor of the gauge connection, (49), can be expanded in terms of the component curvature tensors, since it is valued in the algebra:

$$\begin{aligned}\mathcal{R}_{mn}(X) = R_{mn}{}^{ab}(X) \otimes \Sigma_{ab} + \tilde{R}_{mn}{}^{a}(X) \otimes P_a + R_{mn}{}^{a}(X) \otimes K_a \\ + \tilde{R}_{mn}(X) \otimes D + R_{mn}(X) \otimes 1\!\!1.\end{aligned} \tag{51}$$

All necessary information for the determination of the transformations of the gauge fields and the expressions of the component curvature tensors is obtained. The explicit expressions and calculations can be found in the first paper of ref.[50].

4.3 *The Action and the Constraints for the Symmetry Breaking*

Concerning the action of the theory, it is natural to consider one of Yang-Mills type[5]:

$$\mathcal{S} = \mathrm{Trtr}\{\mathcal{R}_{mn}, \mathcal{R}_{rs}\}\epsilon^{mnrs}, \tag{52}$$

where Tr denotes the trace over the coordinates-$N \times N$ matrices (it replaces the integration of the continuous case) and tr denotes the trace over the generators of the algebra.

However the gauge symmetry of the resulting theory, with which we would like to end up, is the one described by the Lorentz group, in the Euclidean signature, the $SO(4)$. In this direction, one could consider directly a constrained theory in which the only component curvature tensors that would not be imposed to vanish would be the ones that correspond to the Lorentz and the $U(1)$ generators of the algebra, achieving a breaking of the initial $SO(6) \times U(1)$ symmetry to the $SO(4) \times U(1)$. However, counting the degrees of freedom, adopting the above breaking would lead to an overconstrained theory. Therefore, it is more efficient to follow a different procedure and perform the symmetry breaking in a less straightforward way [50]. Accordingly, the first constraint is the torsionless condition:

$$\tilde{R}_{mn}{}^{a}(P) = 0, \tag{53}$$

[5]A Yang-Mills action $\mathrm{tr}F^2$ defined on the fuzzy dS_4 space is gauge invariant, for details see Appendix A of the first paper of [50].

which is also imposed in the cases in which the Einstein and conformal gravity theories are described as gauge theories. The presence of the gauge field $b_m^{\ a}$ would admit an interpretation of a second vielbein of the theory, that would lead to a bimetric theory, which is not what we are after in the present case. Here it would be preferable to have the relation $e_m^{\ a} = b_m^{\ a}$ in the solution of the constraint. This choice leads also in expressing of the spin connection $\omega_m^{\ ab}$ in terms of the rest of the independent fields, $e_m^{\ a}, a_m, \tilde{a}_m$. To obtain the explicit expression of the spin connection in terms of the other fields, the following two identities are employed:

$$\delta^{abc}_{fgh} = \epsilon^{abcd}\epsilon_{fghd} \qquad \text{and} \qquad \frac{1}{3!}\delta^{abc}_{fgh}a^{fgh} = a^{[fgh]}. \tag{54}$$

Solving the constraint $\tilde{R}(P) = 0$, it follows that:

$$\epsilon^{abcd}[e_m^{\ b}, \omega_n^{\ cd}] - i\{\omega_m^{\ ab}, e_{nb}\} = -[D_m, e_m^{\ a}] - i\{e_m^{\ a}, \tilde{a}_m\}, \tag{55}$$

where $D_m = X_m + a_m$ being the covariant coordinate of an Abelian noncommutative gauge theory. Then the above equation leads to the following two:

$$\epsilon^{abcd}[e_m^{\ b}, \omega_n^{\ cd}] = -[D_m, e_m^{\ a}] \qquad \text{and} \qquad \{\omega_m^{\ ab}, e_{nb}\} = \{e_m^{\ a}, \tilde{a}_n\}. \tag{56}$$

Taking into account also the identities, (54), the above equations lead to the desired expression for the spin connection in terms of the rest fields:

$$\omega_n^{\ ac} = -\frac{3}{4}e^m_{\ b}(-\epsilon^{abcd}[D_m, e_{nd}] + \delta^{[bc}\{e_n^{\ a]}, \tilde{a}_m\}). \tag{57}$$

According to [61], the vanishing of the field strength tensor in a gauge theory could lead to the vanishing of the associated gauge field. However, the vanishing of the torsion component tensor, $\tilde{R}(P) = 0$, does not imply $e_\mu^{\ a} = 0$, because such a choice would lead to degeneracy of the metric tensor of the space [12]. The field that can be gauge-fixed to zero is the $\tilde{a}_m$. Then this fixing, $\tilde{a}_m = 0$, will modify the expression of the spin connection, (57), leading to a further simplified expression of the spin connection in terms of the vielbein:

$$\omega_n^{\ ac} = \frac{3}{4}e^m_{\ b}\epsilon^{abcd}[D_m, e_{nd}]. \tag{58}$$

We note that the $U(1)$ field strength tensor, $R_{mn}(1)$, signaling the noncommutativity of the space, is not considered to be vanishing. The $U(1)$ remains unbroken in the resulting theory after the breaking, since we still have a theory on a noncommutative space. However, the corresponding field, a_m, would vanish if we consider the commutative limit of the broken theory, in which noncommutativity is lifted and a_m decouples being super heavy. In this limit, the gauge theory would be just $SO(4)$. Alternatively, another way to break the $SO(6)$ gauge symmetry to the desired $SO(4)$ is to induce a spontaneous symmetry breaking by including two scalar fields in the 6 representation of $SO(6)$ [52], extending the argument developed for the case of the conformal gravity to the noncommutative framework. It is expected that the spontaneous symmetry breaking induced by the scalars would lead to a constrained theory as the one that was obtained above by the imposition of the constraints (53). After the symmetry breaking, i.e. including the constraints, the surviving terms of the action will be:

$$\begin{aligned}\mathcal{S} = 2\mathrm{Tr}(&R_{mn}{}^{ab} R_{rs}{}^{cd} \epsilon_{abcd} \epsilon^{mnrs} + 4\tilde{R}_{mn} R_{rs} \epsilon^{mnrs} \\ &+ \frac{1}{3} H_{mnp}{}^{ab} H^{mnpcd} \epsilon_{abcd} + \frac{4}{3} \tilde{H}_{mnp} H^{mnp}).\end{aligned} \tag{59}$$

Finally replacing with the explicit expressions of the component tensors and writing the ω gauge field in terms of the surviving gauge fields, (58) and then varying with respect to the independent gauge fields would lead to the equations of motion.

5 Summary and Conclusions

In the present review we presented a 4-d gravity model as a gauge theory on a fuzzy version of the 4-d de Sitter space. It should be stressed that the constructed fuzzy dS_4 consists a 4-d covariant noncommutative space, respecting Lorentz invariance, which is of major importance in our case. Next, although we started by gauging the isometry group of dS_4, $SO(5)$, we were led to enlarge it to $SO(6) \times U(1)$ in order to include the anticommutators of its generators that appear naturally in the noncommutative framework and in fixing the representation. Then, following the standard procedure we calculated the transformations of the fields and the expressions of the component curvature tensors. Since our aim was to result with a theory respecting the Lorentz symmetry, we imposed certain constraints in order to break the initial symmetry. After the symmetry breaking, the action takes its final form and its variation will lead to the equations of motion. The latter will be part of our future work. It should be noted that, before the symmetry breaking, the results of the above construction reduce to the ones of the conformal gravity in the commutative limit. Finally, it should be also emphasized that the above is a matrix model giving insight into the gravitational interaction in the high-energy regime and also giving promises for improved UV properties as compared to ordinary gravity. Clearly, the latter, as well the inclusion of matter fields is going to be a subject of further study.

Acknowledgements We would like to thank Ali Chamseddine, Paolo Aschieri, Thanassis Chatzistavrakidis, Evgeny Ivanov, Larisa Jonke, Danijel Jurman, Alexander Kehagias, Dieter Lust, Denjoe O'Connor, Emmanuel Saridakis, Harold Steinacker, Kelly Stelle, Patrizia Vitale and Christof Wetterich for useful discussions. The work of two of us (GM and GZ) was partially supported by the COST Action MP1405, while both would like to thank ESI—Vienna for the hospitality during their participation in the Workshop "Matrix Models for Noncommutative Geometry and String Theory", Jul 09–13, 2018. One of us (GZ) has been supported within the Excellence Initiative funded by the German and States Governments, at the Institute for Theoretical Physics, Heidelberg University and from the Excellent Grant Enigmass of LAPTh. GZ would like to thank the ITP—Heidelberg, LAPTh—Annecy and MPI—Munich for their hospitality.

Last but not least GZ thanks the organisers of the Workshop in Varna for their warm hospitality.

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String Theories and Gravity Theories

Modified Gravity Theories Based on the Non-canonical Volume-Form Formalism

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Abstract We present a concise description of the basic features of gravity-matter models based on the formalism of non-canonical spacetime volume-forms in its two versions: (a) the *method of non-Riemannian volume-forms* (metric-independent covariant volume elements) and (b) the *dynamical spacetime formalism*. Among the principal outcomes we briefly discuss: (i) quintessential universe evolution with a gravity-"inflaton"-assisted suppression in the "early" universe and, respectively, dynamical generation in the "late" universe of Higgs spontaneous electroweak gauge symmetry breaking; (ii) unified description of dark energy and dark matter as manifestations of a single material entity—a second scalar field "darkon"; (iii) unification of dark energy and dark matter with diffusive interaction among them; (iv) explicit derivation of a stable "emergent universe" solution, *i.e.*, a creation without Big Bang; (v) mechanism for suppression of 5-th force without fine-tuning.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_15

1 Introduction—Non-Riemannian Volume-Form Formalism

Extended (modified) gravity theories as alternatives/generalizations of the standard Einstein General Relativity (for detailed accounts, see Refs. [1–4]) are being widely studied in the last decade or so due to pressing motivation from cosmology (problems of dark energy and dark matter), quantum field theory in curved spacetime (renormalization in higher loops) and string theory (low-energy effective field theories).

A broad class of actively developed modified/extended gravitational theories is based on employing alternative non-Riemannian spacetime volume-forms (metric-independent generally covariant volume elements) in the pertinent Lagrangian actions instead of the canonical Riemannian one given by the square-root of the determinant of the Riemannian metric (originally proposed in [5, 6], for a concise geometric formulation, see [7, 8]). A characteristic feature of these extended gravitational theories is that when starting in the first-order (Palatini) formalism the non-Riemannian volume-forms are almost *pure-gauge* degrees of freedom, *i.e.* they *do not* introduce any additional propagating gravitational degrees of freedom except for few discrete degrees of freedom appearing as arbitrary integration constants (for a canonical Hamiltonian treatment, see Appendices A in Refs. [8, 9]).

Let us recall that volume-forms in integrals over differentiable manifolds (not necessarily Riemannian one, so *no* metric is needed) are given by nonsingular maximal rank differential forms ω:

$$\int_{\mathcal{M}} \omega(\ldots) = \int_{\mathcal{M}} dx^D\, \Omega(\ldots),$$
$$\omega = \frac{1}{D!}\omega_{\mu_1\ldots\mu_D} dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_D} \quad , \quad \omega_{\mu_1\ldots\mu_D} = -\varepsilon_{\mu_1\ldots\mu_D}\Omega, \tag{1}$$

(our conventions for the alternating symbols $\varepsilon^{\mu_1,\ldots,\mu_D}$ and $\varepsilon_{\mu_1,\ldots,\mu_D}$ are: $\varepsilon^{01\ldots D-1} = 1$ and $\varepsilon_{01\ldots D-1} = -1$). The volume element density (integration measure density) Ω transforms as scalar density under general coordinate reparametrizations.

In standard generally-covariant theories (with action $S = \int d^D x\sqrt{-g}\mathcal{L}$) the Riemannian spacetime volume-form is defined through the "D-bein" (frame-bundle) canonical one-forms $e^A = e^A_\mu dx^\mu$ ($A = 0,\ldots,D-1$):

$$\omega = e^0\wedge\ldots\wedge e^{D-1} = \det\|e^A_\mu\|\, dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_D}$$
$$\longrightarrow \quad \Omega = \det\|e^A_\mu\|\, d^D x = \sqrt{-\det\|g_{\mu\nu}\|}\, d^D x. \tag{2}$$

Instead of $\sqrt{-g}d^D x$ we can employ another alternative *non-Riemannian* volume element as in (1) given by a non-singular *exact* D-form $\omega = dB$ where:

$$B = \frac{1}{(D-1)!} B_{\mu_1 \ldots \mu_{D-1}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{-1}}$$
$$\longrightarrow \quad \Omega \equiv \Phi(B) = \frac{1}{(D-1)!} \varepsilon^{\mu_1 \ldots \mu_D} \partial_{\mu_1} B_{\mu_2 \ldots \mu_D}. \tag{3}$$

In other words, the non-Riemannian volume element density is defined in terms of the dual field-strength scalar density of an auxiliary rank $D-1$ tensor gauge field $B_{\mu_1 \ldots \mu_{D-1}}$.

The plan of exposition is as follows. In Sect. 2 we describe in some detail the construction and the main properties of extended gravity models, based on the formalism of non-Riemannian volume elements, coupled to a scalar "inflaton" field driving the cosmological evolution and a second scalar "darkon" field responsible for the unification of dark energy and dark matter, as well as coupled to the bosonic sector of the standard electroweak particle model, thus exhibiting a gravity-assisted dynamical generation of the Higgs electroweak spontaneous symmetry breaking in the post-inflationary universe. In particular, we find an "emergent-universe" cosmological solution without Big-Bang singularity (on classical level).

Further, in Sect. 3 we briefly present an alternative mechanism of dark energy - dark matter unification with diffusive interaction among them based on the formalism of "dynamical spacetime" [10, 11]. Section 4 provides a short discussion of the principal new features which arise upon inclusion of fermionic fields in modified gravity models based on the formalism of non-canonical spacetime volume elements as well as on the requirement of global scale invariance, first of all—a plausible solution of the problem of "fifth force" without fine-tuning [12, 13]. The last Section contains our conclusions.

2 Modified Gravity-Matter Models with Non-Riemannian Volume-Forms—Cosmological Implications

To illustrate the main interesting properties of the new class of extended gravity-matter models based on the non-Riemannian volume-form formalism we will consider modified gravity in the Palatini formalism coupled in a non-standard way via non-Riemannian volume elements to [9, 14, 15]: (i) scalar "inflaton" field φ; (ii) a second scalar "darkon" field u; (iii) the bosonic fields of the standard electroweak particle model – $\sigma \equiv (\sigma_a)$ being a complex $SU(2) \times U(1)$ iso-doublet Higgs-like scalar, and the $SU(2) \times U(1)$ gauge fields $\mathcal{A}_\mu, \mathcal{B}_\mu$.

The "inflaton" φ apart from driving the cosmological evolution triggers suppression, respectively, generation of the electroweak (Higgs) spontaneous symmetry breaking in the "early", respectively, in the "late" universe. The "darkon" u is responsible for the unified description of dark energy and dark matter in the "late" universe.

The corresponding action reads (for simplicity we use units with the Newton constant $G_N = 1/16\pi$):

$$
\begin{aligned}
S = & \int d^4x\, \Phi_1(A)\Big[R + L^{(1)}(\varphi,\sigma)\Big] \\
& + \int d^4x\, \Phi_2(B)\Big[L^{(2)}(\varphi,\boldsymbol{\mathcal{A}},\mathcal{B}) + \frac{\Phi_4(H)}{\sqrt{-g}}\Big] \\
& - \int d^4x\, \big(\sqrt{-g} + \Phi_3(C)\big)\frac{1}{2}g^{\mu\nu}\partial_\mu u \partial_\nu u.
\end{aligned} \tag{4}
$$

Here the following notations are used:

(i) $\Phi_1(A)$, $\Phi_2(B)$, $\Phi_3(C)$ are three independent non-Riemannian volume elements as in (3) for $D = 4$; $\Phi_4(H)$ is again of the form (3) for $D = 4$ and it is needed for consistency of (4).
(ii) The scalar curvature R in Palatini formalism is $R = g^{\mu\nu}R_{\mu\nu}(\Gamma)$, where the Ricci tensor is a function of the affine connection $\Gamma^\lambda_{\mu\nu}$ *a priori* independent of $g_{\mu\nu}$.
(iii) The matter field Lagrangians are:

$$
\begin{aligned}
L^{(1)}(\varphi,\sigma) \equiv & -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - f_1 e^{-\alpha\varphi} \\
& -g^{\mu\nu}(\nabla_\mu\sigma)^*_a\nabla_\nu\sigma_a - \frac{\lambda}{4}\left((\sigma_a)^*\sigma_a - \mu^2\right)^2,
\end{aligned} \tag{5}
$$

$$
L^{(2)}(\varphi,\boldsymbol{\mathcal{A}},\mathcal{B}) = -\frac{b}{2}e^{-\alpha\varphi}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + f_2 e^{-2\alpha\varphi} - \frac{1}{4g^2}F^2(\boldsymbol{\mathcal{A}}) - \frac{1}{4g'^2}F^2(\mathcal{B}), \tag{6}
$$

where α, f_1, f_2 are dimensionful positive parameters, whereas b is a dimensionless one (b is needed to obtain a stable "emergent" universe solution, see below (25). $F^2(\boldsymbol{\mathcal{A}})$ and $F^2(\mathcal{B})$ in (6) are the squares of the field-strengths of the electroweak gauge fields, and the last term in (5) is of the same form as the standard Higgs potential.

Let us note that the form of the "inflaton" part of the action (4) is fixed by the requirement of invariance under global Weyl-scale transformations:

$$
\begin{aligned}
& g_{\mu\nu} \to \lambda g_{\mu\nu},\ \Gamma^\mu_{\nu\lambda} \to \Gamma^\mu_{\nu\lambda},\ \varphi \to \varphi + \frac{1}{\alpha}\ln\lambda, \\
& A_{\mu\nu\kappa} \to \lambda A_{\mu\nu\kappa},\ B_{\mu\nu\kappa} \to \lambda^2 B_{\mu\nu\kappa},\ H_{\mu\nu\kappa} \to H_{\mu\nu\kappa}.
\end{aligned} \tag{7}
$$

Scale invariance played an important role in the original papers on the non-canonical volume-form formalism where also fermions were included [6] (see also Sect. 3 below).

The equations of motion of the initial action (4) w.r.t. auxiliary tensor gauge fields $A_{\mu\nu\lambda}$, $B_{\mu\nu\lambda}$, $C_{\mu\nu\lambda}$ and $H_{\mu\nu\lambda}$ yield the following algebraic constraints:

$$
R + L^{(1)} = M_1 = \text{const},\ L^{(2)} + \frac{\Phi_4(H)}{\sqrt{-g}} = -M_2 = \text{const},
$$

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu u\partial_\nu u = M_0 = \text{ const}, \;\; \frac{\Phi_2(B)}{\sqrt{-g}} \equiv \chi_2 = \text{ const}, \tag{8}$$

where M_0, M_1, M_2 are arbitrary dimensionful and χ_2 an arbitrary dimensionless *integration constants*.

The equations of motion of (4) w.r.t. affine connection $\Gamma^\mu_{\nu\lambda}$ yield a solution for $\Gamma^\mu_{\nu\lambda}$ as a Levi-Civita connection $\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda}(\bar{g}) = \frac{1}{2}\bar{g}^{\mu\kappa}\left(\partial_\nu \bar{g}_{\lambda\kappa} + \partial_\lambda \bar{g}_{\nu\kappa} - \partial_\kappa \bar{g}_{\nu\lambda}\right)$ w.r.t. to the a *Weyl-rescaled metric* $\bar{g}_{\mu\nu} = \chi_1 g_{\mu\nu}, \;\; \chi_1 \equiv \frac{\Phi_1(A)}{\sqrt{-g}}$.

The passage to the "Einstein-frame" (EF) is accomplished by a Weyl-conformal transformation to $\bar{g}_{\mu\nu}$ upon using relations (8), so that the EF action with a canonical Hilbert-Einstein gravity part w.r.t. $\bar{g}_{\mu\nu}$ and with the canonical Riemannian volume element density $\sqrt{\det\|-\bar{g}_{\mu\nu}\|}$ reads:

$$S_{\text{EF}} = \int d^4x\sqrt{-\bar{g}}\Big[R(\bar{g}) + L_{\text{EF}}\Big], \tag{9}$$

and where the EF matter Lagrangian turns out to be of a quadratic "k-essence" type [16–19] w.r.t. both the "inflaton" φ and "darkon" u fields:

$$L_{\text{EF}} = \bar{X} - \bar{Y}\Big[f_1 e^{-\alpha\varphi} + \frac{\lambda}{4}\left((\sigma_a)^*\sigma_a - \mu^2\right)^2 + M_1 - \chi_2 b e^{-\alpha\varphi}\bar{X}\Big]$$
$$+\bar{Y}^2\Big[\chi_2(f_2 e^{-2\alpha\varphi} + M_2) + M_0\Big] + L[\sigma, \boldsymbol{\mathcal{A}}, \mathcal{B}], \tag{10}$$

with $L[\sigma, \boldsymbol{\mathcal{A}}, \mathcal{B}] \equiv -\bar{g}^{\mu\nu}(\nabla_\mu\sigma_a)^*\nabla_\nu\sigma_a - \frac{\chi_2}{4g^2}\bar{F}^2(\boldsymbol{\mathcal{A}}) - \frac{\chi_2}{4g'^2}\bar{F}^2(\mathcal{B})$. In (10) all quantities defined in terms of the EF metric $\bar{g}_{\mu\nu}$ are indicated by an upper bar, and the following short-hand notations are used: $\bar{X} \equiv -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi, \;\; \bar{Y} \equiv -\frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu u\partial_\nu u$.

From (10) we deduce the following full effective scalar potential:

$$U_{\text{eff}}\big(\varphi, \sigma\big) = \frac{\Big(f_1 e^{-\alpha\varphi} + \frac{\lambda}{4}\left((\sigma_a)^*\sigma_a - \mu^2\right)^2 + M_1\Big)^2}{4\big[\chi_2(f_2 e^{-2\alpha\varphi} + M_2) + M_0\big]} \tag{11}$$

As discussed in Refs.[14, 15] $U_{\text{eff}}(\varphi, \sigma)$ (11) has few remarkable properties. First, $U_{\text{eff}}(\varphi, \sigma)$ possesses two infinitely large flat regions as function of φ when σ is fixed:

(a) (−) flat "inflaton" region for large negative values of φ corresponding to the evolution of the "early" universe;
(b) (+) flat "inflaton" region for large positive values of φ with σ fixed corresponding to the evolution of the "late" universe".

This is graphically depicted on Fig.1.

In the (−) flat "inflaton" region, *i.e.*, in the "early" universe the effective scalar field potential (11) reduces to (an approximately) constant value

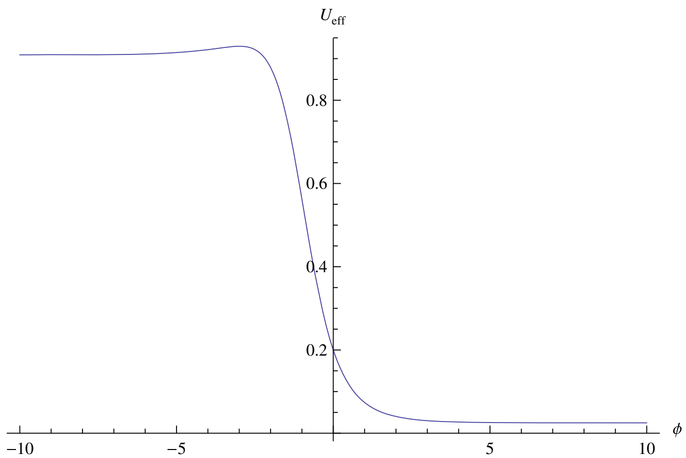

Fig. 1 Qualitative shape of the effective scalar potential $U_{\rm eff}$ (11) as function of the "inflaton" φ for $M_1 > 0$ and fixed Higgs-like σ

$$U_{\rm eff}\big(\varphi, \sigma\big) \simeq U_{(-)} = \frac{f_1^2}{4\chi_2\, f_2} \tag{12}$$

Thus, there is no σ-field potential and, therefore, *no electroweak spontaneous breakdown in the "early" universe*.

On the other hand, in the (+) flat "inflaton" region, *i.e.*, in the "late" universe the effective scalar field potential becomes:

$$U_{\rm eff}\big(\varphi, \sigma\big) \simeq U_{(+)}(\sigma) = \frac{\Big(\frac{\lambda}{4}\left((\sigma_a)^*\sigma_a - \mu^2\right)^2 + M_1\Big)^2}{4\big(\chi_2 M_2 + M_0\big)}, \tag{13}$$

which obviously yields *nontrivial vacuum for the Higgs-like field* $|\sigma_{\rm vac}| = \mu$. Therefore, in the "late" universe we have the standard spontaneous breakdown of electroweak $SU(2) \times U(1)$ gauge symmetry. Moreover, at the Higgs vacuum we obtain from (13) a dynamically generated cosmological constant $\Lambda_{(+)}$ of the "late" Universe:

$$U_{(+)}(\mu) \equiv 2\Lambda_{(+)} = \frac{M_1^2}{4\big(\chi_2 M_2 + M_0\big)}. \tag{14}$$

If we identify the integration constants with the fundamental scales in Nature as $M_{0,1} \sim M_{EW}^4$ and $M_2 \sim M_{Pl}^4$, where $M_{\rm Pl}$ is the Planck mass scale and $M_{\rm EW} \sim 10^{-16} M_{\rm Pl}$ is the electroweak mass scale, then $\Lambda_{(+)} \sim M_{EW}^8/M_{Pl}^4 \sim 10^{-120} M_{Pl}^4$, which is the right order of magnitude for the present epoch's vacuum energy density as already realized in [20].

On the other hand, if we take the order of magnitude of the coupling constants in the effective potential (11) $f_1 \sim f_2 \sim (10^{-2} M_{Pl})^4$, then the order of magnitude of the vacuum energy density of the "early" universe (12) becomes:

$$U_{(-)} \sim f_1^2/f_2 \sim 10^{-8} M_{Pl}^4, \tag{15}$$

which conforms to the Planck Collaboration data [21, 22] implying the energy scale of inflation of order $10^{-2} M_{Pl}$.

Now, let us perform FLRW reduction of the EF action (9). *i.e.*, restricting the metric $\bar{g}_{\mu\nu}$ to the FLRW form $ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\boldsymbol{x}^2$. Thus we obtain in the "late" universe, *i.e.*, for large positive "inflaton" φ values the following results for the density, pressure, the Friedmann scale factor (the solution for $a(t)$ below first appeared in [23]) and the "inflaton" velocity:

$$\rho = \frac{M_1^2}{4(\chi_2 M_2 + M_0)} + \frac{\pi_u}{a^3} \Big[\frac{M_1}{\chi_2 M_2 + M_0}\Big]^{\frac{1}{2}} + \mathrm{O}(\frac{\pi_u^2}{a^6}), \tag{16}$$

$$p = -\frac{M_1^2}{4(\chi_2 M_2 + M_0)} + \mathrm{O}(\frac{\pi_u^2}{a^6}), \tag{17}$$

$$a(t) \simeq \Big(\frac{C_0}{2\Lambda_{(+)}}\Big)^{1/3} \sinh^{2/3}\Big(\sqrt{\frac{3}{4}\Lambda_{(+)}}\, t\Big), \tag{18}$$

$$\dot{\varphi} \simeq \text{const} \sinh^{-2}\Big(\sqrt{\frac{3}{4}\Lambda_{(+)}}\, t\Big), \tag{19}$$

where π_u is the conserved "darkon" canonical momentum, $\Lambda_{(+)}$ is as in (14) and $C_0 \equiv \pi_u \sqrt{M_1(\chi_2 M_2 + M_0)^{-1}}$.

Relations (16)–(17) straightforwardly show that in the "late" universe we have explicit unification of dark energy (given by the dynamically generated cosmological constant (14) – first constant terms on the r.h.sides in (16) and (17), and dark matter given as a "dust" fluid contribution – second term $\mathrm{O}(a^{-3})$ on the r.h.s. of (16).

A further interesting property under consideration is the existence of a stable "emergent" universe solution – a creation without Big Bang (cf. Refs. [25, 26]). It is characterized by the condition on the Hubble parameter H:

$$H = 0 \quad \to \quad a(t) = a_0 = \text{const}, \quad \rho + 3p = 0, \\ \frac{K}{a_0^2} = \frac{1}{6}\rho \;(=\text{const}), \tag{20}$$

and the "inflaton" is on the $(-)$ flat region (large negative values of φ). Then relations (20) together with the "inflaton" and "darkon" equations of motion imply that also "inflaton" velocity $\dot{\varphi} =$ const and the constant density and pressure read:

$$\rho \simeq -\frac{3\chi_2 b^2}{16 f_2}\,\dot{\varphi}^4\,\frac{1}{2}\,\dot{\varphi}^2\,\Bigl(1+\frac{bf_1}{2f_2}\Bigr)+\frac{f_1^2}{4\chi_2 f_2}, \tag{21}$$

$$p \simeq -\frac{\chi_2 b^2}{16 f_2}\,\dot{\varphi}^4\,\frac{1}{2}\,\dot{\varphi}^2\,\Bigl(1+\frac{bf_1}{2f_2}\Bigr)-\frac{f_1^2}{4\chi_2 f_2}. \tag{22}$$

The truncated Friedmann Eqs. (20) yield exact solutions for the constant "inflaton" velocity $\dot{\varphi}_0$ and Friedmann factor a_0:

$$\dot{\varphi}_0^2 = \frac{8 f_2}{3\chi_2 b^2}\Bigl[1+\frac{bf_1}{2f_2}-\sqrt{(1+\frac{bf_1}{2f_2})^2-\frac{3b^2 f_1^2}{16 f_2^2}}\Bigr], \tag{23}$$

and $a_0^2 = 6K/\rho_0$ where:

$$\rho_0 = \frac{f_1^2}{2\chi_2 f_2}-\frac{1}{2}\,\dot{\varphi}_0^2\,\Bigl(1+\frac{bf_1}{2f_2}\Bigr). \tag{24}$$

Studying perturbation $a \to a + \delta a(t)$ of the "emergent" universe condition (20) we obtain a harmonic oscillator equation for $\delta a(t)$ (here $\dot{\varphi}_0^2$ as in (23), and ρ_0 as in (24)):

$$\delta\,\ddot{a} + \omega^2 \delta a = 0,$$
$$\omega^2 \equiv \frac{\rho_0}{6}\Bigl[3\frac{\frac{1}{2}(1+bf_1/2f_2)-\dot{\varphi}_0^2\,\chi_2 b^2/8f_2}{\dot{\varphi}_0^2\,3\chi_2 b^2/8f_2-\frac{1}{2}(1+bf_1/2f_2)}-1\Bigr] > 0 \tag{25}$$

for $-8(1-\frac{1}{2}\sqrt{3})\frac{f_2}{f_1} < b < -\frac{f_2}{f_1}$.

The non-Riemannian volume-form formalism was also successfully applied to propose an qualitatively new mechanism for a *dynamical spontaneous breaking of supersymmetry* in supergravity by constructing modified formulation of standard minimal $N = 1$ supergravity as well as of anti-de Sitter supergravity in terms of non-Riemannian volume elements [7, 24]. This naturally triggers the appearance of a *dynamically generated cosmological constant* as an arbitrary integration constant which signifies *dynamical spontaneous supersymmetry breakdown*. The same formalism applied to anti-de Sitter supergravity allows us to appropriately choose the above mentioned arbitrary integration constant so as to obtain simultaneously a *very small effective observable cosmological constant* as well as a *large physical gravitino mass* as required by modern cosmological scenarios for slowly expanding universe of the present epoch [27–29].

3 Dynamical Spacetime Formulation

Let us now observe that the non-Riemannian volume element density $\Omega = \Phi(B)$ (3) on a Riemannian manifold can be rewritten using Hodge duality (here $D = 4$) in terms of a vector field $\chi^\mu = \frac{1}{3!}\frac{1}{\sqrt{-g}}\varepsilon^{\mu\nu\kappa\lambda}B_{\nu\kappa\lambda}$ so that Ω becomes $\Omega(\chi) = \partial_\mu\big(\sqrt{-g}\chi^\mu\big)$, *i.e.* it is a non-canonical volume element density different from $\sqrt{-g}$, but involving the metric. It can be represented alternatively through a Lagrangian multiplier action term yielding covariant conservation of a specific energy-momentum tensor of the form $\mathcal{T}^{\mu\nu} = g^{\mu\nu}\mathcal{L}$:

$$\mathcal{S}_{(\chi)} = \int d^4x\sqrt{-g}\,\chi_{\mu;\nu}\mathcal{T}^{\mu\nu} = \int d^4x\,\partial_\mu\big(\sqrt{-g}\chi^\mu\big)\big(-\mathcal{L}\big), \tag{26}$$

where $\chi_{\mu;\nu} = \partial_\nu\chi_\mu - \Gamma^\lambda_{\mu\nu}\chi_\lambda$.

The vector field χ_μ is called *"dynamical space time vector"*, because the energy density of $\mathcal{T}^{00}$ is a canonically conjugated momentum w.r.t. χ_0, which is what we expected from a dynamical time.

In what follows we will briefly consider a new class of gravity-matter theories based on the ordinary Riemannian volume element density $\sqrt{-g}$ but involving action terms of the form (26) where now $\mathcal{T}^{\mu\nu}$ is of more general form than $\mathcal{T}^{\mu\nu} = g^{\mu\nu}\mathcal{L}$. This new formalism is called *"dynamical spacetime formalism"* [10, 11] due to the above remark on χ_0.

Ref. [30] describes a unification between dark energy and dark matter by introducing a quintessential scalar field in addition to the dynamical time action. The total Lagrangian reads:

$$\mathcal{L} = \frac{1}{2}R + \chi_{\mu;\nu}\mathcal{T}^{\mu\nu} - \frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi), \tag{27}$$

with energy-momentum tensor $\mathcal{T}^{\mu\nu} = -\frac{1}{2}\phi^{,\mu}\phi^{,\nu}$. From the variation of the Lagrangian term $\chi_{\mu;\nu}\mathcal{T}^{\mu\nu}$ with respect to the vector field χ_μ, the covariant conservation of the energy-momentum tensor $\nabla_\mu\mathcal{T}^{\mu\nu} = 0$ is implemented. The latter within the FLRW framework forces the kinetic term of the scalar field to behave as a dark matter component:

$$\nabla_\mu\mathcal{T}^{\mu\nu} = 0 \quad \Rightarrow \quad \dot{\phi}^2 = \frac{2\Omega_{m0}}{a^3}. \tag{28}$$

where Ω_{m0} is an integration constant. The variation with respect to the scalar field ϕ yields a current:

$$-V'(\phi) = \nabla_\mu j^\mu, \quad j^\mu = \frac{1}{2}\phi_{,\nu}(\chi^{\mu;\nu} + \chi^{\nu;\mu}) + \phi^{,\mu} \tag{29}$$

For constant potential $V(\phi) = \Omega_\Lambda = const$ the current is covariantly conserved.

In the FLRW setting, where the dynamical time ansatz introduces only a time component $\chi_\mu = (\chi_0, 0, 0, 0)$, the variation (29) gives:

$$\dot{\chi}_0 - 1 = \xi\, a^{-3/2}, \tag{30}$$

where ξ is an integration constant. Accordingly, the FLRW energy density and pressure read:

$$\rho = \left(\dot{\chi}_0 - \frac{1}{2}\right)\dot{\phi}^2 + V, \quad p = \frac{1}{2}\dot{\phi}^2(\dot{\chi}_0 - 1) - V. \tag{31}$$

Plugging the relations (28,30) into the density and the pressure terms (31) yields the following simple form of the latter:

$$\rho = \Omega_\Lambda + \frac{\xi\,\Omega_{m0}}{a^{9/2}} + \frac{\Omega_{m0}}{a^3}, \quad p = -\Omega_\Lambda + \frac{\xi\,\Omega_{m0}}{2\,a^{9/2}}. \tag{32}$$

In (32) there are 3 components for the "dark fluid": dark energy with $\omega_\Lambda = -1$, dark matter with $\omega_m = 0$ and an additional equation of state $\omega_\xi = 1/2$. For non-vanishing and negative ξ the additional part introduces a minimal scale parameter, which avoids singularities. If the dynamical time is equivalent to the cosmic time $\chi_0 = t$, we obtain $\xi = 0$ from Eq. (30), whereupon the density and the pressure terms (32) coincide with those from the ΛCDM model precisely. The additional part (for $\xi \neq 0$) fits more to the late time accelerated expansion data, as observed in Ref. [31].

Ref. [32] shows that with higher dimensions, the solution derived from the Lagrangian (27) describes inflation, where the total volume oscillates and the original scale parameter exponentially *grows*.

The dynamical spacetime Lagrangian can be generalized to yield a *diffusive energy-momentum tensor*. Ref. [33] shows that the diffusion equation has the form:

$$\nabla_\mu \mathcal{T}^{\mu\nu} = 3\sigma j^\nu, \quad j^\mu_{;\mu} = 0, \tag{33}$$

where σ is the diffusion coefficient and j^μ is a current source. The covariant conservation of the current source indicates the conservation of the number of the particles. By introducing the vector field χ_μ in a different part of the Lagrangian:

$$\mathcal{L}_{(\chi,A)} = \chi_{\mu;\nu}\mathcal{T}^{\mu\nu} + \frac{\sigma}{2}(\chi_\mu + \partial_\mu A)^2, \tag{34}$$

the energy-momentum tensor $\mathcal{T}^{\mu\nu}$ *gets a diffusive source*. From a variation with respect to the dynamical space time vector field χ_μ we obtain:

$$\nabla_\nu \mathcal{T}^{\mu\nu} = \sigma(\chi^\mu + \partial^\mu A) = f^\mu, \tag{35}$$

a current source $f^\mu = \sigma(\chi^\mu + \partial^\mu A)$ for the energy-momentum tensor. From the variation with respect to the new scalar A, a covariant conservation of the current emerges $f^\mu_{;\mu} = 0$. *The latter relations correspond to the diffusion equation* (33).

Refs. [35–38] study the cosmological solution using the energy-momentum tensor $\mathcal{T}^{\mu\nu} = -\frac{1}{2}g^{\mu\nu}\phi^{,\lambda}\phi_{\lambda}$. The total Lagrangian reads:

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi) + \chi_{\mu;\nu}\mathcal{T}^{\mu\nu} + \frac{\sigma}{2}(\chi_\mu + \partial_\mu A)^2. \tag{36}$$

The FLRW solution unifies the dark energy and the dark matter originating from one scalar field with possible diffusion interaction. Ref. [34] investigates more general energy-momentum tensor combinations and shows that asymptotically all of the combinations yield ΛCDM model as a stable fixed point.

4 Scale Invariance, Fifth Force and Fermionic Matter

The originally proposed theory with two volume element densities (integration measure densities) [5, 6], where at least one of them was a non-canonical one and short-termed “two-measure theory” (TMT), has a number of remarkable properties if fermions are included in a self-consistent way [6]. In this case, the constraint that arises in the TMT models in the Palatini formalism can be represented as an equation for $\chi \equiv \Phi/\sqrt{-g}$, in which the left side has an order of the vacuum energy density, and the right side (in the case of non-relativistic fermions) is proportional to the fermion density. Moreover, it turns out that even cold fermions have a (non-canonical) pressure P_f^{noncan} and the corresponding contribution to the energy-momentum tensor has the structure of a cosmological constant term which is proportional to the fermion density. The remarkable fact is that the right hand side of the constraint coincide with P_f^{noncan}. This allows us to construct a cosmological model [39] of the late universe in which dark energy is generated by a gas of non-relativistic neutrinos without the need to introduce into the model a specially designed scalar field.

In models with a scalar field, the requirement of scale invariance of the initial action [5] plays a very constructive role. It allows to construct a model [40] where without fine tuning we have realized: absence of initial singularity of the curvature; k-essence; inflation with graceful exit to zero cosmological constant.

Of particular interest are scale invariant models in which both fermions and a dilaton scalar field ϕ are present. Then it turns out that the Yukawa coupling of fermions to ϕ is proportional to P_f^{noncan}. As a result, it follows from the constraint, that in all cases when fermions are in states which constitute a regular barionic matter, the Yukawa coupling of fermions to dilaton has an order of ratio of the vacuum energy density to the fermion energy density [12]. Thus, the theory provides a solution of the 5-th force problem without any fine tuning or a special design of the model. Besides, in the described states, the regular Einstein’s equations are reproduced. In the opposite case, when fermions are very diluted, e.g. in the model of the late Universe filled with a cold neutrino gas, the neutrino dark energy appears in such a way that the dilaton ϕ dynamics is closely correlated with that of the neutrino gas [12].

A scale invariant model containing a dilaton ϕ and dust (as a model of matter) [13] possesses similar features. The dilaton to matter coupling “constant” f appears to be dependent of the matter density. In normal conditions, i.e. when the matter energy density is many orders of magnitude larger than the dilaton contribution to the dark energy density, f becomes less than the ratio of the “mass of the vacuum” in the volume occupied by the matter to the Planck mass. The model yields this kind of “Archimedes law” without any special (intended for this) choice of the underlying action and without fine tuning of the parameters. The model not only explains why all attempts to discover a scalar force correction to Newtonian gravity were unsuccessful so far but also predicts that in the near future there is no chance to detect such corrections in the astronomical measurements as well as in the specially designed fifth force experiments on intermediate, short (like millimeter) and even ultrashort (a few nanometer) ranges. This prediction is alternative to predictions of other known models.

More recently other authors have rediscovered the important role of scale invariance in the avoidance of a 5-th force [44]. We should point out that our original work [12, 13] on avoidance of the 5-th force through scale invariance symmetry precedes that of Ref. [44] by a substantial number of years.

5 Conclusions

In the present paper we describe in some details the principal physically interesting features of a specific class on extended (modified) gravitational theories beyond the standard Einstein’s general relativity. They are constructed in terms of non-Riemannian spacetime volume forms (metric-independent non-canonical volume elements). An important role is also being played by the requirement of global scale invariance. We present a modified gravity-matter model where gravity is coupled in a non-canonical way to two scalar fields (“inflaton” and “darkon”) as well as to the bosonic sector of the standard electroweak model of elementary particle physics. The “inflaton” scalar field triggers a quintessential inflationary evolution of the Universe where all energy scales are determined dynamically through free integration constants arising due to the modified gravitational dynamics because of the non-Riemannian volume elements. The “darkon” scalar field on its part creates through its dynamics a unified description of dark energy and dark matter. A particularly notable feature is the gravity-“inflaton”-assisted dynamical generation of Higgs electroweak spontaneous symmetry breaking in the post-inflationary epoch and its suppression in the early-universe stage. Under special initial condition on the Hubble parameter we find (on classical level) an “emergent universe” solution describing early universe evolution without spacetime singularities (no “Big Bang”).

Furthermore, we have briefly discussed a parallel alternative non-canonical spacetime volume element approach based on the concept of “dynamical spacetime” and have demonstrated the appearance of unified description of dark energy and dark matter with a diffusive interaction among them. Finally we briefly outlined, based on our original work [12, 13], how the formalism of non-canonical volume elements in modified gravity-matter models with fermions provides a resolution of the problem of "fifth force" without any fine tunings.

In the above constructions we have employed the first-order (Palatini) formalism in the initial gravity actions. Further physically interesting features are obtained when combining the non-Riemannian spacetime volume element formalism with the second order (metric) gravity formalism. In particular, in the latter case it was recently shown [41] that starting with a pure modified gravity in terms of several non-Riemannian volume elements and without any initial matter fields one creates dynamically (in the "Einstein frame") a canonical scalar field with a non-trivial inflationary potential generalizing the classical Starobinsky potential [42] and yielding results for the cosmological observables (scalar power spectral index and the tensor-to-scalar ratio) fitting very well to the available observational data [43].

Acknowledgements We gratefully acknowledge support of our collaboration through the academic exchange agreement between the Ben-Gurion University in Beer-Sheva, Israel, and the Bulgarian Academy of Sciences. E.N. and E.G. have received partial support from European COST actions MP-1405, CA-16104, CA-18108 and from CA-15117, CA-16104, CA-18108 respectively. E.N. and S.P. are also thankful to Bulgarian National Science Fund for support via research grant DN-18/1.

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Non-linear Symmetries in Maxwell-Einstein Gravity: From Freudenthal Duality to Pre-homogeneous Vector Spaces

Alessio Marrani

Abstract We review the relation between Freudenthal duality and U-duality Lie groups of type E_7 in extended supergravity theories, as well as the relation between the Hessian of the black hole entropy and the pseudo-Euclidean, rigid special (pseudo)Kähler metric of the pre-homogeneous spaces associated to the U-orbits.

1 Freudenthal Duality

We start and consider the following Lagrangian density in four dimensions, (*cf.*, *e.g.*, [1]):

$$\begin{aligned}\mathcal{L} = & -\frac{R}{2} + \frac{1}{2} g_{ij}(\varphi)\, \partial_\mu \varphi^i \partial^\mu \varphi^j + \frac{1}{4} I_{\Lambda\Sigma}(\varphi)\, F^\Lambda_{\mu\nu} F^{\Sigma|\mu\nu} \\ & + \frac{1}{8\sqrt{-G}} R_{\Lambda\Sigma}(\varphi)\, \epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma},\end{aligned} \tag{1}$$

describing Einstein gravity coupled to Maxwell (Abelian) vector fields and to a non-linear sigma model of scalar fields (with no potential); note that $\mathcal{L}$ may -but does not necessarily need to - be conceived as the bosonic sector of $D = 4$ (*ungauged*) supergravity theory. Out of the Abelian two-form field strengths F^Λ's, one can define their duals G_Λ, and construct a symplectic vector:

$$H := \left(F^\Lambda, G_\Lambda\right)^T, \quad {}^*G_{\Lambda|\mu\nu} := 2\frac{\delta\mathcal{L}}{\delta F^{\Lambda|\mu\nu}}. \tag{2}$$

We then consider the simplest solution of the equations of motion deriving from $\mathcal{L}$, namely a static, spherically symmetric, asymptotically flat, dyonic extremal black hole with metric [2]

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_16

$$ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)}\left[\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}\left(d\theta^2 + \sin\theta d\psi^2\right)\right], \tag{3}$$

where $\tau := -1/r$. Thus, the two-form field strengths and their duals can be fluxed on the two-sphere at infinity S^2_∞ in such a background, respectively yielding the electric and magnetic charges of the black hole itself, which can be arranged in a symplectic vector $\mathcal{Q}$:

$$p^\Lambda := \frac{1}{4\pi}\int_{S^2_\infty} F^\Lambda, \quad q_\Lambda := \frac{1}{4\pi}\int_{S^2_\infty} G_\Lambda, \tag{4}$$

$$\mathcal{Q} := \left(p^\Lambda, q_\Lambda\right)^T. \tag{5}$$

Then, by exploiting the symmetries of the background (3), the Lagrangian (6) can be dimensionally reduced from $D = 4$ to $D = 1$, obtaining a 1-dimensional effective Lagrangian ($' := d/d\tau$) [3]:

$$\mathcal{L}_{D=1} = \left(U'\right)^2 + g_{ij}(\varphi)\varphi^{i\prime}\varphi^{j\prime} + e^{2U}V_{BH}(\varphi, \mathcal{Q}) \tag{6}$$

along with the Hamiltonian constraint [3]

$$\left(U'\right)^2 + g_{ij}(\varphi)\varphi^{i\prime}\varphi^{j\prime} - e^{2U}V_{BH}(\varphi, \mathcal{Q}) = 0. \tag{7}$$

The so-called "effective black hole potential" V_{BH} appearing in (6) and (7) is defined as [3]

$$V_{BH}(\varphi, \mathcal{Q}) := -\frac{1}{2}\mathcal{Q}^T\mathcal{M}(\varphi)\mathcal{Q}, \tag{8}$$

in terms of the symplectic and symmetric matrix [1]

$$\mathcal{M} := \begin{pmatrix} \mathbf{I} & -R \\ 0 & \mathbf{I} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & I^{-1} \end{pmatrix}\begin{pmatrix} \mathbf{I} & 0 \\ -R & \mathbf{I} \end{pmatrix} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}, \tag{9}$$

$$\mathcal{M}^T = \mathcal{M}; \ \mathcal{M}\Omega\mathcal{M} = \Omega, \tag{10}$$

where $\mathbf{I}$ denotes the identity, and $R(\varphi)$ and $I(\varphi)$ are the scalar-dependent matrices occurring in (6); moreover, Ω stands for the symplectic metric ($\Omega^2 = -\mathbf{I}$). Note that, regardless of the invertibility of $R(\varphi)$ and as a consequence of the physical consistence of the kinetic vector matrix $I(\varphi)$, $\mathcal{M}$ is negative-definite; thus, the effective black hole potential (8) is positive-definite.

By virtue of the matrix $\mathcal{M}$, one can introduce a (scalar-dependent) *anti-involution* $\mathcal{S}$ in any Maxwell-Einstein-scalar theory described by (6) with a symplectic structure Ω, as follows:

$$\mathcal{S}(\varphi) := \Omega\mathcal{M}(\varphi). \tag{11}$$

Indeed, by (10),

$$\mathcal{S}^2(\varphi) = \Omega \mathcal{M}(\varphi)\, \Omega \mathcal{M}(\varphi) = \Omega^2 = -\mathbf{I}. \tag{12}$$

In turn, this allows to define an anti-involution on the dyonic charge vector $\mathcal{Q}$, which has been called (scalar-dependent) *Freudenthal duality* [4–6]:

$$\mathbf{F}(\mathcal{Q};\varphi) := -\mathcal{S}(\varphi)\,\mathcal{Q}; \tag{13}$$

$$\mathbf{F^2} = -\mathbf{I},\ (\forall\,\{\varphi\}). \tag{14}$$

By recalling (8) and (11), the action of $\mathbf{F}$ on $\mathcal{Q}$, defining the so-called (φ-dependent) Freudenthal dual of $\mathcal{Q}$ itself, can be related to the symplectic gradient of the effective black hole potential V_{BH}:

$$\mathbf{F}(\mathcal{Q};\varphi) = \Omega \frac{\partial V_{BH}(\varphi,\mathcal{Q})}{\partial \mathcal{Q}}. \tag{15}$$

Through the attractor mechanism [7], all this enjoys an interesting physical interpretation when evaluated at the (unique) event horizon of the extremal black hole (3) (denoted below by the subscript "H"); indeed

$$\partial_\varphi V_{BH} = 0 \Leftrightarrow \lim_{\tau\to-\infty} \varphi^i(\tau) = \varphi^i_H(\mathcal{Q}); \tag{16}$$

$$S_{BH}(\mathcal{Q}) = \frac{A_H}{4} = \pi\ V_{BH}|_{\partial_\varphi V_{BH}=0} = -\frac{\pi}{2}\mathcal{Q}^T\mathcal{M}_H(\mathcal{Q})\,\mathcal{Q}, \tag{17}$$

where S_{BH} and A_H respectively denote the Bekenstein-Hawking entropy [8] and the area of the horizon of the extremal black hole, and the matrix horizon value $\mathcal{M}_H$ is defined as

$$\mathcal{M}_H(\mathcal{Q}) := \lim_{\tau\to-\infty} \mathcal{M}(\varphi(\tau)). \tag{18}$$

Correspondingly, one can define the (scalar-independent) horizon Freudenthal duality $\mathbf{F}_H$ as the horizon limit of (13):

$$\widetilde{\mathcal{Q}} \equiv \mathbf{F}_H(\mathcal{Q}) := \lim_{\tau\to-\infty} \mathbf{F}(\mathcal{Q};\varphi(\tau)) = -\Omega\mathcal{M}_H(\mathcal{Q})\,\mathcal{Q} = \frac{1}{\pi}\Omega\frac{\partial S_{BH}(\mathcal{Q})}{\partial\mathcal{Q}}. \tag{19}$$

Remarkably, the (horizon) Freudenthal dual of $\mathcal{Q}$ is nothing but ($1/\pi$ times) the symplectic gradient of the Bekenstein-Hawking black hole entropy S_{BH}; this latter, from dimensional considerations, is only constrained to be an homogeneous function of degree two in $\mathcal{Q}$. As a result, $\widetilde{\mathcal{Q}} = \widetilde{\mathcal{Q}}(\mathcal{Q})$ is generally a complicated (non-linear) function, homogeneous of degree one in $\mathcal{Q}$.

It can be proved that the entropy S_{BH} itself is invariant along the flow in the charge space $\mathcal{Q}$ defined by the symplectic gradient (or, equivalently, by the horizon Freudenthal dual) of $\mathcal{Q}$ itself:

$$S_{BH}(\mathcal{Q}) = S_{BH}(\mathbf{F}_H(\mathcal{Q})) = S_{BH}\left(\frac{1}{\pi}\Omega\frac{\partial S_{BH}(\mathcal{Q})}{\partial\mathcal{Q}}\right) = S_{BH}\left(\widetilde{\mathcal{Q}}\right). \quad (20)$$

It is here worth pointing out that this invariance is pretty remarkable: the (semi-classical) Bekenstein-Hawking entropy of an extremal black hole turns out to be invariant under a generally non-linear map acting on the black hole charges themselves, and corresponding to a symplectic gradient flow in their corresponding vector space.

For other applications and instances of Freudenthal duality, see [9–14].

2 Groups of Type E_7

The concept of Lie groups *of type* E_7 was introduced in the 60s by Brown [15], and then later developed *e.g.* by [16–20].

Starting from a pair $(G, \mathbf{R})$ made of a Lie group G and its faithful representation $\mathbf{R}$, the three axioms defining $(G, \mathbf{R})$ as a group of type E_7 read as follows:

1. Existence of a unique symplectic invariant structure Ω in $\mathbf{R}$:

$$\exists!\Omega \equiv \mathbf{1} \in \mathbf{R} \times_{\mathbf{a}} \mathbf{R}, \quad (21)$$

which then allows to define a symplectic product $\langle\cdot,\cdot\rangle$ among two vectors in the representation space $\mathbf{R}$ itself:

$$\langle\mathcal{Q}_1, \mathcal{Q}_2\rangle := \mathcal{Q}_1^M \mathcal{Q}_2^N \Omega_{MN} = -\langle\mathcal{Q}_2, \mathcal{Q}_1\rangle. \quad (22)$$

2. Existence of a unique rank-4 completely symmetric invariant tensor (K-tensor) in $\mathbf{R}$:

$$\exists!K \equiv \mathbf{1} \in (\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R})_{\mathbf{s}}, \quad (23)$$

which then allows to define a degree-4 invariant polynomial I_4 in $\mathbf{R}$ itself:

$$I_4 := K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q. \quad (24)$$

3. Defining a triple map T in $\mathbf{R}$ as

$$T \ : \mathbf{R}\times\mathbf{R}\times\mathbf{R} \to \mathbf{R}; \quad (25)$$
$$\langle T(\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3), \mathcal{Q}_4\rangle := K_{MNPQ}\mathcal{Q}_1^M\mathcal{Q}_2^N\mathcal{Q}_3^P\mathcal{Q}_4^Q, \quad (26)$$

it holds that

$$\langle T(\mathcal{Q}_1, \mathcal{Q}_1, \mathcal{Q}_2), T(\mathcal{Q}_2, \mathcal{Q}_2, \mathcal{Q}_2)\rangle = \langle\mathcal{Q}_1, \mathcal{Q}_2\rangle K_{MNPQ}\mathcal{Q}_1^M\mathcal{Q}_2^N\mathcal{Q}_2^P\mathcal{Q}_2^Q. \quad (27)$$

This property makes a group of type E_7 amenable to a description as an automorphism group of a *Freudenthal triple system* (or, equivalently, as the conformal groups of an underlying Jordan triple system).

All electric-magnetic duality (U-duality[1]) groups of $\mathcal{N} \geq 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are of type E_7. Among these, degenerate groups of type E_7 are those in which the K-tensor is actually reducible, and thus I_4 is the square of a quadratic invariant polynomial I_2. In fact, in general, in theories with electric-magnetic duality groups of type E_7 holds that

$$S_{BH} = \pi\sqrt{|I_4(\mathcal{Q})|} = \pi\sqrt{\left|K_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q\right|}, \tag{28}$$

whereas in the case of degenerate groups of type E_7 it holds that $I_4(\mathcal{Q}) = (I_2(\mathcal{Q}))^2$, and therefore the latter formula simplifies to

$$S_{BH} = \pi\sqrt{|I_4(\mathcal{Q})|} = \pi\,|I_2(\mathcal{Q})|\,. \tag{29}$$

Simple, non-degenerate groups of type E_7 relevant to $\mathcal{N} \geq 2$-extended $D = 4$ supergravity theories with symmetric scalar manifolds are reported in Table 1.

Semi-simple, non-degenerate groups of type E_7 of the same kind are given by $G = SL(2, R) \times SO(2, n)$ and $G = SL(2, R) \times SO(6, n)$, with $\mathbf{R} = (\mathbf{2}, \mathbf{2} + \mathbf{n})$ and $\mathbf{R} = (\mathbf{2}, \mathbf{6} + \mathbf{n})$, respectively relevant for $\mathcal{N} = 2$ and $\mathcal{N} = 4$ supergravity.

Moreover, degenerate (simple) groups of type E_7 relevant to the same class of theories are $G = U(1, n)$ and $G = U(3, n)$, with complex fundamental representations $\mathbf{R} = \mathbf{n} + \mathbf{1}$ and $\mathbf{R} = \mathbf{3} + \mathbf{n}$, respectively relevant for $\mathcal{N} = 2$ and $\mathcal{N} = 3$ supergravity [19].

The classification of groups of type E_7 is still an open problem, even if some progress have been recently made *e.g.* in [31] (in particular, *cf.* Table D therein).

In all the aforementioned cases, the scalar manifold is a *symmetric* cosets $\frac{G}{H}$, where H is the maximal compact subgroup (with symmetric embedding) of G. Moreover, the K-tensor can generally be expressed as [20]

$$K_{MNPQ} = -\frac{n(2n+1)}{6d}\left[t^{\alpha}_{MN}t_{\alpha|PQ} - \frac{d}{n(2n+1)}\Omega_{M(P}\Omega_{Q)N}\right], \tag{30}$$

where $\dim \mathbf{R} = \mathbf{2n}$ and $\dim G = d$, and t^{α}_{MN} denotes the symplectic representation of the generators of G itself. Thus, the horizon Freudenthal duality can be expressed in terms of the K-tensor as follows [4]:

$$\mathbf{F}_H(\mathcal{Q})_M \equiv \widetilde{\mathcal{Q}}_M = \frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}^M} = \epsilon\frac{2}{\sqrt{|I_4(\mathcal{Q})|}}K_{MNPQ}\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q, \tag{31}$$

[1]Here U-duality is referred to as the "continuous" symmetries of [21]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [22].

Table 1 Simple, non-degenerate groups G related to Freudenthal triple systems $\mathbf{M}(J_3)$ on simple rank-3 Jordan algebras J_3. In general, $G \cong Conf(J_3) \cong Aut(\mathbf{M}(J_3))$ (see *e.g.* [23–25] for a recent introduction, and a list of Refs.). O, H, C and R respectively denote the four division algebras of octonions, quaternions, complex and real numbers, and O_s, H_s, C_s are the corresponding split forms. Note that the G related to split forms O_s, H_s, C_s is the *maximally non-compact* (*split*) real form of the corresponding compact Lie group. $M_{1,2}(O)$ is the Jordan triple system generated by 2×1 vectors over O [26]. Note that the STU model, based on $J_3 = R \oplus R \oplus R$, has a *semi-simple* G_4, but its *triality symmetry* [27] renders it "effectively simple". The $D = 5$ uplift of the T^3 model based on $J_3 = R$ is the *pure* $\mathcal{N} = 2$, $D = 5$ supergravity. J_3^H is related to both 8 and 24 supersymmetries, because the corresponding supergravity theories are *"twin"*, namely they share the very same bosonic sector [26, 28–30].

J_3	G_4	**R**	$\mathcal{N}$
J_3^O	$E_{7(-25)}$	**56**	2
$J_3^{O_s}$	$E_{7(7)}$	**56**	8
J_3^H	$SO^*(12)$	**32**	2, 6
$J_3^{H_s}$	$SO(6,6)$	**32**	0
J_3^C	$SU(3,3)$	**20**	2
$J_3^{C_s}$	$SL(6,R)$	**20**	0
$M_{1,2}(O)$	$SU(1,5)$	**20**	5
J_3^R	$Sp(6,R)$	$\mathbf{14}'$	2
$R \oplus R \oplus R$ (STU)	$SL(2,R)^{\otimes 3}$	$(\mathbf{2},\mathbf{2},\mathbf{2})$	2
R (T^3)	$SL(2,R)$	**4**	2

where $\epsilon := I_4/|I_4|$; note that the horizon Freudenthal dual of a given symplectic dyonic charge vector $\mathcal{Q}$ is well defined only when $\mathcal{Q}$ is such that $I_4(\mathcal{Q}) \neq 0$. Consequently, the invariance (20) of the black hole entropy under the horizon Freudenthal duality can be recast as the invariance of I_4 itself:

$$I_4(\mathcal{Q}) = I_4\left(\widetilde{\mathcal{Q}}\right) = I_4\left(\Omega\frac{\partial\sqrt{|I_4(\mathcal{Q})|}}{\partial\mathcal{Q}}\right). \tag{32}$$

In absence of "flat directions" at the attractor points (namely, of unstabilized scalar fields at the horizon of the black hole), and for $I_4 > 0$, the expression of the matrix $\mathcal{M}_H(\mathcal{Q})$ at the horizon can be computed to read

$$\mathcal{M}_{H|MN}(\mathcal{Q}) = -\frac{1}{\sqrt{I_4}}\left(2\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q + \mathcal{Q}_M\mathcal{Q}_N\right), \tag{33}$$

and it is invariant under horizon Freudenthal duality:

$$\mathbf{F}_H(\mathcal{M}_H)_{MN} := \mathcal{M}_{H|MN}(\widetilde{\mathcal{Q}}) = \mathcal{M}_{H|MN}(\mathcal{Q}). \tag{34}$$

3 Duality Orbits, Rigid Special Kähler Geometry and Pre-homogeneous Vector Spaces

For $I_4 > 0$, $\mathcal{M}_H(\mathcal{Q})$ given by (33) is one of the two possible solutions to the set of equations [32]

$$\begin{cases} M^T(\mathcal{Q})\,\Omega M(\mathcal{Q}) = \epsilon\Omega; \\ M^T(\mathcal{Q}) = M(\mathcal{Q}); \\ \mathcal{Q}^T M(\mathcal{Q})\,\mathcal{Q} = -2\sqrt{|I_4(\mathcal{Q})|}, \end{cases} \tag{35}$$

which describes symmetric, purely $\mathcal{Q}$-dependent structures at the horizon; they are symplectic or anti-symplectic, depending on whether $I_4 > 0$ or $I_4 < 0$, respectively. Since in the class of (super)gravity $D = 4$ theories we are discussing the sign of I4 separates the G-orbits (usually named duality orbits) of the representation space $\mathbf{R}$ of charges into distinct classes, the symplectic or anti-symplectic nature of the solutions to the system (35) is G-invariant, and supported by the various duality orbits of G (in particular, by the so-called "large" orbits, for which I_4 is non-vanishing).

One of the two possible solutions to the system (35) reads [32]

$$M_+(\mathcal{Q}) = -\frac{1}{\sqrt{|I_4|}}\left(2\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6\epsilon K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q + \epsilon\mathcal{Q}_M\mathcal{Q}_N\right),$$

whose corresponding $\mathbf{F}_H\,(M_+)_{MN}$ reads

$$\mathbf{F}_H\,(M_+)_{MN} := M_{+|MN}(\widetilde{\mathcal{Q}}) = \epsilon M_{+|MN}(\mathcal{Q}).$$

For $\epsilon = +1 \Leftrightarrow I_4 > 0$, it thus follows that

$$M_+(\mathcal{Q}) = \mathcal{M}_H\,(\mathcal{Q})\,, \tag{36}$$

as anticipated.

On the other hand, the other solution to system (35) reads [32]

$$M_-\,(\mathcal{Q}) = \frac{1}{\sqrt{|I_4|}}\left(\widetilde{\mathcal{Q}}_M\widetilde{\mathcal{Q}}_N - 6\epsilon K_{MNPQ}\mathcal{Q}^P\mathcal{Q}^Q\right),$$

whose corresponding $\mathbf{F}_H\,(M_-)_{MN}$ reads

$$\mathbf{F}_H\,(M_-)_{MN} := M_{-|MN}(\widetilde{\mathcal{Q}}) = \epsilon M_{-|MN}(\mathcal{Q}).$$

By recalling the definition of I_4 (24), it is then immediate to realize that $M_-\,(\mathcal{Q})$ is the (opposite of the) Hessian matrix of ($1/\pi$ times) the black hole entropy S_{BH}:

$$M_{-|MN}\,(\mathcal{Q}) = -\partial_M\partial_N\sqrt{|I_4|} = -\frac{1}{\pi}\partial_M\partial_N S_{BH}. \tag{37}$$

The matrix $M_-\,(\mathcal{Q})$ is the (opposite of the) pseudo-Euclidean metric of a non-compact, rigid special pseudo-Kä hler manifold related to the duality orbit of the black hole electromagnetic charges (to which $\mathcal{Q}$ belongs), which is an example of pre-homogeneous vector space (PVS) [33]. In turn, the nature of the rigid special manifold may be Kähler or pseudo-Kähler, depending on the existence of a $U(1)$ or $SO(1,1)$ connection.[2]

In order to clarify this statement, let us make two examples within maximal $\mathcal{N} = 8$, $D = 4$ supergravity. In this theory, the electric-magnetic duality group is $G = E_{7(7)}$, and the representation in which the e.m. charges sit is its fundamental $\mathbf{R} = \mathbf{56}$. The scalar manifold has rank-7 and it is the real symmetric coset[3] $G/H = E_{7(7)}/SU(8)$, with dimension 70.

1. The unique duality orbit determined by the G-invariant constraint $I_4 > 0$ is the 55-dimensional non-symmetric coset

$$\mathcal{O}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)}}. \tag{38}$$

[2] For a thorough introduction to special Kähler geometry, see *e.g.* [34].

[3] To be more precise, it is worth mentioning that the actual relevant coset manifold is $E_{7(7)}/[SU(8)/Z_2]$, because spinors transform according to the double cover of the stabilizer of the scalar manifold (see *e.g.* [35, 36], and Refs. therein).

Table 2 Non-generic, nor irregular PVS with simple G, of type 2 (in the complex ground field). To avoid discussing the finite groups appearing, the list presents the Lie algebra of the isotropy group rather than the isotropy group itself [37]. The interpretation (of suitable real, non-compact slices) in $D = 4$ theories of Einstein gravity is added; remaining cases will be investigated in a forthcoming publication

G	V	n	Isotropy alg.	Degree	Interpr. $D = 4$
$SL(2, C)$	S^3C^2	1	0	4	$\mathcal{N} = 2$, $R\ (T^3)$
$SL(6, C)$	Λ^3C^6	1	$sl(3, C)^{\oplus 2}$	4	$\mathcal{N} = 2$, J_3^C $\mathcal{N} = 0$, $J_3^{C_s}$ $\mathcal{N} = 5$, $M_{1,2}(O)$
$SL(7, C)$	Λ^3C^7	1	g_2^C	7	
$SL(8, C)$	Λ^3C^8	1	$sl(3, C)$	16	
$SL(3, C)$	S^2C^3	2	0	6	
$SL(5, C)$	Λ^2C^5	3	$sl(2, C)$	5	
		4	0	10	
$SL(6, C)$	Λ^2C^6	2	$sl(2, C)^{\oplus 3}$	6	
$SL(3, C)^{\otimes 2}$	$C^3 \otimes C^3$	2	$gl(1, C)^{\oplus 2}$	6	
$Sp(6, C)$	$\Lambda_0^3C^6$	1	$sl(3, C)$	4	$\mathcal{N} = 2$, J_3^R
$Spin(7, C)$	C^8	1	g_2^C	2	
		2	$sl(3, C) \oplus so(2, C)$	2	
		3	$sl(2, C) \oplus so(3, C)$	2	
$Spin(9, C)$	C^{16}	1	$spin(7, C)$	2	
$Spin(10, C)$	C^{16}	2	$g_2^C \oplus sl(2, C)$	2	
		3	$sl(2, C) \oplus so(3, C)$	4	
$Spin(11, C)$	C^{32}	1	$sl(5, C)$	4	
$Spin(12, C)$	C^{32}	1	$sl(6, C)$	4	$\mathcal{N} = 2, 6$, J_3^H $\mathcal{N} = 0$, $J_3^{H_s}$
$Spin(14, C)$	C^{64}	1	$g_2^C \oplus g_2^C$	8	
G_2^C	C^7	1	$sl(3, C)$	2	
		2	$gl(2, C)$	2	
E_6^C	C^{27}	1	f_4^C	3	
		2	$so(8, C)$	6	
E_7^C	C^{56}	1	e_6^C	4	$\mathcal{N} = 2$, J_3^O $\mathcal{N} = 8$, $J_3^{O_s}$

By customarily assigning positive (negative) signature to non-compact (compact) generators, the pseudo-Euclidean signature of $\mathcal{O}_{I_4>0}$ is $(n_+, n_-) = (30, 25)$. In this case, $M_-\ (\mathcal{Q})$ given by (37) is the 56 -dimensional metric of the non-compact, rigid special pseudo-Kä hler non-symmetric manifold

$$\mathbf{O}_{I_4>0} = \frac{E_{7(7)}}{E_{6(2)}} \times R^+, \tag{39}$$

with signature $(n_+, n_-) = (30, 26)$, thus with character $\chi := n_+ - n_- = 4$. Through a conification procedure (amounting to modding out[4] $C \cong SO(2) \times SO(1,1) \cong U(1) \times R^+$), one can obtain the corresponding 54-dimensional non-compact, special pseudo-Kähler symmetric manifold

$$\mathbf{O}_{I_4>0}/C \cong \widehat{\mathbf{O}}_{I_4>0} := \frac{E_{7(7)}}{E_{6(2)} \times U(1)}. \tag{40}$$

2. The unique duality orbit determined by the G-invariant constraint $I_4 < 0$ is the 55-dimensional non-symmetric coset

$$\mathcal{O}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)}}, \tag{41}$$

with pseudo-Euclidean signature given by $(n_+, n_-) = (28, 27)$, thus with character $\chi = 0$. In this case, M_- $(\mathcal{Q})$ given by (37) is the 56-dimensional metric of the non-compact, rigid special pseudo-Kähler non-symmetric manifold

$$\mathbf{O}_{I_4<0} = \frac{E_{7(7)}}{E_{6(6)}} \times R^+, \tag{42}$$

with signature $(n_+, n_-) = (28, 28)$. Through a "pseudo-conification" procedure (amounting to modding out $C_s \cong SO(1,1) \times SO(1,1) \cong R^+ \times R^+$), one can obtain the corresponding 54-dimensional non-compact, special pseudo-Kähler symmetric manifold

$$\mathbf{O}_{I_4<0}/C_s \cong \widehat{\mathbf{O}}_{I_4<0} := \frac{E_{7(7)}}{E_{6(6)} \times SO(1,1)}. \tag{43}$$

(39) and (42) are non-compact, real forms of $\frac{E_7}{E_6} \times GL(1)$, which is the type 29 in the classification of regular, pre-homogeneous vector spaces (PVS) worked out by Sato and Kimura in [37]. From its definition, a PVS is a finite-dimensional vector space V together with a subgroup G of $GL(V)$, such that G has a Zariski open dense orbit in V (thus open and dense in V also in the standard topology). PVS are subdivided into two types (type 1 and type 2), according to whether there exists an homogeneous polynomial on V which is invariant under the semi-simple (reductive) part of G itself. For more details, see *e.g.* [33, 38, 39].

In the case of $\frac{E_7}{E_6} \times GL(1)$, V is provided by the fundamental representation space $\mathbf{R} = \mathbf{56}$ of $G = E_7$, and there exists a quartic E_7-invariant polynomial I_4 (24) in the $\mathbf{56}$; $H = E_6$ is the isotropy (stabilizer) group.

Amazingly, simple, non-degenerate groups of type E_7 (relevant to $D = 4$ Einstein (super)gravities with symmetric scalar manifolds) *almost* saturate the list of irreducible PVS with unique G-invariant polynomial of degree 4 (*cf.* Table 2); in

[4] The signature along the R^+-direction is negative [32].

particular, the parameter n characterizing each PVS can be interpreted as the number of centers of the regular solution in the (super)gravity theory with electric-magnetic duality (U-duality) group given by G. This topic will be considered in detail in a forthcoming publication.

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Phase Transitions at High Supersymmetry Breaking Scale in String Theory

Hervé Partouche and Balthazar de Vaulchier

Abstract When supersymmetry is spontaneously broken at tree level, the spectrum of the heterotic string compactified on orbifolds of tori contains an infinite number of potentially tachyonic modes. We show that this implies instabilities of Minkowski spacetime, when the scale of supersymmetry breaking is of the order of the string scale. We derive the phase space structure of vacua in the case where the tachyonic spectrum contains a mode with trivial momenta and winding numbers along the internal directions not involved in the supersymmetry breaking.

1 Introduction

Phase transitions occur in various contexts in high energy physics. The most common setup describing such effects is the Brout-Englert-Higgs mechanism, which occurs when a scalar field ϕ becomes tachyonic. When the squared mass is negative, ϕ sits at a maximum of the scalar potential and therefore condenses. The new vacuum expectation value (vev) of ϕ minimizes (locally) the potential, and the theory has switched from a "wrong" to a "true" vacuum. What we review in the present note is that a similar condensation occurs in string theory, when the scale M_{susy} of spontaneous supersymmetry breaking is of the order of the string scale M_{string} [1].

To be specific, we consider classical string models in Minkowski spacetime, where supersymmetry is spontaneously broken. Because there is only one true constant scale in the theory, which is M_{string}, the scale M_{susy} is a field the tree level potential V depends on. Our assumption on flatness of the classical background amounts to saying that minima of V lie at $V = 0$. It turns out that local supersymmetry implies the latter to be degenerate, and that one of the flat directions is parameterized by the

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_17

field M_{susy} itself. For this reason, the supergravity models describing the spontaneous breaking of supersymmetry in flat space are referred as "no-scale models" [2], since there is no preferred value for the vev $\langle M_{\text{susy}}\rangle$ at tree level. In the framework of string theory, this statement is actually valid up to a critical value M_c of $\langle M_{\text{susy}}\rangle$, which is of the order of M_{string}. Above this bound, the condensation of a tachyonic scalar triggers a second order phase transition from the no-scale phase to a new phase, which is argued to be associated with a non-critical string theory. Even though this phenomenon is physically very different from the Hagedorn phase transition encountered in string theory at finite temperature T, when the latter is of the order of M_{string} [3], it turns out to be similar from a technical point of view [4, 5].

In its usual formulation, string theory is defined in first quantized formalism. This means that what is known (at least in principle) is the massless and massive spectrum that is allowed to populate a consistent vacuum described by a conformal field theory on the worldsheet. In order to find the shape of the potential far from the vacuum under consideration, one should in principle evaluate an infinite number of correlation functions, and resum them in order to reconstruct the full expression of the off-shell tree level potential. Alternatively, we may consider in principle a second quantized formulation of string theory, i.e. string fields theory, in order to derive the potential. However, given the fact that we are only interested in the vacuum structure of the tree level potential, we will analyze the problem at low energy, in the effective supergravity.

In Sect. 2, we introduce a class of string theory no-scale models in four dimensions that realize the $\mathcal{N}=4\to\mathcal{N}=0$ spontaneous breaking of supersymmetry. In Sect. 3, we implement an orbifold action that reduces the initial $\mathcal{N}=4$ supersymmetry to $\mathcal{N}=1$, and we present the necessary ingredients to derive the tree level potential V in presence of super-Higgs mechanism. The final expression of V is presented in Sect. 4, where the different phases of the theory are derived. Our conclusions can be found in Sect. 5.

2 $\mathcal{N}=4\to\mathcal{N}=0$ Heterotic No-Scale Models

Our starting point is the heterotic string compactified on a 6-dimensional torus, where supersymmetry is spontaneously broken by a stringy version [6] of the Scherk-Schwarz mechanism [7, 8]. In field theory, the latter is a refined version of the Kaluza-Klein reduction we first present in its simplest possible realization. Let us consider a field theory in $4+1$ dimensions, where the extra coordinate is compactified on a circle of radius R_4. Assuming the existence of a symmetry with conserved charge Q in $4+1$ dimensions, we may impose Q-dependent boundary conditions for every field φ, which translate into Kaluza-Klein masses M for its Fourier modes $m_4\in\mathbb{Z}$,

$$\varphi(x^\mu,x^4)=\frac{1}{\sqrt{2\pi R^4}}\sum_m\varphi_m(x^\mu)\,e^{i\,\frac{m_4+eQ}{R_4}x^4}\quad\Longrightarrow\quad M^2=\left(\frac{m_4+eQ}{R_4}\right)^2. \tag{1}$$

In the above formulas, $\mu \in \{0, \dots, 3\}$ and we have included a parameter $e = 1$ or 0 in order to describe both Scherk-Schwarz and Kaluza-Klein cases, respectively. When the higher dimensional theory is supersymmetric and we choose $Q \equiv \frac{F}{2} + Q_{\text{susy}}$, where F is the fermionic number and Q_{susy} is a constant charge within each supermultiplet, the boson/fermion degeneracy in four dimensions is lifted and the theory describes a super-Higgs mechanism, with scale $M_{\text{susy}} = e/(2R_4)$.

In the $E_8 \times E_8$ heterotic string compactified on a factorized torus $T^6 \equiv S^1(R_4) \times T^5$, the previous mass formula in string units ($M_{\text{string}} = 1$) is generalized to [6]

$$M^2 = \left(\frac{m_4 + eQ - \frac{n_4}{2}e^2}{R_4} + n_4 R_4 \right)^2 + 2\Big[(Q - en_4)^2 + Q_2^2 + Q_3^2 + Q_4^2 - 1 \Big], \tag{2}$$

where $n_4 \in \mathbb{Z}$ is the winding number of the string along $S^1(R_4)$, and $\mathbf{Q} \equiv (Q, Q_2, Q_3, Q_4)$ is a quadruple of charges arising from the fact that for $e = 0$ the theory is $\mathcal{N} = 4$ supersymmetric. The above equation applies to the lightest modes, which in the bosonic sector have $(Q, Q_2, Q_3, Q_4) = (\pm 1, 0, 0, 0)$ or permutations. Notice the presence of the -1 contribution in the squared brackets, which is the zero point energy arising from the quantization of the fields on the worldsheet. In the supersymmetric case ($e = 0$), we have $M^2 \geq 0$ for all modes, while in the spontaneously broken case ($e = 1$), the dangerous contribution -1 is not canceled when $Q = n_4 = \pm 1$. Looking at this fact more closely, one finds that the pair of scalar states $m_4 = -n_4 = -Q = \epsilon$, where $\epsilon = \pm 1$, are tachyonic when

$$\frac{\sqrt{2} - 1}{\sqrt{2}} \equiv \frac{1}{2R_c} < R_4 < R_c \equiv \frac{\sqrt{2} + 1}{\sqrt{2}}. \tag{3}$$

Therefore, an instability arises in the theory when the supersymmetry breaking scale M_{susy} reaches the critical value $M_{\text{c}} = 1/(2R_c)$.

Moreover, taking into account the fact that the tachyonic modes may also have non-trivial momentum $m_5 \in \mathbb{Z}$ or winding number $n_5 \in \mathbb{Z}$ (but not both, due to the left/right-level matching) along one more internal direction X^5, their mass formula becomes

$$M^2 = \frac{1}{4R_4^2} + R_4^2 - 3 + \left(\frac{m_5}{R_5} \right)^2 \quad \text{or} \quad M^2 = \frac{1}{4R_4^2} + R_4^2 - 3 + (n_5 R_5)^2, \tag{4}$$

where we have assumed for simplicity the internal space to be factorized as $T^6 \equiv S^1(R_4) \times S^1(R_5) \times T^4$. Therefore, the larger (smaller) R_5 is, the larger the number of tachyonic momentum (winding) states along $S^1(R_5)$ is, as shown in Fig. 1. One of our goal is then to see whether the infinity of potentially tachyonic modes yield a multiphase diagram or not, beside the no-scale-phase we started with. Of course, even if we will not do so, this question may be considered in the most general case, where the momenta and winding numbers along the remaining internal radii directions of T^4 are taken into account.

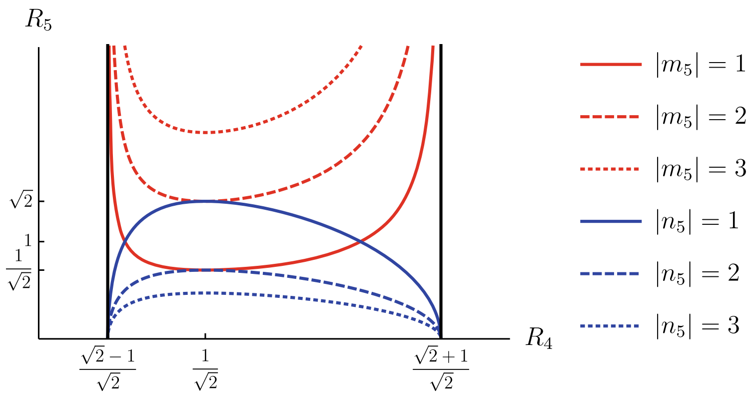

Fig. 1 Boundary curves of the regions of the plan (R_4, R_5), where Kaluza-Klein or winding modes along $S^1(R_5)$ are tachyonic

Before concluding this section, let us specify what conserved charges Q may be used to implement the $\mathcal{N} = 4 \to 0$ Scherk-Schwarz breaking of supersymmetry. On the left-moving supersymmetric side of the heterotic string, we can rotate any pair of worldsheet fermions ψ^a, ψ^b, where $a, b \in \{2, \dots, 9\}$ in light cone gauge. The charges Q are then the eigenvalues of the generator associated with one of the $O(2)$ affine algebra currents $: \psi^a \psi^b :$. Because all ψ^a's have identical boundary conditions on the worldsheet, all pairs (a, b) yield equivalent non-supersymmetric models when $e = 1$.

3 Gauged $\mathcal{N} = 4$ Supergravity Truncated to $\mathcal{N} = 1$

Gauged $\mathcal{N} = 4$ supergravity contains a gravity multiplet coupled to an arbitrary number $6 + k$ of vector multiplets [9–13]. The scalar content is a complex field Φ and $6 \times (6 + k)$ real scalars Z_a^S, $a \in \{4, \dots, 9\}$, $S \in \{4, \dots, 15 + k\}$, defining a non-linear σ-model with target space

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(6, 6+k)}{SO(6) \times SO(6+k)}. \tag{5}$$

The coordinates of the second coset satisfy $\eta_{ST} Z_a^S Z_b^T = -\delta_{ab}$, where $\eta = \mathrm{diag}(-1, \dots, -1, 1, \dots)$ with 6 entries -1. To diminish the number of degrees of freedom and simplify the analysis, we implement from now on a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold action on the parent supersymmetric heterotic model, which reduces $\mathcal{N} = 4$ to $\mathcal{N} = 1$. The generators G_1, G_2 act respectively as twists $X^a \to -X^a$ on the directions X^6, X^7, X^8, X^9 and X^4, X^5, X^8, X^9, thus reducing T^6 to $S^1(R_4) \times S^1(R_5) \times T^2 \times T^2$. In that case, the choice of charge Q must be compatible with the orbifold action. A consistent choice amounts to taking the $O(2)$ current with $a = 6, b = 8$ (i.e. in *distinct* T^2 tori). To convince ourselves, let us note that the tachyonic modes, say with pure momenta along $T^2 \times T^2$, transform consistently into each other under G_1 and G_2[1]:

$$\begin{aligned}
&\frac{\psi^6 + i\epsilon\psi^8}{\sqrt{2}}\, e^{i\epsilon X_{\rm R}^4}\, e^{ip_{5\rm L}X^5}\, e^{i\sum_{I=6}^{9} p_{I\rm L}X^I}\,|0\rangle_{\rm NS}\otimes|\tilde{0}\rangle \\
\longrightarrow\; &-(-1)^{\xi}\,\frac{\psi^6 + i\epsilon\psi^8}{\sqrt{2}}\, e^{i\epsilon X_{\rm R}^4}\, e^{ip_{5\rm L}X^5}\, e^{-i\sum_{I=6}^{9} p_{I\rm L}X^I}\,|0\rangle_{\rm NS}\otimes|\tilde{0}\rangle \\
\longrightarrow\; &\frac{\psi^6 - i\epsilon\psi^8}{\sqrt{2}}\, e^{-i\epsilon X_{\rm R}^4}\, e^{-ip_{5\rm L}X^5}\, e^{i\left(p_{6\rm L}X^6 + p_{7\rm L}X^7 - p_{8\rm L}X^8 - p_{9\rm L}X^9\right)}|0\rangle_{\rm NS}\otimes|\tilde{0}\rangle. \qquad (6)
\end{aligned}$$

On the contrary, with $(a, b) = (6, 7)$ or $(8, 9)$, the generator G_1 would inconsistently send the tachyons into massive superpartners. In the above formula, we have introduced a discrete torsion $\xi = 1$ or 0 that yields two drastically different patterns of tachyonic modes surviving the G_1-orbifold action.[2] In the following, we restrict ourselves to the analysis of the case $\xi = 1$. Notice that since the $O(2)$ generator used to implement the Scherk-Schwarz breaking of $\mathcal{N} = 1$ supersymmetry rotates directions of distinct T^2's, some of the tori deformation moduli are projected out.

Our goal is to derive the $\mathcal{N} = 1$ supergravity potential V that depends on the scalar fields whose masses are given in Eq. (4), and on the radii R_4, R_5 and the dilaton field. This amounts to freezing (artificially) all remaining moduli, which are associated with (i) the internal $T^2 \times T^2 \times T^2$ (ii) or $E_8 \times E_8$ Wilson lines, (iii) or which arise from the twisted sectors. Moreover, as said before, we do not include the potentially tachyonic modes with non-trivial momentum or winding numbers along X^6, X^7, X^8, X^9, which we expect would not change the final phase diagram for the choice of discrete torsion $\xi = 1$ considered in this work. In that case, we find convenient to derive the result by truncating suitably the $\mathcal{N} = 4$ gauged supergravity associated with the parent $\mathcal{N} = 4 \to \mathcal{N} = 0$ heterotic no-scale model. The non-linear σ-model reduces to

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2)}{SO(2)\times SO(2)} \times \frac{SO(2,k_+)}{SO(2)\times SO(k_+)} \times \frac{SO(2,k_-)}{SO(2)\times SO(k_-)}, \qquad (7)$$

whose complex dimension is $1 + 2 + k_+ + k_-$. In these cosets, $k_+ = +\infty$ is the number of real scalars $m_4 = -n_4 = -Q = +1$ with m_5 or n_5 arbitrary. Similarly, $k_- = +\infty$ is the number of "anti-tachyons" $m_4 = -n_4 = -Q = -1$ with $-m_5$ or $-n_5$. Due to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold action (see Eq. 6), we know that tachyons and anti-tachyons are identified. Among the coordinates Z_a^S, those which do not survive

[1] In our notations, $e^{ip_{I\rm L}X_{\rm L}^I + ip_{I\rm R}X_{\rm R}^I}|0\rangle_{\rm NS}\otimes|\tilde{0}\rangle$ stands for $|p_{\rm L}\rangle_{\rm NS}\otimes|\widetilde{p_{\rm R}}\rangle$, where the coordinates and the generalized momenta are divided into their left- and right-moving contributions, $X^I = X_{\rm L}^I + X_{\rm R}^I$, $p_I = p_{I\rm L} + p_{I\rm R}$, and where we have set $R_4 = 1/\sqrt{2}$ for convenience.

[2] See the revised version of arXiv:1903.09116 [1].

the truncation are set to zero. For instance, the third coset is parameterized by Z_a^S, $a \in \{6, 7\}$, where the superscript is restricted to $S \in \{12, \ldots, 11 + k_+\} \equiv \mathcal{I}$, and that satisfy $\sum_{S,T\in\mathcal{I}} \eta_{ST} Z_a^S Z_b^T = -\delta_{ab}$.

Once we know the supermultiplet content of the $\mathcal{N} = 4$ supergravity, we need to specify the gauging, i.e. the non-Abelian interactions between the gauge bosons belonging to the vector multiplets as well as the 6 graviphotons. This amounts to determining the structure constants f_{RST}, totally antisymmetric in their indices $R, S, T \in \{4, \ldots, 9 + (2 + k_+ + k_-)\}$. By supersymmetry, a potential is generated, which is [9–14]

$$V = \frac{|\Phi|^2}{4} Z^{RU} Z^{SV} \Big(\eta^{TW} + \frac{2}{3} Z^{TW}\Big) f_{RST} f_{UVW}, \tag{8}$$

where $Z^{RU} = Z_a^R Z_a^U$. To understand how the structure constants can be determined, it is instructive to consider as an example the supersymmetric case ($e = 0$), for which the left- and right-moving generalized momenta and squared mass for $m_4 = -n_4 = \epsilon$, $m_5 = n_5 = 0$, $\mathbf{Q}^2 = 1$ take the following form:

$$p_{4{\rm L \atop R}} = \frac{\epsilon}{\sqrt{2}}\Big(\frac{1}{R_4} \mp R_4\Big), \quad M^2 = \Big(\frac{1}{R_4} - R_4\Big)^2. \tag{9}$$

When $R_4 = 1$, two vectors multiplets become massless and satisfy $p_{4\rm L} = 0$, $p_{4\rm R} = \epsilon\sqrt{2}$. Recognizing $p_{4\rm R}$ to be the non-Cartan charges of $SU(2)$, one concludes that the massless vector multiplet enhance the $U(1)_{\rm L} \times U(1)_{\rm R}$ gauge symmetry generated by the dimensionally reduced metric and antisymmetric tensor, $(G + B)_{\mu 4}$, $(G - B)_{\mu 4}$, to $U(1)_{\rm L} \times SU(2)_{\rm R}$. As a result, in a supersymmetric string theory model at some given point in moduli space, the structure constants in a Weyl-Cartan basis are nothing but the generalized momenta evaluated in the associated background, $\langle p_{I\rm L}\rangle$, $\langle p_{I\rm R}\rangle$ [14].

The generalization of this result when supersymmetry is spontaneously broken ($e = 1$) is not known. The main difficulty in that within a vector multiplet, the values of the generalized momenta depend on $\mathbf{Q}$. However, because in our case of interest all scalar superpartners of the possible tachyons have masses of order $M_{\rm string}$, they can be safely set to zero and the potential V can be expressed only in terms of the structure constants associated with the generalized momenta of the tachyonic modes. Labelling the latter by an index A or $\bar{A}$,

$$\begin{aligned} A &\equiv (m_4 = -n_4 = -Q = +1, \quad m_5, 0) \text{ or } (m_4 = -n_4 = -Q = +1, 0, \quad n_5)\,, \\ \bar{A} &\equiv (m_4 = -n_4 = -Q = -1, -m_5, 0) \text{ or } (m_4 = -n_4 = -Q = -1, 0, -n_5), \end{aligned}$$

the non-trivial structure constants involving vector multiplets are, up to antisymmetry,

$$f_{4A\bar{A}} = \langle p_{4L} \rangle = \frac{1}{\sqrt{2}}\Big(\frac{1}{2\langle R_4\rangle} - \langle R_4\rangle\Big), \; f_{10A\bar{A}} = \langle p_{4R} \rangle = \frac{1}{\sqrt{2}}\Big(\frac{1}{2\langle R_4\rangle} + \langle R_4\rangle\Big)$$
$$f_{5A\bar{A}} = \langle p_{5L} \rangle = \frac{m_5}{\sqrt{2}\,\langle R_5\rangle} \quad \text{or} \quad \frac{n_5}{\sqrt{2}}\,\langle R_5\rangle\,,$$
$$f_{11A\bar{A}} = \langle p_{5R} \rangle = \frac{m_5}{\sqrt{2}\,\langle R_5\rangle} \quad \text{or} \quad -\frac{n_5}{\sqrt{2}}\,\langle R_5\rangle. \tag{10}$$

Moreover, the non-Abelian structure of the 6 graviphotons of $\mathcal{N}=4$ supergravity must be specified. For this purpose, we consider an ansatz consistent with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold action,

$$f_{468} = e_{\rm L}, \qquad f_{10,68} = e_{\rm R}, \qquad f_{479} = \tilde{e}_{\rm L}, \qquad f_{10,79} = \tilde{e}_{\rm R}, \tag{11}$$

where the right hand sides will be determined by imposing the no-scale supergravity phase to reproduce data of the heterotic model.

4 Tree Level Potential

We are ready to derive the potential of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ truncated $\mathcal{N}=4$ supergravity, by using all ingredients introduced in the previous sections. In Eq. (7), the last three cosets can be reparameterized in terms of "constrained" variables ϕ^S satisfying $Z^{ST} = 4\big(\phi^S\bar{\phi}^T + \bar{\phi}^S\phi^T\big)$. In particular, for the second manifold, we define

$$\phi^4 = \frac{1-TU}{\sqrt{y}}, \quad \phi^5 = \frac{T+U}{\sqrt{y}}, \quad \phi^{10} = \frac{1+TU}{\sqrt{y}}, \quad \phi^{11} = \frac{T-U}{\sqrt{y}},$$
$$y = -(T-\bar{T})(U-\bar{U}) > 0, \tag{12}$$

where T, U are "unconstrained" complex coordinates. Similarly, for the third coset, we take

$$\phi^6 = \frac{1}{2\sqrt{Y}}\Big(1+\sum_A \omega_A\Big)\,, \quad \phi^7 = \frac{i}{2\sqrt{Y}}\Big(1-\sum_A \omega_A\Big)\,, \quad \phi^A = \frac{\omega_A}{\sqrt{Y}}\,,$$
$$Y \equiv 1 - 2\sum_A |\omega_A|^2 + \Big|\sum_A \omega_A^2\Big|^2 > 0, \tag{13}$$

in terms of unconstrained Calabi-Vesentini complex coordinates ω_A. Finally, ϕ^8, ϕ^9, $\phi^{\bar{A}}$ can be expressed in terms of unconstrained coordinates $\omega_{\bar{A}}$ of the fourth manifold.

In order to identify tachyons and anti-tachyons, and to set to zero their massive superpartners (the tachyons belong to chiral multiplets), we impose $\omega \equiv \omega_{\bar{A}} \in \mathbb{R}$. Moreover, because we restrict our analysis to the case where the compact directions X^4, X^5 are factorized circles $S^1(R_4) \times S^1(R_5)$, we take the supergravity variables

T, U to be of the form $T = i\mathcal{R}_4\mathcal{R}_5$, $U = i\mathcal{R}_4/\mathcal{R}_5$. In these conditions, the truncated gauged $\mathcal{N} = 4$ supergravity potential takes the following form [1]

$$V = \frac{|\Phi|^2}{2}\Big(C^{(0)} + C^{(2)}_A \Omega_A^2 + C^{(4)}_{AB}\Omega_A^2\Omega_B^2\Big), \qquad \Omega_A \equiv \frac{\omega_A}{\sqrt{Y}}, \tag{14}$$

where $C^{(0)}, C^{(2)}_A, C^{(4)}_{AB}$ are explicitly given in terms of the moduli $\mathcal{R}_4, \mathcal{R}_5$ and the structure constants of Eqs. (10), (11). The dictionary between the supergravity variables and the string theory moduli may not be trivial. Therefore, we introduce real coefficients $\gamma_{\rm dil}, \gamma_4, \gamma_5$ such that

$$|\Phi|^2 = \gamma_{\rm dil}\, e^{2\phi_{\rm dil}}, \qquad \mathcal{R}_4 = \gamma_4 R_4, \qquad \mathcal{R}_5 = \gamma_5 R_5, \tag{15}$$

where $\phi_{\rm dil}$ is the string theory dilaton field. Imposing that in the no-scale supergravity phase, where all Ω_A's vanish, the cosmological constant is zero, and the mass spectrum matches Eq. (4), we find two solutions ($\sigma = \pm 1$)

$$\begin{aligned} &e_{\rm L} = \langle p_{4\rm L} + \sigma\sqrt{3}p_{4\rm R}\rangle\,, \qquad e_{\rm R} = \langle p_{4\rm R} + \sigma\sqrt{3}p_{4\rm L}\rangle\,, \qquad -\tilde e_{\rm L}^2 + \tilde e_{\rm R}^2 = 2\,,\\ &\gamma_{\rm dil} = \frac{1}{2}\,, \qquad \gamma_4 = \frac{2+\sigma\sqrt{3}}{\langle R_4\rangle}\,, \qquad \gamma_5 = \frac{1}{\langle R_5\rangle}. \end{aligned} \tag{16}$$

In the end, written in terms of the heterotic string theory moduli fields, the potential takes the final form,

$$\begin{aligned} V = e^{2\phi_{\rm dil}}\, 4\Bigg\{&\Big(\frac{1}{4R_4^2} + R_4^2 - 3\Big)\sum_A \Omega_A^2 + \frac{1}{R_5^2}\sum_{m_5} m_5^2\,\Omega_A^2 + R_5^2\sum_{n_5} n_5^2\,\Omega_A^2\\ &+\Big(\frac{1}{R_4^2} + 4R_4^2\Big)\Big(\sum_A \Omega_A^2\Big)^2\\ &+\frac{4}{R_5^2}\Big(\sum_{m_5} m_5\,\Omega_A^2\Big)^2 + 4R_5^2\Big(\sum_{n_5} n_5\,\Omega_A^2\Big)^2\Bigg\}. \end{aligned} \tag{17}$$

Some remarks are in order. First, we note that the duality transformations $R_4 \to 1/(2R_4)$ and $R_4 \to 1/R_5$, which are satisfied by the 1-loop heterotic string partition function, remain valid off-shell, at least at the low energy level, since they are symmetries of V (as well as of the full effective action). Therefore, for the definition of the supersymmetry breaking scale to be valid for arbitrary R_4, we take

$$M_{\rm susy} \equiv \frac{1}{\sqrt{2}\, e^{|\ln(\sqrt{2}R_4)|}}. \tag{18}$$

Second, when the background value $\langle R_4\rangle$ sits outside the range given in Eq. (3), because all mass terms in the first line of Eq. (17) are positive, it is clear that the no-scale phase of the heterotic model is recovered, with its degenerate vacua and flat directions:

$$\langle V\rangle = 0\,,\quad \langle \Omega_A\rangle = 0\,,\,\forall A\,,\quad \langle M_{\rm susy}\rangle < M_{\rm c}\,,\quad \langle R_5\rangle,\ \phi_{\rm dil}\ \text{arbitrary}. \tag{19}$$

Third, when $\langle R_4\rangle$ sits in the range of Eq. (3), one finds two degenerate branches of extrema with respect to the Ω_A's and the radii:

$$\langle \Omega_A\rangle = \pm\frac{1}{2}\,\delta_{m_5,0}\,\delta_{n_5,0}\,,\quad \langle R_4\rangle = \frac{1}{\sqrt{2}},\quad \langle R_5\rangle\ \text{arbitrary}. \tag{20}$$

Only one scalar condenses, which is the tachyon with trivial momentum and winding numbers in all directions other than the Scherk-Schwarz circle $S^1(R_4)$. Expanding the condensing mode as $\pm 1/2+\delta\Omega_0$, and the radius as $R_4 = 1/\sqrt{2}+\delta R_4$, the potential becomes for small fluctuations of the fields

$$V = e^{2\phi_{\rm dil}}\Big(-1+8\delta R_4^2+16\delta\Omega_0^2\ +\frac{4}{R_5^2}\sum_{m_5} m_5^2\,\Omega_A^2 +4R_5^2\sum_{n_5} n_5^2\,\Omega_A^2+\cdots\Big). \tag{21}$$

Therefore, δR_4, $\delta\Omega_0$ and all non-condensing Ω_A's are massive, while R_5 is massless. However, the dilaton field has a tadpole and cannot be stabilized. Actually, writing the effective action in string frame, $\hat g_{\mu\nu} = e^{2\phi_{\rm dil}} g_{\mu\nu}$, where $g_{\mu\nu}$ is the Einstein frame metric, one obtains

$$S_{\rm tree} = \int d^4x\sqrt{-\hat g}\,e^{-2\phi_{\rm dil}}\Big(\frac{\hat{\mathcal{R}}}{2}+2(\partial\phi_{\rm dil})^2+1+\mathcal{O}(\delta)+\text{other fields}\Big), \tag{22}$$

where $\hat{\mathcal{R}}$ is the Ricci curvature. Notice that this expression matches the action of a non-critical string theory with linear dilaton background $\phi_{\rm dil} = \kappa_\mu X^\mu+\phi_0$, where κ_μ is a constant vector. As a result, it may be that the new phase arising from tachyon condensation, and which is characterized by a negative potential, is associated with a new fundamental heterotic string theory in non-critical dimension [1, 4, 5].

5 Conclusion

In this note, we have considered classical heterotic string backgrounds realizing the spontaneous breaking of $\mathcal{N}=1$ supersymmetry in Minkowski spacetime, and we have shown that the scale $M_{\rm susy}$ cannot exceed some critical value $M_{\rm c} = \mathcal{O}(M_{\rm string})$. We have restricted our analysis to the case where the condensing tachyon has vanishing momentum and winding numbers along the internal directions not involved in the Scherk-Schwarz breaking of supersymmetry. However, as can be seen from Eq. 6, another choice of discrete torsion in the model imply all potentially tachyonic states surviving the orbifold action to have non-trivial momentum or winding in these directions. It would be very interesting to apply our approach to this case, in order to find all different regions in moduli space corresponding to new string theory phases.

Another interesting generalization of our work would be to take into account all metric and antisymmetric tensor moduli-dependence of the torus of coordinates X^4, X^5. In that case, the region in moduli space where the tachyon condensation takes place is much more involved.

As a conclusion, let us mention that because in the very early universe the supersymmetry breaking scale is naturally of the order of the string scale, the phenomenon described in the present work may yield an alternative paradigm to inflation or bouncing cosmologies.

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Exotic Branes and Exotic Dualities in Supergravity

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Abstract We show how T-duality in string theory implies the presence of exotic branes, that is branes of the lower-dimensional theory that do not have a geometric higher-dimensional origin. We then move to discuss the potentials under which these branes are electrically charged. We show that these are mixed-symmetry potentials, and we discuss the duality relations among these potentials and the standard potentials of ten-dimensional supergravity. Finally, we discuss how such duality relations can be naturally described within the framework of double field theory, and we show one particular physical consequence of this description.

1 Introduction

Duality symmetries play a crucial role in our understanding of various aspects of string theory. In particular, S and U dualities relate BPS branes with tensions scaling with different powers of the string dilaton, and therefore allow us to gain information on non-perturbative aspects of the theory. In general, these duality symmetries act as discrete subgroups of the global symmetry groups of the low-energy supergravity theory. In this talk we are interested in theories with maximal supersymmetry, that arise as torus reductions of IIA/IIB string theories. The global symmetry group of the theory in $10-d$ dimensions is $E_{d+1(d+1)}$, and the non-perturbative U-duality symmetry of the full quantum theory is conjectured to be its discrete subgroup $E_{d+1(d+1)}(\mathbb{Z})$ [14].

The T-duality group $O(d,d;\mathbb{Z})$, which is a subgroup of U-duality, is a symmetry of the perturbative string spectrum of the theory dimensionally reduced on T^d. Correspondingly, in the low energy supergravity one can consider the maximal subgroup $\mathbb{R}^+ \times O(d,d)$ of $E_{d+1(d+1)}$, where $\mathbb{R}^+$ is a symmetry under shifts of the d-dimensional string dilaton, while $O(d,d)$ leaves the dilaton invariant and it is therefore a perturbative symmetry of the low-energy action. In four dimensions, the $\mathbb{R}^+$

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_18

symmetry is enhanced to $SL(2,\mathbb{R})$, while in three dimensions the full $\mathbb{R}^+ \times O(7,7)$ is enhanced to $SO(8,8)$.

We quickly review how the $O(d,d)$ symmetry acts on the scalar fields of the maximal supergravity theory in $10-d$ dimensions. In particular we are interested in the scalars coming from the metric and the B field, that parametrise the coset space $O(d,d)/[O(d)\times O(d)]$ by forming the $O(d,d)$ matrix

$$\mathcal{M}_{MN} = \begin{pmatrix} g^{mn} & -g^{mp}B_{pn} \\ B_{mp}g^{pn} & g_{mn} - B_{mp}g^{pq}B_{qn} \end{pmatrix}. \tag{1}$$

Under an $O(d,d)$ transformation $\mathcal{O}$, this matrix transforms as

$$\mathcal{M} \to \mathcal{O}^T \mathcal{M} \mathcal{O}. \tag{2}$$

T-duality is the discrete subgroup $O(d,d;\mathbb{Z})$. That is, given background values for the G and B scalars, every $O(d,d;\mathbb{Z})$ transformation, that acts on these background fields as in (2), leaves the string spectrum invariant. One defines the $O(d,d)$ invariant tensor

$$\eta_{MN} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \tag{3}$$

which identifies the "lightlike" $O(d,d)$ coordinates X and $\tilde{X}$. The coordinates X are precisely the coordinates of the d-dimensional torus, and one can ask what is the physical meaning of the coordinates $\tilde{X}$. To answer this question, one writes X in terms of the string coordinates $X_L(\sigma,\tau)$ and $X_R(\sigma,\tau)$ which describe the left and the right modes respectively, as

$$X = X_L + X_R. \tag{4}$$

The factorised T-duality transformation that maps IIA to IIB inverting the compactification radius corresponds to

$$X^a_L \to X^a_L \quad X^a_R \to -X^a_R, \tag{5}$$

where a is the direction one is T-dualising. On the other hand, such transformation is the $O(d,d)$ matrix that maps X^a to $\tilde{X}^a$. This means that the coordinates $\tilde{X}$ are the "winding" coordinates

$$\tilde{X} = X_L - X_R. \tag{6}$$

The fact that T-duality transformations exchange the metric and the B field implies that in string theory one has to generalise the concept of geometry. In particular one can consider compactifications on generalised manifolds such that the transition functions are T-duality transformations [13]. As a simple occurrence of non-geometry, we can consider the IIB theory compactified to six dimensions on the orbifold $T^4/\mathbb{Z}_2$.

The six-dimensional low-energy theory is $\mathcal{N} = (2, 0)$ supergravity coupled to 21 tensor multiplets. Can we interpret this as arising from IIA? We can, but from the point of view of IIA the $\mathbb{Z}_2$ involution will act non-geometrically.

In the following we will first discuss how T-duality implies that in string theory one has to consider, together with "standard" branes, that are the branes of the 10-dimensional IIA or IIB theory, also "exotic" branes, that are branes that arise in the lower-dimensional theory but do not have a clear higher-dimensional origin. We will then move to study the potentials under which these branes are electrically charged, and show that these are in general mixed-symmetry potentials related by "exotic" duality relations to the potentials of the ten-dimensional theories. Finally, we will show how these duality relations are unified in the framework of double field theory (DFT), and we will discuss what information can be gained from the DFT picture.

2 Exotic Branes

We start by considering the IIA or IIB theory compactified on a 2-torus to eight dimensions. In this case the perturbative global symmetry of the supergravity theory is $SO(2, 2)$, which is isomorphic to $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. This means that the G and B scalars parametrise the coset manifold $(SL(2, \mathbb{R})/SO(2))^2$. The scalars can be grouped in two complex scalars τ and ρ each transforming under one of the two $SL(2, \mathbb{R})$'s in a linear fractional way. While the scalar τ is made purely in terms of the metric, the scalar ρ is

$$\rho = B_{89} + i\sqrt{\det G} \tag{7}$$

and therefore a transformation

$$\rho \to \frac{a\rho + b}{c\rho + d} \tag{8}$$

mixes the B field and the determinant of the internal metric.

The NS5-brane solution in the string frame is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + H(r) dy^m dy^m, \tag{9}$$

where the NS-NS 3-form field strength and the dilaton are related to the harmonic function $H(r)$ as

$$H_{mnp} = \epsilon_{mnpq} \partial_q H(r) \qquad e^\phi = H^{1/2}(r). \tag{10}$$

We want to T-dualise along the transverse directions 8 and 9. So we first have to smear the NS5 along these directions. After smearing, the harmonic function becomes logarithmic. The equation for B_{89} becomes $\frac{1}{r}\partial_\theta B_{89} = -\partial_r H(r)$. Hence B_{89} depends linearly on θ, that is

$$B_{89} = \frac{\theta}{2\pi}, \tag{11}$$

and if one rotates around the brane $B_{89} \to B_{89} + 1$. That is, the monodromy is the T-duality transformation

$$\rho \to \rho + 1, \tag{12}$$

which is a symmetry of the eight-dimensional theory.

One can ask what happens to this solution after a generic T-duality transformation. In particular, one can consider the transformation corresponding to two factorised T-dualities in the directions 8 and 9. The action of such transformation on the scalar ρ is

$$\rho \to -1/\rho. \tag{13}$$

Hence, one ends up with a solution with monodromy

$$\beta^{89} \to \beta^{89} + 1 \tag{14}$$

where

$$\beta^{89} = \mathrm{Re}(-1/\rho) = -B_{89}/(B_{89}^2 + \det G). \tag{15}$$

Because of the monodromy, the explicit solution [8] is such that the internal metric is not well-defined. This means that the resulting 5-brane is globally non-geometric, *i.e.* it is "exotic". It is called 5_2^2 in the literature, where the top number denotes the number of isometries (in this case directions 8 and 9), while the bottom number denotes the scaling of the tension with respect to the dilaton (in this case g_S^{-2}). Models constructed introducing these branes had already appeared in the literature [10] well before the work of [8]. In particular, the model of [10] describes IIA 5-branes localised on a 2-sphere S^2, with monodromy $SL(2,\mathbb{Z})_\rho \times SL(2,\mathbb{Z})_\tau$. If the monodromy is non-trivial only with respect to $SL(2,\mathbb{Z})_\tau$, the model has $\mathcal{N} = (1,1)$ supersymmetry and it is geometric, that is it is IIA on K3 where the K3 is elliptically fibered. If the monodromy is non-trivial only with respect to the other $SL(2,\mathbb{Z})$, the model has $\mathcal{N} = (2,0)$ supersymmetry and it is in general non-geometric. Finally, if the monodromy is non-trivial with respect to both groups, supersymmetry is broken to $\mathcal{N} = (1,0)$.

In general, using chains of S and T dualities one finds all the non-geometric solutions of the type of the 5_2^2-brane [16]. Moreover, using the same dualities one derives also the expression for the tension of all such branes as functions of the string coupling and the compactification radii [9, 17, 18]. Following [17], one can consider instead of the tension the mass that arises when one compactifies the brane to a particle in three dimensions. So for instance for the D7-brane one gets $m_{D7} \sim g_s^{-1} R_3 ... R_9$, while for its S-dual we have $m_{SD7} \sim g_s^{-3} R_3 ... R_9$. The NS5 gives a mass $g_s^{-2} R_3 ... R_7$ and the 5_2^2 gives $g_s^{-2} R_3 ... R_7 R_8^2 R_9^2$. The fact that the exotic brane gives a mass proportional to a power of the radius higher than one is completely general and implies that the tension of the exotic brane diverges in the decompactification limit.

We want to associate to each brane the potential under which the brane is electrically charged. We use the following notation: if tension scales like g_s^{-n}, with $n = 1, 2, 3, 4...$, we denote the potentials with letters $C, D, E, F, ...$ That is, n is associated to the order in the alphabet. The indices of these potentials correspond to the directions contributing to that mass formulae for the three-dimensional particles above (plus the time direction). This means that the wrapped D7-brane above is charged with respect to the component $C_{03456789}$ of the RR 8-form C_8, its S-dual is charged with respect to $E_{03456789}$, which is a component of the 8-from E_8, and the NS5 gives D_{034567} (potential D_6). The square dependence on the radii R_8 and R_9 for the 5^2_2 give a potential $D_{03456789,89}$, which is a component of the mixed-symmetry potential $D_{8,2}$ (that is a field in a hook Young Tableau representation made of two columns, one with 8 boxes and one with 2). This gives a precise mapping between exotic branes and mixed-symmetry potentials [5]. What we want to analyse in the following is what are these mixed-symmetry potentials and how can they be related to the standard potentials of supergravity.

3 Exotic Dualities

We start by considering the NS5-brane. This brane is electrically charged under the potential D_6, which is the electromagnetic dual of the NS-NS 2-form B_2. We know how to dualise the NS-NS 2-form potential B_{ab}. We start from the kinetic term of the 2-form,

$$S[B] = \int d^{10}x\Big(-\frac{1}{12}H_{abc}H^{abc}\Big), \tag{16}$$

where $H_3 = dB_2$, and we write the parent action

$$S[D, H] = \int d^{10}x\Big(-\frac{1}{12}H_{abc}H^{abc} - \frac{1}{6}\epsilon^{a_1\ldots a_6 abcd}D_{a_1\ldots a_6}\partial_a H_{bcd}\Big), \tag{17}$$

where now the 3-form H_3 is treated as an independent field. The equation for D_6 gives the Bianchi identity $dH_3 = 0$, which implies $H_3 = dB_2$ and plugging this back into the action (17) gives back Eq. (16). On the other hand, the equation for H_3 gives the duality relation

$$H_{a_1\ldots a_7} = 7\partial_{[a_1}D_{a_2\ldots a_7]} = \frac{1}{6}\epsilon_{a_1\ldots a_7 abc}H_{abc}, \tag{18}$$

and solving this for H_3 in terms of D_6 in Eq. (17) gives the dual action for D_6.

In the full supergravity theory, this potential turns out not only to transform with respect to its own gauge transformations, but also with respect to the gauge transformations of the RR potentials. As a result, the NS5 brane effective action contains couplings to the RR potentials which give information on which type of brane can end on the NS5. In particular, in the IIA theory the NS5 Wess-Zumino term has the form

$$\int [D_6 + \mathcal{G}_1 C_5 + \mathcal{G}_3 C_3 + \mathcal{G}_5 C_1], \tag{19}$$

where $\mathcal{G}_1$ and $\mathcal{G}_5$ are the field strengths of a world-volume scalar and its dual, while $\mathcal{G}_3$ is the field strength of a world-volume self-dual 2-form. The NS5 in IIA is the end-point of D0, D2 and D4 branes. Similar considerations apply to the IIB NS5-brane.

We want to repeat the same analysis in eight dimensions. We want 6-form potentials that couple to the NS5, the KK monopole and the 5^2_2. These potentials are in the $(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$ of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, which is as we already mentioned the perturbative symmetry of the eight-dimensional theory, and they come from the 10-dimensional mixed-symmetry potentials

$$D_6 \qquad D_{7,1} \qquad D_{8,2}. \tag{20}$$

We want to identify the last two potentials as dual to the standard fields of the ten-dimensional theory. As we will show, the $D_{7,1}$ is the dual of the graviton, while the $D_{8,2}$ is the exotic dual of B_2.

We first consider the dual graviton. We dualise linearised gravity in the frame formulation, *i.e.* we dualise the linearised vielbein $e_\mu{}^a = \delta_\mu{}^a + h_\mu{}^a$ [22]. One starts with the linearised EH action written as

$$S_{\rm EH}[h] \;=\; \int d^dx \left[f_{ab}{}^b f^{ac}{}_c - \tfrac{1}{2} f_{abc} f^{acb} - \tfrac{1}{4} f_{abc} f^{abc} \right], \tag{21}$$

where

$$f_{ab}{}^c = \partial_a h_b{}^c - \partial_b h_a{}^c. \tag{22}$$

In terms of f, the linearised Einstein equations are

$$\partial^c f_{c(ab)} + \partial_{(a} f_{b)c}{}^c - \eta_{ab}\, \partial^c f_{cd}{}^d \;=\; 0, \tag{23}$$

where f satisfies the Bianchi identity

$$\partial_{[a} f_{bc]}{}^d \;=\; 0. \tag{24}$$

One then moves to a first order formulation and considers the parent action adding the lagrange multiplier $D_{d-3,1}$ that imposes the Bianchi identity,

$$\int d^dx\, \epsilon^{a_1\ldots a_{d-3}bcd} D_{a_1\ldots a_{d-3},e} \partial_b f_{cd}{}^e. \tag{25}$$

Observe that now you cannot impose that the $(d-3, 1)$ potential is irreducible: there is also a completely antisymmetric part. The equation for D gives the Bianchi identity, while the equation for f gives the duality relation, and using the latter to solve for f in terms of $D_{d-3,1}$ and plugging this back in the action gives the linearised action for the dual graviton. In ten dimensions the potential is $D_{7,1}$.

We now move on to discuss the potential $D_{8,2}$, and show that it is related to B_2 by an exotic duality relation. By suitably integrating by parts, we write the B_2 kinetic action as

$$S[B] \; = \; -\frac{1}{4}\int d^d x\Big(Q_{a,bc}Q^{a,bc} - 2Q_a{}^{ab}Q^c{}_{cb}\Big), \tag{26}$$

where $Q_{a,bc} = \partial_a B_{bc}$ (only antisymmetric in bc). We then introduce the parent action

$$S[Q, D] = -\frac{1}{4}\int d^d x\Big(Q^{a,bc}Q_{a,bc} - 2Q_a{}^{ab}Q^c{}_{cb} + \epsilon^{a_1\ldots a_{d-2}ab}D_{a_1\ldots a_{d-2},cd}\partial_a Q_b{}^{cd}\Big) \tag{27}$$

where the $D_{d-2,2}$ potential imposes the Bianchi identity $\partial_{[a}Q_{b]cd} = 0$, and as before it is in a reducible representation. The equation for D gives the Bianchi identity, while the equation for Q gives the duality relation, and plugging this back into the action one then recovers the second order equation for the dual field [6]. In ten dimensions the exotic dual potential is precisely $D_{8,2}$.

4 Exotic Dualities in DFT

The duality relations described in the previous section have a natural unified description in the framework of double field theory (DFT) [15, 20, 21]. In DFT the coordinates X and $\tilde{X}$ discussed in the introduction are treated on the same footing, and are grouped together in $X^M = (X^m, \tilde{X}_m)$, where M is an $SO(10, 10)$ index. The fields can depend in principle on both sets of coordinates, provided that they satisfy the section condition, that is on any pair of fields on the doubled space one must impose

$$\eta^{MN}\partial_M \otimes \partial_N = 0. \tag{28}$$

We are only interested in linearised field equations, and we employ the formulation of [1, 2], which is the DFT extension of the vierbein formulation of gravity. One introduces the generalised fluxes

$$\mathcal{F}_{ABC} \; = \; 3\,\partial_{[A}h_{BC]}\;, \qquad \mathcal{F}_A \; = \; \partial^B h_{BA} + 2\,\partial_A\phi, \tag{29}$$

where $A, B, \ldots$ are $SO(1, 9) \times SO(1, 9)$ indices, h_{AB} is the generalised vierbein and ϕ is the dilaton. The linearised action is

$$S_{DFT} = \int d^{2d}X\, e^{-2\bar{\phi}} \left(S^{AB} \mathcal{F}_A \mathcal{F}_B + \frac{1}{6} S^{ABCDEF} \mathcal{F}_{ABC} \mathcal{F}_{DEF} \right), \quad (30)$$

where S^{AB} and S^{ABCDEF} are invariant tensors of $SO(1,9) \times SO(1,9)$.

The fluxes obey Bianchi identities, which in a first order formulation we want to obtain as equations for the dual fields. We thus consider a parent action with Lagrange multipliers D_{ABCD} and D_{AB},

$$\int d^{2d}X\, [D^{ABCD}\, \partial_A \mathcal{F}_{BCD} + D^{AB} \big(\partial^C \mathcal{F}_{CAB} + 2\, \partial_A \mathcal{F}_B \big) + D\, \partial^A \mathcal{F}_A], \quad (31)$$

whose field equations are the linearised Bianchi identities

$$\begin{aligned} \partial_{[A} \mathcal{F}_{BCD]} &= 0 \\ \partial^C \mathcal{F}_{CAB} + 2\, \partial_{[A} \mathcal{F}_{B]} &= 0 \\ \partial^A \mathcal{F}_A &= 0. \end{aligned} \quad (32)$$

The equations for the fluxes give the duality relations, and plugging this back in the parent action gives the linearised action for the dual fields. The potentials D_6, $D_{7,1}$ and $D_{8,2}$ of the previous section are the components D^{abcd}, $D^{abc}{}_d$ and $D^{ab}{}_{cd}$ of the DFT potential D_{ABCD}, and this analysis reproduces exactly the duality relations of D_6, $D_{7,1}$ and $D_{8,2}$ [3]. In particular, the standard dualisation and the exotic dualisation of B_2 are unified in DFT.

To go back to the brane effective actions, we want to write down a DFT equivalent of the WZ term in Eq. (19). To do this, one needs a DFT formulation of the RR potentials. This formulation was given in [12], and it consists in collecting the RR potentials in a chiral spinor of $SO(10, 10)$

$$\chi = \sum_{p=0}^{10} \frac{1}{p!} C_{m_1 \ldots m_p}\, \Gamma^{m_1 \cdots m_p} |0\rangle, \quad (33)$$

with the Clifford vacuum $|0\rangle$ annihilated by all the gamma matrices Γ_m. The field strengths of the world-volume potentials describing D-branes ending on the NS5-brane and their T-duals is also a chiral spinor $\mathcal{G}$, and the DFT expression for the WZ term is [4]

$$S_{WZ} = \int d^6\xi\ Q_{MNPQ} [D^{MNPQ} + \overline{\mathcal{G}} \Gamma^{MNPQ} \chi], \quad (34)$$

where $\overline{\mathcal{G}} \Gamma^{MNPQ} \chi$ is an $SO(10, 10)$ spinor bilinear. The charge Q_{MNPQ} selects the type of brane one is considering. In particular, Q_{mnpq} corresponds to the NS5, $Q_{mnp}{}^q$ to the KK monopole and $Q_{mn}{}^{pq}$ to the 5^2_2-brane, while the remaining charges correspond to branes whose solutions are not even locally geometric.

As a nice application of this framework, we can consider the form of this effective action when the IIA Romans mass [19] is turned on. It is known [7] that massive couplings in WZ terms give anomalous creation of branes. For instance, for the D0-brane, one has in the WZ term:

$$S_{\text{massive D0-brane}} \sim \int m\, b_1 \tag{35}$$

which implies that when a D0 crosses a D8, a fundamental string is created:

$$\begin{array}{ll|lllllllll} \text{D0}: & \times & - & - & - & - & - & - & - & - & - \\ \text{D8}: & \times & \times & \times & \times & \times & \times & \times & \times & \times & - \\ \text{F1}: & \times & - & - & - & - & - & - & - & - & \times \end{array}$$

Similarly, for the NS5-brane, one has

$$S_{\text{massive NS5-brane}} \sim \int m\, c_6 \tag{36}$$

giving rise to the creation of a D6 brane when a D8 crosses an NS5:

$$\begin{array}{ll|lllllllll} \text{NS5}: & \times & \times & \times & \times & \times & \times & - & - & - & - \\ \text{D8}: & \times & \times & \times & \times & \times & \times & \times & \times & \times & - \\ \text{D6}: & \times & \times & \times & \times & \times & \times & - & - & - & \times \end{array}$$

What our WZ term shows is that one can similarly consider the T-dual picture, in which a 5^2_2 crosses a D8 giving rise to a D6 [4]:

$$\begin{array}{ll|lllllllll} 5^2_2: & \times & \times & \times & \times & \times & \times & - & - & \otimes & \otimes \\ \text{D8}: & \times & \times & \times & \times & \times & \times & \times & \times & - & \times \\ \text{D6}: & \times & \times & \times & \times & \times & \times & - & - & \times & - \end{array}$$

To conclude, we have shown that at least at the linearised level one can introduce mixed-symmetry potentials which couple to exotic branes and are related to the standard potentials by exotic duality relations. We have also shown how DFT provides a unified framework in which standard dualities and exotic dualities are treated on the same footing. One can then write down unified effective actions. It would be extremely interesting both from a conceptual point of view and from the point of view of model building to understand whether this descriptions could be extended at the interacting level.

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Thermodynamic Information Geometry and Applications in Holography

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Abstract In this report we investigate the space of equilibrium states for the three dimensional warped anti-de Sitter black hole solution ($WAdS_3$) of Topological massive gravity (TMG). Our considerations include the proper thermodynamic Riemannian metrics on the statistical manifold, spanned by the intensive quantities of the black hole, namely its temperature and angular velocity. Analyzing the conditions for thermodynamic stability of the system we identify possible phase transition points and impose several restrictions on the left and right central charges from the dual gauge theory. Finally, by considering the thermodynamic length of geodesic paths we find the optimal paths for quasi-static protocols on the equilibrium statistical manifold of the warped black hole solution.

1 Introduction

The celebrated AdS/CFT correspondence revealed many important and unexpected phenomena in various classical and quantum systems. One of its important feature is that it relates a higher dimensional classical gravitational theory in the bulk of space-time to a quantum field theory without gravity on a lower-dimensional boundary, and

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_19

vice versa. On the other hand, the correspondence is also a duality between weak and strong coupling regimes of both theories, allowing one to do perturbative calculations in one of the theories and consequently translate the results to the non-perturbative strong coupling part of the other theory. The latter feature lifts the correspondence to an extraordinary powerful framework.

When one considers a gauge theory in a finite temperature its holographically dual gravitational theory contains a black hole as a natural thermal source. In this case, the thermodynamic features of the black hole in the bulk theory can be used to impose various restrictions on the parameter space of the dual gauge theory. Besides referring only to the standard Bekenstein-Hawking entropy and related quantities one can resort to more intrinsically geometric methods of Thermodynamic information geometry (TIG), where one defines a proper Riemannian metrics on the statistical manifold of the bulk theory. In Sect. 2 we briefly introduce the basic concepts in TIG.

As an example in this report, we will consider the warped three dimensional anti-de Sitter black hole solution, which is a stable vacuum solution of the 3d Topological massive gravity (TMG), described by the action [1][1]

$$I_{TMG} = \frac{1}{16\pi}\int_{\mathcal{M}} d^3x\,\sqrt{-g}\,\left(R + \frac{2}{L^2}\right) + \frac{1}{\mu}\,I_{CS}\,. \tag{1}$$

In the last expression, the term I_{CS} is the gravitational Chern-Simons action given by

$$I_{CS} = \frac{1}{32\pi}\int_{\mathcal{M}} d^3x\,\sqrt{-g}\,\varepsilon^{\lambda\mu\nu}\,\Gamma^{r}_{\lambda\sigma}\left(\partial_\mu \Gamma^{\sigma}_{r\nu} + \frac{2}{3}\,\Gamma^{\sigma}_{\mu\tau}\,\Gamma^{\tau}_{\nu r}\right), \tag{2}$$

and the coupling μ is the mass of the graviton, $\varepsilon^{\lambda\mu\nu} = \epsilon^{\lambda\mu\nu}/\sqrt{-g}$, $\epsilon^{012} = +1$. Although there are classical AdS_3 solutions for every value of the coupling μ, the only stable case is defined by the condition $\mu L = 1$, which leads to a non-negative energy of the gravitons. However, other stable TMG vacua, namely warped backgrounds, can be constructed, if one considers non-chiral values of μL. The latter are discrete quotients by elements of $SL(2, R) \times U(1)$ of warped AdS_3 space. In this case, the group elements of the quotient define the left and the right temperatures in the dual gauge theory. With a certain choice for the central charges the density of states in the gauge theory exactly matches the entropy of the corresponding black hole solution, thus a duality between both theories can be conjectured.

The structure of the paper is as follows. In Sect. 2 we briefly introduce the basic concepts of Thermodynamic information geometry. In Sect. 3 we present the relevant features of the $WAdS_3$ black hole solution and its dual warped gauge theory. In Sect. 4 we investigate the thermodynamic stability of the $WAdS_3$ solution in the space of intensive thermodynamic parameters (T, Ω), which enables us to identify any phase transition points of the model. Consequently, we analyze several metric approaches to the equilibrium of the system and determine the admissible thermodynamic metrics,

[1]Although it is not relevant for our considerations, one should also note that a certain boundary term was introduced in [2] in order to make the variational principle well-defined.

which can be used to describe the statistical manifold of the black hole solution. In Sect. 5 the thermodynamic length of geodesic paths and the corresponding optimal quasi static processes on the statistical manifold are considered. Finally, in Sect. 6 we make a brief discussion on our results.

2 Basics of Thermodynamic Information Geometry

The first geometric approaches to thermodynamics of a given system were introduced by Weinhold [3] and Ruppeiner [4]. Weinhold showed that the empirical laws of equilibrium thermodynamics can be related to the axioms of an abstract metric space. In his approach the Hessian of the internal energy U with respect to the extensive parameters of the system plays a central role in defining the proper Riemannian metric on the space of macro states,

$$g_{ab}^{(W)} = \partial_a \partial_b U(\mathbf{E})\,. \tag{3}$$

Here $\mathbf{E} = (E^1, E^2, \ldots, E^n)$ are the other extensive parameters of the system besides U. On the other hand, Ruppeiner developed his Thermodynamic geometry within fluctuation theory, where the Hessian of the entropy S is used to define the proper thermodynamic metric:

$$g_{ab}^{(R)} = -\partial_a \partial_b S(\mathbf{E})\,. \tag{4}$$

Of course, due to the relation between the intensive and the extensive parameters for systems with well defined first law, one can equivalently use the intensive thermodynamic parameters as coordinates on the equilibrium manifold. Sometimes this is considered even more natural. As it turns out, both Hessian metrics (3) and (4) are conformally related to each other with the temperature T of the system being the conformal factor, $ds^2_{(R)} = ds^2_{(W)}/T$.

In order to understand why Hessian geometry plays an important role in Thermodynamic geometry, let us consider an open finite volume system A enclosed by a larger thermal reservoir. The system A exchanges energy through fluctuations. The microcanonical ensemble requires all microstates of A to be selected with equal probabilities. Therefore, the probability of finding the internal energy $u = U/V$ per volume of A between u and $u + du$ is proportional to the number of microstates of A corresponding to this range

$$P(u, V)du = C\Omega(u, V)du, \tag{5}$$

where Ω is the density of states, and C is a normalization factor. On the other hand, one has Boltzmann's expression for the entropy $S = k_B \ln \Omega$, which yields Einstein's relation for the probability ($k_B = 1$)

$$P(u, V)du = Ce^{S(u,V)}du. \tag{6}$$

This formula can be easily generalized in the presence of more fluctuating variables $\mathbf{E} = (E^1, E^2, \ldots, E^n)$:

$$P(\mathbf{E})d^nE = Ce^{S(\mathbf{E})}d^nE. \tag{7}$$

The next step is to expand the entropy S up to quadratic terms in E^a:

$$S(\mathbf{E}) - S_0 = \frac{V}{2}\frac{\partial^2 S}{\partial E^a \partial E^b}\Delta E^a \Delta E^b + \cdots$$

where $S_0 = S(\langle E^a \rangle)$ and $\Delta E^a = E^a - \langle E^a \rangle$. At equilibrium $\partial_a S = 0$ and S is maximized, thus $\partial_a \partial_b S < 0$. Now, one can define the quantity

$$g_{ab} = -\frac{\partial^2 S}{\partial E^a \partial E^b} = -\text{Hess}(S(\mathbf{E})), \tag{8}$$

which is the Ruppeiner thermodynamic information metric (4). Therefore, one arrives at the Gaussian approximation for the probability

$$P(\mathbf{E})d^nE = \frac{1}{2\pi}\exp\left(-\frac{V}{2}g^{(R)}_{ab}\Delta E^a \Delta E^b\right)\sqrt{|g|}d^nE, \tag{9}$$

which is useful in calculating the average values of any given quantity. A breakdown of the Gaussian approximation occurs when

$$V < |R|, \quad R \sim \xi^d, \tag{10}$$

where V is the volume of the system, R is the thermodynamic scalar curvature, ξ is the correlation length of the system, and d is the dimension of the system. In this case, the singularities of the scalar curvature R correspond to the possible spinodal curves and phase transition points of the model.

Although Weinhold and Ruppeiner metrics have been successfully applied to describe the phase structure of condensed matter systems, when utilized for black holes they do not often agree with each other. One of the reasons is due to the fact that Hessian metrics are not Legendre invariant, thus they do not preserve the geometric structure under a change of the thermodynamic potential. For this reason, in Ref. [5] H. Quevedo considered the $(2n + 1)$-dimensional thermodynamic phase space, spanned by the thermodynamic potential Φ, the set of extensive variables $\mathbf{E}$, and the set of intensive variables $\mathbf{I}$, to find the general Legendre invariant form of the metric on the space of equilibrium states:

$$g^{(Q)} = \Omega_\Phi\, \Phi(\mathbf{E})\, \chi_a{}^b \frac{\partial^2 \Phi}{\partial E^b\, \partial E^c}\, dE^a\, dE^c\,, \tag{11}$$

where $\chi_a{}^b = \chi_{af}\,\delta^{fb}$ is a constant diagonal matrix and $\Omega_\Phi \in \mathbb{R}$ is the degree of generalized homogeneity, $\Phi(\lambda^{\beta_1}\, E^1, \ldots, \lambda^{\beta_N}\, E^N) = \lambda^{\Omega_\Phi}\, \Phi(E^1, \ldots, E^N),\ \beta_a \in \mathbb{R}$.

In this case, the Euler identity for homogeneous functions should hold

$$\beta_{ab}\, E^a \,\frac{\partial \Phi}{\partial E^b} = \Omega_\Phi \,\Phi\,, \tag{12}$$

where $\beta_{ab} = \mathrm{diag}(\beta_1,\ \beta_2, \ldots, \beta_N)$. From the first law $d\Phi = I_a\, dE^a$, one notes that $I_a = \partial\Phi/\partial E^a$. When $\beta_{ab} = \delta_{ab}$, one recovers the standard Euler identity. If we choose $\beta_{ab} = \delta_{ab}$, for complicated systems this may lead to some non-trivial conformal factor, which is no longer proportional to the potential Φ. On the other hand, the choice $\chi_{ab} = \eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$ applies to systems with second-order phase transitions, while the choice $\chi_{ab} = \delta_{ab}$ is suitable for the description of systems with at least a first order phase transition.

Although the Legendre invariant proposal of Quevedo is very general, it allows one to choose from various conformal factors, which can be used to reduce the number of any redundant singularities in the thermodynamic curvature, coming from working in non-physical reference frames. A specific proposal in this line, which seems to work well in many cases, was given in [6], where the authors introduce a special metric, known as the HPEM metric.

Finally, a recent approach to Thermodynamic geometry was considered by Mirza and Mansoori in [7–9], which is based on conjugate thermodynamic potentials, specifically chosen to reflect the relevant thermodynamic properties of system under consideration.

Some applications of these approaches to different gravitational systems can be found for example in [10–12]. In order to identify the admissible thermodynamic metrics for a given black hole solution, a case by case study is required.

3 The WAdS$_3$/WCFT$_2$ Correspondence

The warped AdS$_3$ black hole solution is given by [1]

$$ds^2 = L^2\left(dt^2 + D(r)dr^2 + N(r)d\theta^2 + 2F(r)dtd\theta\right), \tag{13}$$

where $r \in [0, \infty]$, $t \in [-\infty, \infty]$, $\theta \sim \theta + 2\pi$, and

$$F(r) = \nu r - \frac{1}{2}\sqrt{r_+ r_-(\nu^2+3)}, \quad D(r) = \frac{1}{(\nu^2+3)(r-r_+)(r-r_-)}, \tag{14}$$

$$N(r) = \frac{r}{4}\left(3r(\nu^2-1) + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+ r_-(\nu^2+3)}\right). \tag{15}$$

The horizons are located at r_+ and r_-, where g^{rr} vanishes. Here, we also introduced the parameter $\nu = \mu L/3$. Notice that (13) reduces to the BTZ black hole in a rotating frame, when $\nu^2 = 1$. The physical black hole solutions exist only for $\nu^2 > 1$, as long as r_+ and r_- stay positive. For $\nu^2 < 1$, we always encounter closed time-like curves and such geometries posses no interest. Therefore, without loss of generality it is natural to choose $L > 0$ and $1 < \nu^2 < \infty$ for the non-chiral case.

The entropy S and the ADT conserved charges M and J of the warped AdS black hole are given by

$$M = \frac{(\nu^2+3)}{24}\left(r_+ + r_- - \frac{1}{\nu}\sqrt{r_+ r_- (\nu^2+3)}\right), \tag{16}$$

$$S = \frac{\pi L}{24\,\nu}\left((9\,\nu^2+3)\,r_+ - (\nu^2+3)\,r_- - 4\,\nu\sqrt{(\nu^2+3)\,r_+ r_-}\right), \tag{17}$$

$$J = \frac{\nu L(\nu^2+3)}{96}\left[\left(r_+ + r_- - \frac{1}{\nu}\sqrt{r_+ r_-(\nu^2+3)}\right)^2 - \frac{(5\nu^2+3)}{4\nu^2}(r_+ - r_-)^2\right]. \tag{18}$$

One also has the Hawking temperature and the angular velocity

$$T = \frac{\left(\nu^2+3\right)\,(r_+ - r_-)}{4\,\pi\,L\left(2\,\nu\,r_+ - \sqrt{\left(\nu^2+3\right)r_+ r_-}\right)}, \quad \Omega = \frac{2}{L\left(2\,\nu\,r_+ - \sqrt{\left(\nu^2+3\right)r_+ r_-}\right)}. \tag{19}$$

In this case, the first law of thermodynamics holds

$$dM = T\,dS + \Omega\,dJ\,. \tag{20}$$

Let us comment on the admissible values of the thermodynamic quantities. Assuming $r_+ \geq r_-$ one can reach $S = 0$ only if $-1 \leq \nu \leq 1$, which is not our case. The same condition holds for $M = 0$. Thus, we consider only positive $S > 0,\ M > 0$. From Eq. (19), one notes that the angular velocity Ω never reaches zero, while the temperature T is zero for coincident horizons, $r_+ = r_-$, which is the extremal case. On the other hand, the laws of thermodynamics forbid us from ever reaching the absolute zero, thus $T > 0$.

Instead of r_+ and r_-, it is more useful to work with the left and right temperatures from the dual gauge theory, namely

$$T_R = \frac{(\nu^2+3)(r_+ - r_-)}{8\pi L}, \quad T_L = \frac{\nu^2+3}{8\pi L}\left(r_+ + r_- - \frac{\sqrt{(\nu^2+3)r_+ r_-}}{\nu}\right), \tag{21}$$

and the left and right central charges[2]

$$c_R = \frac{L(5\,\nu^2+3)}{\nu\,(\nu^2+3)}, \quad c_L = \frac{4\,\nu\,L}{\nu^2+3}, \quad c_L - c_R = -\frac{L}{\nu}. \tag{22}$$

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[2]Although we are going to consider only positive central charges throughout the paper, which lead to unitary CFTs, one should keep in mind that negative charges can play vital role in anomaly cancellations, when considering the total central charge.

From Eqs. (22), under the requirement of positive central charges and $\nu^2 > 1$, one can restrict only to $\nu > 1$. Therefore, it immediately follows that $c_L < L$ and $c_R < 2\,L$. For large $\nu \to \infty$ one has vanishing central charges, which is physically excluded due to the divergence of the Kretschmann invariant of the metric

$$K = \frac{18 - 12\nu^2 + 6\nu^4}{L^2}. \tag{23}$$

Furthermore, the third expression in Eq. (22) clearly forbids the case $c_L = c_R$, while its negative sign suggests that $c_R > c_L$. Putting everything together one finds

$$0 < c_L < L, \quad L < c_R < 2\,L\,. \tag{24}$$

On the other hand, from Eq. (22), one finds the ratio of the central charges

$$\frac{c_L}{c_R} = \frac{4\,\nu^2}{3 + 5\,\nu^2}\,. \tag{25}$$

It depends only on ν and certain limits can be considered. For $\nu \to \infty$, the ratio reaches a maximum value of $4/5$. One has to exclude this value due to Eq. (23). When $\nu = 1$, the ratio is $1/2$, which is also excluded from our considerations. Therefore,

$$\frac{1}{2} < \frac{c_L}{c_R} < \frac{4}{5}\,. \tag{26}$$

In terms of the dual CFT temperatures and charges the entropy takes the Cardy form

$$S = \frac{\pi^2\,L}{3}\,\left(c_L\,T_L + c_R\,T_R\right)\,. \tag{27}$$

One can also define the following left and right moving energies,

$$E_L = \frac{\pi^2\,L}{6}\,c_L\,T_L^2\,, \quad E_R = \frac{\pi^2\,L}{6}\,c_R\,T_R^2\,, \tag{28}$$

which allow us to write the mass M, the angular momentum J and the Hawking temperature in the following way

$$M = \sqrt{\frac{2\,L\,E_L}{3\,c_L}}\,, \quad J = L\,(E_L - E_R), \quad \frac{1}{T} = \frac{4\,\pi\,\nu\,L}{\nu^2 + 3}\,\frac{T_L + T_R}{T_R}\,. \tag{29}$$

W have everything to proceed with finding the proper thermodynamic metrics on the space of equilibrium states of the warped black hole solution.

4 Thermodynamic Information Geometry of $WAdS_3$

In order to study the thermodynamic properties of the $WAdS_3$ black hole within the formalism of Thermodynamic information geometry, we express all extensive parameters (M, S, J) of the solution in terms of the intensive ones (T, Ω) and the left and right central charges (c_L, c_R) from the dual gauge theory, i.e.

$$M = \frac{1 - \pi c_L T}{3 c_L \Omega}, \quad S = \pi \frac{1 - \pi T (c_L - c_R)}{3\Omega}, \quad J = \frac{1 + \pi c_L T (\pi T (c_L - c_R) - 2)}{6 c_L \Omega^2}. \tag{30}$$

One naturally requires $S > 0$, which is always satisfied due to Eq. (24), while imposing $M, J > 0$ leads to

$$T < \frac{1}{\pi \left(c_L + \sqrt{c_L c_R}\right)}. \tag{31}$$

The local thermodynamic stability of the $WAdS_3$ black hole in (T, Ω) space can be determined by the explicit form of its heat capacities

$$C_\Omega = \frac{\pi^2 c_R T}{3(\pi c_L T \Omega (\pi T (c_L - c_R) - 2) + \Omega)}, \qquad C_J = \frac{\pi^2 T (c_R - c_L)}{3\Omega}, \tag{32}$$

where imposing $C_{\Omega, J} > 0$ leads to condition (31). The Davies critical points are given by the singularities of the heat capacities together with the points where they change sign, namely

$$T_c = \frac{1}{\pi \left(c_L + \sqrt{c_L c_R}\right)}. \tag{33}$$

This is the same critical temperature found in [11], where the authors consider different equilibrium manifolds.

The simplest metric one can define on the equilibrium manifold is the Ruppeiner metric given by the Hessian of the entropy with respect to the intensive parameters (T, Ω):

$$\hat{g}^{(R)} = -\text{Hess}(S(T, \Omega)) = \begin{pmatrix} 0 & \frac{(c_R - c_L)\pi^2}{3\Omega^2} \\ \frac{(c_R - c_L)\pi^2}{3\Omega^2} & -\frac{2\pi((c_R - c_L)\pi T + 1)}{3\Omega^3} \end{pmatrix}. \tag{34}$$

Due to the probabilistic interpretation of the Hessian thermodynamic metrics we additionally require their positive definiteness (Sylvetser's criterion), where all principal minors of the metric should be positive,

$$p_1 = g^{(R)}_{TT} > 0, \quad p_2 = g^{(R)}_{\Omega\Omega} > 0, \quad p_3 = \det(\hat{g}^{(R)}) > 0. \tag{35}$$

However, one immediately notes that this is not possible for (34) due to the fact that its third principal minor is always negative, i.e.

$$p_3 = \det(\hat{g}^{(R)}) = -\frac{(c_L - c_R)^2 \pi^4}{9\Omega^4} < 0. \tag{36}$$

Therefore, one can suggest that Ruppeiner's approach is not suitable for the description of the equilibrium space of the warped black hole.

On the other hand, Weinhold's approach takes the Hessian of the internal energy of the system instead of the entropy. In the case of black holes their internal energy is equivalent to the conserved ADT mass M. Therefore

$$\hat{g}^{(W)} = \text{Hess}(M(T, \Omega)) = \begin{pmatrix} 0 & \frac{\pi}{3\Omega^2} \\ \frac{\pi}{3\Omega^2} & \frac{2(1-c_L \pi T)}{3c_L \Omega^3} \end{pmatrix}. \tag{37}$$

Once again, it is not possible to impose Sylvester's criterion due to the fact $p_3 = \det(\hat{g}^{(W)}) = -\pi^2/(9\Omega^4) < 0$ is negative. Therefore, Weinhold information metric also fails to produce a viable thermodynamic metric.

One can find similar results for the Hessians of other thermodynamic potential, thus the conclusion is that the Hessian approach is not powerful enough to describe the equilibrium manifold of the warped black hole. The latter is also supported by the fact that the corresponding Hessian thermodynamic curvatures do not account for the relevant phase transition points of the system.

In order to overcome these issues one can consider Legendre invariant metrics. For example, choosing $\chi_{ab} = \eta_{ab} = \text{diag}(-1, 1)$, the Quevedo metric (11) in (T, Ω) space becomes

$$\hat{g}^{(Q)} = \begin{pmatrix} \frac{(c_R - c_L)\pi^3 T((c_L - c_R)\pi T - 1)}{9\Omega^2} & 0 \\ 0 & \frac{\pi T((c_L - c_R)\pi T - 1)(c_L \pi T((c_L - c_R)\pi T - 2) + 1)}{9c_L \Omega^4} \end{pmatrix}. \tag{38}$$

This metric cannot be positive definite for any values of T and Ω, but in this case, due to the unclear physical meaning of its components, the requirement of positive definiteness is not necessary, although it would be preferable if a probabilistic interpretation is found to be true. Nevertheless, the thermodynamic curvature accounts for the relevant phase transition points. However, it also introduces additional ones, as it is obvious from the denominator of the scalar thermodynamic curvature

$$\text{den}(R^{(Q)}) \propto T^3 (\pi T(c_L - c_R) - 1)^3 (\pi c_L T(\pi T(c_L - c_R) - 2) + 1)^2, \tag{39}$$

where only one of the roots of the expression is positive and coincides with the critical temperature (33).

On the other hand, the HPEM metric in (S, J) space is given by

$$ds^2_{(H)} = S \frac{\partial_S M}{(\partial_J^2 M)^3} (-\partial_S^2 M dS^2 + \partial_J^2 M dJ^2). \tag{40}$$

Transforming this metric to (T, Ω) space, one finds

$$\hat{g}^{(H)} = \begin{pmatrix} \frac{cR^3\pi^3 T((c_R-c_L)\pi T+1)}{243c_L^3(c_L-c_R)^2\Omega^{11}} & 0 \\ 0 & \frac{cR^3\pi T((c_R-c_L)\pi T+1)(c_L\pi T((c_R-c_L)\pi T+2)-1)}{243c_L^4(c_L-c_R)^3\Omega^{13}} \end{pmatrix}. \tag{41}$$

In contrast to Quevedo's case, the HPEM metric is be positive definite in the region of local thermodynamic stability (below the critical temperature $T < T_c$), which makes it a better choice for a viable thermodynamic metric on the equilibrium manifold. The denominator of its scalar curvature also accounts for all relevant critical points

$$\text{den}(R^{(H)}) \propto T^3(\pi T(c_R - c_L) + 1)^3(\pi c_L T(\pi T(c_R - c_L) + 2) - 1)^2, \tag{42}$$

where only one of the roots of the expression is positive and coincides with the critical temperature (33).

Similar positive definite metric can be constructed via the conjugate potential $K = M - \Omega J$. In this case, the coefficients of the MM metric can be calculated easily in (S, Ω) space via

$$ds^2_{(M)} = \frac{1}{T}\left(-\frac{\partial^2 K}{\partial S^2}dS^2 + 2\frac{\partial^2 K}{\partial S \partial \Omega}dS d\Omega + \frac{\partial^2 K}{\partial \Omega^2}d\Omega^2\right). \tag{43}$$

The result in (T, Ω) space is given by

$$\hat{g}^{(M)} = \begin{pmatrix} \frac{(c_R-c_L)\pi^2}{3T\Omega} & \frac{\pi((c_L-c_R)\pi T-1)}{3T\Omega^2} \\ \frac{\pi((c_L-c_R)\pi T-1)}{3T\Omega^2} & -\frac{cR}{3c_L T\Omega^3(c_L-c_R)} \end{pmatrix}, \tag{44}$$

where the MM metric is positive definite when $T < T_c$. A nice feature of this approach is that by construction it admits only the relevant singularities of the system, as one can see from its scalar curvature

$$R^{(M)} = \frac{3c_R\Omega(3\pi c_L T(\pi T(c_R - c_L) + 1) - 1)}{\pi^2 T(c_L - c_R)^2(\pi c_L T(\pi T(c_L - c_R) - 2) + 1)^2}. \tag{45}$$

Let us summarize the results in this section. We have considered Hessian thermodynamic metrics, namely Ruppeiner and Weinhold metrics and established that they are not suitable for the description of the equilibrium thermodynamic space of the WAdS$_3$ black hole. On the other hand, considering two Legendre invariant metric approaches, namely Quevedo and HPEM ones, we found that they correctly account for the phase structure of the black hole solution. However, only HPEM metric can be made positive definite in a subregion of the equilibrium manifold. Finally, using the method of conjugate thermodynamic potentials, we were able to construct a positive definite metric, which also correctly describes the thermodynamic features of the system in equilibrium.

5 Thermodynamic Length and Quasi-Static Processes

The viable thermodynamic metrics, found in the previous section, can be used to study geodesics on the equilibrium manifold, spanned by the intensive parameters (T, Ω). This allows one to calculate the thermodynamic length (the shortest distance) between two macro states, which can be used to optimize the implementation of quasi-static protocols[3] within a given statistical ensemble. The action for the thermodynamic geodesics is written by [13]

$$\mathcal{L} = \int_0^\tau \sqrt{g_{ab}(\boldsymbol{\lambda})\frac{d\lambda^a}{dt}\frac{d\lambda^b}{dt}}dt, \tag{46}$$

where t is an affine parameter on the geodesics,[4] $\lambda^a(t) = (T(t), \Omega(t))$ are the set of intensive thermodynamic parameters. We can vary the action to obtain the system of coupled geodesic equations

$$\ddot{\lambda}^c(t) + \Gamma^c_{ab}(\hat{g})\dot{\lambda}^a(t)\dot{\lambda}^b(t) = 0, \tag{47}$$

where the dot denotes a derivative with respect to t. By definition the solutions of equations (47) extremizes the thermodynamic length $\mathcal{L}$ between two equilibrium states. The latter given by the on-shell value of the action (46) for the geodesic curve, connecting those states. We can also define a related quantity, called the thermodynamic divergence of the path,

$$\mathcal{J} = \tau \int_0^\tau g_{ab}(\boldsymbol{\lambda})\frac{d\lambda^a}{dt}\frac{d\lambda^b}{dt}dt, \tag{48}$$

which is a measure of the energy dissipation or entropy production for a transition between two equilibrium points at particular rates of change of the control parameters. In other words, $\mathcal{J}$ measures the efficiency of the quasi-static protocols and satisfies the following bound

$$\mathcal{J} \geq \mathcal{L}^2. \tag{49}$$

The latter follows from the Cauchy-Schwarz inequality for integrals and provides a formal definition of the degree of irreversibility of the process[5] (see [14] and references therein). In what follows, we are going to consider only cases, which can be solved analytically.

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[3] A quasi-static protocol, or a quasi-static process, is a process applied to a given physical system, which on every step awaits for equilibration of the parameters. Thus, the systems is never taken out of equilibrium, when going from one macro state to a different one.

[4] In thermodynamics the parameter $t \in [0, \tau]$, where $(t = 0, t = \tau)$ denote the initial and final states, does not necessarily correspond to time. It can be any well-defined order parameter of the system.

[5] With reversibility only for $\mathcal{J} = 0$.

The thermodynamic geodesics in (T, Ω) parameter space for the Quevedo metric (38) can be solved analytically if one considers a constant temperature profile $T(t) = T = const$. In this case, the system of coupled geodesic equations (47) reduces to one second order ordinary differential equation for the function $\Omega(t)$:

$$\Omega(t)\Omega''(t) - 2\Omega'(t)^2 = 0, \tag{50}$$

and a cubic algebraic equation for the constant temperature $T = const$:

$$4\pi^3 c_L (c_L - c_R)^2 T^3 + 9\pi^2 c_L (c_R - c_L) T^2 + 2\pi(3c_L - c_R)T - 1 = 0. \tag{51}$$

In the last algebraic equation, the only allowed solutions are real roots with $T > 0$, which do not coincide with any critical points of the system. In this particular case, one notes that the discriminant of the cubic equation (51)

$$\Delta = 4\pi^6 c_L c_R (c_L - c_R)^2 \left(27c_L^2 + 9c_L c_R + 32c_R^2\right) > 0 \tag{52}$$

is positive, thus there are always three distinct real roots. The solution to the differential equation (50) is given by

$$\Omega(t) = \frac{\Omega_0^2}{\Omega_0 - t\mathcal{W}_0}, \quad \Omega(0) = \Omega_0, \quad \Omega'(0) = \mathcal{W}_0, \tag{53}$$

where $\Omega(0) = \Omega_0$ and $\Omega'(0) = \mathcal{W}_0$ are the initial value and the initial rate of change of the angular velocity, respectively. Substituting this solution and the metric coefficients in Eqs. (46) and (48) one can compute the geodesic length and the corresponding geodesic divergence between a state at $t = 0$ and a state at $t = \tau$, namely

$$\mathcal{L} = \frac{|\mathcal{W}_0|\sqrt{\pi T\left((c_L - c_R)\pi T - 1\right)\left(c_L \pi T\left((c_L - c_R)\pi T - 2\right) + 1\right)}}{3\Omega_0^2\sqrt{c_L}}\tau, \tag{54}$$

$$\mathcal{J} = \frac{\pi T \mathcal{W}_0^2\left((c_L - c_R)\pi T - 1\right)\left(c_L \pi T\left((c_L - c_R)\pi T - 2\right) + 1\right)}{9c_L\Omega_0^4}\tau. \tag{55}$$

In this case, for the given geodesic path, one has $\mathcal{J} = \mathcal{L}^2$, which saturates the bound in Eq. (49). The latter means that in Quevedo's case a quasi-static process, along a constant temperature profile, is the optimal one with smallest energy cost.

Similar analysis can also be conducted in HPEM's case. Here, the constant temperature geodesic profile leads to the same cubic algebraic equation (51), while the equation for the angular velocity now becomes

$$2\Omega(t)\Omega''(t) - 13\Omega'(t)^2 = 0. \tag{56}$$

Its solution is given by

$$\Omega(t) = \Omega_0\left(\frac{11\mathcal{W}_0}{2\Omega_0}t - 1\right)^{-2/11}, \quad \Omega(0) = \Omega_0, \quad \Omega'(0) = W_0. \tag{57}$$

The corresponding thermodynamic length and divergence are written by

$$\mathcal{L} = \frac{|\mathcal{W}_0| c_R \sqrt{c_R \pi T\left(\left(c_R - c_L\right)\pi T + 1\right)\left(c_L \pi T\left(\left(c_R - c_L\right)\pi T + 2\right) - 1\right)}}{9\Omega_0^6 c_L^2 (c_R - c_L)\sqrt{3\Omega_0(c_L - c_R)}}\tau, \tag{58}$$

$$\mathcal{J} = \frac{\pi T c_R^3 \mathcal{W}_0^2\left(\left(c_R - c_L\right)\pi T + 1\right)\left(c_L \pi T\left(\left(c_R - c_L\right)\pi T + 2\right) - 1\right)}{243 c_L^4 (c_L - c_R)^3 \Omega_0^{13}}\tau. \tag{59}$$

One notes that for the given geodesic path the bound in Eq. (49) is saturated, which defines any quasi-static process along a constant temperature profile to be the optimal one in HPEM's case.

6 Conclusion

In this report, we have considered the thermodynamic stability over the space of the intensive parameters for the $WAdS_3$ black hole solution of $3d$ Topological massive gravity. Restricting ourselves only to non-chiral values of the graviton's mass parameter, we found several conditions (24) and (26) on the central charges from the dual warped gauge theory. Consequently, considering positive values of the relevant thermodynamic quantities, such as the ADT mass and the Hawking-Beckenstein entropy of the black hole, we constrained the possible values of the Hawking temperature only to a subregion in (T, Ω) parameter space, namely Eq. (31). The latter was also shown to be the condition for local thermodynamic stability of the model.

We have extended our analysis of the WAdS solution by considering several geometric approaches to its equilibrium thermodynamic space. In this case, we have shown that the simpler Hessian approaches of Ruppeiner and Weinhold are not suitable for the description of the equilibrium space of the $WAdS_3$ black hole. On the other hand, the Legendre invariant metric approaches of Quevedo and HPEM, and the recent approach based on conjugate thermodynamic potentials (the MM approach), correctly accounted for the relevant thermodynamic features of the black hole solution. However, only HPEM and MM metrics can be made positive definite in a subregion of the equilibrium manifold.

Finally, investigating geodesics on the equilibrium manifold, equipped with Quevedo and HPEM metrics respectively, we have found analytic solutions (Eqs. (53) and (57)) to the system of coupled geodesic equations (47) in the case of constant temperature profiles. By calculating the thermodynamic lengths (Eqs. (54) and (58)) and the corresponding thermodynamic divergences (Eqs. (55) and (59)) in the cases mentioned above, we have also shown that the obtained solutions lead to the implementation of optimal quasi-static protocols (requiring minimal energy cost) on the space of equilibrium states of the warped black hole solution.

Acknowledgements This work was supported by the Bulgarian NSF grants № DM18/1 and № N28/5, and the Sofia university grant № 80-10-149. T. V. gratefully acknowledge the support of the Bulgarian national program "Young Scientists and Postdoctoral Research Fellow".

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The Role of the Slope in the Multi-measure Cosmological Model

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Abstract In this work, we report some results on the numerical exploration of the model of Guendelman-Nissimov-Pacheva. This model has been previously applied to cosmology, but there were open questions regarding its parameters. Here we demonstrate the existence of families of solutions on the slope of the effective potential which preserve the duration of the inflation and its power. For this solutions, one can see the previously reported phenomenon of the inflaton scalar field climbing up the slope, with the effect more pronounced when starting lower on the potential slope. Finally we compare the dynamical and the potential slow-roll parameters for the model and we find that the latter describe the numerically observed inflationary period better.

1 Cosmology Today

Some of the most defining features of the Universe we live in are that it is isotropic, homogeneous and flat. They have been confirmed to great precision by cosmological probes (WMAP, Planck). Another important observation is that the universe is currently expanding in an accelerated way (confirmed by the data from SNIa and the Cepheids) which requires the introduction of dark energy. A model which describe all of those fundamental properties is the $\Lambda - CDM$ model, in which different components of the energy density contribute to the evolution of the universe as different powers of the scale factor.

Explicitly, in the Friedman-Lemaitre-Robertson-Walker (FLRW) metric $\tilde{g}_{\mu\nu} = \mathrm{diag}\{-1, a(t)^2, a(t)^2, a(t)^2\}$, we have for the first Friedman equation: $H = \frac{\dot{a}}{a} = H_0\sqrt{\Omega_m a^{-3} + \Omega_{rad} a^{-4} + \Omega_\Lambda}$.

Here $H = \dot{a}(t)/a$ is the Hubble parameter and $a(t)$ is the scale factor parametrizing the expansion of the Universe. H_0 is the current Hubble constant, Ω_m is the critical

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*, Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_20

matter density (dark matter and baryonic matter), Ω_{rad} is the critical radiation density, and Ω_Λ is the critical density of the cosmological constant (i.e. dark energy). In our units ($G = 1/16\pi$), $\rho_{crit} = 6H_0^2$, therefore $\Omega_x = \rho_x/\rho_{crit} = \rho_x/(6H_0^2)$ for $X = \{m, rad, \Lambda\}$.

While the $\Lambda - CDM$ model offers a rather simple explanation of the evolution of the Universe (the minimal $\Lambda - CDM$ has only 6 parameters), it still has its problems. Some of the oldest ones—the horizon problem, the flatness problem, the missing monopoles problem and the large-structures formation problem, require the introduction of a new stage of the development of the Universe—the inflation. The inflation is an exponential expansion of the Universe lasting between 10^{-36} s and 10^{-32} s after the Bing Bang, which however increases the volume of the Universe 10^{70} times.

The simplest way to produce inflation [1] is to introduce a scalar field ϕ which is moving in a potential $V_{infl}(\phi)$. Inflation is generated by the exchange of potential energy for kinetic energy. In this case, the evolution of the Universe will be described by two differential equations:

$$H^2 = \frac{8\pi}{3m_{Pl}^2}(V_{infl}(\phi) + \frac{1}{2}\dot{\phi}^2) \tag{1}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'_{infl}(\phi) = 0, \tag{2}$$

where the first one is the Friedman equation and the second is the inflaton equation. Inflation occurs when $\ddot{a}(t) > 0$ which happens in this simple system when $\dot{\phi}^2 < V(\phi)$, i.e. when the potential energy dominates over the kinetic one. The pressure and the energy density are:

$$p_\phi = \dot{\phi}^2/2 - V_{infl}(\phi), \rho_\phi = \dot{\phi}^2/2 + V_{infl}(\phi).$$

One can consider different forms for the effective potential, but those simplistic inflationary theories have the problem of not being able to reproduce the graceful transition from inflation to the other observed epochs.

2 The Multimeasure Model

There are different ways to obtain a model with richer structure. Here we follow the model developed by Guendelman, Nissimov and Pacheva [2–8]. The idea is to couple two scalar fields (the inflaton ϕ and the darkon u) to both standard Riemannian metric and to another non-Riemannian volume form, so that the model can describe simultaneously early inflation, the smooth exit to modern times, and the existence of dark matter and dark energy.

The action of the model: $S = S_{darkon} + S_{inflaton}$ is (for more details [8–10]):

$$S_{darkon} = \int d^4x(\sqrt{-g} + \Phi(C))L(u, X_u)$$

$$S_{inflaton} = \int d^4x \Phi_1(A)(R + L^{(1)}) + \int d^4x \Phi_2(B)\left(L^{(2)} + \frac{\Phi(H)}{\sqrt{-g}}\right)$$

where $\Phi_i(Z) = \frac{1}{3}\epsilon^{\mu\nu\kappa\lambda}\partial_\mu Z_{\nu\kappa\lambda}$ for $Z = A, B, C, H$, are the non-Riemannian measures, constructed with the help of 4 auxiliary completely antisymmetric rank-3 tensors and we have the following Lagrangians for the two scalar fields u and ϕ:

$$\begin{aligned}
L(u) &= -X_u - W(u) \\
L^{(1)} &= -X_\phi - V(\phi),\ \ V(\phi) = f_1 e^{-\alpha\phi} \\
L^{(2)} &= -b_0 e^{-\alpha\phi} X_\phi + U(\phi),\ \ U(\phi) = f_2 e^{-2\alpha\phi}
\end{aligned}$$

where $X_c = \frac{1}{2} g^{\mu\nu}\partial_\mu c \partial_\nu c$ are the standard kinetic terms for $c = u, \phi$.

Trough the use of variational principle, for this model, it has been found that there exists a transformation

$$\tilde{g}_{\mu\nu} = \frac{\Phi(A)}{\sqrt{-g}} g_{\mu\nu} \tag{3}$$

$$\frac{\partial \tilde{u}}{\partial u} = (W(u) - 2M_0)^{-\frac{1}{2}}, \tag{4}$$

for which for the Weyl-rescaled metric $\tilde{g}$, the action becomes

$$S^{(eff)} = \int d^4x \sqrt{-\tilde{g}}(\tilde{R} + L^{(eff)}). \tag{5}$$

For the rescaled metric $\tilde{g}$ and the derived effective Lagrangian, L_{eff}, the Einstein Field equations are satisfied.

The action in the FLRW metric becomes ($v = \dot{u}$):

$$S^{(eff)} = \int dt\ a(t)^3\Big(- 6\,\frac{\dot{a}(t)^2}{a(t)^2} + \frac{\dot{\phi}^2}{2} - \frac{v^2}{2}\left(V + M_1 - \chi_2 b_0 e^{-\alpha\phi}\dot{\phi}^2/2\right) + \frac{v^4}{4}\left(\chi_2(U + M_2) - 2M_0\right)\Big).$$

from which one can obtain the equations of motion in the standard way.

Explicitly, the equations of motion are:

$$v^3 + 3\mathbf{a}v + 2\mathbf{b} = 0 \tag{6}$$

$$\dot{a}(t) = \sqrt{\frac{\rho}{6}} a(t), \tag{7}$$

$$\frac{d}{dt}\left(a(t)^3\dot{\phi}(1+\frac{\chi_2}{2}b_0e^{-\alpha\phi}v^2)\right) + a(t)^3(\alpha\frac{\dot{\phi}^2}{2}\chi_2 b_0 e^{-\alpha\phi} + V_\phi - \chi_2 U_\phi \frac{v^2}{2})\frac{v^2}{2} = 0 \tag{8}$$

Here $\mathbf{a} = \frac{-1}{3}\frac{V(\phi)+M_1-\frac{1}{2}\chi_2 b e^{-\alpha\phi}\dot{\phi}^2}{\chi_2(U(\phi)+M_2)-2M_0}$, $\mathbf{b} = \frac{-p_u}{2a(t)^3(\chi_2(U(\phi)+M_2)-2M_0)}$ and

$$\rho = \frac{1}{2}\dot{\phi}^2(1+\frac{3}{4}\chi_2 b e^{-\alpha\phi}v^2) + \frac{v^2}{4}(V+M_1) + \frac{3p_u v}{4a(t)^3}$$

is the energy density.

3 The Numerical Solutions

One can see that the parameters of this system are 12: 4 free parameters $\{\alpha, b_0, f_1, f_2\}$, 5 integration constants $\{M_0, M_1, M_2, \chi_2, p_u\}$ and 3 initial conditions $\{a(0), \phi(0), \dot{\phi}(0)\}$.

We use the following initial conditions:

$$a(0) = 10^{-10}, \phi(0) = \phi_0, \dot{\phi}(0) = 0. \tag{9}$$

To narrow down the parameter-space, we add also $\{a(1) = 1, \ddot{a}(0.71) = 0\}$. The consequences of these choices are as follow:

1) The initial condition $a(0) = 0$ introduces a singularity at the beginning of the evolution.
2) The normalization $a(1) = 1$ fixes the age of the Universe.
3) The condition $\ddot{a}(0.71) = 0$ sets the end of the matter-domination epoch.

Defined like this, we have an initial value problem (Eq. 9), which we solve using the shooting method, starting the integration from $t = 0$.

It is possible to also start the integration backwards, from $t = 1$, using as initial conditions: $a(1) = 1, \phi(1) = \phi_{end}, \dot{\phi}(1) = 0$ and aim for $a(0) = 0$. Here $\dot{\phi}(1) = 0$ guarantees that the evolution of the inflaton field has stopped and the universe is expanding in an accelerated fashion. While both approaches work, integrating forward has the benefit of dealing with the singularity at $a(0) = 0$ at the beginning of the integration, rather than at its end. Moving our initial point of integration away from $a(0) = 0$ decreases the significance of the term $p_u/a(t)^3$. This effectively means putting $p_u = 0$, which we do not want, because p_u is the conserved Noether charge of the "dust" dark matter current (see [8]).

The initial velocity of the scalar field $\dot{\phi}(0)$ is not a free parameter of the system, because its value is quickly fixed by the inflaton equation, i.e the results do not depend on $\dot{\phi}(0)$ in a very large interval.

An important feature of the model, is that the type of evolution one would obtain, depends critically on the starting position on the effective potential. We consider as physically "realistic" only the evolution with four epochs—short first deceleration epoch (FD), early inflation (EI), second deceleration (SD) which we interpret as radiation and matter dominated epochs together and finally—slowly accelerating expansion (AE). In terms of the equation of state(EOS) parameter $w(t) = p/\rho$, those are solutions for which: 1) $w_{FD} \to 1/3$, corresponding to the EOS of ultra-relativistic matter, 2) $w_{EI} \to -1$—EOS of dark energy, 3) $w_{SD} > -1/3$—EOS of matter-radiation domination, 4) $w_{AE} < -1/3$—accelerating expansion period. One obtains this type of solution only for specific choice of the parameters and when starting on the slope of the effective potential. Starting anywhere else results in a non-physical solution (with less epochs). Here we will work only with the "realistic" solutions.

Numerically, the times of the different epochs are defined by the three points in which second derivative of the scale factor becomes zero, i.e. $\ddot{a}(t_i) = 0$ for $t_i = t_{EI}, t_{SD}, t_{AE}$. In the units we use, $t_{\text{SD}} \sim 10^{-50}$ and $t_{\text{AE}} \sim 0.71$. We have already reported [11] a study on how the choice of the parameters affects t_{SD}. Here we will discuss some additional features of the model.

In [11] we used the parameter b_0 to set $t_{\text{AE}} \sim 0.71$ and parameter f_1 to ensure $a(1) = 1$. Changing f_1 however changes the effective potential defined by:

$$U_{eff}(\phi) = \frac{1}{4} \frac{(f_1 e^{-\alpha\phi} + M_1)^2}{\chi_2(f_2 e^{-2\alpha\phi} + M_2) - 2M_0}. \tag{10}$$

thus making it harder to study how the solutions depend on the starting position on the slope.

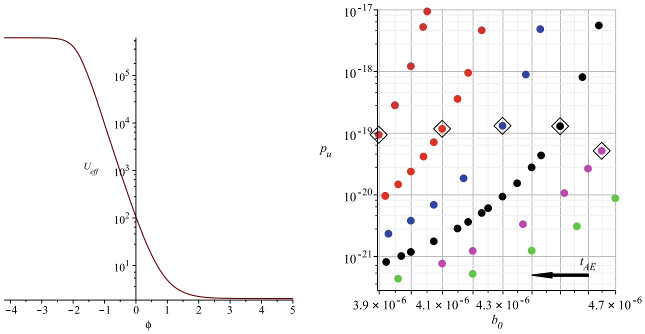
In the current article, we will go a different route and we will use b_0 to set $t_{\text{AE}} \sim 0.71$ and p_u to ensure $a(1) = 1$. This will simplify our problem significantly, since now we will have only 3 parameters to consider $\{b_0, p_u, \phi_0\}$. It will also enable us to study how the solutions depend on ϕ_0, as the effective potential does not depend on b_0 and p_u.

Numerically, we work with the following solution:

$$\{\chi_2 = 1,\ M_0 = -0.034,\ M_1 = 0.8,\ M_2 = 0.01, \alpha = 2.4,\ f_1 = 5,\ f_2 = 10^{-5}\}.$$

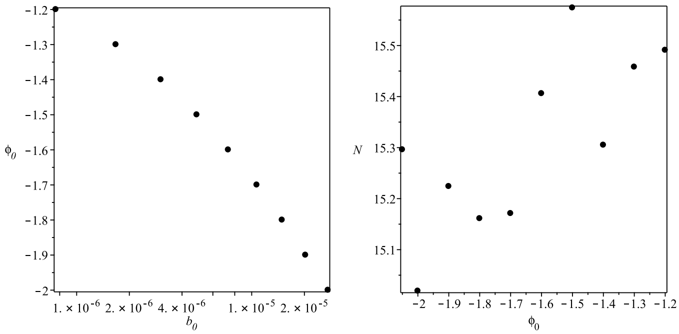
For these values of the parameters, the effective potential is step-like, as seen on Fig. 1(a) . The effective potential reaches its asymptotic values for the plateaus for $\phi_- < -4.5$, $\phi_+ > 1.7$ (i.e. where $U'_{eff} \to 0$). The slope can be defined as the region $\phi \in (-2.5, -1.2)$, with an inflexion point at $\phi_0 = -1.87$.

For this effective potential, one can find different sets of solutions. A family of solutions is shown on Fig. 1(b). One can see the different branches of the solutions corresponding to different ϕ_0 (in this case $\phi_0 \in [-1.5, -1.45]$). As we have mentioned, we require from our solutions to fulfill $a''(0.71) = 0$, i.e. $t_{AE} = 0.71$. The points on Fig. 1(b) do that with precision of 10^{-2}, the diamonds show the points



a b

Fig. 1 On the panels one can see: **a** the effective potential **d** the plane $p_u(b_0)$, the different branches correspond to different ϕ_0, the points are solutions, the diamonds are solutions with $t_{AE} = 0.709 - 0.711$



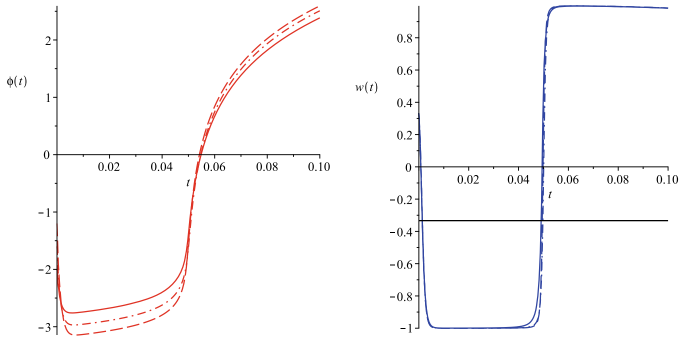
a b

Fig. 2 On the panels is **a** the dependence $\phi_0(b_0)$ **b** the dependence $N(\phi_0)$

which satisfy it with precision of 10^{-3} and the different branches correspond to different ϕ_0. From this plot, one can gain insight on the dependence of $b_0(p_u)$ for the solutions (here different points on the branches correspond to $t_{AE} = 0.705..0.715$) and how solutions corresponding to different ϕ_0 depend on t_{AE} which increases with the decrease of b_0 along each branch.

On Fig. 2(a) we show the dependence $\phi_0(b_0)$ for solutions with $t_{AE} = 0.71$ with precision of 10^{-3}. We do not show the dependence $\phi_0(p_u)$ as it appears chaotic.

A very interesting feature of these solutions, plotted on Fig. 2(a) and (b), is that they keep t_{EI} and t_{SD} approximately constant. I.e. while moving up and down the slope, we do not change the physical properties of our solutions. This can be seen on Fig. 2(b), where we have plotted the e-folds parameter defined as $N = \ln(a_{SD}/a_{EI})$ for the solutions corresponding to different ϕ_0. One can see that it remains more or less the same under the precision we are working with. In our results, only $\phi(t)$ is sensitive to the changes in ϕ_0.



a b

Fig. 3 Zoom in on the evolution in the interval $t \in (0, 0.1)$ of: **a** the inflaton field $\phi(t)$, **b** the equation of state $w(t) = p(t)/\rho(t)$. Here $\phi_0 = -1.2, -1.6, -2$ are with dashed, dashed-dotted and solid lines respectively

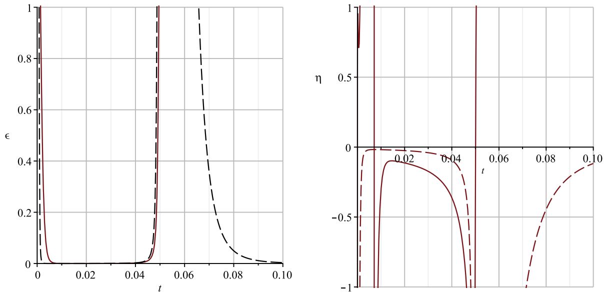
On Fig. 3(a) and (b) we have plotted how $w(t)$ and $\phi(t)$ vary for $\phi_0 = \{-1.2, -1.6, -2\}$ and we have zoomed on the interval $t = 0..0.1$ where inflation should take place. One can see that there is very little difference in the times of inflation (corresponding to $w(t) = -1$), but there is more visible difference in the evolution of $\phi(t)$. This, however leads to a very small deviation in $\phi(t = 1)$—less than 5% from the lowest to the highest point on the slope. This result is highly unexpected since one could expect that the power of the inflation and end values of $a(t)$, $\phi(t)$ will depend more strongly on ϕ_0, which we do not observe here. Note, we could not integrate further up than $\phi_0 = -2.05$ which is far from the upper end of the slope $\phi = -2.5$. This is because after this point, $\ddot{\phi}(t)$ becomes infinite and the numerical system hits a singularity.

Finally, an important note to make is that on Fig. 3(a) one can see the reported before [11, 12] climbing up the slope. Somewhat unexpectedly, it is strongest for points with lowest ϕ_0. For them, the highest value of $\phi(t)$ is reaching $\phi = -3.10$ which corresponds to the upper plateau. This is a further confirmation of our observation in [11] that the movement of the inflaton doesn't correspond to the classical exchange of potential energy for kinetic one, but instead it is closer to the stability of the L4 and L5 Lagrange points.

This follows from the fact that the effective potential does not bring the kinetic energy in standard form (Eq. 2). In the slow roll approximation (neglecting the terms $\sim \dot{\phi}^2,\ \dot{\phi}^3,\ \dot{\phi}^4$) the inflaton equation has the form:

$$(A + 1)\ddot{\phi} + 3H(A + 1)\dot{\phi} + U'_{eff} = 0, \tag{11}$$

where $A = \frac{1}{2} b_0 e^{-\alpha\phi} \frac{V+M_1}{U+M_2}$.



a b

Fig. 4 A zoom in on the slow-roll parameters (**a** ϵ and **b** η) in the interval where infation occurs ($t = 0.00138..0.04953$). With a solid black line are plotted the dynamical parameters, with the black dashed one—the potential ones. The values of $\{b_0, p_u, \phi_0\}$ are $\{0.76 \times 10^{-6}, 0.53 \times 10^{-19}, -1.2\}$

For this reason, we find it interesting to compare the kinetic slow-roll parameters (also called dynamical) defined as:

$$\epsilon_H = -\frac{\dot{H}}{H^2}, \quad \eta_H = -\frac{\ddot{\phi}}{H\dot{\phi}} \tag{12}$$

with the potential slow-roll parameters which can be derived to be [4, 13] (Note that our A is different from the one used in [4]):

$$\epsilon_V = \frac{1}{1+A}\left(\frac{U'_{eff}}{U_{eff}}\right)^2, \quad \eta_V = \frac{2}{1+A}\frac{U''_{eff}}{U_{eff}} \tag{13}$$

On Fig. 4 we present an example of the evolution of the "slow-roll" parameters for both the kinetic and the potential definitions [4, 13]. One can see from the plots that the two definitions in this interval are very similar—both give mostly very small values of the slow-roll parameters. One also notices that the intervals on t for which the slow-roll parameters of both kinds are very small (say, $| < 0.1|$) are shorter than the numerically obtained one, given by $t_{EI}..t_{SD}$.[1] In general, it seems that the potential slow-roll parameters give intervals closer to the numerical ones. However, the conclusion is that if the slow-roll parameters are used to estimate inflation theoretically, those small deviations in the intervals may lead to misestimations of N.

Finally, a note on the e-folds parameter, which measures the power of the inflation. The theoretical estimation for the number of e-folds needed to solve the horizon problem is model-dependent but is $N > 70$. In our example, we get $N \approx 15$. It is important to note, that there is a numerical maximum of the number of e-folds of about $N \approx 22$, due to the fact we are starting our integration at $a(0) \sim 10^{-10}$. In

[1]To be precise, the intervals are: for ϵ_V : 0.0012..0..0473, for ϵ_H : 0.0034..0.0481 for η_V := 0.0021..0.041 for η_H : 0.013..0.033.

order to get a higher N, one needs to start with smaller $a(0)$, but to do so, we need to improve significantly the precision of the integration. Parameters-wise, the best way to get powerful early inflation is trough increasing α or decreasing f_2 [11].

4 Conclusions

In this work, we have explored numerically the model of Guendelman-Nissimov-Pacheva in a specific part of its parametric space related to the different initial conditions on the slope of the effective potential. Even though it is impossible to study the entire parameter space, we have shown some important properties of the model.

Most importantly, we have shown that there exist families of solutions on the slope which preserve the initial and the ending times of the inflation and also, that they give similar number of e-folds. Furthermore, one can see that the main difference between starting at the top of the slope and at its bottom is the behavior of the inflaton scalar field, which climbs up all the way to the top of the slope before entering in inflation regime. This mechanism requires further study.

Finally, we have considered the dynamical and the potential slow-roll parameters for the model and we have shown that the potential slow-roll parameters seem to describe the inflationary period better. They, however, do not match entirely with the numerically obtained duration of the inflation.

Acknowledgments It is a pleasure to thank E. Nissimov, S. Pacheva and M. Stoilov for discussions.

The work is supported by Bulgarian NSF grant DN-18/1/10.12.2017 and by Bulgarian NSF grant 8-17. D.S. is also partially supported by COST Actions CA18108.

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Gauge Theory of Gravity Based on the Correspondence Between the 1^{st} and the 2^{nd} Order Formalisms

D. Benisty, E. I. Guendelman, and J. Struckmeier

Abstract A covariant canonical gauge theory of gravity free from torsion is studied. Using a metric conjugate momentum and a connection conjugate momentum, which takes the form of the Riemann tensor, a gauge theory of gravity is formulated, with form-invariant Hamiltonian. Through the introduction of the metric conjugate momenta, a correspondence between the Affine-Palatini formalism and the metric formalism is established. When the dynamical gravitational Hamiltonian $\tilde{H}_{Dyn}$ does not depend on the metric conjugate momenta, a metric compatibility is obtained from the equations of motion and the equations of motion correspond to the solution is the metric formalism.

1 Introduction

General Relativity is one of the well tested theories in physics, with many excellent predictions. A search of a rigorous derivation of General Relativity on the basis of the action principle and the requirement that the description of any system should be form-invariant under general space time transformations has been constructed in the framework of the Covariant Canonical Gauge theory of Gravity.

The Covariant Canonical Gauge theory of Gravity [1, 2, 7] is formulated within the framework of the covariant Hamiltonian formalism of classical field theories. The latter ensures by construction that the action principle is maintained in its form requiring all transformations of a given system to be canonical. The imposed requirement of invariance of the original action integral with respect to local transformations in curved space time is achieved by introducing additional degrees of freedom, the gauge fields. In the basis of the formulation there are two independent fields: the metric $g^{\alpha\beta}$, which contains the information about lengths and angles of the space time, and the connection $\gamma^{\lambda}{}_{\alpha\beta}$, which contains the information how a vector transforms

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_21

under parallel displacement. In this formulation, these two fields are assumed to be independent dynamical quantities in the action and referred to as the Affine-Palatini formalism (or the 1^{st} order formalism).

Using the structure of metric and the affine connection independently, with their conjugate momenta, yields a new formulation of gauge theories of gravity. In [1] the discussion was with the presence of torsion, and here we discuss about the formulation with no torsion from the beginning, that has a link between the metric affine and the metric formalism [3].

2 A Basic Formulation

The Covariant Canonical Gauge theory of Gravity is a well defined formulation derived from the canonical transformation theory in the covariant Hamiltonian picture of classical field theories [1]. It identifies two independent fundamental fields, which form the basis for a description of gravity: the metric $g^{\alpha\beta}$ and the connection $\gamma^{\lambda}{}_{\alpha\beta}$. In the Hamiltonian description, any fundamental field has a conjugate momentum: the metric conjugate momentum is $\tilde{k}^{\alpha\lambda\beta}$ and the connection conjugate momentum is $\tilde{q}_{\eta}{}^{\alpha\xi\beta}$:

$$S = \int_R \left(\tilde{k}^{\alpha\lambda\beta}\, g_{\alpha\lambda,\beta} - \tfrac{1}{2}\tilde{q}_{\eta}{}^{\alpha\xi\beta}\gamma^{\eta}{}_{\alpha\xi,\beta} - \tilde{\mathscr{H}}_0 \right) d^4x \tag{1}$$

where the "tilde" sign denotes a tensor density, which multiplies the tensor with $\sqrt{-g}$. As the conjugate momentum components of the fields are the duals of the complete set of the derivatives of the field, the formulation is referred to as "covariant canonical". A closed description of the coupled dynamics of fields and space-time geometry has been derived in [1], where the gauge formalism yields:

$$S = \int_R \left(\tilde{k}^{\alpha\lambda\beta}\, g_{\alpha\lambda;\beta} - \tfrac{1}{2}\tilde{q}_{\eta}{}^{\alpha\xi\beta} R^{\eta}{}_{\alpha\xi\beta} - \tilde{\mathscr{H}}_{\mathrm{Dyn}}(\tilde{q}, \tilde{k}, g) \right) d^4x \tag{2}$$

As a result of the gauge procedure, all partial derivatives of tensors in Eq. (1) reappear as covariant derivatives. The partial derivative of the (non-tensorial) connection changes into the tensor $R^{\eta}{}_{\alpha\xi\beta}$, which was shown to be the Riemann-Christoffel curvature tensor:

$$R^{\eta}{}_{\alpha\xi\beta} = \frac{\partial\gamma^{\eta}{}_{\alpha\beta}}{\partial x^{\xi}} - \frac{\partial\gamma^{\eta}{}_{\alpha\xi}}{\partial x^{\beta}} + \gamma^{\tau}{}_{\alpha\beta}\,\gamma^{\eta}{}_{\tau\xi} - \gamma^{\tau}{}_{\alpha\xi}\,\gamma^{\eta}{}_{\tau\beta}\,. \tag{3}$$

The "dynamics" Hamiltonian $\tilde{H}_{Dyn}$—which is supposed to describe the dynamics of the free (uncoupled) gravitational field—is to be built from a combination of the metric conjugate momentum, the connection conjugate momentum, and the metric itself.

3 A Correspondence Between the 1^{st} and the 2^{nd} Order Formalism

In addition to the foundations of the gauge theory of gravity, it turned out that the part of the action: $\tilde{k}^{\alpha\beta\gamma} g_{\alpha\beta;\gamma}$, which contains the metric conjugate momentum, has a strong impact as a connector between the affine-Palatini formalism (or the 1^{st} order formalism) and the metric formalism (or the 2^{nd} order formalism):

$$\mathscr{L}(g,\gamma)_{\ 1\mathbf{order}} + k^{\alpha\beta\gamma} g_{\alpha\beta;\gamma} \Leftrightarrow \mathscr{L}(g)_{\ 2\mathbf{order}} \tag{4}$$

In the 1^{st} order formalism, one assumes that there are two independent fields: the metric $g^{\mu\nu}$ and the connection $\gamma^{\mu}_{\alpha\beta}$. In contrast to that, in the 2^{nd} order formalism the connection is assumed to be the Levi Civita or Christoffel symbol:

$$\gamma^{\rho}{}_{\mu\nu} = \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} = \frac{1}{2} g^{\rho\lambda}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}) \tag{5}$$

and appears in the action directly in this way. In general, only for Lovelock theories, which includes Einstein Hilbert action, both formulations will yield the same equations of motion and the connection will be in both cases the Christoffel symbol. In Ref [3], it was proved that for any general action which starts in the 1^{st} order formalism in addition to the term $k^{\alpha\beta\gamma} g_{\alpha\beta;\gamma}$ the energy momentum tensor will be the same as it would be calculated in the 2^{nd} order formalism. The main reason for that correspondence is the metric compatibility constraint. The variation with respect to $k^{\alpha\beta\gamma}$ gives the metricity condition:

$$g_{\alpha\beta;\gamma} = 0 \quad \Rightarrow \quad \gamma^{\rho}{}_{\mu\nu} = \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\}, \tag{6}$$

which cause the connection to be the Christoffel symbol. The variation with respect to the connection gives the tensors:

$$\frac{\delta}{\delta\gamma^{\rho}_{\mu\nu}} k^{\alpha\beta\gamma} g_{\alpha\beta;\gamma} = -k^{\alpha\mu\nu} g_{\rho\alpha} - k^{\alpha\nu\mu} g_{\rho\alpha} \tag{7}$$

with a symmetrization between the components μ and ν. The variation with respect to the metric is:

$$\frac{\delta}{\delta g_{\mu\nu}} k^{\alpha\beta\gamma} g_{\alpha\beta;\gamma} = -k^{\mu\nu\lambda}{}_{;\lambda}. \tag{8}$$

Because of the new contribution to the field equation $k^{\mu\nu\lambda}{}_{;\lambda}$, the complete field equation will contains additional terms which make the first order field equations to be equivalent to the field equation under the second order formalism. Indeed, isolating the tensor $k^{\mu\nu\lambda}$ and inserting it back into Eq. (8) gives the relation:

$$\frac{\partial \mathscr{L}(\kappa)}{\partial g_{\sigma\nu}} = \frac{1}{2}\nabla_{\mu}\left(g^{\rho\sigma}\frac{\partial \mathscr{L}(\kappa)}{\partial \gamma^{\rho}{}_{\mu\nu}} + g^{\rho\nu}\frac{\partial \mathscr{L}(\kappa)}{\partial \gamma^{\rho}{}_{\mu\sigma}} - g^{\rho\mu}\frac{\partial \mathscr{L}(\kappa)}{\partial \gamma^{\rho}{}_{\nu\sigma}}\right) \tag{9}$$

where $\mathscr{L}(\kappa) = k^{\alpha\beta\gamma} g_{\alpha\beta;\gamma}$. The terms in the right hand side represents the additional terms that appear in the second order formalism. One option for obtain the contributions into the field equation is to solve $k^{\alpha\beta\gamma}$. The direct way is by using this equation, that gives the new contributions for the second order formalism into the field equation, from the variation with respect to the connection $\gamma^{\rho}_{\mu\nu}$. An application for this correspondence is from the Covariant Canonical Gauge theory of gravity action (2).

4 Path to Gauge Theories

From the correspondence between the 1^{st} and the 2^{nd} order formalisms theorem, we obtain a basic link between the dependence of the $\mathscr{H}_{\mathbf{Dyn}}$ with the metric conjugate momentum $\tilde{k}^{\alpha\beta\gamma}$ and the structure of the metric energy momentum tensor. In the first case, $\mathscr{H}_{\mathbf{Dyn}}$ does not depend on the metric conjugate momentum $\tilde{k}^{\alpha\beta\gamma}$. A variation with respect to the metric conjugate momentum $\tilde{k}^{\alpha\beta\gamma}$ gives the metric compatibility condition. According to the theorem (4) the gravitational energy momentum tensor is the same as the gravitational energy momentum tensor in the second order formalism. In the second case $\mathscr{H}_{\mathbf{Dyn}}$ does depend on the metric conjugate momentum $\tilde{k}^{\alpha\beta\gamma}$. A variation with respect to the metric conjugate momentum $\tilde{k}^{\alpha\beta\gamma}$ breaks the metric compatibility condition. This basic framework is not a special feature only for the Covariant Canonical Gauge Theory of Gravity, but leads to a fundamental correlation for many options of $\mathscr{H}_{Dyn}$.

In analogy to the definition of the metric energy-momentum tensor density of the given system Hamiltonian, the metric energy-momentum tensor density is being define as the variation of the $\tilde{\mathscr{L}}_m$ with respect to the metric:

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial \tilde{\mathscr{L}}_m}{\partial g_{\mu\nu}} \tag{10}$$

Therefore the complete action takes the form:

$$\tilde{\mathscr{L}} = \tilde{k}^{\alpha\beta\gamma} g_{\alpha\beta;\gamma} - \frac{1}{2}\tilde{q}_{\lambda}{}^{\alpha\beta\gamma} R^{\lambda}{}_{\alpha\beta\gamma} - \tilde{\mathscr{H}}_{\mathbf{dyn}}(\tilde{q}, \tilde{k}, g) + \tilde{\mathscr{L}}_m \tag{11}$$

The variation with respect to the metric conjugate momenta:

$$g_{\alpha\beta;\gamma} = \frac{\partial \tilde{H}_{\mathbf{Dyn}}}{\partial \tilde{k}^{\alpha\beta\gamma}} \tag{12}$$

which presents the existence of non-metricity if $\mathscr{H}_{Dyn}$ depends on $k^{\alpha\beta\gamma}$. The second variation is the variation with respect to the connection:

$$-\left(\tilde{k}^{\alpha\mu\nu}+\tilde{k}^{\alpha\nu\mu}\right)g_{\alpha\rho}=\frac{1}{2}\nabla_{\beta}\left(\tilde{q}_{\rho}{}^{\mu\beta\nu}+\tilde{q}_{\rho}{}^{\nu\beta\mu}\right), \tag{13}$$

which contracts the relation between the momenta of the metric and the connection. The third variation is with respect to the connection conjugate momentum $\tilde{q}_{\sigma}{}^{\mu\nu\rho}$, which gives:

$$\frac{\partial\tilde{H}_{\mathbf{Dyn}}}{\partial\tilde{q}_{\sigma}{}^{\mu\nu\rho}}=-\frac{1}{2}R^{\sigma}{}_{\mu\nu\rho} \tag{14}$$

If $\tilde{H}_{\mathbf{Dyn}}$ is not depend on $\tilde{q}$, the Riemann tensor will be zero. Therefore the contribution for the stress energy tensor comes from the Dynamical Hamiltonian and from the metric conjugate momenta:

$$T^{\mu\nu}=g^{\mu\nu}(k^{\alpha\beta\gamma}g_{\alpha\beta;\gamma}-\frac{1}{2}q_{\lambda}^{\alpha\beta\gamma}R^{\lambda}_{\alpha\beta\gamma})-2k^{\mu\nu\gamma}_{;\gamma}+\frac{2}{\sqrt{-g}}\frac{\partial\tilde{\mathscr{H}}_{\mathbf{Dyn}}}{\partial g_{\mu\nu}} \tag{15}$$

From the variation with respect the connection (13), the value of the momentum $\tilde{k}^{\alpha\beta\gamma}$. As an example, we consider a Dynamical Hamiltonian which has no dependence with the metric conjugate momentum.

5 Sample $\tilde{\mathscr{H}}_{\mathbf{dyn}}$ Without Breaking Metricity

Our starting point is a dynamical Hamiltonian with the connection conjugate momentum up to the second order, without a dependence on the metric conjugate momentum:

$$\tilde{\mathscr{H}}_{\mathbf{dyn}}=\frac{1}{4g_1}\tilde{q}_{\eta}{}^{\alpha\epsilon\beta}q_{\alpha}{}^{\eta\tau\lambda}g_{\epsilon\tau}g_{\beta\lambda}-g_2\,\tilde{q}_{\eta}{}^{\alpha\tau\beta}g_{\alpha\beta}\delta^{\eta}_{\tau}+g_3\sqrt{-g} \tag{16}$$

This Hamiltonian was investigated in [4] under the original formalism for non-zero torsion (which is finally set to zero), and led to resolving the cosmological constant problem. In our case, assuming that there is no torsion, the formalism demands that the energy momentum tensor is covariantly conserved, as is supposed to be in the second order formalism.

The variation with respect to the metric conjugate momenta $\tilde{k}^{\alpha\beta\gamma}$ give the metricity condition. The variation with respect to the connection conjugate momenta $\tilde{q}_{\sigma}{}^{\mu\nu\rho}$ gives

$$q_{\eta\alpha\epsilon\beta}=g_1\left(R_{\eta\alpha\epsilon\beta}-\hat{R}_{\eta\alpha\epsilon\beta}\right) \tag{17}$$

where:

$$\hat{R}_{\eta\alpha\epsilon\beta} = g_2 \left(g_{\eta\epsilon} g_{\alpha\beta} - g_{\eta\beta} g_{\epsilon\alpha} \right) \tag{18}$$

refers to the ground state geometry of space-time which is the de Sitter (dS) or the anti-de Sitter (AdS) space-time for the positive or the negative sign of g_2, respectively. The last variation is with respect to the metric. In order to isolate the tensor $k^{\mu\nu\gamma}$ one can use the following process: First, we multiply by the metric $g^{\rho\sigma}$ and sum over the index σ:

$$-\tilde{k}^{\sigma\mu\nu} - \tilde{k}^{\sigma\nu\mu} = \frac{1}{2}\nabla_\alpha \left(\tilde{q}^{\ \sigma\mu\alpha\nu} + \tilde{q}^{\ \sigma\nu\alpha\mu}\right) \tag{19}$$

Switching the indices $\sigma \leftrightarrow \nu$ and the indices $\mu \leftrightarrow \nu$ gives a new combination of the $k^{\alpha\beta\gamma}$ tensor. By summing the Eqs. the isolated value gives:

$$-\tilde{k}^{\sigma\nu\mu} = \frac{1}{2}\nabla_\alpha \left(\tilde{q}^{\sigma\mu\alpha\nu} + \tilde{q}^{\nu\mu\alpha\sigma}\right) \tag{20}$$

Therefore, the contribution for the stress energy momentum comes from the covariant derivative of (20):

$$T^{\mu\nu} = -\frac{1}{2} g^{\mu\nu} q_\lambda^{\ \ \alpha\beta\gamma} R^\lambda_{\ \ \alpha\beta\gamma} + \nabla_\gamma \nabla_\alpha \left(q^{\mu\gamma\nu\alpha} + q^{\nu\gamma\mu\alpha}\right) + \frac{2}{\sqrt{-g}} \frac{\partial \tilde{\mathscr{H}}_{\mathbf{Dyn}}}{\partial g_{\mu\nu}} \tag{21}$$

Plugging in the value of the tensor $q_{\alpha\beta\gamma\delta}$ from Eq. (17) gives the result:

$$\begin{aligned} T^{\mu\nu} &= \frac{1}{8\pi G} G^{\mu\nu} + g^{\mu\nu} \Lambda \\ +g_1 \Big(R^{\mu\alpha\beta\gamma} R^\nu_{\alpha\beta\gamma} &- \frac{1}{4} g^{\mu\nu} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} + (\nabla_\alpha \nabla_\beta + \nabla_\beta \nabla_\alpha) R^{\mu\alpha\nu\beta} \Big) \end{aligned} \tag{22}$$

where $G^{\mu\nu}$ is the Einstein tensor. The coupling constants relate to the physical quantities with the relations:

$$g_1 g_2 = \frac{1}{16\pi G}, \quad 6 g_1 g_2^2 + g_3 = \frac{\Lambda}{8\pi G} \tag{23}$$

This stress energy momentum tensor is exactly the same metric energy momentum tensor if our starting point was the effective Lagrangian:

$$\mathscr{L} = g_1 R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - \frac{1}{16\pi G}(R - 2\Lambda) \tag{24}$$

and the stress energy momentum tensor is the same stress energy momentum tensor for this Lagrangian in the second order formalism. One from the big benefits of this formulation is common in many gauge theories of gravity [5], where the starting point is with additional variables with no higher derivatives in the action, and the equations of motion are equivalent to actions with higher derivatives of metric. In this specific case, the starting point is with the quartic momentum q and at the end is equivalent to an action with quadratic Riemann term.

6 Discussion

In this paper we investigated the formulation of the covariant canonical gauge theory of gravity free from torsion. Diffeomorphisms appear as canonical transformations. A tensor field which plays the role of the canonical conjugate of the metric is introduced. It enforces the metricity condition provided that the "Dynamics" Hamiltonian does not depend on this field. The resulting theory has a direct correspondence with our recent work concerning the correspondence between the first order formalism and the second order formalism through the introduction of a Lagrange multiplier field which in this case corresponds with the field that is used to provide the metric with a canonically conjugate momentum. The procedure is exemplified by using a "Dynamics" Hamiltonian which consists of a quadratic term of the connection conjugate momentum. The effective stress energy momentum tensor that emerged from the canonical equations of motion were equivalent to Einstein Hilbert tensor in addition to quadratic Riemann term. [8] derives the complete combination for the quadratic theories of gravity with R^2 and $R_{\mu\nu}R^{\mu\nu}$. [9] derives inflation from fermios based on the Covariant Canonical Gauge theories of Gravity approach to spinors [10].

Acknowledgements We gratefully acknowledge support of our collaboration through the Exchange Agreement between Ben-Gurion University, Beer-Sheva, Israel and Bulgarian Academy of Sciences, Sofia, Bulgaria. D.B. partially supported by COST Actions CA15117, CA16104 and the action CA18108.

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Integrable Systems

Nested Bethe Ansatz for RTT–Algebra of $U_q(\mathrm{sp}(2n))$ Type

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Abstract We study the highest weight representations of the RTT–algebras for the R–matrix of $\mathrm{sp}_q(2n)$ type by the nested algebraic Bethe ansatz. It is a generalization of our study for R–matrix of $\mathrm{sp}(2n)$ and $\mathrm{so}(2n)$ type.

1 Introduction

The formulation of the quantum inverse scattering method, or algebraic Bethe ansatz, by the Leningrad school [1] provides eigenvectors and eigenvalues of the transfer matrix. The latter is the generating function of the conserved quantities of a large family of quantum integrable models. The transfer matrix eigenvectors are constructed from the representation theory of the RTT–algebras. In order to construct these eigenvectors, one should first prepare Bethe vectors, depending on a set of complex variables. The first formulation of the Bethe vectors for the $\mathrm{gl}(n)$–invariant models was given by P.P. Kulish and N.Yu. Reshetikhin in [2] where the nested algebraic Bethe ansatz was introduced. These vectors are given by recursion on the rank of the algebra. Our calculation is some q–generalization of the construction which we published in recent works [3–6] for the non-deformed case of $\mathrm{sp}(2n)$, $\mathrm{so}(2n)$ and $\mathrm{sp}(4)$.

Our construction of Bethe vectors used the new RTT–algebra $\tilde{\mathcal{A}}_n$ which is defined in Sect. 3 and is not the RTT–subalgebra of $\mathrm{sp}_q(2n)$.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_22

This algebra has two RTT–subalgebras of $\mathrm{gl}_q(n)$ type and the study of the nested Bethe ansatz for this RTT–algebra is in progress. The simplest case for $n = 2$ was really solved and we will publish in the next paper.

Our construction of Bethe vectors is in any sense a generalization of Reshetikhin's results [7]. Another approach to the nested Bethe ansatz for very special representations of RTT–algebras of $\mathrm{sp}(2n)$ type was given by Martin and Ramas [8].

In this note, due to the lack of space, we omit the proofs of many claims. Mostly, it is possible to prove them similarly as the corresponding claims in [6].

2 Basic Definitions and Notation

Let indices go trough the set $\{\pm 1, \pm 2, \ldots, \pm n\}$. We will denote by $\mathbf{E}_i^k$ the matrices that have all elements equal to zero with the exception of the element on the i-th row and k-th column that is equal to one. Then $\mathbf{I} = \sum_{k=-n}^{n} \mathbf{E}_k^k$ is the unit matrix and $\mathbf{E}_i^k \mathbf{E}_r^s = \delta_r^k \mathbf{E}_i^s$ is valid.

We will consider the R–matrix of $U_q\big(\mathrm{sp}(2n)\big)$ which has the shape

$$\begin{aligned}\mathbf{R}(x) = \frac{1}{\alpha(x)}\Big(&\sum_{i,k;\, i\neq \pm k} \mathbf{E}_i^i \otimes \mathbf{E}_k^k + f(x)\sum_i \mathbf{E}_i^i \otimes \mathbf{E}_i^i \\ &+ f(x^{-1}q^{-n-1})\sum_i \mathbf{E}_i^i \otimes \mathbf{E}_{-i}^{-i} + g(x)\sum_{k<i} \mathbf{E}_k^i \otimes \mathbf{E}_i^k - g(x^{-1})\sum_{i<k} \mathbf{E}_k^i \otimes \mathbf{E}_i^k \\ &- g(xq^{n+1})\sum_{k<i} q^{k-i}\epsilon_i\epsilon_k \mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i} + g(x^{-1}q^{-n-1})\sum_{i<k} q^{k-i}\epsilon_i\epsilon_k \mathbf{E}_k^i \otimes \mathbf{E}_{-k}^{-i}\Big)\end{aligned}$$

where $\epsilon_i = \mathrm{sign}(i)$ and

$$f(x) = \frac{xq - x^{-1}q^{-1}}{x - x^{-1}}\,, \qquad g(x) = \frac{x(q - q^{-1})}{x - x^{-1}}\,, \qquad \alpha(x) = 1 + \frac{q - q^{-1}}{x - x^{-1}}\,.$$

This R–matrix satisfies the Yang-Baxter equation

$$\mathbf{R}_{1,2}(x)\mathbf{R}_{1,3}(xy)\mathbf{R}_{2,3}(y) = \mathbf{R}_{2,3}(y)\mathbf{R}_{1,3}(xy)\mathbf{R}_{1,2}(x)$$

and is invertible.

The RTT–algebra of $U_q\big(\mathrm{sp}(2n)\big)$ type is an associative algebra $\mathcal{A}$ with unit, which is generated by $T_k^i(x)$, for which the monodromy operator

$$\mathbf{T}(x) = \sum_{i,k=-n}^{n} \mathbf{E}_i^k \otimes T_k^i(x)$$

fulfills the RTT–equation

$$\mathbf{R}_{1,2}(xy^{-1})\mathbf{T}_1(x)\mathbf{T}_2(y) = \mathbf{T}_2(y)\mathbf{T}_1(x)\mathbf{R}_{1,2}(xy^{-1})\,.$$

From the invertibility of the R–matrix we have that the operator

$$H(x) = \mathrm{Tr}\big(\mathbf{T}(x)\big) = \sum_{i=-n}^{n} T_i^i(x)$$

fulfills the equation $H(x)H(y) = H(y)H(x)$ for any x and y.

We suppose that in the representation space $\mathcal{W}$ of the RTT–algebra $\mathcal{A}$ there exists a vacuum vector $\omega \in \mathcal{W}$, for which $\mathcal{W} = \mathcal{A}\omega$ and

$$T_k^i(x)\omega = 0 \quad \text{pro} \quad i < k\,, \qquad T_i^i(x)\omega = \lambda_i(x)\omega \quad \text{pro} \quad i = \pm 1,\ \pm 2,\ \ldots,\ \pm n\,.$$

In the vector space $\mathcal{W} = \mathcal{A}\omega$, we will look for eigenvectors of $H(x)$.

3 RTT–Algebra $\tilde{\mathcal{A}}_n$

In the RTT–algebra $\mathcal{A}$, we have the RTT–subalgebras $\mathcal{A}^{(+)}$ and $\mathcal{A}^{(-)}$ that are generated by the elements $T_k^i(x)$ and $T_{-k}^{-i}(x)$, where $i,\ k = 1,\ 2,\ \ldots,\ n$. First, we will study the subspace

$$\mathcal{W}_0 = \mathcal{A}^{(+)}\mathbf{A}^{(-)}\omega \subset \mathcal{W} = \mathcal{A}\omega\,.$$

Lemma 1. For any $i,\ k = 1,\ 2,\ \ldots,\ n$ and any $\Omega \in \mathcal{W}_0$ $T_k^{-i}(x)\Omega = 0$ is valid.

Lemma 2. If we denote

$$\mathbf{T}^{(+)}(x) = \sum_{i,k=1}^{n} \mathbf{E}_i^k \otimes T_k^i(x)\,, \qquad \mathbf{T}^{(-)}(x) = \sum_{i,k=1}^{n} \mathbf{E}_{-i}^{-k} \otimes T_{-k}^{-i}(x)\,,$$

then on the space $\mathcal{W}_0$ for any $\epsilon_1,\ \epsilon_2 = \pm$

$$\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(xy^{-1})\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{T}_2^{(\epsilon_2)}(y) = \mathbf{T}_2^{(\epsilon_2)}(y)\mathbf{T}_1^{(\epsilon_1)}(x)\mathbf{R}_{1,2}^{(\epsilon_1,\epsilon_2)}(xy^{-1}) \tag{1}$$

where

$$\mathbf{R}^{(+,+)}_{1,2}(x) = \frac{1}{f(x)}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^i_i \otimes \mathbf{E}^k_k \;+\; f(x)\sum_{i=1}^{n}\mathbf{E}^i_i \otimes \mathbf{E}^i_i$$
$$+\, g(x)\sum_{1\le k<i\le n} \mathbf{E}^i_k \otimes \mathbf{E}^k_i - g(x^{-1})\sum_{1\le i<k\le n} \mathbf{E}^i_k \otimes \mathbf{E}^k_i\Big)$$
$$\mathbf{R}^{(-,-)}_{1,2}(x) = \frac{1}{f(x)}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^{-i}_{-i} \otimes \mathbf{E}^{-k}_{-k} + f(x)\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i} \otimes \mathbf{E}^{-i}_{-i}$$
$$+g(x)\sum_{1\le i<k\le n} \mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^{-k}_{-i} - g(x^{-1})\sum_{1\le k<i\le n} \mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^{-k}_{-i}\Big)$$

$$\mathbf{R}^{(+,-)}_{1,2}(x) = \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^i_i \otimes \mathbf{E}^{-k}_{-k} + f(x^{-1}q)\sum_{i=1}^{n}\mathbf{E}^i_i \otimes \mathbf{E}^{-i}_{-i}$$
$$-\, g(xq^{-1})\sum_{1\le k<i\le n} q^{k-i}\mathbf{E}^i_k \otimes \mathbf{E}^{-i}_{-k} + g(x^{-1}q)\sum_{1\le i<k\le n} q^{k-i}\mathbf{E}^i_k \otimes \mathbf{E}^{-i}_{-k}$$
$$\mathbf{R}^{(-,+)}_{1,2}(x) = \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^{-i}_{-i} \otimes \mathbf{E}^k_k + f(x^{-1}q^{-n-1})\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i} \otimes \mathbf{E}^i_i$$
$$-\, g(xq^{n+1})\sum_{1\le i<k\le n} q^{i-k}\mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^i_k + g(x^{-1}q^{-n-1})\sum_{1\le k<i\le n} q^{i-k}\mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^i_k$$

is valid.

Proposition 1. If we define

$$\tilde{\mathbf{R}}_{1,2}(x) = \mathbf{R}^{(+,+)}_{1,2}(x) + \mathbf{R}^{(+,-)}_{1,2}(x) + \mathbf{R}^{(-,+)}_{1,2}(x) + \mathbf{R}^{(-,-)}_{1,2}(x)$$
$$\tilde{\mathbf{T}}(x) = \mathbf{T}^{(+)}(x) + \mathbf{T}^{(-)}(x)\,,$$

the RTT–equation

$$\tilde{\mathbf{R}}_{1,2}(xy^{-1})\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{T}}_2(y) = \tilde{\mathbf{T}}_2(y)\tilde{\mathbf{T}}_1(x)\tilde{\mathbf{R}}_{1,2}(xy^{-1})$$

is valid on the space $\mathcal{W}_0$.

Also, the R–matrix $\tilde{\mathbf{R}}(x)$ fulfills the Yang–Baxter equation

$$\tilde{\mathbf{R}}_{1,2}(x)\tilde{\mathbf{R}}_{1,3}(xy)\tilde{\mathbf{R}}_{2,3}(y) = \tilde{\mathbf{R}}_{2,3}(y)\tilde{\mathbf{R}}_{1,3}(xy)\tilde{\mathbf{R}}_{1,2}(x)$$

and has the inverse matrix

$$\big(\tilde{\mathbf{R}}_{1,2}(x)\big)^{-1} = \big(\mathbf{R}^{(+,+)}_{1,2}(x)\big)^{-1} + \big(\mathbf{R}^{(+,-)}_{1,2}(x)\big)^{-1} + \big(\mathbf{R}^{(-,+)}_{1,2}(x)\big)^{-1} + \big(\mathbf{R}^{(-,-)}_{1,2}(x)\big)^{-1}$$

where

$$\begin{aligned}
\big(\mathbf{R}^{(+,+)}_{1,2}(x)\big)^{-1} &= \frac{1}{f(x^{-1})}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^i_i \otimes \mathbf{E}^k_k + f(x^{-1})\sum_{i=1}^{n}\mathbf{E}^i_i \otimes \mathbf{E}^i_i \\
&\quad - g(x)\sum_{1\le k<i\le n}\mathbf{E}^i_k \otimes \mathbf{E}^k_i + g(x^{-1})\sum_{1\le i<k\le n}\mathbf{E}^i_k \otimes \mathbf{E}^k_i\Big) \\
\big(\mathbf{R}^{(-,-)}_{1,2}(x)\big)^{-1} &= \frac{1}{f(x^{-1})}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^{-i}_{-i} \otimes \mathbf{E}^{-k}_{-k} + f(x^{-1})\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i} \otimes \mathbf{E}^{-i}_{-i} \\
&\quad - g(x)\sum_{1\le i<k\le n}\mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^{-k}_{-i} + g(x^{-1})\sum_{1\le k<i\le n}\mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^{-k}_{-i}\Big) \\
\big(\mathbf{R}^{(+,-)}_{1,2}(x)\big)^{-1} &= \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^i_i \otimes \mathbf{E}^{-k}_{-k} + f(xq^{-n-1})\sum_{i=1}^{n}\mathbf{E}^i_i \otimes \mathbf{E}^{-i}_{-i} \\
&\quad + g(xq^{-n-1})\sum_{1\le k<i\le n} q^{i-k}\mathbf{E}^i_k \otimes \mathbf{E}^{-i}_{-k} - g(x^{-1}q^{n+1})\sum_{1\le i<k\le n} q^{i-k}\mathbf{E}^i_k \otimes \mathbf{E}^{-i}_{-k} \\
\big(\mathbf{R}^{(-,+)}_{1,2}(x)\big)^{-1} &= \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^{-i}_{-i} \otimes \mathbf{E}^k_k + f(xq)\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i} \otimes \mathbf{E}^i_i \\
&\quad + g(xq)\sum_{1\le i<k\le n} q^{k-i}\mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^i_k - g(x^{-1}q^{-1})\sum_{1\le k<i\le n} q^{k-i}\mathbf{E}^{-i}_{-k} \otimes \mathbf{E}^i_k
\end{aligned}$$

The validity of the RTT–equation is Lemma 2. The Yang–Baxter equation that is equivalent to the equations

$$\mathbf{R}^{(\epsilon_1,\epsilon_2)}_{1,2}(x)\mathbf{R}^{(\epsilon_1,\epsilon_3)}_{1,3}(xy)\mathbf{R}^{(\epsilon_2,\epsilon_3)}_{2,3}(y) = \mathbf{R}^{(\epsilon_2,\epsilon_3)}_{2,3}(y)\mathbf{R}^{(\epsilon_1,\epsilon_3)}_{1,3}(xy)\mathbf{R}^{(\epsilon_1,\epsilon_2)}_{1,2}(x) \tag{2}$$

and the conditions for the inverse R–matrix, i.e. the relations

$$\mathbf{R}^{(\epsilon_1,\epsilon_2)}_{1,2}(x)\big(\mathbf{R}^{(\epsilon_1,\epsilon_2)}_{1,2}(x)\big)^{-1} = \mathbf{I}_{\epsilon_1}\otimes\mathbf{I}_{\epsilon_2}\,, \quad \text{where} \quad \mathbf{I}_+ = \sum_{i=1}^{n}\mathbf{E}^i_i\,, \quad \mathbf{I}_- = \sum_{i=1}^{n}\mathbf{E}^{-i}_{-i}\,,$$

can be shown by direct calculation.

Definition. We denote the RTT–algebra defined by the R–matrix $\tilde{\mathbf{R}}(x)$ as $\tilde{\mathcal{A}}_n$.

We find out by the standard procedure from the RTT–equation (1) that in the RTT–algebra $\tilde{\mathcal{A}}_n$ mutually commutate not only the operators $\tilde{H}(x)$ and $\tilde{H}(y)$, where

$$\tilde{H}(x) = \mathrm{Tr}_{(+,-)}\big(\tilde{\mathbf{T}}(x)\big) = \mathrm{Tr}_+\big(\mathbf{T}^{(+)}(x)\big) + \mathrm{Tr}_-\big(\mathbf{T}^{(-)}(x)\big) = \sum_{i=1}^{n}\big(T^i_i(x) + T^{-i}_{-i}(x)\big)$$

but also all operators $\tilde{H}^{(\pm)}(x)$ a $\tilde{H}^{(\pm)}(y)$, where

$$\begin{aligned}
\tilde{H}^{(+)}(x) &= \mathrm{Tr}_+\big(\mathbf{T}^{(+)}(x)\big) = \sum_{i=1}^{n} T^i_i(x)\,, \\
\tilde{H}^{(-)}(x) &= \mathrm{Tr}_-\big(\mathbf{T}^{(-)}(x)\big) = \sum_{i=1}^{n} T^{-i}_{-i}(x)\,.
\end{aligned}$$

4 General Shape of Eigenvectors

Let $\mathbf{u} = (u_1, u_2, \ldots, u_M)$ be an ordered set of mutually different complex numbers. We will look for eigenvectors in the form

$$\mathfrak{V}(\mathbf{u}) = \sum_{i_1,\ldots,i_M,k_1,\ldots,k_M=1}^{n} T^{i_1}_{-k_1}(u_1)T^{i_2}_{-k_2}(u_2)\ldots T^{i_M}_{-k_M}(u_M)\Phi^{k_1,k_2,\ldots,k_M}_{i_1,i_2,\ldots i_M}$$

where $\Phi^{k_1,k_2,\ldots,k_M}_{i_1,i_2,\ldots i_M} \in \mathcal{W}_0$. Let us denote

$$\mathbf{B}(u) = \sum_{i,k=1}^{n} \mathbf{e}_i \otimes \mathbf{f}^{-k} \otimes T^{i}_{-k}(u) \in \mathcal{V}_+ \otimes \mathcal{V}^*_- \otimes \mathcal{A}$$

where $\mathbf{e}_i$ is the basis of the space $\mathcal{V}_+$ and $\mathbf{f}^{-k}$ is the basis of the space $\mathcal{V}^*_-$ and define

$$\begin{aligned}\mathbf{B}_{1,\ldots,M}(\mathbf{u}) &= \mathbf{B}_1(u_1) \otimes \mathbf{B}_2(u_2) \otimes \ldots \otimes \mathbf{B}_M(u_M)\\ &= \sum_{i_1,\ldots,k_M}^{n} \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_M} \otimes \mathbf{f}^{-k_1} \otimes \ldots \otimes \mathbf{f}^{-k_M} \otimes T^{i_1}_{-k_1}(u_1)\ldots T^{i_M}_{-k_M}(u_M)\end{aligned}$$

If $\mathbf{f}^r$ is the dual basis with respect to $\mathbf{e}_i$ in the space $\mathcal{V}^*_+$ and $\mathbf{e}_{-s}$ is the dual basis with respect to $\mathbf{f}^{-k}$ in the space $\mathcal{V}_-$ and we denote

$$\mathbf{\Phi} = \sum_{r_1,\ldots,r_M,s_1,\ldots,s_M} \mathbf{f}^{r_1} \otimes \ldots \otimes \mathbf{f}^{r_M} \otimes \mathbf{e}_{-s_1} \otimes \ldots \otimes \mathbf{e}_{-s_M} \otimes \mathbf{\Phi}^{s_1,\ldots,s_M}_{r_1,\ldots,r_M}$$

we can write the general shape of Bethe vectors in the form

$$\mathfrak{V}(\mathbf{u}) = \Big\langle \mathbf{B}_{1,\ldots,M}(\mathbf{u}), \mathbf{\Phi} \Big\rangle.$$

5 Commutation Relations $\mathbf{T}^{(\pm)}_0(x)\mathbf{B}_{1,\ldots,M}(\mathbf{u})$

On the space $\mathcal{V}_0 \otimes \mathcal{V}^*_{1_+} \otimes \mathcal{V}_{1_-} \otimes \mathcal{A}$ we define

$$\begin{aligned}\widehat{\mathbf{T}}^{(+)}_{0;1}(x;u) &= \left(\widehat{\mathbf{R}}^{(+,+)}_{0,1^*}(xu^{-1})\right)^{-1}\mathbf{T}^{(+)}_0(x)\widehat{\mathbf{R}}^{(+,-)}_{0,1}(xu^{-1})\\ \widehat{\mathbf{T}}^{(-)}_{0;1}(x;u) &= \left(\widehat{\mathbf{R}}^{(-,+)}_{0,1^*}(xu^{-1})\right)^{-1}\mathbf{T}^{(-)}_0(x)\widehat{\mathbf{R}}^{(-,-)}_{0,1}(xu^{-1})\end{aligned}$$

where

$$\begin{aligned}
\big(\widehat{\mathbf{R}}^{(+,+)}_{0,1^*}(x)\big)^{-1} &= \frac{1}{f(x^{-1})}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^i_i\otimes\mathbf{F}^k_k\otimes\mathbf{I}_- + f(x^{-1})\sum_{i=1}^{n}\mathbf{E}^i_i\otimes\mathbf{F}^i_i\otimes\mathbf{I}_- \\
&\quad + g(x^{-1})\sum_{1\le i<k\le n}\mathbf{E}^i_k\otimes\mathbf{F}^k_i\otimes\mathbf{I}_- - g(x)\sum_{1\le k<i\le n}\mathbf{E}^i_k\otimes\mathbf{F}^k_i\otimes\mathbf{I}_-\Big)\\
\big(\widehat{\mathbf{R}}^{(-,+)}_{0,1^*}(x)\big)^{-1} &= \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^{-i}_{-i}\otimes\mathbf{F}^k_k\otimes\mathbf{I}_- + f(xq)\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i}\otimes\mathbf{F}^i_i\otimes\mathbf{I}_- \\
&\quad + g(xq)\sum_{1\le i<k\le n} q^{k-i}\mathbf{E}^{-i}_{-k}\otimes\mathbf{F}^i_k\otimes\mathbf{I}_- \\
&\quad - g(x^{-1}q^{-1})\sum_{1\le k<i\le n} q^{k-i}\mathbf{E}^{-i}_{-k}\otimes\mathbf{F}^i_k\otimes\mathbf{I}_- \\
\widehat{\mathbf{R}}^{(+,-)}_{0,1}(x) &= \sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^i_i\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-k}_{-k} + f(x^{-1}q)\sum_{i=1}^{n}\mathbf{E}^i_i\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-i}_{-i} \\
&\quad + g(x^{-1}q)\sum_{1\le i<k\le n} q^{k-i}\mathbf{E}^i_k\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-i}_{-k} \\
&\quad - g(xq^{-1})\sum_{1\le k<i\le n} q^{k-i}\mathbf{E}^i_k\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-i}_{-k} \\
\widehat{\mathbf{R}}^{(-,-)}_{0,1}(x) &= \frac{1}{f(x)}\Big(\sum_{i,k=1;\, i\neq k}^{n} \mathbf{E}^{-i}_{-i}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-k}_{-k} + f(x)\sum_{i=1}^{n}\mathbf{E}^{-i}_{-i}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-i}_{-i} \\
&\quad + g(x)\sum_{1\le i<k\le n}\mathbf{E}^{-i}_{-k}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-k}_{-i} - g(x^{-1})\sum_{1\le k<i\le n}\mathbf{E}^{-i}_{-k}\otimes\mathbf{I}^*_+\otimes\mathbf{E}^{-k}_{-i}\Big)
\end{aligned}$$

Lemma 3. In the RTT–algebra of $U_q\big(\mathrm{sp}(2n)\big)$ type the relations

$$\begin{aligned}
\mathbf{T}^{(+)}_0(x)\Big\langle\mathbf{B}_1(u),\mathbf{f}^r\otimes\mathbf{e}_{-s}\Big\rangle &= f(x^{-1}u)\Big\langle\mathbf{B}_1(u),\widehat{\mathbf{T}}^{(+)}_{0;1}(x;u)\big(\mathbf{I}\otimes\mathbf{f}^r\otimes\mathbf{e}_{-s}\big)\Big\rangle\\
&\quad + g(xu^{-1})\Big\langle\mathbf{B}_1(x),\widehat{\mathbf{T}}^{(+)}_{0;1}(u;u)\big(\mathbf{I}\otimes\mathbf{f}^r\otimes\mathbf{e}_{-s}\big)\Big\rangle\\
\mathbf{T}^{(-)}_0(x)\Big\langle\mathbf{B}_1(u),\mathbf{f}^r\otimes\mathbf{e}_{-s}\Big\rangle &= f(xu^{-1})\Big\langle\mathbf{B}_1(u),\widehat{\mathbf{T}}^{(-)}_{0;1}(x;u)\big(\mathbf{I}\otimes\mathbf{f}^r\otimes\mathbf{e}_{-s}\big)\Big\rangle\\
&\quad - g(xu^{-1}))\Big\langle\mathbf{B}_1(x),\widehat{\mathbf{T}}^{(-)}_{0;1}(u;u)\big(\mathbf{I}\otimes\mathbf{f}^r\otimes\mathbf{e}_{-s}\big)\Big\rangle
\end{aligned}$$

are valid.

For ordered M–tuples $\mathbf{u}=(u_1,\dots,u_M)$, let $\overline{u}$ denote the set $\overline{u}=\{u_1,\dots,u_M\}$. We define

$$\begin{aligned}
&\mathbf{u}_k = (u_1,\dots,u_{k-1},u_{k+1},\dots,u_M)\,,\\
&\overline{u}_k = \overline{u}\setminus\{u_k\} = \{u_1,\dots,u_{k-1};u_{k+1},\dots,u_M\}\,,\\
&F(x;\overline{u}^{-1}) = \prod_{k=1}^{M} f(xu_k^{-1})\,, \qquad\qquad F(x^{-1},\overline{u}) = \prod_{k=1}^{M} f(x^{-1}u_k)\,.
\end{aligned}$$

and introduce operators

$$\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\mathbf{u}) = \big(\widehat{\mathbf{R}}^{(+,+)}_{0,1^*}(xu_1^{-1})\big)^{-1}\dots\big(\widehat{\mathbf{R}}^{(+,+)}_{0,M^*}(xu_M^{-1})\big)^{-1}\mathbf{T}^{(+)}_0(x)$$
$$\widehat{\mathbf{R}}^{(+,-)}_{0,M}(xu_M^{-1})\dots\widehat{\mathbf{R}}^{(+,-)}_{0,1}(xu_1^{-1})$$
$$\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\mathbf{u}) = \big(\widehat{\mathbf{R}}^{(-,+)}_{0,1^*}(xu_1^{-1})\big)^{-1}\dots\big(\widehat{\mathbf{R}}^{(-,+)}_{0,M^*}(xu_M^{-1})\big)^{-1}\mathbf{T}^{(-)}_0(x)$$
$$\widehat{\mathbf{R}}^{(-,-)}_{0,M}(xu_M^{-1})\dots\widehat{\mathbf{R}}^{(-,-)}_{0,1}(xu_1^{-1})$$
$$\mathbf{B}_{k;1,\dots,M}(x;\mathbf{u}_k) = \mathbf{B}_k(x)\otimes\mathbf{B}_1(u_1)\otimes\dots\otimes\mathbf{B}_{k-1}(u_{k-1})$$
$$\otimes\,\mathbf{B}_{k+1}(u_{k+1})\otimes\dots\otimes\mathbf{B}_M(u_M)$$

Proposition 2. The following relationships are applied:

$$\mathbf{T}^{(+)}_0(x)\Big\langle\mathbf{B}_{1,\dots,M}(\mathbf{u}),\,\mathbf{\Phi}\Big\rangle = F(x^{-1};\overline{u})\Big\langle\mathbf{B}_{1,\dots,M}(\mathbf{u}),\,\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\mathbf{u})\mathbf{\Phi}\Big\rangle$$
$$+\sum_{u_k\in\overline{u}} g(xu_k^{-1})F(u_k^{-1};\overline{u}_k)\Big\langle\mathbf{B}_{k;1,\dots,M}(x;\mathbf{u}_k),$$
$$\big(\widehat{\mathbf{R}}^{(+,+)}_{1^*,\dots,k^*}(\mathbf{u})\big)^{-1}\widehat{\mathbf{R}}^{(-,-)}_{1,\dots,k}(\mathbf{u})\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(u_k;\mathbf{u})\mathbf{\Phi}\Big\rangle$$

$$\mathbf{T}^{(-)}_0(x)\Big\langle\mathbf{B}_{1,\dots,M}(\mathbf{u}),\,\mathbf{\Phi}\Big\rangle = F(x;\overline{u}^{-1})\Big\langle\mathbf{B}_{1,\dots,M}(\mathbf{u}),\,\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\mathbf{u})\mathbf{\Phi}\Big\rangle$$
$$-\sum_{u_k\in\overline{u}} g(xu_k^{-1})F(u_k;\overline{u}_k^{-1})\Big\langle\mathbf{B}_{k;1,\dots,M}(x;\mathbf{u}_k),$$
$$\big(\widehat{\mathbf{R}}^{(+,+)}_{1^*,\dots,k^*}(\mathbf{u})\big)^{-1}\widehat{\mathbf{R}}^{(-,-)}_{1,\dots,k}(\mathbf{u})\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(u_k;\mathbf{u})\mathbf{\Phi}\Big\rangle$$

where

$$\widehat{\mathbf{R}}^{(+,+)}_{1^*,\dots,k^*}(\mathbf{u}) = \widehat{\mathbf{R}}^{(+,+)}_{(k-1)^*,k^*}(u_{k-1}u_k^{-1})\dots\widehat{\mathbf{R}}^{(+,+)}_{2^*,k^*}(u_2u_k^{-1})\widehat{\mathbf{R}}^{(+,+)}_{1^*,k^*}(u_1u_k^{-1})$$
$$\widehat{\mathbf{R}}^{(-,-)}_{1,\dots,k}(\mathbf{u}) = \widehat{\mathbf{R}}^{(-,-)}_{1,k}(u_1u_k^{-1})\widehat{\mathbf{R}}^{(-,-)}_{2,k}(u_2u_k^{-1})\dots\widehat{\mathbf{R}}^{(-,-)}_{k-1,k}(u_{k-1}u_k^{-1})$$
$$\widehat{\mathbf{R}}^{(+,+)}_{1^*,2^*}(x) = \frac{1}{f(x)}\Big(\sum_{i,k=1;\,i\neq k}^{n}\mathbf{F}^i_i\otimes\mathbf{F}^k_k + f(x)\sum_{i=1}^{n}\mathbf{F}^i_i\otimes\mathbf{F}^i_i$$
$$-\,g(x^{-1})\sum_{1\le i<k\le n}\mathbf{F}^i_k\otimes\mathbf{F}^k_i + g(x)\sum_{1\le k<i\le n}\mathbf{F}^i_k\otimes\mathbf{F}^k_i\Big)$$

6 Bethe Conditions and Eigenvectors of the Operator $H(x)$

Let us denote by $\widehat{T}^i_k(x;\mathbf{u})$ and $\widehat{T}^{-i}_{-k}(x;\mathbf{u})$ the operators defined by the relations

$$\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\mathbf{u}) = \sum_{i,k=1}^{n}\mathbf{E}^k_i\otimes\widehat{T}^i_k(x;\mathbf{u})\,,$$
$$\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\mathbf{u}) = \sum_{i,k=1}^{n}\mathbf{E}^{-k}_{-i}\otimes\widehat{T}^{-i}_{-k}(x;\mathbf{u})\,.$$

The following statement, which gives part of the Bethe conditions, follows from the previous part.

Theorem 1. Let $\mathbf{\Phi}$ be common eigenvector of the operators

$$\widehat{H}^{(+)}_{1,\dots,M}(x;\mathbf{u}) = \mathrm{Tr}_0\Big(\widehat{\mathbf{T}}^{(+)}_{0;1,\dots,M}(x;\mathbf{u})\Big),$$
$$\widehat{H}^{(-)}_{1,\dots,M}(x;\mathbf{u}) = \mathrm{Tr}_0\Big(\widehat{\mathbf{T}}^{(-)}_{0;1,\dots,M}(x;\mathbf{u})\Big)$$

with eigenvalues $\widehat{E}^{(+)}_{1,\dots,M}(x;\mathbf{u})$ and $\widehat{E}^{(-)}_{1,\dots,M}(x;\mathbf{u})$. If for each $u_k \in \overline{u}$ the relations

$$\widehat{E}^{(+)}_{1,\dots,M}(u_k;\mathbf{u})F(u_k^{-1};\overline{u}_k) = \widehat{E}^{(-)}_{1,\dots,M}(u_k;\mathbf{u})F(u_k;\overline{u}_k^{-1}) \tag{3}$$

are true, then $\Big\langle \mathbf{B}_{1,\dots,M}(\mathbf{u}),\,\mathbf{\Phi}\Big\rangle$ is the eigenvector of the operator $H(x) = H^{(+)}(x) + H^{(-)}(x)$, where $H^{(\pm)}(x) = \mathrm{Tr}\big(\mathbf{T}^{(\pm)}_0(x)\big)$ with the eigenvalue

$$E_{1,\dots,M}(x;\mathbf{u}) = \widehat{E}^{(+)}_{1,\dots,M}(x;\mathbf{u})F(x^{-1};\overline{u}) + \widehat{E}^{(-)}_{1,\dots,M}(x;\mathbf{u})F(x;\overline{u}^{-1})\,.$$

Thus, to find the eigenvectors of the operators $H(x)$, it is sufficient to find common eigenvectors of the operators $\widehat{H}^{(+)}_{1,\dots,M}(x;\mathbf{u})$ and $\widehat{H}^{(-)}_{1,\dots,M}(x;\mathbf{u})$.

Theorem 2. The operators $\widehat{\mathbf{T}}^{(\pm)}_{0;1,\dots,M}(x;\mathbf{u})$ fulfill the RTT–equation

$$\begin{aligned}\mathbf{R}^{(\epsilon,\epsilon')}_{0,0'}(xy^{-1})\widehat{\mathbf{T}}^{(\epsilon)}_{0;1,\dots,M}(x;\mathbf{u})\widehat{\mathbf{T}}^{(\epsilon')}_{0';1,\dots,M}(y;\mathbf{u})&\\ = \widehat{\mathbf{T}}^{(\epsilon')}_{0';1,\dots,M}(y;\mathbf{u})\widehat{\mathbf{T}}^{(\epsilon)}_{0;1,\dots,M}(x;\mathbf{u})\mathbf{R}^{(\epsilon,\epsilon')}_{0,0'}(xy^{-1})&\end{aligned}$$

for any $\mathbf{u}$ and $\epsilon,\ \epsilon' = \pm$. Thus, they generate RTT–algebra $\tilde{A}_n$.

Theorem 3. The vector

$$\widehat{\Omega} = \underbrace{\mathbf{f}^1 \otimes \ldots \otimes \mathbf{f}^1}_{M\times} \otimes \underbrace{\mathbf{e}_{-1} \otimes \ldots \otimes \mathbf{e}_{-1}}_{M\times} \otimes \omega$$

is a vacuum vector for representation of the RTT–algebra $\tilde{\mathcal{A}}_n$ with the weights

$$\begin{aligned}&\mu_1(x;\mathbf{u}) = \lambda_1(x)F(x^{-1}q;\overline{u})\,,\\ &\mu_{-1}(x;\mathbf{u}) = \lambda_{-1}(x)F(xq;\overline{u}^{-1})\,,\\ &\mu_k(x;\mathbf{u}) = \lambda_k(x)F(xq^{-1};\overline{u}^{-1})\,, \qquad k = 2,\ \ldots,\ n\,,\\ &\mu_{-k}(x;\mathbf{u}) = \lambda_{-k}(x)F(x^{-1}q^{-1};\overline{u})\,,\ k = 2,\ \ldots,\ n\,.\end{aligned}$$

So to find eigenvectors of the operators $H(x)$ for the RTT–algebra of $U_q\big(\mathrm{sp}(2n)\big)$ type, it is enough to formulate the Bethe ansatz for the RTT–algebra $\tilde{\mathcal{A}}_n$.

Acknowledgements The authors acknowledge financial support by the Ministry of Education, Youth and Sports of the Czech Republic, project no. CZ.02.1.01/0.0/0.0/16_019/0000778.

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Equivalence Groupoid and Enhanced Group Classification of a Class of Generalized Kawahara Equations

Olena Vaneeva, Olena Magda, and Alexander Zhalij

Abstract Transformation properties of a class of generalized Kawahara equations with time-dependent coefficients are studied. We construct the equivalence groupoid of the class and prove that this class is not normalized but can be presented as a union of two disjoint normalized subclasses. Using the obtained results and properly gauging the arbitrary elements of the class, we carry out its complete group classification, which covers gaps in the previous works on the subject.

1 Introduction

The equations of Kawahara type are important models appearing in solitary waves theory. In the usual sense, solitary waves are nonlinear waves of constant form which decay rapidly in their tail regions. The rate of this decay is usually exponential. However, under critical conditions in dispersive systems (e.g., the magneto-acoustic waves in plasmas, the waves with surface tension, etc.), unexpected rise of weakly nonlocal solitary waves occurs. These waves consist of a central core which is similar to that of classical solitary waves, but they are accompanied by copropagating oscillatory tails which extend indefinitely far from the core with a nonzero constant amplitude. In order to describe and clarify the properties of these waves Kawahara introduced generalized nonlinear dispersive equations which have a form of the KdV equation with an additional fifth order derivative term, namely,

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_23

$$u_t + \alpha u u_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0,$$

where α, β and σ are nonzero constants [4]. This equation was heavily studied from various points of view (see the related discussion and references in [6]). We note that neither the classical Kawahara equation nor its generalization adduced below are integrable by the inverse scattering transform method [9].

Generalized constant coefficient models related to the Kawahara equation have appeared later. For example, long waves in a shallow liquid under ice cover in the presence of tension or compression were described in [7, 17] by the equation

$$u_t + u_x + \alpha u u_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0.$$

This equation is similar to the classical Kawahara equation with respect to the simple changes of variables: $\tilde{x} = x - t$, where t and u are not transformed, or $\tilde{u} = 1 + \alpha u$, where t and x are not transformed. So, all results on symmetries, conservation laws and exact solutions for this equation can be easily derived from the analogous results for the classical Kawahara equation.

Generalized and formal symmetries as well as local conservation laws of the constant coefficient Kawahara equations with arbitrary nonlinearity

$$u_t + f(u)u_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0, \quad f_u \beta\sigma \neq 0,$$

were classified recently in [20].

In the last years special attention is paid to variable coefficient models of the Kawahara type. This is due to the fact that variable coefficient equations model certain real-world phenomena with more accuracy than their constant coefficient counterparts. To the best of our knowledge, Lie symmetries of variable coefficient Kawahara equations were studied for the first time in [3]. Namely, the equations

$$u_t + \alpha(t)u^n u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \quad n\alpha\beta\sigma \neq 0,$$

for $n = 1$ and $n = 2$ were considered therein. The presence of four arbitrary elements, $n, \alpha(t), \beta(t)$, and $\sigma(t)$, made the problem of classifying Lie symmetries too difficult to get complete results without usage of equivalence transformations. Due to this reason a limited progress was made in [3]. It was shown in [6] that right choice of gauging the arbitrary elements by equivalence transformations is a cornerstone for the complete solution of the problem. As a result, the extended group analysis of the above equations for arbitrary $n \neq 0$ was performed in [6], in particular, admissible transformations, Lie symmetries and Lie reductions were classified exhaustively as well as some solutions and local conservation laws were found therein.

Lie symmetries and local conservation laws of generalized Kawahara equations with arbitrary nonlinearity and time-dependent coefficients,

$$u_t + \alpha(t)f(u)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \quad f_u\alpha\beta\sigma \neq 0, \tag{1}$$

where f, α, β and σ being smooth nonvanishing functions of their variables, were studied recently in [1]. The gauging $\sigma = 1$ was performed therein to simplify the problem, however the complete classification of Lie symmetries was not derived, in particular, because the gauging $\sigma = 1$ is not optimal. It was shown in [18] that knowledge of normalization properties of a class of PDEs allows one to choose the optimal gauging not by guessing but algorithmically. That is why we begin our investigation with the study of the equivalence groupoid and normalization properties of the class (1) (Sect. 2). Then the optimal gauging of the arbitrary elements of the class is performed (Sect. 3). Finally, the complete Lie symmetry classification of such equations is carried out (Sect. 4).

2 Equivalence Groupoid

It is widely known that there is no general theory for integration of nonlinear partial differential equations (PDEs). Nevertheless, many special cases of complete integration or finding particular exact solutions are related to appropriate changes of variables. The transformation methods, which include in particular Lie symmetry method, are among the most powerful analytical tools currently available for the study of nonlinear PDEs.

The systematic study of transformation properties of classes of nonlinear PDEs was initiated in 1991 by J. G. Kingston and C. Sophocleous [5]. These authors later named the transformations related two particular equations in a class of PDEs *form-preserving transformations*, because such transformations preserve the form of the equation in a class and change only its arbitrary elements. Only a year later in 1992 J. P. Gazeau and P. Winternitz also began to investigate such transformations in classes of PDEs calling them *allowed transformations* [21]. Rigorous definitions and developed theory on the subject was proposed later by R. O. Popovych [13, 15]. As formalization of notion of form-preserving (allowed) transformations the term *admissible transformations* was proposed. In brief, an admissible transformation is a triple consisting of two fixed equations from a given class and a nondegenerate transformation that links these equations. The set of admissible transformations considered with the standard operation of composition of transformations is also called *equivalence groupoid* [14]. Equivalence groupoids can be used not only in group classification problems but also in other problems related to classes of PDEs like, for example, finding exact solutions and conservation laws, the study of integrability [16, 19].

Equivalence transformations, which are invaluable tools of group analysis of differential equations, generate a subset in a set of admissible transformations. It is important that admissible transformations are not necessarily related to a group structure, but equivalence transformations always form a group. An equivalence transformation applied to any equation from the class always maps it to another equation from the same class, while an admissible transformation may exist only for a specific pair of equations from the class under consideration.

By L. V. Ovsiannikov, the equivalence group consists of the nondegenerate point transformations of the independent and dependent variables and of the arbitrary elements of the class, where transformations for independent and dependent variables are projectable on the space of these variables [12]. After appearance of other kinds of equivalence group the one with properties described above is called now *usual equivalence group*. If the transformations for independent and/or dependent variables involve arbitrary elements, then the corresponding equivalence group is called *generalized equivalence group* [8]. If new arbitrary elements appear to depend on old ones in a nonlocal way (e.g., new arbitrary elements are expressed via integrals of old ones), then the corresponding equivalence group is called *extended equivalence group* [2]. Simultaneously weakening the conditions of locality and projectability, leads to the notion of *extended generalized equivalence group*. The notion of *effective generalized equivalence group* was proposed recently in [11], this is a minimal subgroup of the entire generalized equivalence group of a given class of PDEs which generates the same equivalence subgroupoid of the class as the entire group does.

If any admissible transformation in a given class is induced by a transformation from its usual equivalence group, then this class is called *normalized* in the usual sense. In analogous way, the notions of normalization of a class in generalized, extended and extended generalized senses are formulated [15]. If a given class is normalized in the generalized sense, then its effective generalized equivalence group generates the equivalence groupoid of the class.

Once it is proved that certain class is normalized, finding the equivalence groupoids of its subclasses becomes essentially simpler since they are always subgroupoids of the equivalence groupoid of the initial class. It was shown in [19] that the class

$$u_t = F(t)u_n + G(t, x, u, u_1, \dots, u_{n-1}), \tag{2}$$

where $F \neq 0, G_{u_i u_{n-1}} = 0, i = 1, \dots, n-1, n \geqslant 2, u_n = \frac{\partial^n u}{\partial x^n}$, F and G are arbitrary smooth functions of their variables, is normalized in the usual sense. The transformation components for the variables t, x and u of admissible transformations for the class (2) are of the form

$$\tilde{t} = T(t), \quad \tilde{x} = X^1(t)x + X^0(t), \quad \tilde{u} = U^1(t,x)u + U^0(t,x), \tag{3}$$

where T, X^1, X^0, U^1 and U^0 are arbitrary smooth functions of their variables with $T_t X^1 U^1 \neq 0$.

As class (1) is a subclass of class (2) with $n = 5$, transformation components for the variables t, x, and u of admissible transformations in (1) can be sought in the form (3). Following the direct method [5], we suppose that Eq. (1) is similar to an equation from the same class,

$$\tilde{u}_{\tilde{t}} + \tilde{\alpha}(\tilde{t})\tilde{f}(\tilde{u})\tilde{u}_{\tilde{x}} + \tilde{\beta}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{\sigma}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0, \tag{4}$$

and these two equations are connected by a nondegenerate point transformation of the form (3). Rewriting (4) in terms of the untilded variables, we further substitute

$u_t = -\alpha(t)f(u)u_x - \beta(t)u_{xxx} - \sigma(t)u_{xxxxx}$ to the derived equation. Splitting the obtained identity with respect to the derivatives of u leads to the determining equations on the functions T, X^1, X^0, U^1 and U^0, which result in the system

$$U^1_x = 0, \quad \tilde{\beta}T_t - \beta(X^1)^3 = 0, \quad \tilde{\sigma}T_t - \sigma(X^1)^5 = 0, \tag{5}$$

$$U^0_x\alpha f + U^1_t u + \sigma U^0_{xxxxx} + \beta U^0_{xxx} + U^0_t = 0, \tag{6}$$

$$\tilde{\alpha}\tilde{f}T_t = \alpha f X^1 + X^1_t x + X^0_t. \tag{7}$$

At first we find usual equivalence group of the class (1). The following assertion is true.

Theorem 1. *The usual equivalence group of the class (1) consists of the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u + \delta_4,$$
$$\tilde{f} = \delta_0 f, \quad \tilde{\alpha} = \frac{\delta_1}{\delta_0 T_t}\alpha, \quad \tilde{\beta} = \frac{{\delta_1}^3}{T_t}\beta, \quad \tilde{\sigma} = \frac{{\delta_1}^5}{T_t}\sigma,$$

where δ_j, $j = 0, 1, \ldots, 4$, *are arbitrary constants with* $\delta_0\delta_1\delta_3 \neq 0$, *and* $T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$.

We proceed with the classification of all admissible transformations in the class (1). Equation (6) implies that such transformations essentially differ for the cases $f_{uu} \neq 0$ and $f_{uu} = 0$.

I. If $f(u)$ is a nonlinear function, then Eqs. (6), (7) imply the conditions

$$U^0_x = U^0_t = U^1_t = X^1_t = 0, \quad \tilde{\alpha}\tilde{f}T_t = \alpha f X^1 + X^0_t.$$

Solving these equations together with Eqs. (5) we find that X^0_t is proportional to the arbitrary element α, so the explicit form of x-component of admissible transformations for the subclass of the class (1) singled out by the condition $f_{uu} \neq 0$ will be nonlocal with respect to the arbitrary element α. To keep equivalence transformations to be point and well defined, we extend the tuple of arbitrary elements $(f, \alpha, \beta, \sigma)$ with the additional arbitrary element A, that satisfies the auxiliary condition $A_t = \alpha$. We derive the following statement.

Theorem 2. *The subclass of the class (1) singled out by the constraint* $f_{uu} \neq 0$ *is normalized in the extended generalized sense. Its extended generalized equivalence group* $\hat{G}^{\sim}$ *consists of the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1(x + \delta_2 A) + \delta_3, \quad \tilde{u} = \delta_4 u + \delta_5,$$
$$\tilde{f} = \delta_0\,(f + \delta_2)\,, \quad \tilde{\alpha} = \frac{\delta_1}{\delta_0 T_t}\alpha, \quad \tilde{A} = \frac{\delta_1}{\delta_0}A + \varepsilon_0, \quad \tilde{\beta} = \frac{{\delta_1}^3}{T_t}\beta, \quad \tilde{\sigma} = \frac{{\delta_1}^5}{T_t}\sigma,$$

where ε_0 *and* δ_j, $j = 0, 1, \ldots, 5$, *are arbitrary constants with* $\delta_0\delta_1\delta_4 \neq 0$, *and* $T(t)$ *is an arbitrary smooth function with* $T_t \neq 0$. *The additional arbitrary element* A *satisfies the auxiliary equation* $A_t = \alpha$.

Remark 1. The group $\hat{G}^{\sim}$ is also the extended generalized equivalence group for the entire class (1), which is not normalized in contrast to its subclass with $f_{uu} \neq 0$. The effective generalized equivalence group $\hat{G}^{\sim}$ for reparameterized class (1) with the extended tuple of arbitrary elements $(f, \alpha, \beta, \sigma, A)$ is a nontrivial extended generalized equivalence group for the initial class (1). Indeed, this group induces the maximal subgroupoid in the equivalence groupoid of the class (1) among all the equivalence groups of possible reparameterizations of this class. Moreover, the x-components of transformations from $\hat{G}^{\sim}$ depend on the new arbitrary element A.

II. If $f(u) = au + b$, where a is a nonzero constant and b is an arbitrary constant, then we can set $a = 1$ by a simple gauging transformation of the arbitrary elements, namely, the transformation $\tilde{f} = \delta_0 f$, $\tilde{\alpha} = \alpha/\delta_0$ with δ_0 being a nonzero constant. Without loss of generality we further consider the class

$$u_t + \alpha(t)(u + b)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \tag{8}$$

where the real constant b and the nonvanishing smooth functions $\alpha(t)$, $\beta(t)$ and $\sigma(t)$ are arbitrary elements. In this case splitting of Eqs. (6) and (7) with respect to u results in the following conditions

$$U^0_x\alpha + U^1_t = 0, \quad U^0_x\alpha b + U^0_t = 0, \quad \tilde{\alpha}U^1T_t = \alpha X^1,$$
$$\tilde{\alpha}T_t(U^0 + \tilde{b}) = \alpha b X^1 + X^1_t x + X^0_t.$$

Solving these equations together with Eqs. (5) we get the statement.

Theorem 3. *The class (8) is normalized in the extended generalized sense. Its extended generalized equivalence group* $\hat{G}^{\sim}_1$ *is constituted by the transformations of the form*

$$\tilde{t} = T(t), \quad \tilde{x} = \frac{\varepsilon_2 x + \varepsilon_1 A + \varepsilon_0}{\delta_2 A + \delta_1}, \quad \tilde{u} = \frac{\varepsilon_2}{\Delta}\left((\delta_2 A + \delta_1)u - \delta_2 x + \delta_2 b A + \varepsilon_3\right),$$
$$\tilde{A} = \frac{\delta_2' A + \delta_1'}{\delta_2 A + \delta_1}, \quad \tilde{\alpha} = \frac{\Delta}{T_t(\delta_2 A + \delta_1)^2}\alpha, \quad \tilde{\beta} = \frac{{\varepsilon_2}^3}{T_t(\delta_2 A + \delta_1)^3}\beta,$$
$$\tilde{\sigma} = \frac{{\varepsilon_2}^5}{T_t(\delta_2 A + \delta_1)^5}\sigma, \quad \tilde{b} = \frac{1}{\Delta}\left(b\delta_1\varepsilon_2 + \delta_1\varepsilon_1 - \delta_2\varepsilon_0 - \varepsilon_3\varepsilon_2\right),$$

where δ_j, δ_j' $j = 1, 2$, *and* ε_i, $i = 0, 1, 2, 3$, *are arbitrary constants defined up to a nonzero multiplier, with* $\Delta = \delta_2'\delta_1 - \delta_1'\delta_2 \neq 0$ *and* $\varepsilon_2 \neq 0$. *The additional arbitrary element* A *satisfies the auxiliary equation* $A_t = \alpha$.

The equivalence groupoid of the subclass of the class (1) singled out by the condition $f_{uu} \neq 0$ (resp. of the class (8)) is generated by its extended generalized equivalence group $\hat{G}^{\sim}$ (resp. $\hat{G}_1^{\sim}$). Therefore, the class (1) is not normalized but it can be partitioned into two disjoint normalized subclasses each of which is normalized. Note that we have found only effective generalized equivalence groups of the corresponding reparameterized classes. The question whether these groups are entire generalized equivalence groups of these classes needs an additional study.

Using the results of Theorems 2 and 3 we derive the criterion of reducibility of variable coefficient generalized Kawahara equations to their constant coefficient counterparts. The following statement is true.

Proposition 1. *An equation from the class (1) with variable coefficients α, β and σ is reducible to a constant coefficient equation from the same class if and only if the coefficients satisfy the conditions*

$$\left(\frac{\beta}{\alpha}\right)_t = \left(\frac{\sigma}{\alpha}\right)_t = 0, \quad \textit{for} \quad f_{uu} \neq 0,$$
$$\left(\frac{1}{\alpha}\left(\frac{\beta}{\alpha}\right)_t\right)_t = 0, \quad \left(\frac{\sigma\alpha^2}{\beta^3}\right)_t = 0, \quad \textit{for} \quad f_{uu} = 0.$$

3 Gauging of Arbitrary Elements

The presence of the arbitrary function $T(t)$ in the equivalence transformations from the group $\hat{G}^{\sim}$ of the entire class (1) allows one to gauge either α or β or σ to a simple constant value, e.g., to 1. An important question is which one of the three potential gaugings is the optimal one. It was shown in [6, 18] how to choose the optimal gauging using the normalization property of the class under consideration. The classes normalized in the usual sense are most convenient for investigation and the classes normalized in the extended generalized sense is most complicated among normalized classes. If the class is not normalized one should look for the possibility of a partition of such class into normalized subclasses. In our case the class (1) that is not normalized can be partitioned into two normalized subclasses singled out by the conditions $f_{uu} \neq 0$ and $f_{uu} = 0$. Gauging of the arbitrary elements will be performed separately in each subclass.

I. If $f_{uu} \neq 0$, then Theorem 2 implies that the subclasses of the class (1) singled out by the conditions $\beta = 1$ or $\sigma = 1$ will stay normalized only in the extended generalized sense, since equivalence transformations for such subclasses will still involve A. After gauging $\alpha = 1$ in the subclass of (1) with $f_{uu} \neq 0$ we obtain the class normalized in the usual sense, as in this case up to gauging equivalence transformations we can set $A = t$ and this nonlocality disappears.

The gauging $\alpha = 1$ is realized by the family of point transformations from the equivalence group $\hat{G}^{\sim}$ with $\tilde{t} = \int_{t_0}^{t} \alpha(y)\,\mathrm{d}y$, $\tilde{x} = x$, $\tilde{u} = u$, that maps equations from the class (1) with $f_{uu} \neq 0$ to equations from the same class with $\tilde{\alpha} = 1$, $\tilde{\beta} = \beta/\alpha$ and $\tilde{\sigma} = \sigma/\alpha$. All results on symmetries, conservation laws, classical solutions and other related objects for Eqs. (1) with $f_{uu} \neq 0$ can be found using the similar results derived for equations from its subclass

$$u_t + f(u)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \quad f_{uu}\beta\sigma \neq 0. \tag{9}$$

We derive the equivalence groupoid of the class (9) and formulate the following statement.

Theorem 4. *The class (9) is normalized in the usual sense. Its usual equivalence group $G^{\sim}$ consists of the transformations*

$$\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_3 x + \delta_4 t + \delta_5, \quad \tilde{u} = \delta_6 u + \delta_7,$$
$$\tilde{f} = \frac{1}{\delta_1}\left(\delta_3 f + \delta_4\right), \quad \tilde{\beta} = \frac{{\delta_3}^3}{\delta_1}\beta, \quad \tilde{\sigma} = \frac{{\delta_3}^5}{\delta_1}\sigma,$$

where δ_j, $j = 1, \ldots, 7$, are arbitrary constants with $\delta_1\delta_3\delta_6 \neq 0$.

II. If $f_{uu} = 0$, then Theorem 3 implies that after the gauging $\alpha = 1$ the respective subclass of the class (8) will be normalized in the generalized sense, since the dependence on the constant arbitrary element b will remain in the transformations. So, the optimal gauging in this case is the simultaneous gauging of two arbitrary elements, $\alpha = 1$ and $b = 0$. This gauging is performed by the family of point transformations from the equivalence group $\hat{G}_1^{\sim}$ with $\tilde{t} = \int_{t_0}^{t}\alpha(y)\,\mathrm{d}y$, $\tilde{x} = x$, $\tilde{u} = u + b$, that maps equations from the class (8) to equations from the same class with $\tilde{\alpha} = 1$, $\tilde{b} = 0$, $\tilde{\beta} = \beta/\alpha$ and $\tilde{\sigma} = \sigma/\alpha$. Without loss of generality we can restrict ourselves with the investigation of the following class

$$u_t + uu_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0, \tag{10}$$

instead of its superclass (8).

The study of the equivalence groupoid of the class (10) results in the following assertion.

Theorem 5. *The class (10) is normalized in the usual sense. Its usual equivalence group $G_1^{\sim}$ consists of the transformations of the form*

$$\tilde{t} = \frac{\delta_4 t + \delta_3}{\delta_2 t + \delta_1}, \; \tilde{x} = \frac{\varepsilon_2 x + \varepsilon_1 t + \varepsilon_0}{\delta_2 t + \delta_1}, \; \tilde{u} = \frac{\varepsilon_2(\delta_2 t + \delta_1)u - \varepsilon_2\delta_2 x + \varepsilon_1\delta_1 - \varepsilon_0\delta_2}{\Delta},$$
$$\tilde{\beta} = \frac{{\varepsilon_2}^3}{(\delta_2 t + \delta_1)\Delta}\beta, \quad \tilde{\sigma} = \frac{{\varepsilon_2}^5}{(\delta_2 t + \delta_1)^3\Delta}\sigma,$$

where ε_i, $i = 0, 1, 2$, and δ_j, $j = 1, \ldots, 4$, are arbitrary constants defined up to a nonzero multiplier; $\varepsilon_2 \neq 0$ and $\Delta = \delta_1\delta_4 - \delta_2\delta_3 \neq 0$.

In the next section we demonstrate that the chosen gaugings allow us to solve exhaustively the group classification problems for both derived normalized subclasses of the class (1).

4 Group Classification

As in previous section we consider separately two normalized subclasses of the class (1), that are singled out by the conditions $f_{uu} \neq 0$ and $f_{uu} = 0$.

I. The group classification problem for the class (1) with $f_{uu} \neq 0$ up to $\hat{G}^{\sim}$-equivalence reduces to the similar problem for the class (9) up to $G^{\sim}$-equivalence. The group classification of the class (9) is performed using the classical algorithm based on direct integration of the determining equations implied by the infinitesimal invariance criterion [10, 12]. The symmetry generators of the form $Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ must satisfy the criterion of infinitesimal invariance,

$$Q^{(5)}\{u_t + f(u)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx}\}\Big|_{u_t = f(u)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx}} = 0,$$

where $Q^{(5)}$ is the fifth prolongation of the vector field Q [10, 12].

The infinitesimal invariance criterion implies the determining equations simplest of which result in the following forms of τ, ξ, and η,

$$\tau = \tau(t), \quad \xi = \xi^1(t)x + \xi^0(t), \quad \eta = \big(2\xi^1(t) + \mu(t)\big)\, u + \eta^0(t,x),$$

where τ, ξ^1, ξ^0, μ and η^0 are arbitrary smooth functions of their variables.

Then the rest of the determining equations have the form

$$\eta^0_x f + (2\xi^1_t + \mu_t)u + \eta^0_t + \eta^0_{xxx}\beta + \eta^0_{xxxxx}\sigma = 0, \tag{11}$$
$$\big((2\xi^1 + \mu)u + \eta^0\big)\, f_u = (\xi^1 - \tau_t) f + \xi^1_t x + \xi^0_t, \tag{12}$$
$$\tau\sigma_t = (5\xi^1 - \tau_t)\sigma, \quad \tau\beta_t = (3\xi^1 - \tau_t)\beta. \tag{13}$$

Below we give the sketch of the proof. To find the kernel $A^{\cap}$ of the maximal Lie invariance algebras $A^{\max}$ of Eqs. (9) we split the determining equations with respect to the arbitrary elements and their derivatives, which results in $\tau = \eta = 0$, $\xi = c_1$, where c_1 is an arbitrary constant. Therefore, $A^{\cap}$ is the one-dimensional algebra $\langle \partial_x \rangle$ (Case 0 of Table 1). At the next step we assume that f is arbitrary and look for specifications of β and σ for which Eqs. (9) possess Lie symmetry extensions. These are the cases $(\beta,\sigma) = (\lambda,\delta)$ and $(\beta,\sigma) = (\lambda(pt+q)^2, \delta(pt+q)^4)$, where λ, δ, p and q are constants and $\lambda\delta p \neq 0$. Using equivalence transformations from the group $G^{\sim}$ we can set $\delta = \pm 1$, $p = 1$ and $q = 0$. The respective bases of the maximal Lie invariance algebras are adduced in Cases 1 and 2 of Table 1.

As $f_{uu} \neq 0$, Eq. (11) implies $\eta^0_x = \eta^0_t = 0$, so, $\eta^0 = c_0$ and $\mu = -2\xi^1 + c_1$, where c_0 and c_1 are arbitrary constants. Then Eq. (12) leads to the condition $\xi^1_t = 0$ and reduces to

$$(c_1 u + c_0) f_u = (\xi^1 - \tau_t) f + \xi^0_t. \tag{14}$$

Table 1 The group classification of the class $u_t+\alpha(t)f(u)u_x+\beta(t)u_{xxx}+\sigma(t)u_{xxxxx}=0$, $f_{uu}\alpha\beta\sigma\neq 0$.

No.	$f(u)$	$\beta(t)$	$\sigma(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	$\forall$	∂_x
1	$\forall$	λt^2	δt^4	$\partial_x,\ t\partial_t+x\partial_x$
2	$\forall$	λ	δ	$\partial_x,\ \partial_t$
3	$\ln u$	$\forall$	$\forall$	$\partial_x, t\partial_x+u\partial_u$
4	$\ln u$	λt^2	δt^4	$\partial_x, t\partial_x+u\partial_u, t\partial_t+x\partial_x$
5	$\ln u$	λ	δ	$\partial_x,\ t\partial_x+u\partial_u,\ \partial_t$
6	u^n	λt^ρ	$\delta t^{\frac{5\rho+2}{3}}$	$\partial_x,\ 3nt\partial_t+(\rho+1)nx\partial_x+(\rho-2)u\partial_u$
7	u^n	λe^t	$\delta e^{\frac{5}{3}t}$	$\partial_x,\ 3n\partial_t+nx\partial_x+u\partial_u$
8	e^u	λt^ρ	$\delta t^{\frac{5\rho+2}{3}}$	$\partial_x,\ 3t\partial_t+(\rho+1)x\partial_x+(\rho-2)\partial_u$
9	e^u	λe^t	$\delta e^{\frac{5}{3}t}$	$\partial_x,\ 3\partial_t+x\partial_x+\partial_u$

Here $\alpha(t)=1 \bmod \hat{G}^{\sim}$, ρ and n are arbitrary constants, $n\neq 0,1$; δ and λ are nonzero constants, $\delta=\pm 1 \bmod \hat{G}^{\sim}$.

The equations from the class (9) possess Lie symmetry extensions, when f takes one of the following forms: $f=\nu(au+b)^n+\kappa$, $n\neq 0,1$, $f=\nu e^{au}+\kappa$, and $f=\nu\ln(au+b)+\kappa$, where ν and a are nonzero constants, κ and b are arbitrary constants. Up to the $G^{\sim}$-equivalence this list is exhausted by the cases 1. $f=u^n$, $n\neq 0,1$, 2. $f=e^u$ and 3. $f=\ln u$. Each of these forms of f should be substituted to Eq. (14), then the final forms of the coefficients τ, ξ, and η are found and the classifying Eqs. (13) give the possible forms of β and σ for which Eqs. (9) possess Lie symmetry extensions. The detailed proof for the case $f=u^n$, $n\neq 0$, was given in [6]. The consideration of the other cases is analogous. Due to the luck of space we omit the detailed proof for these cases and formulate the final result in the following statement.

Theorem 6. *The kernel of the maximal Lie invariance algebras of equations from the class (1) with $f_{uu}\neq 0$ coincides with the one-dimensional algebra $\langle\partial_x\rangle$. All possible $\hat{G}^{\sim}$-inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by Cases 1–9 of Table 1.*

II. In the previous section we have shown that the group classification problem for the class (8) up to $\hat{G}_1^{\sim}$-equivalence reduces to such a problem for the class (10) up to $G_1^{\sim}$-equivalence. The group classification of the class (10) up to $G_1^{\sim}$-equivalence was carried out exhaustively in [6]. So we use the results derived therein and formulate the following statement.

Theorem 7. *The kernel of the maximal Lie invariance algebras of equations from the class (8) coincides with the two-dimensional algebra $\langle\partial_x,\ t\partial_x+\partial_u\rangle$. All possible $\hat{G}_1^{\sim}$-inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by Cases 1–4 of Table 2.*

Table 2 The group classification of the class $u_t + \alpha(t)(u+b)u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxxx} = 0$, $\alpha\beta\sigma \neq 0$.

No.	$\beta(t)$	$\sigma(t)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	$\partial_x,\ t\partial_x + \partial_u$
1	λt^ρ	$\delta t^{\frac{5\rho+2}{3}}$	$\partial_x,\ t\partial_x + \partial_u,\ 3t\partial_t + (\rho+1)x\partial_x + (\rho-2)u\partial_u$
2	λe^t	$\delta e^{\frac{5}{3}t}$	$\partial_x,\ t\partial_x + \partial_u,\ 3\partial_t + x\partial_x + u\partial_u$
3	λ	δ	$\partial_x,\ t\partial_x + \partial_u,\ \partial_t$
4	$\dfrac{\lambda(t^2+1)^{\frac{1}{2}}}{e^{3\nu\arctan t}}$	$\dfrac{\delta(t^2+1)^{\frac{3}{2}}}{e^{5\nu\arctan t}}$	$\partial_x,\ t\partial_x + \partial_u,\ (t^2+1)\partial_t + (t-\nu)x\partial_x + (x-(t+\nu)u)\partial_u$

Here $\alpha = 1 \bmod \hat{G}_1^{\sim}$, $b = 0 \bmod \hat{G}_1^{\sim}$, ρ and ν are arbitrary constants, $\rho \geqslant 1/2 \bmod \hat{G}_1^{\sim}$, $\nu \leqslant 0 \bmod \hat{G}_1^{\sim}$; δ and λ are nonzero constants, $\delta = \pm 1 \bmod \hat{G}_1^{\sim}$.

The presented group classification reveals equations of the form (1) that are of more potential interest for applications and for which the classical Lie reduction method can be used. The complete result was achieved using the knowledge of transformation properties of the class namely due to partition of the class into two normalized subclasses and choosing the optimal gauging for each of these subclasses.

Acknowledgements O.V. would like to thank all the Organizing Committee of LT-13 and especially Prof. Vladimir Dobrev for the hospitality and support. The authors are grateful to Prof. Roman Popovych for invaluable discussions on the topic and also to the referee and the editor for their suggestions on the improvement of the manuscript. OV acknowledges the financial support of her research within the L'Oréal-UNESCO For Women in Science International Rising Talents Programme.

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Interplay between Minimal Gravity and Intersection Theory

Chaiho Rim

Abstract We present the relation between the two-dimensional minimal quantum gravity with boundaries and open intersection theory. The boundary minimal gravity is defined in terms of boundary cosmological constant μ_B and the open intersection theory in terms of boundary marked point generating parameter s. It is demonstrated that a generating function of a theory is the Laplace transform of the other, if the right solution is to be carefully chosen among many others.

1 Introduction

The role of the KdV hierarchy was proposed in [1] and proved in [2] between the two-dimensional minimal quantum gravity (MG) and intersection theory (IT) on the moduli spaces of closed Riemann surfaces. Recently, similar relation holds on the Riemann surfaces with boundaries [3–5], *i.e.* between boundary minimal gravity (BMG) and open intersection theory (OIT) [6]. We sketch some of the recent results, details of which can be found in [3–5].

This paper is organized as follows. Section 2 is to compare the KdV hierarchy and string equation of one theory with the other on the Riemann surfaces with/without boundaries. Section 3 provides an elegant way to find generating function of OIT from BMG using the Laplace transform. Section 4 is the conclusion.

2 KdV Hierarchy

We summarize KdV hierarchy and string equation of IT and of MG on the Riemann surfaces with/without boundaries.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_24

2.1 Riemann Surfaces Without Boundary

The generating function (GF) F^c of intersection theory on the moduli space of Riemann surfaces depends on the infinitely many parameters, $t = (t_0, t_1, \cdots)$, and also satisfies the KdV hierarchy and the string equation [1]:

$$\frac{1}{\lambda^2}\frac{2n+1}{2}\frac{\partial^3 F^c}{\partial t_0^2 \partial t_n} = \frac{\partial^2 F^c}{\partial t_0^2}\frac{\partial^3 F^c}{\partial t_0^2 \partial t_{n-1}} + \frac{1}{2}\frac{\partial^3 F^c}{\partial t_0^3}\frac{\partial^2 F^c}{\partial t_0 \partial t_{n-1}} + \frac{1}{8}\frac{\partial^5 F^c}{\partial t_0^4 \partial t_{n-1}} \tag{1}$$

$$\frac{\partial F^c}{\partial t_0} = \sum_{n\geq 0} t_{n+1}\frac{\partial F^c}{\partial t_n} + \frac{t_0^2}{2\lambda^2}\,. \tag{2}$$

F^c has the natural expansion in λ, the genus expansion parameter:

$$F^c = \sum_{g=0}^{\infty} \lambda^{2g-2} F^c_{(g)}. \tag{3}$$

The minimal quantum gravity (MG) is described by the scaling limit of one-matrix model [7, 8] for the Lee-Yang series $M(2, 2p+1)$ on the closed Riemann surface. MG also satisfies the KdV hierarchy and the string equation in terms of p number of parameters $\tau_0, \cdots, \tau_{p-1}$ [9–12], which are the same as (1) and (2) if t_i is identified with τ_i but is truncated: $t_k = 0$ for $k > p-1$ except $t_{p+1} = 1$. We will reserve the notation F for the GF of IT and $\mathcal{F}$ for MG when two distinction is necessary. It is noted that MG is also described by Liouville field theory coupled to conformal matter [13] but its correlation number looks different due to the resonance transformations (so called Liouville frame in contrast with the KdV frame) [14, 15].

Even though GF satisfies the similar KdV and string equation, the explicit form of GF of one theory differs from the other. This can be clearly seen in the lowest genus case ($g = 0$). The KdV hierarchy becomes the dispersionless limit and has the simple form

$$\frac{\partial^3 \mathcal{F}^c_{(0)}}{\partial \tau_n \partial \tau_0^2} = \frac{\partial v}{\partial \tau_n} = \frac{v^n}{n!}\frac{\partial v}{\partial \tau_0} \quad \text{for} \quad 1 \leq n \leq p-1. \tag{4}$$

Here $\mathcal{F}^c_{(0)}$ is the GF on the fluctuating sphere and $v \equiv \partial^2 \mathcal{F}^c_{(0)}/\partial \tau_0^2$. Then the string equation together with the KdV hierarchy can be reduced to a polynomial equation of v,

$$\mathcal{P}(\tau, v) = 0; \qquad \mathcal{P}(\tau, v) \equiv \sum_{m=0}^{p-1} \tau_m \frac{v^m}{m!} + \frac{v^{p+1}}{(p+1)!}. \tag{5}$$

Here $\tau_{p-1} \propto \mu$ is the basic parameter where μ is the bulk cosmological constant. If one switches off all the other parameters except μ, then, the right solution is chosen as $v = \sqrt{-\tau_{p-1}}$.

The equation and solutions are compared with those of IT. For comparison, we restrict ourselves to the subspace $t = (t_0, t_1, \cdots t_{p-1})$ of the parameters of IT by demanding $t_n = 0$ ($n \geq p$). Then, the string polynomial equation for IT is obtained from by the above described between τ and t,

$$P(t, v) = 0; \quad P(t, v) = \sum_{m=0}^{p-1} t_m \frac{v^m}{m!} - v, \tag{6}$$

where the linear power of v is added due to the t_1 shift. In this case, t_0 becomes the basic parameter and the others are treated as deformation parameters, so that the un-deformed solution is $v = t_0$.

In addition, note that v^{p+1}-term in (5) and v-term in (6) is not coupled to the parameter t_i or τ_i. This allows the different gravitation scaling to the parameter. As a result, the KdV parameters of IT couple to the gravitational descendants in topological gravity and the parameters of MG couple to the primary operators of the Lee-Yang series [5].

2.2 Riemann Surfaces with Boundaries

The intersection theory on the moduli space of the Riemann surfaces with boundaries was developed in [6], see also [16, 17]. The GF, which depends on the KdV parameters t_k and an additional parameter s, associated with the insertion of the marked points on the boundary, has a natural topological expansion

$$F^o = \sum_{\bar{g}=0}^{\infty} \lambda^{\bar{g}-1} F^o_{(\bar{g})}. \tag{7}$$

Here $\bar{g}$ is the genus of the doubled Riemann surface. This expansion is equivalent to the expansion of powers of the Euler characteristic $\chi = 2 - 2g - k$ in the presence of boundaries, where $g \geq 0$ is the number of handles, $k \geq 1$ the number of boundaries and therefore, one has $\bar{g}$ as $\bar{g} = 1 - \chi$.

A generalization of the Virasoro constraints introduces a parameter s, a generator of mark-points on the boundary and reduces to the open KdV hierarchy [6];

$$\frac{2n+1}{2} \frac{\partial F^o}{\partial t_n} = \lambda \frac{\partial F^o}{\partial s} \frac{\partial F^o}{\partial t_{n-1}} + \lambda \frac{\partial^2 F^o}{\partial s \partial t_{n-1}} + \frac{\lambda^2}{2} \frac{\partial F^o}{\partial t_0} \frac{\partial^2 F^c}{\partial t_0 \partial t_{n-1}} - \frac{\lambda^2}{4} \frac{\partial^3 F^c}{\partial t_0^2 \partial t_{n-1}} \tag{8}$$

and open string equation

$$\frac{\partial F^o}{\partial t_0} = \sum_{n \geq 0} t_{n+1} \frac{\partial F^o}{\partial t_n} + \frac{s}{\lambda}. \tag{9}$$

One may find an additional consistent s-flow equation [18]:

$$\frac{\partial F^o}{\partial s} = \lambda \left\{ \frac{1}{2}\left(\frac{\partial F^o}{\partial t_0}\right)^2 + \frac{1}{2}\frac{\partial^2 F^o}{\partial t_0^2} + \frac{\partial^2 F^c}{\partial t_0^2} \right\}, \tag{10}$$

which provides a useful check for the solution of the open KdV hierarchy.

On the other hand, GF of BMG satisfies the similar open KdV hierarchy which has no s parameter but instead μ_B parameter:

$$\frac{2n+1}{2}\frac{\partial \mathcal{F}^o}{\partial \tau_n} = -\mu_B \frac{\partial \mathcal{F}^o}{\partial \tau_{n-1}} + \frac{\lambda^2}{2}\frac{\partial \mathcal{F}^o}{\partial \tau_0}\frac{\partial^2 \mathcal{F}^c}{\partial \tau_0 \partial \tau_{n-1}} - \frac{\lambda^2}{4}\frac{\partial^3 \mathcal{F}^c}{\partial \tau_0^2 \partial \tau_{n-1}} \quad \text{for} \quad n \geq 1. \tag{11}$$

The open string equation becomes

$$0 = \sum_{n\geq 0} \tau_{n+1}\frac{\partial \mathcal{F}^o}{\partial \tau_n} + \frac{\partial \mathcal{F}^o}{\partial \mu_B}. \tag{12}$$

In addition, the s-flow equation has a new form in the μ_B-space:

$$-\mu_B = \lambda \left\{ \frac{1}{2}\left(\frac{\partial \mathcal{F}^o}{\partial \tau_0}\right)^2 + \frac{1}{2}\frac{\partial^2 \mathcal{F}^o}{\partial \tau_0^2} + \frac{\partial^2 \mathcal{F}^c}{\partial \tau_0^2} \right\}, \tag{13}$$

which we call boundary condition equation (BCE), noting that the boundary cosmological constant plays a role of the boundary condition.

GF of OIT turns out to be the Laplace (or Fourier) transform of the one of BMG [4]:

$$e^{\mathcal{F}^o(s)} = \frac{1}{\sqrt{2\pi\lambda}} \int d\mu_B \, e^{-\frac{s\mu_B}{\lambda}} \, e^{\mathcal{F}^o(\mu_B)}. \tag{14}$$

To avoid confusion we indicate explicitly the variable s or μ_B. Here we continue to use the notation $\mathcal{F}$ after the Laplace transform and, depending on the context, we assume it to depend on the KdV parameters τ or t, the reason of which is that we expect this relation still to hold beyond the proper parameter range of the OIT indicated in [6]. Furthermore, as we will show in the following section, the right GF of OIT is to be carefully chosen to be a very particular $\mathcal{F}^o(\mu_B)$, which corresponds to a certain topological branch of BMG.

3 Boundary Minimal Gravity and Open Intersection Theory

We provide a simple example how to find GF of OIT from GF of BMG on a disk.

3.1 GF of BMG on a Disk

We start with the GF of BMG on a disk

$$\mathcal{F}^{o}_{(0)}(\tau, \mu_B) = \frac{i}{\sqrt{2\pi}} \int_0^\infty \frac{dl}{l^{3/2}} e^{-l\mu_B} \int_{\tau_0}^\infty dx\, e^{-lw(x)}, \tag{15}$$

which is symbolically the same as $\langle \mathrm{Tr} \log(\mu_B + Q_2) \rangle$ [19]. Here, $w(x)$ is a solution of the string polynomial equation (5) with τ_0 replaced by x. We select the solution $w(x) \propto \sqrt{\mu}$ as noted in (5). We encode the choice of the solution more explicitly by changing the integration variable x to v and by including the string polynomial $\mathcal{P}(\tau, v)$ in (5) to the integral representation (15):

$$\mathcal{F}^{o}_{(0)}(\tau, \mu_B) = -\frac{i}{\sqrt{2\pi}} \int_0^\infty \frac{dl}{l^{3/2}} e^{-l\mu_B} \int_{w}^\infty dv\, \mathcal{P}^{(1)}(\tau, v)\, e^{-lv}. \tag{16}$$

Here $\mathcal{P}^{(1)}(\tau, v) = d\mathcal{P}/dv$ plays the role of the Jacobian factor $dx/dv = -d\mathcal{P}/dv$.

3.2 GF of OIT on a Disk

One may find the GF of OIT using the open KdV hierarchy (8) and the open string Eq. (9) with the initial conditions

$$F^{o}(t_0, t_{i>0} = 0, s) = \frac{1}{\lambda}\left(st_0 + \frac{s^3}{3!}\right). \tag{17}$$

It turns out that its solution is unique [20]. We, however, adopt a different but powerful method, the Laplace transform (14) to find the GF of OIT from that of BMG.

Note that the Laplace transform (14) reduces to the Legendre transform when $\bar{g} = 0$ [4]:

$$F^{o}_{(0)}(t, s) = F^{o}_{(0)}(t, \mu_B) - s\mu_B, \tag{18}$$

and the boundary parameters, s and μ_B, are related through

$$s = \frac{\partial F^o_{(0)}(t, \mu_B)}{\partial \mu_B} \quad \text{or} \quad \mu_B = -\frac{\partial F^o_{(0)}(t, s)}{\partial s}. \tag{19}$$

To find the right GF, one needs to make sure that GF of OIT satisfies the its open string Eq. (9) whereas GF of BMG satisfies the its open string Eq. (12). This can be done by replacing $\mathcal{P}(\tau, v)$ in (16) with $P(\tau, v)$ of (6):

$$F^o_{(0)}(t, \mu_B) = -\frac{i}{\sqrt{2\pi}} \int_0^\infty \frac{dl}{l^{3/2}} e^{-l\mu_B} \int_w^\infty dv\, P^{(1)}(t, v)\, e^{-lv}. \tag{20}$$

Here w is the right solution of $P(t, v) = 0$.

Let us check if this OIT solution coincides with the GF in s-space. If one tries with $p = 1$ case, then one notes $P^{(1)} = -1$ and $w = t_0$. Thus, (20) has the simple form:

$$F^o_{(0)}(t_0, \mu_B) = \frac{i}{\sqrt{2\pi}} \int_0^\infty \frac{dl}{l^{5/2}} e^{-l(\mu_B + w)}. \tag{21}$$

Note that this integral depends only on the sum $\mu_B + w$ but is divergent as $l \to 0$, which needs regularization. By differentiating the integral, we have a finite result:

$$\frac{\partial^2 F^o_{(0)}}{\partial \mu_B^2} = \frac{i}{\sqrt{2\pi}} \int_0^\infty \frac{dl}{l^{1/2}} e^{-l(\mu_B + t_0)} = \frac{i}{\sqrt{2}} (\mu_B + w)^{-1/2}. \tag{22}$$

After integration over μ_B once, we have $s = \partial F^o_{(0)}/\partial \mu_B = i\sqrt{2}(\mu_B + w)^{1/2}$ where we discard μ_B independent term. Likewise, one more integration gives $F^o_{(0)}(t_0, \mu_B) = i\frac{2\sqrt{2}}{3}(\mu_B + w)^{3/2}$ which reproduces the initial condition (17) after the Legendre transformation (18).

For general p, we need more complicated algebraic manipulation. One has the relation between μ_B and s from (23)

$$s = \frac{\partial F^o_{(0)}}{\partial \mu_B} = \sqrt{2}\, i \sum_{n=0}^{p-2} \frac{(-2)^n \xi_{n+1} (\mu_B + w)^{n+1/2}}{(2n+1)!!}. \tag{23}$$

We may put μ_B in power series of s. Putting $h_0 = -2\xi_1^2(\mu_B + w)/s^2$ and $z_i = s^{2i}\xi_{i+1}/\Big((2i+1)!!\xi_1^{2i+1}\Big)$, we represent (23) as follows:

$$\frac{1}{h_0} = \left(1 + \sum_{n=1}^{p-2} z_n\, h_0^n\right)^2 \tag{24}$$

which has the power series of z_k's:

$$h_0 = \sum_{n_k \geq 0} a_{n_1, \cdots, n_{p-2}}\, z_1^{n_1} \cdots z_{p-2}^{n_{p-2}}, \tag{25}$$

where $a_{n_1,n_2,\ldots,n_{p-2}} = 2\frac{(-1)^{n_1+n_2+\cdots+n_{p-2}}(1+3n_1+5n_2+\cdots+(2p-3)n_{p-2})!}{n_1!\,n_2!\ldots n_{p-2}!\,(2+2n_1+4n_2+\cdots+2(p-2)\,n_{p-2})!}$.

$F^o_{(0)}(t,\mu_B)$ is obtained by integrating (23) over μ_B:

$$F^o_{(0)}(t,\mu_B) = -\sqrt{2}\,i\sum_{n=0}^{p-2}\frac{(-2)^{n+1}\xi_{n+1}(\mu_B+w)^{n+3/2}}{(2n+3)!!} \tag{26}$$

or in the power series of z_k's,

$$F^o_{(0)}(t,s) = sw + \frac{s^3}{2\xi_1^2}\sum_{n_i=0}^{\infty}\frac{a_{n_1,\cdots,n_{p-2}}\,z_1^{n_1}\cdots z_{p-2}^{n_{p-2}}}{(3+2n_1+4n_2+\cdots+2(p-2)\,n_{p-2})}. \tag{27}$$

To get the complete GF of OIT one extends p to infinity in (27). One may obtain the special case by restricting $t_0 \to 0$, GF of the following form:

$$F^o_{(0)}\Big|_{t_0=0} = \sum_{n_i=0}^{\infty} b_{n_1,n_2,\cdots}\frac{s^{3+2n_1+4n_2+\ldots}}{(1-t_1)^{2+3n_1+5n_2+\ldots}}\prod_{j=1}^{\infty}\left(\frac{t_{j+1}}{n_j!(2j+1)!!}\right)^{n_j}. \tag{28}$$

where $b_{n_1,n_2,\cdots} = \frac{(1+3n_1+5n_2+\ldots)!}{(1-t_1)^{2+3n_1+5n_2+\ldots}}$. This provides the correlation numbers

$$\langle O_{\alpha_1}\ldots O_{\alpha_\ell}\sigma^k\rangle^o_0 = \frac{\partial^{\ell+k}F^o_{(0)}}{\partial t_{\alpha_1}\ldots\partial t_{\alpha_\ell}\partial s^k}\Bigg|_{(t,s)=0} = \frac{(1+\sum_{i=1}^{\ell}(2\alpha_i-1))!}{\prod_{i=1}^{\ell}(2\alpha_i-1)!!} \tag{29}$$

where k is the number of marked points on the boundary and is related with α_i, $k = 3+\sum_{i=1}^{\ell}2(\alpha_i-1)$.

4 Conclusion

We demonstrate that GF of OIT and GF of BMG are related through the Laplace transform, with the emphasis on the choice of the right solution to the open KdV and open string equation. As an explicit example, we present how to find the right GF of OIT on a disk using GF of BMG. One may extend the calculation straightforwardly to higher genus expansion as can be found in [5]

Acknowledgements The author has greatly benefited from co-works with A. Belavin, G. Ishiki, A. Bawane, H. Muraki and A. Alexandrov. This work is partially supported by by National Research Foundation of Korea grant number 2017R1A2A2A05001164 and by Sogang University Grant number 201919025.01.

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Generalized Macdonald Functions, AGT Correspondence and Intertwiners of DIM Algebra

Yusuke Ohkubo

Abstract In this paper, we explain the 5D AGT correspondence using the Ding-Iohara-Miki algebra (DIM algebra) and generalized Macdonald functions. Furthermore, we describe a certain duality formula for the intertwining operators of the DIM algebra. This paper is based on the presentation by the author in the International Workshop "Lie Theory and its applications in physics" held in Varna.

1 Introduction

In 2009, Alday, Gaiotto and Tachikawa proposed a surprising relationship between 2D conformal field theories and 4D gauge theories (AGT correspondence). The original AGT correspondence states that the 4-point Virasoro conformal block coincides with the Nekrasov partition function of 4-dimensional $\mathcal{N} = 2$ supersymmetric $SU(2)$ gauge theory with 4 matters [1]. To understand this relationship, Alba, Fateev, Litvinov and Tarnopolsky gave a significant basis (AFLT basis) in the representation space of the tensor product of the Virasoro algebra and some Heisenberg algebra [2]. The matrix elements of the Primary field are factorized with respect to the AFLT basis, and they correspond to the Nekrasov factors. Therefore, the expansion of the conformal block (with respect to the algebra (Virasoro) $\otimes$ (Heisenberg)) by the AFLT basis naturally reproduces the Nekrasov partition functions.

In this paper, We give a q-deformation of this context in the case of the W_N algebra, which was conjectured in [3] and proved in [4]. The tensor product of the q-deformed W_N algebra and the Heisenberg algebra is derived from some representation of the DIM algebra. The q-deformation of the AFLT basis can be regarded as a generalization of Macdonald functions. Furthermore, our main operator can be realized by intertwining operators of the DIM algebra. Let us also mention that their intertwining operators reproduce Iqbal, Kozcaz and Vafa's or Awata and Kanno's refined topological vertices [5].

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_25

This paper is organized as follows. In Sect. 2, we start with the definition and representations of the DIM algebra. In Sect. 3, the generalized Macdonald functions are defined, and the q-deformed AGT correspondence is explained. At last, in Sect. 4, we briefly describe a certain duality formula for intertwining operators of the DIM algebra.

2 Ding-Iohara-Miki Algebra and Its Representation

First of all, we explain the definition of the DIM algebra [6, 7].

Definition 1. Let q and t be generic complex parameters. The DIM algebra is the unital associative algebra generated by the four currents $x^{\pm}(z) = \sum_{n\in\mathbb{Z}} x_n^{\pm} z^{-n}$, $\psi^{\pm}(z) = \sum_{\pm n\in\mathbb{Z}_{\geq 0}} \psi_n^{\pm} z^{-n}$ and central elements $\gamma^{\pm 1/2}$ satisfying the defining relations

$$
\begin{aligned}
&G^{\mp}(z/w)x^{\pm}(z)x^{\pm}(w) = G^{\pm}(z/w)x^{\pm}(w)x^{\pm}(z),\\
&[x^{+}(z), x^{-}(w)] = \frac{(1-q)(1-1/t)}{1-p}\big(\delta(\gamma^{-1}\frac{z}{w})\psi^{+}(\gamma^{\frac{1}{2}}w) - \delta(\gamma\frac{z}{w})\psi^{-}(\gamma^{-\frac{1}{2}}w)\big),\\
&\psi^{\pm}(z)\psi^{\pm}(w) = \psi^{\pm}(w)\psi^{\pm}(z), \qquad \psi^{+}(z)\psi^{-}(w) = \frac{g(\gamma^{+1}w/z)}{g(\gamma^{-1}w/z)}\psi^{-}(w)\psi^{+}(z),\\
&\psi^{+}(z)x^{\pm}(w) = g(\gamma^{\mp 1/2}w/z)^{\mp 1}x^{\pm}(w)\psi^{+}(z),\\
&\psi^{-}(z)x^{\pm}(w) = g(\gamma^{\mp 1/2}z/w)^{\pm 1}x^{\pm}(w)\psi^{-}(z).
\end{aligned}
$$

Here $g(z) := G^{+}(z)/G^{-}(z)$, $G^{\pm}(z) := (1-q^{\pm 1}z)(1-t^{\mp 1}z)(1-p^{\mp 1}z)$ and $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$.

Remark 1. We may include the Serre-like relations with respect to $x_n^{\pm}$ in the defining relations. Actually, Miki introduced the algebra divided by the Serre relations [7]. However, our results are not affected by them because the representations used in this paper automatically satisfy the Serre relations. Hence, we omit them in this paper.

The DIM algebra is a Hopf algebra. Let us omit the formulas for the coproduct, counit and antipode. We only explain the following formulas for the coproduct Δ required in this paper:

$$
\Delta(x^{+}(z)) = x^{+}(z)\otimes 1 + \psi^{-}(\gamma_{(1)}^{1/2}z)\otimes x^{+}(\gamma_{(1)}z), \quad \Delta(\gamma) = \gamma\otimes\gamma.
$$

Here, $\gamma_{(1)} := \gamma\otimes 1$, $\gamma_{(2)} := 1\otimes\gamma$. Note that $\psi_0^{\pm}$ are also central elements. A representation ρ of the DIM algebra is called level (n, m) representation if $\gamma \overset{\rho}{\mapsto} (t/q)^{\frac{n}{2}}$, $(\psi_0^{+}/\psi_0^{-})^{1/2} \overset{\rho}{\mapsto} (q/t)^{\frac{m}{2}}$. In this paper, we use a level $(1, m)$ representation, which is a free field representation by the following Heisenberg algebra. Let $\{a_n\}$

be the Heisenberg algebra satisfying $[a_n, a_m] = n\frac{1-q^{|n|}}{1-t^{|n|}}\delta_{n+m,0}$. The Fock space generated by the vacuum state $|0\rangle$ ($a_n|0\rangle = 0$ ($\forall n \geq 0$)) is denoted by $\mathcal{F}$.

Definition 2. Define

$$\eta(z) := \exp\Big(\sum_{n=1}^{\infty}\frac{1-t^{-n}}{n}z^n a_{-n}\Big)\exp\Big(-\sum_{n=1}^{\infty}\frac{1-t^{n}}{n}z^{-n}a_n\Big),$$
$$\xi(z) := \exp\Big(-\sum_{n=1}^{\infty}\frac{1-t^{-n}}{n}(t/q)^{n/2}z^n a_{-n}\Big)\exp\Big(\sum_{n=1}^{\infty}\frac{1-t^{n}}{n}(t/q)^{n/2}z^{-n}a_n\Big),$$
$$\varphi_{\pm}(z) := \exp\Big(\mp\sum_{n=1}^{\infty}\frac{1-t^{\pm n}}{n}(1-t^nq^{-n})(t/q)^{-n/4}z^{\mp n}a_{\pm n}\Big).$$

Fact 3 **([8]).** *Let u be an indeterminate. The following homomorphism $\rho_u^{(1,m)}$ is a representation of the DIM algebra:*

$$x^+(z) \mapsto uz^{-m}(t/q)^{m/2}\cdot\eta(z), \quad x^-(z)\mapsto u^{-1}z^m(t/q)^{-m/2}\cdot\xi(z),$$
$$\psi^{\pm}(z)\mapsto (q/t)^{\pm m/2}\cdot\varphi^{\pm}(z), \qquad \gamma\mapsto (t/q)^{1/2}.$$

The Fock space endowed with the level $(1, m)$ module structure is denoted by $\mathcal{F}_u^{(1,m)}$. Hereafter, we mainly treat the $m = 0$ case. Moreover, we can extend this representation to the level $(N, 0)$ representation by using the coproduct.

Definition 4. For an N-tuple of parameters $\mathbf{u} = (u_1, \ldots, u_N)$, define

$$\rho_{\mathbf{u}}^{(N,0)} := (\rho_{u_1}^{(1,0)}\otimes\rho_{u_2}^{(1,0)}\otimes\cdots\otimes\rho_{u_N}^{(1,0)})\circ\Delta^{(N)},$$
$$\Delta^{(1)} := \mathrm{id}, \quad \Delta^{(N)} := (\underbrace{\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{N-2}\otimes\Delta)\circ\cdots\circ(\mathrm{id}\otimes\Delta)\circ\Delta, \quad N\geq 2.$$

In what follows, we fix a positive integer N, and write for short $\mathcal{F}_{\mathbf{u}} = \mathcal{F}_{u_1}^{(1,0)}\otimes\cdots\otimes\mathcal{F}_{u_N}^{(1,0)}$. In this spaces $\mathcal{F}_{\mathbf{u}}$, a sort of W-algebra can be obtained.

Definition 5. Set the generators $X^{(i)}(z) = \sum_n X_n^{(i)}z^{-n}$ ($i = 1, \ldots, N$) by

$$\Lambda_i(z) = \varphi_-((q/t)^{-\frac{1}{4}}z)\otimes\cdots\otimes\varphi_-((q/t)^{-\frac{2i-3}{4}}z)\otimes\eta((q/t)^{-\frac{N-i}{2}}z)\otimes 1\otimes\cdots\otimes 1$$

$$X^{(i)}(z(q/t)^{\frac{i-1}{2}}) := \sum_{1\leq j_1<\cdots<j_i\leq N} :\Lambda_{j_1}(z)\cdots\Lambda_{j_i}((q/t)^{i-1}z): u_{j_1}\cdots u_{j_i}.$$

Note that $X^{(1)}(z) = \rho_{\mathbf{u}}^{(N,0)}(x^+(z))$. The algebra generated by $X_n^{(i)}$ can be regarded as the tensor product of the deformed W_N algebra and some Heisenberg algebra [9]. The structure of representation of the algebra $\langle X_n^{(i)} \rangle$ say the PBW type basis and singular vectors, is investigated in [10].

3 Generalized Macdonald Functions and AGT Correspondence

The generalized Macdonald functions are defined to be the eigenfunctions of $X_0^{(1)}$. The eigenvalues of $X_0^{(1)}$ are parametrized by N-tuples of partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$, and it is known that they are in the form

$$\epsilon_{\boldsymbol{\lambda}} = \sum_{i=1}^{N} u_i \big(1 + (t-1) \sum_{k \geq 1} (q^{\lambda_k^{(i)}} - 1) t^{-i} \big).$$

Definition 6. Define $|P_{\boldsymbol{\lambda}}\rangle \in \mathcal{F}_{\mathbf{u}}$ by two conditions:

- $X_0^{(1)} \, |P_{\boldsymbol{\lambda}}\rangle = \epsilon_{\boldsymbol{\lambda}} \, |P_{\boldsymbol{\lambda}}\rangle$;
- $|P_{\boldsymbol{\lambda}}\rangle = 1 \cdot |m_{\boldsymbol{\lambda}}\rangle + \cdots$.

Here, $|m_{\boldsymbol{\lambda}}\rangle := |m_{\lambda^{(1)}}\rangle \otimes \cdots \otimes |m_{\lambda^{(N)}}\rangle \in \mathcal{F}_{\mathbf{u}}$ is the tensor product of vectors corresponding to the monomial symmetric functions.

Remark 2. The existence theorem of $|P_{\boldsymbol{\lambda}}\rangle$ follows from the triangulation of the operator $X_0^{(1)}$ similarly to the ordinary Macdonald functions. The triangulation of $X_0^{(1)}$ can be proved by choosing the tensor product of the ordinary Macdonald functions as a basis. Then, the triangulation and the fact that the eigenvalues are not degenerate show that there exists an unique function $|P_{\boldsymbol{\lambda}}\rangle$ satisfying the above condition. For more details, see [11].

In the $N = 1$ case, $|P_\lambda\rangle$'s correspond to the ordinary Macdonald functions. In the case of general N, $|P_{\boldsymbol{\lambda}}\rangle$ are called generalized Macdonald functions. Furthermore, we give the following renormalization.

Definition 7. Define $|K_{\boldsymbol{\lambda}}\rangle \in \mathcal{F}_{\mathbf{u}}$ by

$$|K_{\boldsymbol{\lambda}}\rangle := (\text{monomial}) \cdot \prod_{1 \leq i < j \leq N} N_{\lambda^{(i)},\lambda^{(j)}}(qu_i/tu_j) \cdot \prod_{k=1}^{N} c_{\lambda^{(k)}} \cdot |P_{\boldsymbol{\lambda}}\rangle ,$$

$$c_\lambda := \prod_{(i,j)\in\lambda} (1 - q^{a_\lambda(i,j)} t^{\ell_\lambda(i,j)+1}),$$

$$N_{\lambda\mu}(u) := \prod_{(i,j)\in\lambda} \big(1 - uq^{a_\lambda(i,j)} t^{\ell_\mu(i,j)+1}\big) \prod_{(i,j)\in\mu} \big(1 - uq^{-a_\mu(i,j)-1} t^{-\ell_\lambda(i,j)}\big) .$$

where (monomial) means a certain monomial in parameters q, t and u_i, but let us omit an explicit form. $a_\lambda(i, j)$ and $\ell_\lambda(i, j)$ are the arm length and leg length, respectively.

This renormalization is obtained by a conjecture about relationship between $|K_{\boldsymbol{\lambda}}\rangle$ and the PBW type basis of the algebra $\langle X_n^{(i)} \rangle$ [4, Conjecture 3.38]. Moreover, define $\langle K_{\boldsymbol{\lambda}}|$ similarly.

Now, we explain the 5D AGT correspondence. Firstly, we defined an operator corresponding to the q-deformation of the Primary field.

Definition 8. Define the linear operator $\mathcal{V}(w) = \mathcal{V}\left(\begin{matrix}\mathbf{v}\\ \mathbf{u}\end{matrix}; w\right) : \mathcal{F}_{\mathbf{u}} \to \mathcal{F}_{\mathbf{v}}$ by

$$\left(X_n^{(i)} - wX_{n-1}^{(i)}\right) \mathcal{V}(w) = \mathcal{V}(w) \left(X_n^{(i)} - (t/q)^i wX_{n-1}^{(i)}\right) \quad (i = 1, \dots, N),$$

and $\langle 0| \mathcal{V}(w) |0\rangle = 1$.

We showed that this operator exists uniquely and proved the following formula for the matrix elements [4].

Theorem 1.

$$\langle K_{\boldsymbol{\lambda}}| \mathcal{V}\left(\begin{matrix}\mathbf{v}\\ \mathbf{u}\end{matrix}; x\right) |K_{\boldsymbol{\mu}}\rangle = (\mathit{monomial}) \times \prod_{i,j=1}^{N} N_{\lambda^{(i)},\mu^{(j)}}(qv_i/tu_j).$$

This formula was conjectured in [3]. The proof is given by a realization of $\mathcal{V}(z)$ in terms of screened vertex operators and Kajihara and Noumi's transformation formula for the q-multiple hypergeometric series [12]. By virtue of the above theorem, inserting the identity $1 = \sum_{\boldsymbol{\lambda}} \frac{|K_{\boldsymbol{\lambda}}\rangle\langle K_{\boldsymbol{\lambda}}|}{\langle K_{\boldsymbol{\lambda}}|K_{\boldsymbol{\lambda}}\rangle}$, we obtain

$$\begin{aligned} &\langle \mathbf{0}| \mathcal{V}\left(\begin{matrix}\mathbf{w}\\ \mathbf{v}\end{matrix}; z_1\right)\mathcal{V}\left(\begin{matrix}\mathbf{v}\\ \mathbf{u}\end{matrix}; z_2\right) |\mathbf{0}\rangle \\ &= \sum_{\boldsymbol{\lambda}} \left(\frac{u_1 \cdots u_N z_2}{v_1 \cdots v_N z_1}\right)^{|\boldsymbol{\lambda}|} \prod_{i,j=1}^{N} \frac{N_{\emptyset,\lambda^{(j)}}(qw_i/tv_j) N_{\lambda^{(i)},\emptyset}(qv_i/tu_j)}{N_{\lambda^{(i)},\lambda^{(j)}}(qv_i/tv_j)}. \end{aligned}$$

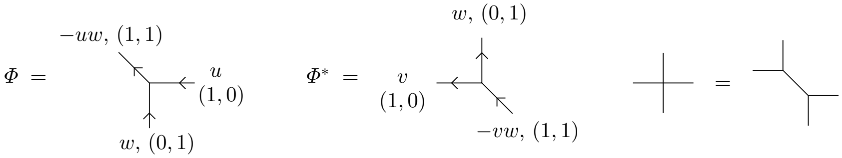
This equation is a 5D AGT correspondence, i.e., the LHS can be regarded as q-deformation of the 4-point conformal block, and the RHS is the Nekrasov partition function of the 5-dimensional $U(N)$ gauge theory with $2N$-matters.

4 Intertwiners of the DIM Algebra

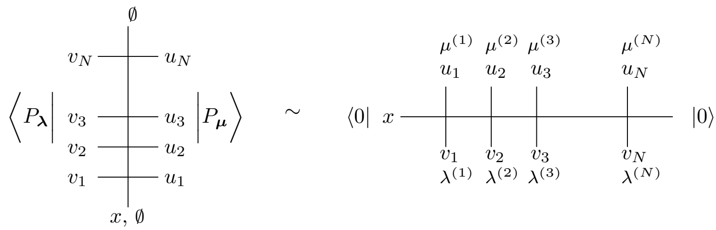
At last, we describe a certain duality formula for intertwining operators of the DIM algebra. Due to the number of pages, we will give only pictorial explanation. For more details, see [4]. To introduce intertwining operators of the DIM algebra, we require the level (0, 1) representation [5]. This representation is defined over the space $\mathcal{F}_u^{(0,1)} = Spann\{|\lambda\rangle \,|\lambda : \text{partition}\}$, where $|\lambda\rangle$'s are abstract vectors labeled by partitions, and u is some spectral parameter. Let us omit its definition in this paper. By these two representations, we can construct intertwining operators of the DIM algebra.

Fact 9 **([5]).** *There exists an unique intertwiner* $\Phi : \mathcal{F}_w^{(0,1)} \otimes \mathcal{F}_u^{(1,0)} \to \mathcal{F}_{-wu}^{(1,1)}$ *such that* $a\Phi = \Phi\Delta(a)$ *for all elements* a *in the DIM algebra. Similarly, there exists an unique intertwiner* $\Phi^* : \mathcal{F}_{-vw}^{(1,1)} \to \mathcal{F}_v^{(1,0)} \otimes \mathcal{F}_w^{(0,1)}$ *such that* $\Delta(a)\Phi^* = \Phi^* a$ $(\forall a)$

These operators are often expressed by the following trivalent diagrams. In this paper, we denote $\Phi^* \circ \Phi$ by the cross diagram for simplicity.



We consider the compositions of this cross diagram and take the matrix elements. Then $\mathcal{V}\left(\begin{matrix}\mathbf{v}\\ \mathbf{u}\end{matrix}; x\right)$ corresponds to the operator in the LHS of the picture below, and we can prove the following duality formula, which corresponds to the S-duality of the string theory:



Acknowledgements The author would like to thank the organizers of the International Workshop "Lie Theory and its applications in physics" in Varna for their kind hospitality. The author is partially supported by Grant-in-Aid for JSPS Research Fellow (18J00754).

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Representation Theory

Closures of K-orbits in the Flag Variety for $\mathrm{Sp}(2n, \mathbb{R})$

William M. McGovern

Abstract We give a pattern avoidance criterion to classify the orbits of $\mathbb{GL}(n, \mathbb{C})$ on the flag variety of $\mathrm{Sp}(2n, \mathbb{C})$ with smooth or rationally smooth closure, showing that orbits whose clans avoid the bad patterns have closures which fiber in a nice way over an orbit closure in a smaller flag variety.

1 Introduction

Suppose G is a complex connected reductive algebraic group and let θ denote an involutive automorphism of G. Write K (or K' or K'') for the fixed points of θ, and $\mathcal{B}$ for variety of Borel subalgebras of the Lie algebra $\mathfrak{g}$ of G. (Henceforth we call this variety simply the flag variety of G and identify it with G/B, B a Borel subgroup of G.) Then K acts with finitely many orbits on $\mathcal{B}$ via the restriction of the adjoint action.

In previous work we have considered the special cases $G = GL(p+q, \mathbb{C})$, $K' = GL(p, \mathbb{C}) \times GL(q, \mathbb{C})$ and $G = \mathrm{Sp}(2p+2q, \mathbb{C}$, $K'' = \mathrm{Sp}(2p, \mathbb{C}) \times \mathrm{Sp}(2q, \mathbb{C})$ (corresponding to the real forms $U(p, q)$ and $\mathrm{Sp}(p, q)$ of $GL(p+q, \mathbb{C})$ and $\mathrm{Sp}(2p+2q, \mathbb{C})$, respectively [3, 4]. Here we treat the case $G = \mathrm{Sp}(2n, \mathbb{C})$, $K = GL(n, \mathbb{C})$, corresponding to the split real form $\mathrm{Sp}(2n, \mathbb{R})$ of G. The pattern avoidance characterization of rationally smooth K-orbit closures that we will give is very similar to that of [4], but in this setting smoothness and rational smoothness of orbit closures are not equivalent; we give a pattern avoidance characterization of smoothness as well. We also take this opportunity to correct an error in [3] and [4].

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_26

2 Preliminaries

We begin by recalling that K-orbits in G/B are parametrized by skew-symmetric clans of length $2n$, that is, by sequences $c = c_1 \ldots c_{2n}$ such that each c_i is either a sign ($+$ or $-$) or a natural number, every natural number occurs either exactly twice among the c_i or not at all, and we have $c_{2n+1-i} = -c_i$ if c_i is a sign, while $c_i = c_j \in \mathbb{N}$ if and only if $c_{2n+1-i} = c_{2n+1-j} \in \mathbb{N}$ [10, §3]. We identify any two clans if they have the same signs in the same positions and pairs of equal numbers in the same positions, regardless of the sizes of the numbers; thus for example the clan $1 + -1$ is identified with $2 + -2$. The dimension of the orbit $\mathcal{O}_c$ corresponding to the clan c is given [10, §3.4] by

$$d(c) = \frac{n(n-1)}{2} + \frac{C_1}{2} + \frac{C_2}{2} \tag{1}$$

where

$$C_1 = \sum_{c_i = c_j \in \mathbb{N}, i<j} (j - i - \#\{k \in \mathbb{N} : c_s = c_t = k \text{ for some } s < i < t < j\}) \tag{2}$$

and

$$C_2 = \#\{t \in \mathbb{N} : c_s = c_t \in \mathbb{N} \text{ and } s \leq n < t \leq 2n + 1 - s\} \tag{3}$$

while the clan corresponding to the open orbit is

$$\gamma_0 = (1, \ldots, n, n \ldots, 1). \tag{4}$$

We have the action defined on [5] of noncompact imaginary root reflections s_α on clans. Given a clan $c = c_1, \ldots, c_{2n}$ with corresponding orbit $\mathcal{O}_c$, the root $\alpha = e_i - e_j$ is noncompact imaginary if and only if either c_i and c_j are opposite signs or they are unequal natural numbers with $(c_i, c_j) = (c_{2n+1-j}, c_{2n+1-i})$; in the first case $s_\alpha \cdot \mathcal{O}_c$ corresponds to the clan obtained from c by replacing c_i and c_j by equal natural numbers and c_{2n+1-i}, c_{2n+1-j} by a different pair of equal numbers (where in both cases the numbers do not appear elsewhere in c); in the second case $s_\alpha \cdot \mathcal{O}_c$ is obtained from c by interchanging c_i, c_j) but *not* c_{2n+1-i}, c_{2n+1-j}) [8]. Similarly, $\beta = e_i + e_j$ is noncompact imaginary for $\mathcal{O}_c$ if and only if c_i and c_{2n+1-j} are opposite signs, and in that case $s_\beta \cdot \mathcal{O}_c$ corresponds to the clan obtained from c by replacing c_i, c_{2n+1-j} by a pair of equal numbers and c_{2n+1-i}, c_j by a different pair of equal numbers (again not appearing elsewhere in c). Finally, the root $\gamma = 2e_i$ is noncompact imaginary for $\mathcal{O}_c$ if and only if c_i, c_{2n+1-i} are opposite signs, in which case $s_\gamma \cdot \mathcal{O}_c$ corresponds to the clan obtained from c by replacing c_i, c_{2n+1-i} by a pair of equal numbers not occurring elsewhere in c.

In all cases the orbit $s_\alpha \cdot \mathcal{O}_c$, $s\beta \cdot \mathcal{O}_c$, or $s_\gamma \cdot \mathcal{O}_c$ lies above $\mathcal{O}_c$ in the partial order by containment of closures. The general description of this order was given by Wyser for K'-orbits in $GL(p+q, \mathbb{C})/B$ [9]; using ideas from [10] we can extend it to K-orbits in G/B, as follows. Following the discussion before [9, Theorem 1.2], define $\gamma(i:+)$ for every clan $\gamma = c_1 \dots c_{2n}$ and every index $i \le n$ to be the total number of plus signs and pairs of equal natural numbers among $c_1 \dots c_i$; similarly define $\gamma(i;-)$ to be the total number of minus signs and pairs of equal natural numbers among $c_1 \dots c_i$. For all indices $i < j$ let $\gamma(i:j)$ be the number of pairs of equal natural numbers c_s, c_t with $s \le i < j < t$. Finally, for every index $i > n$ let $\gamma(i)$ be the number of pairs of equal natural numbers c_s, c_t with $s < n < t \le 2n+1-s, s \le i$. Partially ordering clans by containment of the corresponding orbit closures, we then have

Theorem 1 *We have $\gamma \le \tau$ in the above partial order if and only if the conditions $\gamma(i:+) \ge \tau(i:+), \gamma(i:-) \ge \tau(i:-), \gamma(i;j) \le \tau(i:j)$, and $\gamma(i) \le \tau(i)$, hold for all indices i, j in the appropriate range.*

The proof follows that of [9, Theorem 1.2] closely, in particular using the list of "moves" generating the partial order on skew-symmetric clans given in [9, Theorem 2.8] for the corresponding order on ordinary clans (parametrizing K'-orbits in the flag variety for $GL(p+q,\mathbb{C})$), but with some modifications and additions. Specifically, the moves replacing a pattern $+11-$ or $-11+$ by 1212 must be added to the list whenever the pairs $+-$, $-+$, or 11 in the patterns on the left occur symmetrically about the midpoint of the clan, that is, in positions i, $2n+1-i$, where $2n$ is the length of the clan. Similarly, the move replacing a pattern $+-+-$ or $-+-+$ by 1212 must be added whenever the outermost and innermost pairs of opposite signs in the patterns on the left occur symmetrically about the midpoint of the clan. Furthermore, whenever one of Wyser's moves is applied to a pair of entries not symmetric about the midpoint, the same move must simultaneously be applied to the corresponding pair of entries on the other side of the midpoint so as to preserve the skew-symmetry of the clan. The additional moves are justified by exhibiting a smooth curve of flags indexed by a parameter $t \in \mathbb{C}*$ lying in the first clan whose limit as $t \to 0$ lies in the second, as in Cases $1-9$ of the proof of [9, Theorem 1.2].

We now correct the main results of [3] and [4]. Recall (as mentioned in the previous paragraph) that K'-orbits in the flag variety of $GL(p+q,\mathbb{C})$ are parametrized by (ordinary) clans of length $p+q$ and signature (p,q); that is, by sequences $c = c_1 \dots c_{p+q}$ such that each c_i is either a sign or a natural number, every natural number occurs exactly twice among the c_i or not at all, and the number of pairs of equal numbers and $+$ signs among the c_i is p (so that the number of pairs of equal numbers and—signs among the c_i is q) [10, §2.2]. No further symmetry or skew-symmetry conditions on the c_i are imposed. The dimension of the orbit $\mathcal{O}_c$ corresponding to the clan $c = c_1 \dots c_{p+q}$ is given by

$$d_{p,q} + \sum_{c_i = c_j \in \mathbb{N}, i<j} (j - i - \#\{k \in \mathbb{N} : c_s = c_t = k \text{ for some } s < i < t < j\}). \quad (5)$$

where $d_{p,q} = \frac{1}{2}(p(p-1) + q(q-1))$, the dimension of all closed orbits. Furthermore, K''-orbits in the flag variety for $\mathrm{Sp}(2p+2q,\mathbb{C})$ are parametrized by symmetric clans of length $2p+2q$, that is, by clans $c = c_1 \dots c_{2p+2q}$ such that $c_i =$

$c_{2p+2q+1-i}$ whenever c_i is a sign and if $c_i = c_j \in \mathbb{N}$, then $j \neq 2p + 2q + 1 - i$ and $c_{2p+2q+1-i} = c_{2p+2q+1-j} \in \mathbb{N}$. The dimension of the orbit $\mathcal{O}_c$ corresponding to the clan c is given by the same formula as in the K-orbit case above. Define pattern inclusion and avoidance as in [4, Definition 2.1].

Theorem 2 *The K'-orbit $\mathcal{O}_c$ corresponding to the (ordinary) clan c has rationally smooth closure if and only if it has smooth closure, or if and only if c avoids the patterns* $1+-1$, $1-+1$, 1212, $1+221$, $1-221$, $122+1$, $122-1$, *and* 122331. *The K''-orbit $\mathcal{O}_c$ corresponding to the symmetric clan c has rationally smooth closure if and only if it has smooth closure, or if and only if there are nonnegative integers p', r, q', s such that $p' + r = p$, $q' + s = q$, and c takes the form $\gamma_1\gamma_0\gamma_1^r$, where γ_1 is an ordinary clan of signature (r, s) avoiding all bad patterns, γ_1^r is obtained from γ_1 by reversing it as a string and replacing every pair of equal numbers by a pair of different equal numbers, and the clan γ_0 is the one parametrizing the open orbit in the flag variety for* $\mathrm{Sp}(2p' + 2q', \mathbb{C})$ *([4, Equation (13)]); it is empty if $p' = q' = 0$.*

The last bad pattern 122331 is missing from the list in both [3, §3] and [4, Theorems 3.2, 4.2]; I would like to thank Ben Wyser for bringing it to my attention.

Proof We proceed as in [3] and [4]: if c is an ordinary clan and contains one of the bad patterns, replace every pair of equal numbers in the bad pattern by $-+$, in that order and in the same positions; replace every other pair of equal numbers by $+-$, in that order. We obtain a clan γ corresponding to a closed orbit $\mathcal{O}_\gamma$ in the partial order. In the action of noncompact root reflections on closed orbits (which is the same as in the K-orbit case described above) we easily check that more than $\dim \mathcal{O}_c - d_{p,q}$ such reflections send $\mathcal{O}_\gamma$ to another orbit lying between it and $\mathcal{O}_c$, whence the closure of $\mathcal{O}_c$ is not rationally smooth [7, 3.2, 3.3]. The same is true for symmetric clans c containing a bad pattern but not of the form specified above, by the proof of [4, Theorem 3.2]. If conversely c avoids all bad patterns and c is ordinary, then the intervals $[s, t]$ of indices s, t with $c_s = c_t \in \mathbb{N}$ either have one contained in the other or are disjoint. All signs lying between a pair of equal numbers in c are the same. If a sign lies between a pair of equal numbers, then it also lies between every pair of equal numbers enclosed by the first pair. Finally, if one pair of equal numbers lies inside another, then the pairs of equal numbers lying inside this pair form a single nested chain. The orbit $\mathcal{O}_c$ must then be a derived functor orbit in the sense of [3] and so has smooth closure. A similar argument applies if instead c is symmetric and takes the above form, by the proof of [4, Theorem 3.2]; in this case the closure $\bar{\mathcal{O}}_c$ of $\mathcal{O}_c$ is a fiber bundle with smooth fiber over a partial flag variety and so is smooth. If c does not take the above form then the criterion of [7, 3.2, 3.3] applies, as in [4], and shows that $\bar{\mathcal{O}}_c$ is not rationally smooth. □

Similarly, the list of bad patterns in the statement of [4, Theorem 4.2] should be amended to include 122331 as well.

3 Main Result

Our main result is a characterization of smooth and rationally smooth K-orbits in G/B, very much along the lines of the previous result.

Theorem 3 *The closure $\bar{\mathcal{O}}_c$ of the K-orbit corresponding to the skew-symmetric clan c has rationally smooth closure if and only if c takes the form $\gamma_1\gamma_0\gamma_1^r$, where γ_0 is either empty, $1+-1$, $1-+1$, or 1212, γ_1 avoids the bad patterns of Theorem 2.2, and γ_1^r is obtained from the ordinary clan γ_1 by reversing it as a string, changing all the signs, and replacing every pair of equal numbers by a pair of different equal numbers. There is no restriction on the signature of γ_1. The closure $\bar{\mathcal{O}}_c$ is smooth if and only if it is rationally smooth and γ_0 is either empty or 1212.*

Proof We argue as in [4] to show that $\bar{\mathcal{O}}_c$ is rationally singular if c does not take the above form. If it does take this form then once again $\bar{\mathcal{O}}_c$ is seen to be a fiber bundle with smooth fiber of the closure of the orbit with clan γ_0. By the Leray-Hirsch Theorem [6, p. 258] it suffices to verify the conclusion for the three nonempty possibilities for γ_0 itself. So take $G = \mathrm{Sp}(4, \mathbb{C})$ and fix a basis $\{e_1, e_2\}$ for a maximal isotropic subspace of $\mathbb{C}^4$, equipped with a nondegenerate skew-symmetric bilinear form $(\cdot, \cdot)$. Let $\{f_1, f_2\}$ be the corresponding basis of an isotropic dual to this space, so that $(e_i, f_j) = \delta_{ij}$. Then G/B may be identified with the space of maximal isotropic flags $0 \subset V_1 \subset V_2$ in $\mathbb{C}^4$. First let $\bar{\mathcal{O}}_\gamma$ be the orbit closure corresponding to the clan $\gamma = 1+-1$. The set of maximal isotropic flags for which V_1 is spanned by $e_1 + af_1 + bf_2$ and V_2 is spanned by V_1 and $ce_2 + bcf_1 + df_2$ for some affine coordinates $a, b \in \mathbb{C}^2$ and projective coordinates $c, d \in \mathbb{P}^2$ is a slice in the sense of [2, Definition 2.1] to $\bar{\mathcal{O}}_\gamma$ at the point corresponding to $a = b = d = 0, c = 1$ if the coordinates a, b, c, d are constrained to satisfy the equation $ad - b^2c = 0$. A similar argument of course applies to the clan $\gamma' = 1-+1$. Finally, the orbit closure $\bar{\mathcal{O}}''_\gamma$ corresponding to the clan $\gamma'' = 1212$ admits such a slice with the constraint $ad - b^2c = 0$ replaced by $a = 0$. Then the result follows by inspection. □

In particular we see that smoothness and rational smoothness are not equivalent for K-orbit closures. Wyser has shown that orbit closures corresponding to clans avoiding the pattern 1212 are Richardson varieties (intersections of Schubert varieties and opposite Schubert varieties) [8] Billey and Lakshmibai have characterized smooth and rationally smooth Schubert varieties in type C (and all other classical types) in [1].

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Action of the Restricted Weyl Group on the L-Invariant Vectors of a Representation

Ilia Smilga

Abstract This note constitutes a brief survey of our recent work on the problem of determining, for a given real Lie group G, the set of representations V in which the longest element w_0 of the restricted Weyl group W acts nontrivially on the subspace V^L of V formed by vectors that are invariant by L, the centralizer of a maximal split torus of G.

1 Introduction

1.1 Basic Notations and Statement of Problem

Let G be a semisimple real Lie group, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}^{\mathbb{C}}$ the complexification of $\mathfrak{g}$. We start by establishing the notations for some well-known objects related to $\mathfrak{g}$.

- We choose in $\mathfrak{g}$ a *Cartan subspace* $\mathfrak{a}$ (an abelian subalgebra of $\mathfrak{g}$ whose elements are diagonalizable over $\mathbb{R}$ and which is maximal for these properties).
- We choose in $\mathfrak{g}^{\mathbb{C}}$ a *Cartan subalgebra* $\mathfrak{h}^{\mathbb{C}}$ (an abelian subalgebra of $\mathfrak{g}^{\mathbb{C}}$ whose elements are diagonalizable and which is maximal for these properties) that contains $\mathfrak{a}$.
- We denote $L := Z_G(\mathfrak{a})$ the centralizer of $\mathfrak{a}$ in G, and $\mathfrak{l}$ its Lie algebra.
- Let Δ be the set of roots of $\mathfrak{g}^{\mathbb{C}}$ in $(\mathfrak{h}^{\mathbb{C}})^*$. We shall identify $(\mathfrak{h}^{\mathbb{C}})^*$ with $\mathfrak{h}^{\mathbb{C}}$ via the Killing form. We call $\mathfrak{h}_{(\mathbb{R})}$ the $\mathbb{R}$-linear span of Δ; it is given by the formula $\mathfrak{h}_{(\mathbb{R})} = \mathfrak{a} \oplus i\mathfrak{t}$, where $\mathfrak{t}$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{h}^{\mathbb{C}} \cap \mathfrak{g}$.
- We choose on $\mathfrak{h}_{(\mathbb{R})}$ a lexicographical ordering that "puts $\mathfrak{a}$ first", i.e. such that every vector whose orthogonal projection onto $\mathfrak{a}$ is positive is itself positive. We call Δ^+ the set of roots in Δ that are positive with respect to this ordering, and we let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots in Δ^+. Let $\varpi_1, \dots, \varpi_r$ be the corresponding fundamental weights.
- We introduce the *dominant Weyl chamber*

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*, Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_27

$$\mathfrak{h}^+ := \{X \in \mathfrak{h} \mid \forall i = 1, \dots, r, \quad \alpha_i(X) \geq 0\},$$

and the *dominant restricted Weyl chamber*

$$\mathfrak{a}^+ := \mathfrak{h}^+ \cap \mathfrak{a}.$$

- We introduce the *restricted Weyl group* $W := N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ of G. Then $\mathfrak{a}^+$ is a fundamental domain for the action of W on $\mathfrak{a}$. We define the *longest element* of the restricted Weyl group as the unique element $w_0 \in W$ such that $w_0(\mathfrak{a}^+) = -\mathfrak{a}^+$.

Our goal is to study the action of W, and more specifically of w_0, on various representations V of G. Note however that this action is ill-defined: indeed a representative $\tilde{w_0} \in N_G(\mathfrak{a}) \in G$ of the abstract element $w_0 \in W = N_G(\mathfrak{a})/Z_G(\mathfrak{a})$ is defined only up to multiplication by an element of $Z_G(\mathfrak{a}) = L$, whose action on V can of course be nontrivial.

This naturally suggests the idea of restricting to L-invariant vectors. Given a representation V of $\mathfrak{g}$, we denote

$$V^L := \{v \in V \mid \forall l \in L, \quad l \cdot v = v\}$$

the L-invariant subspace of V: then W, and in particular w_0, has a well-defined action on V^L. Now our goal is to solve the following problem:

Problem 1. Given a semisimple real Lie group G, characterize the representations V of G for which the action of w_0 on V^L is nontrivial.

In this note, we shall present our recent work on this problem.

1.2 Background and Motivation

This problem arose from the author's work in geometry. The interest of this particular algebraic property is that it furnishes a sufficient, and presumably necessary, condition for another, geometric property of V. Namely, the author obtained the following result:

Theorem 1. *[15] Let G be a semisimple real Lie group, V a representation of G. Suppose that the action of w_0 on V^L is nontrivial. Then there exists, in the affine group $G \ltimes V$, a subgroup Γ whose linear part is Zariski-dense in G, which is free of rank at least 2, and acts properly discontinuously on the affine space corresponding to V.*

He, and other people, also proved the converse statement in some special cases:

Theorem 2. *The converse holds, for irreducible V:*

- *[14] if G is split, but not of type A_n ($n \geq 2$), D_{2n+1} or E_6;*

- *[14] if G is split, has one of these types, and V satisfies a very restrictive additional assumption (see [14] for the precise statement);*
- *[3] if $G = \mathrm{SO}(p, q)$ for arbitrary p and q, and $V = \mathbb{R}^{p+q}$ is the standard representation.*

Moreover, it seems plausible that, by combining the approaches of [14] and [3], we might prove the converse in all generality. This geometric property is related to the so-called Auslander conjecture [4], which is an important conjecture that has stood for more than fifty years and generated an enormous amount of work: see e.g. [2, 6, 7, 11, 12] and many many others. For the statement of the conjecture as well as a more comprehensive survey of past work on it, we refer to [1].

The author would like to acknowledge the support of the European Research Council (ERC) under the European Union Horizon 2020 research and innovation programme, grant 647133 (ICHAOS).

2 Basic Properties

2.1 Reduction from Groups to Algebras

First of all, note that, without loss of generality, we may restrict our attention to irreducible representations: indeed, plainly, the condition $w_0|_{V^L} \neq \mathrm{Id}$ holds for a direct sum $V = V_1 \oplus V_2$ if and only if it holds for either V_1 or V_2.

Let us start by recalling the classification of the irreducible representations of G, $\mathfrak{g}$ and $\mathfrak{g}^{\mathbb{C}}$. We introduce the notation P (resp. Q) for the *weight lattice* (resp. *root lattice*), i.e. the abelian subgroup of $\mathfrak{h}_{(\mathbb{R})}$ (or, technically, its dual) generated by $\varpi_1, \ldots, \varpi_r$ (resp. by Δ).

Theorem 3 (see e.g. [9, Theorem 5.5] **or** [8, Theorems 9.4 and 9.5]**).** *To every irreducible representation of $\mathfrak{g}^{\mathbb{C}}$ we may associate, in a bijective way, a vector $\lambda \in P \cap \mathfrak{h}^+$ called its* highest weight.

Theorem 4 (see e.g. [9, Proposition 7.15]**).** *The restriction map $\rho \mapsto \rho|_{\mathfrak{g}}$ induces a bijection between irreducible representations of $\mathfrak{g}^{\mathbb{C}}$ and those of $\mathfrak{g}$.*

Theorem 5. *Every irreducible representation of G yields, by derivation, an irreducible [9, Proposition 7.15] representation of $\mathfrak{g}$. The converse is true if $G = \tilde{G}$ is simply-connected. For arbitrary G, the representation $\rho_\lambda(\mathfrak{g})$ lifts to G if and only if [9, § 5.8] λ lies in some lattice Λ_G satisfying $Q \subset \Lambda_G \subset P$.*

For every dominant weight $\lambda \in P \cap \mathfrak{h}^+$, we denote by V_λ the representation of $\mathfrak{g}^{\mathbb{C}}$, of $\mathfrak{g}$ or (if it exists) of G with highest weight λ.

To reformulate Problem 1 in terms of algebras, it remains to note two things. First, we note that the action of w_0 on $V^{\mathfrak{l}}$ depends only on $\mathfrak{g}$, not on G: indeed G is in general the quotient of the (unique) simply-connected group $\tilde{G}$ with algebra $\mathfrak{g}$

by some subgroup of its center; but the center of $\tilde{G}$ is in particular contained in the centralizer L of $\mathfrak{a}$, so is irrelevant when acting on V^L. Second, it is easy to see that the space V^L always coincides with the space

$$V^{\mathfrak{l}} := \{v \in V \mid \forall X \in \mathfrak{l}, \quad X \cdot v = 0\}$$

of $\mathfrak{l}$-invariant vectors of V. So Problem 1 is in fact equivalent to the following:

Problem 2. Given a semisimple real Lie algebra $\mathfrak{g}$, characterize the set of weights $\lambda \in P \cap \mathfrak{h}^+$ for which the action of w_0 on the $\mathfrak{l}$-invariant subspace $V_\lambda^{\mathfrak{l}}$ of the representation V_λ of $\mathfrak{g}$ with highest weight λ is nontrivial.

2.2 Additivity Properties

A first step towards the solution of Problem 2 is given by the following characterization:

Proposition 1.

(i) The set of weights $\lambda \in P \cap \mathfrak{h}^+$ *such that* $V_\lambda^{\mathfrak{l}} \neq 0$ *is contained in* $Q \cap \mathfrak{h}^+$.
(ii) The set of weights λ *such that* $V_\lambda^{\mathfrak{l}} \neq 0$ *is in fact a submonoid of the additive monoid* $Q \cap \mathfrak{h}^+$, *i.e.*

$$\forall \lambda, \mu \in Q \cap \mathfrak{h}^+, \quad \begin{cases} V_\lambda^{\mathfrak{l}} \neq 0 \\ V_\mu^{\mathfrak{l}} \neq 0 \end{cases} \implies V_{\lambda+\mu}^{\mathfrak{l}} \neq 0.$$

(iii) In this monoid, the subset of weights λ *such that* $w_0|_{V_\lambda^{\mathfrak{l}}} \neq \pm\mathrm{Id}$ *is an ideal, i.e.*

$$\forall \lambda, \mu \in Q \cap \mathfrak{h}^+, \quad \begin{cases} w_0|_{V_\lambda^{\mathfrak{l}}} \neq \pm\mathrm{Id} \\ V_\mu^{\mathfrak{l}} \neq 0 \end{cases} \implies w_0|_{V_{\lambda+\mu}^{\mathfrak{l}}} \neq \pm\mathrm{Id}.$$

Proof. Point (i) is straightforward: indeed, since $\mathfrak{h}$ is an abelian algebra containing $\mathfrak{a}$, we have $\mathfrak{h} \subset \mathfrak{l}$, hence $V_\lambda^{\mathfrak{l}}$ is contained in $V_\lambda^{\mathfrak{h}}$, which is just the zero-weight space of V_λ. By well-known theory (see e.g. [8], Theorem 10.1), the latter is nontrivial, or in other terms 0 is a weight of V_λ, if and only if λ lies in the root lattice Q.

Points (ii) and (iii) rely on the following classical theorem. Let G be a simply-connected complex Lie group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$, let N be a maximal unipotent subgroup of G. Let $\mathbb{C}[G/N]$ denote the space of regular (i.e. polynomial) functions on G/N. Pointwise multiplication of functions is G-equivariant and makes $\mathbb{C}[G/N]$ into a $\mathbb{C}$-algebra without zero divisors (as G/N is irreducible as an algebraic variety).

Theorem 6 (see e.g. [13, (3.20)–(3.21)]). *Each finite-dimensional irreducible representation of G (or equivalently of its Lie algebra $\mathfrak{g}^{\mathbb{C}}$) occurs exactly once as a direct summand of the representation $\mathbb{C}[G/N]$. The $\mathbb{C}$-algebra $\mathbb{C}[G/N]$ is graded by the highest weight λ, in the sense that the product of a vector in V_λ by a vector in V_μ lies in $V_{\lambda+\mu}$ (where V_λ stands here for the subrepresentation of $\mathbb{C}[G/N]$ with highest weight λ).*

For given λ and μ, we call *Cartan product* the induced bilinear map $\odot : V_\lambda \times V_\mu \to V_{\lambda+\mu}$. Given $u \in V_\lambda$ and $v \in V_\mu$, this defines $u \odot v \in V_{\lambda+\mu}$ as the projection of $u \otimes v \in V_\lambda \otimes V_\mu = V_{\lambda+\mu} \oplus \dots$. Since $\mathbb{C}[G/N]$ has no zero divisor, $u \odot v \neq 0$ whenever $u \neq 0$ and $v \neq 0$. We may now finish the proof:

Proof. [Proof of Proposition 1, continued]

(ii) Let λ and μ be two weights with this property. Choose any two nonzero vectors u_1 and u_2 in $V^{\mathfrak{l}}_{\lambda_1}$ and $V^{\mathfrak{l}}_{\lambda_2}$ respectively. Then the vector $u_1 \odot u_2$ is in $V_{\lambda_1+\lambda_2}$, is invariant by $\mathfrak{l}$, and is still nonzero.

(iii) Let λ be such that $w_0|_{V^{\mathfrak{l}}_\lambda} \neq \pm\mathrm{Id}$, and μ be such that $V^{\mathfrak{l}}_\mu \neq 0$. We can then choose nonzero vectors u_+ and u_- in $V^{\mathfrak{l}}_\lambda$ such that $w_0 \cdot u_+ = u_+$ and $w_0 \cdot u_- = -u_-$, and a nonzero vector $v \in V^{\mathfrak{l}}_\mu$ such that $w_0 \cdot v = \pm v$ for some sign (indeed since $w_0^2 = \mathrm{Id}$ in the Weyl group and the action of the Weyl group on $V^{\mathfrak{l}}$ is well-defined, $w_0|_{V^{\mathfrak{l}}_\mu}$ is a linear involution). Then $u_+ \odot v$ and $u_- \odot v$ are nonzero elements of $V^{\mathfrak{l}}_{\lambda+\mu}$ on which w_0 acts by opposite signs.

2.3 Reduction from Semisimple to Simple Algebras

Proposition 1 suggests the decomposition of Problem 2 into two subproblems, as follows:

Problem 3. Given a semisimple Lie algebra $\mathfrak{g}$ and a weight $\lambda \in P \cap \mathfrak{h}^+$, give a simple necessary and sufficient condition for having $V^{\mathfrak{l}}_\lambda \neq 0$.

Problem 4. Given a semisimple Lie algebra $\mathfrak{g}$ and a weight $\lambda \in P \cap \mathfrak{h}^+$, assuming that $V^{\mathfrak{l}}_\lambda \neq 0$, give a simple necessary and sufficient condition for having $w_0|_{V^{\mathfrak{l}}_\lambda} = \mathrm{Id}$.

Let us now reduce these two problems to the case where $\mathfrak{g}$ is simple. This can be done by using the following theorem, whose proof is straightforward:

Theorem 7. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be a semisimple Lie algebra. Let $\lambda = \lambda_1 + \cdots + \lambda_k$ be a weight of $\mathfrak{g}$, with every component λ_i lying in the Cartan subalgebra of $\mathfrak{g}_i$. Then we have, with the obvious notations: $\mathfrak{l} = \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_k$; $V^{\mathfrak{l}}_\lambda = V^{\mathfrak{l}_1}_{1,\lambda_1} \otimes \cdots \otimes V^{\mathfrak{l}_k}_{k,\lambda_k}$; $W = W_1 \times \cdots \times W_k$; $w_0 = w_{0,1} \cdots w_{0,k}$; and $w_0|_{V^{\mathfrak{l}}_\lambda} = w_{0,1}\big|_{V^{\mathfrak{l}_1}_{1,\lambda_1}} \otimes \cdots \otimes w_{0,k}\big|_{V^{\mathfrak{l}_k}_{k,\lambda_k}}$.*

So Problem 3 completely reduces to the simple case, as $V_\lambda^{\mathfrak{l}}$ is nontrivial if and only if each of its factors $V_{i,\lambda_i}^{\mathfrak{l}_i}$ is nontrivial. For Problem 4, things are slightly more subtle: if the tensor product of k linear maps is the identity map, this does not imply that all factors are identity maps—only that all factors are scalars, with the coefficients having product 1. Since $w_0^2 = \mathrm{Id}$ in the Weyl group, these coefficients can only be ± 1. So Problem 4 reduces to the following, slightly more general problem in the simple case:

Problem 5. Given a simple Lie algebra $\mathfrak{g}$ and a weight $\lambda \in P \cap \mathfrak{h}^+$, assuming that $V_\lambda^{\mathfrak{l}} \neq 0$, give a simple necessary and sufficient condition for having $w_0|_{V_\lambda^{\mathfrak{l}}} = \pm\mathrm{Id}$, as well as a criterion to determine the actual sign.

3 Known Results

3.1 *Case Where $\mathfrak{g}$ is Split*

We start by focusing on the particular case where $\mathfrak{g}$ is split. In this case, we have $\mathfrak{a} = \mathfrak{h}$ hence $\mathfrak{l} = \mathfrak{h} = \mathfrak{a}$ (by maximality of $\mathfrak{h}$), so that $V_\lambda^{\mathfrak{l}}$ is simply the zero-weight space of V_λ, that we shall denote by V_λ^0. As for the restricted Weyl group, it coincides in this case with the ordinary Weyl group. So we may actually forget that $\mathfrak{g}$ is a real Lie algebra, and simply work over $\mathbb{C}$.

Problem 3 then becomes trivial: in fact, the containment given in Proposition 1 (i) now becomes an equality, so the condition is just that $\lambda \in Q$.

As for Problem 5, it has been solved by the author together with Le Floch; this was the work presented at the conference talk from which this proceedings paper is derived. We proved the following result:

Theorem 8. *[10] Let $\mathfrak{g}$ be a simple Lie algebra, $\lambda \in Q \cap \mathfrak{h}^+$ a dominant weight of $\mathfrak{g}$ that lies in the root lattice.*

- *Suppose that λ is of the form $\lambda = kp_i\varpi_i$, where ϖ_i is one of the fundamental weights of $\mathfrak{g}$, p_i is the smallest positive integer such that $p_i\varpi_i \in Q$, and k does not exceed some constant $m_i \in \{0, 1, 2, +\infty\}$ depending on $\mathfrak{g}$ and ϖ_i.*
  Then $w_0|_{V_\lambda^0} = \sigma_i^k \mathrm{Id}$, where σ_i is some sign depending on $\mathfrak{g}$ and ϖ_i. The specific values of the constants p_i, m_i and σ_i are all listed in Table 1 of [10].
- *If λ does not have this form, then $w_0|_{V_\lambda^0} \neq \pm\mathrm{Id}$.*

3.2 *Case Where $\mathfrak{g}$ is Arbitrary*

In the general case, Problem 3 has just recently been solved by the author [16]. The answer is as follows:

Theorem 9. *[16] Let $\mathfrak{g}$ be a real simple Lie algebra. Then the set X of dominant weights $\lambda \in P \cap \mathfrak{h}^+$ such that $V_\lambda^{\mathfrak{l}} \neq 0$ has one of the following forms:*

$$X = Q \cap \mathscr{C} \quad \text{or} \quad X = \Lambda \cap \mathscr{C} \quad \text{or} \quad X = (Q \cap \mathfrak{h}^+ \setminus \mathscr{C}) \cup (\Lambda \cap \mathscr{C}),$$

where $\mathscr{C}$ is a closed polyhedral convex cone (i.e. a set determined by a finite number of inequalities of the form $\phi_i(\lambda) \geq 0$ where each ϕ_i is a linear form) contained in the dominant Weyl chamber $\mathfrak{h}^+$, and Λ (when applicable) is a sublattice of Q of index 2.

We refer to [16] for an exhaustive table listing the sets $\mathscr{C}$ and Λ for all the real simple Lie algebras. Here we will just give a brief overview:

- The sublattice Λ intervenes only when $\mathfrak{g}$ is isomorphic to some $\mathfrak{so}(p, q)$ with $p + q$ odd. In all other cases, we simply have $X = Q \cap \mathscr{C}$.
- For split groups, quasi-split groups, all non-compact exceptional groups and some of the classical groups, $\mathscr{C}$ is actually the whole dominant Weyl chamber.
- However in the remaining cases, $\mathscr{C}$ does not always have nonempty interior. In fact, often $\mathscr{C}$ is the intersection of the dominant Weyl chamber with a vector subspace of $\mathfrak{h}$. When $\mathfrak{g}$ is compact (and only then), we have $\mathscr{C} = \{0\}$.

For Problem 4, we only have experimental results so far. Explicit computation of $w_0|_{V^{\mathfrak{l}}}$ for all sufficiently low-dimensional representations V of all sufficiently low-rank simple Lie algebras, combined with Proposition 1. (iii), suggests the following generalization of Theorem 8:

Conjecture 1. Let $\mathfrak{g}$ be any simple real Lie algebra of rank r, and $\lambda \in Q \cap \mathfrak{h}^+$ a dominant weight of $\mathfrak{g}$ that lies in the root lattice. Let us decompose λ into its coordinates on the basis formed by the fundamental weights: $\lambda = \sum_{i=1}^r c_i \varpi_i$. Suppose that $V_\lambda^{\mathfrak{l}} \neq 0$: then $w_0|_{V_\lambda^{\mathfrak{l}}} = \pm \mathrm{Id}$ can happen only when at most K of the coordinates c_i are nonzero, where $K = 3$.

(Compare this with the split case, where this statement holds for $K = 1$.) Moreover, these experimental results allow us to conjecture an explicit description of the set of weights satisfying this property, for almost all simple Lie groups $\mathfrak{g}$. Here are a few examples.

Conjecture 2. Let $\mathfrak{g}$ and $\lambda = \sum_{i=1}^r c_i \varpi_i$ be as previously, with the fundamental weights $\varpi_1, \ldots, \varpi_r$ labelled in the Bourbaki ordering [5]. Then:

(i) If $\mathfrak{g} = \mathfrak{so}(2, q)$ with $q = 7$ or $q \geq 9$, then [16] tells us that $V_\lambda^{\mathfrak{l}} \neq 0$ if and only if $\lambda \in Q$ and $c_i = 0$ for all $i > 4$; and assuming this is the case, we have $w_0|_{V_\lambda^{\mathfrak{l}}} = \pm \mathrm{Id}$ if and only if $\lambda = x\varpi_i + y\varpi_4$ with $i \in \{1, 2, 3\}$ and x, y arbitrary nonnegative integers.

(ii) If $\mathfrak{g} = EIV$ (the real form of E_6 with maximal compact subalgebra F_4, also known as E_6^{-26}), then [16] tells us that $V_\lambda^{\mathfrak{l}} \neq 0$ always holds for $\lambda \in Q$; and then $w_0|_{V_\lambda^{\mathfrak{l}}} = \pm \mathrm{Id}$ if and only if $\lambda = x\varpi_i + y\varpi_2$ with $i \in \{1, 3, 5, 6\}$ and x, y arbitrary nonnegative integers.

(iii) If $\mathfrak{g}$ is the Lie algebra variously called $\mathfrak{sp}(12, 4)$ (by some authors, such as Bourbaki [5]) or $\mathfrak{sp}(6, 2)$ (by some authors, such as Knapp [9]), then [16] tells us that $V_\lambda^{\mathfrak{l}} \neq 0$ always holds for $\lambda \in Q$; and then we have $w_0|_{V_\lambda^{\mathfrak{l}}} = \pm \mathrm{Id}$ if and only if $\lambda = x\varpi_i + y\varpi_8$ with $1 \leq i \leq 7$ and x, y some nonnegative integers, with the additional condition $x \leq 2$ if $i = 3, 4$ or 5.

However, there are a few pairs $(\mathfrak{g}, V)$ where the dimension of V is so large that brute-force computations become intractable, but where analogous representations in smaller rank are not sufficiently well-behaved to establish a general pattern. For instance, take $\mathfrak{g} = \mathfrak{sp}(9, 3)$ (or $\mathfrak{sp}(18, 6)$ with the other convention) and $\lambda = 4\varpi_{11}$. We know, from [16], that we then have $V_\lambda^{\mathfrak{l}} \neq 0$; but we do not currently know (and cannot easily guess) whether $w_0|_{V_\lambda^{\mathfrak{l}}}$ is scalar or not.

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Hom-Lie Structures on Complex 4-Dimensional Lie Algebras

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Abstract The space of possible Hom-Lie structures on complex 4-dimensional Lie algebras is considered in terms of linear maps that turn the Lie algebras into Hom-Lie algebras. Hom-Lie structures on the representatives of isomorphism classes of complex 4-dimensional Lie algebras are described.

MSC2020 Classification: 17B61, 17D30

1 Introduction

The q-deformed Jacobi identities observed for the q-deformed algebras in physics, along with q-deformed versions of homological algebra and discrete modifications of differential calculi have been an initial motivation for the general quasi-Hom-Lie quasi-deformations and discretizations of Lie algebras of vector fields using more general σ-derivations (twisted derivations) developed in [3] and introduction of the general abstract quasi-Lie algebras and the subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras as well as their general colored (graded) counterparts in [3–6, 12]. Subsequently, Hom-Lie admissible algebras have been considered in [8] where also the Hom-associative algebras have been introduced and shown to be Hom-Lie admissible natural generalizations of associative algebras corresponding to Hom-Lie algebras, and also several other interesting classes of Hom-Lie admissible algebras generalising some non-associative algebras have been introduced. Moreover in [8] some classes of low-dimensional Hom-Lie algebras have been described initiating the investigation of the classification problems for Hom-Lie algebras of a given finite dimension. Since these pioneering works [3–8], Hom-algebra structures have become a popular area with increasing number of publications in various directions.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_28

Hom-Lie algebras are generalizations of Lie algebras with a twist α, a linear mapping twisting Jacobi identity. The algebraic varieties of such structures can be defined by a system of polynomial equations in terms of their structure constants [3, 11]. In [11] it was proven that any 3-dimensional skew-symmetric algebra admits a Hom-Lie algebra structure by considering the kernel space of matrix involving sums of structure constants of the bilinear map. In [9], the authors independently proceed with constructing such linear maps and give a condition for the kernel space to be of minimum dimension 6 for 3-dimension case. The result on having non-trivial vector subspace of endomorphisms α for given skew-symmetric bilinear map is also given in [2] where 3-dimensional complex Hom-Lie algebras are studied. The authors in [2] also classify 3-dimensional complex Hom-Lie algebras by beginning with representatives of the skew-symmetric bilinear map and give a complete set of canonical forms for linear maps α which lead to a Hom-Lie algebra. The space of such linear maps that turn any skew-symmetric algebra into a Hom-Lie algebra forms a vector subspace, known as the space of Hom-Lie structures, consisting of all such twisting linear maps that make the product to satisfy Hom-Jacobi identity, the generalised version of Jacobi identity. In [11], it is proved that every 3-dimensional skew-symmetric algebra can be turned into a Hom-Lie algebra. Hom-Lie structures in 3-dimensional case have also been studied in [10] and [9]. In [10] some results obtained showing correspondence between the dimension of the space of Hom-Lie structures and the rank of a 3×3 matrix having structure constants of the bilinear map. However, not all complex 4-dimensional skew-symmetric algebras can be turned into a Hom-Lie algebra. An example of such a non-Lie algebra is given in [11].

This work considers complex 4-dimensional Lie algebras as a step towards classifying Hom-Lie structures on complex 4-dimensional skew-symmetric algebras. The first section gives preliminaries and polynomial equations arising from the Hom-Jacobi identity for the dimension 4 case and a list of all representatives of classification for 4-dimensional Lie algebras. In the second section we give the values of α that turn the Lie algebras into a Hom-Lie algebra and their respective automorphism groups. The third section gives non-isomorphic forms of Hom-Lie structures for some of the Lie algebras. These forms are constructed via group action of the automorphism groups on the Hom-Lie structures.

2 Preliminaries

We begin by giving the definition of a Hom-Lie algebra.

Definition 1 A Hom-Lie algebra is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, bilinear map $[\cdot,\cdot] : V \times V \longrightarrow V$ and a linear space homomorphism $\alpha : V \to V$ satisfying

$$[x, y] = -[y, x] \qquad (skew - symmetry) \tag{1}$$

$$[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0 \qquad (Hom - Jacobi\ identity) \tag{2}$$

for all $x, y, z \in V$.

Let the structure constants associated to both the bilinear product and linear map be given by $\{C_{ij}^k\}_{i<j}$ and $\{a_{it}\}$ respectively. We have the following equations involving the structure constants in general for n-dimensional case:

$$[e_i, e_j] = \sum_{s=1}^{n} C_{ij}^s e_s \text{ and } \alpha(e_i) = \sum_{t=1}^{n} a_{it} e_t \tag{3}$$

Hom-Lie structures, is the linear space of all linear endomorphisms that satisfy the Hom-Jacobi identity for some skew-symmetric algebra $\mathfrak{L}$ with bilinear map μ, that is $HomLie(\mu) = \{\alpha \in End\ V \mid \circlearrowleft_{x,y,z} \mu(\alpha(x), \mu(y,z)) = 0\}$ where $\circlearrowleft_{x,y,z} = \mu(\alpha(x), \mu(y,z)) + \mu(\alpha(y), \mu(z,x)) + \mu(\alpha(z), \mu(x,y))$. Thus the following system of polynomial equations arise for 4-dimensional case:

$$\sum_{\substack{m,n=1 \\ m<n}}^{4} \Big(\circlearrowleft_{i,j,k} (a_{im} C_{jk}^n C_{mn}^r - a_{in} C_{jk}^m C_{mn}^r) \Big) = 0 \text{ for } r = 1, 2, 3, 4 \tag{4}$$

where the symbol $\circlearrowleft_{i,j,k}$ denotes a summation over the cyclic permutation on i, j, k with $i, j, k = 1, 2, 3, 4$. We end up with 16 equations after expanding each of the four Hom-Jacobi identity expressions. Equation (4) linear in a_{ij} is given as;

$$\begin{aligned}
a_{i1}(C_{jk}^2 C_{12}^r + C_{jk}^3 C_{13}^r + C_{jk}^4 C_{14}^r) + a_{j1}(C_{ki}^2 C_{12}^r + C_{ki}^3 C_{13}^r + C_{ki}^4 C_{14}^r) \\
+ a_{k1}(C_{ij}^2 C_{12}^r + C_{ij}^3 C_{13}^r + C_{ij}^4 C_{14}^r) + a_{i2}(-C_{jk}^1 C_{12}^r + C_{jk}^3 C_{23}^r + C_{jk}^4 C_{24}^r) \\
+ a_{j2}(-C_{ki}^1 C_{12}^r + C_{ki}^3 C_{23}^r + C_{ki}^4 C_{24}^r) + a_{k2}(-C_{ij}^1 C_{12}^r + C_{ij}^3 C_{23}^r + C_{ij}^4 C_{24}^r) \\
+ a_{i3}(-C_{jk}^1 C_{13}^r - C_{jk}^2 C_{23}^r + C_{jk}^4 C_{34}^r) + a_{j3}(-C_{ki}^1 C_{13}^r - C_{ki}^2 C_{23}^r + C_{ki}^4 C_{34}^r) \\
+ a_{k3}(-C_{ij}^1 C_{13}^r - C_{ij}^2 C_{23}^r + C_{ij}^4 C_{34}^r) + a_{i4}(-C_{jk}^1 C_{14}^r - C_{jk}^2 C_{24}^r - C_{jk}^3 C_{34}^r) \\
+ a_{j4}(-C_{ki}^1 C_{14}^r - C_{ki}^2 C_{24}^r - C_{ki}^3 C_{34}^r) + a_{k4}(-C_{ij}^1 C_{14}^r - C_{ij}^2 C_{24}^r - C_{ij}^3 C_{34}^r) = 0
\end{aligned}$$

Complex 4-dimensional Lie algebras have been classified in [1]. We use the list of representatives to describe the Hom-Lie structures on 4-dimensional non-abelian Lie algebras with basis $\{e_1, e_2, e_3, e_4\}$.

Lemma 1 ([1]) *Every complex* 4*-dimensional Lie algebra (non-abelian) is isomorphic to one and only one Lie algebra of the following list.*

Table 1 Classification of complex 4-dimensional Lie algebras

$\mathfrak{L}$	Lie brackets
$\mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_3$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^2$	$[e_1, e_2] = e_1$
$\mathfrak{r}_3(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_2, \ \ [e_1, e_3] = e_1 + e_3$
$\mathfrak{r}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3, \ \ \lambda \in \mathbb{C}, 0 < \lvert\lambda\rvert \leq 1$
$\mathfrak{r}_2(\mathbb{C}) \oplus \mathfrak{r}_2(\mathbb{C})$	$[e_1, e_2] = e_1, \ \ [e_3, e_4] = e_3$
$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$	$[e_1, e_2] = e_3, \ \ [e_1, e_3] = -2e_1, \ \ [e_2, e_3] = 2e_2$
$\mathfrak{n}_4(\mathbb{C})$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4$
$\mathfrak{g}_1(\gamma)$	$[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = \gamma e_4, \ \ \gamma \in \mathbb{C}^*$
$\mathfrak{g}_2(\gamma, \beta)$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = \gamma e_2 - \beta e_3 + e_4, \gamma \in \mathbb{C}^*, \beta \in \mathbb{C}$ or $\gamma, \beta = 0$
$\mathfrak{g}_3(\gamma)$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, \ \ [e_1, e_4] = \gamma(e_2 + e_3), \ \ \gamma \in \mathbb{C}^*$
$\mathfrak{g}_4$	$[e_1, e_2] = e_3, [e_1, e_3] = e_4, \ \ [e_1, e_4] = e_2$
$\mathfrak{g}_5$	$[e_1, e_2] = \frac{1}{3}e_2 + e_3, [e_1, e_3] = \frac{1}{3}e_3, [e_1, e_4] = \frac{1}{3}e_4$
$\mathfrak{g}_6$	$[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_1, e_4] = 2e_4, [e_2, e_3] = e_4$
$\mathfrak{g}_7$	$[e_1, e_2] = e_3, [e_1, e_3] = e_2, [e_2, e_3] = e_4$
$\mathfrak{g}_8(\gamma)$	$[e_1, e_2] = e_3, [e_1, e_3] = -\gamma e_2 + e_3, [e_1, e_4] = e_4, [e_2, e_3] = e_4, \ \ \gamma \in \mathbb{C}$

3 Hom-Lie Structures and Automorphisms

Let $Aut(\mathfrak{L})$ denote the automorphism group of representative Lie algebra $\mathfrak{L}$, that is $Aut(\mathfrak{L}) = \{g \in GL(4, \mathbb{C}) \mid g[e_i, e_j] = [g(e_i), g(e_j)], 1 \leq i < j \leq 4\}$, where $g(e_i) = \sum_{k=1}^{4} g_{ik} e_k$. For each of the representatives given in Table 1, we give the values of α that turn the Lie algebras into a Hom-Lie algebra and the automorphism group $g \in Aut(\mathfrak{L})$.

$\underline{\mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ 0 & 0 & g_{11}g_{22} - g_{12}g_{21} & 0 \\ 0 & 0 & g_{43} & g_{44} \end{pmatrix}, \begin{matrix} g_{44} \neq 0 \\ g_{12}g_{21} \neq g_{11}g_{22} \end{matrix}$$

$\underline{\mathfrak{t}_2(\mathbb{C}) \oplus \mathbb{C}^2}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ g_{21} & 1 & g_{23} & g_{24} \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & g_{43} & g_{44} \end{pmatrix}, \quad \begin{matrix} g_{11} \neq 0 \\ g_{34}g_{43} \neq g_{33}g_{44} \end{matrix}$$

$\underline{\mathfrak{t}_3(\mathbb{C}) \oplus \mathbb{C}}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & 0 & 0 & g_{14} \\ 0 & g_{22} & 0 & 0 \\ 0 & g_{32} & 1 & -g_{14} \\ 0 & 0 & 0 & g_{44} \end{pmatrix}, \quad g_{22}g_{44} \neq 0$$

$\underline{\mathfrak{t}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C}}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & 0 \\ 0 & 0 & 0 & g_{44} \end{pmatrix}, \quad g_{22}g_{33}g_{44} \neq 0$$

$\underline{\mathfrak{t}_2(\mathbb{C}) \oplus \mathfrak{t}_2(\mathbb{C})}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ g_{12} & 1 & 0 & 0 \\ 0 & 0 & g_{33} & 0 \\ 0 & 0 & g_{43} & 1 \end{pmatrix}, \quad g_{11}g_{33} \neq 0$$

$\underline{\mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{11} & a_{23} & a_{24} \\ 2a_{23} & 2a_{13} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}; g = \begin{pmatrix} g_{11} & 0 & 0 & g_{14} \\ 0 & \frac{1}{g_{11}} & 0 & g_{24} \\ 0 & 0 & 1 & g_{34} \\ 0 & 0 & 0 & g_{44} \end{pmatrix}, \quad g_{11}, g_{44} \neq 0$$

$\underline{\mathfrak{n}_4(\mathbb{C})}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ 0 & \frac{g_{33}}{g_{11}} & g_{23} & g_{24} \\ 0 & 0 & g_{33} & g_{23} \\ 0 & 0 & 0 & g_{11}g_{33} \end{pmatrix}, \quad g_{11}g_{33} \neq 0$$

$\underline{\mathfrak{g}_1(\gamma)}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & g_{22} & g_{23} & g_{24} \\ 0 & g_{32} & g_{33} & g_{23} \\ 0 & 0 & 0 & g_{44} \end{pmatrix}, \quad \begin{matrix} g_{44} \neq 0 \\ g_{23}g_{32} \neq g_{22}g_{33} \end{matrix}$$

$\underline{\mathfrak{g}_2(\gamma,\beta), \gamma, \beta = 0}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{21} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & g_{33} & g_{23} & g_{44} - g_{23} - g_{33} \\ 0 & 0 & g_{33} & g_{44} - g_{33} \\ 0 & 0 & 0 & g_{44} \end{pmatrix}, g_{33}g_{44} \neq 0$$

$\underline{\mathfrak{g}_2(\gamma,\beta), \gamma \neq 0, \beta = 0}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & g_{44} - g_{34} & g_{34} - g_{24} & g_{24} \\ 0 & g_{24}\gamma & g_{44} - g_{34} & g_{34} \\ 0 & g_{34}\gamma & g_{24}\gamma & g_{44} \end{pmatrix}, \det g \neq 0$$

$\underline{\mathfrak{g}_2(\gamma,\beta), \gamma, \beta \neq 0}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & \frac{(\gamma-\beta)a_{14}}{\beta} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & g_{44} - g_{34} + g_{24}\beta & g_{34} - g_{24} & g_{24} \\ 0 & g_{24}\gamma & g_{44} - g_{34} & g_{34} \\ 0 & g_{34}\gamma & g_{24}\gamma - g_{34}\beta & g_{44} \end{pmatrix},$$

$$\det g \neq 0$$

$\underline{\mathfrak{g}_3(\gamma)}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{14} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & g_{44} - g_{24}\gamma & g_{34} & g_{24} \\ 0 & g_{24}\gamma & g_{44} & g_{34} \\ 0 & g_{34}\gamma & (g_{24} + g_{34})\gamma & g_{44} \end{pmatrix}, \det g \neq 0$$

$\underline{\mathfrak{g}_4}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & g_{22} & g_{23} & g_{24} \\ 0 & g_{24} & g_{22} & g_{23} \\ 0 & g_{23} & g_{24} & g_{22} \end{pmatrix}, \det g \neq 0$$

$\underline{\mathfrak{g}_5}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & g_{22} & g_{23} & \frac{3\, g_{34}}{2} \\ 0 & 0 & g_{22} & g_{34} \\ 0 & 0 & g_{43} & g_{44} \end{pmatrix}, g_{22} \neq 0, g_{43}g_{34} \neq g_{22}g_{44}$$

$\underline{\mathfrak{g}_6}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & 2a_{11} - a_{22} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & g_{22} & g_{23} & g_{13}g_{22} - g_{12}g_{23} \\ 0 & g_{32} & g_{33} & g_{13}g_{32} - g_{12}g_{33} \\ 0 & 0 & 0 & g_{22}g_{33} - g_{23}g_{32} \end{pmatrix}, g_{23}g_{32} \neq g_{22}g_{33}$$

$\underline{\mathfrak{g}_7}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & -a_{23} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & g_{22} & g_{23} & g_{12}g_{22} - g_{13}g_{32} \\ 0 & g_{32} & g_{22} & g_{12}g_{32} - g_{13}g_{22} \\ 0 & 0 & 0 & g_{22}^2 - g_{32}^2 \end{pmatrix}, g_{32} \neq g_{22}$$

$\underline{\mathfrak{g}_8(\gamma), \gamma = 0}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{21} & a_{22} - a_{11} & a_{33} & a_{34} \\ 0 & a_{21} & 0 & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & 0 & g_{13} & g_{14} \\ 0 & g_{22} & -g_{22} + g_{33} & g_{13}g_{22} + g_{34} \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & 0 & g_{22}g_{33} \end{pmatrix}, g_{22}g_{33} \neq 0$$

$\underline{\mathfrak{g}_8(\gamma), \gamma \neq 0}$

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & \frac{a_{11}-a_{22}+a_{32}}{\gamma} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}; g = \begin{pmatrix} 1 & g_{12} & g_{13} & g_{14} \\ 0 & -g_{23} + g_{33} & g_{23} & \frac{-g_{12}g_{33} - g_{13}g_{23}\gamma}{\gamma} \\ 0 & -g_{23}\gamma & g_{33} & \frac{-g_{12}g_{33} + (g_{12}g_{23} - g_{13}g_{33})\gamma}{\gamma} \\ 0 & 0 & 0 & g_{23}^2\gamma + g_{33}^2 - g_{22}g_{33} \end{pmatrix},$$

$$\det g \neq 0$$

Remark 1 The limit of some parametric Lie algebras approach other non-isomorphic Lie algebras as the parameter tends to zero. We observe that such Lie algebras also have coinciding automorphism groups. For example, $\mathfrak{g}_3(\gamma) \to \mathfrak{n}_4(\mathbb{C})$ as $\gamma \to 0$ and $\text{Aut}(\mathfrak{g}_3(\gamma)) \subseteq \text{Aut}\ (\mathfrak{n}_4(\mathbb{C}))$ when $\gamma = 0$. The two automorphism groups coincide when $g_{11} = 1$. Also, $\mathfrak{g}_1(\gamma) \to \mathfrak{t}_{3,1}(\mathbb{C}) \oplus \mathbb{C}$ as $\gamma \to 0$ and $\text{Aut}(\mathfrak{g}_1(\gamma)) \subseteq \text{Aut}\ (\mathfrak{t}_{3,\lambda}(\mathbb{C}) \oplus \mathbb{C})$. The two automorphism groups coincide when $g_{23}, g_{24}, g_{32} = 0$.

If two skew-symmetric algebras (L, μ) and (L', μ') are isomorphic, then HomLie (μ) is isomorphic to HomLie(μ') (see [10]). Let $\phi : L \to L'$ be the isomorphism of the algebras. Let $\alpha \in$ HomLie(μ) and $\beta \in$ HomLie(μ'). The isomorphism between the Hom-Lie structures $\varphi : HomLie(\mu) \to HomLie(\mu')$ is defined by $\beta = \varphi(\alpha) := \phi \circ \alpha \circ \phi^{-1}$. Given $x, y, z \in L$, then there exists $x', y', z' \in L'$ as images under the isomorphism ϕ, that is $\phi(x) = x'$, $\phi(y) = y'$ and $\phi(z) = z'$. $\beta = \phi \circ \alpha \circ \phi^{-1}$ implies $\beta(x') = \phi(\alpha(x))$, $\beta(y') = \phi(\alpha(y))$ and $\beta(z') = \phi(\alpha(z))$. If $\alpha \in$ HomLie(μ), then $\beta \in$ HomLie(μ') since

$$\begin{aligned} &\mu'(\beta(x'), \mu'(y', z')) + \mu'(\beta(y'), \mu(z', x')) + \mu'(\beta(z'), \mu'(x', y')) \\ &= \mu'(\phi(\alpha(x)), \mu'(\phi(y), \phi(z)) + \mu'(\phi(\alpha(y)), \mu(\phi(z), \phi(x)) + \mu'(\phi(\alpha(z)), \mu'(\phi(x), \phi(y)) \\ &= \phi[\mu((\alpha(x)), \mu(y, z) + \mu((\alpha(y)), \mu(z, x) + \mu((\alpha(z)), \mu(x, y)] = 0 \end{aligned}$$

In particular, we can use the representatives of the isomorphism classes of algebras to describe the dimension of the space of possible linear endomorphisms that turn any skew-symmetric algebras, including Lie algebras, into a Hom-Lie algebra.

Proposition 1 *Let* (L, μ) *be a* 4*-dimensional complex Lie algebra. Then* $9 \leq \dim$ *Hom-Lie* $(\mu) \leq 16$.

4 Non-isomorphic Forms of Hom-Lie Structures

The Hom-Lie algebras that arise from the 4-dimensional complex Lie algebra representatives are infinitely many. Such Hom-Lie algebras are isomorphic if and only if their respective linear maps α are pairwise equivalent up to conjugation. We give the non-isomorphic forms of only two Hom-Lie structures of 4-dimensional complex Lie algebras under the left $Aut(\mathfrak{L})$-action given by

$$\alpha \mapsto g \cdot \alpha := g \circ \alpha \circ g^{-1}, \; g \in Aut(\mathfrak{L}), \quad \alpha \in \text{Hom-Lie}(\mu)$$

Proposition 2 *We have the following non-isomorphic forms of Hom-Lie structures for the following Lie algebra representatives. For Lie algebra* $(\mathfrak{t}_3(\mathbb{C}) \oplus \mathbb{C}, \mu)$, *any* $\alpha \in$ *Hom-Lie*(μ) *can be decomposed into the following non-isomorphic forms.*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}, a_{43} \neq 0 \,; \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}, a_{43} = 0, a_{23}, a_{42} \neq 0$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & 0 & a_{44} \end{pmatrix}, \begin{matrix} a_{23}, a_{43} = 0 \\ a_{42} \neq 0 \end{matrix} ; \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}, a_{23} \neq 0, a_{42}, a_{43} = 0$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}, a_{23}, a_{42}, a_{43} = 0$$

For Lie algebra, $(\mathfrak{t}_2(\mathbb{C}) \oplus \mathbb{C}^2, \mu)$, *any* $\alpha \in$ *Hom-Lie*(μ) *can be decomposed into the following non-isomorphic forms:*

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{pmatrix}, a_{12} \neq 0; \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & 0 & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{pmatrix}, a_{12} = 0$$

Acknowledgements Elvice Ongong'a is grateful to the International Science Program, Uppsala University for the financial support within the network of the Eastern Africa Universities Mathematics Programme (EAUMP). He also thanks the research environment in Mathematics and Applied Mathematics (MAM), the Division of Applied Mathematics of the School of Education, Culture and Communication at Mälardalen University for hospitality and excellent conditions for research and research education.

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Multiplet Classification and Invariant Differential Operators over the Lie Algebra F'_4

V. K. Dobrev

Abstract In the present paper we continue the project of systematic construction of invariant differential operators on the example of the non-compact exceptional Lie algebra F'_4 which is split real form of the exceptional Lie algebra F_4. We consider induction from a maximal parabolic algebra. We classify the reducible Verma modules over F_4 which are compatible with this induction. Thus, we obtain the classification of the corresponding invariant differential operators.

1 Introduction

Invariant differential operators play very important role in the description of physical symmetries. The general scheme for constructing these operators was given some time ago [1]. In recent papers [2, 3] we started the systematic explicit construction of the invariant differential operators.

The first task in the construction is to make the multiplet classification of the reducible Verma modules over the algebra in consideration following [4]. Such classification provides the weights of embeddings between the Verma modules via the singular vectors, and thus, by [1], the weights of the invariant differential operators.

We have done the multiplet classification for many real non-compact algebras, first from the class of algebras that have discrete series representations, see [5]. In the present paper we focus on the complex exceptional Lie algebra F_4 and on its split real form algebra F'_4. Our scheme requires that we use induction from parabolic subalgebras. In the present paper we choose a parabolic subalgebra containing the factor $\mathcal{M} \oplus \mathcal{A}$, where $\mathcal{M} = sl(3, \mathbb{R}) \oplus sl(2, \mathbb{R})$, $\mathcal{A} = o(2)$. This choice is motivated by the fact that the complexification of $\mathcal{M} \oplus \mathcal{A}$ and the corresponding compact form

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_29

$\mathcal{M}_c \oplus \mathcal{A}_c = su(3) \oplus su(2) \oplus u(1)$ have applications in physics being the Lie algebra symmetry of the standard model of elementary particles [6] (see also [7]).[1]

We present the multiplet classification of the reducible Verma modules over F_4 which are compatible with the chosen parabolic of F'_4. We give also the weights of the singular vectors between these modules. By the scheme of [1] these singular vectors will produce the invariant differential operators.

2 Preliminaries

2.1 Lie Algebra

We start with the complex exceptional Lie algebra $\mathcal{G}^{\mathbb{C}} = F_4$. We use the standard definition of $\mathcal{G}^{\mathbb{C}}$ given in terms of the Chevalley generators $X_i^{\pm}, H_i, i = 1, 2, 3, 4 (=\text{rank}\, F_4)$, by the relations:

$$[H_j\,,\,H_k] = 0\,,\ \ [H_j\,,\,X_k^{\pm}] = \pm a_{jk} X_k^{\pm}\,,\ \ [X_j^{+}\,,\,X_k^{-}] = \delta_{jk}\, H_j\,, \tag{1}$$
$$\sum_{m=0}^{n} (-1)^m \binom{n}{m} \left(X_j^{\pm}\right)^m X_k^{\pm} \left(X_j^{\pm}\right)^{n-m} = 0\,,\ j \neq k\,,\ n = 1 - a_{jk}\,,$$

where

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}; \tag{2}$$

is the Cartan matrix of $\mathcal{G}^{\mathbb{C}}$, $\alpha_j^{\vee} \equiv \frac{2\alpha_j}{(\alpha_j,\alpha_j)}$ is the co-root of α_j, $(\cdot,\cdot)$ is the scalar product of the roots, so that the nonzero products between the simple roots are: $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2(\alpha_3, \alpha_3) = 2(\alpha_4, \alpha_4) = 2,\ (\alpha_1, \alpha_2) = -1,\ (\alpha_2, \alpha_3) = -1,\ (\alpha_3, \alpha_4) = -1/2$. The elements H_i span the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}^{\mathbb{C}}$, while the elements $X_i^{\pm}$ generate the subalgebras $\mathcal{G}^{\pm}$. We shall use the standard triangular decomposition

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}_+ \oplus \mathcal{H} \oplus \mathcal{G}_-\,, \qquad \mathcal{G}_{\pm} \equiv \underset{\alpha\in\Delta^{\pm}}{\oplus} \mathcal{G}_{\alpha}\,, \tag{3}$$

where Δ^+, Δ^-, are the sets of positive, negative, roots, resp. Explicitly we have that there are roots of two lengths with length ratio 2 : 1. The long roots of length 2 in terms of the simple roots are:

[1] More precisely the symmetry is presented on the group level as : $G = SU(3) \times SU(2) \times U(1)/Z$, where Z belongs to the center of G.

$$\begin{aligned}
&\alpha_1, \alpha_2, \alpha_{12} \equiv \alpha_1 + \alpha_2, \alpha_{23+3} \equiv \alpha_2 + 2\alpha_3, \alpha_{13+3} \equiv \alpha_1 + \alpha_2 + 2\alpha_3, \\
&\alpha_{13+23} \equiv \alpha_1 + 2\alpha_2 + 2\alpha_3,\ \alpha_{24+34} \equiv \alpha_2 + 2\alpha_3 + 2\alpha_4, \\
&\alpha_{14+34} \equiv \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4,\ \alpha_{14+24} \equiv \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \\
&\alpha_{14+24+3+3} \equiv \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4,\ \alpha_{14+24+23+3} \equiv \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \\
&\alpha_{14+14+24+3} \equiv 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4
\end{aligned} \tag{4}$$

where we have introduced short-hand notation that will be useful below.
The short roots have length 1 and they are:

$$\begin{aligned}
&\alpha_3, \alpha_4, \alpha_{34} \equiv \alpha_3 + \alpha_4,\ \alpha_{23} \equiv \alpha_2 + \alpha_3,\ \alpha_{13} \equiv \alpha_1 + \alpha_2 + \alpha_3, \\
&\alpha_{24} \equiv \alpha_2 + \alpha_3 + \alpha_4,\ \alpha_{14} \equiv \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_{24+3} \equiv \alpha_2 + 2\alpha_3 + \alpha_4, \\
&\alpha_{14+23} \equiv \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4,\ \alpha_{14+3} \equiv \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \\
&\alpha_{14+23+3} \equiv \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4,\ \alpha_{14+24+3} \equiv \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4
\end{aligned} \tag{5}$$

(Note that the short roots are exactly those which contain α_3 and/or α_4 with odd coefficient, while the long roots contain α_3 and α_4 with even coefficients).

Thus, F_4 is 52–dimensional ($52 = |\Delta| +$ rank F_4).

In terms of the normalized basis $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ we have:

$$\begin{aligned}
\Delta^+ = \{&\varepsilon_i, 1 \le i \le 4; \varepsilon_j \pm \varepsilon_k, 1 \le j < k \le 4; \\
&\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4), \text{all signs}\}.
\end{aligned} \tag{6}$$

The simple roots are:

$$\pi = \{\alpha_1 = \varepsilon_2 - \varepsilon_3, \alpha_2 = \varepsilon_3 - \varepsilon_4, \alpha_3 = \varepsilon_4,\ \alpha_4 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}. \tag{7}$$

Note that the 16 roots on the first line of (6) form the positive root system of B_4 with simple roots $\varepsilon_i - \varepsilon_{i+1}$, $i = 1, 2, 3, \varepsilon_4$.

The Weyl group of F_4 is the semidirect product of S_3 with a group which itself is the semidirect product of S_4 with $(\mathbb{Z}/2\mathbb{Z})^3$, thus, $|W| = 2^7\, 3^2 = 1152$.

2.2 *Verma Modules*

Let us recall that a *Verma module* V^Λ is defined as the HWM over $\mathcal{G}^{\mathbb{C}}$ with highest weight $\Lambda \in \mathcal{H}^*$ and highest weight vector $v_0 \in V^\Lambda$, induced from the one-dimensional representation $V_0 \cong \mathbb{C}v_0$ of $U(\mathcal{B})$, where $\mathcal{B} = \mathcal{H} \oplus \mathcal{G}_+$ is a Borel subalgebra of $\mathcal{G}^{\mathbb{C}}$, such that:

$$X\, v_0 = 0, \quad \forall X \in \mathcal{G}_+$$

$$H v_0 = \Lambda(H) v_0 \,, \quad \forall H \in \mathcal{H} \tag{8}$$

Verma modules are generically irreducible. A Verma module V^Λ is reducible [8] iff there exists a root $\beta \in \Delta^+$ and $m \in \mathbb{N}$ such that

$$(\Lambda + \rho, \beta^\vee) = m \tag{9}$$

holds, where $\rho = \frac{1}{2} \sum_{\alpha\in\Delta^+} \alpha$, $(\rho = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 11\alpha_4)$.

If (9) holds then the reducible Verma module V^Λ contains an invariant submodule which is also a Verma module $V^{\Lambda'}$ with shifted weight $\Lambda' = \Lambda - m\beta$. This statement is equivalent to the fact that V^Λ contains a *singular vector* $v_s \in V^\Lambda$, such that $v_s \neq \xi v_0$, $(0 \neq \xi \in \mathbb{C})$, and:

$$\begin{aligned} &X v_s = 0, \quad \forall X \in \mathcal{G}_+ \\ &H v_s = \Lambda'(H) v_s \,, \quad \Lambda' = \Lambda - m\beta, \ \forall H \in \mathcal{H} \end{aligned} \tag{10}$$

More explicitly, [1],

$$v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0 \ . \tag{11}$$

The general reducibility conditions (9) for V^Λ spelled out for the simple roots in our situation are:

$$\begin{aligned} m_1 \equiv m_{\alpha_1} &= (\Lambda + \rho, \alpha_1), \quad m_2 \equiv m_{\alpha_2} = (\Lambda + \rho, \alpha_2), \\ m_3 \equiv m_{\alpha_3} &= (\Lambda + \rho, 2\alpha_3), \quad m_4 \equiv m_{\alpha_4} = (\Lambda + \rho, 2\alpha_4) \end{aligned} \tag{12}$$

The numbers m_i from (12) corresponding to the simple roots are called Dynkin labels, while the more general Harish-Chandra parameters are:

$$m_\beta = (\Lambda + \rho, \beta^\vee), \quad \beta \in \Delta^+ \tag{13}$$

2.3 Structure Theory of the Real Form

The split real form of F_4 is denoted as F'_4, sometimes as $F_{4(4)}$. This real form has discrete series representations since $\mathrm{rank} F'_4 = \mathrm{rank}\, \mathcal{K}$, where $\mathcal{K}$ is the maximal compact subalgebra $\mathcal{K} = sp(3) \oplus su(2)$. Its complexification $\mathcal{K}^{\mathbb{C}}$ may be embedded most easily in F_4 as the Lie algebra generated by the subalgebras with simple roots $\{\alpha_2, \alpha_3, \alpha_4\}$ and $\{\alpha_1\}$. The long roots of $sp(3, \mathbb{C})$ in this embedding are: α_2, $\alpha_2 + 2\alpha_3$, $\alpha_2 + 2\alpha_3 + 2\alpha_4$. The short roots are: α_3, α_4, $\alpha_2 + \alpha_3$, $\alpha_3 + \alpha_4$, $\alpha_2 + \alpha_3 + \alpha_4$, $\alpha_2 + 2\alpha_3 + \alpha_4$.
For F'_4, we can use the same basis as for F_4 (but over $\mathbb{R}$) and the same root system.

The Iwasawa decomposition of the real split form $\mathcal{G} \equiv F_4'$, is:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{A}_0 \oplus \mathcal{N}_0 , \qquad (14)$$

the Cartan decomposition is:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{Q}, \qquad (15)$$

where we use: the maximal compact subgroup $\mathcal{K} \cong sp(3) \oplus su(2)$, $\dim_{\mathbb{R}} \mathcal{Q} = 28$, $\dim_{\mathbb{R}} \mathcal{A}_0 = 4$, $\mathcal{N}_0 = \mathcal{N}_0^+$, or $\mathcal{N}_0 = \mathcal{N}_0^- \cong \mathcal{N}_0^+$, $\dim_{\mathbb{R}} \mathcal{N}_0^\pm = 24$.

Since $\mathcal{G}$ is maximally split, then the centralizer $\mathcal{M}_0$ of $\mathcal{A}_0$ in $\mathcal{K}$ is zero, thus, the minimal parabolic $\mathcal{P}_0$ and the corresponding Bruhat decomposition are:

$$\mathcal{P}_0 = \mathcal{A}_0 \oplus \mathcal{N}_0 , \qquad \mathcal{G} = \mathcal{A}_0 \oplus \mathcal{N}_0^+ \oplus \mathcal{N}_0^- \qquad (16)$$

The importance of the parabolic subgroups comes from the fact that the representations induced from them generate all (admissible) irreducible representations of the group under consideration [9–11].

We recall that in general a parabolic subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ is any subalgebra of $\mathcal{G}$ which contains a minimal parabolic subalgebra $\mathcal{M}_0$. In general, $\mathcal{M}$ contains the subalgebra $\mathcal{M}_0$, while $\mathcal{A}$ is contained in $\mathcal{A}_0$, $\mathcal{N}$ is contained in $\mathcal{N}_0$.

On the other extreme are the maximal parabolic subalgebras for which $\dim \mathcal{A} = 1$.

2.4 Elementary Representations

Further, let G, K, P, M, A, N are Lie groups with Lie algebras $\mathcal{G}_0, \mathcal{K}, \mathcal{P}, \mathcal{M}, \mathcal{A}, \mathcal{N}$.

Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$. Let μ fix a finite-dimensional (non-unitary) representation D^μ of M on the space V_μ. In the case when M is cuspidal then we may use also the discrete series representation of M with the same Casimirs as D^μ. (We ignore a possible discrete center of M since its representations are not relevant for the construction of invariant differential operators [5]).

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [12]. (These are called *generalized principal series representations* (or *limits thereof*) in [13]). Their spaces of functions are:

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1})\, \mathcal{F}(g)\} \qquad (17)$$

where $a = \exp(H) \in A$, $H \in \mathcal{A}$, $m \in M$, $n \in N$. The representation action is the *left regular action*:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g') , \quad g, g' \in G . \qquad (18)$$

An important ingredient in our considerations are the highest/lowest weight representations of $\mathcal{G}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}$, where $\Lambda \in (\mathcal{H})^*$, the weight $\Lambda = \Lambda(\chi)$ being determined uniquely from χ [1].

As we have seen when a Verma module is reducible and (9) holds then there is a singular vector (11). Relatedly, then there exists [1] an *invariant differential operator*

$$\mathcal{D}_{m,\beta} \; : \; \mathcal{C}_{\chi(\Lambda)} \longrightarrow \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{19}$$

given explicitly by:

$$\mathcal{D}_{m,\beta} \; = \; \mathcal{P}_{m,\beta}(\widehat{\mathcal{N}^-}) \tag{20}$$

where $\widehat{\mathcal{N}^-}$ denotes the *right action* on the functions $\mathcal{F}$.

Actually, since our ERs are induced from finite-dimensional representations of $\mathcal{M}$ the corresponding Verma modules are always reducible. Thus, it is more convenient to use *generalised Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu \, v_0$.

Algebraically, the above is governed by the notion of $\mathcal{M}$-compact roots of $\mathcal{G}^\mathbb{C}$. These are the roots of $\mathcal{G}^\mathbb{C}$ which can be identified as roots of $\mathcal{M}^\mathbb{C}$ as the latter is a subalgebra of $\mathcal{G}^\mathbb{C}$. The consequence of this is that (9) is always fulfilled for the $\mathcal{M}$-compact roots of $\mathcal{G}^\mathbb{C}$. That is why we consider generalised Verma modules. Relatedly, the invariant differential operators corresponding to $\mathcal{M}$-compact roots are trivial.

3 Invariant differential operators for F'_4

The real form F'_4 has several parabolic subalgebras [2]. We shall consider the maximal parabolic subalgebra [2]:

$$\begin{aligned} \mathcal{P} \; &= \; \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \, , \\ \mathcal{M} \; &= \; sl(3,\mathbb{R}) \oplus (2,\mathbb{R}) \, , \\ \dim \mathcal{A} = 1, & \qquad \dim \mathcal{N} = 20 \end{aligned} \tag{21}$$

such that the embedding of $\mathcal{M}$ and $\mathcal{M}^\mathbb{C}$ in $\mathcal{G}^\mathbb{C}$ is given by:

$$sl(3,\mathbb{R})^\mathbb{C} : \{\alpha_3, \alpha_4, \alpha_{34} = \alpha_3 + \alpha_4\}, \qquad sl(2,\mathbb{R})^\mathbb{C} : \{\alpha_1\} \tag{22}$$

Remark: Note that F'_4 has a another maximal parabolic subalgebra that is also written as (21) but the embedding of $\mathcal{M}$ and $\mathcal{M}^\mathbb{C}$ flips the short and long roots [2]:

$$sl(3,\mathbb{R})^\mathbb{C} : \{\alpha_1, \alpha_2, \alpha_{12} = \alpha_1 + \alpha_2\}, \qquad sl(2,\mathbb{R})^\mathbb{C} : \{\alpha_4\} \tag{23}$$

That case is also very interesting and was considered in [14]. ◇

Further we classify the generalized Verma modules (GVM) relative to the maximal parabolic subalgebra (21). This also provides the classification of the P-induced ERs with the same Casimirs. The classification is done as follows. We group the reducible Verma modules (also the corresponding ERs) related by nontrivial embeddings in sets called *multiplets* [1, 4]. These multiplets may be depicted as a connected graph, the vertices of which correspond to the GVMs and the lines between the vertices correspond to the GVM embeddings (and also the invariant differential operators between the ERs). The explicit parametrization of the multiplets and of their Verma modules (and ERs) is important for understanding of the situation.

The result of our classification is a follows. The multiplets of GVMs (and ERs) induced from (21) are parametrized by four positive integers $\chi = [m_1, m_2, m_3, m_4]$. Each multiplet contains 96 GVMs (ERs).

(The positive integers $\{m_1, m_2, m_3, m_4\}$ parametrize the finite-dimensional nonunitary irreps of F'_4 (also the unitary finite-dimensional irreps of the compact Lie algebra f_4)).

The explicit parametrization of the signatures $\chi(\Lambda)$ of the generalized Verma modules V^Λ in the multiplets is given below. For each GVM module we give also the GVM(s) that are embedded in it by the corresponding arrow on which we indicate the level of the embedding $m_n\,\beta$. (Recall that $\Lambda' = \Lambda - m_n\,\beta$ (10)). Explicitly, we have:

$$\begin{aligned}
\chi_0^- &= [m_1, m_2, m_3, m_4] \rightarrow_{m_2\alpha_2} \chi_1^- \\
\chi_1^- &= [m_{12}, -m_2, m_{23}+m_2, m_4] \rightarrow_{m_3\alpha_{23}} \chi_{2,1}^-, \ \rightarrow_{m_1\alpha_{12}} \chi_{2,2}^- \\
\chi_{2,1}^- &= [m_{13}, -m_{23}, m_{23}+m_2, m_{34}] \rightarrow_{m_1\alpha_{12}} \chi_3^-, \ \rightarrow_{m_4\alpha_{24}} \chi_{3,1}^-, \\
&\qquad \rightarrow_{m_2\alpha_{23+2}} \chi_{3,2}^- \\
\chi_{2,2}^- &= [m_2, -m_{12}, m_{13}+m_{12}, m_4] \rightarrow_{m_3\alpha_{23}} \chi_3^- \\
\chi_3^- &= [m_{23}, -m_{13}, m_{13}+m_{12}, m_{34}] \rightarrow_{m_4\alpha_{24}} \chi_{4,1}^-, \ \rightarrow_{m_2\alpha_{13+23}} \chi_{4,2}^- \\
\chi_{3,1}^- &= [m_{14}, -m_{24}, m_{24}+m_2, m_3] \rightarrow_{m_1\alpha_{12}} \chi_{4,1}^-, \ \rightarrow_{m_2\alpha_{23+2}} \chi_{4,3}^- \\
\chi_{3,2}^- &= [m_{13}+m_2, -m_{23}, m_3, m_{24}+m_2] \rightarrow_{m_4\alpha_{24}} \chi_{4,3}^-, \ \rightarrow_{m_1\alpha_{13+23}} \chi_{4,4}^- \\
\chi_{4,1}^- &= [m_{24}, -m_{14}, m_{14}+m_{12}, m_3] \rightarrow_{m_2\alpha_{13+23}} \chi_5^- \\
\chi_{4,2}^- &= [m_{23}, -m_{13}-m_2, m_{13}+m_{12}, m_{24}+m_2] \rightarrow_{m_4\alpha_{24}} \chi_5^-, \\
&\qquad \rightarrow_{m_3\alpha_{13}} \chi_{5,1}^-, \ \rightarrow_{m_1\alpha_{23+2}} \chi_{5,2}^- \\
\chi_{4,3}^- &= [m_{14}+m_2, -m_{24}, m_{34}, m_{23}+m_2] \rightarrow_{m_1\alpha_{13+23}} \chi_{5,3}^-, \\
&\qquad \rightarrow_{m_3\alpha_{24+3}} \chi_{5,4}^- \\
\chi_{4,4}^- &= [m_{13}+m_2, -m_{13}, m_3, m_{14}+m_{12}] \rightarrow_{m_2\alpha_{12}} \chi_{5,2}^-, \ \rightarrow_{m_4\alpha_{24}} \chi_{5,3}^-
\end{aligned} \tag{24}$$

$$\begin{aligned}
\chi_5^- &= [m_{24}, -m_{14}-m_2, m_{14}+m_{12}, m_{23}+m_2] \ \rightarrow_{m_3\alpha_{14+23}} \ \chi_{6,1}^-, \\
&\quad \rightarrow_{m_1\alpha_{23+2}} \ \chi_{6,2}^- \\
\chi_{5,1}^- &= [m_2, -m_{13}-m_2, m_{13}+m_{12}, m_{24}+m_{23}] \ \rightarrow_{m_1\alpha_{23+2}} \ \chi_{6,3}^-, \\
&\quad \rightarrow_{m_4\alpha_{14+23}} \ \chi_{6,4}^- \\
\chi_{5,2}^- &= [m_{13}, -m_{13}-m_2, m_2+m_{23}, m_{14}+m_{12}] \ \rightarrow_{m_4\alpha_{24}} \ \chi_{6,2}^-, \\
&\quad \rightarrow_{m_3\alpha_{13}} \ \chi_{6,3}^- \\
\chi_{5,3}^- &= [m_{14}+m_2, -m_{14}, m_{34}, m_{12}+m_{13}] \ \rightarrow_{m_2\alpha_{12}} \ \chi_{6,2}^-, \quad \rightarrow_{m_3\alpha_{24+3}} \ \chi_{6,5}^- \\
\chi_{5,4}^- &= [m_{14}+m_{23}, -m_{24}, m_4, m_{23}+m_2] \ \rightarrow_{m_1\alpha_{13+23}} \ \chi_{6,5}^-, \\
&\quad \rightarrow_{m_2\alpha_{24+34}} \ \chi_{6,6}^- \\
\chi_{6,1}^- &= [m_{24}, -m_{14}-m_{23}, m_{14}+m_{13}, m_{23}+m_2] \ \rightarrow_{m_1\alpha_{23+2}} \ \chi_7^-, \\
&\quad \rightarrow_{m_4\alpha_{13}} \ \chi_{7,1}^-, \quad \rightarrow_{m_2\alpha_{14+24}} \ \chi_{7,2}^- \\
\chi_{6,2}^- &= [m_{14}, -m_{14}-m_2, m_{24}+m_2, m_{13}+m_{12}] \ \rightarrow_{m_3\alpha_{14+23}} \ \chi_7^- \\
\chi_{6,3}^- &= [m_{12}, -m_{13}-m_2, m_{23}+m_2, m_{14}+m_{13}] \ \rightarrow_{m_2\alpha_{13+3}} \ \chi_{7,4}^-, \\
&\quad \rightarrow_{m_4\alpha_{14+23}} \ \chi_{7,5}^- \\
\chi_{6,4}^- &= [m_2, -m_{14}-m_2, m_{14}+m_{12}, m_{24}+m_{23}] \ \rightarrow_{m_3\alpha_{24}} \ \chi_{7,1}^-, \\
&\quad \rightarrow_{m_1\alpha_{23+2}} \ \chi_{7,5}^- \\
\chi_{6,5}^- &= [m_{14}+m_{23}, -m_{14}, m_4, m_{12}+m_{13}] \ \rightarrow_{m_2\alpha_{14+24+23+3}} \ \chi_{7,3}^- \\
\chi_{6,6}^- &= [m_{14}+m_{23}+m_2, -m_{24}, m_4, m_3] \ \rightarrow_{m_1\alpha_{14+24+23+3}} \ \chi_{7,6}^- \\
\chi_7^- &= [m_{14}, -m_{14}-m_{23}, m_{24}+m_{23}, m_{13}+m_{12}] \ \rightarrow_{m_2\alpha_{14+24+23+3}} \ \chi_{8,1}^-, \\
&\quad \rightarrow_{m_4\alpha_{13}} \ \chi_{8,2}^- \\
\chi_{7,1}^- &= [m_{23}, -m_{14}-m_{23}, m_{14}+m_{13}, m_{24}+m_2] \ \rightarrow_{m_1\alpha_{23+2}} \ \chi_{8,2}^-, \\
&\quad \rightarrow_{m_2\alpha_{14+24}} \ \chi_{8,7}^- \\
\chi_{7,2}^- &= [m_{24}, -m_{14}-m_{23}-m_2, m_{14}+m_{13}+2m_2, m_3] \\
&\quad \rightarrow_{m_1\alpha_{14+24+23+3}} \ \chi_{8,6}^-, \quad \rightarrow_{m_4\alpha_{13}} \ \chi_{8,7}^- \\
\chi_{7,3}^- &= [m_{14}+m_{23}+m_2, -m_{14}-m_2, m_4, m_{12}+m_{13}] \ \rightarrow_{m_3\alpha_{14+23}} \ \chi_{8,3}^-, \\
&\quad \rightarrow_{m_1\alpha_{24+34}} \ \chi_{8,4}^- \\
\chi_{7,4}^- &= [m_1, -m_{13}, m_3, m_{14}+m_{13}+2m_2] \ \rightarrow_{m_4\alpha_{14+23}} \ \chi_{8,5}^- \\
\chi_{7,5}^- &= [m_{12}, -m_{14}-m_2, m_{24}+m_2, m_{14}+m_{13}] \ \rightarrow_{m_3\alpha_{24}} \ \chi_{8,2}^- \\
&\quad \rightarrow_{m_2\alpha_{13+3}} \ \chi_{8,5}^- \\
\chi_{7,6}^- &= [m_{14}+m_{13}+m_2, -m_{14}, m_4, m_3] \ \rightarrow_{m_2\alpha_{13+23}} \ \chi_{8,4}^-
\end{aligned}$$

$$\begin{aligned}
\chi^-_{8,1} &= [m_{14}+m_2, -m_{14}-m_{23}-m_2, m_{24}+m_{23}, m_{13}+m_{12}] \\
&\quad \to_{m_4\alpha_{13}} \chi^-_9, \quad \to_{m_3\alpha_{24+3}} \chi^-_{9,1}, \quad \to_{m_1\alpha_{14+24}} \chi^-_{9,3} \\
\chi^-_{8,2} &= [m_{13}, -m_{14}-m_{23}, m_{24}+m_{23}, m_{14}+m_{12}] \to_{m_2\alpha_{14+24+23+3}} \chi^-_9 \\
\chi^-_{8,3} &= [m_{14}+m_{23}+m_2, -m_{14}-m_{23}, m_{34}, m_{12}+m_{13}] \to_{m_2\alpha_{12}} \chi^-_{9,1}, \\
&\quad \to_{m_4\alpha_{14+23+3}} \chi^-_{9,4}, \quad \to_{m_1\alpha_{24+34}} \chi^-_{9,7} \\
\chi^-_{8,4} &= [m_{14}+m_{13}+m_2, -m_{14}-m_2, m_4, m_{23}+m_2] \to_{m_3\alpha_{14+23}} \chi^-_{9,7} \\
\chi^-_{8,5} &= [m_1, -m_{14}, m_{34}, m_{14}+m_{13}+2m_2] \to_{m_3\alpha_{14+23+3}} \chi^-_{9,2} \\
\chi^-_{8,6} &= [m_{14}, -m_{14}-m_{13}-m_2, m_{14}+m_{13}+2m_2, m_3] \to_{m_2\alpha_{23+3}} \chi^-_{9,3}, \\
&\quad \to_{m_4\alpha_{13}} \chi^-_{9,6} \\
\chi^-_{8,7} &= [m_{23}, -m_{14}-m_{23}-m_2, m_{14}+m_{13}+2m_2, m_{34}] \to_{m_3\alpha_{14}} \chi^-_{9,5}, \\
&\quad \to_{m_1\alpha_{14+24+23+3}} \chi^-_{9,6} \\
\chi^-_9 &= [m_{13}+m_2, -m_{14}-m_{23}-m_2, m_{24}+m_{23}, m_{14}+m_{12}] \\
&\quad \to_{m_3\alpha_{14+23+3}} \chi^-_{10,1}, \quad \to_{m_1\alpha_{14+24}} \chi^+_{10,1} \\
\chi^-_{9,1} &= [m_{14}+m_{23}, -m_{14}-m_{23}-m_2, m_{24}+m_2, m_{12}+m_{13}] \\
&\quad \to_{m_4\alpha_{14+23+3}} \chi^-_{10,3}, \quad \to_{m_1\alpha_{14+24}} \chi^+_{10,3} \\
\chi^-_{9,2} &= [m_1, -m_{14}, m_4, m_{14}+m_{13}+m_{23}+m_2] \to_{m_2\alpha_{14+24+23+3}} \chi^+_{10,4} \\
\chi^-_{9,3} &= [m_{14}+m_2, -m_{14}-m_{13}-m_2, m_{14}+m_{13}, m_{23}+m_2] \\
&\quad \to_{m_4\alpha_{13}} \chi^+_{10,1}, \quad \to_{m_3\alpha_{24+3}} \chi^+_{10,3} \\
\chi^-_{9,4} &= [m_{14}+m_{23}+m_2, -m_{14}-m_{23}, m_3, m_{14}+m_{12}] \to_{m_1\alpha_{24+34}} \chi^-_{10,2}, \\
&\quad \to_{m_2\alpha_{12}} \chi^-_{10,3} \\
\chi^-_{9,5} &= [m_2, -m_{14}-m_{23}-m_2, m_{14}+m_{13}+m_{23}+m_2, m_4] \\
&\quad \to_{m_1\alpha_{14+24+23+3}} \chi^-_{10,4} \\
\chi^-_{9,6} &= [m_{13}, -m_{14}-m_{13}-m_2, m_{14}+m_{13}+2m_2, m_{34}] \to_{m_3\alpha_{14}} \chi^-_{10,4}, \\
&\quad \to_{m_2\alpha_{23+3}} \chi^+_{10,1} \\
\chi^-_{9,7} &= [m_{14}+m_{13}+m_2, -m_{14}-m_{23}, m_{34}, m_{23}+m_2] \\
&\quad \to_{m_4\alpha_{14+23+3}} \chi^-_{10,2}, \quad \to_{m_2\alpha_{14+24}} \chi^+_{10,2} \\
\chi^-_{10,1} &= [m_{13}+m_2, -m_{14}-m_{23}-m_2, m_{24}+m_2, m_{14}+m_{13}] \\
&\quad \to_{m_1\alpha_{14+24}} \chi^+_9, \quad \to_{m_4\alpha_{24+3}} \chi^+_{9,3}, \quad \to_{m_2\alpha_{13+3}} \chi^+_{9,6} \\
\chi^-_{10,2} &= [m_{14}+m_{13}+m_2, -m_{14}-m_{23}, m_3, m_{24}+m_2] \to_{m_2\alpha_{14+24}} \chi^+_{9,7} \\
\chi^-_{10,3} &= [m_{14}+m_{23}, -m_{14}-m_{23}-m_2, m_{23}+m_2, m_{14}+m_{12}] \\
&\quad \to_{m_1\alpha_{14+24}} \chi^+_{9,1}, \quad \to_{m_3\alpha_{13}} \chi^+_{9,3} \\
\chi^-_{10,4} &= [m_{12}, -m_{14}-m_{13}-m_2, m_{14}+m_{13}+m_{23}+m_2, m_4] \\
&\quad \to_{m_2\alpha_{14+14+23+3}} \chi^+_{9,2}
\end{aligned}$$

$$\begin{aligned}
\chi^+_{10,1} &= [m_{13}+m_2, -m_{14}-m_{13}-m_2, m_{14}+m_{13}, m_{24}+m_2] \\
&\quad \to_{m_3\alpha_{14+23+3}} \chi^+_9 \\
\chi^+_{10,2} &= [m_{14}+m_{13}+m_2, -m_{14}-m_{23}-m_2, m_{24}+m_2, m_3] \to_{m_1\alpha_{12}} \chi^+_{9,4}, \\
&\quad \to_{m_4\alpha_{14+23+3}} \chi^+_{9,7} \\
\chi^+_{10,3} &= [m_{14}+m_{23}, -m_{14}-m_{13}-m_2, m_{14}+m_{12}, m_3+2m_2] \\
&\quad \to_{m_4\alpha_{14+23+3}} \chi^+_{9,1}, \quad \to_{m_2\alpha_{24+34}} \chi^+_{9,4} \\
\chi^+_{10,4} &= [m_{12}, -m_{14}-m_2, m_4, m_{14}+m_{13}+m_{23}+m_2] \\
&\quad \to_{m_1\alpha_{14+14+23+3}} \chi^+_{9,5}, \quad \to_{m_3\alpha_{24}} \chi^+_{9,6} \\
\chi^+_9 &= [m_{13}+m_2, -m_{14}-m_{13}-m_2, m_{14}+m_{12}, m_{24}+m_{23}] \\
&\quad \to_{m_4\alpha_{24+3}} \chi^+_{8,1}, \quad \to_{m_2\alpha_{14+14+23+3}} \chi^+_{8,2} \\
\chi^+_{9,1} &= [m_{14}+m_{23}, -m_{14}-m_{13}-m_2, m_{13}+m_{12}, m_{24}+m_2] \\
&\quad \to_{m_3\alpha_{13}} \chi^+_{8,1}, \quad \to_{m_2\alpha_{24+34}} \chi^+_{8,3} \\
\chi^+_{9,2} &= [m_1, -m_{14}-m_{13}-m_2, m_{14}+m_{13}+m_{23}+m_2, m_4] \\
&\quad \to_{m_3\alpha_{14+23+3}} \chi^+_{8,5} \\
\chi^+_{9,3} &= [m_{14}+m_2, -m_{14}-m_{23}-m_2, m_{23}+m_2, m_{14}+m_{13}] \\
&\quad \to_{m_1\alpha_{14+24}} \chi^+_{8,1}, \quad \to_{m_2\alpha_{13+3}} \chi^+_{8,6} \\
\chi^+_{9,4} &= [m_{14}+m_{23}+m_2, -m_{14}-m_{13}-m_2, m_{14}+m_{12}, m_3] \\
&\quad \to_{m_4\alpha_{14+23+3}} \chi^+_{8,3} \\
\chi^+_{9,5} &= [m_2, -m_{14}-m_2, m_4, m_{14}+m_{13}+m_{23}+m_2] \ \to_{m_3\alpha_{24}} \chi^+_{8,7} \\
\chi^+_{9,6} &= [m_{13}, -m_{14}-m_{23}, m_{34}, m_{14}+m_{13}+2m_2] \\
&\quad \to_{m_4\alpha_{24+3}} \chi^+_{8,6}, \quad \to_{m_1\alpha_{14+14+23+3}} \chi^+_{8,7} \\
\chi^+_{9,7} &= [m_{14}+m_{13}+m_2, -m_{14}-m_{23}-m_2, m_{23}+m_2, m_{34}] \\
&\quad \to_{m_1\alpha_{12}} \chi^+_{8,3}, \quad \to_{m_3\alpha_{14+24+3}} \chi^+_{8,4} \\
\chi^+_{8,1} &= [m_{14}+m_2, -m_{14}-m_{13}-m_2, m_{13}+m_{12}, m_{24}+m_{23}] \\
&\quad \to_{m_2\alpha_{14+14+23+3}} \chi^+_7 \\
\chi^+_{8,2} &= [m_{13}, -m_{14}-m_{13}-m_2, m_{14}+m_{12}, m_{24}+m_{23}] \ \to_{m_4\alpha_{24+3}} \chi^+_7 \\
&\quad \to_{m_1\alpha_{13+3}} \chi^+_{7,1}, \quad \to_{m_3\alpha_{14}} \chi^+_{7,5} \\
\chi^+_{8,3} &= [m_{14}+m_{23}+m_2, -m_{14}-m_{13}-m_2, m_{12}+m_{13}, m_{34}] \\
&\quad \to_{m_3\alpha_{14+24+3}} \chi^+_{7,3} \\
\chi^+_{8,4} &= [m_{14}+m_{13}+m_2, -m_{14}-m_{23}-m_2, m_{23}+m_2, m_4] \\
&\quad \to_{m_1\alpha_{12}} \chi^+_{7,3}, \quad \to_{m_2\alpha_{14+24+3+3}} \chi^+_{7,6}
\end{aligned}$$

$$\chi^+_{8,5} = [m_1, -m_{14} - m_{13} - m_2, m_{14} + m_{13} + 2m_2, m_{34}]$$
$$\to_{m_4\alpha_{14+24+3}} \chi^+_{7,4}, \quad \to_{m_2\alpha_{23+3}} \chi^+_{7,5}$$
$$\chi^+_{8,6} = [m_{14}, -m_{14} - m_{23}, m_3, m_{14} + m_{13} + 2m_2] \to_{m_1\alpha_{14+14+23+3}} \chi^+_{7,2}$$
$$\chi^+_{8,7} = [m_{23}, -m_{14} - m_{23}, m_{34}, m_{14} + m_{13} + 2m_2] \to_{m_2\alpha_{14+24}} \chi^+_{7,1},$$
$$\to_{m_4\alpha_{23+3}} \chi^+_{7,2}$$
$$\chi^+_7 = [m_{14}, -m_{14} - m_{13} - m_2, m_{13} + m_{12}, m_{24} + m_{23}]$$
$$\to_{m_1\alpha_{13+3}} \chi^+_{6,1}, \quad \to_{m_3\alpha_{14+24+3}} \chi^+_{6,2}$$
$$\chi^+_{7,1} = [m_{23}, -m_{14} - m_{23} - m_2, m_{24} + m_2, m_{14} + m_{13}]$$
$$\to_{m_4\alpha_{24+3}} \chi^+_{6,1}, \quad \to_{m_3\alpha_{14}} \chi^+_{6,4}$$
$$\chi^+_{7,2} = [m_{24}, -m_{14} - m_{23}, m_3, m_{14} + m_{13} + 2m_2] \to_{m_2\alpha_{14+24}} \chi^+_{6,1}$$
$$\chi^+_{7,3} = [m_{14} + m_{23} + m_2, -m_{14} - m_{13} - m_2, m_{12} + m_{13}, m_4]$$
$$\to_{m_2\alpha_{14+14+23+3}} \chi^+_{6,5}$$
$$\chi^+_{7,4} = [m_1, -m_{14} - m_{13} - m_2, m_{14} + m_{13} + 2m_2, m_3] \to_{m_2\alpha_{23+3}} \chi^+_{6,3}$$
$$\chi^+_{7,5} = [m_{12}, -m_{14} - m_{13} - m_2, m_{14} + m_{13}, m_{24} + m_2]$$
$$\to_{m_4\alpha_{14+24+3}} \chi^+_{6,3}, \quad \to_{m_1\alpha_{13+3}} \chi^+_{6,4}$$
$$\chi^+_{7,6} = [m_{14} + m_{13} + m_2, -m_{14} - m_{23}, m_3, m_4] \to_{m_1\alpha_{14+14+23+3}} \chi^+_{6,6}$$
$$\chi^+_{6,1} = [m_{24}, -m_{14} - m_{23} - m_2, m_{23} + m_2, m_{14} + m_{13}] \to_{m_3\alpha_{14+24+3}} \chi^+_5$$
$$\chi^+_{6,2} = [m_{14}, -m_{14} - m_{13} - m_2, m_{13} + m_{12}, m_{24} + m_2]$$
$$\to_{m_1\alpha_{13+3}} \chi^+_5, \quad \to_{m_4\alpha_{14}} \chi^+_{5,2}, \quad \to_{m_2\alpha_{24+34}} \chi^+_{5,3}$$
$$\chi^+_{6,3} = [m_{12}, -m_{14} - m_{13} - m_2, m_{14} + m_{13}, m_{23} + m_2]$$
$$\to_{m_1\alpha_{13+3}} \chi^+_{5,1}, \quad \to_{m_3\alpha_{24+3}} \chi^+_{5,2}$$
$$\chi^+_{6,4} = [m_2, -m_{14} - m_{23} - m_2, m_{24} + m_{23}, m_{14} + m_{12}] \to_{m_4\alpha_{14+24+3}} \chi^+_{5,1}$$
$$\chi^+_{6,5} = [m_{14} + m_{23}, -m_{14} - m_{13} - m_2, m_{12} + m_{13}, m_4]$$
$$\to_{m_3\alpha_{13}} \chi^+_{5,3}, \quad \to_{m_1\alpha_{14+24+3+3}} \chi^+_{5,4}$$
$$\chi^+_{6,6} = [m_{14} + m_{23} + m_2, -m_{14} - m_{23}, m_3, m_4] \to_{m_2\alpha_{12}} \chi^+_{5,4}$$
$$\chi^+_5 = [m_{24}, -m_{14} - m_{23} - m_2, m_{23} + m_2, m_{14} + m_{12}]$$
$$\to_{m_2\alpha_{14+24+3+3}} \chi^+_{4,1}, \quad \to_{m_4\alpha_{14}} \chi^+_{4,2}$$
$$\chi^+_{5,1} = [m_2, -m_{14} - m_{23} - m_2, m_{24} + m_{23}, m_{13} + m_{12}] \to_{m_3\alpha_{24+3}} \chi^+_{4,2}$$

$$\begin{aligned}
\chi^+_{5,2} &= [m_{13}, -m_{14} - m_{13} - m_2, m_{14} + m_{12}, m_2 + m_{23}] \\
&\quad \to_{m_1\alpha_{13+3}} \chi^+_{4,2}, \quad \to_{m_2\alpha_{24+34}} \chi^+_{4,4} \\
\chi^+_{5,3} &= [m_{14} + m_2, -m_{14} - m_{13} - m_2, m_{12} + m_{13}, m_{34}] \\
&\quad \to_{m_1\alpha_{14+24+3+3}} \chi^+_{4,3}, \quad \to_{m_4\alpha_{14}} \chi^+_{4,4} \\
\chi^+_{5,4} &= [m_{14} + m_{23}, -m_{14} - m_{23} - m_2, m_{23} + m_2, m_4] \to_{m_3\alpha_{13}} \chi^+_{4,3} \\
\chi^+_{4,1} &= [m_{24}, -m_{14} - m_{23}, m_3, m_{14} + m_{12}] \\
&\quad \to_{m_4\alpha_{14}} \chi^+_3, \quad \to_{m_1\alpha_{24+34}} \chi^+_{3,1} \\
\chi^+_{4,2} &= [m_{23}, -m_{14} - m_{23} - m_2, m_{24} + m_2, m_{13} + m_{12}] \to_{m_2\alpha_{14+24+3+3}} \chi^+_3 \\
\chi^+_{4,3} &= [m_{14} + m_2, -m_{14} - m_{23} - m_2, m_{23} + m_2, m_{34}] \\
&\quad \to_{m_2\alpha_{13+3}} \chi^+_{3,1}, \quad \to_{m_4\alpha_{14}} \chi^+_{3,2} \\
\chi^+_{4,4} &= [m_{13} + m_2, -m_{14} - m_{13} - m_2, m_{14} + m_{12}, m_3] \to_{m_1\alpha_{14+24+3+3}} \chi^+_{3,2} \\
\chi^+_3 &= [m_{23}, -m_{14} - m_{23}, m_{34}, m_{13} + m_{12}] \\
&\quad \to_{m_1\alpha_{24+34}} \chi^+_{2,1}, \quad \to_{m_3\alpha_{14+3}} \chi^+_{2,2} \\
\chi^+_{3,1} &= [m_{14}, -m_{14} - m_{23}, m_3, m_{24} + m_2] \to_{m_4\alpha_{14}} \chi^+_{2,1} \\
\chi^+_{3,2} &= [m_{13} + m_2, -m_{14} - m_{23} - m_2, m_{24} + m_2, m_3] \to_{m_2\alpha_{13+3}} \chi^+_{2,1} \\
\chi^+_{2,1} &= [m_{13}, -m_{14} - m_{23}, m_{34}, m_{23} + m_2] \to_{m_3\alpha_{14+3}} \chi^+_1 \\
\chi^+_{2,2} &= [m_2, -m_{14} - m_2, m_4, m_{13} + m_{12}] \to_{m_1\alpha_{24+34}} \chi^+_1 \\
\chi^+_1 &= [m_{12}, -m_{14} - m_2, m_4, m_{23} + m_2] \to_{m_2\alpha_{14+34}} \chi^+_0
\end{aligned}$$

The multiplets are presented also on Fig. 1. Since the figure is overcrowded on each arrow we have marked only the number $n, n = 1, 2, 3, 4$, from the level $m_n\beta$ of the embedding.

Next we point out to an additional symmetry w.r.t. to the central point of the diagram marked by a bullet. It is relevant for the ERs and marks integral intertwining Knapp-Stein (KS) operators acting between the ERs. Due to this symmetry in the actual parametrization we use the conformal weight $d = 5/2 + c$, actually the parameter c, instead of the non-compact Dynkin label m_2. The parameter c is more convenient since the KS operators flip its sign. The KS operators also involve $sl(3)$ flip of the Dynkin labels m_3, m_4 (see below). Thus, the entries are:

$$\chi = \{n_1, c, n_3, n_4] \tag{25}$$

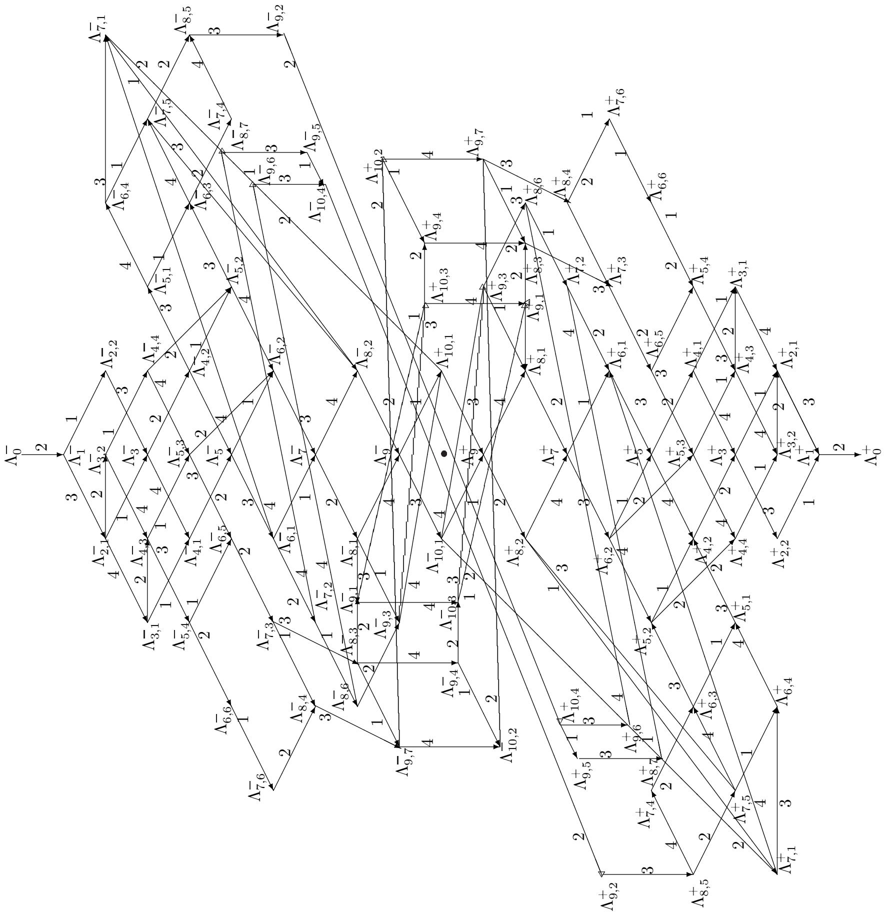

Fig. 1 Multiplets for the real split form F_4' using maximal parabolic with $\mathcal{M} = sl(3,\mathbb{R})_{\text{short roots}} \oplus sl(2,\mathbb{R})_{\text{long roots}}$

so that for the top ER (GVM) on the figure Λ_0^- we have:

$$\chi_0^- \;=\; \{n_1 = m_1, c = -\tfrac{1}{2}(m_1 + 2m_2 + m_3 + m_4), n_3 = m_3, n_4 = m_4] \qquad (26)$$

The mentioned $sl(3)$ flip $(n_3, n_4)^{\pm}$ will be given below by:

$$(n_3, n_4)^+ \;=\; (n_3, n_4), \qquad (n_3, n_4)^- \;=\; (n_4, n_3) \qquad (27)$$

Thus, we can give more compactly the parametrization of the multiplet:

$$\begin{aligned}
\chi_0^{\mp} &= \{\, m_1, \mp\tfrac{1}{2}(m_{14} + m_2), (m_3, m_4)^{\pm} \,\} \qquad (28)\\
\chi_1^{\mp} &= \{\, m_{12}, \mp\tfrac{1}{2}m_{14}, (m_{23} + m_2, m_4)^{\pm} \,\}\\
\chi_{2,1}^{\mp} &= \{\, m_{13}, \mp\tfrac{1}{2}m_{14}, (m_{23} + m_2, m_{34})^{\pm} \,\}\\
\chi_{2,2}^{\mp} &= \{\, m_2, \mp\tfrac{1}{2}m_{24}, (m_{13} + m_{12}, m_4)^{\pm} \,\}\\
\chi_3^{\mp} &= \{\, m_{23}, \mp\tfrac{1}{2}m_{24}, (m_{13} + m_{12}, m_{34})^{\pm} \,\}\\
\chi_{3,1}^{\mp} &= \{\, m_{14}, \mp\tfrac{1}{2}m_{13}, (m_{24} + m_2, m_3)^{\pm} \,\}\\
\chi_{3,2}^{\mp} &= \{\, m_{13} + m_2, \mp\tfrac{1}{2}(m_{14} + m_2), (m_3, m_{24} + m_2)^{\pm} \,\}\\
\chi_{4,1}^{\mp} &= \{\, m_{24}, \mp\tfrac{1}{2}m_{23}, (m_{14} + m_{12}, m_3)^{\pm} \,\}\\
\chi_{4,2}^{\mp} &= \{\, m_{23}, \mp\tfrac{1}{2}m_{24}, (m_{13} + m_{12}, m_{24} + m_2)^{\pm} \,\}\\
\chi_{4,3}^{\mp} &= \{\, m_{14} + m_2, \mp\tfrac{1}{2}(m_{13} + m_2), (m_{34}, m_{23} + m_2)^{\pm} \,\}\\
\chi_{4,4}^{\mp} &= \{\, m_{13} + m_2, \mp\tfrac{1}{2}(m_{14} + m_2), (m_3, m_{14} + m_{12})^{\pm} \,\}\\
\chi_5^{\mp} &= \{\, m_{24}, -m_{14} - m_2, m_{14} + m_{12}, m_{23} + m_2 \,\}\\
\chi_{5,1}^{\mp} &= \{\, m_2, \mp\tfrac{1}{2}m_{24}, (m_{13} + m_{12}, m_{24} + m_{23})^{\pm} \,\}\\
\chi_{5,2}^{\mp} &= \{\, m_{13}, \mp\tfrac{1}{2}m_{14}, (m_2 + m_{23}, m_{14} + m_{12})^{\pm} \,\}\\
\chi_{5,3}^{\mp} &= \{\, m_{14} + m_2, \mp\tfrac{1}{2}(m_{13} + m_2), (m_{34}, m_{12} + m_{13})^{\pm} \,\}\\
\chi_{5,4}^{\mp} &= \{\, m_{14} + m_{23}, \mp\tfrac{1}{2}(m_{13} + m_2), (m_4, m_{23} + m_2)^{\pm} \,\}\\
\chi_{6,1}^{\mp} &= \{\, m_{24}, \mp\tfrac{1}{2}m_2, (m_{14} + m_{13}, m_{23} + m_2)^{\pm} \,\}\\
\chi_{6,2}^{\mp} &= \{\, m_{14}, \mp\tfrac{1}{2}m_{13}, (m_{24} + m_2, m_{13} + m_{12})^{\pm} \,\}\\
\chi_{6,3}^{\mp} &= \{\, m_{12}, \mp\tfrac{1}{2}m_{14}, (m_{23} + m_2, m_{14} + m_{13})^{\pm} \,\}\\
\chi_{6,4}^{\mp} &= \{\, m_2, \mp\tfrac{1}{2}m_{23}, (m_{14} + m_{12}, m_{24} + m_{23})^{\pm} \,\}\\
\chi_{6,5}^{\mp} &= \{\, m_{14} + m_{23}, \mp\tfrac{1}{2}(m_{13} + m_2), (m_4, m_{12} + m_{13})^{\pm} \,\}\\
\chi_{6,6}^{\mp} &= \{\, m_{14} + m_{23} + m_2, \mp\tfrac{1}{2}m_{13}, (m_4, m_3)^{\pm} \,\}\\
\chi_7^{\mp} &= \{\, m_{14}, \mp\tfrac{1}{2}m_{12}, (m_{24} + m_{23}, m_{13} + m_{12})^{\pm} \,\}\\
\chi_{7,1}^{\mp} &= \{\, m_{23}, \mp\tfrac{1}{2}m_2, (m_{14} + m_{13}, m_{24} + m_2)^{\pm} \,\}
\end{aligned}$$

$$\chi^{\mp}_{7,2} = \{ m_{24}, \pm\tfrac{1}{2}m_2, (m_{14}+m_{13}+2m_2, m_3)^{\pm} \}$$
$$\chi^{\mp}_{7,3} = \{ m_{14}+m_{23}+m_2, \mp\tfrac{1}{2}m_{13}, (m_4, m_{12}+m_{13})^{\pm} \}$$
$$\chi^{\mp}_{7,4} = \{ m_1, \mp\tfrac{1}{2}(m_{14}+m_2), (m_3, m_{14}+m_{13}+2m_2)^{\pm} \}$$
$$\chi^{\mp}_{7,5} = \{ m_{12}, \mp\tfrac{1}{2}m_{13}, (m_{24}+m_2, m_{14}+m_{13})^{\pm} \}$$
$$\chi^{\mp}_{7,6} = \{ m_{14}+m_{13}+m_2, \mp\tfrac{1}{2}m_{23}, (m_4, m_3)^{\pm} \}$$
$$\chi^{\mp}_{8,1} = \{ m_{14}+m_2, \mp\tfrac{1}{2}m_1, (m_{24}+m_{23}, m_{13}+m_{12})^{\pm} \}$$
$$\chi^{\mp}_{8,2} = \{ m_{13}, \mp\tfrac{1}{2}m_{12}, (m_{24}+m_{23}, m_{14}+m_{12})^{\pm} \}$$
$$\chi^{\mp}_{8,3} = \{ m_{14}+m_{23}+m_2, \mp\tfrac{1}{2}m_{12}, (m_{34}, m_{12}+m_{13})^{\pm} \}$$
$$\chi^{\mp}_{8,4} = \{ m_{14}+m_{13}+m_2, \mp\tfrac{1}{2}m_{23}, (m_4, m_{23}+m_2)^{\pm} \}$$
$$\chi^{\mp}_{8,5} = \{ m_1, \mp\tfrac{1}{2}(m_{13}+m_2), (m_{34}, m_{14}+m_{13}+2m_2)^{\pm} \}$$
$$\chi^{\mp}_{8,6} = \{ m_{14}, \pm\tfrac{1}{2}m_{12}, (m_{14}+m_{13}+2m_2, m_3)^{\pm} \}$$
$$\chi^{\mp}_{8,7} = \{ m_{23}, \pm\tfrac{1}{2}m_2, (m_{14}+m_{13}+2m_2, m_{34})^{\pm} \}$$
$$\chi^{\mp}_{9} = \{ m_{13}+m_2, \mp\tfrac{1}{2}m_1, (m_{24}+m_{23}, m_{14}+m_{12})^{\pm} \}$$
$$\chi^{\mp}_{9,1} = \{ m_{14}+m_{23}, \mp\tfrac{1}{2}m_1, (m_{24}+m_2, m_{12}+m_{13})^{\pm} \}$$
$$\chi^{\mp}_{9,2} = \{ m_1, \mp\tfrac{1}{2}(m_{13}+m_2), (m_4, m_{14}+m_{13}+m_{23}+m_2)^{\pm} \}$$
$$\chi^{\mp}_{9,3} = \{ m_{14}+m_2, \pm\tfrac{1}{2}m_1, (m_{14}+m_{13}, m_{23}+m_2)^{\pm} \}$$
$$\chi^{\mp}_{9,4} = \{ m_{14}+m_{23}+m_2, \mp\tfrac{1}{2}m_{12}, (m_3, m_{14}+m_{12})^{\pm} \}$$
$$\chi^{\mp}_{9,5} = \{ m_2, \pm\tfrac{1}{2}m_{23}, (m_{14}+m_{13}+m_{23}+m_2, m_4)^{\pm} \}$$
$$\chi^{\mp}_{9,6} = \{ m_{13}, \pm\tfrac{1}{2}m_{12}, (m_{14}+m_{13}+2m_2, m_{34})^{\pm} \}$$
$$\chi^{\mp}_{9,7} = \{ m_{14}+m_{13}+m_2, \mp\tfrac{1}{2}m_2, (m_{34}, m_{23}+m_2)^{\pm} \}$$
$$\chi^{\mp}_{10,1} = \{ m_{13}+m_2, \mp\tfrac{1}{2}m_1, (m_{24}+m_2, m_{14}+m_{13})^{\pm} \}$$
$$\chi^{\mp}_{10,2} = \{ m_{14}+m_{13}+m_2, \mp\tfrac{1}{2}m_2, (m_3, m_{24}+m_2)^{\pm} \}$$
$$\chi^{\mp}_{10,3} = \{ m_{14}+m_{23}, \mp\tfrac{1}{2}m_1, (m_3+2m_2, m_{14}+m_{12})^{\pm} \}$$
$$\chi^{\mp}_{10,4} = \{ m_{12}, \pm\tfrac{1}{2}m_{13}, (m_{14}+m_{13}+m_{23}+m_2, m_4)^{\pm} \}$$

The integral intertwining KS operators act between the spaces $\mathcal{C}_{\chi^{\mp}}$ in opposite directions:

$$G^{+}_{KS} : \mathcal{C}_{\chi^-} \longrightarrow \mathcal{C}_{\chi^+} , \qquad G^{-}_{KS} : \mathcal{C}_{\chi^+} \longrightarrow \mathcal{C}_{\chi^-} \tag{29}$$

4 Concluding Remarks

We expect that the discrete series are contained in the representation χ_0^+ since it is dual to χ_0^- where are contained the finite-dimensional (non-unitary) irreps. Following the Harish-Chandra criterion we must check which $\mathcal{M}$-non-compact entries are negative. We recall that the $\mathcal{M}$-compact entries are $m'_1, m'_3, m'_4, m'_{34}$, all other are non-compact. We give the Harish-Chandra parameters of χ_0^+ in the same order as the roots in (4) and (5)

$$m_1, \ -m_{14}, \ -m_{24}, \ -m_{13}, \ -m_{13}, \ -m_{14}-m_{23}, \ -m_{12}, \\ -m_2, \ -m_{14}, \ -m_{13}-m_2, \ -m_{14}-m_{13}-m_2, \ -m_{14}-m_{23}-m_2, \tag{30}$$

$$m_4, \ m_3, \ m_{34}, \ -m_{14}-m_{13}, \ -m_{14}-m_{23}, \ m_{14}-m_{12}, \\ -m_{14}-m_2, \ -m_{13}-m_{12}, \ -2m_{14}-m_{23}-m_2, \\ -m_{23}-m_2, \ -m_{14}-m_{13}-m_{23}-m_2, \ -m_{14}-m_{13}-2m_2 \tag{31}$$

It is easy to see that all the $\mathcal{M}$-non-compact entries of χ_0^+ are negative. (We repeat that the compact entries are the first one in (30) and the first three in (31)).

The discrete series irrep with lowest possible conformal weight $d = 5$ ($c = 5/2$) is contained in χ_0^+ for $m_1 = m_2 = m_3 = m_4 = 1$. It corresponds to the one-dimensional irrep contained in χ_0^- with $d = 0$ ($c = -5/2$).

Acknowledgments The author has received partial support from Bulgarian NSF Grant DN-18/1.

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An Exceptional Symmetry Algebra for the 3D Dirac–Dunkl Operator

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Abstract We initiate the study of an algebra of symmetries for the 3D Dirac–Dunkl operator associated with the Weyl group of the exceptional root system G_2. For this symmetry algebra, we give both an abstract definition and an explicit realisation. We then construct ladder operators, using an intermediate result we prove for the Dirac–Dunkl symmetry algebra associated with arbitrary finite reflection group acting on a three-dimensional space.

1 Introduction

In the present paper, we initiate the study of an algebra of symmetries for the Dirac–Dunkl operator associated with the exceptional root system G_2. The latter is primarily known from the classification of simple Lie algebras. The associated Lie group and algebra continue to spark interest, see for instance the recent paper of Dobrev [4] and references therein. Our purpose is related instead to the action of the Weyl group associated with G_2 on a (two-dimensional subspace of a) three-dimensional space. Though G_2 is indeed a root system of rank 2, the arising symmetry algebra associated with three-dimensional space portrays interesting non-trivial relations, which are not present when considering the two-dimensional analogue.

We will briefly recall how the symmetry algebra in question arises. For a finite reflection group W acting on a finite dimensional vector space, there exists a rational Cherednik algebra (RCA) [6] that can be viewed as a deformation of the algebra of polynomial differential operators on the vector space. An explicit realisation is given by means of differential-difference operators called Dunkl operators [5].

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_30

A generalisation of the Dirac operator is defined abstractly inside the tensor product of the RCA and a Clifford algebra, or explicitly by using Dunkl operators in lieu of partial derivatives in the ordinary definition of the Dirac operator.

In this way, the Dirac–Dunkl operator squares to a Dunkl version of the Laplace operator whose invariance is restricted to the group W as opposed to the full orthogonal invariance of its classical counterpart. Moreover, together with its dual partner, the Dirac–Dunkl operator generates a Lie superalgebra isomorphic to $\mathfrak{osp}(1|2)$. The latter's (super)centraliser inside the tensor product of RCA and Clifford algebra gives an algebra of symmetries (super)commuting with the Dirac–Dunkl operator. Structurally it can be seen as a deformation of the orthogonal Lie algebra representing total angular momentum in the non-deformed case.

In previous work [1], explicit expressions for the elements of the symmetry algebra and the generated algebraic structure were determined for arbitrary finite reflection group. Subsequently, the study was specialised to the A_2 root system with Coxeter group S_3 acting on a three-dimensional Euclidean space [2]. In this case it was possible to classify all irreducible representations and give conditions for when they are unitarisable. An important tool was the construction of ladder operators.

A natural follow-up question is whether this approach extends to settings with other reflection groups. The existence of ladder operators will in general depend on the root system under consideration. One of our aims is to work out in detail the conditions for their existence. The full analysis goes beyond the scope of this contribution; here we will already present some preliminary results pertaining to three-dimensional spaces and focus in particular on the exceptional root system G_2, embedded herein.

In Sect. 2 the required definitions of the exceptional root system G_2 and Dirac–Dunkl operator are introduced and we present the symmetry algebra both abstractly and as an explicit realisation. In Sect. 3, we prove an intermediate result for arbitrary root system in $\mathbb{R}^3$ and show that this leads to the existence of ladder operators for the symmetry algebra associated with G_2.

2 An Exceptional Symmetry Algebra

We consider the Euclidean space $\mathbb{R}^3$ with coordinates x_1, x_2, x_3. The 2-dimensional root system G_2 is realised in a plane and is generated by two simple roots $\alpha_1 = (0, 1, -1)$ and $\alpha_2 = (1, -2, 1)$. The Coxeter group linked to G_2 is the dihedral group $D_{2\cdot 6}$ that we will present by: $D_{12} = \langle \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^2 = (\sigma_1\sigma_2)^6 = (\sigma_2\sigma_1)^6 = 1\rangle$ with the reflections σ_1 connected to the short root α_1, and σ_2 to the long root α_2. Their actions on $\mathbb{R}^3$ are expressed matricially by:

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}. \tag{1}$$

A set of positive roots is given by

$$R_+ = \{\alpha_1 = (0, 1, -1), \alpha_2 = (1, -2, 1), \alpha_3 = (1, -1, 0), \\ \alpha_4 = (1, 1, -2), \alpha_5 = (1, 0, -1), \alpha_6 = (2, -1, -1)\}\,. \tag{2}$$

To each root α_i, a reflection σ_i is paired. The reflections have the following decompositions in terms of the simple reflections σ_1, σ_2:

$$\sigma_3 = \sigma_2\sigma_1\sigma_2, \quad \sigma_4 = \sigma_1\sigma_2\sigma_1, \quad \sigma_5 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1, \quad \sigma_6 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2. \tag{3}$$

We introduce a D_{12}-invariant weight function $\kappa : G_2 \to \mathbb{C}$, which is defined by two complex numbers κ_1 and κ_2 linked respectively to the short and long roots. With this, it is possible to define Dunkl operators [5] for the root system G_2; for example the one associated with the coordinate x_2 is given by

$$\mathscr{D}_2 = \frac{\partial}{\partial x_2} + \kappa_1\left(\frac{1-\sigma_1}{x_2 - x_3} + \frac{1-\sigma_3}{x_1 - x_2}\right) \\ + \kappa_2\left(-2\frac{1-\sigma_2}{x_1 - 2x_2 + x_3} + \frac{1-\sigma_4}{x_1 + x_2 - x_3} - \frac{1-\sigma_6}{2x_1 - x_2 - x_3}\right), \tag{4}$$

while $\mathscr{D}_1$ and $\mathscr{D}_3$ are defined similarly.

Next, we consider the Clifford algebra with three anticommuting generators e_1, e_2, e_3 that all square to $\varepsilon \in \{+1, -1\}$. The Dirac–Dunkl operator associated with our embedding of G_2 in $\mathbb{R}^3$ is realised explicitly by $\mathscr{D} = \mathscr{D}_1 e_1 + \mathscr{D}_2 e_2 + \mathscr{D}_3 e_3$. Together with its dual partner $x_1e_1 + x_2e_2 + x_3e_3$, it generates a realisation of $\mathfrak{osp}(1|2)$. For ease of notation, we shall not make explicit mention of the tensor product, trusting the reader to add it whenever Clifford elements e_i are involved.

The elements of the symmetry algebra were obtained in previous work [1] (that they indeed generate the full centraliser is the subject of [8]) and we will go over them now. First, we need a double cover of the Weyl group D_{12}. The orthogonal group $O(3)$ has two non-isomorphic double covers. These correspond to the two choices of ε in the definition of the Clifford algebra [7]. For either choice of ε, we obtain a double cover $\widetilde{D}_{12}^{\varepsilon}$ by viewing D_{12} as a subgroup of the orthogonal group $O(3)$, through the pullback of the projection of the $\mathrm{Pin}^{\varepsilon}(3)$ double cover onto $O(3)$. In this way, we obtain the $\widetilde{D}_{12}^{\varepsilon}$ elements (together with their additive inverses):

$$\widetilde{\sigma}_1 = \frac{\sigma_1(e_2 - e_3)}{\sqrt{2}}, \qquad \widetilde{\sigma}_3 = \frac{\sigma_3(e_1 - e_2)}{\sqrt{2}}, \qquad \widetilde{\sigma}_5 = \frac{\sigma_5(e_1 - e_3)}{\sqrt{2}},$$
$$\widetilde{\sigma}_2 = \frac{\sigma_2(e_1 - 2e_2 + e_3)}{\sqrt{6}}, \quad \widetilde{\sigma}_4 = \frac{\sigma_4(e_1 + e_2 - 2e_3)}{\sqrt{6}}, \quad \widetilde{\sigma}_6 = \frac{\sigma_6(2e_1 - e_2 - e_3)}{\sqrt{6}}.$$

Note that the group relations depend on the choice of ε. By direct computation we find $\widetilde{D}_{12}^{\varepsilon} = \langle \widetilde{\sigma}_1, \widetilde{\sigma}_2 \mid \widetilde{\sigma}_1^2 = \widetilde{\sigma}_2^2 = \varepsilon, (\widetilde{\sigma}_1\widetilde{\sigma}_2)^6 = (\widetilde{\sigma}_2\widetilde{\sigma}_1)^6 = -1\rangle$, which also follows from [7, Thm 4.2]. The order of this group is 24, and for $\varepsilon = +1$ it is again a dihedral group, while for $\varepsilon = -1$ it is a dicyclic group. Regardless of the choice of ε, all elements of $\widetilde{D}_{12}^{\varepsilon}$ will supercommute with the Dunkl-Dirac operator when taking into account the $\mathbb{Z}_2$-grading inherited from the Clifford algebra. Both $\mathscr{D}$ and $\pm\sigma_i$ are odd elements with respect to this grading, so they will in fact anticommute. In the following, we will use the standard notation for anticommutator ($\{-, -\}$) and commutator ($[-, -]$).

Furthermore, there are three analogues of the total angular momentum operators that commute with the Dirac operator: O_{12}, O_{23}, O_{13}. Classically (non-Dunkl) they generate a realisation of the orthogonal Lie algebra $\mathfrak{so}(3)$, though here it will be a deformation thereof. An explicit realisation is given by

$$O_{ij} = L_{ij} + \varepsilon e_i e_j/2 + O_i e_j - O_j e_i, \tag{5}$$

where $L_{ij} = x_i \mathscr{D}_j - x_j \mathscr{D}_i$ is a Dunkl analogue of angular momentum, and for ease of notation we denote some specific linear combinations of elements of $\widetilde{D}_{12}^{\varepsilon}$ as follows:

$$\begin{aligned} O_1 &= \kappa_1(\widetilde{\sigma}_3 + \widetilde{\sigma}_5) + \kappa_2(\widetilde{\sigma}_2 + \widetilde{\sigma}_4 + 2\widetilde{\sigma}_6), \\ O_2 &= \kappa_1(\widetilde{\sigma}_1 - \widetilde{\sigma}_3) + \kappa_2(-2\widetilde{\sigma}_2 + \widetilde{\sigma}_4 - \widetilde{\sigma}_6), \\ O_3 &= \kappa_1(-\widetilde{\sigma}_1 - \widetilde{\sigma}_5) + \kappa_2(\widetilde{\sigma}_2 - 2\widetilde{\sigma}_4 - \widetilde{\sigma}_6). \end{aligned} \tag{6}$$

It is immediate to see that the sum $O_1 + O_2 + O_3 = 0$. Moreover, we will denote $\mathscr{E} = [O_1, O_2]$, and by direct but slightly tedious computations, we can also see that $[O_2, O_3] = \mathscr{E} = -[O_1, O_3]$. From the realisation (5), it is clear that $O_{ij} = -O_{ji}$, and it is convenient to abide by this convention also when defining the algebra elements abstractly.

The interaction of the two simple reflections $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ with the two-index symmetries of equation (5) are given by:

$$\begin{aligned} \widetilde{\sigma}_1 O_{12} &= O_{13}\widetilde{\sigma}_1, & \widetilde{\sigma}_2 O_{12} &= (-2/3 O_{12} + 2/3 O_{13} + 1/3 O_{23})\widetilde{\sigma}_2, \\ \widetilde{\sigma}_1 O_{13} &= O_{12}\widetilde{\sigma}_1, & \widetilde{\sigma}_2 O_{13} &= (2/3 O_{12} + 1/3 O_{13} + 2/3 O_{23})\widetilde{\sigma}_2, \\ \widetilde{\sigma}_1 O_{23} &= -O_{23}\widetilde{\sigma}_1, & \widetilde{\sigma}_2 O_{23} &= (1/3 O_{12} + 2/3 O_{13} - 2/3 O_{23})\widetilde{\sigma}_2; \end{aligned} \tag{7}$$

from which the entire action of $\widetilde{D}_{12}^{\varepsilon}$ follows.

The final generator of our symmetry algebra is a central element O_{123}, of which an explicit realisation is given by

$$O_{123} = \varepsilon e_1 e_2 e_3 + O_1 e_2 e_3 - O_2 e_1 e_3 + O_3 e_1 e_2 + L_{12} e_3 - L_{13} e_2 + L_{23} e_1. \tag{8}$$

As a consequence of the relations in the general case, see [1, Thm 3.12] or [2, eq. (1.7)], the two-index symmetries (5) respect

$$\begin{aligned} [O_{13}, O_{12}] &= O_{23} + 2 O_{123} O_1 + \mathscr{E}; \\ [O_{23}, O_{12}] &= -O_{13} + 2 O_{123} O_2 + \mathscr{E}; \\ [O_{23}, O_{13}] &= O_{12} + 2 O_{123} O_3 + \mathscr{E}. \end{aligned} \tag{9}$$

These relations can be proved specifically for the G_2 case, in a similar manner as was done for S_3 [3].

In the right-hand sides of (9) appear the linear combinations of elements of $\widetilde{D}^{\varepsilon}_{12}$ given by (6) and $\mathscr{E}$. When the deformation parameters κ_1, κ_2 are chosen to be zero, these all vanish and the relations (9) reduce to those of the orthogonal Lie algebra $\mathfrak{so}(3)$.

3 Ladder Operators

The result we prove next holds for arbitrary root system in $\mathbb{R}^3$. Hereto, one should use the appropriate definitions for O_1, O_2, O_3 as given in [1, eq. (3.8) and Ex. 4.2] and the relations analogous to (9) given by [2, eq. (1.7)]. What we obtain in this way are not yet the desired ladder operators, though we will show that they do lead to ladder operators for the G_2 case at hand.

Proposition 1. *Let $\omega = e^{2i\pi/3}$ and consider the following linear combinations:*

$$\begin{aligned} O_0 &= -i/\sqrt{3}(O_{12} + O_{23} - O_{13}), \\ O_+ &= -i\sqrt{2/3}(O_{12} + \omega O_{23} - \omega^2 O_{13}), \\ O_- &= -i\sqrt{2/3}(O_{12} + \omega^2 O_{23} - \omega O_{13}). \end{aligned} \tag{10}$$

Denoting $\omega^+ = \omega$ and $\omega^- = \omega^2$, they satisfy

$$\begin{aligned} \left[O_0, O_\pm\right] &= \pm O_\pm \mp i\sqrt{2/3}(2O_{123}(O_3 + \omega^\pm O_1 + \omega^\mp O_2) \\ &\quad + [O_1, O_2] + \omega^\pm [O_2, O_3] + \omega^\mp [O_3, O_1]); \\ \left[O_+, O_-\right] &= 2O_0 - 2i/\sqrt{3}(2O_{123}(O_1 + O_2 + O_3) \\ &\quad + [O_1, O_2] + [O_2, O_3] + [O_3, O_1]). \end{aligned} \tag{11}$$

Proof. Using the definitions (10) and grouping the terms appropriately we obtain

$$\begin{aligned} \left[O_0, O_\pm\right] = -\sqrt{2}/3 \big(&(1-\omega^\pm)[O_{23}, O_{12}] \\ &+ (\omega^\mp - 1)[O_{12}, O_{31}] + (\omega^\pm - \omega^\mp)[O_{31}, O_{23}]\big). \end{aligned}$$

Noticing that $(\omega^\pm - \omega^\mp) = \pm i\sqrt{3}$, and $(1-\omega^\pm) = 3/2 \mp i\sqrt{3}/2 = \pm i\sqrt{3}\omega^\mp$, and $(\omega^\mp - 1) = -3/2 \mp i\sqrt{3}/2 = \pm i\sqrt{3}\omega^\pm$, and applying [2, eq. (1.7)] results in

$$\begin{aligned} = \mp i\sqrt{2}/\sqrt{3}\big(&\omega^\mp(O_{31} + \{O_{123}, O_2\} + [O_3, O_1]) \\ &+ \omega^\pm(O_{23} + \{O_{123}, O_1\} + [O_2, O_3]) \\ &+ O_{12} + \{O_{123}, O_3\} + [O_1, O_2]\big), \end{aligned}$$

and finally using again the definition (10) one arrives at the desired expression.

In the same manner for the second equation, we find

$$\begin{aligned}\left[O_+, O_-\right] &= -2/3(\omega - \omega^2)\,([O_{23}, O_{12}] + [O_{12}, O_{31}] + [O_{31}, O_{23}]) \\ &= -2i/\sqrt{3}\,(O_{31} + \{O_{123}, O_2\} + [O_3, O_1] + O_{23} + \{O_{123}, O_1\} + [O_2, O_1] \\ &\qquad + O_{12} + \{O_{123}, O_1\} + [O_1, O_2]) \\ &= 2O_0 - 2i/\sqrt{3}\,(\{O_{123}, O_1 + O_2 + O_3\} + [O_1, O_2] + [O_2, O_3] + [O_3, O_1])\,.\end{aligned}$$

As O_{123} is central, this proves the second equality. □

When the root system satisfies some specific properties, we can use the previous result to obtain ladder operators.

Proposition 2. *For the root system G_2, the elements O_0, O_+ and O_- satisfy*

$$\begin{aligned}\left[O_0, O_\pm\right] &= \pm O_\pm \mp 2i\sqrt{2/3}\,O_{123}\left(O_3 + \omega^\pm O_1 + \omega^\mp O_2\right); \\ \left[O_+, O_-\right] &= 2O_0 - 2i\sqrt{3}\mathscr{E}.\end{aligned} \tag{12}$$

Moreover, the quadratic elements $K_\pm = 1/2\{O_0, O_\pm\}$ fulfill the ladder operator relations $[O_0, K_\pm] = \pm K_\pm$.

Proof. Starting from the relations (11), we can use $1 + \omega + \omega^2 = 0$, and $O_1 + O_2 + O_3 = 0$, while $[O_1, O_2] = [O_2, O_3] = [O_3, O_1] = \mathscr{E}$, to arrive at (12).

In addition, we have $\left[O_0, K_\pm\right] = 1/2\left[O_0, \{O_0, O_\pm\}\right] = 1/2\left\{O_0, \left[O_0, O_\pm\right]\right\}$. By the first relation (12), this becomes

$$\left[O_0, K_\pm\right] = \pm 1/2\,\{O_0, O_\pm\} \mp i\sqrt{2/3}\left\{O_0, O_{123}(O_3 + \omega^\pm O_1 + \omega^\mp O_2)\right\} = \pm K_\pm.$$

In the last step we used the fact that O_{123} is central, and that all elements of $\widetilde{D}^\varepsilon_{12}$ anticommute with O_0, which is clear from the action (7). □

These ladder operators can now be used in the study of the representation theory of the symmetry algebra in a similar vein as was done in the S_3 case [2], which we aim to do in future work. In addition, we will investigate the construction of ladder operators for other reflection groups.

Acknowledgements We wish to thank Hendrik De Bie and Joris Van der Jeugt for helpful discussions and support. This research was supported in part by EOS Research Project number 30889451. ALR also holds a scholarship from the Fonds de recherche du Québec – Nature et technologies number 270527. RO was supported by the Research Project KP-06-N28/6 from the Bulgarian National Science Fund and by a postdoctoral fellowship, fundamental research, of the Research Foundation – Flanders (FWO), number 12Z9920N. This support is gratefully acknowledged.

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Graphene Dots via Discretizations of Weyl-Orbit Functions

Jiří Hrivnák and Lenka Motlochová

Abstract The application of two fundamental discretizations of Weyl-orbit functions to an electron propagation on the graphene triangular dots are presented. Symmetries of the point and label sets inside dual weight and root lattices of root systems are provided by affine and extended affine Weyl groups. The discrete orthogonality relations of the Weyl-orbit functions over the dual weight and root point sets induce four types of complex discrete Fourier-Weyl transforms. Subtractively combining the transforms of the A_2 group induces two types of extended Weyl-orbit functions and their corresponding discrete transforms on the fragment of the honeycomb lattice. Special types of extended Weyl-orbit functions represent stationary states of the electron propagation on the triangular graphene dot with armchair boundaries.

1 Extensions of Weyl Groups

The current notation is based on papers [2, 4, 5]. Each simple Lie algebra from the four series A_n $(n \geq 1)$, B_n $(n \geq 3)$, C_n $(n \geq 2)$, D_n $(n \geq 4)$ and each exceptional algebra E_6, E_7, E_8, F_4 and G_2 determine their root system Π together with the set $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Pi$ of the simple roots [1]. The set of simple roots Δ spans the Euclidean space $\mathbb{R}^n$ with the scalar product $\langle\, , \rangle$. For simple Lie algebras with two different root-lengths, the set Δ is disjointly decomposed into the set of short simple roots Δ_s and the set of long simple roots Δ_l. The following quantities are standardly deduced [1] from the entire root system Π: the Cartan matrix C, the highest root $\xi = m_1\alpha_1 + \dots + m_n\alpha_n$, the root lattice $Q = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n$, the dual weight lattice $P^\vee = \mathbb{Z}\omega_1^\vee + \dots + \mathbb{Z}\omega_n^\vee$, the dual root lattice $Q^\vee = \mathbb{Z}\alpha_1^\vee + \dots + \mathbb{Z}\alpha_n^\vee$, where

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*, Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_31

$\alpha_i^\vee = 2\alpha_i/\langle\alpha_i, \alpha_i\rangle$ and the weight lattice $P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n$. The set of vectors $\Delta^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ also constitutes a root system and determines the highest dual root $\eta = m_1^\vee\alpha_1^\vee + \cdots + m_n^\vee\alpha_n^\vee$.

The Weyl group W is generated by n reflections r_α, $\alpha \in \Delta$ and the affine Weyl group is expressed as the semidirect product $W^{\text{aff}} = Q^\vee \rtimes W$ that induces the retraction homomorphism $\psi : W^{\text{aff}} \to W$. The fundamental domain $F \subset \mathbb{R}^n$ of W^{aff} contains exactly one point from each $W^{\text{aff}}-$orbit. The dual affine Weyl group W_Q^{aff} is also given as a semidirect product, $W_Q^{\text{aff}} = Q \rtimes W$, and its dual fundamental domain F_Q that contains exactly one point from each $W_Q^{\text{aff}}-$orbit is given explicitly by

$$F_Q = \left\{b_1\omega_1 + \cdots + b_n\omega_n \,|\, b_0 + b_1 m_1^\vee + \cdots + b_n m_n^\vee,\ b_i \geq 0, i = 0, \ldots, n\right\}$$

The discrete counting $\varepsilon-$ and $h-$functions are defined for $a \in \mathbb{R}^n$ and $M \in \mathbb{N}$ via relations,

$$\begin{aligned} \varepsilon(a) &= |W| \cdot |\text{Stab}_{W^{\text{aff}}}(a)|^{-1}, \\ h_M(a) &= \left|\text{Stab}_{W_Q^{\text{aff}}}\left(\frac{a}{M}\right)\right|. \end{aligned}$$

The $n + 1$ numbers that determine each point $b \in F_Q$ are Kac coordinates $b = [b_0, \ldots, b_n]$. The extended dual affine Weyl group extends the dual affine Weyl group W_Q^{aff} via the following semidirect product,

$$W_P^{\text{aff}} = P \rtimes W,$$

and induces the dual retraction homomorphism $\widehat{\psi} : W_P^{\text{aff}} \to W$.

The finite abelian subgroup $\Gamma \subset W_P^{\text{aff}}$ contains maps which stabilize the fundamental domain F_Q,

$$\Gamma = \left\{\gamma \in W_P^{\text{aff}} \,|\, \gamma \cdot F_Q = F_Q\right\}.$$

The group Γ acts on the fundamental domain F_Q and its magnified version MF_Q, $M \in \mathbb{N}$ as a certain permutation on the Kac coordinates $[b_0, \ldots, b_n]$ and $[Mb_0, \ldots, Mb_n]$. Considering the lexicographic ordering $>_{\text{lex}}$ on the Kac coordinates $[b_0, \ldots, b_n]$, the set $F_P \subset \mathbb{R}^n$ defined in [2] by

$$F_P = \left\{b \in F_Q \,|\, b = \max_{>_{\text{lex}}} \Gamma b\right\},$$

forms a fundamental domain of W_P^{aff}. Examples of the tiling of the space $\mathbb{R}^3$ by the shifted copies of the domains WF_P are depicted in Fig. 1.

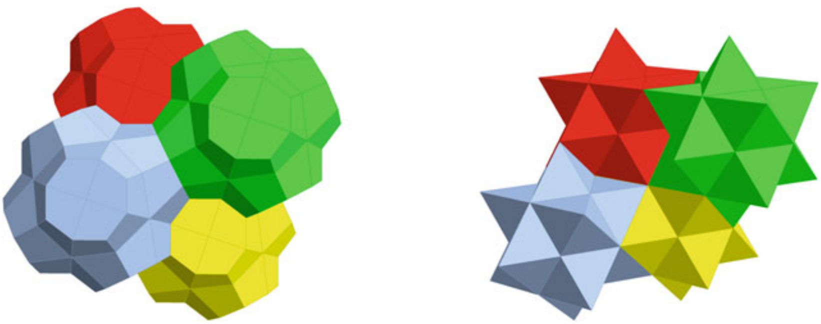

Fig. 1 The tiling of the space $\mathbb{R}^3$ by the shifted copies of the domains WF_P for the root systems of algebras A_3 (left) and B_3 (right)

2 Discretizations of Weyl-Orbit Functions

The four sign homomorphisms $\imath, \sigma^e, \sigma^s, \sigma^l : W \to \{\pm 1\}$ that are utilized in [4] and given on the generators of the Weyl group W by

$$\imath(r_\alpha) = 1, \quad \sigma^e(r_\alpha) = -1,$$
$$\sigma^s(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_l, \\ -1, & \alpha \in \Delta_s, \end{cases} \qquad \sigma^l(r_\alpha) = \begin{cases} 1, & \alpha \in \Delta_s, \\ -1, & \alpha \in \Delta_l \end{cases}$$

induce the signed fundamental domains $F^\sigma \subset F$, $F_Q^\sigma \subset F_Q$ and $F_P^\sigma \subset F_P$ by relations

$$F^\sigma = \left\{ a \in F \;\middle|\; \sigma \circ \psi \left(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)\right) = \{1\} \right\},$$
$$F_Q^\sigma = \left\{ b \in F_Q \;\middle|\; \sigma \circ \widehat{\psi} \left(\mathrm{Stab}_{W_Q^{\mathrm{aff}}}(b)\right) = \{1\} \right\},$$
$$F_P^\sigma = \left\{ b \in F_P \mid \sigma \circ \widehat{\psi}(\mathrm{Stab}_{W_P^{\mathrm{aff}}}(b)) = \{1\} \right\}.$$

For any sign homomorphism $\sigma \in \{\imath, \sigma^e, \sigma^s, \sigma^l\}$ and any weight $b \in P$, the Weyl-orbit complex functions $\varphi_b^\sigma : \mathbb{R}^n \to \mathbb{C}$ are given by [1, 6, 7]

$$\varphi_b^\sigma(a) = \sum_{w \in W} \sigma(w)\, e^{2\pi \mathrm{i} \langle wb, a \rangle}, \quad a \in \mathbb{R}^n.$$

The point set $F_{P^\vee, M}^\sigma$ of the dual weight discretization contains points from the refined dual weight lattice and labels are taken from the set $\Lambda_{Q,M}^\sigma$,

$$F^{\sigma}_{P^\vee,M} = \frac{1}{M} P^\vee \cap F^\sigma ,$$
$$\Lambda^{\sigma}_{Q,M} = P \cap M F^{\sigma}_{Q}.$$

Theorem 1 (Discrete orthogonality on $F^{\sigma}_{P^\vee,M}$ [4, 5]). *For any $b, b' \in \Lambda^{\sigma}_{Q,M}$ it holds that*

$$\sum_{a\in F^{\sigma}_{P^\vee,M}} \varepsilon(a)\varphi^{\sigma}_{b}(a)\overline{\varphi^{\sigma}_{b'}(a)} = \det C \cdot |W|\, M^n h_M(b)\, \delta_{b,b'}.$$

The point set $F^{\sigma}_{Q^\vee,M}$ of the dual root lattice discretization contains points from refined dual root lattice and labels are taken from the set $\Lambda^{\sigma}_{P,M}$,

$$F^{\sigma}_{Q^\vee,M} = \frac{1}{M} Q^\vee \cap F^\sigma ,$$
$$\Lambda^{\sigma}_{P,M} = P \cap M F^{\sigma}_{P}.$$

Theorem 2 (Discrete orthogonality on $F^{\sigma}_{Q^\vee,M}$ [2]). *For any $b, b' \in \Lambda^{\sigma}_{P,M}$ it holds that*

$$\sum_{a\in F^{\sigma}_{Q^\vee,M}} \varepsilon(a)\varphi^{\sigma}_{b}(a)\overline{\varphi^{\sigma}_{b'}(a)} = |W|\, M^n \left|\mathrm{Stab}_{W^{\mathrm{aff}}_{P}}\left(\frac{b}{M}\right)\right| \delta_{b,b'}.$$

3 Triangular Armchair Graphene Dots

For the description of the armchair graphene dots, the root system A_2 with its two sign homomorphisms $\sigma = 1, \sigma^e$ is considered [3]. Note that the dual roots $\alpha_1^\vee, \alpha_2^\vee$ and dual weights $\omega_1^\vee, \omega_2^\vee$ coincide with the roots α_1, α_2 and weights ω_1, ω_2, respectively. Armchair fragment of the honeycomb lattice $H^{\sigma}_{P^\vee,M}$ is given subtractively as

$$H^{\sigma}_{P^\vee,M} = F^{\sigma}_{P^\vee,M} \setminus F^{\sigma}_{Q^\vee,M}.$$

The triangular graphene dot is also described classically as a union of its two sublattices $H^{\sigma}_{P^\vee,M} = H^{(\sigma,1)}_{P^\vee,M} \cup H^{(\sigma,2)}_{P^\vee,M}$,

$$H^{(\sigma,1)}_{P^\vee,M} = \tfrac{1}{M}(\omega_1^\vee + Q^\vee) \cap F^\sigma ,$$
$$H^{(\sigma,2)}_{P^\vee,M} = \tfrac{1}{M}(\omega_2^\vee + Q^\vee) \cap F^\sigma .$$

The explicit form of the weight set $\Lambda^{1}_{Q,M}$ for A_2 is the following,

$$\Lambda^{1}_{Q,M} = \left\{\lambda_1\omega_1 + \lambda_2\omega_2 \mid \lambda_0, \lambda_1, \lambda_2 \in \mathbb{Z}^{\geq 0}, \lambda_0 + \lambda_1 + \lambda_2 = M\right\}$$

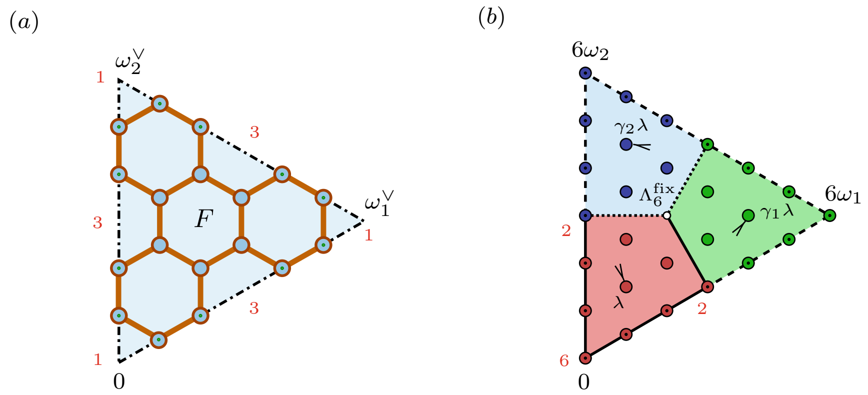

Fig. 2 (a) The fundamental domain F of A_2 contains 18 nodes of the point set $H^{1}_{P^\vee,6}$. The points of the set $H^{1}_{P^\vee,6}$, without the twelve dotted points on the boundary of F, form the point set $H^{\sigma^e}_{P^\vee,6}$. The brown lines depict possible jumps of the electron between two points of the graphene dot. (b) The magnified fundamental domain $6F_Q$ contains the red kite-shaped magnified fundamental domain $6F_P$. The domain $6F_P$ contains the nine red nodes of the weight set $L^{1}_{P,6}$. The weights of the set $L^{1}_{P,6}$, without the six dotted weights, form the weight set $L^{\sigma^e}_{P,6}$

and action of the group $\Gamma = \{1, \gamma_1, \gamma_2\}$ on a weight in Kac coordinates $[\lambda_0, \lambda_1, \lambda_2] \in \Lambda^{1}_{Q,M}$ is given cyclically as

$$\gamma_1[\lambda_0, \lambda_1, \lambda_2] = [\lambda_2, \lambda_0, \lambda_1],$$
$$\gamma_2[\lambda_0, \lambda_1, \lambda_2] = [\lambda_1, \lambda_2, \lambda_0].$$

The subset $\Lambda^{\mathrm{fix}}_M \subset \Lambda^{1}_{P,M}$ that contains the fixed Dirac point,

$$\Lambda^{\mathrm{fix}}_M = \left\{\lambda \in \Lambda^{1}_{P,M} \mid \Gamma\lambda = \lambda\right\},$$

is utilized to define the weight set $L^{\sigma}_{P,M} \subset \Lambda^{\sigma}_{P,M}$ as

$$L^{\sigma}_{P,M} = \Lambda^{\sigma}_{P,M} \setminus \Lambda^{\mathrm{fix}}_M.$$

The honeycomb point and weight sets are for $M = 6$ depicted in Fig. 2.

Extended Weyl-orbit functions $\varphi^{\sigma,\pm}_\lambda$, $\lambda \in L^{\sigma}_{P,M}$,

$$\varphi^{\sigma,\pm}_\lambda(x) = \mu^{\pm,0}_\lambda \varphi^{\sigma}_\lambda(x) + \mu^{\pm,1}_\lambda \varphi^{\sigma}_{\gamma_1\lambda}(x) + \mu^{\pm,2}_\lambda \varphi^{\sigma}_{\gamma_2\lambda}(x)$$

admit non-constant values of the extension coefficients $\mu^{\pm,0}_\lambda, \mu^{\pm,1}_\lambda, \mu^{\pm,2}_\lambda \in \mathbb{C}$,

$$\mu_\lambda^{\pm,0} = \mathrm{Re}\left\{(3+\sqrt{3}\,\mathrm{i})\varphi_\lambda^{\mathbb{1}}\left(\tfrac{\omega_1^\vee}{M}\right)\right\},$$
$$\mu_\lambda^{\pm,1} = 0,$$
$$\mu_\lambda^{\pm,2} = \mathrm{Re}\left\{(3-\sqrt{3}\,\mathrm{i})\varphi_\lambda^{\mathbb{1}}\left(\tfrac{\omega_1^\vee}{M}\right)\right\} \pm 3\left|\varphi_\lambda^{\mathbb{1}}\left(\tfrac{\omega_1^\vee}{M}\right)\right|.$$

The normalization functions $\mu^\pm$ are introduced as

$$\mu^\pm(\lambda) = 9\left|\varphi_\lambda^{\mathbb{1}}\left(\tfrac{\omega_1^\vee}{M}\right)\right|\left(2\left|\varphi_\lambda^{\mathbb{1}}\left(\tfrac{\omega_1^\vee}{M}\right)\right| \pm \mathrm{Re}\left\{(1-\sqrt{3}\,\mathrm{i})\varphi_\lambda^{\mathbb{1}}\left(\tfrac{\omega_1^\vee}{M}\right)\right\}\right).$$

Theorem 3 (Discrete orthogonality on $H^\sigma_{P^\vee,M}$ [3]). *For any $\lambda, \lambda' \in L^\sigma_{P,M}$ it holds that*

$$\sum_{a\in H^\sigma_{P^\vee,M}} \varepsilon(a)\varphi_\lambda^{\sigma,\pm}(a)\overline{\varphi_{\lambda'}^{\sigma,\pm}(a)} = 12M^2 h_M(\lambda)\mu^\pm(\lambda)\delta_{\lambda\lambda'},$$
$$\sum_{a\in H^\sigma_{P^\vee,M}} \varepsilon(a)\varphi_\lambda^{\sigma,\pm}(a)\overline{\varphi_{\lambda'}^{\sigma,\mp}(a)} = 0.$$

The scalar product of two complex discrete functions $f, g : H^\sigma_{P^\vee,M} \to \mathbb{C}$ is given as

$$\langle f, g\rangle_{H^\sigma_{P^\vee,M}} = \sum_{a\in H^\sigma_{P^\vee,M}} \varepsilon(a) f(a)\overline{g(a)}$$

and determines the finite-dimensional Hilbert spaces of complex valued functions $\mathcal{H}^\sigma_{P^\vee,M}$. The first orthonormal basis $|a\rangle$, $a \in H^\sigma_{P^\vee,M}$ of $\mathcal{H}^\sigma_{P^\vee,M}$ is formed by the functions

$$|a\rangle a' = \varepsilon^{-\frac{1}{2}}(a)\,\delta_{a,a'}, \quad a' \in H^\sigma_{P^\vee,M}.$$

The second orthonormal basis $|\lambda\rangle^{\sigma,\pm}$, $\lambda \in L^\sigma_{P,M}$ is formed by the functions

$$|\lambda\rangle^{\sigma,\pm} a' = (12M^2 h_M(\lambda)\mu^\pm(\lambda))^{-\frac{1}{2}}\varphi_\lambda^{\sigma,\pm}(a'), \quad a' \in H^\sigma_{P^\vee,M}.$$

The A_2 triangular armchair graphene electron propagation model is illustrated in Fig. 2. The points from the sets $H^\sigma_{P^\vee,M}$ represent atoms of the graphene dot with armchair boundaries and the set of possible positions of an electron. Jumps of the electron between the nearest atoms are possible with the amplitude $iA/\hbar$ per unit time. Each base state $|a\rangle$ represents the electron positioned at the atom $a \in H^\sigma_{P^\vee,M}$. For the identity homomorphism $\sigma = \mathbb{1}$, the boundaries of F represent ideal mirrors and for $\sigma = \sigma^e$ the boundaries of F represent ideal barriers. The refined orbits of the fundamental weights $\omega_1^\vee$ and $\omega_2^\vee$ are given as

$$O(\omega_1^\vee, M) = \frac{1}{M}\{\omega_1^\vee, -\omega_1^\vee + \omega_2^\vee, -\omega_2^\vee\},$$
$$O(\omega_2^\vee, M) = \frac{1}{M}\{\omega_2^\vee, \omega_1^\vee - \omega_2^\vee, -\omega_1^\vee\}.$$

For any $a \in \mathbb{R}^n$ there exists $a' \in F$ and $w^{\mathrm{aff}}(a) \in W^{\mathrm{aff}}$ such that $a = w^{\mathrm{aff}}(a)a'$. The function $\chi^\sigma : \mathbb{R}^n \to \{-1, 0, 1\}$ is given by

$$\chi^\sigma(a) = \begin{cases} \sigma \circ \psi \left(w^{\mathrm{aff}}(a)\right), & \sigma \circ \psi \left(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)\right) = \{1\}, \\ 0, & \sigma \circ \psi \left(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)\right) = \{\pm 1\}. \end{cases}$$

The matrix elements of the Hamiltonian $\widehat{H}^\sigma_{P^\vee,M}$, $a \in H^{(\sigma,k)}_{P^\vee,M}$ are determined via relation

$$\left\langle a \left| \widehat{H}^\sigma_{P^\vee,M} \right| a' \right\rangle = -A\, \varepsilon^{\frac{1}{2}}(a)\varepsilon^{-\frac{1}{2}}(a') \sum_{\nu \in W^{\mathrm{aff}} a' \cap (a + O(\omega_k^\vee, M))} \chi^\sigma(\nu).$$

The Schrödinger equation admits the stationary solutions $|\lambda\rangle^{\sigma,\pm}$, $\lambda \in L^\sigma_{P,M}$, i.e.

$$\widehat{H}^\sigma_{P^\vee,M} |\lambda\rangle^{\sigma,\pm} = E^{\sigma,\pm}_{P^\vee,M}(\lambda)\, |\lambda\rangle^{\sigma,\pm},$$

with the eigenenergies $E^{\sigma,\pm}_{P^\vee,M}(\lambda)$ determined as

$$E^{\sigma,\pm}_{P^\vee,M}(\lambda) = \pm\frac{1}{2} A \left| \varphi^{\mathbb{1}}_\lambda \left(\frac{\omega_1^\vee}{M} \right) \right|, \quad \lambda \in L^\sigma_{P,M}.$$

The case of the triangular graphene dot surrounded by barriers, $\sigma = \sigma^e$, is detailed in a different approach already in [8] and yields similar stationary solutions and the same eigenenergies.

Acknowledgements The authors acknowledge support from the Czech Science Foundation (GAČR), Grant No. 19-19535S.

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A Construction of $(\mathfrak{g}, K)$-modules over Commutative Rings

Takuma Hayashi

Abstract This manuscript is a summary of the author's work [6] and [7] on $(\mathfrak{g}, K)$-modules over commutative rings.

1 Introduction

A $(\mathfrak{g}, K)$-module over the field $\mathbb{C}$ of complex numbers is a complex vector space, equipped with compatible actions of a complex Lie algebra $\mathfrak{g}$ and a complex algebraic group K, which is an algebraic model of a representation of a real reductive Lie group. For example, let G be a real reductive Lie group. Let $\mathfrak{g}$ (resp. K) be the complexification of the Lie algebra (resp. a maximal compact subgroup) of G. Then $\mathfrak{g}$ and K form a Harish-Chandra pair $(\mathfrak{g}, K)$ over $\mathbb{C}$. The subspace of K-finite vectors of an admissible Hilbert representation of G is a $(\mathfrak{g}, K)$-module in a natural way.

In representation theory of real reductive Lie groups, cohomological induction is one of the most important ways to construct $(\mathfrak{g}, K)$-modules. To explain it, we start with induction of representations of finite groups. Let G be a finite group, H be a subgroup of G, and V be a representation of H over $\mathbb{C}$. Then the induction $\mathrm{Ind}_H^G V$ is the space of H-invariant V-valued functions on G. The point is that $\mathrm{Ind}_H^G V$ satisfies the adjointness property $\mathrm{Hom}_H(-, V) \cong \mathrm{Hom}_G(-, \mathrm{Ind}_H^G V)$. Similarly, for a morphism $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ of Harish-Chandra pairs, the forgetful functor from the category of $(\mathfrak{g}, K)$-modules to that of $(\mathfrak{q}, M)$-modules has a right adjoint functor $I_{\mathfrak{q},M}^{\mathfrak{g},K}$. Unlike the finite group setting, we have a nontrivial right derived functor $\mathbb{R}I_{\mathfrak{q},M}^{\mathfrak{g},K}$. Roughly speaking, $\mathbb{R}I_{\mathfrak{q},M}^{\mathfrak{g},K}$ is called cohomological induction. The important fact is that cohomological induction produces the $A_{\mathfrak{q}}(\lambda)$-modules which form a special class of irreducible unitary $(\mathfrak{g}, K)$-modules. See [12] for the precise definitions and details.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_32

During the 2010s works on two new directions in the theory of $(\mathfrak{g}, K)$-modules over commutative rings appeared. For applications to rationality (integrality) of special values of automorphic L-functions and Rankin-Selberg L-functions, M. Harris, G. Harder, and F. Januszewski work over number fields and localization of their rings of integers (see [3, 4, 9–11]). Motivated by mathematical physics, J. Bernstein, N. Higson, and E. Subag introduced contraction families as Harish-Chandra pairs over the polynomial ring $\mathbb{C}[z]$ (or the complex projective line $\mathbb{P}^1$) in [1] and [2]. As an application, Subag (resp. Higson and Subag) explained how the hidden symmetry of the Schrödinger equation of dimension 2 (resp. 3) can be obtained from the contraction family in [13] (resp. [8]). More primitively, to work over polynomial rings $\mathbb{C}[z_1, z_2, \cdots, z_n]$ should give a clear insight into generic properties of representations with continuous parameters like principal series representations.

In [6] and [7], the author studied general theory of $(\mathfrak{g}, K)$-modules over commutative rings. He introduced basic definitions, and constructed the functor $I^{\mathfrak{g},K}_{\mathfrak{q},M}$ in his general setting. Explicitly calculating the modules that are obtained via $\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}$ is a nontrivial task. In [6], the author proved that the ith derived functor $R^i I^{\mathfrak{g},K}_{\mathfrak{q},M}$ commutes with flat base change functors under certain conditions which should be satisfied in practice. This is called the flat base change theorem. This result is helpful when we study generic properties of families of $(\mathfrak{g}, K)$-modules. In this manuscript, we outline them.

2 Basic Definitions

Fix a ground commutative ring k. In principle, we can define the notions of Harish-Chandra pairs and $(\mathfrak{g}, K)$-modules over k in a similar way to the complex case. However, we have technical remarks on working over commutative rings. Let K be an affine group scheme over k. We denote the category of representations of K by K-mod. Then K-mod does not have nice properties in general. For example, kernels of K-mod are not necessarily the kernels in the category of k-modules unless K is flat over k. Another issue is about the definition of the adjoint representation. This happens since the normal k-module might have a bad behavior with respect to base changes. See [7] Example 2.1.4. We can avoid it when K is flat and finitely presented over k. To describe a more general condition, let K be a flat affine group scheme over k. Let $\omega_{K/k} = I_e/I_e^2$ be the conormal k-module along the unit, where I_e is the kernel of the counit of the coordinate ring of K. Recall that the Lie algebra of K is the dual of $\omega_{K/k}$ as a k-module. For a flat affine group scheme, we denote its Lie algebra by the corresponding small German letter.

Definition 1 **([6] Condition 1.1.4, Condition 1.1.6).** We say that a flat affine group scheme K over k satisfies (♡) if K enjoys the following conditions:

1. For every flat commutative k-algebra R, the canonical homomorphism

$$\mathrm{Hom}_k(\omega_{K/k}, k) \otimes_k R \to \mathrm{Hom}_k(\omega_{K/k}, R)$$

is an isomorphism.

2. For every k-module W and every flat commutative k-algebra R , the canonical homomorphism

$$\mathrm{Hom}_k(\mathfrak{k}, W) \otimes_k R \to \mathrm{Hom}_k(\mathfrak{k}, W \otimes_k R)$$

is an isomorphism.

If a flat affine group scheme K satisfies condition 1, we can define the adjoint representation on $\mathfrak{k}$ (see [7] Sect. 2.1 and [6] Lemma 2.2.1). Condition 2 is helpful for the proof of Theorem 1 3. In the rest, we put the adjoint action on $\mathfrak{k}$ unless otherwise noted.

For the action map ϕ of a representation of K, we denote its differential by $d\phi$. For a homomorphism $f : K \to L$ of flat affine group schemes, let $df : \mathfrak{k} \to \mathfrak{l}$ denote the differential of f.

Definition 2 ([6] **Sect. 1.1,** [7] **Sect. 2.1).**

1. A Harish-Chandra pair is a pair of a flat affine group scheme K over k satisfying (♡) and a Lie algebra over k, equipped with an action ϕ of K on $\mathfrak{g}$ preserving the Lie bracket and a K-equivariant Lie algebra homomorphism $\psi : \mathfrak{k} \to \mathfrak{g}$ such that $d\phi(\xi) = [\psi(\xi), -]$ for every $\xi \in \mathfrak{k}$.
2. A map $(\mathfrak{g}, K, \phi, \psi) \to (\mathfrak{h}, L, \phi', \psi')$ of Harish-Chandra pairs consists of a group scheme homomorphism $f_k : K \to L$ and a K-equivariant Lie algebra homomorphism $f_a : \mathfrak{g} \to \mathfrak{h}$ such that $f_a \circ \psi = \psi' \circ df_k$, where $\mathfrak{h}$ is regarded as a representation of K for the restriction of the action along f_k.

Example 1 ([1, 2]). Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair over $\mathbb{C}$, equipped with a K-equivariant involution θ of $\mathfrak{g}$. Let $\mathfrak{g}^{\theta=1}$ (resp. $\mathfrak{g}^{\theta=-1}$) denote the eigenspace of θ with eigenvalue 1 (resp. -1). Assume that the Lie algebra $\mathfrak{k}$ of K is contained in $\mathfrak{g}^{\theta=1}$. Then $\boldsymbol{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z]$ is a Lie algebra over $\mathbb{C}[z]$ for the bracket summand wisely defined by

$$\left[\eta z^m, \xi z^n\right] = \begin{cases} [\eta, \xi]\, z^{m+n+1} & (\eta, \xi \in \mathfrak{g}^{\theta=-1}) \\ [\eta, \xi]\, z^{m+n} & (\text{otherwise}). \end{cases}$$

Then $\boldsymbol{\mathfrak{g}}$ and $K \otimes \mathbb{C}[z]$ form a Harish-Chandra pair over $\mathbb{C}[z]$ which is called the contraction family.

Definition 3 ([6] **Sect. 1.1,** [7] **Sect. 2.1).** For a Harish-Chandra pair $(\mathfrak{g}, K)$ over k, a $(\mathfrak{g}, K)$-module is a representation (V, ν) of K, equipped with a K-equivariant action π of $\mathfrak{g}$ such that $d\nu = \pi \circ \psi$. We denote the category of $(\mathfrak{g}, K)$-modules by $(\mathfrak{g}, K)$ -mod.

Let $f = (f_a, f_k) : (\mathfrak{q}, M) \to (\mathfrak{g}, K)$ be a map of Harish-Chandra pairs, and (V, π_2, ν_2) be a $(\mathfrak{g}, K)$-module. Then it is easy to see that V is a $(\mathfrak{q}, M)$-modules for $\pi_1 = \pi_2 \circ f_a$ and $\nu_1 = \nu_2 \circ f_k$. This determines a functor

$$\mathcal{F}_{\mathfrak{g},K}^{\mathfrak{q},M} : (\mathfrak{g}, K)\text{-mod} \to (\mathfrak{q}, M)\text{-mod}\,.$$

Theorem 1 ([6] **Lemma 1.1.9,** [7] **Theorem 1.2.2, Corollary 2.2.11).**

1. *The category* $(\mathfrak{g}, K)$ *-mod is a Grothendieck abelian category whose small colimits and finite limits are computed in the category of* k*-modules.*
2. *The functor* $\mathcal{F}^{\mathfrak{q},M}_{\mathfrak{g},K}$ *admits a right adjoint functor* $I^{\mathfrak{g},K}_{\mathfrak{q},M}$.
3. *If* $K = M$ *and* f_a *is the identity map, the functor* $\mathcal{F}^{\mathfrak{q},M}_{\mathfrak{g},M}$ *admits a left adjoint functor* $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{q}}$.

For the proof, we replace $\mathfrak{g}$ by a k-algebra $\mathcal{A}$ through $\mathcal{A} = U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. We then think of similar assertions for weak $(\mathcal{A}, K)$-modules which are defined by dropping the condition $d\nu = \pi \circ \psi$. They formally follow from general theory of monoidal categories since the notion of weak $(\mathcal{A}, K)$-modules and all statements can be described in terms of the closed symmetric monoidal category of representations of K (and that of M). Finally, we deduce Theorem 1 by showing that the canonical embedding of $(\mathcal{A}, K)$ -mod to the category of weak $(\mathcal{A}, K)$-modules has left and right adjoint functors.

3 Flat Base Change Theorems

In this section, fix a Noetherian commutative ring k.

Theorem 2 ([6] **Theorem D).** *Let* k' *be a flat* k*-algebra, and* $(\mathfrak{q}, M) \to (\mathfrak{g}, K)$ *be a map of Harish-Chandra pairs over* k*. Suppose that the following conditions are satisfied:*

1. *The summation map* $\mathfrak{k} \oplus \mathfrak{q} \to \mathfrak{g}$ *is surjective.*
2. $\mathfrak{k}$ *and* $\mathfrak{q}$ *are finitely generated as* k*-modules.*

Then there is a canonical isomorphism

$$(\mathbb{R}I^{\mathfrak{g},K}_{\mathfrak{q},M}-) \otimes_k k' \simeq \mathbb{R}I^{\mathfrak{g}\otimes_k k', K\otimes_k k'}_{\mathfrak{q}\otimes_k k', M\otimes_k k'}(- \otimes_k k')$$

on the derived category $D^+(\mathfrak{q}, M)$ *of cochain complexes bounded below of* $(\mathfrak{q}, M)$*-modules.*

Theorem 3 ([6] **Proposition 4.1.3).** *Let* $(f_a, f_k) : (\mathfrak{q}, M) \to (\mathfrak{g}, M)$ *be a map of pairs over* k*, and* Z *be a* $(\mathfrak{q}, M)$*-module. Suppose that the following conditions are satisfied:*

1. *The map* f_a *is injective, and* f_k *is the identity map.*
2. *For* $x \in \mathfrak{g}$*, we have* $[x, x] = 0$*. This holds if 2 is a unit of* k*.*
3. *There is an* M*-invariant Lie subalgebra* $\mathfrak{u}^- \subset \mathfrak{g}$ *such that the summation map* $\mathfrak{q} \oplus \mathfrak{u}^- \to \mathfrak{g}$ *is an isomorphism of* k*-modules.*
4. *There are free bases of* $\mathfrak{q}$ *and* $\mathfrak{u}^-$.

5. *Regard* $U(\mathfrak{u}^-)$ *as a representation of* M *in the canonical way. Then there exist finitely generated subrepresentations* $U(\mathfrak{u}^-)_{\mathcal{O}} \subset U(\mathfrak{u}^-)$ *such that* $U(\mathfrak{u}^-) = \oplus_{\mathcal{O}} U(\mathfrak{u}^-)_{\mathcal{O}}$.
6. *For any finitely generated* M*-module* Q*,* $\mathrm{Hom}_M(Q, \mathrm{Hom}(U(\mathfrak{u}^-)_{\mathcal{O}}, Z))$ *vanishes for all but finitely many* $\mathcal{O}$.

Then we have an isomorphism as an M*-module*

$$\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(Z) \cong \oplus_{\mathcal{O}} \mathrm{Hom}_k(U(\mathfrak{u}^-)_{\mathcal{O}}, Z).$$

In particular, the base change formula $\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}(Z) \otimes k' \cong \mathrm{pro}_{\mathfrak{q}\otimes k'}^{\mathfrak{g}\otimes k'}(Z \otimes k')$ *along a ring homomorphism* $k \to k'$ *between Noetherian rings holds if either*

1. *the homomorphism* $k \to k'$ *is flat, or*
2. *for any finitely generated* $M \otimes k'$*-module* Q*,* $\mathrm{Hom}_{M\otimes k'}(Q, \mathrm{Hom}(U(\mathfrak{u}^-)_{\mathcal{O}} \otimes k', Z \otimes k'))$ *vanishes for all but finitely many* $\mathcal{O}$.

Theorem 2 follows from the flat base change theorem of Hom which is essentially reduced to the following elementary fact: for a finitely presented k-module V, a k-module W, and a flat k-algebra k', there is a canonical isomorphism $\mathrm{Hom}_k(V, W) \otimes_k k' \cong \mathrm{Hom}_{k'}(V \otimes_k k', W \otimes_k k')$. The basic idea for Theorem 3 is that every finitely generated representation of M contributes to only finitely many entries in the product $\prod_{\mathcal{O}} \mathrm{Hom}_k(U(\mathfrak{u}^-)_{\mathcal{O}}, Z)$ as a representation of M by conditions 5 and 6.

4 Contraction Analog of Cohomological Induction

We end this manuscript with an application of the above results to the contraction setting. See [6] Example 1.2.3 and Theorem H for applications to integral models of $(\mathfrak{g}, K)$-modules over $\mathbb{C}$. Let $(\mathfrak{g}, K)$ be a reductive pair in the sense of [12] Definition 4.30, and θ be its Cartan involution. We regard $(\mathfrak{g}, K)$ as a Harish-Chandra pair over $\mathbb{C}$ in our sense by identifying K with its complexification. Let $(\mathfrak{q}, K_L)$ be a θ-stable parabolic subpair in the sense of [12]. Remark that we implicitly assume that $\mathfrak{q}$ contains a θ-stable real Cartan subalgebra. We denote the Levi part (resp. nilradical) of $\mathfrak{q}$ by $\mathfrak{l}$ (resp. $\mathfrak{u}$). Notice that $(\mathfrak{l}, K_L) \subset (\mathfrak{q}, K_L)$ are stable under θ in $(\mathfrak{g}, K)$. Since $\mathfrak{q}$ is a θ-stable parabolic subalgebra, $\mathfrak{u}$ is stable under θ in $\mathfrak{g}$. Therefore we obtain maps of contraction families

$$(\mathfrak{l}, K_L \otimes \mathbb{C}[z]) \leftarrow (\mathfrak{q}, K_L \otimes \mathbb{C}[z]) \to (\mathfrak{g}, K \otimes \mathbb{C}[z]).$$

We call $\mathbb{R}I_{\mathfrak{q},K_L\otimes\mathbb{C}[z]}^{\mathfrak{g},K\otimes\mathbb{C}[z]} \mathcal{F}_{\mathfrak{l},K_L\otimes\mathbb{C}[z]}^{\mathfrak{q},K_L\otimes\mathbb{C}[z]}(- \otimes_{\mathbb{C}[z]} \wedge^{\dim \mathfrak{u}} \mathfrak{u})$ the contraction analog of cohomological induction. Note that $\mathrm{pro}_{\mathfrak{q}}^{\mathfrak{g}}$ is exact by [5]. Let Z be a torsion-free $(\mathfrak{l}, K_L \otimes \mathbb{C}[z])$-module. Fix a fundamental Cartan subalgebra of $\mathfrak{l}$, and let $h_{\rho(\mathfrak{u})}$ be as in [12] Proposition 4.70. Suppose that the element $h_{\rho(\mathfrak{u})}$ acts on Z by a scalar.

If $Z \otimes \mathbb{C}(z)$ is admissible, cohomological induction enjoys the flat base change formula to the algebraic closure $\overline{\mathbb{C}(z)}$ of the field of rational functions $\mathbb{C}(z)$

$$\mathbb{R}I^{\mathfrak{g}, K\otimes\mathbb{C}[z]}_{\mathfrak{q}, K_L\otimes\mathbb{C}[z]}(\mathcal{F}^{\mathfrak{q}, K_L\otimes\mathbb{C}[z]}_{\mathfrak{l}, K_L\otimes\mathbb{C}[z]}(Z \otimes_{\mathbb{C}[z]} \wedge^{\dim \mathfrak{u}} \mathfrak{u})) \otimes \overline{\mathbb{C}(z)}$$
$$\cong \mathbb{R}I^{\mathfrak{g}\otimes\overline{\mathbb{C}(z)}, K\otimes\overline{\mathbb{C}(z)}}_{\mathfrak{q}\otimes\overline{\mathbb{C}(z)}, K_L\otimes\overline{\mathbb{C}(z)}}\mathcal{F}^{\mathfrak{q}\otimes\overline{\mathbb{C}(z)}, K_L\otimes\overline{\mathbb{C}(z)}}_{\mathfrak{l}\otimes\overline{\mathbb{C}(z)}, K_L\otimes\overline{\mathbb{C}(z)}}(Z \otimes \wedge^{\dim \mathfrak{u}} \mathfrak{u} \otimes \overline{\mathbb{C}(z)}).$$

Acknowledgements The author encountered many great researchers and had stimulating discussions with them during this conference. He is grateful to the organizers for giving him this opportunity. He also thanks many participants for helping him during this conference.

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Lie Groups Actions on Non Orientable Klein Surfaces

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Abstract It is known that any Lie Group is an orientable manifold. Thus, a non-orientable surface cannot have a structure of Lie group. However, actions of Lie groups on non-orientable Klein surfaces exist. We deal in this paper with such actions.

1 Introduction

We dealt in [2] with Lie groups actions on the Möbius strip. The rationals of such a study resided in the fact that non orientable surfaces became lately an important topic in surface topology due to the numerous applications they have found in some fields of science like quantum physics, chemistry and biology. At every instance their presence was related to some "energetic reasons" implying the existence of potential functions related to them. It is known that a minimal condition for a surface to support harmonic functions is that the respective surface has a dianalytic structure. Abstract surfaces with this property are known as Klein surfaces. The category of Klein surfaces includes that of (bordered and border free) Riemann surfaces. Klein surfaces can be orientable, as well as non orientable. Any Klein surface can be obtained factorizing a symmetric Riemann surface, i.e. a Riemann surface endowed with an antianalytic involution k, by the two element group $< k >$ generated by k. The dianalytic structure obtained in this way is, in particular, real analytic and therefore it make sense to consider analytic actions of Lie groups on Klein surfaces. The case of Möbius strip we treated in [2] is just a particular example, yet it can be used as a guide in dealing with more general situations. This gives us the motivation to reiterate those results as an introduction to a more general theory. We also believe that it is useful to present some basic notions on Lie groups we will use in the next paragraphs.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_33

2 Basic Notions on Lie Groups

We introduce in this section some basic notions and results on Lie groups which will be used in what follows. The proofs of the affirmations made here can be found for example in [3, 4] and [5].

Definition 1. A *Lie group* over $\mathbb{R}$ is a group G endowed with a structure of C^∞ differentiable manifold over $\mathbb{R}$ such that the group operation $(x, y) \to xy$ from $G \times G$ to G and the mapping $x \to x^{-1}$ from G to G are C^∞ mappings.

If G_1 and G_2 are two Lie groups, a mapping $f : G_1 \to G_2$ is called a *Lie group homomorphism* if it is simultaneously an isomorphism of abstract groups and a diffeomorphism of manifolds.

Definition 2. A *left action* of a Lie group G on a differentiable manifold X is a C^∞ mapping

$$\alpha : G \times X \to X \quad \text{such that} \quad :$$

(i) If e is the identity of G, then $\alpha(e, x) = x$ for every $x \in X$
(ii) For every $x \in X$ and every couple $g_1, g_2 \in G$ we have $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$

Right actions are mappings $(x, g) \to \alpha(g^{-1}, x)$ where α is a left action of G on X. For every Lie group G the mappings $L(g, x) = gx$ and $R(g, x) = xg^{-1}$, $g, x \in G$, represent left (respectively right) actions of the group on itself. They are called *actions by left (and right) translations*.

A left action α of G on X induces a homomorphism $t_\alpha : g \to \alpha(g)$ from the group G to the group *DiffX* of diffeomorphisms of X such that $(g, x) \to \alpha(g)x$ is of class C^∞. If t_α is injective we say that the action is *effective*. If $gx = x$ for some $x \in X$ implies $g = e$, then we say that the action is *free*. If there is $x \in X$ such that $\{\alpha(g, x) \mid g \in G\} = X$ we say that α is *transitive*.

A discrete Lie group is a countable set with the discrete topology endowed with group operation. For a discrete group Γ to verify that $\alpha : \Gamma \times X \to X$ is a Lie group action it is sufficient to show that α_γ is a diffeomorphism for every $\gamma \in \Gamma$.

Definition 3. A discrete group Γ is said to act *properly discontinuously* on a manifold X if the action is of class C^∞ and:

(i) Every $x \in X$ has a neighborhood U such that the set $\{\gamma \in \Gamma \mid \gamma U \cap U \neq \varnothing\}$ is finite
(ii) If $x_1, x_2 \in X$ are such that for every $\gamma \in \Gamma$, $x_2 \neq \gamma x_1$, then there are neighborhoods U_1 and U_2 of x_1 and x_2 such that $U_1 \cap \Gamma U_2 = \varnothing$.

It is known that any discrete subgroup of a Lie group G acts freely and properly discontinuously on G by left (and by right) translations.

3 A Group of Bi-Möbius Transformations

Let us deal with the following function defined on $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$

$$f(z_1, z_2) = \frac{Az_1z_2 + a(1 - z_1 - z_2)}{a(z_1z_2 - z_1 - z_2) + A}, z_1, z_2 \in \mathbb{C}, a \in \mathbb{C}\backslash\{0, 1\}, A = a^2 - a + 1, \quad (1)$$

$$f(z, \infty) = f(\infty, z) = \frac{(a - 1 + 1/a)z - 1}{z - 1}, \qquad f(\infty, \infty) = a - 1 + 1/a$$

Theorem 1. The function $f(z_1, z_2)$ satisfies the following relations:

(a) $f(z_1, z_2) = f(z_2, z_1)$ for every $z_1, z_2 \in \overline{\mathbb{C}}$

(b) $z_1 \to f(z_1, z_2)$ and $z_2 \to f(z_2, z_1)$ are Möbius transformations for every $z_2 \in \overline{\mathbb{C}}\backslash\{a, 1/a\}$, respectively $z_1 \in \overline{\mathbb{C}}\backslash\{a, 1/a\}$.

For this reason we call $f(z_1, z_2)$ a *bi-Möbius transformation.*

(c) $f(z_1, 1) = z_1, \qquad f(1, z_2) = z_2$ for every $z_1 \in \overline{\mathbb{C}}$, respectively every $z_2 \in \overline{\mathbb{C}}$

(d) $f(z, \frac{1}{z}) = 1$ for every $z \in \overline{\mathbb{C}}$

(e) $f(\frac{1}{z_1}, \frac{1}{z_2}) = \frac{1}{f(z_1, z_2)}$ for every $z_1, z_2 \in \overline{\mathbb{C}}$

(f) $f(z_1, f(z_2, z_3)) = f(f(z_1, z_2), z_3)$ for every $z_1, z_2, z_3 \in \overline{\mathbb{C}}$

(g) $f(z, a) = f(a, z) = a$ and $f(z, 1/a) = f(1/a, z) = 1/a$ for every $z \in \overline{\mathbb{C}}$

Proof. The proof requires only elementary computation, although for (f) it may look quite tedious □.

It results that the composition law $z_1 \circ z_2 = f(z_1, z_2)$ defines a structure of Abelian group on $\overline{\mathbb{C}}$ with the unit element 1 and such that the inverse element of every $z \in \overline{\mathbb{C}}$ is $1/z$. Moreover:

Theorem 2. If for a $z \in \overline{\mathbb{C}}$ we define $g_z(w) = f(z, w)$ then $G = \{g_z \mid z \in \overline{\mathbb{C}}\backslash\{a, 1/a\}\}$ with the composition law $g_z \times g_\zeta = g_{z \circ \zeta}$ is a group of Möbius transformations.

Proof. The formula (1) implies:

$$g_z(w) = \frac{(Az - a)w - a(z - 1)}{a(z - 1)w + A - az} \tag{2}$$

and $(Az - a)(A - az) + a^2(z - 1)^2 \neq 0$ as long as $z \neq a$, and $z \neq 1/a$, hence g_z is a Möbius transformation, as stated in Theorem 1 (b).

Moreover, by Theorem 2 we have $g_z(w) \times g_1(w) = g_{z\circ 1}(w) = g_z(w)$, therefore the unit of this composition law is $e = g_1$. Also $g_z(w) \times g_{1/z}(w) = g_{z\circ\frac{1}{z}}(w) = g_1(w) = e$, i.e. every element g_z has an inverse, which is $g_{1/z}$. We have as well $g_z(w) \times g_\zeta(w) = g_{z\circ\zeta}(w) = g_{\zeta\circ z}(w) = g_\zeta(w) \times g_z(w)$ and finally, $[g_z(w) \times g_\zeta(w)] \times g_\eta(w) = g_{z\circ\zeta}(w) \times g_\eta(w) = g_{(z\circ\zeta)\circ\eta}(w) = g_{z\circ(\zeta\circ\eta)}(w) = g_z(w) \times g_{\zeta\circ\eta}(w) = g_z(w) \times [g_\zeta(w) \times g_\eta(w)]$, thus G with this composition law is an Abelian group □.

It can be easily checked that the mapping $z \to g_z$ of $\overline{\mathbb{C}}\setminus\{a, 1/a\}$ into G is bijective, hence it confers to G a Lie group structure.

The actions by left and right translations of G on itself are defined as:

$L(g_z, g_\zeta) = g_{z\circ\zeta}$, respectively $R(g_z, g_\zeta) = g_{\zeta\circ z^{-1}}$.

Theorem 1 implies that G acts freely and transitively on itself by left and by right translations.

Let $z \in \overline{\mathbb{C}}$ be arbitrary and for every $n \in \mathbb{Z}$ we denote

$$z^{(n+1)} = z^{(n)} \circ z, \text{ where } z^{(0)} = 1. \tag{3}$$

It is obvious that for every $n, m \in \mathbb{Z}$ we have $z^{(n)} \circ z^{(m)} = z^{(n+m)}$, therefore

$g_{z^{(n)}} \times g_{z^{(m)}} = g_{z^{(n+m)}}$

In particular $g_{z^{(n)}} \times g_{z^{(-n)}} = g_{z^{(0)}} = g_1 = e$, hence the group $< g_z >$ generated by g_z is a subgroup of G.

Theorem 3. For every $z \in \overline{\mathbb{C}} \setminus\{a, 1/a\}$ the group $< g_z >$ is a discrete subgroup of G.

Proof. Let us notice that $g_z(w) = w$ if and only if $w = a$ or $w = 1/a$. In particular if $z \notin \{a, 1/a\}$ an easy induction argument shows that $z^{(n+1)} = g_z(z^n) \neq z^{(n)}$. By repeating this argument we find that for every $m, n \in \mathbb{Z}$, $z^{(m)} \neq z^{(n)}$, hence $g_{z^{(m)}} \neq g_{z^{(n)}}$. In other words the terms of the sequence $\{g_{z^{(n)}}\}$ are all distinct. Suppose that for a given $z \notin \{a, 1/a\}$ we have $\lim_{n\to\infty} z^{(n)} = z_0$. Then $z_0 = \lim_{n\to\infty} z^{(n+1)} = \lim_{n\to\infty} g_{z^{(n)}}(z) = g_{z_0}(z) \neq z_0$, which is a contradiction. Thus the set $< g_z >$ has no cluster point □.

Corollary. For every $z \in \overline{\mathbb{C}} \setminus\{a, 1/a\}$, $< g_z >$ acts freely and properly discontinuously on G by left (and by right) translations.

4 Dianalytic Structures on Möbius Strip and on the Real Projective Plane

It is known (see [2] that any annulus $A = \{z \in \mathbb{C} | 0 < R_1 < |z| < R_2\}$ is conformal equivalent with $A_R = \{z \in \mathbb{C} | 1/R < |z| < R\}$, where $R = \sqrt{R_1/R_2}$. The antianalytic involution $k(z) = -1/\overline{z}$ maps A_R onto itself and ds defined by

$$ds(z) = \frac{1}{2}(1 + \frac{1}{|z|^2})|dz| \tag{4}$$

is a conformal metric on A_R, which is symmetric with respect to k, i. e. $ds(z) = ds(k(z))$ for every $z \in A_R$. It is also known (see [1]) that every conformal metric on a oriented surface induces a conformal structure on that surface. The annulus A_R can be considered as a bordered Riemann surface with the conformal structure induced by the conformal metric ds. A Riemann surface S endowed with an antianalytic involution k is called symmetric Riemann surface and if $< k >$ is the two element group generated by k, then $S/ < k >$ has a dianalytic structure , i. e. a structure such that any transition mapping is either conformal or has a conformal complex conjugate on every connected component of its domain. Such a surface is called Klein surface. Thus

$$M_R = A_R / < k > \tag{5}$$

which is the Möbius strip, endowed with the dianalytic structure induced by ds is a non orientable bordered Klein surface.

Similarly, ds induces on $\overline{\mathbb{C}}/ < k >$ a dianalytic structure and the non orientable Klein surface obtained is the real projective plane.

For every $z_k \in A_R$, $z_k = \rho_k e^{i\theta_k}$, $1/R < \rho_k < R$, $k = 1, 2$, let us define

$$z_1 * z_2 = f(\rho_1, \rho_2) e^{i(\theta_1 + \theta_2)} \tag{6}$$

where f is the function defined by (1) with $a = R > 1$. Then it can be easily checked that $1/R < f(\rho_1, \rho_2) < R$ and therefore $z_1 * z_2 \in A_R$.

Theorem 4. The operation $*$ defines a structure of Abelian group on A_R.

Proof. Since $f(\rho_1, \rho_2) = f(\rho_2, \rho_1)$ we have that $z_1 * z_2 = z_2 * z_1$. Moreover, for $z = \rho e^{i\theta} \in A_R$ we have by Theorem 1 (c) that $z * 1 = f(\rho, 1)e^{i\theta} = \rho e^{i\theta} = z$, hence $1 \in A_R$. is the unit of this operation. Then, if $z = \rho e^{i\theta} \in A_R$, we have by Theorem 1 (d) that $z * \frac{1}{z} = f(\rho, 1/\rho) = 1$, hence $1/z$ is the inverse element of z. Finally, for $z_k = \rho_k e^{i\theta_k} \in A_R$, $k = 1, 2, 3$ Theorem 1 (f) implies $z_1 * (z_2 * z_3) = \rho_1 e^{i\theta_1} * [f(\rho_2, \rho_3) e^{i(\theta_2 + \theta_3)}] = f(\rho_1, f(\rho_2, \rho_3)) e^{i(\theta_1 + (\theta_2 + \theta_3))} = f(f(\rho_1, \rho_2), \rho_3) e^{i(\theta_1 + \theta_2) + \theta_3)}$ $= (z_1 * z_2) * z_3$ □

Let us denote by $G_1 = \{g_z \mid z \in A_R\}$, where $g_z(w) = z * w$ for every $z \in A_R$.

Theorem 5. G_1 is a Lie group acting freely and transitively on itself by left (and by right) translations.

Proof. Indeed A_R with the topology induced by the complex plane is a C^∞ manifold and the mapping $z \to g_z$ from A_R to G_1 is bijective since it is obviously surjective and $g_{z_1} = g_{z_2}$ implies $z_1 * w = z_2 * w$ for every $w \in A_R$, hence $z_1^{-1} * (z_1 * w) = z_1^{-1} * (z_2 * w)$, i.e. $(z_1^{-1} * z_1) * w = (z_1^{-1} * z_2) * w$, or $w = (z_1^{-1} * z_2) * w$, which means $(z_1^{-1} * z_2) = 1$ and finally $z_1 = z_2$. Then, by Theorem 4, G_1 is a Lie group. Its action on itself by left translations defined by $L(g_{z_1}, g_{z_2}) = g_{z_1 * z_2}$ is free, since $g_{z*w} = g_w$ for a $z \in A_R$ implies $z = 1$, i.e. $g_z = e$. It is also transitive since $G_1 g_z = G_1$ for every $z \in A_R$.The case of right translations action can be treated similarly.□

Given $z \in A_R$ let us denote $z^{[0]} = 1$ and for every integer n, $z^{[n+1]} = z^{[n]} * z$. Then, by Theorem 4, $[g_z] = \{g_{z^{[n]}} \mid n \in \mathbb{Z}\}$, where $g_{z^{[n]}}(w) = z^{[n]} * w$, $w \in A_R$ is a discrete subgroup of G_1. Indeed, by Theorem 2, this subgroup has no cluster point in A_R.

Theorem 6. For every $z \in A_R$, $z \neq 1$, the group $[g_z]$ acts freely and properly discontinuously on G_1 by left (and by right) translations.

Proof. Since G_1 acts freely on itself by left (and right) translations, so does $[g_z]$. Being discrete, $[g_z]$ acts properly discontinuously on G_1 by left (and right) translations. If $z = e^{i\theta}$, then g_z are rotations of A_R, and $[g_z]$ is still a discrete subgroup of G_1. When θ is congruent to 0 modulo an integer divisor of 2π this subgroup is a cyclic one.□

5 Lie Groups Acting on the Möbius Strip

The elements $\widetilde{z} \in M_R$ are defined as couples $\{z, k(z)\}$, $z \in A_R$. Let $H = \{z \in A_R \mid |z| > 1$, or $|z| = 1$ and $0 \leq \arg z < \pi\}$ and $K = A_R \setminus H$. Then $z \in H$ if and only if $k(z) \in K$. We make the convention to always denote by z that element of the couple $\{z, k(z)\}$ which belongs to H. Then we can define an internal operation on M_R as follows. Given $\widetilde{z}_j \in M_R$, $j = 1, 2$ we denote

$$\widetilde{z}_1 \cdot \widetilde{z}_2 = z_1 \widetilde{* k(-}z_2) \tag{7}$$

It is obvious that $z_2 \in A_R$ implies $k(-z_2) \in A_R$ and by Theorem 4, $z_1 * k(-z_2) \in A_R$ and finally $\widetilde{z}_1 \cdot \widetilde{z}_2 \in M_R$, hence indeed this is an internal operation on M_R. The identity of this operation is $\widetilde{1} = (1, -1)$ since $\widetilde{z} \cdot \widetilde{1} = z \widetilde{* k(-}1) = \widetilde{z * 1} = \widetilde{z}$. Let us show that the inverse element of $\widetilde{z}$ is $\widetilde{\overline{z}}$. We have $\widetilde{z} \cdot \widetilde{\overline{z}} = z \widetilde{* k(-}\overline{z}) = z * \widetilde{(-1/-z)} = z \widetilde{* (1/z)} = \widetilde{1}$. We do not expect this operation to be transitive, since in the affirmative case it would make M_R into a non orientable Lie group, and such a thing, by [4] (page 140), does not exist. However, we can prove:

Theorem 7. The mapping $\alpha : G_1 \times M_R \to M_R$ defined by

$$\alpha(g_z, \widetilde{w}) = \widetilde{z * w}, \qquad w \in H \tag{8}$$

represents a left action of the Lie group G_1 on the Möbius strip M_R. If $z_2/z_1 \notin \mathbb{R}$ then for every $\widetilde{w} \in M_R$ we have that $\alpha(g_{z_1}, \widetilde{w}) \neq \alpha(g_{z_2}, \widetilde{w})$. Moreover, the action is transitive.

Proof. Obviously, α is a C^∞ mapping. Since the unit element of G_1 is $e = g_1$, we have

$$\alpha(g_1, \widetilde{w}) = \widetilde{g_1 * w} = \widetilde{w} \tag{9}$$

and for arbitrary $z_1, z_2 \in A_R$ the following equalities are true:

$$\begin{aligned}\alpha(g_{z_1}, \alpha(g_{z_2}, \widetilde{w})) = \alpha(g_{z_1}, \widetilde{z_2 * w}) = z_1 * \widetilde{(z_2 * w)} = \widetilde{(z_1 * z_2)} * w \\ = \alpha(g_{z_1 * z_2}, w) = \alpha(g_{z_1} g_{z_2}, w)\end{aligned} \tag{10}$$

and indeed (8) represents a left action of G_1 on M_R.

Suppose that $\alpha(g_z, \widetilde{w}_1) = \alpha(g_z, \widetilde{w}_2)$, i.e. $\widetilde{z * w_1} = \widetilde{z * w_2}$. Then $z * w_1 = z * w_2$ or $z * w_1 = -1/(\overline{z * w_2})$. If $z = \rho e^{i\theta}$ and $w_k = r_k e^{i\varphi_k}$ then the first equality implies $f(\rho, r_1)e^{i(\theta+\varphi_1)} = f(\rho, r_1)e^{i(\theta+\varphi_2)}$ i.e. $f(\rho, r_1)/f(\rho, r_2) = e^{i(\varphi_2-\varphi_1)}$ and the second equality implies $f(\rho, r_1)e^{i(\theta+\varphi_1)} = \frac{e^{ii(\theta+\varphi_2+\pi)}}{f(\rho, r_2)}$, or $f(\rho, r_1)f(\rho, r_2) = e^{i(\varphi_2-\varphi_1+\pi)}$. In both of these equalities, on the left hand side we have real numbers, hence $\varphi_2 - \varphi_1 \equiv 0 (\mathrm{mod}\, \pi)$, i.e. $z_2/z_1 \in \mathbb{R}$.

Having in view that $G_1 w = A_R$ for every $w \in A_R$, we conclude that $\{\alpha(g_z, \widetilde{w}) \mid z \in A_R\} = M_R$, hence α acts transitively on M_R. □

Corollary. For every $z \in A_R$, $|z| \neq 1$, the left (and right) action of $[g_z]$ on M_R is free and properly discontinuous.

Theorem 8. For every $z \in A_R$, $|z| \neq 1$, the space $A_R \,/\, [g_z]$ (respectively $M_R \,/\, [g_z]$) is isomorphic to a torus (respectively a Klein bottle).

One can find the proof of this theorem in [2].

6 A Lie Group Acting on the Non Orientable Real Projective Plane

The conformal metric (4) defined on the Riemann sphere $\overline{\mathbb{C}}$ induces a structure of Riemann surface on $\overline{\mathbb{C}}$. An analytic atlas on this Riemann surface is, for example, the atlas formed with two charts : $\varphi_1 : \mathbb{C} \to \mathbb{C}$ defined by $\varphi_1(z) = z$ and $\varphi_2 : \overline{\mathbb{C}} \backslash \{0\} \to \mathbb{C}$

defined by $\varphi_2(z) = 1/z$. The Riemann sphere is a symmetric Riemann surface with respect to the symmetry $k(z) = -1/\overline{z}$. The points z and $k(z)$ belong to different hemispheres and if we denote again by $H = \{z \in \overline{\mathbb{C}} \mid \ |z| > 1 \text{ or } |z| = 1 \text{ and } 0 \leq \arg z < \pi\}$, $K = \overline{\mathbb{C}} \setminus H$ then $z \in H$ if and only if $k(z) \in K$. We continue to keep the notation z for that element of the couple $\widetilde{z} = \{z, k(z)\}$ which belongs to H and the notation $< k >$ for the two element group generated by k. The real projective plan $\wp_2$ is the non orientable Klein surface obtained factorizing $\overline{\mathbb{C}}$ by $< k >$. The points of $\wp_2$ are $\widetilde{z} = \{z, k(z)\}$, $z \in H$ and the topology of $\wp_2$ is the trace topology of $\overline{\mathbb{C}}$, hence $\wp_2$ is a C^∞ differentiable manifold. By $\widetilde{0}$ we understand $\{0, \infty\}$.

We define, as for M_R, an internal operation on $\wp_2$ by $\widetilde{z_1} * \widetilde{z_2} = \widetilde{z_1 z_2}$, where $z_1, z_2 \in H$. This operation is commutative. Indeed, $\widetilde{z_1} * \widetilde{z_2} = \widetilde{z_1 z_2} = \widetilde{z_2 z_1} = \widetilde{z_2} * \widetilde{z_1}$. It has the unit element $\widetilde{1}$ since for every $\widetilde{z} \in \wp_2$ we have $\widetilde{z} * \widetilde{1} = \widetilde{z1} = \widetilde{z}$ and the inverse of $\widetilde{z}$ is $\widetilde{1/z}$ since $\widetilde{z} * \widetilde{1/z} = \widetilde{z(1/z)} = \widetilde{1}$. Due to the fact that $\wp_2$ is non orientable, the operation $*$ cannot be associative.

Let us notice that $\mathbb{C} \setminus \{0\}$ with the usual multiplication is a Lie group G_2 with the identity element 1 and the inverse of $z \in \mathbb{C} \setminus \{0\}$ being $1/z$. We can prove the following:

Theorem 9. The mapping $\alpha : G_2 \times \wp_2 \to \wp_2, \quad \alpha(z, \widetilde{w}) = \widetilde{z \cdot w}$ is a left action of the group G_2 on the non orientable manifold $\wp_2$.

Proof. The mapping α is obviously of class C^∞. Moreover, $\alpha(1, \widetilde{w}) = \widetilde{1 \cdot w} = \widetilde{w}$ for every $\widetilde{w} \in \wp_2$. Finally, $\alpha(z_1, (\alpha(z_2, \widetilde{w})) = \alpha(z_1, \widetilde{z_2 \cdot w}) = z_1 \cdot \widetilde{(z_2 \cdot w)} = \widetilde{(z_1 \cdot z_2)} \cdot w = \alpha(z_1 \cdot z_2, \widetilde{w})$ □

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Quantum Groups and Deformations

Quantum Diagonal Algebra and Pseudo-Plactic Algebra

Todor Popov

Abstract The subalgebra of diagonal elements of a quantum matrix group has been conjectured by Daniel Krob and Jean-Yves Thibon to be isomorphic to a cubic algebra, coined the quantum pseudo-plactic algebra. We present a functorial approach to the conjecture through the quantum Schur-Weyl duality between the quantum group and the Hecke algebra. The relations of the quantum diagonal subalgebra are found to be the image of the braid relations of the underlying Hecke algebra by an appropriate Schur functor which gives a straightforward proof of the conjecture.

1 Introduction

The diagonal elements of a quantum matrix group $\mathbb{C}[GL_q(V)]$ form a subalgebra which is the noncommutative avatar of the algebra of the functions on the torus. The resulting quantum diagonal algebra $\mathbb{C}[GL_q(V)]^\Delta \subset \mathbb{C}[GL_q(V)]$ provides a noncommutative character theory of quantum group comodules which is a lifting of the commutative symmetric functions. We identify the functions on the quantum torus $\mathbb{T}$ with the subspace of $End(V)^*$ stable by the transposition τ, $\tau(x^i_j) = x^j_i$. Krob and Thibon conjectured [4] that the algebra $\mathbb{C}[GL_q(V)]^\Delta$ spanned by $x^i_i \in \mathbb{C}[GL_q(V)]$ is isomorphic to the *quantum pseudo-plactic algebra* defined as the quotient $\mathfrak{PP}_q(\mathbb{T}) \cong \mathbb{C}(q)\langle\mathbb{T}\rangle/(\mathfrak{L}^\Delta_q(\mathbb{T}))$ of the free diagonal algebra $\mathbb{C}(q)\langle\mathbb{T}\rangle$ by the ideal $(\mathfrak{L}^\Delta_q(\mathbb{T}))$ generated by

$$\begin{array}{llll}
\mathfrak{L}^{\Delta\, i_1,i_2}_{q\ \ i_3} := [[x^{i_1}_{i_1}, x^{i_3}_{i_3}], x^{i_2}_{i_2}] & \text{with} & i_1 < i_2 < i_3 & \\
\mathfrak{L}^{\Delta\, i_1,i_1}_{q\ \ i_2} := [[x^{i_1}_{i_1}, x^{i_2}_{i_2}], x^{i_1}_{i_1}]_{q^2} & \text{with} & i_1 < i_2 & . \\
\mathfrak{L}^{\Delta\, i_1,i_2}_{q\ \ i_2} := [x^{i_2}_{i_2}, [x^{i_1}_{i_1}, x^{i_2}_{i_2}]]_{q^2} & \text{with} & i_1 < i_2 &
\end{array} \qquad (1)$$

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_34

Stated differently, the ideal of relations of the diagonal subalgebra $\mathbb{C}[GL_q(V)]^{\Delta}$ is generated by the cubic relations (1) and there are no relations in higher order which are independent from the cubic ones, Eq. (1).

We introduce a Schur bifunctor from the tower of Hecke algebras $\mathcal{H}^q_r$ into the coordinate ring of the quantum group $\mathbb{C}[GL_q(V)]$. It maps $(\mathcal{H}^q_r, \mathcal{H}^q_r)$-modules into $(U_q\mathfrak{gl}(V), U_q\mathfrak{gl}(V))$-modules. The polarization functor denoted by $\rightsquigarrow$ is the adjoint functor of the Schur bifunctor. The polarization of the diagonal subalgebra $\mathbb{C}[GL_q(V)]^{\Delta}$ is defined to be the *diagonal Hecke algebra* $\mathcal{H}^{q\,\Delta}$

$$\begin{array}{ccccc}
\mathfrak{PP}_q(\mathbb{T}) & \cong & \mathbb{C}[GL_q(V)]^{\Delta} & \hookrightarrow & \mathbb{C}[GL_q(V)] = \bigoplus_{r\geq 0}\mathbb{C}[GL_q(V)]_r \\
\wr\downarrow & & \wr\downarrow & & \wr\downarrow \\
\mathfrak{PP}_q & \cong & \mathcal{H}^{q\,\Delta} & \hookrightarrow & \mathcal{H}^q = \bigoplus_{r\geq 0}\mathcal{H}^q_r\,.
\end{array}$$

The quantum Weyl action [6] is stabilizing the diagonal $\mathbb{C}[GL_q(V)]^{\Delta}_r$ in $\mathbb{C}[GL_q(V)]_r$. It has a counterpart, an "adjoint" $\mathcal{H}^q_r$-action on $\mathcal{H}^{q\,\Delta}_r$. We consider also the polarization $\mathfrak{PP}_q$ of the quantum pseudo-plactic algebra $\mathfrak{PP}_q(\mathbb{T})$ which we refer to as the pre-plactic algebra defined as a factor algebra. We prove that the ideal of $\mathfrak{PP}_q$ induced by the unique polarized pseudo-plactic relation, namely

$$\mathfrak{L}^{\Delta}_q = [[x^1_1, x^3_3], x^2_2] \quad \Leftrightarrow \quad \mathfrak{L}^{\Delta}_q = (x^1_1x^3_3x^2_2 - x^3_3x^1_1x^2_2) - (x^2_2x^1_1x^3_3 - x^2_2x^3_3x^1_1) \quad (3)$$

contains all relations of the diagonal Hecke algebra $\mathcal{H}^{q\,\Delta}$, that is, one has the isomorphism of $\mathcal{H}^q$-modules between the pre-plactic algebra and the diagonal Hecke algebra

$$\mathfrak{PP}_q \cong \mathcal{H}^{q\,\Delta}\,.$$

The pre-plactic relation (3) is the difference of the polarized Knuth relations of the plactic monoid [5] therefore the pre-plactic algebra $\mathfrak{PP}_q$ is a lifting of the Poirier-Reutenauer algebra [13]. Both Poirier-Reutenauer $\mathfrak{PR}$ and pre-plactic algebra $\mathfrak{PP}_q$ [15] are Hopf algebra quotients of the Malvenuto-Reutenauer algebra $\mathfrak{MR}$ [10]

$$\mathfrak{PR} \subset \mathfrak{PP}_q \subset \mathfrak{MR}\,.$$

There exists also a Schur functor PS_q [9] mapping the q-deformation of $\mathfrak{PR}$ to q-deformation $PS_q(V)$ of the plactic algebra (deformed parastatistics algebra [2, 8]).

In Chap. 4 by applying a Schur bifunctor to the $\mathcal{H}^q$-modules $\mathfrak{PP}_q \cong \mathcal{H}^{q\,\Delta}$ we prove the conjecture of Krob and Thibon

$$\mathfrak{PP}_q(\mathbb{T}) \cong \mathbb{C}[GL_q(V)]^{\Delta}\,.$$

The cubic relations $\mathfrak{L}^{\Delta}_q(\mathbb{T}) = 0$ are playing a role similar to the Knuth relations of the plactic algebra in the theory of noncommutative symmetric functions. Moreover the pre-plactic relation $\mathfrak{L}_q^{\Delta} = 0$ acquire in the process of our proof clear geometrical meaning, it is nothing but the braid relation of the Hecke algebra.

2 Schur Functor and its Adjoint

The algebra of functions on the quantum group $\mathbb{C}[GL_q(V)]$ coacts on itself, it is a naturally a $(\mathbb{C}[GL_q(V)], \mathbb{C}[GL_q(V)])$-comodule. The "regular representation" of $\mathbb{C}[GL_q(V)]$ according to the Peter-Weyl theorem has a decomposition into a product of left and right irreducible $[GL_q(V)]$-comodules

$$\mathbb{C}[GL_q(V)]_r \cong \bigoplus_{\lambda \vdash r} S_\lambda(V^*) \otimes S^\lambda(V) \ .$$

In their seminal paper Faddeev-Reshetikhin-Takhtajan [3] defined the algebra of functions on the general linear quantum group $\mathbb{C}[GL_q(V)]$ as the commutant of the action of the Hecke algebra $\mathcal{H}_2(q)$. The Hecke algebra $\mathcal{H}_2(q)$ is represented by the Drinfeld-Jimbo quantum R-matrix $\hat{R}_q \in End(V^{\otimes 2})$

$$\mathbb{C}[GL_q(V)] = \mathbb{C}(q)\left\langle W^*\right\rangle/(\hat{R}_q W \otimes W - W \otimes W\hat{R}_q) \quad W = End(V)^* \cong V^* \otimes V \tag{4}$$

In other words the homogeneous elements of $\mathbb{C}[GL_q(V)]$ are the coinvariants of the natural $\mathcal{H}_r^q$-action

$$\mathbb{C}[GL_q(V)]_r \cong (W^{\otimes r})^{\mathcal{H}_r^q} \cong (V^*)^{\otimes r} \otimes_{\mathcal{H}_r^q} V^{\otimes r} \ .$$

Here the decorated tensor product $\otimes_{\mathcal{H}_r^q}$ stands for the quotient relating the right and left $\mathcal{H}_r^q$-action, e.g. $(W^{\otimes 2})^{\mathcal{H}_2^q} := W^{\otimes 2}/(W \otimes W\hat{R}_q - \hat{R}_q W \otimes W)$.

By duality the two-sided comodules $\mathbb{C}[GL_q(V)]_r$ are bimodules of the quantum universal enveloping algebra $U_q(\mathfrak{g})$. The quantum Schur-Weyl duality is the double commutant property of the action of the quantum universal enveloping algebra $U_q\mathfrak{gl}(V)$ and the action of the Hecke algebra $\mathcal{H}_r^q$. The Schur-Weyl duality allows to build the Schur functor which maps the category of representations of the Hecke algebra $\mathcal{H}_r^q - mod$ to the category of representations of the quantum universal enveloping algebra $U_q\mathfrak{gl}(V) - mod$.

Orthogonal Idempotents in $\mathcal{H}_r^q$. An orthogonal idempotent $e_\lambda(T)$ in $\mathcal{H}_r^q$ is parametrized by a partition λ of r, $\lambda \vdash r$ and a Standard Young Tableau T with shape λ, $T \in STab(\lambda)$. Different idempotents are orthogonal

$$e_\lambda(T)e_\mu(T') = e_\lambda(T)\delta_{\lambda\mu}\delta_{TT'} \ .$$

A system of orthogonal idempotents provides a partition of unity

$$\sum_{\lambda\vdash r}\sum_{sh(T)=\lambda} e_\lambda(T) = \mathbb{1}_{\mathcal{H}_r^q}\ .$$

An irreducible right $\mathcal{H}_r^q$-module is constructed as the ideal given by multiplication with an idempotent e_λ from the left, $S^\lambda = e_\lambda \mathcal{H}_r^q$. Idempotents $e_\lambda(T)$ and $e_\lambda(T')$ with different Young Tableaux $T, T' \in STab(\lambda)$ having same shape λ lead to isomorphic $\mathcal{H}_r^q$-modules $S^\lambda \cong S^{\lambda(T)} \cong S^{\lambda(T')}$, so we often suppress T.

The Schur functor $S^\lambda(V) = S^\lambda \otimes_{\mathcal{H}_r^q} V^{\otimes r}$ maps the irreducible right $\mathcal{H}_r^q$-module $S^\lambda = e_\lambda\mathcal{H}_r^q$ into an irreducible right $U_q\mathfrak{gl}(V)$-module. Similarly on defines and irreducible left $\mathcal{H}_r^q$-module $S_\lambda = \mathcal{H}_r^q e_\lambda$ and irreducible left $U_q\mathfrak{gl}(V)$-module $S_\lambda(V^*) = V^{*\otimes r} \otimes_{\mathcal{H}_r^q} S_\lambda$. The polarization is a functor adjoint to the Schur functor, it maps a $U_q\mathfrak{gl}(V)$-module $S^\lambda(V)$ with $|\lambda| = r$ into its underlying $\mathcal{H}_r^q$-module S^λ.

The coordinate ring is $\mathbb{N}^d$-bigraded by the weight[1] of the multi-indices A and B

$$\mathbb{C}[GL_q(V)]_{r\,B}^{A} \cong \bigoplus_{\lambda\vdash r} S_\lambda^A(V^*) \otimes_{\mathcal{H}_r^q} S_B^\lambda(V) \qquad A, B \in \{1,\ldots,d = \dim V\}^r\ .$$

The polarization of $[GL_q(V)]_r$ is its component of weight 1^r (we suppose that we have chosen V such that $\dim V = r$). Equivalently the multi-indices α and β of weight 1^r are words of permutations $\alpha, \beta \in \mathfrak{S}_r$. The polarization of the $(U_q\mathfrak{gl}(V), U_q\mathfrak{gl}(V))$-module $\mathbb{C}[GL_q(V)]_r$ is isomorphic to the $(\mathcal{H}_r^q, \mathcal{H}_r^q)$-bimodule $\mathcal{H}_r^q$ yielding the decomposition of the regular representation of $\mathcal{H}_r^q$

$$(\mathcal{H}_r^q)_\beta^\alpha := \mathbb{C}[GL_q(V)]_{r\,\beta}^{\alpha} = \bigoplus_{\lambda\vdash r} S_\lambda^\alpha \otimes_{\mathcal{H}_r^q} S_\beta^\lambda\ . \tag{5}$$

Every left (right) $\mathcal{H}_r^q$-module S_λ (S^λ) appears in the regular representation with multiplicity equal to its dimension $f_\lambda = \dim S_\lambda$.

For generic q the regular representation $\mathcal{H}_r^q$ is isomorphic to its specialization at $q = 1$, *i.e.*, to the regular representation of $\mathbb{C}[\mathfrak{S}_r]$. Hence $\mathbb{C}[\mathfrak{S}_r]$ can be seen as the polarization of the commutative algebra $\mathbb{C}[GL(V)]$. A permutation $\alpha \in \mathfrak{S}_r$ is given by the two row bijective correspondence (with commuting biletters) or equivalently by its word

$$\alpha = \begin{pmatrix} 1 & \ldots & r \\ \alpha_1 & \ldots & \alpha_r \end{pmatrix} \qquad \alpha = (\alpha_1 \ldots \alpha_r)\ .$$

The inverse permutation α^{-1} is simply obtained by exchanging the two rows

$$\alpha^{-1} = \begin{pmatrix} \alpha_1 & \ldots & \alpha_r \\ 1 & \ldots & r \end{pmatrix} = \begin{pmatrix} 1 & \ldots & r \\ \alpha_1^{-1} & \ldots & \alpha_r^{-1} \end{pmatrix}$$

[1]The weight of the multi-index A is the d-dimensional vector $(w_1(A),\ldots,w_d(A))$ defined as $w_i(A) = \#\{a_k = i \mid 1 \le k \le r\}$.

in which we have rearranged the commuting biletters. More generally any two row bijection yields a permutation representable as a product

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\alpha_1 \dots \alpha_r\\ 1 \dots r\end{pmatrix}\begin{pmatrix}1 \dots r\\ \beta_1 \dots \beta_r\end{pmatrix} = \alpha^{-1}\beta\ . \tag{6}$$

The permutation $\alpha^{-1}\beta$ can be thought as an element in the double coset where the right $\mathbb{C}[\mathfrak{S}_r]$-action is by place permutation and the left $\mathbb{C}[\mathfrak{S}_r]$-action is by substitution

$$\alpha^{-1}\beta = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{C}[\mathfrak{S}_r] \cong \mathbb{C}[\mathfrak{S}_r] \otimes_{\mathbb{C}[\mathfrak{S}_r]} \mathbb{C}[\mathfrak{S}_r]\ .$$

The commutativity of the biletters is expressed by the coset notation $\otimes_{\mathbb{C}[\mathfrak{S}_r]}$.

We now come back to the Hecke algebra $\mathcal{H}_r^q$ isomorphic to the permutation group algebra $\mathcal{H}_r^q \cong \mathbb{C}[\mathfrak{S}_r]$ for generic q. It has a basis $T_\sigma \in \mathcal{H}_r^q$ indexed by the permutations $\sigma \in S_r$. We introduce another basis $T^\sigma := T_{\sigma^{-1}}$

In parallel with $\mathbb{C}[\mathfrak{S}_r]$ one has the double coset for $(\mathcal{H}_r^q)_\beta^\alpha$, cf. Eq. (5) with basis

$$T_\beta^\alpha \in \mathcal{H}_r^q \cong \mathcal{H}_r^q \otimes_{\mathcal{H}_r^q} \mathcal{H}_r^q \qquad T_\beta^\alpha := T^\alpha \otimes_{\mathcal{H}_r^q} T_\beta = T_{\alpha^{-1}} \otimes_{\mathcal{H}_2^q} T_\beta \qquad \alpha, \beta \in \mathfrak{S}_r\ . \tag{7}$$

Polarization of $\mathbb{C}[GL_q(V)]$ Relations. By evaluation of the Drinfeld-Jimbo R-matrix $\hat{R}_q$ with indices i, j running in the range $1 \le i, j \le d = \dim V$

$$\hat{R}_q = \sum_{i,j} q^{\delta_{ij}} e_j^i \otimes e_i^j + (q - q^{-1}) \sum_{i<j} e_j^i \otimes e_j^i \qquad e_j^i \in \mathfrak{gl}(V)\ . \tag{8}$$

we get the Faddeev-Reshetikhin-Takhtajan relations [3] $\hat{R}_q W \otimes W = W \otimes W \hat{R}_q$ of the coordinate ring of the quantum group $\mathbb{C}[GL_q(V)]$

$$\begin{array}{lll} x_k^j x_k^i = q x_k^i x_k^j & x_j^k x_i^k = q x_i^k x_j^k & j > i \\ x_l^j x_k^i = x_k^i x_l^j + (q - q^{-1}) x_l^i x_k^j & x_k^j x_l^i = x_l^i x_k^j & j > i \quad l > k \end{array}\ . \tag{9}$$

These relations span an ideal which is also a coideal for the coaction $\Delta x_j^i = \sum_k x_k^i \otimes x_j^k$. By duality it is a $(U_q\mathfrak{gl}(V), U_q\mathfrak{gl}(V))$-module. The polarization of the ideal generators $\hat{R}_q W \otimes W - W \otimes W \hat{R}_q$, that is, the weight 1^2 relations of $\mathbb{C}[GL_q(2)]$, yields a $(\mathcal{H}_2^q, \mathcal{H}_2^q)$-module $\mathcal{H}_2^q$

$$T_{21}^{21} = T_{12}^{12} + (q - q^{-1}) T_{21}^{12} \qquad T_{21}^{12} = T_{12}^{21}\ . \tag{10}$$

With the help of the identification (7) one has $T_{12}^{12} = \mathbb{1} \otimes \mathbb{1}$

$$T_{21}^{21} = T_{s_1} \otimes_{\mathcal{H}_2^q} T_{s_1} = \mathbb{1} \otimes_{\mathcal{H}_2^q} (T_{s_1})^2 \qquad T_{21}^{12} = \mathbb{1} \otimes_{\mathcal{H}_2^q} T_{s_1} = T_{s_1} \otimes_{\mathcal{H}_2^q} \mathbb{1} = T_{12}^{21}\ .$$

Hence the polarization (10) of Eq. (9) is equivalent to the two factorizations of the Hecke relation (taken together)

$$(q^{-1}\mathbb{1} + T_{s_1}) \otimes_{\mathcal{H}_2^q} (q\mathbb{1} - T_{s_1}) = 0 = (q\mathbb{1} - T_{s_1}) \otimes_{\mathcal{H}_2^q} (q^{-1}\mathbb{1} + T_{s_1}) \tag{11}$$

Hence by applying the polarization functor to the quantum matrix group $\mathbb{C}[GL_q(V)]$ relations $\hat{R}_q W \otimes W - W \otimes W \hat{R}_q$ one gets the relations of the Hecke algebra $\mathcal{H}_2^q$.

Conversely, the normalization of the factorization of the Hecke relations, cf. Eq. (11) (after division by $[2] = q + q^{-1} \neq 0$ for $q^2 \neq -1$) leads to two orthogonal idempotents in $\mathcal{H}_2^q$

$$e_2 e_{1^2} = 0 = e_{1^2} e_2 \qquad (\mathbb{1}_{\mathcal{H}_2^q} = e_2 + e_{1^2})$$

where the q-symmetrizer and the q-antisymmetrizer are respectively

$$e_2 := \frac{1}{[2]}\left(q^{-1}\mathbb{1} + T_{s_1}\right) , \qquad e_{1^2} := \frac{1}{[2]}\left(q\mathbb{1} - T_{s_1}\right) .$$

Half of the relations of the quantum matrix group $\mathbb{C}[GL_q(V)]$, cf. Eq. (4) are obtained through the Schur bifunctors

$$e_2 W \otimes W e_{1^2} = S_2(V^*) \otimes S^{1^2}(V) = V^{*\otimes 2} \otimes_{\mathcal{H}_2^q} S_2 \otimes S^{1^2} \otimes_{\mathcal{H}_2^q} V^{\otimes 2}$$

where $S^{1^2}(V) = e_{1^2} V^{\otimes 2} = S^{1^2} \otimes_{\mathcal{H}_2^q} V^{\otimes 2}$ and $S_2(V^*) = V^{*\otimes 2} e_2 = V^{\otimes 2} \otimes_{\mathcal{H}_2^q} S^2$. The $\mathcal{H}_2^q$-action ρ of the projectors e_2 and e_{1^2} is through the multiplication by the R-matrix Eq.(8), $\rho(T_{s_1}) = \hat{R}_q \in \mathrm{End}(V^{\otimes 2})$. The other half of the relations in Eq. (4), (the "missing relation" after Yuri Manin [11])

$$e_{1^2} W \otimes W e_2 = S_{1^2}(V^*) \otimes S^2(V) = V^{*\otimes 2} \otimes_{\mathcal{H}_2^q} S_{1^2} \otimes S^2 \otimes_{\mathcal{H}_2^q} V^{\otimes 2} .$$

The above relations define the so called left and right quantum semi-groups [11], respectively. Taken together they span the ideal of the quantum group relations Eq. (4). Equivalently one has the short exact sequence of $(U_q\mathfrak{gl}(V), U_q\mathfrak{gl}(V))$-modules

$$0 \to (S_2(V^*) \otimes S^{1^2}(V) \oplus S_{1^2}(V^*) \otimes S^2(V)) \to \mathbb{C}(q)\left\langle W^* \right\rangle \to \mathbb{C}[GL_q(V)] \to 0 \tag{12}$$

whose polarization yields the short exact sequence of $(\mathcal{H}_r^q, \mathcal{H}_r^q)$-modules

$$0 \to (S_2 \otimes S^{1^2} \oplus S_{1^2} \otimes S^2) \to \mathcal{H}_r^q \otimes \mathcal{H}_r^q \xrightarrow{p} \mathcal{H}_r^q \cong \mathcal{H}_r^q \otimes_{\mathcal{H}_r^q} \mathcal{H}_r^q \to 0 .$$

The projection $p : \mathcal{H}_r^q \otimes \mathcal{H}_r^q \to \mathcal{H}_r^q$ acts by $p(T^\alpha \otimes T_\beta) = T^\alpha \otimes_{\mathcal{H}_r^q} T_\beta$. Clearly $p(S_2 \otimes S^{1^2}) = S_2 \otimes_{\mathcal{H}_2^q} S^{1^2} = 0 = p(S_{1^2} \otimes S^2)$. The one-dimensional $\mathcal{H}_2^q$-bimodules $S_2 \otimes S^{1^2}$ and $S_{1^2} \otimes S^2$ can be visualized by the braid diagrams [2]

3 Quantum Diagonal Algebra $\mathbb{C}[GL_q(V)]^\Delta$

Definition 1. The quantum diagonal algebra $\mathbb{C}[GL_q(V)]^\Delta$ is the subalgebra of the quantum matrix algebra $\mathbb{C}[GL_q(V)]$ generated by the elements $x_1^1, x_2^2, \ldots, x_d^d$.

The restriction of the commutative ring $\mathbb{C}[GL(V)]$ to the subring of the diagonal matrix elements x_i^i yields $\mathbb{C}[\mathbb{T}]$, the *commutative* functions on the torus $\mathbb{T}$. We now derive the relations in the restriction $\mathbb{C}[GL_q(V)]^\Delta$ to the diagonal of the noncommutative ring $\mathbb{C}[GL_q(V)]$.

Lemma 1. *Let* $\mathbb{T}^* = \sum_{i=1}^d \mathbb{C}x_i^i = W^\Delta$ *be the span of the diagonal generators in* $\mathbb{C}[GL_q(V)]$. *The subspace* $\mathfrak{L}_q^\Delta(\mathbb{T}) \subset \mathbb{T}^{*\otimes 3}$ *of the cubic relations of the quantum diagonal algebra* $\mathbb{C}[GL_q(V)]^\Delta$ *is generated by the pseudo-plactic relations (1). Its dimension is* $\dim \mathfrak{L}_q^\Delta(\mathbb{T}) = \binom{d}{3} + 2\binom{d}{2}$ *where* $d = \dim V$.

Proof. By expanding $x_i^i x_j^j x_k^k =: x_{ijk}^{ijk}$ in a basis $x_{ijk} := x_i^1 x_j^2 x_k^3$ of monomials in $\mathbb{C}[GL_q(V)]_3$ one gets the linear dependences between the elements in $\mathbb{C}[GL_q(V)]^\Delta$, the nontrivial solutions of the equation $x_\sigma^\sigma c^\sigma = 0$.

Expanding the diagonal monomials x_{ijk}^{ijk} in the basis x_{ijk} yields the matrix

$$\begin{pmatrix} x_{123}^{123} \\ x_{132}^{132} \\ x_{213}^{213} \\ x_{231}^{231} \\ x_{312}^{312} \\ x_{321}^{321} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \omega & 0 & 0 & 0 & 0 \\ 1 & 0 & \omega & 0 & 0 & 0 \\ 1 & 0 & \omega & 0 & 0 & \omega \\ 1 & \omega & 0 & 0 & 0 & \omega \\ 1 & \omega & \omega & \omega^2 & \omega^2 & \omega^3 + \omega \end{pmatrix} \begin{pmatrix} x_{123} \\ x_{132} \\ x_{213} \\ x_{231} \\ x_{312} \\ x_{321} \end{pmatrix} \qquad \omega = q - q^{-1} \, . \tag{13}$$

Thus one has a transformation with a singular matrix, $x_\sigma^\sigma = x_\rho M_\sigma^\rho$ with $\rho, \sigma \in S_3$. Since the monomials x_ρ are a true basis in 1^3-graded part of $\mathbb{C}[GL_q(V)]_3$ we have

$$x_\sigma^\sigma c^\sigma = 0 \quad \Leftrightarrow \quad M_\sigma^\rho c^\sigma = 0 \qquad \mathrm{Ker} M = \mathfrak{L}_{q\,3}^{\Delta\,12} = \mathbb{C}\,[[x_1^1, x_3^3], x_2^2] \, ,$$

[2]The span $S_{1^2} \otimes S^2 \oplus S_{1^2} \otimes S^2$ is equivalent to the Hecke relations eqs. (10,19), see also [12].

where the combination of columns (rows in Eq. (13)) of M^{ρ}_{σ} yields the nontrivial solution $c^{132} = c^{231} = -c^{312} = -c^{213} = 1$ and $c^{123} = 0 = c^{321}$.

For $x^{\sigma}_{\sigma} \in \mathbb{C}[GL_q(V)]^{\Delta}$ of weight (2, 1) we have a basis $x_{112} := x^{112}_{112}$ and $x_{121} := x^{112}_{121}$. Hence by expanding the diagonal elements we get a 3×2 matrix M

$$\begin{pmatrix} x^{112}_{112} \\ x^{121}_{121} \\ x^{211}_{211} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & \omega \\ 1 & (q^2+1)\omega \end{pmatrix} \begin{pmatrix} x_{112} \\ x_{121} \end{pmatrix} \qquad \mathrm{Ker} M = \mathfrak{L}^{\Delta\,11}_{q\;2} = \mathbb{C}\,[[x^1_1, x^2_2], x^1_1]_{q^2}$$

Similarly from $x^{\sigma}_{\sigma} \in \mathbb{C}[GL_q(V)]^{\Delta}$ of weight (1, 2) one obtains $\mathrm{Ker} M = \mathfrak{L}^{\Delta\,12}_{q\;2}$. □

Remark. The origin of the kernel of M is the existence of two "homotopic" expressions of maximal element x_{321} whose difference $[[x^1_1, x^3_3], x^2_2]\omega^{-1} \in \mathfrak{L}^{\Delta\,1,2}_{q\;3}$

$$x^{123}_{321} = x^1_3 x^3_1 x^2_2 = [x^3_3, x^1_1] x^2_2 \omega^{-1} \qquad x^{123}_{321} = x^2_2 x^1_3 x^3_1 = x^2_2 [x^3_3, x^1_1]\omega^{-1} \, . \tag{14}$$

In the same vein, the maximal element of weight (2, 1), that is, x^{112}_{211} while expressed in $\mathbb{C}[GL_q(V)]^{\Delta}$ can be written in two different ways

$$x^{112}_{211} = q^{-1} x^{121}_{211} = q^{-1}[x^2_2, x^1_1] x^1_1 \omega^{-1} \qquad x^{112}_{211} = q x^{112}_{121} = q x^1_1 [x^2_2, x^1_1]\omega^{-1} \tag{15}$$

and the difference of two expressions yields $-\omega^{-1} q^{-1}[[x^1_1, x^2_2], x^1_1]_{q^2} \in \mathfrak{L}^{\Delta\,1,1}_{q\;2}$. Similarly the maximal element of weight (1, 2), *i.e.*, x^{122}_{221} forks and leads to $\mathfrak{L}^{\Delta\,1,2}_{q\;2}$.

4 A Functorial Way to the Diagonal Algebra $\mathbb{C}[GL_q(V)]^{\Delta}$

Definition 2. Let $\mathbb{C}[GL_q(V)]^{\Delta}$ be the diagonal algebra generated by $x^i_i \in \mathbb{C}[GL_q(V)]$. The diagonal Hecke algebra $\mathcal{H}^{q\,\Delta} = \bigoplus_r \mathcal{H}^{q\,\Delta}_r$ is the polarization of $\mathbb{C}[GL_q(V)]^{\Delta}$

$$\mathbb{C}[GL_q(V)]^{\Delta}_r = \sum_{A \in \{1,\dots,d\}^r} \mathbb{C}(q) x^A_A \rightsquigarrow \mathcal{H}^{q\,\Delta}_r = \sum_{\alpha \in \mathfrak{S}_r} \mathbb{C}(q) T^{\alpha}_{\alpha} \, .$$

We fix a partition of the unit in $1\!\!1_{\mathcal{H}^q_3}$ by orthogonal idempotents

$$1\!\!1_{\mathcal{H}^q_3} = e_3 + e^+_{21} + e^-_{21} + e_{1^3}$$

the two idempotents $e^{\pm}_{21}$ corresponding to the two Standard Young Tableaux with shape $\lambda = 21$, $\dim S_{21} = 2$. The idempotent e^+_{21} is a deformation of the Eulerian idempotent $e^{[1]}_3$ considered by Jean-Louis Loday [7]. Eulerian idempotents $e^{[1]}_n$ split the Harisson homology from the Hochschild homology.

Lemma 2 ([8]). *The splitting of the central idempotent $E_{21} = e_{21}^{+} + e_{21}^{-}$ into two minimal idempotents $e_{21}^{+}e_{21}^{-} = 0$ is uniquely chosen by the θ-multiplication eigenvalues*

$$\theta e_{21}^{\pm} = \pm e_{21}^{\pm} = e_{21}^{\pm}\theta \qquad \theta = T_{s_1}T_{s_2}T_{s_1} \ . \tag{16}$$

These minimal idempotents are polynomials of $T_\sigma \in \mathcal{H}_3^q$ (for details see [8])

$$\begin{aligned} e_{21}^{\pm} = &\frac{1}{[3]}\left(T_{123} - \frac{1}{2}(T_{231} \pm T_{213} \pm T_{132} + T_{312}) \pm T_{321}\right) \\ &+ \frac{\omega}{2[3]}\,(T_{213} \mp T_{312} \mp T_{231} + T_{132}) \qquad \omega = q - q^{-1} \ . \end{aligned} \tag{17}$$

The projector e_{21}^{-} is obtained from e_{21}^{+} by the involution $T_\sigma \to (-1)^\sigma T_\sigma$, $q \to q^{-1}$.

We prove in the appendix the following important Lemma

Lemma 3. ([14]). *Let us denote by $\mathfrak{L}_q^{\pm}(W)$ the $U_q\mathfrak{gl}(V)$-bimodule*

$$\mathfrak{L}_q^{\pm}(W) = e_{21}^{\pm} W^{\otimes 3} e_{21}^{\mp} = S_{21}^{\pm}(V^*) \otimes S_{\mp}^{21}(V) \ .$$

The relations of the quantum pseudo-plactic algebra $\mathfrak{PP}_q(\mathbb{T})$ are the image of the restriction of $\mathfrak{L}_q^{\pm}(W)$ to the diagonal $\mathbb{T}^*$

$$\mathfrak{L}_q^{\Delta}(\mathbb{T}) = \mathfrak{L}_q^{+}(W)|_{\mathbb{T}^*} = \mathfrak{L}_q^{-}(W)|_{\mathbb{T}^*} \ . \tag{18}$$

Definition 3. The pre-plactic algebra $\mathfrak{PP}_q$ is the graded algebra $\mathfrak{PP}_q = \bigoplus_{r\geq 0} \mathfrak{PP}_q(r)$ with degrees given by the quotient

$$\mathfrak{PP}_q(r) \cong (\mathcal{H}_r^q \otimes \mathcal{H}_r^q)^{\Delta} / (\mathfrak{L}_q^{\Delta})_r$$

where $(\mathfrak{L}_q^{\Delta})_r$ stays for the degree r of the ideal $(\mathfrak{L}_q^{\Delta})$ generated by the polarization of the pseudo-plactic relations $\mathfrak{L}_q^{\Delta}(\mathbb{T})$ cf. Eq. (18)

$$\mathfrak{L}_q^{\Delta} := [[13]2] := \widetilde{T}_{132}^{132} - \widetilde{T}_{312}^{312} - \widetilde{T}_{213}^{213} + \widetilde{T}_{231}^{231} \qquad \widetilde{T}_{\alpha}^{\alpha} := T^{\alpha} \otimes T_{\alpha} \ .$$

It is clear that the pre-plactic algebra $\mathfrak{PP}_q \subset \mathcal{H}^q = \bigoplus_{r\geq 0} \mathcal{H}_r^q$ is the polarization of the quantum pseudo-plactic algebra $\mathfrak{PP}_q(\mathbb{T})$. We obtain now the key result;

Theorem 1. *The diagonal Hecke algebra $(\mathcal{H}^q)^{\Delta} = \bigoplus_{r\geq 0} (\mathcal{H}_r^q)^{\Delta}$ is isomorphic to the pre-plactic algebra $\mathfrak{PP}_q = \bigoplus_{r\geq 0} \mathfrak{PP}_q(r)$*

$$(\mathcal{H}^q)^{\Delta} \cong \mathfrak{PP}_q \ .$$

Proof. The $r!$ elements $\widetilde{T}^{\alpha}_{\alpha} := T^{\alpha} \otimes T_{\alpha}$ freely generate the diagonal space

$$(\mathcal{H}^q_r \otimes \mathcal{H}^q_r)^{\Delta} := \bigoplus_{\alpha \in \mathfrak{S}_r} \mathbb{C}(q) \widetilde{T}^{\alpha}_{\alpha} \,.$$

We are going to show now that $\mathfrak{L}^{\Delta}_q$ is the unique combination in $(\mathcal{H}^q_3 \otimes \mathcal{H}^q_3)^{\Delta}$ which is projected out by $p : \mathcal{H}^q_3 \otimes \mathcal{H}^q_3 \to \mathcal{H}^q_3 \otimes_{\mathcal{H}^q_3} \mathcal{H}^q_3$, by a pictorial proof

$$p(\mathfrak{L}^{\Delta}_q) = T^{132}_{132} - T^{312}_{312} - T^{213}_{213} + T^{231}_{231} = 0$$

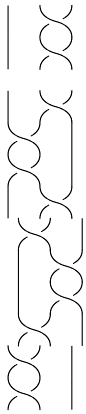
We attach to each generator in $(\mathcal{H}_3(q) \otimes \mathcal{H}_3(q))^{\Delta}$ its braid using coset notation (6) putting the left factor in the upper half-plane (above the horizon) and right factor under the horizon

$$T^{132}_{132} = T^{132}_{123} \otimes T^{123}_{132} = T_{s_2} \otimes T_{s_2} =$$

$$T^{312}_{312} = T^{312}_{123} \otimes T^{123}_{312} = T_{s_2 s_1} \otimes T_{s_1 s_2} =$$

$$T^{231}_{231} = T^{231}_{123} \otimes T^{123}_{231} = T_{s_1 s_2} \otimes T_{s_2 s_1} =$$

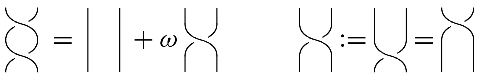
$$T^{213}_{213} = T^{213}_{123} \otimes T^{123}_{213} = T_{s_1} \otimes T_{s_1} =$$



These braids are symmetric with respect to the horizon, the left factor being the braid indexed by the inverse permutation of the right. The projection p is gluing the upper and the lower braids allowing generators to flow across the tensor product

$$p(\widetilde{T}^{\alpha}_{\alpha}) = T_{\alpha^{-1}} \otimes_{\mathcal{H}^q_3} T_{\alpha} = \mathbb{1} \otimes_{\mathcal{H}^q_3} T_{\alpha^{-1}} T_{\alpha}$$

and allows to reduce the number of crossings, by reducing the word written with braid generators. For instance, in the Hecke relation $T^2_{s_1} = \mathbb{1} + (q - q^{-1}) T_{s_1}$, the "bubble" $(T_{s_1})^2$ being reduced to braids in $\mathcal{H}^q$ with lower length number of crossings

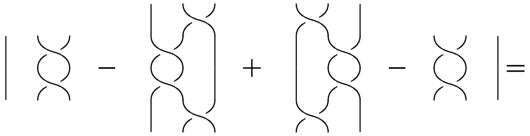


$$\omega = (q - q^{-1}) \qquad (19)$$

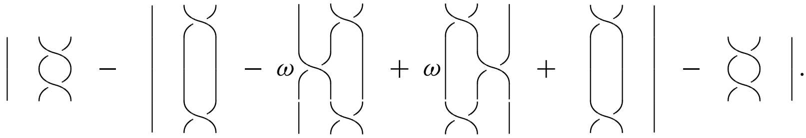
a move that we are referring as Hecke move.

The "standardized" pseudo-plactic relation is pictorially represented by sum of "diagonal" diagrams

$$T^{132}_{132} - T^{312}_{312} - T^{213}_{213} + T^{231}_{231} =$$

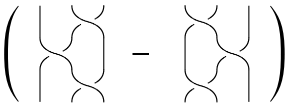


which we simplify by reducing terms T^{312}_{312} and T^{231}_{231}



The rightmost and leftmost terms with bubbles are homotopic and cancel hence

$$T^{132}_{132} - T^{312}_{312} - T^{213}_{213} + T^{231}_{231} = -(q - q^{-1})$$



$$= 0$$

and we got the transparent result: the pre-plactic relation $\mathfrak{L}^{\Delta}_q$ is equivalent to the braid relation. The reduced word 321 has two representatives and the two braids cancel

$$p(\mathfrak{L}^{\Delta}_q) = \omega(T_{s_1}T_{s_2}T_{s_1} - T_{s_2}T_{s_1}T_{s_2}) = 0\ .$$

Lemma 4. *All symmetric elements in* $(\mathcal{H}^q \otimes \mathcal{H}^q)^{\Delta}$ *with vanishing projection in* $\mathcal{H}^{q\,\Delta}$ *belong to the ideal generated by the braid relation* $(\mathfrak{L}^{\Delta}_q)$*,*

$$(\mathfrak{L}^{\Delta}_q) = \{x \in (\mathcal{H}^q \otimes \mathcal{H}^q)^{\Delta} | p(x) = 0\}\ .$$

Proof of the Lemma. Reducing a generator x in $\mathcal{H}^{q\,\Delta} \subset \mathcal{H}^q$ to its minimal length is a "normal ordering" such that we can't apply the Hecke moves the braid diagram of x any more. Any non-zero element in $\mathcal{H}^{q\,\Delta}$ is represented by a combination of reduced words, but there is a remaining "gauge freedom", $T_{s_i s_{i+1} s_i} \sim T_{s_{i+1} s_i s_{i+1}}$ which is the "mutation" of the reduced word with respect to the braid relation.

Assume that in degree $n > 3$ we have a relation $\mathcal{H}^{q\,\Delta}$ not generated by $\mathfrak{L}^{\Delta}_q$. We conclude that there exists an independent relation between the reduced words in $\mathcal{H}^q$ which is of degree higher than 3 which is a contradiction since any reduced word can be brought to any other by a sequence of braiding mutations $T_{s_i s_{i+1} s_i} \sim T_{s_{i+1} s_i s_{i+1}}$. The Lemma is proved.

When restricted to the diagonal $\mathbb{T}$ the sequence of $(U_q\mathfrak{gl}(V), U_q\mathfrak{gl}(V))$-modules cf. Eq. (12) yields the sequence of spaces

$$0 \to (\mathfrak{L}^{\Delta}_q(\mathbb{T}))_r \to \mathbb{C}(q)\,\langle\mathbb{T}\rangle \to \mathbb{C}[GL_q(V)]^{\Delta} \to 0\ . \tag{20}$$

These spaces are stable under the quantum Weyl action [6] (see also [1]) which lives in a completion $\widehat{U_q\mathfrak{gl}}(V)$.

The diagonal restriction $(\mathcal{H}^q\otimes\mathcal{H}^q)^\Delta := \bigoplus_{r\geq 0}\left(\sum_{\alpha\in\mathfrak{S}_r}\mathbb{C}\widetilde{T}^\alpha_\alpha\right)$ of the $\mathcal{H}^q$-bimodule $\mathcal{H}^q\otimes\mathcal{H}^q$ is a $\mathcal{H}^q$-module with a left action

$$T_\rho\cdot\widetilde{T}^\sigma_\sigma := T_{\rho^{-1}}T_{\sigma^{-1}}\otimes T_\sigma T_\rho \qquad \text{(similarly for } \ T^\sigma_\sigma\in\mathcal{H}^{q\,\Delta}) \ .$$

The polarization of this sequence of $\widehat{U_q\mathfrak{gl}}(V)$-modules provides the sequence of $\mathcal{H}^q_r$-modules

$$0\to(\mathfrak{L}^\Delta_q)_r\to(\mathcal{H}^q_r\otimes\mathcal{H}^q_r)^\Delta\xrightarrow{p}\mathcal{H}^{q\,\Delta}_r\to 0\ . \tag{21}$$

According to Lemma 4 the sequence of $\mathcal{H}^q_r$-modules cf. (21) is exact for all $r\geq 0$. The exactness implies the isomorphism $(\mathcal{H}^q)^\Delta\cong\mathfrak{P}\mathfrak{P}_q$. The theorem is proven. □

Theorem 1 implies the conjecture of Daniel Krob and Jean-Yves Thibon [4].

Corollary 1. *The diagonal algebra* $\mathbb{C}[GL_q(V)]^\Delta$ *is isomorphic to the quantum pseudo-plactic algebra*

$$\mathbb{C}[GL_q(V)]^\Delta\cong\mathfrak{P}\mathfrak{P}_q(\mathbb{T})\ .$$

The isomorphism $\mathfrak{P}\mathfrak{P}_q(\mathbb{T})\cong\mathbb{C}[GL_q(V)]^\Delta$ holds true if and only if the sequence of $\widehat{U_q\mathfrak{gl}}(V)$-modules (20) is exact

$$\mathbb{C}[GL_q(V)]^\Delta\cong\mathbb{C}(q)\left\langle\mathbb{T}\right\rangle/(\mathfrak{L}^\Delta_q(\mathbb{T}))\ .$$

By functoriality the exactness of the sequence of $\widehat{U_q\mathfrak{gl}}(V)$-modules (20) follows from the exactness of the sequence of $\mathcal{H}^q_r$-modules (21). The latter exactness is due the isomorphism $\mathfrak{P}\mathfrak{P}_q\cong\mathcal{H}^{q\,\Delta}$ (Theorem 1).

Acknowledgements It is my pleasure to thank Peter Dalakov, Tekin Dereli, Vladimir Dobrev, Michel Dubois-Violette, Gérard Duchamp, Ludmil Hadjiivanov, Nikolay Nikolov, Petko Nikolov, Oleg Ogievetsky and Ivan Todorov for their encouraging interest in that work and many enlightening discussions. This work has been supported by the Bulgarian National Science Fund research grant DN 18/3 and in part by the TUBITAK 2221 program.

Appendix

Proof of Lemma 3. By abuse of notation we will write $e^{\mp}_{21}$ for the image in the $\mathcal{H}^q_3$-representation $\pi(e^{\mp}_{21})$, *i.e.*, a polynomial of the matrices $\pi(T_{s_1})=(\hat{R}_q)_{12}$ and $\pi(T_{s_2})=(\hat{R}_q)_{23}$ in $\mathrm{End}(V^{\otimes 3})$. The matrices $\pi(T_{s_i})$ commute with quantum matrix elements

$$(\hat{R}_q)_{12}W^{\otimes 3}=W^{\otimes 3}(\hat{R}_q)_{12}\qquad(\hat{R}_q)_{23}W^{\otimes 3}=W^{\otimes 3}(\hat{R}_q)_{23}$$

thus the orthogonality $e^+_{21}e^-_{21}=0$ implies $p(\mathfrak{L}^{\pm}_q(W))=p(e^+_{21}W^{\otimes 3}e^-_{21})=0$.

The proof is by brute force, a direct check using the $n^3 \times n^3$ matrix $[e_{21}^{\pm}]^{i_1 i_2 i_3}_{j_1 j_2 j_3}$.

The grading of the Drinfeld-Jimbo matrix Eq. (8) implies that the matrices in $\mathrm{End}(V^{\otimes 3})$ have zero entries $[e_{21}^{\pm}]^{i_1 i_2 i_3}_{j_1 j_2 j_3} = 0$ if $\{i_1 i_2 i_3\} \neq \{j_1 j_2 j_3\}$ as multisets thus it is enough to restrict our attention to $\dim V = 3$. The matrix of the idempotent $[e_{21}^{\pm}]$ has 6×6 blocks for 3 different indices and 3×3 blocks for 2 different indices. The 6×6 blocks are indexed by $\sigma, \rho \in \{123, 132, 213, 231, 312, 321\}$

$$[e_{21}^{\pm}]^{\sigma}_{\rho} = \frac{1}{[3]} \begin{pmatrix} 1 & \frac{\omega\mp1}{2} & \frac{\omega\mp1}{2} & -\frac{1\pm\omega}{2} & -\frac{1\pm\omega}{2} & \pm1 \\ \frac{\omega\mp1}{2} & \frac{\omega^2\mp\omega+2}{2} & -\frac{1\pm\omega}{2} & \pm1 & \mp\frac{\omega^2+1}{2} & -\frac{1\mp\omega}{2} \\ \frac{\omega\mp1}{2} & -\frac{1\pm\omega}{2} & \frac{\omega^2\mp\omega+2}{2} & \mp\frac{\omega^2+1}{2} & \pm1 & -\frac{1\mp\omega}{2} \\ -\frac{1\pm\omega}{2} & \pm1 & \mp\frac{\omega^2+1}{2} & \frac{\omega^2\pm\omega+2}{2} & -\frac{1\mp\omega}{2} & -\frac{\omega\pm1}{2} \\ -\frac{1\pm\omega}{2} & \mp\frac{\omega^2+1}{2} & \pm1 & -\frac{1\mp\omega}{2} & \frac{\omega^2\pm\omega+2}{2} & -\frac{\omega\pm1}{2} \\ \pm1 & -\frac{1\mp\omega}{2} & -\frac{1\mp\omega}{2} & -\frac{\omega\pm1}{2} & -\frac{\omega\pm1}{2} & 1 \end{pmatrix}$$

while the 3×3 blocks are indexed by $\lambda, \mu \in \{112, 121, 211\}$ or $\{122, 212, 221\}$.

$$[e_{21}^{\pm}]^{\lambda}_{\mu} = \frac{1}{2(q + q^{-1} \pm 1)} \begin{pmatrix} q & -1 \mp q & \pm1 \\ -1 \mp q & q \pm 2 + q^{-1} & -1 \mp q^{-1} \\ \pm1 & -1 \mp q^{-1} & q^{-1} \end{pmatrix}$$

Applying the Einstein summation convention over repeating indices k and l but not on i's we get the sum

$$\mathfrak{L}_q^{\pm}(W)^{i_1 i_2 i_3}_{i_1 i_2 i_3} = [e^{\pm}(q)]^{i_1 i_2 i_3}_{k_1 k_2 k_3} x^{k_1}_{l_1} x^{k_2}_{l_2} x^{k_3}_{l_3} [e^{\mp}(q)]^{l_1 l_2 l_3}_{i_1 i_2 i_3} \qquad \text{no summation on } i_1, i_2, i_3\ .$$

For multi-indices $1 = i_1 < i_2 = i_3 = 2$ of weight $(1, 2)$ we get

$$\mathfrak{L}_q^{\pm}(W)^{122}_{122} = [e^{\pm}(q)]^{122}_{abc} x^a_i x^b_j x^c_k [e^{\mp}(q)]^{ijk}_{122} = \frac{[2]}{4\omega[3]} [x^2_2, [x^1_1, x^2_2]]_{q^2} \in \mathfrak{L}_q^{\Delta}(\mathbb{T})^{12}_2$$

where the off-diagonal terms have replaced by the substitutions

$$\begin{array}{lll} x^{122}_{221} \to q x^{122}_{212} & x^{221}_{122} \to q x^{212}_{122} & x^{122}_{212} \to [x^2_2, x^1_1] x^2_2/\omega \\ x^{221}_{212} \to x^{212}_{221} & x^{212}_{122} \to x^{122}_{212} & x^{212}_{221} \to x^2_2 [x^2_2, x^1_1]/\omega \end{array} . \tag{22}$$

By similar substitutions for indices of weight $(2, 1)$ we get $\mathfrak{L}_q^{\pm}(W)^{112}_{112} \in \mathfrak{L}_q^{\Delta}(\mathbb{T})^{11}_2$.

For multi-indices of weight $(1, 1, 1)$ we "diagonalize" the expression

$$\mathfrak{L}_q^{\pm}(W)^{123}_{123} = [e^{\pm}(q)]^{123}_{\sigma} x^{\sigma}_{\rho} [e^{\mp}(q)]^{\rho}_{123} \qquad \sigma, \rho \in S_3$$

by rewriting it in diagonal monomials $x^{\sigma}_{\sigma} := x^i_i x^j_j x^k_k \in (\mathbb{T}^*)^{\otimes 3}$ where $\sigma \in S_3$ is the word ijk of the permutation $\sigma(1) = i, \sigma(2) = j, \sigma(3) = k$. The restriction to the diagonal subalgebra $\mathfrak{L}_q^{\Delta}(\mathbb{T})$ can be done by the substitutions

$$\begin{array}{lll} x^{213}_{321} \to x^{123}_{231} + \omega x^{213}_{231} & x^{132}_{321} \to x^{123}_{312} + \omega x^{132}_{312} & x^{231}_{312} \to x^{123}_{231} + \omega x^{213}_{231} \\ x^{213}_{312} \to x^{123}_{132} + \omega x^{123}_{312} & x^{312}_{231} \to x^{123}_{312} + \omega x^{123}_{321} & x^{321}_{213} \to x^{123}_{312} + \omega x^{231}_{213} \\ x^{312}_{213} \to x^{132}_{123} + \omega x^{132}_{213} & x^{231}_{132} \to x^{213}_{123} + \omega x^{123}_{312} & x^{132}_{231} \to x^{123}_{213} + \omega x^{123}_{231} \\ x^{132}_{213} \to x^{123}_{231} & x^{321}_{132} \to x^{123}_{132} + \omega x^{123}_{312} & x^{213}_{132} \to x^{123}_{312} \\ x^{321}_{123} \to x^{231}_{213} & x^{312}_{123} \to x^{123}_{231} & x^{231}_{123} \to x^{123}_{312} \end{array}$$

where the terms in RHS are either lowest in lexicographical order , *i.e.,* x^{123}_{abc} or with one rightmost (leftmost) diagonal entry , $x^i_k x^k_i x^j_j$ and $x^j_j x^i_k x^k_i$. In the latter case one can systematically replace the off-diagonal terms by commutators of diagonal terms

$$x^i_k x^k_i x^j_j \to [x^i_i, x^k_k] x^j_j/\omega \qquad\qquad x^j_j x^i_k x^k_i \to x^j_j [x^i_i, x^k_k]/\omega \, .$$

For instance, the terms $x_{213} := x^{123}_{213}$ and $x_{132} := x^{123}_{132}$ which are also lowest in lexicographical order are rewritten as

$$x^{123}_{213} = [x^2_2, x^1_1] x^3_3/\omega \qquad x^{123}_{132} = x^1_1 [x^3_3, x^2_2]/\omega \, . \tag{23}$$

After imposing all substitution above we are left with only three non-diagonal terms x^{123}_{231}, x^{123}_{312} and x^{123}_{321}.

The last equation in the system (13) is the only one containing x^{123}_{231} and x^{123}_{312}

$$x^{321}_{312} - x^{123}_{123} - \omega x^{123}_{213} - \omega x^{123}_{132} - (\omega^3 + \omega) x^{123}_{321} = \omega^2 (x^{123}_{231} + x^{123}_{312})$$

hence it provides an obstruction of an expression to be reducible to a sum of $x^i_i x^j_j x^k_k \in (\mathbb{T}^*)^{\otimes 3}$: it is clear that a sum $\sum_{\sigma \in S_3} d_\sigma x^{123}_\sigma \in (\mathbb{T}^*)^{\otimes 3}$ if and only if $d_{231} = d_{312}$.

The direct check shows that indeed in the sum $\mathfrak{L}^{\pm}_q(W)^{123}_{123}$ the coefficient of x^{123}_{231} is the same as the coefficient of x^{123}_{312} thus the substitution[3]

$$x^{123}_{312} \to -x^{123}_{231} + \frac{1}{\omega^2} \left\{ x^{321}_{312} - x^{123}_{123} - [x^2_2, x^1_1] x^3_3 - x^1_1 [x^3_3, x^2_2] - (\omega^3 + \omega) x^{123}_{321} \right\}$$

will cancel term x^{123}_{312} with x^{123}_{231}. Finally we can eliminate the last off-diagonal term x^{123}_{321} by one of the two possible ways (14) or a combination thereof. We are going to choose the "gauge"

$$x^{123}_{321} \to \omega^{-1} [x^3_3, x^1_1] x^2_2 \, .$$

By direct calculation after replacing all off-diagonal terms by the above substitution we recover the pseudo-Knuth relations $\mathfrak{L}^{\Delta}_q(\mathbb{T})$ for three different indices in Eq. (1)

$$[\mathfrak{L}^{\pm}_q(\mathbb{T})]^{i_1 i_2 i_3}_{i_1 i_2 i_3} = -\frac{\omega^3 \mp \omega^2 \mp 2}{2\omega [3]^2} [[x^{i_1}_{i_1}, x^{i_3}_{i_3}], x^{i_2}_{i_2}] \in \mathfrak{L}^{\Delta}_q(\mathbb{T})^{i_1 i_2}_{i_3} \qquad i_1 < i_2 < i_3 \, .$$

[3]We have used the substitution (23) to eliminate off-diagonal terms $-\omega x^{123}_{213} - \omega x^{123}_{132}$.

Remark. It is tempting to obtain the pseudo-plactic relations in Lemma 3 not as an image of the idempotents $e^{\pm}_{21}$ but by restricting to the diagonal $\mathbb{T}^*$ the image of the q-(anti)symmetrizers e_{1^3} and e_3 instead, e.g.

$$\mathfrak{L}^{\Delta}_q(\mathbb{T}) \stackrel{?}{=} (e_{1^3} W^{\otimes 3} e_3)|_{\mathbb{T}^*} \oplus (e_3 W^{\otimes 3} e_{1^3})|_{\mathbb{T}^*}$$

in parallel with the manner of obtaining the quantum group relations, cf. Eq. (12)

$$\hat{R}_q W \otimes W - W \otimes W \hat{R}_q = e_{1^2} W^{\otimes 2} e_2 \oplus e_2 W^{\otimes 2} e_{1^2} \ .$$

It turns out that indeed we can obtain that way the pseudo-plactic relations with three different indices, e.g., $\mathfrak{L}^{\Delta}_q{}^{1,2}_{\ 3}$ but we, can't obtain the relations with repeating indices, like $\mathfrak{L}^{\Delta}_q{}^{1,1}_{\ 2}$.

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Contractions of Realizations

Maryna Nesterenko and Severin Pošta

Abstract The direct application of the parameterized linear transformations (contraction matrices) to the Lie vector fields that realize a Lie algebra leads to improper (zero operators) realizations. This can be avoided by the contraction of the equivalent realization with the same structure constants. Several illustrative examples are considered.

1 Introduction

Realizations of Lie algebras (representations of Lie algebras by vector fields on manifolds) are widely applicable in modern group analysis of differential equations [1–6], in classification of gravity fields [7], in geometric control theory [8], in difference schemes for numerical solutions of differential equations [9], etc.

To study limit processes between different theories, models or equations it is useful to parameterize respective realization so that as the parameter tends to zero, the realization converges to the realization of another (nonisomorphic) Lie algebra. This problem originates from the contractions of abstract Lie algebras that were introduced in 1951 by I. Segal and later in the work [10] of E. Inönü and E. Wigner it was shown that different physical theories are connected by the contractions of their underlying symmetry algebras.

Unfortunately, the direct application of the known contraction to a realization or representation of a Lie algebra gives several zero operators, what makes it impossible for further application, here we propose the practical method for the construction of the realization contraction, which uses the Shirokov's approach to the construction of left-invariant vector fields [11].

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_35

The paper is arranged as follows. In Sect. 2 we give all necessary notions and definitions on contractions of abstract Lie algebras, then in Sect. 3 we introduce a definition of realizations and general method for their construction. Finally, in Sect. 4 we propose the general method for the contraction of the realization in the case of known contraction matrix and consider several examples. In Conclusions we discuss the possibility of extension of the given algorithm for Lie algebra representations.

2 Contractions of Abstract Lie Algebras

Let $\mathcal{L}_n(V)$ be the variety of n-dimensional Lie algebras (set of Lie brackets) on a vector space V over a field $\mathbb{F}$, then each n-dimensional Lie algebra $\mathfrak{g} = (V, [.,.])$ corresponds to a multiplication rule $\mu \in \mathcal{L}_n$: $\forall\, x, y \in V \quad [x, y] = \mu(x, y)$.

General linear group $\mathrm{GL}(V)$ acts on the variety of Lie brackets as follows:

$$\forall A \in \mathrm{GL}(V),\ \forall \mu \in \mathcal{L}_n \quad (A\mu)(x, y) = A^{-1}(\mu(Ax, Ay)) \quad \forall x, y \in V.$$

To define contractions of abstract Lie algebras over an arbitrary field we consider $\mathrm{GL}(V)$-orbits of $\mu \in \mathcal{L}_n$ and their closures in Zariski topology (closed sets are the algebraic subsets of the variety $\mathcal{L}_n$). In this way Lie algebra $\mathfrak{g} = (V, \mu_0)$ is a *contraction* of $\mathfrak{g} = (V, \mu)$ if μ_0 belongs to the orbit closure of μ.

In practice mainly the fields of real ($\mathbb{R}$) and complex ($\mathbb{C}$) numbers are used, therefore we reformulate the notion of contraction in terms of contraction matrices and structure constants conjugations.

Consider a continuous function $U(\varepsilon) = U\colon (0, 1] \to \mathrm{GL}(V)$ and a parameterized family of Lie algebras $\mathfrak{g}_\varepsilon = (V, [.,.]_\varepsilon)$ with the Lie product defined for arbitrary elements of the vector space $[x, y]_\varepsilon = {U_\varepsilon}^{-1}[U_\varepsilon x, U_\varepsilon y]$. All such algebras are isomorphic to the initial algebra $\mathfrak{g} = (V, [.,.])$.

If $\forall x, y \in V$ there exists a limit

$$[x, y]_0 := \lim_{\varepsilon \to +0} [x, y]_\varepsilon = \lim_{\varepsilon \to +0} {U_\varepsilon}^{-1}[U_\varepsilon x, U_\varepsilon y]$$

then $[.,.]_0$ is a well-defined Lie bracket and Lie algebra $\mathfrak{g}_0 = (V, \mu_0)$ is called a *contraction* of the Lie algebra $\mathfrak{g}$.

Let $\{e_1, \ldots, e_n\}$ be a basis of V, then there is one-to-one correspondence between a Lie algebra $\mathfrak{g} = (V, [.,.])$ and a structure constants tensor $(C_{ij}^k) \in \mathbb{R}^{n^3}$ (or $(C_{ij}^k) \in \mathbb{C}^{n^3}$) given by the commutation relations $[e_i, e_j] = C_{ij}^k e_k$; here and after $i, j, k = 1, 2, \ldots, n$ and the summation w.r.t. the repeated indices is implied.

The operator U_ε is defined by the respective *contraction matrix* and structure constants tensors of the parameterised Lie algebras $\mathfrak{g}_\varepsilon$ are

$$C_{\varepsilon, i'j'}^{k'} := (U_\varepsilon)_{i'}^i (U_\varepsilon)_{j'}^j ({U_\varepsilon}^{-1})_k^{k'} C_{ij}^k.$$

Therefore, the definition of contraction is reduced to the convergence of structure constants $\forall\, i',\ j'$ and k'

$$C^{k'}_{0,i'j'} := \lim_{\varepsilon\to+0} C^{k'}_{\varepsilon,i'j'} = \lim_{\varepsilon\to+0} (U_\varepsilon)^i_{i'} (U_\varepsilon)^j_{j'} (U_\varepsilon{}^{-1})^{k'}_k C^k_{ij}.$$

The study of the contractions of abstract Lie algebras is usually provided in frames of two problems: description of all possible contractions of a fixed Lie algebra or description of all contractions of Lie algebras of a fixed dimension. Both these problems of are rather complicated (e.g., up to now the complete classification of contractions is known only for dimensions not greater then four [12], or for some subsets closed with respect to contractions).

The effective method that allows one to study all inequivalent contractions consists of two key steps: i) to exclude all the pairs of Lie algebras that do not admit any contraction (i.e. that cannot serve as a pair of parent and contracted Lie algebra), ii) to construct explicitly the contraction matrix for the rest of the pairs (of parent and their respective contracted Lie algebras).

To provide the step i) necessary conditions of the contraction existence are used, for the lists of conditions see e.g. [12, 13]. And the step ii) can be realized in many cases by means of the isomorphism transformations of the both initial and resulting Lie algebras and by means of the diagonal contraction matrix $U_\varepsilon = \mathrm{diag}(\varepsilon^{p_1}, \ldots, \varepsilon^{p_n})$, $p_1, \ldots, p_n \in \mathbb{Z}$.

To apply contractions of abstract Lie algebras to some physical theories, models or equations we have to introduce a contraction to some representation of the Lie algebra. In this paper we focus on the representations of Lie algebras by Lie vector fields.

3 Realizations of Lie Algebras

From this moment we suppose that $F = \mathbb{R}$ and V is a real n-dimensional vector space with a basis $\{e_1, e_2, \ldots, e_n\}$ and the structure constants of considered Lie algebras are real. (Note that all considerations are valid for the complex field as well.)

Let $\mathrm{Aut}(\mathfrak{g}) \subset GL_n$ be the whole automorphism group of $\mathfrak{g}$ and let $\mathrm{Int}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$ be the inner automorphism group of $\mathfrak{g}$. Let $M \subset \mathbb{R}^m$ be an open domain. Let us denote the Lie algebra of smooth vector fields on M by $\mathrm{Vect}(M)$.

A *realization* of a Lie algebra $\mathfrak{g}$ in vector fields on M is a homomorphism

$$R\colon \mathfrak{g} \to \mathrm{Vect}(M).$$

The realization is *faithful* if $\ker R = \{0\}$ and *unfaithful* otherwise.

Denote local coordinates of a point $x \in M$ as $(x_1, \ldots, x_m)$, then a realization $R(\mathfrak{g})$ in coordinate form is performed by the images $\Xi_i(x)$ of the basis elements e_i of a general form

$$\Xi_i(x) = R(e_i) = \sum_{l=1}^{m} \xi_{il}(x_1, x_2, \dots, x_m)\partial_l,$$

hereafter $\partial_l = \frac{\partial}{\partial x_l}$ and the coefficients $\xi_{il}(x_1, x_2, \dots, x_m)$ are smooth (analytic) functions.

Let G be the local Lie group that corresponds to a realization R. Then the realization R of a Lie algebra $\mathfrak{g}$ is called *transitive* if the action of the group G is transitive. *Rank of realization* at some point is the rank of the matrix formed by the coefficients of differential operators at this point. For transitive realizations the rank $R_x = m$ for all $x \in M$. If G acts on M transitively, and rank equals n then the corresponding realization is called a *generic* realization.

Formally generic realizations do coincide with the left-invariant vector fields and can be constructed directly from the structure constants [11] in the following way: for each $x \in M$ the coefficients of the map $\xi(x)$ are recovered (by taking the inverse linear transformation) from the dual left-invariant differential one-forms with the coefficients

$$\omega_i^j(x) = \left(e^{(-x_1 \mathrm{ad} e_1)} e^{(-x_2 \mathrm{ad} e_2)} \dots e^{(-x_{i-1} \mathrm{ad} e_{i-1})}\right)_i^j, \quad i, j = 1, 2, \dots, n,$$

where $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$ and $\mathrm{Der}(\mathfrak{g})$ is the algebra of differentiations.

For non-generic case situation is a bit different, but it can be solved by the combination of the above method and projections [14]. Non-transitive cases are constructed by means of the introduction of the arbitrary functions in the place of parameters.

4 Contractions of Realizations

First let us test several known contractions [12] and realizations [15] and combine them directly.

Consider the following three-dimensional Lie algebras: Abelian $3A_1$ [16–19], Heisenberg $A_{3.1}$ with the non-zero commutation relation $[e_2, e_3] = e_1$, and $A_{3.2}$ with the non-zero commutators $[e_1, e_3] = e_1$ and $[e_2, e_3] = e_1 + e_2$.

Each of these algebras has the unique generic realization:

$$\begin{aligned}
&3A_1\colon \; R(e_1) = \partial_1, \; R(e_2) = \partial_2, \; R(e_3) = \partial_3;\\
&A_{3.1}\colon \; R(e_1) = \partial_1, \; R(e_2) = \partial_2, \; R(e_3) = x_2\partial_1 + \partial_3;\\
&A_{3.2}\colon \; R(e_1) = \partial_1, \; R(e_2) = \partial_2, \; R(e_3) = (x_1 + x_2)\partial_1 + x_2\partial_2 + \partial_3.
\end{aligned}$$

There are three possible inequivalent contractions between these algebras: $A_{3.2} \to A_{3.1}$ with the contraction matrix $U(\varepsilon) = \mathrm{diag}(\varepsilon, 0, \varepsilon)$, $A_{3.2} \to 3A_1$ with the contraction matrix $U(\varepsilon) = \mathrm{diag}(\varepsilon, \varepsilon, \varepsilon)$ and $A_{3.1} \to 3A_1$ by $U(\varepsilon) = \mathrm{diag}(\varepsilon, \varepsilon, \varepsilon)$.

Applying these contractions to the above realizations we get unfaithful realizations in all cases, indeed

$$A_{3.2} \to A_{3.1}\colon R(e_1) \to 0,\ R(e_2) \to \partial_2,\ R(e_3) \to 0;$$
$$A_{3.2} \to 3A_1 \text{ and } A_{3.1} \to 3A_1\colon R(e_1) \to 0,\ R(e_2) \to 0,\ R(e_3) \to 0.$$

Such a result is predictable and it was already indicated in the first paper by Inönü and Wigner concerning representations of Lie algebras. In the case of representations the problem of zero matrices can be overcome by additional ε-dependent similarity transformations of the matrices, see e.g. [20].

Generalizing this approach we have to act additionally on the realizations by the ε-dependent transformations from $\mathrm{Aut}(\mathfrak{g})$ and ε-dependent diffeomorphisms of the manifold M, but this results in cumbersome and unsolvable calculations.

Yet another approach to contractions of realizations was successfully used in [21] and [22], namely that was the idea of ε-dependent Lie group transformations proposed by Inönü and Wigner. The disadvantage of this method is that the contraction result should be previously known.

The simple idea proposed here is to construct realization from the structure constants parameterized respectively to the contraction of the abstract Lie algebra. To do this one should follow the steps:

1. Construct parameterized structure constants using the contraction matrix U, that do realize the desired contraction $C^{k'}_{\varepsilon,i'j'} := (U_\varepsilon)^i_{i'}(U_\varepsilon)^j_{j'}(U_\varepsilon{}^{-1})^{k'}_k C^k_{ij}$, where C^k_{ij} are structure constants of the initial Lie algebra.
2. Calculate ε-dependent adjoint actions (using the structure constants $C^{k'}_{\varepsilon,i'j'}$), exponents and differential 1-forms: $\mathrm{ad}^\varepsilon e_i$, $\exp(-x_i \mathrm{ad}^\varepsilon e_i)$, $\omega^\varepsilon(x)$.
3. Find the inverse transformation to obtain the vector fields $\xi^\varepsilon(x) = (\omega^\varepsilon(x))^{-1}$, that are the parameterized realization that do contracts to the realization of the contracted Lie algebra.

Applying the steps 1–3 to the above Lie algebras $3A_1$, $A_{3.1}$ and $A_{3.2}$ and their generic realizations we get well-defined faithful contracted realizations:

- $A_{3.1} \to 3A_1$ by $U(\varepsilon) = \mathrm{diag}(\varepsilon, \varepsilon, \varepsilon)$, therefore $R^\varepsilon(e_1) = \partial_1 \to \partial_1$, $R^\varepsilon(e_2) = \partial_2 \to \partial_2$ and $R^\varepsilon(e_3) = \varepsilon x_2 \partial_1 + \partial_3 \to \partial_3$,
- $A_{3.2} \to A_{3.1}$ by $U(\varepsilon) = \mathrm{diag}(\varepsilon, 0, \varepsilon)$, therefore $R^\varepsilon(e_1) = \partial_1 \to \partial_1$, $R^\varepsilon(e_2) = \partial_2 \to \partial_2$ and $R^\varepsilon(e_3) = (\varepsilon x_1 + x_2)\partial_1 + \varepsilon x_2 \partial_2 + \partial_3 \to x_2\partial_1 + \partial_3$,
- $A_{3.2} \to 3A_1$ by $U(\varepsilon) = \mathrm{diag}(\varepsilon, \varepsilon, \varepsilon)$, therefore $R^\varepsilon(e_1) = \partial_1 \to \partial_1$, $R^\varepsilon(e_2) = \partial_2 \to \partial_2$ and $R^\varepsilon(e_3) = (\varepsilon x_1 + \varepsilon x_2)\partial_1 + \varepsilon x_2 \partial_2 + \partial_3 \to \partial_3$.

5 Conclusions

To conclude let us mention that contraction of the given realization of a Lie algebra will be a bit more complicated, since first we have to define it's subalgebra (studying the kernel of the linear operator in the initial point), then we have to find the equivalence transformations to the canonical (Shirokov's) realization. Now steps 1–3 have to be applied and followed by the inverse of the equivalence transformations.

The proposed method for contraction of realization can also be extended to the representations of Lie algebras.

If the realizations with linear coefficients $e_i = \sum_{k=1}^{m} \left(\sum_{l=1}^{m} a_{ik}^{l} x_l \right) \partial_k$ do exist for a given Lie algebra, then the connection between these realizations and representations is established in the following way.

Using the matrices

$$A_i = \begin{pmatrix} a_{i1}^1 & a_{i2}^1 & \cdots & a_{im}^1 \\ a_{i1}^2 & a_{i2}^2 & \cdots & a_{im}^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}^m & a_{i2}^m & \cdots & a_{im}^m \end{pmatrix}, \quad D = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \vdots \\ \partial_m \end{pmatrix} \quad \text{and} \quad X = (x_1, x_2, \ldots, x_m)$$

we can rewrite the basis elements as $e_i = X A_i D$. Then, as far as $[e_i, e_j] = C_{ij}^k e_i$, the matrixes A_i satisfy the same commutation relations $[A_i, A_j] = C_{ij}^k A_i$ and, therefore, form a representation of the initial Lie algebra and vice versa.

So, the construction of the contraction of the linear realization immediately gives us the contraction of the representation.

We have tested this approach on the adjoint representations of the low-dimensional simple Lie algebras and we have met the necessity to solve rather huge systems of functional equations connected with the equivalence transformations of realizations. This will be a subject of our further investigations.

Acknowledgements MN is grateful for the hospitality extended to her at the Department of mathematics, FNSPE, Czech Technical University in Prague, where part of this work was done. SP acknowledges the support of SGS18/189/OHK4/3T/14, project of the Grant Agency of Czech Technical University in Prague.

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Multi-parameter Formal Deformations of Ternary Hom-Nambu-Lie Algebras

Per Bäck

Abstract In this note, we introduce a notion of multi-parameter formal deformations of ternary hom-Nambu-Lie algebras. Within this framework, we construct formal deformations of the three-dimensional Jacobian determinant and of the cross-product in four-dimensional Euclidean space. We also conclude that the previously defined ternary q-Virasoro-Witt algebra is a formal deformation of the ternary Virasoro-Witt algebra.

1 Introduction

The notion of n-ary hom-Nambu-Lie algebras was introduced by Ataguema, Makhlouf, and Silvestrov in [3], including n-ary Nambu-Lie algebras as a special case. Generalizing an algebraic structure to a so-called hom-algebraic structure is often motivated by the fact that algebras that are rigid in terms of the former structure may be deformed when viewed as algebras in terms of the latter structure. For the simplest case $n = 2$ of hom-Lie algebras, there seem to be many examples of formal deformations of Lie algebras into hom-Lie algebras (see e.g. [4–6, 8]). For the case $n = 3$ of ternary hom-Nambu-Lie algebras, examples of formal deformations seem to be lacking, however. The purpose of this short, but self-contained note is to provide examples of formal deformations of ternary Nambu-Lie algebras into ternary hom-Nambu-Lie algebras, to devise a concrete method on how to construct these, and to define a framework in which they all fit.

2 Preliminaries

We denote by $\mathbb{N}$ the natural numbers including zero, and by K a field of characteristic zero. The symmetric group on p letters is written S_p.

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_36

Definition 1 (Ternary hom-Nambu algebra). A *ternary hom-Nambu algebra*, written $(V, [\cdot,\cdot,\cdot], (\alpha, \beta))$, consists of a K-vector space V, two K-linear maps $\alpha, \beta\colon V \to V$ called *twisting maps*, and a K-trilinear map $[\cdot,\cdot,\cdot]\colon V \times V \times V \to V$ called the *hom-Nambu bracket*. For all $x_1, \ldots, x_5 \in V$, they are required to satisfy the *ternary hom-Nambu identity*:

$$[\alpha(x_1), \beta(x_2), [x_3, x_4, x_5]] = [[x_1, x_2, x_3], \alpha(x_4), \beta(x_5)] \\ + [\alpha(x_3), [x_1, x_2, x_4], \beta(x_5)] + [\alpha(x_3), \beta(x_4), [x_1, x_2, x_5]].$$

Definition 2 (Ternary hom-Nambu-Lie algebra). A *ternary hom-Nambu-Lie algebra* is a ternary hom-Nambu algebra $(V, [\cdot,\cdot,\cdot], (\alpha, \beta))$ where the bracket $[\cdot,\cdot,\cdot]$ is skew-symmetric, meaning that for all $x_1, x_2, x_3 \in V$, $\sigma \in S_3$,

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = \operatorname{sgn}(\sigma)[x_1, x_2, x_3].$$

Remark 1. If we put $\alpha = \beta = \mathrm{id}_V$ in the above definitions, we get a *ternary Nambu algebra* and a *ternary Nambu-Lie algebra*, respectively. Instead of writing $(V, [\cdot,\cdot,\cdot], (\mathrm{id}_V, \mathrm{id}_V))$, we then use the shorthand notation $(V, [\cdot,\cdot,\cdot])$.

Definition 3 (Morphisms of ternary hom-Nambu-(Lie) algebras). A *morphism* from a ternary hom-Nambu-(Lie) algebra $A := (V, [\cdot,\cdot,\cdot], (\alpha, \beta))$ to a ternary hom-Nambu-(Lie) algebra $A' := (V', [\cdot,\cdot,\cdot]', (\alpha', \beta'))$ is a K-linear map $f\colon V \to V'$ such that for all $x_1, x_2, x_3 \in V$, $f([x_1, x_2, x_3]) = [f(x_1), f(x_2), f(x_3)]'$, $f \circ \alpha = \alpha' \circ f$, and $f \circ \beta = \beta' \circ f$. We denote the set of all such maps by $\mathrm{Hom}_K(A, A')$, and put $\mathrm{End}_K(A) := \mathrm{Hom}_K(A, A)$ for the set of *endomorphisms*.

A ternary hom-Nambu-(Lie) algebra $A := (V, [\cdot,\cdot,\cdot], (\alpha, \beta))$ is *multiplicative* whenever $\alpha = \beta \in \mathrm{End}_K(A)$.

Proposition 1 ([3]). *Let $A := (V, [\cdot,\cdot,\cdot])$ be a ternary Nambu(-Lie) algebra and $\rho \in \mathrm{End}_K(A)$. Then $(V, \rho \circ [\cdot,\cdot,\cdot], (\rho, \rho))$ is a multiplicative, ternary hom-Nambu (-Lie) algebra.*

Let us now take a look at some classical examples of ternary (hom-)Nambu-Lie algebras that can be found in the literature.

Example 1 (The cross-product in $\mathbb{R}^4$). Denote by $\mathbb{R}^4$ be the four-dimensional Euclidean vector space over $\mathbb{R}$, and by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ its standard basis. Using the Einstein summation convention, let $\mathbf{x} := x^i\mathbf{e}_i$, $\mathbf{y} := y^i\mathbf{e}_i$, and $\mathbf{z} := z^i\mathbf{e}_i$ where $x^i, y^i, z^i \in \mathbb{R}$ are arbitrary and $1 \le i \le 4$. We then get a ternary Nambu-Lie algebra $(\mathbb{R}^4, [\cdot,\cdot,\cdot])$ over $\mathbb{R}$ by defining the bracket as the cross-product in $\mathbb{R}^4$,

$$[\mathbf{x}, \mathbf{y}, \mathbf{z}] := \mathbf{x} \times \mathbf{y} \times \mathbf{z} := \begin{vmatrix} x^1 & y^1 & z^1 & \mathbf{e}_1 \\ x^2 & y^2 & z^2 & \mathbf{e}_2 \\ x^3 & y^3 & z^3 & \mathbf{e}_3 \\ x^4 & y^4 & z^4 & \mathbf{e}_4 \end{vmatrix}.$$

Example 2 (The three-dimensional Jacobian determinant [3]*).* Let q_1, q_2, and q_3 be elements of the polynomial K-algebra $K[x_1, x_2, x_3]$ in three indeterminates x_1, x_2, and x_3. Denote by q the triplet (q_1, q_2, q_3), by x (x_1, x_2, x_3), and by $J(q(x)) = (\partial q_i/\partial x_j)_{1\le i,j\le 3}$, the three-dimensional Jacobian. By defining

$$[q_1(x), q_2(x), q_3(x)] := \det(J(q(x))) = \begin{vmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \frac{\partial q_1}{\partial x_3} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \frac{\partial q_2}{\partial x_3} \\ \frac{\partial q_3}{\partial x_1} & \frac{\partial q_3}{\partial x_2} & \frac{\partial q_3}{\partial x_3} \end{vmatrix},$$

we get a ternary Nambu-Lie algebra $(K[x_1, x_2, x_3], [\cdot,\cdot,\cdot])$ over K. Any $\rho \in \mathrm{End}_K((K[x_1, x_2, x_3], [\cdot,\cdot,\cdot]))$ gives rise to a ternary hom-Nambu-Lie algebra by Proposition 1. Let $\gamma_1, \gamma_2, \gamma_3 \in K[x_1, x_2, x_3]$, $\gamma := (\gamma_1, \gamma_2, \gamma_3)$, and define $\rho_\gamma\colon K[x_1, x_2, x_3] \to K[x_1, x_2, x_3]$ by $q(x) \mapsto q(\gamma)$. As $J(\rho_\gamma(q_1), \rho_\gamma(q_2), \rho_\gamma(q_3)) = J(q(\gamma))J(\gamma(x))$, $\det(J(\rho_\gamma(q_1), \rho_\gamma(q_2), \rho_\gamma(q_3))) = \det(J(q(\gamma)))\det(J(\gamma(x)))$. We have that $\det(J(q(\gamma)) = \rho_\gamma(\det(J(q(x))))$, so by means of Proposition 1, we have a ternary hom-Nambu-Lie algebra $(K[x_1, x_2, x_3], [\cdot,\cdot,\cdot]_\gamma, (\rho_\gamma, \rho_\gamma))$ where $[\cdot,\cdot,\cdot]_\gamma := \rho_\gamma \circ [\cdot,\cdot,\cdot]$, if $\det(J(\gamma(x))) = 1$.

Example 3 (The ternary q-Virasoro-Witt algebra [2]*).* Let W be a vector space over $\mathbb{C}$ with generating set $\{Q_n, R_n\}_{n\in\mathbb{Z}}$, and $[\cdot,\cdot,\cdot]\colon W \times W \times W \to W$ the $\mathbb{C}$-trilinear, skew-symmetric bracket defined on the generators of W by

$$\begin{aligned} [Q_k, Q_m, Q_n] &= (k-m)(m-n)(k-n)R_{k+m+n}, \\ [Q_k, Q_m, R_n] &= (k-m)(Q_{k+m+n} + znR_{k+m+n}), \\ [Q_k, R_m, R_n] &= (n-m)R_{k+m+n}, \\ [R_k, R_m, R_n] &= 0. \end{aligned}$$

Whenever $z = \pm 2i$, these relations define a ternary Nambu-Lie algebra $(W, [\cdot,\cdot,\cdot])$ over $\mathbb{C}$ called the *ternary Virasoro-Witt algebra*, introduced by Curtright, Fairlie and Zachos in [7]. Ammar, Makhlouf, and Silvestrov found in [2] that a $\mathbb{C}$-linear map $\rho_q\colon W \to W$ defined on the generators of W by $\rho_q(Q_n) := q^n Q_n$ and $\rho_q(R_n) := q^n R_n$ for some $q \in \mathbb{C}$ is an endomorphism of the ternary Virasoro-Witt algebra. Hence, by Proposition 1, we have a ternary hom-Nambu-Lie algebra $(W, [\cdot,\cdot,\cdot]_q, (\rho_q, \rho_q))$, $[\cdot,\cdot,\cdot]_q := \rho_q \circ [\cdot,\cdot,\cdot]$ whenever $z = \pm 2i$. This algebra is called the *ternary q-Virasoro-Witt algebra*, including the ternary Virasoro-Witt algebra in the case $q = 1$. When $q \neq 1$, one does in general not get a ternary Nambu-Lie algebra.

Remark 2. When $z \neq \pm 2i$, the ternary Virasoro-Witt algebra is in fact a hom-Nambu-Lie algebra [2].

3 Examples

Guided by the way of constructing the ternary q-Virasoro-Witt algebra in Example 3 by means of Proposition 1, we will here apply the same method to the cross-product in $\mathbb{R}^4$ defined in Example 1 and the three-dimensional Jacobian determinant defined in Example 2.

Example 4 (A deformed cross-product in $\mathbb{R}^4$). Using the same notation as in Example 1, we would like to find all $\mathbb{R}$-linear maps $\rho\colon \mathbb{R}^4 \to \mathbb{R}^4$ such that $\rho([\mathbf{x}, \mathbf{y}, \mathbf{z}]) = [\rho(\mathbf{x}), \rho(\mathbf{y}), \rho(\mathbf{z})]$. To this end, let $\rho(\mathbf{e}_l) = a_l^i \mathbf{e}_i$ where $a_l^i \in \mathbb{R}$, $1 \le i, l \le 4$. By using the Kronecker delta δ_j^i and a Levi-Civita symbol ε_{ijk}^l,

$$\varepsilon_{ijk}^l := \begin{cases} +1 & \text{if } (i, j, k, l) \text{ is an even permutation of } (1, 2, 3, 4), \\ -1 & \text{if } (i, j, k, l) \text{ is an odd permutation of } (1, 2, 3, 4), \\ 0 & \text{if any two indices are equal,} \end{cases}$$

$$\begin{aligned} \rho\left([\mathbf{e}_l, \mathbf{e}_m, \mathbf{e}_n]\right) =& \rho\left(\begin{vmatrix} \delta_l^1 & \delta_m^1 & \delta_n^1 & \mathbf{e}_1 \\ \delta_l^2 & \delta_m^2 & \delta_n^2 & \mathbf{e}_2 \\ \delta_l^3 & \delta_m^3 & \delta_n^3 & \mathbf{e}_3 \\ \delta_l^4 & \delta_m^4 & \delta_n^4 & \mathbf{e}_4 \end{vmatrix}\right) = \rho\left(\varepsilon_{pqr}^s \delta_l^p \delta_m^q \delta_n^r \mathbf{e}_s\right) = \rho\left(\varepsilon_{lmn}^s \mathbf{e}_s\right) \\ =& \varepsilon_{lmn}^s \rho(\mathbf{e}_s) = \varepsilon_{lmn}^s a_s^t \mathbf{e}_t, \\ [\rho(\mathbf{e}_l), \rho(\mathbf{e}_m), \rho(\mathbf{e}_n)] =& \begin{vmatrix} a_l^1 & a_m^1 & a_n^1 & \mathbf{e}_1 \\ a_l^2 & a_m^2 & a_n^2 & \mathbf{e}_2 \\ a_l^3 & a_m^3 & a_n^3 & \mathbf{e}_4 \\ a_l^4 & a_m^4 & a_n^4 & \mathbf{e}_4 \end{vmatrix} = \varepsilon_{pqr}^s a_l^p a_m^q a_n^r \mathbf{e}_s. \end{aligned}$$

By comparing coefficients, we get the following system of equations: $\varepsilon_{lmn}^s a_s^t = \varepsilon_{pqr}^t a_l^p a_m^q a_n^r$, $1 \le l, m, n, t \le 4$. A solution ρ_θ, $\theta := (\theta_1, \theta_2)$, is $a_1^1 = a_3^3 = \cos\theta_1$, $a_2^2 = a_4^4 = \cos\theta_2$, $a_1^3 = -a_3^1 = \sin\theta_1$, $a_2^4 = -a_4^2 = \sin\theta_2$, as the only nonzero elements, $\theta_1, \theta_2 \in \mathbb{R}$. In matrix form,

$$\begin{aligned} \rho_\theta =& \begin{pmatrix} a_1^1 & a_2^1 & a_3^1 & a_4^1 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 \\ a_1^4 & a_2^4 & a_3^4 & a_4^4 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \\ =& \begin{pmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix}. \end{aligned}$$

Thus, we see that ρ_θ is the product of two rotation matrices (which commute): one in the $\mathbf{e}_1\mathbf{e}_3$-plane by an angle θ_1, and one in the $\mathbf{e}_2\mathbf{e}_4$-plane by an angle θ_2. We put $[\cdot,\cdot,\cdot]_\theta := \rho_\theta \circ [\cdot,\cdot,\cdot]$ and call the hom-Nambu-Lie algebra $(\mathbb{R}^4, [\cdot,\cdot,\cdot]_\theta, (\rho_\theta, \rho_\theta))$ a *deformed cross-product in* $\mathbb{R}^4$ (in the last section, we will justify the name *deformed*). In general, this is not a Nambu-Lie algebra. Put e.g. $\theta_1 = \theta_2 = \pi/2$ and $\mathbf{e}_5 := \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4$. Then $[\mathbf{e}_1, \mathbf{e}_2, [\mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5]_\theta]_\theta = \mathbf{e}_1 + \mathbf{e}_2$, while $[[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]_\theta, \mathbf{e}_4, \mathbf{e}_5]_\theta + [\mathbf{e}_3, [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4]_\theta, \mathbf{e}_5]_\theta + [\mathbf{e}_3, \mathbf{e}_4, [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5]_\theta]_\theta = -\mathbf{e}_1 - \mathbf{e}_2$.

Example 5 (A deformed three-dimensional Jacobian determinant). Using the same notation as in Example 2, we would like to find a nontrivial $\gamma(x)$ with $\det(J(\gamma(x))) = 1$. We start with the simple, but nontrivial assumption that $J(\gamma(x))$ is an upper triangular matrix (the case when $J(\gamma(x))$ is a lower triangular matrix is analogous). Using that $\det(J(\gamma(x)))$ is the product of all the diagonal entries of $J(\gamma(x))$, one readily verifies that $\det(J(\gamma(x))) = 1$ if and only if $\gamma(x_1, x_2, x_3) = (k_1x_1 + p_1(x_2, x_3), k_2x_2 + p_2(x_3), k_3x_3 + k_4)$ for some $p_1(x_2, x_3) \in K[x_2, x_3]$, $p_2(x_2) \in K[x_2]$, and $k_1, k_2, k_3, k_4 \in K$ where $k_1k_2k_3 = 1$. A basis of $K[x_1, x_2, x_3]$ as a K-vector space consists of all monomials $x_1^l x_2^m x_3^n$ where $l, m, n \in \mathbb{N}$, and so if we define $\rho_\gamma(x_1^l x_2^m x_3^n) := (k_1x_1 + p_1(x_2, x_3))^l (k_2x_2 + p_2(x_3))^m (k_3x_3 + k_4)^n$ and then extend the definition linearly, we have a ternary hom-Nambu-Lie algebra $(K[x_1, x_2, x_3], [\cdot,\cdot,\cdot]_\gamma, (\rho_\gamma, \rho_\gamma))$ where $[\cdot,\cdot,\cdot]_\gamma := \rho_\gamma \circ [\cdot,\cdot,\cdot]$. We refer to it as a *deformed three-dimensional Jacobian determinant* (again, we will justify the name *deformed* in the last section). In general this is not a Nambu-Lie algebra. Take e.g. $p_1(x_2, x_3) = p_2(x_3) = 0, k_1 = k_2 = k_3 = 1$, and $q_1 := x_1, q_2 := x_2, q_3 := x_3^3, q_4 := x_1^2, q_5 := x_2x_3$. Then $[q_1, q_2, [q_3, q_4, q_5]_\gamma]_\gamma = 18x_1(x_3 + 2k_4)^2$, while $[[q_1, q_2, q_3]_\gamma, q_4, q_5]_\gamma + [q_3, [q_1, q_2, q_4]_\gamma, q_5]_\gamma + [q_3, q_4, [q_1, q_2, q_5]_\gamma]_\gamma = 6x_1(x_3 + k_4)(3x_3 + 5k_4)$, so the two expressions are equal if and only if $k_4 = 0$, in which case we have the original three-dimensional Jacobian determinant.

4 Multi-parameter Formal Deformations

One-parameter formal ternary hom-Nambu-Lie deformations were defined in [1]. Here, we generalize that notion to multi-parameter analogues.

Definition 4 (Multi-parameter formal ternary hom-Nambu(-Lie) deformation). An *n-parameter formal ternary hom-Nambu(-Lie) deformation* of a ternary hom-Nambu(-Lie) algebra $(V, [\cdot,\cdot,\cdot]_0, (\alpha_0, \beta_0))$ over K is a ternary hom-Nambu(-Lie) algebra $(V[[t_1, t_2, \ldots, t_n]], [\cdot,\cdot,\cdot]_t, (\alpha_t, \beta_t))$ over $K[[t_1, t_2, \ldots, t_n]]$ where $n \in \mathbb{N}_{>0}$, $t := (t_1, t_2, \ldots, t_n)$, and

$$[\cdot,\cdot,\cdot]_t = \sum_{i\in\mathbb{N}^n} [\cdot,\cdot,\cdot]_i t^i, \quad \alpha_t = \sum_{i\in\mathbb{N}^n} \alpha_i t^i, \quad \beta_t = \sum_{i\in\mathbb{N}^n} \beta_i t^i.$$

Here, $i := (i_1, i_2, \ldots, i_n) \in \mathbb{N}^n$, $t^i := t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$, and $[\cdot,\cdot,\cdot]_i : V \times V \times V \to V$ is a K-trilinear operation, $\alpha_i, \beta_i : V \to V$ K-linear maps, extended homogeneously to a $K[[t_1, t_2, \ldots, t_n]]$-trilinear operation, $[\cdot,\cdot,\cdot]_i\colon V[[t_1, t_2, \ldots, t_n]] \times V[[t_1, t_2, \ldots, t_n]] \times V[[t_1, t_2, \ldots, t_n]] \to V[[t_1, t_2, \ldots, t_n]]$, and $K[[t_1, t_2, \ldots, t_n]]$-linear maps $\alpha_i, \beta_i\colon V[[t_1, t_2, \ldots, t_n]] \to V[[t_1, t_2, \ldots, t_n]]$, respectively.

In the above definition, a map $f\colon V \to V$ is said to be *extended homogeneously* to a map $f\colon V[[t_1, t_2, \ldots, t_n]] \to V[[t_1, t_2, \ldots, t_n]]$ when $f(at^i) := f(a)t^i$ for all $a \in V$ and $i \in \mathbb{N}^n$. The case for ternary maps is analogous.

Remark 3. Note that there is some slight abuse of notation in Definition 4. The maps $[\cdot,\cdot,\cdot]_0$, α_0, and β_0 are the maps $[\cdot,\cdot,\cdot]_i$, α_i, and β_i where $i = (0, 0, \ldots, 0) \in \mathbb{N}^n$.

Proposition 2. *The ternary q-Virasoro-Witt algebra is a one-parameter formal ternary hom-Nambu-Lie deformation of the ternary Virasoro-Witt algebra.*

Proof. Put $t := q - 1$ and regard it as a formal parameter.

Proposition 3. *The deformed cross-product is a two-parameter formal ternary hom-Nambu-Lie deformation of the cross-product in $\mathbb{R}^4$.*

Proof. Put $t_1 := \theta_1$ and $t_2 := \theta_2$ and regard them as formal parameters. Replace $\cos t_k$ and $\sin t_k$ for $1 \le k \le 2$ with their corresponding formal power series $\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} t_k^{2i}$ and $\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} t_k^{2i+1}$, respectively.

Proposition 4. *The deformed three-dimensional Jacobian determinant is a multi-parameter formal ternary hom-Nambu-Lie deformation of the three-dimensional Jacobian determinant.*

Proof. Choose k_1, k_2, k_3 such that $k_1 k_2 k_3 = 1$ and regard k_4 and all the coefficients in the polynomials $p_1(x_2, x_3)$, $p_2(x_3)$ as formal parameters.

Acknowledgements I wish to thank Joakim Arnlind for some initial discussions.

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Applications to Quantum Theory

Meta-conformal Invariance in the Directed Glauber-Ising Chain

Stoimen Stoimenov and Malte Henkel

Abstract Meta-conformal transformations arise from the description of non-equilibrium relaxational dynamics in directed spin systems. The two-point correlation function of the biased Glauber-Ising chain, quenched to a vanishing temperature, is analysed. If in addition sufficiently long-ranged initial spin-spin correlations are admitted, the dynamical exponent changes from the value $z = 2$ of diffusive transport to $z = 1$, typical of ballistic transport. Then the exactly derived two-time spin-spin correlator coïncides with the prediction of meta-conformal invariance.

1 Introduction

In a two-dimensional time-space with points $(t, r) \in \mathbb{R}^2$, meta-conformal transformations have the infinitesimal generators [18]

$$
\begin{aligned}
X_n &= -t^{n+1}\partial_t - \mu^{-1}[(t+\mu r)^{n+1} - t^{n+1}]\partial_r - (n+1)\frac{\gamma}{\mu}[(t+\mu r)^n - t^n] - (n+1)\delta t^n \\
Y_n &= -(t+\mu r)^{n+1}\partial_r - (n+1)\gamma(t+\mu r)^n
\end{aligned}
\tag{1}
$$

where δ, γ are constants and μ^{-1} is a constant universal velocity.

The generators $X_{-1} = -\partial_t$ and $Y_{-1} = -\partial_r$ of time/space-translations, and the generator $X_0 = -t\partial_t - r\partial_r - \delta$ of dilatations are the same as for usual $2D$ conformal

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_37

algebra.[1] The other generators are different,[2] hence meta-conformal transformations (1) are in general not angle-preserving. Their Lie algebra $\langle X_n, Y_n\rangle_{n\in\mathbb{Z}}$ obeys

$$[X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m] = (n-m)Y_{n+m}, \quad [Y_n, Y_m] = \mu(n-m)Y_{n+m} \tag{2}$$

The maximal finite-dimensional sub-algebra is denoted $\mathfrak{meta}(1,1) := \langle X_{\pm1,0}, Y_{\pm1,0}\rangle$. Indeed, if $\mu \neq 0$, (2) is isomorphic to $2D$ ortho-conformal algebra. To see this, let $X_n = \ell_n + \bar{\ell}_n$ and $Y_n = \mu\bar{\ell}_n$. This gives

$$\begin{aligned} \ell_n &= -t^{n+1}\left(\partial_t - \frac{1}{\mu}\partial_r\right) - (n+1)\left(\delta - \frac{\gamma}{\mu}\right)t^n \\ \bar{\ell}_n &= -\frac{1}{\mu}(t+\mu r)^{n+1}\partial_r - (n+1)\frac{\gamma}{\mu}(t+\mu r)^n. \end{aligned} \tag{3}$$

The reduction of (3) to the standard form of Virasoro generators, in 'complex' light-cone coordinates $z, \bar{z}$ is achieved by setting $z = t$ and $\bar{z} = t + \mu r$, and identifying the conformal weights $\Delta = \delta - \gamma/\mu$ and $\overline{\Delta} = \gamma/\mu$. In $1+1$ time-space dimensions, the meta-conformal transformations (1) and the ortho-conformal transformations are two representations of the same conformal Lie algebra. They both extend (in different way) a global dynamical scaling with dynamical exponent $z = 1$. However in more than one space dimensions the structure of meta-conformal algebra differs from that of corresponding ortho-conformal algebra [22, 26].

The generators (1) are dynamical symmetries of the equation of motion

$$\hat{S}\phi(t,r) = (-\mu\partial_t + \partial_r)\varphi(t,r) = 0. \tag{4}$$

Indeed, since (with $n \in \mathbb{Z}$)

$$[\hat{S}, X_n] = -(n+1)t^n\hat{S} + n(n+1)\mu\left(\delta - \frac{\gamma}{\mu}\right)t^{n-1} \quad , \quad [\hat{S}, Y_n] = 0 \tag{5}$$

a solution φ of (4) with scaling dimension $\delta_\varphi = \delta = \gamma/\mu$ is mapped onto another solution of (4). Hence *the space of solutions of the equation* (4) *is meta-conformally invariant.* This is the analogue of the ortho-conformal invariance of the $2D$ Laplace equation. The equation of motion (4), with a directional bias, motivates to look for physical applications in the kinetics of spin systems with directed dynamics.

Meta-conformally co-variant two-point functions read [18], up to normalisation

$$C(t,r) = \langle\varphi_1(t,r)\varphi_2(0,0)\rangle = \delta_{\delta_1,\delta_2}\delta_{\gamma_1,\gamma_2}\, t^{-2\delta_1}\left(1 + \mu\frac{r}{t}\right)^{-2\gamma_1/\mu} \tag{6}$$

If $\mu \to 0$, the algebra (2) contracts to the *'conformal galilean algebra'* CGA(1).

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[1] We shall call it *ortho-conformal algebra* in order to distinguish it from meta-conformal one.

[2] The above generators can be readily extended [26], but this fact is more important for representations with more than one space dimensions.

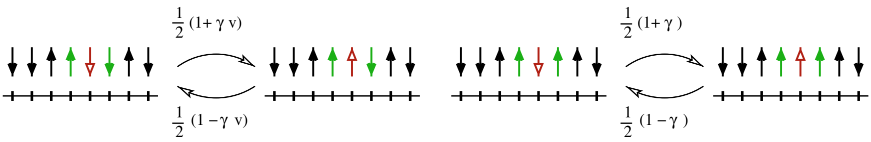

Fig. 1 The flipping rates of the central spin (open arrow, in red) in the directed Glauber-Ising model. They depend on the bias v only if (a) the two nearest neighbours (in green) have different orientation, but not if (b) the orientation of the two neighbours is the same

Physical systems with dynamical exponent $z = 1$ are quite common. First, the dynamical symmetries of the Jeans-Vlassov equation [4, 9, 24] in one space dimension are given by a representation of (2), distinct from (1) [25]. Second, the non-equilibrium dynamics of closed quantum systems generically has $z = 1$, related to ballistic spreading of signals, see [2, 3, 6, 8]. Third, Eq. (4) arises in generalised hydrodynamics description of strongly interacting non-equilibrium quantum systems [1, 5, 7, 23].

Here, we shall focus on a new application of meta-conformal invariance in the non-equilibrium relaxational dynamics in directed spin systems,[3] such as the directed Glauber-Ising model [13, 15, 16].

2 The Directed Glauber-Ising Chain

The directed Glauber-Ising chain is defined as follows, on an infinitely long chain. Ising spins $\sigma_n = \pm 1$ are attached to each site n. To each configuration $\{\sigma\}$ of spins the energy $\mathscr{H}[\sigma] = -\sum_n \sigma_n \sigma_{n+1}$ is associated. The dynamics proceeds through flips of individual spins, with the rates given by [13, 15]

$$w_n(\sigma_n) = \frac{1}{2}\left[1 - \frac{\gamma}{2}(1-v)\sigma_{n-1}\sigma_n - \frac{\gamma}{2}(1+v)\sigma_n\sigma_{n+1}\right] \tag{7}$$

where $\gamma = \tanh(2/T)$ parametrises the temperature and the left-right bias of the dynamics is described by the parameter v, see Fig. 1 for illustration. Such a directed dynamics does no longer obey the condition of detailed balance, although global balance still holds. Therefore, with the rates (7), the equilibrium Gibbs-Boltzmann state is still a stationary state of the dynamics [13]. For either a fully disordered or else a thermalised initial state, the consequences of a non-vanishing bias $v \neq 0$ on the long-time relaxational properties, especially on the precise way how the equilibrium fluctuation-dissipation theorem is broken, have been studied in great detail [13, 15]. Analogous studies have also been carried out in the $2D$ directed Glauber-Ising model [16] and the directed d-dimensional spherical model [14]. Important observables of interest are the two-time and single-time spin-spin correlators

[3] Most of the results discussed here are taken from our original paper [22].

$$C_n(t,s) := \langle \sigma_n(t)\sigma_0(s)\rangle \quad , \quad C_n(s) := C_n(s,s) = \langle \sigma_n(s)\sigma_0(s)\rangle \tag{8}$$

Spatial translation-invariance will be admitted throughout. We focus here on how a meta-conformal dynamical symmetry is realised in this model. Long-ranged correlations in the initial state, viz. $C_n(0) \sim |n|^{-\aleph}$ for $|n| \gg 1$, will become essential.[4]

From the rates (7), the equations of motion of the correlators are found [13]

$$\partial_s C_n(s) = -2(1-\gamma)C_n(s) + \gamma\Big(C_{n-1}(s) + C_{n+1}(s) - 2C_n(s)\Big) + \delta_{n,0}\, Z(s) \tag{9}$$

$$\begin{aligned}\partial_\tau C_n(\tau+s,s) = &-(1-\gamma)C_n(\tau+s,s) \\ &+ \frac{\gamma}{2}\Big(C_{n-1}(\tau+s,s) + C_{n+1}(\tau+s,s) - 2C_n(\tau+s,s)\Big) \\ &+ \frac{\gamma v}{2}\Big(C_{n+1}(\tau+s,s) - C_{n-1}(\tau+s,s)\Big)\end{aligned} \tag{10}$$

where the Lagrange multiplier $Z(s)$ is fixed by the condition $C_0(s) = 1$. One has the compatibility condition $C_n(s,s) = C_n(s)$.

It is known that the requirement of meta-conformal co-variance determines the scaling form of *correlators* [21]. From (9) it follows that the single-time correlator $C_n(s)$ is independent of the bias v. Hence we study the two-time correlator $C_n(t,s)$.

For illustration, consider first the infinite-temperature limit $\gamma \to 0$ but such that $\gamma v \to \upsilon$ remains finite [15]. Take the continuum limit of (10) and let $C(\tau+s,s;r) = e^{-\tau}\mathscr{C}(\tau+s,s;r)$. This gives the equation $\big(\partial_\tau - \upsilon\partial_r\big)\mathscr{C}(\tau+s,s;r) = 0$, analogous to (4), and with the solution $\mathscr{C}(\tau+s,s;r) = \mathfrak{C}(s;r+\upsilon\tau)$. In the special case $s=0$ of a vanishing waiting time, one has $C(0,0;r) = \mathscr{C}(0,0;r) = \mathfrak{C}(0;r)$. Hence, for a spatially long-ranged initial correlator $C(0,0;r) \sim |r|^{-\aleph}$, with $\aleph > 0$, the two-time correlator $C(\tau,0;r) \sim e^{-\tau}(r+\upsilon\tau)^{-\aleph}$ has indeed the form (6) predicted by meta-conformal invariance, up to an exponential pre-factor.

We now analyse the long-time behaviour in more detail, and for any temperature $T \geq 0$. The equation of motion (10) is solved through a Fourier transformation

$$\widetilde{C}(\tau+s,s;k) = \sum_{n\in\mathbb{Z}} C_n(\tau+s,s)\, e^{-\mathrm{i}nk} \quad , \quad C_n(\tau+s,s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}k\, e^{\mathrm{i}kn}\, \widetilde{C}(\tau+s,s;k) \tag{11}$$

which in Fourier space leads to

$$\widetilde{C}(\tau+s,s;k) = \widetilde{C}(s;k)\exp\big(-\big[1-\gamma\cos k - \mathrm{i}\gamma v\sin k\big]\tau\big) \tag{12}$$

Tauberian theorems [10] state that the long-time behaviour follows from the form of $\widetilde{C}(\tau+s,s;k)$ around $k \approx 0$. Here, we want to look at a 'ballistic' scaling regime where $k\tau$ is being kept fixed, rather than that regime $k^2\tau = \text{cste}$ typical for diffusive motion. Indeed, for diffusive scaling, the momenta $k \sim \tau^{-1/2} \gg \tau^{-1}$ are much larger

[4]For *unbiased* dynamics with $v = 0$, any long-ranged initial conditions with $\aleph > 0$ do not modify the leading long-time relaxation behaviour of the Glauber-Ising chain [19].

that the ones to be considered here. From now on, we consider a long-ranged initial correlator of the form $C_n(0) \sim |n|^{-\aleph}$, for $|n| \to \infty$ and with $\aleph > 0$. As an example, we consider the following explicit form: it is symmetric in n, has the required asymptotic behaviour and is normalised to $C_0(0) = 1$. It reads [17, 22]

$$C_n(0) = \frac{\Gamma\big(|n| + (1-\aleph)/2\big)}{\Gamma\big(|n| + (1+\aleph)/2\big)} \frac{\Gamma\big((1+\aleph)/2\big)}{\Gamma\big((1-\aleph)/2\big)} \;, \tag{13}$$
$$\widetilde{C}(0;k) = \frac{\Gamma\big((1+\aleph)/2\big)^2}{\Gamma\big(\aleph\big)} \left(2\sin\frac{|k|}{2}\right)^{\aleph-1}$$

such that indeed $\widetilde{C}(0;k) \simeq \widetilde{C}_0 |k|^{\aleph-1}$, for $|k|$ sufficiently small.

(a) The most simple case arises when the waiting time $s = 0$. We can directly insert the initial correlator (13) into (12) and read off the two-point correlator in the requested scaling limit, and for the range $0 < \aleph < 1$,

$$\begin{aligned} C_n(\tau, 0) &\simeq \frac{\widetilde{C}_0}{2\pi} \int_{\mathbb{R}} \mathrm{d}k\, |k|^{\aleph-1} \left(1 - \frac{\gamma}{2} k^2 \tau + \ldots\right) e^{\mathrm{i}k(n+\gamma v \tau)}\, e^{-(1-\gamma)\tau} \\ &\simeq \frac{\widetilde{C}_0 \Gamma(\aleph) \cos(\pi\aleph/2)}{\pi} \frac{1}{(n+\gamma v \tau)^{\aleph}}\, e^{-(1-\gamma)\tau} \\ &= \frac{\Gamma\big((1+\aleph)/2\big)^2 \cos(\pi\aleph/2)}{\pi} \frac{1}{(n+\gamma v \tau)^{\aleph}}\, e^{-(1-\gamma)\tau} \end{aligned} \tag{14}$$

where the integral is taken from [11, Eq. (2.3.12)] and (13) was used. The unbiased diffusive terms merely lead to corrections to scaling. Equation (14) reproduces indeed the prediction (6) of meta-conformal invariance, with $\delta_1 = \frac{\gamma_1}{\mu} = \frac{\aleph}{2}$, and up to an exponentially decaying pre-factor[5] and a choice of scale of spatial distances. Clearly, both the bias $v \neq 0$ as well as long-ranged initial conditions with $0 < \aleph < 1$ are necessary ingredients for the meta-conformal symmetry to arise.

(b) For arbitrary waiting times $s > 0$, we must now show, under suitable conditions and at least for s sufficiently large and for $|k|$ sufficiently small, that $\widetilde{C}(s;k) \simeq \widetilde{\mathfrak{C}}(s)|k|^{\aleph-1}$. If that is so, then the two-time correlator $C_n(\tau+s, s)$, see Eq. (12), will be of the same form as in (14), with a pre-factor $\widetilde{\mathfrak{C}}(s)$ which might still depend on the waiting time s.

The proof of this property requires to solve (9). Define the Laplace transform $\overline{\widetilde{C}}(p;k) := \int_0^\infty \mathrm{d}t\, e^{-ps}\, \widetilde{C}(s;k)$. The solution of (9) reads in Laplace-Fourier space

$$\overline{\widetilde{C}}(p;k) = \frac{\overline{Z}(p) + \widetilde{C}(0;k)}{p + 2(1-\gamma\cos k)} \tag{15}$$

[5] Such non-universal exponential factors also arise in other problems, for example the number $\mathfrak{N}_{\rm saw} \sim e^{N \ln \mathfrak{z}} N^{\bar{\gamma}-1}$ of a self-avoiding random walk (SAW) of $N \gg 1$ steps contains a non-universal fugacity $\mathfrak{z}$ and an universal exponent $\bar{\gamma}$ [12].

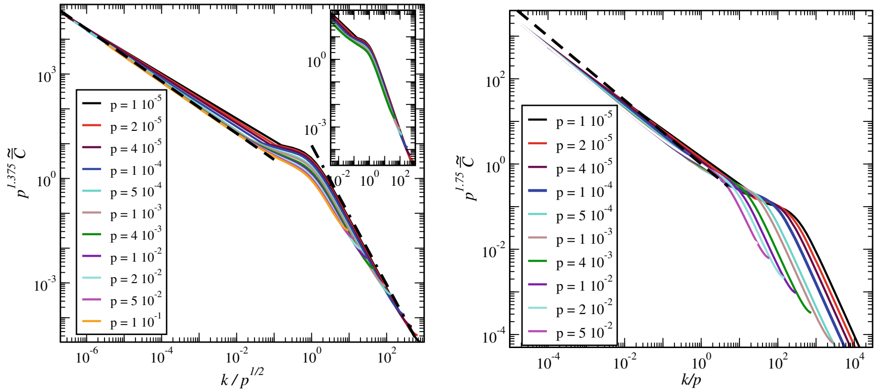

Fig. 2 Dynamical scaling (16) for the assumed value $z = 2$ (left panel) and $z = 1$ (right panel), at $\gamma = 1$ and for $\aleph = \frac{1}{4}$. The dash-dotted and dashed lines gives the asymptotic forms (17)

The generic expected scaling form, for $\gamma = 1$ and the initial condition (13) is [20]

$$\widetilde{\widetilde{C}}(p; k) = p^{-1-(1-\aleph)/z} g_C \left(kp^{-1/z}\right) \tag{16}$$

where $g_c(u)$ is a scaling function such that

$$g_C(u) \sim \begin{cases} u^{\aleph-1} & ; \text{ for } u \to 0 \\ u^{-2} & ; \text{ for } u \to \infty \end{cases} \tag{17}$$

For $\aleph > 0$ sufficiently strong, this implies the existence of two *distinct* scaling regimes, one for $u \to \infty$ which reproduces the well-known Porod law of phase-ordering kinetics in Ising models, see [20], and a new regime which arises for $u \to 0$. These two regimes are illustrated in Fig. 2. The dynamical exponent $z = 2$ describes well the case of larger values of the scaling variables $u = kp^{-1/z}$, whereas the dynamical exponent $z = 1$ leads to an excellent data collapse for u small but fails for u large enough.

In order to understand how these regimes arise, begin by finding the Lagrange multiplier $\overline{Z}(p)$ from the condition $\overline{C}_0(p) = 1/p$, which gives

$$\frac{1}{p} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}k \left[\frac{\overline{Z}(p)}{p + 2(1 - \gamma \cos k)} + \frac{\widetilde{C}(0; k)}{p + 2(1 - \gamma \cos k)} \right] \tag{18}$$

Herein, the first integral can be taken from [13]. To analyse the second integral, we use again the explicit form (13) and consider the leading small-p behaviour of

$$J(p; \gamma, \aleph) := \frac{\Gamma\big((1+\aleph)/2\big)^2}{\pi\, \Gamma(\aleph)} \int_0^{\pi} \mathrm{d}k\, \frac{(2\sin k/2)^{\aleph-1}}{p + 2(1 - \gamma \cos k)} \overset{p\to 0}{\simeq} \begin{cases} J(0; \gamma, \aleph) & ; \text{ if } \gamma < 1 \\ J_\infty\, p^{\aleph/2-1} & ; \text{ if } \gamma = 1 \end{cases} \tag{19}$$

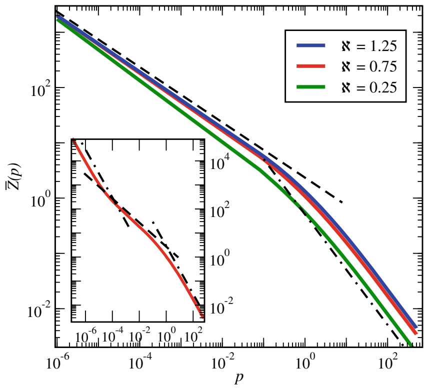

Fig. 3 Cross-over of the function $\overline{Z}(p)$, according to (20), for $\gamma = 1$ and three values of $\aleph$. The inset shows several cross-overs which occur for $\gamma = 0.99999$ and $\aleph = 0.75$. The dashed and dash-dotted lines give the curves $\sim p^{-1/2}$ and $\sim p^{-1}$

where $J_\infty = \Gamma\big((1+\aleph)/2\big)\Gamma\big(1-\aleph/2\big)\big/\, 2^{\aleph}\sqrt{\pi}$. For $\gamma < 1$, $J(0;\gamma,\aleph)$ is a finite constant. From the constraint (18), and since $\aleph > 0$, this implies for the leading small-p behaviour of the Lagrange multiplier

$$\begin{aligned}\overline{Z}(p) &= \big(p+2(1-\gamma)\big)^{1/2}\big(p+2(1+\gamma)\big)^{1/2}\left(\frac{1}{p} - J(p;\gamma,\aleph)\right)\\ &\overset{p\to 0}{\simeq} \begin{cases} 2\sqrt{1-\gamma^2}\; p^{-1}\big(1+\mathrm{o}(p)\big) & ;\ \text{if } \gamma < 1 \\ 2\, p^{-1/2}\big(1+\mathrm{o}(p)\big) & ;\ \text{if } \gamma = 1 \end{cases}\end{aligned} \tag{20}$$

where the estimates (19) for $J(p;\gamma,\aleph)$ were used. This is illustrated in Fig. 3. We see that the leading behaviour of $\overline{Z}(p)$ is independent of the initial condition.

Using (20), we now examine the correlator (15) in the asymptotic double limit $p \to 0$ and $k \to 0$. Because of the dynamical exponent $z = 1$ of meta-conformal invariance, we expect that this limit should be taken such that p/k is being kept fixed. First, for $\gamma < 1$, we find

$$\widetilde{\widetilde{C}}(p;k) \simeq \frac{2\sqrt{1-\gamma^2}\; p^{-1} + \widetilde{C}_0|k|^{\aleph-1}}{2(1-\gamma)+\mathrm{O}(p,k^2)} \simeq \sqrt{\frac{1+\gamma}{1-\gamma}}\; p^{-1}\big(1+\mathrm{o}(1)\big)\ \ ;\ \ \text{if } \gamma < 1 \tag{21}$$

because for $\aleph > 0$, the second term in the numerator is less singular than the first one. Hence, going back to sufficiently long waiting times $s \gg 1$, we obtain $\widetilde{C}(s;k) \simeq \sqrt{\frac{1+\gamma}{1-\gamma}}$ which is constant and independent of the long-range initial conditions. Hence for $\gamma < 1$ there is no meta-conformal invariance of the two-time correlator in the limit of large waiting times. Second, for $\gamma = 1$ we have instead

$$\widetilde{\widetilde{C}}(p;k) \simeq \frac{2\,p^{-1/2} + \widetilde{C}_0|k|^{\aleph-1}}{p+k^2} \simeq \begin{cases} \frac{\Gamma\left((1+\aleph)/2\right)^2}{\Gamma(\aleph)}\frac{|k|^{\aleph-1}}{p} & ;\ \text{if } \aleph < \frac{1}{2} \\ 2\,p^{-3/2} & ;\ \text{if } \aleph > \frac{1}{2} \end{cases}\ \ ,\ \ \text{and if } \gamma = 1 \tag{22}$$

This is the formula which describes what is shown in Fig. 2.

Hence, if $\aleph < \frac{1}{2}$, we have the leading long-time behaviour $\widetilde{C}(s;k) \simeq \widetilde{C}_0|k|^{\aleph-1}$, with $\widetilde{C}_0$ given in (13), for the single-time correlator. We have therefore verified a sufficient condition that the form of the two-time correlator $C_n(\tau+s,s)$ is in agreement with the expected form (6) of meta-conformal invariance, also in agreement with the dynamical exponent $z=1$.

On the other hand, if $\aleph > \frac{1}{2}$, one returns to the case of diffusive scaling with $z=2$ and short-ranged initial conditions such that effectively $\aleph=0$, see (16).

Our main result can be stated as follows.

Proposition: [22] *At zero temperature $T=0$, the two-time spin-spin correlator $C_n(\tau,s)$ in the directed Glauber-Ising chain, with the long-ranged initial correlators $C_n(0) \sim |n|^{-\aleph}$ with $0<\aleph<\frac{1}{2}$, takes for large waiting times $s \gg 1$ and large time differences $\tau = t-s \gg 1$ the form (6), predicted by meta-conformal invariance.*

3 Conclusions

Meta-conformal transformations are constructed as dynamical symmetries of a simple linear equation of ballistic transport. Here, we describe an explicit example from statistical physics: the long-time, large-distance relaxation of non-equilibrium spin systems whose dynamics contains a directional bias, and quenched to zero temperature. While for the habitual initial conditions with short-ranged initial correlations, diffusive transport with a dynamical exponent $z=2$ arises, for any bias v, we have studied here the case of a long-ranged initial spin-spin correlator $C_n(0) \sim |n|^{-\aleph}$, with $\aleph > 0$. Our explicit solution of the two-time spin-spin correlator has shown that for $\aleph < \frac{1}{2}$, there is a new scaling regime, at extremely long times and extremely small momenta, where the dynamical exponent crosses over to the value $z=1$, characteristic of ballistic transport. Then the form of the two-time correlator is indeed described by the prediction of meta-conformal invariance.

Acknowledgments This work was supported by Bulgarian National Science Fund, Grant KP-06-RILA/7.

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Relativistic Three-Body Harmonic Oscillator

Igor Salom and V. Dmitrašinović

Abstract We discuss the relativistic three-body harmonic oscillator problem, and show that in the extreme relativistic limit its energy spectrum is closely related to that of the non-relativistic three-body problem in the Δ-string potential, which blurs the distinction between relativistic and confinement effects. This, perhaps unexpected, feature can be understood in terms of permutation properties of the two problems, and as such can be expected to persist in confining potentials other than the harmonic oscillator.

1 Introduction

The harmonic oscillator is the quintessential example of a solvable/integrable problem in physics, both classical and quantum. It is (always) understood to be non-relativistic, the (special) relativistic version being neither easily solvable, nor integrable in the formal sense of the word.[1] Consequently, the relativistic harmonic oscillator is mentioned only rarely in the literature. Nevertheless, it can be an instructive example, as we shall show below, especially in the (quantum mechanical) three-quark problem, where relativity is expected to play an important role, particularly in excited states.

Over the past five years, or so, we have developed an O(6) hyper-spherical harmonics (HSH) approach to the quantum-mechanical non-relativistic three-body problem [1–3]. This approach relies on the fact that the non-relativistic kinetic energy has an O(6) symmetry, which is generally not shared by the three-body potential. In the case

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[1] Indeed, so-called relativistic "No Interaction Theorems" suggest that the interaction must vanish in a relativistic setting, see Sect. 5.

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_38

of the harmonic oscillator interaction, the potential shares the same O(6) symmetry, and in conjunction with the non-relativistic kinetic energy, leads to the higher U(6) dynamical symmetry, which ensures its super-integrability. What happens to this system when the motion becomes relativistic? We shall explore this question here, and show that (in the ultra-relativistic limit) the energy spectrum is equivalent to that of the non-relativistic three-body system in a linearly rising potential.

We start from the (obvious) observation that under the transition from the configuration-space to the momentum-space representation, one does a Fourier transformation of the whole Schrödinger equation, the wave function included. A necessary condition for this transformation to hold is that all of the functions involved be square-integrable. Whereas that is certainly true of the bound-state wave functions, it does not hold for the harmonic oscillator, or any other infinitely rising (i.e., confining) potential. Therefore, the Fourier transform of the harmonic-oscillator Schrödinger equation is not the usual Lippmann-Schwinger integro-differential equation, but rather, (the same) differential equation, except for the fact that the variables are in the momentum space. In Boukraa and Basdevant's [4] words: "These integral equations appear to be singular for confining potentials and hence need particular treatment." For an arbitrary confining potential this "particular treatment" has been presented in some detail in Ref. [4].

In the special case of the harmonic oscillator potential, however, this "particular treatment" turns out to be simple and yields an elegant solution: instead of an integro-differential equation, one ends up with a purely differential equation of the second order that looks just like an ordinary nonrelativistic Schrödinger equation, see problem V.17, Chapter V in Ref. [5], albeit with a (different) potential, that is the Fourier transform of the relativistic kinetic energy operator. Indeed the relativistic 1-body harmonic oscillator was treated in this way by Li et al. [6]. The Fourier transform of the relativistic kinetic energy be evaluated in closed form with the (possibly not-so-surprising) result that it asymptotically grows linearly with the resulting "separation of quarks in momentum space", which corresponds to a linearly confining potential.

Thus the *relativistic* harmonic oscillator three-body problem is reduced to a *non-relativistic* three-body problem in a linearly growing potential in momentum space.[2] In this light, we do not seem to have gained much, just trading one three-body problem for another. The actual advantage is that the new three-body problem is non-relativistic, which allows us to use the O(6) hyper-spherical harmonics.

From this point onwards, one can employ our previously developed hyper-spherical harmonics methods [1–3] to solve the resulting equations of motion. The new potential is expanded in hyper-spherical harmonics and the resulting expansion coefficients inserted into the (reduced) hyper-radial Schrödinger equations, that can be solved numerically at least in the extreme relativistic limit (zero masses).

The resulting energy spectrum is curious and instructive: instead of the highly degenerate spectrum of the non-relativistic harmonic oscillator, we find that the degeneracy has been maximally lifted, the only remnants of degeneracy being those decreed by the S_3 permutation symmetry. Moreover, degeneracy-lifting effects have

[2] This was noted by Hall et al [7].

the same amplitudes, and (sometimes) the opposite sign to those in the Δ-string potential, at least in the extreme relativistic limit.

Of course, the more realistic case of non-zero masses is expected to interpolate between the two extreme cases of non-relativity and extreme relativity, but that cannot be handled as simply as in the case shown here.

2 Semi-relativistic Three-Body Harmonic Oscillator in Momentum Space

The semi-relativistic three-quark Hamiltonian in configuration space is

$$H = \sum_a \sqrt{m_a^2 + \mathbf{p}_i^2} + V_{3b}\left(|\boldsymbol{\rho}|, |\boldsymbol{\lambda}|, \boldsymbol{\rho}\cdot\boldsymbol{\lambda}\right), \tag{1}$$

where the confining 3-body harmonic oscillator potential reads (for the sake of simplicity we shall work with three identical particles):

$$\begin{aligned} V_{3b}\left(|\boldsymbol{\rho}|, |\boldsymbol{\lambda}|, \boldsymbol{\rho}\cdot\boldsymbol{\lambda}\right) = V_{\mathrm{HO}} &= \frac{k}{2}\left(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2\right) \\ &= \frac{k'}{2}\left((\mathbf{x_1} - \mathbf{x_2})^2 + (\mathbf{x_2} - \mathbf{x_3})^2 + (\mathbf{x_3} - \mathbf{x_1})^2\right) \end{aligned} \tag{2}$$

and $k' = 3k$. Here we used the standard notation for Jacobi coordinates $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$, defined as the following linear combinations of the positions of the three identical particles $\mathbf{x_i}$:

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x_1} - \mathbf{x_2}), \qquad \boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x_1} + \mathbf{x_2} - 2\mathbf{x_3}).$$

The Fourier transform of the harmonic oscillator potential, as is well known, see problem V.17, Chapter V in Ref. [5], is proportional to the Laplacian operator, i.e., to the kinetic energy (in the configuration representation). This property holds in the three-body problem, as well, with the distinction that the Laplacian operates in six-dimensional space

$$\tilde{V}_{\mathrm{HO}} = -\frac{k}{2}\left(\frac{\partial^2}{\partial \mathbf{p}_{\rho}^2} + \frac{\partial^2}{\partial \mathbf{p}_{\lambda}^2}\right).$$

The major difference is in what happens to the relativistic kinetic energy in the momentum space: the kinetic energy

$$T = \sum_{a=1}^{3} \sqrt{m_a^2 + \mathbf{p}_i^2},$$

is a differential operator in configuration representation (because $\mathbf{p}_j = -i\hbar\frac{\partial}{\partial \mathbf{x}_j}$ is a differential operator), but a multiplicative operator in the momentum representation. In the center-of-momentum (CM) frame it holds:

$$\sum_{i=1}^{3} \mathbf{p}_i = 0$$

and, subjected to this constraint, it is possible to express all three spatial momenta $\mathbf{p}_i$ as linear combinations of just two Jacobi vectors $\mathbf{p}_\rho$ and $\mathbf{p}_\lambda$.

Note that kinetic energy T does not have the O(6) symmetry of its non-relativistic analog. Moreover, in the extreme relativistic limit $m_a \to 0$, or $T_{\rm CM} \to \infty$, we can write center-of-momentum kinetic energy as:

$$T_{\rm CM} = \sum_{i=1}^{3} |\mathbf{p}_i| = \sqrt{\frac{2}{3}\mathbf{p}_\lambda^2} + \sqrt{\frac{1}{2}}\left(\left|\mathbf{p}_\rho + \frac{\mathbf{p}_\lambda}{\sqrt{3}}\right| + \left|\mathbf{p}_\rho - \frac{\mathbf{p}_\lambda}{\sqrt{3}}\right|\right), \tag{3}$$

and it is only a linearly (rather than quadratically as in the non relativistic case) rising function of momenta magnitudes.

Thus, the Hamiltonian in momentum space and CM frame reads

$$\tilde{H} = -\frac{k}{2}\left(\frac{\partial^2}{\partial \mathbf{p}_\rho^2} + \frac{\partial^2}{\partial \mathbf{p}_\lambda^2}\right) + \sum_{i=1}^{3} |\mathbf{p}_i|,$$

which, after the substitutions $\mathbf{p}_\rho \leftrightarrow \boldsymbol{\rho}$ and $\mathbf{p}_\lambda \leftrightarrow \boldsymbol{\lambda}$, is equivalent to the Schrödinger equation

$$\tilde{H}\tilde{\Psi} = \tilde{E}\tilde{\Psi}$$

for three identical particles with a mass $m = k$ and interacting with a CM-string potential with unit string tension $\sigma = 1$. We have developed hyper-spherical harmonic methods [1–3, 9] for the solution of such three-body Schrödinger equations.

3 Barycentric-String Three-Body Potential

The potential:

$$V_{\rm CM} = \sigma_{\rm CM}\left(\sqrt{\frac{2}{3}\boldsymbol{\lambda}^2} + \sqrt{\frac{1}{2}}\left(\left|\boldsymbol{\rho} + \frac{\boldsymbol{\lambda}}{\sqrt{3}}\right| + \left|\boldsymbol{\rho} - \frac{\boldsymbol{\lambda}}{\sqrt{3}}\right|\right)\right) \tag{4}$$

corresponding to the expression (3) is known as the barycentric-junction (instead of Torricelli-junction) string

$$V_{\rm CM} = \sigma_{\rm CM} \sum_{i=1}^{3} |\mathbf{x_i} - \mathbf{x}_{\rm CM}|,$$

in the literature [8]. In terms of Iwai-Smith angles it reads

$$\begin{aligned} V_{\rm CM}(R,\alpha,\phi) = \frac{\sigma_{\rm CM}}{\sqrt{3}} R \Bigg(& \sqrt{1 - \sin(\alpha)\cos\left(\phi - \frac{\pi}{3}\right)} \\ & + \sqrt{1 - \sin(\alpha)\cos\left(\phi + \frac{\pi}{3}\right)} \\ & + \sqrt{1 + \sin(\alpha)\cos(\phi)} \Bigg). \end{aligned} \tag{5}$$

The barycentric string 3-body potential is closely related to the Δ-string one,

$$V_{\Delta} = \sigma_{\Delta} \sum_{i>j=1}^{3} |\mathbf{x}_i - \mathbf{x}_j|, \tag{6}$$

which, written in terms of Jacobi vectors reads

$$\begin{aligned} V_{\Delta} = \sigma_{\Delta} \Bigg(& \sqrt{2\boldsymbol{\rho}^2} + \sqrt{\frac{1}{2}\left(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}\right)} \\ & + \sqrt{\frac{1}{2}\left(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}\right)} \Bigg). \end{aligned} \tag{7}$$

Note that the expression Eq. (4) for $V_{\rm CM}$ can be obtained from the expression Eq. (7) for V_{Δ} by exchanging $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ (together with setting $\sigma_{\Delta} = \sigma_{\rm CM}/\sqrt{3}$).

The Δ-string potential Eq. (7), in terms of Iwai-Smith angles, reads

$$\begin{aligned} V_{\Delta}(R,\alpha,\phi) = \sigma_{\Delta} R \Bigg(& \sqrt{1 + \sin(\alpha)\sin\left(\frac{\pi}{6} - \phi\right)} \\ & + \sqrt{1 + \sin(\alpha)\sin\left(\phi + \frac{\pi}{6}\right)} \\ & + \sqrt{1 - \sin(\alpha)\cos(\phi)} \Bigg) \end{aligned} \tag{8}$$

which is just a rotation of angle ϕ through π as compared with Eq. (5). An interchange of the $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$ vectors has exactly such the effect of a ϕ-rotation through π on the hyper-spherical harmonic expansion coefficients, $v^{\Delta}_{K,Q}$, defined in Eq. (9).

In order to find the general hyper-spherical harmonic expansion of the Δ-string potential we note that it factors into the hyper-radial $V_{\Delta}(R) = \sigma_{\Delta} R$ and the hyper-angular part $V_{\Delta}(\alpha,\phi)$, and thus:

Table 1 Expansion coefficients v_{KQ} of the Y- and Δ-string as well as of the Coulomb and Logarithmic potentials in terms of O(6) hyper-spherical harmonics $\mathcal{Y}_{0,0}^{K,0,0}$, for $K = 0, 4, 8, 12$, respectively, and of the hyper-spherical harmonics $\mathcal{Y}_{0,0}^{6,\pm 6,0}$

(K, Q)	v_{KQ}(Y − string)	$v_{KQ}(\Delta)$	v_{KQ}(CM − string)	v_{KQ}(Coulomb)	v_{KQ}(Log)
(0, 0)	8.22	16.04	$16.04/\sqrt{3}$	20.04	−6.58
(4, 0)	−0.398	−0.445	$-0.445/\sqrt{3}$	2.93	−1.21
(6, ±6)	−0.027	−0.14	$0.14/\sqrt{3}$	1.88	−0.56
(8, 0)	−0.064	−0.04	$-0.04/\sqrt{3}$	1.41	−0.33
(12, 0)	−0.01	0	0	0	−0.17

$$V_{\Delta}(R, \alpha, \phi) = V_{\Delta}(R) V_{\Delta}(\alpha, \phi)$$
$$= \sigma_{\Delta} R \sum_{K,Q}^{\infty} v_{K,Q}^{\Delta} \mathcal{Y}_{00}^{KQ\nu}(\alpha, \phi). \quad (9)$$

This circumstance, of course, leaves consequences for the hyper-spherical harmonic expansion coefficients of these two potentials: (1) the $Q = 0$ coefficients $v_{K,Q=0}$ of these two potentials are identical (modulo $\sqrt{3}$ in the definitions of the string tensions); (2) the $Q \neq 0$ coefficients $v_{K,Q\neq 0}$ of these two potentials have the same absolute value, with opposite signs.

We have already calculated the expansion coefficients $v_{K,Q}^{\Delta}$ in Ref. [10], and we can use our above conclusions to directly infer values of the corresponding coefficients in the hyper-spherical decomposition of the potential $V_{\rm CM}$ (4). All these coefficients (and some additional for comparison) are tabulated in Table 1.

4 Results: Energy Spectra

This means that the relativistic 3-body harmonic oscillator spectrum (Table 2) is closely related to that of the non-relativistic Δ-string one, which has been studied in detail Ref. [10], see also Basdevant and Boukraa, [4, 11]. The precise energy eigenvalues depend on the absolute value of the (1-body) mass, and can be bounded from above and below as shown in [6]. In the extreme relativistic limit ($m \to 0$) the ground-state energy is equal to the (rescaled) Δ-string ground state energy (cf. Table 5 in Ref. [8]).

Thus we find that the degeneracy of energy levels has been maximally lifted, the only remnants of degeneracy being those allowed/decreed by the S_3 permutation symmetry and by angular momentum conservation. Moreover, degeneracy-lifting effects have the same amplitudes, and in some cases the opposite sign(s) to those in the Δ-string potential, at least in the extreme ultra-relativistic limit. This stands

Table 2 The eigen-energies (in units of $\left(\frac{\sigma_{\#}}{\sqrt{2m}}\right)^{\frac{2}{3}}$) of the #-string potentials (where # = Y, Δ, CM) for all K = 0,1,2 states. The CM-string eigen-energies are obtained from the Δ by dividing the latter by $\sqrt[3]{3}$, which is equivalent to the substitution $\sigma_{\Delta} = \sqrt{3}\sigma_{\mathrm{CM}}$ in the above formula for units

K	N_K	$[SU(6), L^P]$	$E^{(Y)}_{N_K,K,L}$	$E^{(\Delta)}_{N_K,K,L}$	$E^{(CM)}_{N_K,K,L}$
0	0	$[56, 0^+]$	5.1761	6.1348	4.2536
1	0	$[70, 1^-]$	6.3160	7.4858	5.1904
0	1	$[56, 0^+]$	7.1360	8.4577	5.8643
2	0	$[70, 0^+]$	7.1733	8.6322	5.9852
2	0	$[56, 2^+]$	7.2437	8.6691	6.0108
2	0	$[70, 2^+]$	7.3968	8.7430	6.0621
2	0	$[20, 1^+]$	7.5550	8.8168	6.1132

in stark contrast to the highly degenerate spectrum of the non-relativistic harmonic oscillator.

Moreover, this shows that the relativistic effects can either enhance, or reduce the $(K, Q) = (6, \pm 6)$ coefficient, which is a benchmark signal of the Y-string confinement potential, thus possibly confusing the issue of Δ- vs. Y-string further still.

Of course, the more realistic case of non-zero masses is expected to interpolate between the two extreme cases of non-relativistic and extremely relativistic motion, but that cannot be handled as simply as in the case shown here. Once the masses are turned on, the problem's Hamiltonian loses its homogeneity, and has to be (re)calculated separately at every mass value.

Next we compare our ground state energy (see the right-most column in Table 5 in Ref. [8]) $E_{00} = 5.3592 \times \sqrt[3]{\frac{3}{2}} k^{1/3} = 6.13475 k^{1/3}$ which is slightly (0.12%) larger than Hall et al's [7] (rigorous) lower bound of $2.33810741 \times \sqrt[3]{18} k^{1/3} = 6.12757 k^{1/3}$. The difference may well be down to the numerical inaccuracy (4th significant digit) in the numerical evaluation of the eigenvalue in Table 11 in Ref. [8].

5 Discussion and Conclusions

In this paper we have explicitly established a direct relation between ultra-relativistic three-particle harmonic oscillator problem, and the non-relativistic three particle problem in a linear Δ potential. This relation allowed us to directly infer the harmonic oscillator spectrum from the already known properties of the three particle non-relativistic system in the Δ potential.

In conferences one can still occasionally hear comments to the effect that the Dirac's formulation(s) of (special) relativistic mechanics [12] precludes any and all interaction within few-body systems—the so-called "No Interaction Theorems" of Currie, Jordan, and Sudarshan [13]. It is far less known that in a series of papers

[14, 15], Luis Bel has reviewed which assumption(s) lay at the root of this apparent difficulty, as well as several assumptions supplanting the Currie-Jordan-Sudarshan ones, which permit one to circumvent this "No Interaction theorem". In this regard, see also Jordan's own more recent views [16]. Thus there are no formal grounds for viewing our present results with suspicion.

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An Introduction to Spectral Regularization for Quantum Field Theory

John Mashford

Abstract A spectral calculus for the computation of the spectrum of Lorentz invariant Borel complex measures on Minkowski space is introduced. It is shown how problematical objects in quantum field theory (QFT), such as Feynman integrals associated with loops in Feynman graphs, can be given well defined existence as Lorentz invariant tempered Borel complex measures. Their spectral representation can be used to compute an equivalent density which can be used in QFT calculations. As an application the contraction of the vacuum polarization tensor is considered. The spectral vacuum polarization function is shown to have close agreement (up to finite renormalization) with the vacuum polarization function obtained using dimensional regularization/renormalization. The spectral running coupling for QED is computed and shown to manifest no Landau poles.

1 Introduction

In perturbation theory the Feynman amplitude $\mathcal{M}$ (from which can be computed all quantities of physical interest) associated with any process is given by (the asymptotically convergent) series

$$\mathcal{M} = \sum_{n=1}^{\infty} \mathcal{M}_n, \tag{1}$$

where $\mathcal{M}_n$ is the Feynman amplitude associated with some graph G_n. When G_n is a tree graph, i.e. has no loops, then $\mathcal{M}_n$ can be computed in a fairly straightforward manner. However, when G_n contains loops then the well known divergences of QFT become manifest. The techniques of regularization and renormalization circumvent these problems.

Two of the principal methods of regularization are Pauli-Villars regularization and dimensional regularization. Both of these methods involve modifying a divergent integral to form an integral which exists in a manner depending on a parameter

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_39

where the parameter is a fictitious photon mass Λ for Pauli-Villars regularization or a perturbation $\epsilon > 0$ of the space-time dimension $D = 4 - \epsilon$ for dimensional regularization. The parameter is then varied towards the value that it would have if the divergent integral existed ($\Lambda \to \infty$ for Pauli-Villars, $\epsilon \to 0$ for dimensional) and the badly behaved contributions (e.g. terms of the order of $\log(\Lambda)$ or ϵ^{-1}) are subtracted out or ignored to obtain finite answers which can be compared with experiment. The process whereby the badly behaved contributions are dealt with or compensated for is called renormalization.

Renormalization is only feasible in some theories called renormalizable theories. Many theories of considerable interest such as the IVB theory of the weak force, and quantum gravity, are non-renormalizable.

In the present paper we introduce a method of regularization in which problematical objects in QFT are viewed as Lorentz invariant tempered Borel complex measures on Minkowski space whose spectral representation, and hence density, can be computed using a certain spectral calculus. The method is generally applicable, e.g. to loops involving the vector boson propagator or the graviton propagator.

2 The Spectral Calculus

Consider the following general form of a complex measure μ on Minkowski space.

$$\mu(\Gamma) = c\delta(\Gamma) + \int_{m=0}^{\infty} \Omega_m^+(\Gamma)\,\sigma_1(dm) + \int_{m=0}^{\infty} \Omega_m^-(\Gamma)\,\sigma_2(dm) + \int_{m=0}^{\infty} \Omega_{im}(\Gamma)\,\sigma_3(dm), \tag{2}$$

where $c \in \mathbf{C}$ (the complex numbers), δ is the Dirac delta function (measure), $\sigma_1, \sigma_2, \sigma_3 : \mathcal{B}([0,\infty)) \to \mathbf{C}$ are Borel complex measures (where $\mathcal{B}([0,\infty))$ denotes the Borel algebra of $[0,\infty)$), Ω_m^+ is the standard Lorentz invariant measure on Minkowski space concentrated on the mass shell H_m^+ [3], Ω_m^- is the standard Lorentz invariant measure concentrated on the mass shell H_m^- and Ω_{im} is the standard Lorentz invariant measure concentrated on the imaginary mass hyperboloid H_{im}. Then μ is a Lorentz invariant Borel complex measure. Conversely [1] leads to the following.

Theorem 1. *The Spectral Theorem. Let $\mu : \mathcal{B}(\mathbf{R}^4) \to \mathbf{C}$ be a Lorentz invariant Borel complex measure. Then μ has the form of Eq. 2 for some $c \in \mathbf{C}$ and Borel spectral measures σ_1, σ_2 and σ_3.*

We will now present a spectral calculus whereby the spectrum of a causal Lorentz invariant Borel complex measure on Minkowski space can be calculated, where by causal is meant that the support of the measure is contained in the closed future null cone of the origin.

Any measure of the form

$$\mu(\Gamma) = \int_{m=0}^{\infty} \sigma(m)\Omega_m^+(\Gamma)\,dm, \tag{3}$$

where σ is a locally integrable function and the integration is carried out with respect to the Lebesgue measure, is a causal Lorentz invariant Borel complex measure. If σ is polynomially bounded then μ is a tempered measure.

The spectral calculus that we will now explain is a way to compute the spectrum σ of a Lorentz invariant measure μ if we know that μ can be written in the form of Eq. 3 and σ is continuous.

For $m > 0$ and $\epsilon > 0$ let $S(m, \epsilon)$ be the hyperbolic (hyper-)disc defined by

$$S(m, \epsilon) = \{p \in \mathbf{R}^4 : p^2 = m^2, |\vec{p}| < \epsilon, p^0 > 0\}, \tag{4}$$

where $\vec{p} = \pi(p) = \pi(p^0, p^1, p^2, p^3) = (p^1, p^2, p^3)$. For $a, b \in \mathbf{R}$ with $0 < a < b$ let $\Gamma(a, b, \epsilon)$ be the hyperbolic cylinder defined by

$$\Gamma(a, b, \epsilon) = \bigcup_{m\in(a,b)} S(m, \epsilon). \tag{5}$$

Now suppose that we have a measure in the form of Eq. 3 where σ is continuous. Then we can write (using the notation of [3]),

$$\begin{aligned}
\mu(\Gamma(a, b, \epsilon)) &= \int_{m=0}^{\infty} \sigma(m)\Omega_m(\Gamma(a, b, \epsilon))\,dm \\
&= \int_{m=0}^{\infty} \sigma(m) \int_{\pi(\Gamma(a,b,\epsilon)\cap H_m^+)} \frac{d\vec{p}}{\omega_m(\vec{p})}\,dm \\
&= \int_a^b \sigma(m) \int_{B_\epsilon(\vec{0})} \frac{d\vec{p}}{\omega_m(\vec{p})}\,dm \\
&\approx \frac{4}{3}\pi\epsilon^3 \int_a^b \frac{\sigma(m)}{m}\,dm.
\end{aligned}$$

where $B_\epsilon(\vec{0}) = \{\vec{p} \in \mathbf{R}^3 : |\vec{p}| < \epsilon\}$.

The approximation $\approx$ in the last line comes about because ω_m is not constant over $B_\epsilon(\vec{0})$.

Thus if we define

$$g_a(b) = g(a, b) = \lim_{\epsilon\to 0} \epsilon^{-3}\mu(\Gamma(a, b, \epsilon)), \tag{6}$$

then we can retrieve σ using the formula

$$\sigma(b) = \frac{3}{4\pi} b g_a'(b). \tag{7}$$

Thus we have proved the following fundamental theorem of the spectral calculus of causal Lorentz invariant measures.

Theorem 2. *Suppose that μ is a causal Lorentz invariant measure on Minkowski space with continuous spectrum σ. Then σ can be calculated from the formula*

$$\sigma(b) = \frac{3}{4\pi} b g_a'(b), \tag{8}$$

where, for $a, b \in \mathbf{R}$, $0 < a < b$, $g_a : (a, \infty) \to \mathbf{R}$ is given by Eq. 6.

To make the proof of this theorem rigorous we have the following.

Lemma 1. *Let $a, b \in \mathbf{R}$, $0 < a < b$. Then*

$$\lim_{\epsilon \to 0} \epsilon^{-3} \int_{B_\epsilon(0)} \frac{d\vec{p}}{\omega_m(\vec{p})} = \frac{4\pi}{3} \frac{1}{m}, \tag{9}$$

uniformly for $m \in [a, b]$.

This lemma justifies the step of taking the limit under the integral sign (indicated by the symbol $\approx$) in the proof of Theorem 2.

3 Spectral Regularization

3.1 Definition of the Contraction of the Vacuum Polarization Tensor as a Lorentz Invariant Tempered Complex Measure Π

The vacuum polarization tensor is written as

$$\Pi^{\mu\nu}(k) = -e^2 \int \frac{dp}{(2\pi)^4} \mathrm{Tr}(\gamma^\mu \frac{1}{\not p - m + i\epsilon} \gamma^\nu \frac{1}{\not p - \not k - m + i\epsilon}), \tag{10}$$

([2], p. 319). This can be rewritten as

$$\Pi^{\mu\nu}(k) = -\frac{e^2}{(2\pi)^4} \int \frac{\mathrm{Tr}(\gamma^\mu(\not p + m)\gamma^\nu(\not p - \not k + m))}{(p^2 - m^2 + i\epsilon)((p-k)^2 - m^2 + i\epsilon)} dp. \tag{11}$$

Therefore, contracting with the Minkowski space metric tensor, the "function" that we are interested in computing is

$$\Pi(k) = -\frac{e^2}{(2\pi)^4} \int \frac{\mathrm{Tr}(\eta_{\mu\nu}\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}-\not{k}+m))}{(p^2-m^2+i\epsilon)((p-k)^2-m^2+i\epsilon)}\, dp. \tag{12}$$

As is well known, the integral defining this "function" is divergent for all $k \in \mathbf{R}^4$ and all the machinery of regularization and renormalization has been developed to get around this problem.

We propose that the object defined by Eq. 12 exists when viewed as a measure on Minkowski space. To show this, suppose that Π were a density for a measure which we also denote as Π. Then we may make the following formal computation, for $\Gamma \in \mathcal{B}(\mathbf{R}^4)$ relatively compact,

$$\begin{aligned}
\Pi(\Gamma) &= \int_\Gamma \Pi(k)\, dk \\
&= \int \chi_\Gamma(k)\Pi(k)\, dk \\
&= -\frac{e^2}{(2\pi)^4} \int \chi_\Gamma(k) \frac{\mathrm{Tr}(\eta_{\mu\nu}\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}-\not{k}+m))}{(p^2-m^2+i\epsilon)((p-k)^2-m^2+i\epsilon)}\, dp\, dk \\
&= -\frac{e^2}{(2\pi)^4} \int \chi_\Gamma(k) \frac{\mathrm{Tr}(\eta_{\mu\nu}\gamma^\mu(\not{p}+m)\gamma^\nu(\not{p}-\not{k}+m))}{(p^2-m^2+i\epsilon)((p-k)^2-m^2+i\epsilon)}\, dk\, dp \\
&= -\frac{e^2}{(2\pi)^4} \int \chi_\Gamma(k+p) \frac{\mathrm{Tr}(\eta_{\mu\nu}\gamma^\mu(\not{p}+m)\gamma^\nu(-\not{k}+m))}{(p^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}\, dk\, dp \\
&= \frac{e^2}{(2\pi)^4} \int \chi_\Gamma(k+p) \frac{\mathrm{Tr}(\eta_{\mu\nu}\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}-m))}{(p^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}\, dk\, dp,
\end{aligned}$$

where, for any set Γ, χ_Γ is the characteristic function of Γ defined by

$$\chi_\Gamma(p) = \begin{cases} 1 \text{ if } p \in \Gamma \\ 0 \text{ otherwise.} \end{cases} \tag{13}$$

Now the propagators in QFT can be viewed in a rigorous fashion as measures on Minkowski space and we make the identification (ansatz)

$$\frac{1}{p^2-m^2+i\epsilon} \to -\pi i \Omega_m^\pm(p), m \ge 0, \tag{14}$$

where $\Omega_m^\pm$ is the standard Lorentz invariant measure on the mass shell hyperboloid (cone) $H_m^\pm$ corresponding to mass $m > 0$ $(m = 0)$ (see [3] for explanation). Therefore the outcome of our formal computations is that

$$\Pi(\Gamma) = -\frac{e^2}{16\pi^2} \int \chi_\Gamma(k+p)\mathrm{Tr}(\eta_{\mu\nu}\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}-m))\, \Omega_m(dk)\, \Omega_m(dp), m > 0. \tag{15}$$

(We use the symbol Ω_m to denote Ω_m^+ if $m > 0$ or $\Omega_{|m|}^-$ if $m < 0$.)

The important thing is that the object defined by Eq. 15 exists as a Borel complex tempered measure (i.e. when its argument Γ is a relatively compact Borel set in $\mathbf{R}^4$). This is because

$$\int \chi_\Gamma(k+p)|\mathrm{Tr}(\eta_{\mu\nu}\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}-m))|\,\Omega_m(dk)\,\Omega_m(dp) < \infty, \tag{16}$$

for all relatively compact $\Gamma \in \mathcal{B}(\mathbf{R}^4)$.

Thus Π exists as a Borel tempered measure. Hence we have in a few lines of formal argument arrived at an object which has a well defined existence and can investigate the properties of this object Π without any further concern about ill-definedness or the fear of propagating ill-definedness through our calculations.

3.2 Computation of the Spectral Vacuum Polarization Function

It can be shown that Π is Lorentz invariant tempered complex measure with support contained in $C_{2m} = \{p \in \mathbf{R}^4 : p^2 \geq 4m^2, p^0 > 0\}$ [4]. Therefore by the Spectral Theorem Π must have a spectral representation of the form

$$\Pi(\Gamma) = \int_{m'=2m}^{\infty} \sigma(dm')\Omega_{m'}(\Gamma). \tag{17}$$

We would like to compute the spectral measure σ. Using the spectral calculus (see [4] for details) one can compute the spectrum of Π to be

$$\sigma(m') = \begin{cases} \frac{2}{\pi}e^2m^3Z(m')(3+2Z^2(m')) \text{ if } m' \geq 2m \\ 0 \text{ otherwise,} \end{cases} \tag{18}$$

where $Z : [2m, \infty) \to [0, \infty)$ is defined by

$$Z(m') = (\frac{m'^2}{4m^2} - 1)^{\frac{1}{2}}. \tag{19}$$

It follows [4] that the density defining Π is given by

$$\Pi(q) = \begin{cases} (q^2)^{-\frac{1}{2}}\sigma((q^2)^{\frac{1}{2}}) \text{ if } q^2 > 0, q^0 > 0 \\ 0 \text{ otherwise.} \end{cases} \tag{20}$$

We therefore define the spectral vacuum polarization function π on $\{q \in \mathbf{R}^4 : q^2 > 0, q^0 > 0\}$ by

$$\pi(q) = \frac{\Pi(q)}{3q^2} = \frac{1}{3}\begin{cases} (q^2)^{-\frac{3}{2}}\sigma((q^2)^{\frac{1}{2}}) \text{ if } q^2 > 4m^2, q^0 > 0 \\ 0 \text{ otherwise.} \end{cases} \tag{21}$$

π is a function on $\mathbf{R}^4$ supported on C_{2m} but its value for argument q only depends on q^2. Therefore, with no fear of confusion, one may define the vacuum polarization function $\pi : [2m, \infty) \to [0, \infty)$ by

$$\pi(s) = \frac{1}{3}s^{-3}\sigma(s) = \frac{2}{3\pi}s^{-3}e^2m^3Z(s)(3 + 2Z^2(s)), \tag{22}$$

where

$$Z(s) = (\frac{s^2}{4m^2} - 1)^{\frac{1}{2}}. \tag{23}$$

Thus we can write

$$\pi(\rho) = \frac{2}{3\pi}e^2\rho^{-3}Z(\rho)(3 + 2Z^2(\rho)), \text{ where } \rho = s/m, Z(\rho) = (\frac{\rho^2}{4} - 1)^{\frac{1}{2}}. \tag{24}$$

Let π_s denote our spectral vacuum polarization function and π_r denote the vacuum polarization function obtained using dimensional regularization/renormalization [2]. We find that π_r and π_s coincide approximately when $\pi_r(q)$ is rescaled by a factor (λ say) and then shifted by an amount (τ say). We have the following.

Lemma 2. *Suppose that*

$$\widetilde{\pi}_r(\rho) = \tau + \lambda\pi_r(\rho), \tag{25}$$

where $\tau, \lambda \in \mathbf{R}$ *(and* $\rho = q/m$*) is such that*

$$\widetilde{\pi}_r(2) = \pi_s(2), \tag{26}$$

and

$$\lim_{\rho\to\infty} \widetilde{\pi}_r(\rho) = \lim_{\rho\to\infty} \pi_s(\rho). \tag{27}$$

Then

$$\tau = -\frac{2e^2}{9\pi^2}\lambda, \textit{ with } \lambda = -\frac{2\pi}{\pi - 1}. \tag{28}$$

A graph of our vacuum polarization function versus the vacuum polarization function obtained using dimensional regularization/renormalization is shown in Fig. 1. It can be seen that (up to finite renormalization) the spectral vacuum polarization function agrees well with the dimensional regularized/renormalized vacuum polarization function even though these functions are obtained using quite different methods

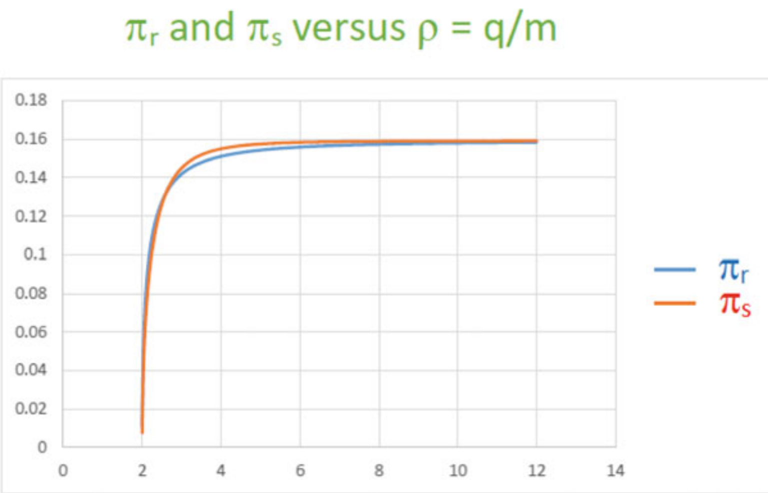

Fig. 1 Renormalized/spectral vacuum polarization versus rho = q/m (= s/m)

and are defined by quite different analytic expressions. It is important to note that no divergence is involved in the definition or computation of the spectral vacuum polarization function.

4 The Spectral Running Coupling for QED and Its Manifestation of No Landau Poles

Now using the Born approximation the change in potential as a result of the Uehling correction is given by

$$\Delta V(q) = \pi(q)V_0(q) = \pi(q)A_0(q).(1, \vec{0}), \tag{29}$$

where A_0 is the 4-potential associated with the Coulomb potential and $V_0(q) = A_0(q).(1, \vec{0})$ is the scalar potential. By Maxwell's equations

$$(\Box A_0)(x) = j(x), \tag{30}$$

where $j(x) = (j(\vec{x}), 0)$ with $j(\vec{x}) = Ze\delta(\vec{x})$ is the 4-current associate with a stationary point charge of magnitude Ze. Thus, in momentum space, we have

$$A_0(q) = -\frac{j(q)}{q^2}, \tag{31}$$

and so (see [4] for details),

$$(\Delta V)(q) = -\pi(q)\frac{j(q)}{q^2}.(1, \vec{0}). \tag{32}$$

Since we are using a non-relativistic approximation (the Born approximation) we take π to be defined by its spacelike form

$$\pi(q) = -\begin{cases} \pi((-q^2)^{\frac{1}{2}}) \text{ if } q^2 < 0, q^0 > 0, \\ 0 \text{ otherwise.} \end{cases} \tag{33}$$

It can be shown [4] that this leads to, in configuration space,

$$(\Delta V)(r) = -\frac{Ze}{r}\int \frac{\pi(s)}{s}\sin(rs)\, ds. \tag{34}$$

Therefore the total equivalent potential for an electron-proton system (H atom) in the Born approximation is

$$V(r) = -\frac{e^2}{4\pi r} - \frac{e^2}{2\pi^2 r}\int \frac{\pi(s)}{s}\sin(rs)\, ds. \tag{35}$$

At range r the potential is equivalent to that produced by an effective charge or running coupling constant e_r given by

$$\begin{aligned} -\frac{e_r^2}{4\pi r} &= -\frac{e^2}{4\pi r}(1 + \frac{2}{\pi}\int \frac{\pi(s)}{s}\sin(rs)\, ds) \\ &= -\frac{e^2}{4\pi r}(1 + \frac{2}{\pi}\int \pi(\frac{s}{r})\frac{\sin(s)}{s}\, ds). \end{aligned}$$

Therefore the running fine structure "constant" at energy μ is given by

$$\alpha(\mu) = \alpha(0)(1 + \frac{2}{\pi}\int \pi(\mu s)\mathrm{sinc}(s)\, ds). \tag{36}$$

$\alpha(0) \approx 1/137$ and α increases with increasing energy having been measured to have a value of $\alpha(\mu) \approx 1/127$ for $\mu = 90$ GeV. Given this explicit expression for the running coupling it is not necessary to use the techniques of the renormalization group equation involving a beta function to investigate its behavior.

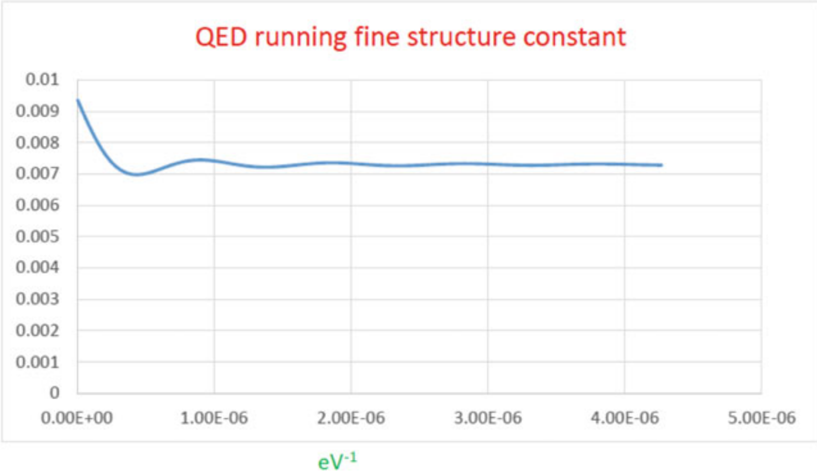

Fig. 2 Spectral QED running fine structure constant on the basis of vacuum polarization

It can be shown [4] that when π is obtained using dimensional regularization/renormaiization then the integral Eq. 36 is divergent for all $\mu > 0$ but that when π is the spectral vacuum polarization function the integral is convergent for all $\mu \geq 0$. The graph of the spectral running coupling is shown in Fig. 2. The spectral running coupling has no Landau poles.

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Relative Quantum States, Observations and Moduli Stacks

Michael Hewitt

Abstract We explore a relationship between observation processes in quantum mechanics and the construction of algebras from entangled pairs of Lie representations, and point to common properties that this method shares with the construction of moduli stacks of algebraic curves.

1 Introduction

The origin of new information and of thermodynamically irreversible processes such as measurement which produce a permanent but unpredictable change is a challenge to quantum physics. An approach based on a progressive construction of quantum states is suggested below in which observation is characterised by embedding rather than projection, and a similarity is noted with the mathematics of moduli spaces of complex curves. The approach to building from simpler components is similar in spirit to that of spin network theory [1].

2 Conventional Observation Process

In a conventional observation process [2], we begin with a state $|\psi\rangle \in H$ of some system S with Hilbert space H and some observable Ω represented by a Hermitian operator on H. Now we have the decomposition $|\psi\rangle = \Sigma_i \alpha_i |\Omega, \psi\rangle_i$ into unit eigenstates $|\Omega, \psi\rangle_i$ of Ω, so that $\Omega|\Omega, \psi\rangle_i = \omega_i |\Omega, \psi\rangle_i$ with the $\omega_i \in \mathrm{Spec}(\Omega)$. The system S is supposed to interact with apparatus A so that the state $|\psi\rangle$ collapses into one of the states $|\Omega, \psi\rangle_i$ with probabilities given by $|\alpha_i|^2$, provided that A is macroscopic. The outcome may be described by the density object

$$\rho(\psi, \Omega) = \Sigma_i |\alpha_i|^2 |\Omega, \psi\rangle_i \langle\Omega, \psi|_i \tag{1}$$

with component density matrix

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_40

$$\rho_{ij} = \delta_{ij}|\alpha_i|^2 \tag{2}$$

and the average information generated by the observation is given by

$$S = -\mathrm{Tr}\rho \ln\rho \tag{3}$$

If we suppose that the apparatus A is also described by quantum mechanics as having state ϕ initially, then during the interaction between S and A we would have

$$|\psi\rangle\,|\phi\rangle \rightarrow \Sigma_i\, \alpha_i|\Omega\psi\rangle_i\,|\Omega\phi\rangle_i \tag{4}$$

for some states $|\Omega\phi\rangle_i$ of A i.e. a final state of S and A that is entangled but still pure. Indeed, even an impure state evolves in such a way that the density matrix ρ maintains $\dot{S}(\rho) = 0$. The apparent infinite regress in accounting for observation within quantum mechanics was the motivation for the many worlds interpretation [3]. An alternative proposal is outlined below.

3 Relative States and State Manifolds

The conventional account (e.g. [2]) takes the observables to be properties of a single system, built ultimately on the properties of single particles, as in the construction of Fock space [4]. With the aim of removing the asymmetry between S and A we will take the view that all observables relate to the relationship between two systems, with the basic building block a pair of particles. This appeal to symmetry is similar to the introduction of Galilean relativity, which challenged the Aristotelian assumption that the overwhelmingly macroscopic nature of the Earth implies that it is rooted in absolute space.

The basic principle of our system is that all observations are the result of new encounters between systems, and that every observation is thus of a new relative state observable for the combined system. This relies on the finite age of the universe, and the main source of information will be based on encounters between particle pairs produced during inflation [5] which will be the basic building blocks from which information is generated through time.

In the framework of quantum field theory we can take Fock space as a universal space in which the Hilbert space for all local systems S are embedded. If S contains energy E and has linear size R then the information content or effective dimension of the Hilbert space $H(S)$ of S is bounded by the Bekenstein limit [6, 7]

$$S \leq 2\pi R\, E \tag{5}$$

in natural units. The time evolution of S is determined by the Hamiltonian through $i\mathcal{H}(S)$, and the space of adiabatic deformations of $i\mathcal{H}(S)$ is given by $\mathcal{A}$, the Lie algebra of anti-Hermitian operators on $H(S)$. $i\mathcal{H}(S)$ and its adiabatic deformations also act on the state manifold $Q(s) = H^*(S)/C$ (as a complex manifold) by Lie dragging. Eigenstates of $i\Omega$ correspond to fixed points f, and the action on the tangent space at f is essentially $i\alpha$ where α is the eigenvalue of Ω at f. Note that there is no loss of information in dealing with $Q(S)$ instead of $H(S)$ as $\rho(\psi, \Omega)$ can be evaluated using the projections of the $|\Omega, \psi\rangle_i$ onto $Q(S)$ as the phases of the α_i cancel.

If we now consider the pair of systems S and A, before interaction their joint state is a simple product $|\psi\rangle|\phi\rangle$ as above. Time evolution acts separately on $H(S)$ and $H(A)$ and maintains the combined state in a subset of $H(S) \otimes H(A)$ which is closed under addition (superposition) only within the factors $H(S)$ and $H(A)$. The state manifold $Q(S, A)$ however is simply the product of the complex manifolds $Q(S)$ and $Q(A)$.

4 Observations and Embeddings

The production of new information is illustrated by experiments with entangled photon pairs e.g. [8] that have been designed to test quantum locality [9–12] and which provide a promising method of secure key generation for cryptography. Typically, a new entangled pseudoscalar photon pair is produced, which then interacts with remote polarization filters, and statistical tests show that the photon polarizations cannot be predicted in advance although correlations can. For the present purpose what is essential here is that since the total angular momentum J_z along the line of flight (z axis) of the photon pair before the interaction is 0, the relative orientation of the polarizations to the filters is not simply unknown, but *cannot exist* before the encounters.

In a case where such a new encounter takes place, there will typically be several possible values for a new relative configuration operator Δ in the joint final state. The eigenstates of Δ will be entangled states of S and A as in the discussion above of conventional observation, but the relevant new information is not of the individual entangled eigenstates, which are regarded as pure states, but that of their relative probability.

Let $|\psi\rangle$ and $|\phi\rangle$ be the initial states of S and A so the combined state

$$|\chi\rangle = |\psi\rangle\, |\phi\rangle \in H(S) \otimes H(A) \tag{6}$$

can be decomposed in eigenstates $|\Delta = \beta_k\rangle$ of Δ as

$$|\chi\rangle = \Sigma_k \chi_k |\beta_k\rangle \tag{7}$$

The density matrix in the basis $|\beta_k\rangle$ is

$$\rho_{km} = \chi_k \chi_m^* \tag{8}$$

During the encounter the Hamiltonian $\mathcal{H}(S,T)$ undergoes the adiabatic deformation

$$\delta\mathcal{H}(S,T) = \varepsilon(t)\Delta \tag{9}$$

which acts on $H(S,T)$ and $Q(S,T)$. We now suppose that ρ_{km} evolves according to

$$\dot{\rho}_{km} = \varepsilon(t)[-i(\beta_k - \beta_m) - |\beta_k - \beta_m|]\rho_{km} \tag{10}$$

The extra exponential decay term in $|E_k - E_m|$ for the off-diagonal elements is conjectured here to enforce the uncertainty principle. Although initially the value of Δ *cannot exist*, as it contributes to the energy of the S, A system the ambiguity should be suppressed.

The density matrix conjectured above interpolates between that of a pure state ($S = 0$) and a mixed or decoherent one on a timescale set by the energy splitting. This is consistent with a random choice as well as a many-worlds interpretation of quantum mechanics. Note that there is no collapse in either $H(S)$ or $H(A)$ but rather a choice of how the limiting final state is embedded in $H(S,A)$ and $Q(S,A)$. We have

$$\dim Q(S,A) = \dim Q(S) + \dim Q(A) + 1 \tag{11}$$

with an expansion of the dimension of the state manifold as would be expected in an irreversible observation process. This will be discussed further in Sect. 6.

5 An Example

Consider two pairs of spin 1/2 particles (A,B) and (C,D) such that each pair has total spin angular momentum $J^2 = 0$ which interact for the first time with B encountering C. The initial states are $|(A,B) : J^2 = 0\rangle$ and $|(C,D) : J^2 = 0\rangle$. Before the encounter we can decompose the product state P into parts where (B,C) has spin 0 and 1 as

$$P = \frac{1}{2}|(B,C) : J^2 = 0\rangle\,|(A,D) : J^2 = 0\rangle + \frac{1}{2}|(B,C) : J^2 = 2\rangle \cdot |(A,D) : J^2 = 2\rangle \tag{12}$$

An interaction between B and C proportional to $J^2(B,C)$ will split the system into two states with evolving density matrix

$$\begin{pmatrix} \frac{1}{4} & e^{-\kappa(t)}\frac{\sqrt{3}}{4} \\ e^{-\kappa(t)}\frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix} \tag{13}$$

where $\kappa(t) = \int \varepsilon(t)dt$ in terms of the energy splitting $\varepsilon(t)$. The average informational entropy ultimately generated is $S = -(\frac{1}{4}\ln\frac{1}{4} + \frac{3}{4}\ln\frac{3}{4})$.

In this case the state spaces $Q(A, B)$ and $Q(C, D)$ are both zero dimensional (single points) and the final space Q is a one complex dimensional Riemann sphere.

6 Analogy to Moduli Stacks of Complex Curves

The state manifolds $Q(S)$ would arise through progressive contact between smaller systems, ultimately entangled particle pairs. New pairs can be produced experimentally, and the early universe would have been populated by such pairs produced by de Sitter horizons during inflation, and which would not yet had time to come into causal contact. This process of construction is reminiscent of Grothendieck's game of Teichmuller Lego [13], in which the moduli spaces of complex curves are built from those for spaces of lower genus g. The moduli space M for a family of disjoint curves C_α is $\Pi M(C_\alpha)$ while the state manifold Q for a family of unconnected systems S_α is $\Pi Q(S_\alpha)$. The complex stack dimension of $M(C)$ is $3(g-1)$, whereas that of $Q(S)$ is $d-1$ where $d = \dim H(S)$. Here the stack dimension counts the number of deformation moduli minus the number of symmetry generators, so is 0 for a torus even though there is a 1 parameter family of complex structure deformations. Similarly the dimension of Q is zero for a pair of spin 1/2 particles with total spin 0 due to symmetry. When a pair of curves are joined by a single bridge, g is additive, while if a pair of systems come into contact d is additive.

This parallelism between the behaviour of state manifolds and moduli spaces is an encouraging sign that both may be examples of some common mathematical structure.

Acknowledgements The author would like to thank the organisers of Lie Theory 13 in Varna for their kind hospitality.

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Dynamical Generation of Flavour Vacuum

Massimo Blasone, Petr Jizba, and Luca Smaldone

Abstract In this paper we review dynamical mixing generation in a generic quantum field theoretical model with global $SU(2)_L \times SU(2)_R \times U(1)_V$ symmetry. By purely algebraic means we analyze the vacuum structure for different patterns of symmetry breaking and show explicitly how the non-trivial flavour vacuum condensate characterizes dynamical mixing generation.

1 Introduction

Neutrino mixing and oscillations are known since long time [4] and many basic features of this phenomenon are now quite well established [3, 14]. The properties of mixing transformation in quantum field theory (QFT) were firstly discussed in Ref. [5] and then developed in many subsequent works (see e.g. Refs. [6, 11, 16, 17]). In particular, it was understood that flavour and mass representations are unitarily inequivalent representations of canonical anticommutation relations [7] and thus *flavour vacuum* is not the same as *mass vacuum*. The former is a condensate of neutrino-antineutrino pairs of particles with different masses.

The idea of dynamical generation of flavour vacuum was firstly introduced in string inspired scenarios [18–21] and then analyzed within Nambu–Jona Lasinio model [8, 9]. In particular, in Ref. [10] it was shown that the spontaneous symmetry breaking (SSB) of $SU(2)_L \times SU(2)_R \times U(1)_V$ chiral flavour symmetry nat-

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_41

urally leads to flavour vacuum which allows for mixing to be dynamically generated. Moreover, Ward–Takahashi (WT) identities were employed in order to count Nambu–Goldstone (NG) modes.

In this paper we review basic facts on SSB and in particular dynamical generation of mixing in the approach of Ref. [10].

2 Spontaneous Symmetry Breaking and ε-Term

Let us consider a Lagrangian $\mathcal{L}$, invariant under the action of an internal group G. We now add a symmetry breaking term:

$$\mathcal{L}_\varepsilon(x) \;=\; \varepsilon\, \Phi(x). \tag{1}$$

This procedure is known under the name ε-term prescription [25, 26]. At the end of the computation the limit $\varepsilon \to 0$ has to be taken.

The Noether charges Q_k play the rôle of group generators. Therefore, a generic transformation can be expressed as

$$g \;=\; \exp\left(\sum_{k=1}^{n} \theta_k\, Q_k\right), \tag{2}$$

and the field Φ, is transformed as:

$$\Phi'(x) \;=\; g\, \Phi(x) \;=\; \Phi(x) \,+\, \sum_{k=1}^{n} \theta_k \delta_k \Phi(x), \tag{3}$$

If the Lagrangian is invariant under the action of G, the WT identity between one point and two points Green' s functions reads [25, 26]

$$i\langle \delta_j \delta_k \Phi(x)\rangle \;=\; \varepsilon \int \mathrm{d}^4 y \langle \mathrm{T}\left\{\left[\delta_j \Phi(y) - c_j\right]\, \delta_k \Phi(x)\right\}\rangle, \tag{4}$$

where $\langle \ldots \rangle \equiv \langle \Omega | \ldots | \Omega \rangle$ and $c_j = \langle \delta_j \Phi(x)\rangle$. Here state $|\Omega\rangle$ represents the vacuum state of the broken phase.

When

$$\lim_{\varepsilon\to 0}\langle \delta_i \delta_j \Phi(x)\rangle \;=\; v_{ij} \;\neq\; 0. \tag{5}$$

for some i and j, we say that the original symmetry G is *spontaneously broken* [15, 22, 25, 26] and v_{ij} are known as *order parameters*. In situations when Φ is a composite field the SSB phenomenon is known as *dynamical symmetry breaking*. The group H generated by that Q_j which annihilate the vacuum is named *vacuum stability subgroup*. Clearly $\tilde{G} = G/H$. The set of vectors:

$$|\Omega(g)\rangle = G(g)|\Omega\rangle, \tag{6}$$

with $g \in G/H$ is known as *vacuum manifold*. These states are energetically degenerate. In fact, $G(g)$ is expressed in terms of Q_j, with $j = 1, \ldots, m$ and then:

$$[G(g),\ H] = 0. \tag{7}$$

However the Fock spaces built on these vacua are all unitarily inequivalent.

A central result named *Goldstone theorem* [22] affirms that Eq. (5) can be satisfied only if massless scalar fields appear among physical fields. These take the name of *Nambu-Goldstone* bosons. In fact, because the LHS of Eq. (4) differs from zero, the Umezawa–Källén–Lehmann spectral representation [25, 26] of RHS implies that the quantity $\varepsilon\rho(\boldsymbol{k} = \boldsymbol{0}, m_k)/m_k^2$ (ρ is the spectral distribution) is non-vanishing for $\varepsilon \to 0$ and thus masses $m_k^2 \propto \varepsilon$ due to positive definiteness of ρ. More specifically, the dynamical map of the scalar Heisenberg fields $\delta_j \Phi(y) - c_j$ reads:

$$\delta_j \Phi(y) - c_j = \sum_{k=0}^{n} C_{jk}(\partial)\, \varphi_k(y) + \ldots, \tag{8}$$

where φ_k are the NG fields.

3 Dynamical Mixing Generation via Flavour Vacuum Condensate

Let us take the symmetry group G as the global *chiral-flavour* group $G = SU(2)_L \times SU(2)_R \times U(1)_V$. Let the fermion field be a flavour doublet $\boldsymbol{\psi} = \left[\tilde{\psi}_1\ \tilde{\psi}_2\right]^T$. The action of the chiral-group transformations $\mathbf{g}$ on $\boldsymbol{\psi}$ is defined by:

$$\boldsymbol{\psi} \to \boldsymbol{\psi}' = \mathbf{g}\boldsymbol{\psi} = \exp\left[i\left(\phi + \boldsymbol{\omega}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\omega}_5\cdot\frac{\boldsymbol{\sigma}}{2}\gamma_5\right)\right]\boldsymbol{\psi}, \tag{9}$$

where $\sigma_j,\ j = 1, 2, 3$ are the Pauli matrices and $\phi, \boldsymbol{\omega}, \boldsymbol{\omega}_5$ are real-valued transformation parameters of G. Corresponding Noether charges are

$$Q = \int \mathrm{d}^3\mathbf{x}\, \boldsymbol{\psi}^\dagger \boldsymbol{\psi}, \quad \boldsymbol{Q} = \int \mathrm{d}^3\mathbf{x}\, \boldsymbol{\psi}^\dagger \frac{\boldsymbol{\sigma}}{2}\boldsymbol{\psi}, \quad \boldsymbol{Q}_5 = \int \mathrm{d}^3\mathbf{x}\, \boldsymbol{\psi}^\dagger \frac{\boldsymbol{\sigma}}{2}\gamma_5\boldsymbol{\psi}. \tag{10}$$

By analogy with quark condensation in QCD [1, 2, 13, 22, 24], we will limit our considerations to order parameters that are condensates of fermion-antifermion pairs. To this end we introduce the following composite operators

$$\Phi_k = \overline{\boldsymbol{\psi}}\, \sigma_k\, \boldsymbol{\psi}, \quad \Phi_k^5 = \overline{\boldsymbol{\psi}}\, \sigma_k\, \gamma_5 \boldsymbol{\psi}, \quad \sigma_0 \equiv \mathbb{1}, \tag{11}$$

with $k = 0, 1, 2, 3$. For simplicity we now assume $\langle \boldsymbol{\Phi}^5 \rangle = 0$. This assumption does not substantially changes our main result.

Let us now look at three SSB schemes $G \to H$:

i) The SSB sequence [12, 22]

$$SU(2)_L \times SU(2)_R \times U(1)_V \longrightarrow SU(2)_V \times U(1)_V, \tag{12}$$

corresponds to dynamical mass generation and it is characterized by the order parameter

$$\langle \Phi_0 \rangle \;=\; v_0 \;\neq 0, \qquad \langle \Phi_k \rangle \;=\; 0, \quad k = 1, 2, 3. \tag{13}$$

One can easily check that this expression is invariant under the residual group $H \equiv U(2)_V$ but not under the full chiral group G.

In order to discuss the NG modes it is convenient to employ the WT identity within the framework of ε-prescription, by taking $\mathcal{L}_\varepsilon = \varepsilon \Phi_0$. We thus find WT identity (4) in the form:

$$i\, v_0 \;=\; \lim_{\varepsilon \to 0} \varepsilon \int \mathrm{d}^4 y \left\langle T \left[\Phi_k^5(y)\, \Phi_k^5(0) \right] \right\rangle, \tag{14}$$

where $k = 1, 2, 3$. Therefore, Φ_k^5 will contain the gapless NG fields as linear terms of their respective dynamical maps (see Eq. (8)).

Let us now analyze the structure of the vacuum manifold. This would be formed by the vectors

$$|\boldsymbol{\theta}_5\rangle \;=\; e^{i \sum_{\alpha=1}^{3} \theta_k^5 Q_{5,k}} |\Omega\rangle. \tag{15}$$

However

$$\langle \theta_k^5 | \Phi_k^5 | \theta_k^5 \rangle \;=\; i \sin \theta_k^5 \; v_0 \,, \qquad k = 1, 2, 3, \tag{16}$$

which, because of Eq. (13), are different from zero when $\theta_k^5 \neq 0$, in contrast with our assumption. The only permitted vacuum is thus the one with $\theta_k^5 = 0$ for each k.

ii) As a second case we consider the SSB pattern

$$SU(2)_L \times SU(2)_R \times U(1)_V \longrightarrow U(1)_V \times U(1)_V^3, \tag{17}$$

which is responsible for the dynamical generation of different masses m_1 and m_2. In this case the order parameters take the form

$$\langle \Phi_0 \rangle \;=\; v_0 \;\neq\; 0, \qquad \langle \Phi_3 \rangle \;=\; v_3 \;\neq\; 0. \tag{18}$$

If we take the ε-term as $\mathcal{L}_\varepsilon = \varepsilon(\Phi_0 + \Phi_3)$, we can derive WT identities and check that the dynamical maps of Φ_1, Φ_2, Φ_1^5, Φ_2^5 and $\Phi_3^5 + \Phi_0^5$ will contain NG fields as linear terms in their Haag expansion [10].

To study the structure of the vacuum manifold, we introduce the charges $Q_\pm = Q_1 \pm i\, Q_2$. The vacuum manifold is thus isomorphic to $SU(2)_V/U(1)_V^3$:

$$|\theta\rangle \;\equiv\; \exp\left(\theta\, Q_+ - \theta^*\, Q_-\right)|\Omega\rangle. \tag{19}$$

These are iso-energetic vacua, which present the structure of $SU(2)$ *coherent states* [23]. If we further consider the case $\theta = \theta^*$ is now easy to get:

$$\langle\theta|\Phi_1(x)|\theta\rangle \;=\; \sin 2\theta\, v_3, \tag{20}$$

which is different from zero for all $\theta \neq 0$. However, this choice does not change the physics, because the residual symmetry here is still $U(1)_V \times U(1)_V^3$ and not $U(1)_V$ which corresponds to conservation of total flavour charge [3]. This fact makes apparent that dynamical mixing generation requires some deeper argument, because any of the order parameters considered until now, did not change the situation producing new physical effects. We will see in the next case that the answer requires the concept of the flavour vacuum.

iii) Finally, we consider the SSB scheme

$$SU(2)_L \times SU(2)_R \times U(1)_V \longrightarrow U(1)_V \times U(1)_V^3 \longrightarrow U(1)_V, \tag{21}$$

which thus corresponds to dynamical generation of field mixing.

Let us introduce

$$\Phi_{k,m} \;=\; \overline{\boldsymbol{\psi}}\,\sigma_k\,\boldsymbol{\psi}, \qquad k = 1, 2, 3, \tag{22}$$

where m indicates that $\boldsymbol{\psi}$ is now a doublet of fields $\boldsymbol{\psi} = [\psi_1\ \psi_2]^T$ in the mass basis. The SSB condition now reads

$$\langle\Phi_{1,m}\rangle \;\equiv\; v_{1,m} \;\neq\; 0. \tag{23}$$

Hence we find that a *necessary condition* for a dynamical generation of field mixing within chiral symmetric systems, is the presence of exotic pairs in the vacuum, made up from fermions and antifermions with different masses[1]:

$$\langle\overline{\psi}_i(x)\,\psi_j(x)\rangle \neq\; 0, \qquad i \;\neq\; j. \tag{24}$$

In other words, *field mixing requires mixing at the level of the vacuum condensate structure*. This structure is the same of *flavour vacuum* [5], consistently with analogous results of Ref. [18].

[1]Generally also diagonal condensate may be present.

4 Conclusions

Here we reviewed the arguments of Ref. [10], where we proved that flavour vacuum exotic condensate structure [5] naturally and inescapably arises in $SU(2)_L \times SU(2)_R \times U(1)_V$ chirally symmetric models, when mixing is dynamically generated. As it was shown in Ref. [10] this vacuum structure is exactly the one encountered in neutrino physics, once we limit ourselves to mean field approximation.

Acknowledgements P. J. was supported by the Czech Science Foundation Grant No. 17-33812L.

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A Representation of the Wave-Function on 3-Dimensional Space

O. C. Stoica

Abstract In nonrelativistic Quantum Mechanics (NRQM), the wavefunction is defined on the 3**N**-dimensional configuration space. Schrödinger, Lorentz, Einstein, and others, were concerned that it was not an object defined on the 3-dimensional (3D) physical space. So far, the task of representing the wave function as an object defined on the 3-dimensional space could only be achieved partially. Based on ref. Phys. Rev. A **100**, 042115, 2019 [1], I describe a fully equivalent representation of the many-particle states as multi-layered fields defined on the 3D physical space. They are shown to evolve locally under Schrödinger's equation, nonlocality entering only if the wavefunction collapses.

1 Introduction

Since the discovery of quantum mechanics (QM), a problem seemed to disturb physicists, in addition to the deeper and more difficult measurement problem and the problem of why the world appears classical at macroscopic level. It is the fact that the many-particle wavefunction (WF) is defined on the 3**N**-dimensional configuration space, rather than on the 3-dimensional physical space. Worries about this were expressed in particular by Schrödinger, Lorentz, Einstein, and others [2, 3], and continue to be expressed today.

Approaches to resolve the problem range from accepting the situation (which is not so unreasonable after all) [4, 5], to the proposal of 3D-space fields that only capture partially the full data contained in the WF [4, 6–8]. These approaches and others are reviewed in [1], as well as various arguments.

I will present a straightforward, although complicated and not aesthetically appealing, representation of the WF on the 3D-space, which is fully equivalent to the WF. Moreover, it turns out that as long as there is no WF collapse and potentials propagate with limited velocity, it evolves locally in space. The purpose is to prove that it is possible, which may be of use to researchers working at interpretations of QM

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_42

which assume the WF to be ontic (although I will not take sides in the debates on interpretations). It also gives a justification to continue working on the configuration space, knowing that there is a way to make sense of the WF on the 3D-space, without worries of criticism from philosophers working on foundations. This representation is defined pointwisely using sections of an infinite-dimensional fiber bundle, so it is locally separable, and can recover completely the WF.

The essential reason why the WF is defined on the configuration space is that it involves tensor products of Hilbert spaces of functions defined on the Euclidean 3D-space $\mathbb{E}_3$, for example the tensor product of two states represented by the wavefunctions $\psi_1(\mathbf{x})$ and $\psi_2(\mathbf{x})$, defined on $\mathbb{E}_3$, is represented by the WF $\psi_1(\mathbf{x}_1) \otimes \psi_2(\mathbf{x}_2)$ defined on $\mathbb{E}_3^2$. Thus, to obtain the representation, we need to represent the tensor product of such functions, with the usual natural operations with tensors, in terms of fields defined on $\mathbb{E}_3$. But this is not straightforward, since our fields have to be local.

2 The 3D Space Representation of Wavefunctions

The main result is (also see [1]).

Theorem 1. *The space of many-particle wavefunctions defined on the configuration space admits a representation as vector fields defined on the* 3*D space* $\mathbb{E}_3$.

Let $\mathbf{D}$ be the set of possible internal states and spin degrees of freedom (normally $|\mathbf{D}|$ is finite). Let $\mathbf{D}$ identify the type of the particle. The set $\mathbf{D}$ can be used to span a complex vector space $\mathbb{V}_{\mathbf{D}}$, representing the quantum internal states. For instance, spin $1/2$ systems have, classically, two degrees of freedom, to which a two dimensional complex space corresponds after quantization. Let $\mathbb{V}_{\mathbb{E}_3}$ be the vector space of complex functions defined on $\mathbb{E}_3$. It can be the Hilbert space of square integrable functions, or we can work with the rigged Hilbert space, but we will keep it general. $\mathbb{V}_{\mathbb{E}_3,\mathbf{D}} \cong \mathbb{V}_{\mathbb{E}_3} \otimes \mathbb{V}_{\mathbf{D}}$ is the vector space of complex functions defined on the one-particle configuration space $\mathbb{E}_3 \times \mathbf{D}$, but by duality in $\mathbb{V}_{\mathbf{D}}$ it can also be seen as the vector space of $\mathbb{V}_{\mathbf{D}}$-valued vector fields on $\mathbb{E}_3$. It represents the state space of a particle of type $\mathbf{D}$. The state space of $\mathbf{N}$ particles of types $\mathbf{D}_1, \ldots, \mathbf{D}_\mathbf{N}$ is denoted by $\mathbb{V}_{\mathbb{E}_3^\mathbf{N},\mathbf{D}_1,\ldots,\mathbf{D}_\mathbf{N}}$, and is the tensor product of the individual state spaces.

We define a binary relation $\sim$ on $\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \oplus \ldots \oplus \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_\mathbf{N}}$,

$$(\psi_1, \ldots, \psi_\mathbf{N}) \sim (\psi_1', \ldots, \psi_2') \tag{1}$$

iff $(\psi_1', \ldots, \psi_2') = T\,(\psi_1, \ldots, \psi_\mathbf{N})$, where T is a linear transformation of $\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \oplus \ldots \oplus \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_\mathbf{N}}$ of the form

$$T = \begin{pmatrix} c_1 1_{|\mathbf{D}_1|} & 0 \ \ldots & 0 \\ \ldots & \ldots\ldots & \ldots \\ 0 & \ldots \ 0 & c_\mathbf{N} 1_{|\mathbf{D}_\mathbf{N}|} \end{pmatrix}, \tag{2}$$

where $c_1 \dots c_{\mathbf{N}} = 1$. Such transformations form a group $\mathcal{G}_{|\mathbf{D}_1|,\dots,|\mathbf{D}_\mathbf{N}|}$. If $\mathbf{N} = 1$, we define it by $(\psi) \sim (\psi')$ iff $\psi = \psi'$. The equivalence class $[\psi_1, \dots, \psi_\mathbf{N}]_\sim$ is the orbit $\mathcal{G}_{|\mathbf{D}_1|,\dots,|\mathbf{D}_\mathbf{N}|}(\psi_1, \dots, \psi_\mathbf{N})$. We denote it by $\psi_1 \boxtimes \dots \boxtimes \psi_\mathbf{N}$. We define

$$\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \boxtimes_s \dots \boxtimes_s \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_\mathbf{N}} := \left(\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \oplus \dots \oplus \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_\mathbf{N}}\right) / \sim .$$

The relation $\sim$ is an equivalence relation (see [1]).

At this point, we can only represent product states. Only a small difference exists between $[\psi_1, \dots, \psi_\mathbf{N}]_\sim$ and the tensor products $\psi_1 \otimes \dots \otimes \psi_\mathbf{N}$ – we can use the same position $\mathbf{x}$ as argument for all fields instead of $\mathbf{N}$ different positions $\mathbf{x}_j$. But momentarily the vector fields of the form $[\psi_1, \dots, \psi_\mathbf{N}]_\sim$ are too rigid, since the transformations T have to be globally constant. The field representing the WF should be definable locally, and at this point it is not.

In the parlance of quantum foundations, we have to implement *local separability*. This means that, for any field representation $\widetilde{\Psi} = [\psi_1, \dots, \psi_\mathbf{N}]_\sim$, and any two regions $A, B \subseteq \mathbb{E}_3$, we should be able to recover $\widetilde{\Psi}|_{A\cup B}$ from $\widetilde{\Psi}|_A$ and $\widetilde{\Psi}|_B$. This is achieved in [1] by promoting the global symmetry $\mathcal{G}_{|\mathbf{D}_1|,\dots,|\mathbf{D}_\mathbf{N}|}$ to a local symmetry, as usually done in QM when the local gauge symmetry of electromagnetism is obtained from the global complex phase. Here is how.

The typical fiber of the vector bundle $\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_j}$ is the vector space $\mathbb{V}_{\mathbf{D}_j}$, so the structure group is $\mathrm{GL}(\mathbb{V}_{\mathbf{D}_j})$. We take as associated principal bundle the trivial bundle $\mathbb{E}_3 \times \mathrm{GL}(\mathbb{V}_{\mathbf{D}_j})$ (since the base manifold $\mathbb{E}_3$ has trivial topology). To express our fields, we need to fix a gauge of $\mathbb{E}_3 \times \mathrm{GL}(\mathbb{V}_{\mathbf{D}_j})$, or a *frame field* of $\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_j}$. Then, any representation of $\psi \in \Gamma\left(\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_j}\right)$ can only make sense if also the frame field in which is expressed is given.

In general, for $\mathbf{N}$ particles, the structure group is $\mathrm{GL}(\mathbb{V}_{\mathbf{D}_1} \oplus \dots \oplus \mathbb{V}_{\mathbf{D}_\mathbf{N}})$. We ignore here the linear transformations of the bundles $\mathbb{V}_{\mathbf{D}_j}$, which can be treated separately. So we can consider that the frame field is fixed everywhere, and we can treat separately the action of the group $\mathcal{G}_{|\mathbf{D}_1|,\dots,|\mathbf{D}_\mathbf{N}|}$. This group is isomorphic to the commutative multiplicative group $\mathcal{G}_\mathbf{N} := (\mathbb{C} \setminus \{0\})^{\mathbf{N}-1}$. Hence, when we fix the frames of each one-particle vector bundle, the principal bundle associated to our representation is $\mathbb{E}_3 \times \mathcal{G}_\mathbf{N}$. Its sections $g(\mathbf{x}) = (c_1(\mathbf{x}), \dots, c_{\mathbf{N}-1}(\mathbf{x}))$ of $\mathbb{E}_3 \times \mathcal{G}_\mathbf{N}$ act on $\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \oplus \dots \oplus \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_\mathbf{N}}$ by local transformations

$$T(g)(\mathbf{x}) := \begin{pmatrix} c_1(\mathbf{x})1_{|\mathbf{D}_1|} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & c_{\mathbf{N}-1}(\mathbf{x})1_{|\mathbf{D}_{\mathbf{N}-1}|} & 0 \\ 0 & \dots & 0 & c_1^{-1}\dots c_{\mathbf{N}-1}^{-1}1_{|\mathbf{D}_\mathbf{N}|} \end{pmatrix}. \tag{3}$$

We can now define the bundle whose sections represent the many-particle WFs, by applying the associated bundle construction to the product bundle $(\mathbb{E}_3 \times \mathcal{G}_\mathbf{N}) \times \left(\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \oplus \dots \oplus \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_\mathbf{N}}\right)$, where the group $\mathcal{G}_\mathbf{N}$ acts by

$$(p(\mathbf{x}), (\psi_1(\mathbf{x}), \dots, \psi_\mathbf{N}(\mathbf{x}))) := \left(p(\mathbf{x})g(\mathbf{x}), T(g^{-1})(\mathbf{x})(\psi_1(\mathbf{x}), \dots, \psi_\mathbf{N}(\mathbf{x}))\right). \tag{4}$$

The orbits $[p(\mathbf{x}), (\psi_1(\mathbf{x}), \ldots, \psi_{\mathbf{N}}(\mathbf{x}))]_\sim$ define the associated bundle

$$\begin{aligned}\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1\boxtimes_s\ldots\boxtimes_s\mathbf{D_N}} &:= (\mathbb{E}_3 \times \mathcal{G}_{\mathbf{N}}) \times_T \left(\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \oplus \ldots \oplus \mathbb{V}_{\mathbb{E}_3,\mathbf{D_N}}\right) \\ &= \left((\mathbb{E}_3 \times \mathcal{G}_{\mathbf{N}}) \times \left(\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \oplus \ldots \oplus \mathbb{V}_{\mathbb{E}_3,\mathbf{D_N}}\right)\right) / \mathcal{G}_{\mathbf{N}},\end{aligned} \tag{5}$$

which is a bundle over $\mathbb{E}_3$. Thus, the degrees of freedom of the associated bundle can be treated as unphysical, but we need them for local separability. We endow this bundle with the flat connection defined by the trivialization, and this connection transforms under $\mathcal{G}_{\mathbf{N}}$ too, as in usual gauge theories. So, our construction is a fiber bundle over the 3D space $\mathbb{E}_3$, endowed with a flat connection, and $\psi_1 \boxtimes \ldots \boxtimes \psi_{\mathbf{N}}$ are its sections. The rigidity still exists, but local separability is realized.

Note that the internal additions of the vector fields is different from the representation of the sums or linear combinations of states, which will be given in the following. The first step is to define one-dimensional vector spaces of fields. The multiplication with a complex number $c \in \mathbb{C}$ is thus defined by $c\psi_1 \boxtimes \ldots \boxtimes \psi_{\mathbf{N}} := (c\psi_1) \boxtimes \ldots \boxtimes \psi_{\mathbf{N}}$. The two sections $\psi_1 \boxtimes \ldots \boxtimes \psi_{\mathbf{N}}$ and $c\psi_1 \boxtimes \ldots \boxtimes \psi_{\mathbf{N}}$ are called *collinear*. Addition of fields having collinear equivalence classes is defined by $c_1\psi_1 \boxtimes \ldots \boxtimes \psi_{\mathbf{N}} + c_2\psi_1 \boxtimes \ldots \boxtimes \psi_{\mathbf{N}} := (c_1 + c_2)\psi_1 \boxtimes \ldots \boxtimes \psi_{\mathbf{N}}$. Thus, the set of all collinear fields becomes a one-dimensional vector space.

It is easy to see that we can apply recursively the constructions defined so far, since we are talking now again about vector fields. Hence, for one-particle or many-particle states ψ_1, ψ_2, ψ_3, it follows that $(\psi_1 \boxtimes \psi_2) \boxtimes \psi_3 = |\psi_1\rangle \boxtimes (|\psi_2\rangle \boxtimes \psi_3) = \psi_1 \boxtimes \psi_2 \boxtimes \psi_3$. The proof is given in detail in [1]. Therefore, we can drop the brackets in such products.

What we did so far works only for separable states, or product states. We need to extend our construction to include linear combinations of such states. A step was done by defining multiplication with scalars, and addition of collinear sections. But we need to extend this to general linear combinations of product states, in a way which is not redundant, so that no WF is represented by more distinct fields. This redundancy can happen for example if we take the bundle spanned by the representations of product states, or of the rays they define, since for instance $|\psi_1\rangle \otimes |\psi_2\rangle + |\psi_1\rangle \otimes |\psi_2'\rangle = |\psi_1\rangle \otimes \left(|\psi_2\rangle + |\psi_2'\rangle\right)$ is again separable. To prevent this, we need to construct the product space representation for a basis, rather than for all separable states, and then use the resulting rays to span the bundle whose sections will represent all states.

Let (ξ_α) be a basis of $\mathbb{V}_{\mathbb{E}_3}$, indexed by α. When possible, it can be orthonormal. For each type of particle $\mathbf{D}_j$, let $\left(\mathbf{d}^k_{(j)}\right)_k = \left(\mathbf{d}^1_{(j)}, \ldots, \mathbf{d}^{|\mathbf{D}_j|}_{(j)}\right)$ be a basis of $\mathbb{V}_{\mathbf{D}_j}$, indexed by $k \in \{1, \ldots, |\mathbf{D}_j|\}$. Then, $\left(\xi^{(j)}_{\alpha k}\right)$ is a basis of $\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_j} := \mathbb{V}_{\mathbb{E}_3} \otimes \mathbb{V}_{\mathbf{D}_j}$, indexed by α and k, where $\xi^{(j)}_{\alpha k} := \xi_\alpha \mathbf{d}^k_{(j)}$.

The first step is to apply the construction we gave in general for separable states $\psi_1 \otimes \ldots \otimes \psi_{\mathbf{N}}$, but this time only to $\mathbf{N}$ particle states of the form $\xi^{(j)}_{\alpha k_1} \otimes \ldots \otimes \xi^{(j)}_{\alpha k_{\mathbf{N}}}$. We represent such states by fields $\xi^{(j)}_{\alpha k_1} \boxtimes \ldots \boxtimes \xi^{(j)}_{\alpha k_{\mathbf{N}}} := \left[\xi^{(j)}_{\alpha k_1}, \ldots, \xi^{(j)}_{\alpha k_{\mathbf{N}}}\right]_\sim$, as above, for each $\mathbf{N} \geq 1$. The second step is to construct the direct sum of all one-dimensional vector spaces spanned by vectors of the form $\xi^{(j)}_{\alpha k_1} \boxtimes \ldots \boxtimes \xi^{(j)}_{\alpha k_{\mathbf{N}}}$. More about this construction in [1]. In particular, to represent $[\psi_1, \ldots, \psi_n]_\sim$, we have first to express each ψ_j in the basis $\left(\xi^{(j)}_{\alpha k_{(j)}}\right)$ of $\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_j}$,

$$\psi_j = \sum_{k_{(j)}} c^{(j)}_{\alpha_{k_{(j)}}} \xi^{(j)}_{\alpha k_{(j)}}, \tag{6}$$

where $c^{(j)}_{\alpha_{k_{(j)}}}$ are complex numbers. Then, we *define*

$$\psi_1 \boxtimes \ldots \boxtimes \psi_{\mathbf{N}} := \sum_{k_{(1)}} \ldots \sum_{k_{(\mathbf{N})}} c^{(1)}_{\alpha_{k_{(1)}}} \ldots c^{(\mathbf{N})}_{\alpha_{k_{(\mathbf{N})}}} \xi^{(1)}_{\alpha k_{(1)}} \boxtimes \ldots \boxtimes \xi^{(\mathbf{N})}_{\alpha k_{\mathbf{N}}}. \tag{7}$$

I know it was defined already earlier, but to move to nonseparable states while avoiding redundancy, we had to redo the definition, so that we apply it first to get fields of the form $\xi^{(1)}_{\alpha k_{(1)}} \boxtimes \ldots \boxtimes \xi^{(\mathbf{N})}_{\alpha k_{\mathbf{N}}}$, and then extend it by linearity to more general separable states. Now, it is immediate to see that such linear combinations also work for nonseparable states that are linear combinations of states of the form $\psi_1 \otimes \ldots \otimes \psi_{\mathbf{N}}$. It is also immediate to see that the operation $\boxtimes$ is distributive over the addition,

$$\psi_1 \boxtimes (\psi_2 + \psi_3) = \psi_1 \boxtimes \psi_2 + \psi_1 \boxtimes \psi_3 \tag{8}$$

$$(\psi_2 + \psi_3) \boxtimes \psi_1 = \psi_2 \boxtimes \psi_1 + \psi_3 \boxtimes \psi_1, \tag{9}$$

where $\psi_1 \in \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1}$ and $\psi_2, \psi_3 \in \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_2}$ [1].

The fields of the form

$$\sum_{k_{(1)}} \ldots \sum_{k_{(\mathbf{N})}} c_{\alpha_{k_{(1)}} \ldots \alpha_{k_{(\mathbf{N})}}} \xi^{(1)}_{\alpha k_{(1)}} \boxtimes \ldots \boxtimes \xi^{(\mathbf{N})}_{\alpha k_{\mathbf{N}}} \tag{10}$$

with $c_{\alpha_{k_{(1)}} \ldots \alpha_{k_{(\mathbf{N})}}} \in \mathbb{C}$, form a vector space, which we denote by $\mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \boxtimes \ldots \boxtimes \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_{\mathbf{N}}}$.

Our representation is thus defined first on separable states

$$\begin{cases} \rho_\boxtimes : \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \otimes \ldots \otimes \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_{\mathbf{N}}} \to \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_1} \boxtimes \ldots \boxtimes \mathbb{V}_{\mathbb{E}_3,\mathbf{D}_{\mathbf{N}}} \\ \rho_\boxtimes(|\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle) = \psi_1 \boxtimes \ldots \boxtimes \psi_n \end{cases} \tag{11}$$

and then extended by linearity.

Now, the fiber bundle construction that we used for local separability can be extended to this direct sum. The result is a bundle whose sections represent the WF of any many-particle state, as a field on $\mathbb{E}_3$.

Also note that, while we used in this construction a particular basis for each one-particle Hilbert space, we can change the basis by using the identity (7). So the construction of the final bundle is independent on the chosen basis, even though that of intermediate bundles does not seem so.

In reference [1], the representation was extended to the Hilbert spaces of variable numbers of particles, and in particular to fermionic and bosonic Fock spaces. It was also shown that Schrödinger's equation translates to an equivalent partial differential equation on fields on $\mathbb{E}_3$, which is local when interactions have limited velocity. As long as only unitary evolution is involved, everything is local, nonlocality appearing only when the WF collapses. This was exemplified with the EPR experiment [9, 10]. Also, although the entire construction was made for NRQM, it was shown how can be applied to represent Fock spaces in quantum field theory. The implications on ontology were also discussed in [1], in connection to various interpretations of QM.

This representation is rather a proof of existence. It is more complicated than the WF on the configuration space $\mathbb{E}_3^{\mathbf{N}}$, just like in the classical mechanics of $\mathbf{N}$ point-particles, the formulation on $\mathbb{E}_3$ is more complicated than that on $\mathbb{E}_3^{\mathbf{N}}$. There should probably exist a more natural way, which in particular gives the right (anti-)symmetries in the case of $\mathbf{N}$ identical particles.

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Various Mathematical Results

On a Grothendieck-Brieskorn Relation for Elliptic Calabi-Yaus

James Fullwood

Abstract For a number of elliptic Calabi-Yau 3-folds $X \to B$, it has been discovered by string theorists that the Euler characteristic of X may be given in terms of the representation theory of a certain Lie algebra $\mathfrak{g}_X$ associated with X. In such a case, we say $X \to B$ satisfies the *Grothendieck-Brieskorn* relation for elliptic Calabi-Yau 3-folds. In this note, we define two invariants $\mathfrak{R}_X$ and $\mathfrak{L}_X$ for an elliptic Calabi-Yau of arbitrary dimension, such that in the case of 3-folds the Grothendieck-Brieskorn relation is given by $\mathfrak{R}_X = \mathfrak{L}_X$. Beyond the 3-fold case, the equation $\mathfrak{R}_X = \mathfrak{L}_X$ then yields a notion of Grothendieck-Brieskorn relation for elliptic Calabi-Yaus of arbitrary dimension.

1 Introduction

By the mid-20th century it was known that simple surface singularities admit an ADE classification. In particular, if one considers the minimal resolution of a simple surface singularity by a sequence of blowups, then the exceptional locus of the resolution consists of a chain of $\mathbb{P}^1$s, whose dual graph then coincides with a Dynkin diagram of ADE-type. In the 1960s, Grothendieck sought a deeper connection between simple surface singularities and Lie algebras beyond that as a mere labeling device, which led him to conjecture a precise relationship between the universal deformation of a simple surface singularity and the representation theory of the associated Lie algebra. Such inquiry then resulted in a theorem proved by Brieskorn in 1970 [3], which paved the way for further development of the relationship between singularities and representations of Lie algebras.

There is also an evident connection between the theory of elliptic fibrations and Lie theory, which has been illuminated by field-theoretic considerations of physicists studying string compactifications on elliptic Calabi-Yaus [9, 11]. If B is a compact complex manifold, then an elliptic fibration is a proper, flat, surjective morphism

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_43

$X \to B$ with a section, such that the generic fiber is an elliptic curve. The elliptic fibers degenerate to singular curves which are unions of rational curves over a hypersurface $\Delta \subset B$ referred to as the *discriminant* of $X \to B$, and it is the singular fibers over Δ which establish a connection between $X \to B$ and a certain Lie algebra $\mathfrak{g}_X$, referred to as the *gauge algebra* of X. In particular, the discriminant Δ may be written as a union of its irreducible components

$$\Delta = \Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_m,$$

and over each Δ_i the generic fiber of $X \to B$ is a chain of rational curves, whose associated dual graph is the Dynkin diagram of a Lie algebra $\mathfrak{g}_i$ (all Dynkin diagrams may occur). By convention we take Δ_0 as the component over which the generic fiber is irreducible, in which case $\mathfrak{g}_0$ is trivial. The gauge algebra $\mathfrak{g}_X$ is then given by

$$\mathfrak{g}_X = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_m \oplus \mathfrak{u}(1)^{\oplus r}, \tag{1}$$

where r is the rank of the Mordell-Weil group of rational sections of $X \to B$.

In the case X is a smooth Calabi-Yau 3-fold (so that B is necessarily a smooth rational surface), let $\mathfrak{R}_X$ be given by

$$\mathfrak{R}_X = \frac{1}{2}\chi(X) + 30K_B^2, \tag{2}$$

where $\chi(X)$ denotes the topological Euler characteristic of X. While $\mathfrak{R}_X$ only depends on the topology of X, it has been shown in a number of examples coming from string theory that a formula for $\mathfrak{R}_X$ may be given in terms of the representation theory of $\mathfrak{g}_X$ [8–10]. In particular, if ρ is a representation of a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, the *charged dimension* of ρ is given by

$$(\dim\rho)_{\mathrm{ch}} = \dim\rho - \dim(\ker\rho|_{\mathfrak{h}}).$$

Under the assumption $\Delta = \Delta_0 \cup \Delta_1$ with Δ_1 smooth of genus g (so that $\mathfrak{g}_X = \mathfrak{g}_1 \oplus \mathfrak{u}(1)^{\oplus r}$), the aforementioned formula for $\mathfrak{R}_X$ takes the form

$$\mathfrak{R}_X = (g-1)\dim(\mathrm{adj})_{\mathrm{ch}} + (g'-g)\dim(\rho_0)_{\mathrm{ch}} + \sum_{p \in A} \dim(\rho_p)_{\mathrm{ch}}, \tag{3}$$

where g' is the genus of a certain branched covering of Δ_1 determined by the singular points of the general fiber over Δ_1, ρ_0 is a certain representation of the Lie algebra associated with the fiber that appears generically over Δ_1, A is the set of points in $\Delta_0 \cap \Delta_1$ such that the fiber over p is of Kodaira type, and ρ_P is a certain representation of the Lie algebra associated with the Dynkin diagram of the Kodaira fiber over p.[1]

[1]For the precise definitions of g', ρ_0 and the ρ_P, one may consult [8, 10].

From here on, we'll refer to formula (3) as the *Grothendieck-Brieskorn relation* associated with $X \to B$, or rather, the GB-relation for short.

At present, there is no proof of a general GB-relation for elliptic 3-folds. While the relation has been verified by direct computation for a number of examples on a case-by-case basis, a purely mathematical explanation has yet to take form. Moreover, from both a mathematical and a physical perspective it is natural to ask whether such a relation exists in (complex) dimension greater than 3. In particular, dimension 4 is particularly relevant for string theory, as elliptic 4-folds yield spacetime models with 3 (uncompactified) dimensions of space and 1 time, while elliptic 3-folds yield models with 5 (uncompactified) dimensions of space and 1 time. In any case, it is the issue of generalizing the GB-relation to dimension greater than 3 with which we concern ourselves in this note.

In what follows, we define two integer-valued invariants $\mathfrak{R}_X$ and $\mathfrak{L}_X$ for an elliptic Calabi-Yau in arbitrary dimension. While $\mathfrak{R}_X$ generalizes the $\mathfrak{R}_X$ given by (2), the invariant $\mathfrak{L}_X$ generalizes the RHS of (3). Moreover, there has recently appeared a generalization of $\mathfrak{R}_X$ and $\mathfrak{L}_X$ to singular elliptic 3-folds [10], and we adapt these invariants to the setting of arbitrary dimension as well.

2 The Invariant $\mathfrak{R}_X$

We now rewrite the invariant $\mathfrak{R}_X$ given by (2) in a dimension-independent way. For this, let B be a compact complex manifold of arbitrary dimension, and let $\mathscr{L} \to B$ be a line bundle. For $\mathscr{E} = \mathscr{O}_B \oplus \mathscr{L} \oplus \mathscr{L}^2$, denote by $\pi : \mathbb{P}(\mathscr{E}) \to B$ the projective bundle of *lines* in $\mathscr{E}$, and denote its tautological bundle by $\mathscr{O}(-1)$. A *smooth Weierstrass fibration* is then an elliptic fibration $W \to B$, whose total space W is a smooth hypersurface in $\mathbb{P}(\mathscr{E})$ given by

$$W : (y^2 z = x^3 + f x z^2 + g z^3) \subset \mathbb{P}(\mathscr{E}). \tag{4}$$

The coefficients f and g are sections of $\pi^*\mathscr{L}^4$ and $\pi^*\mathscr{L}^6$ respectively, and the fiber coordinates x, y and z are sections of $\mathscr{O}(1) \otimes \mathscr{L}^2$, $\mathscr{O}(1) \otimes \mathscr{L}^3$, and $\mathscr{O}(1)$ respectively, so that W then corresponds to the zero-scheme of a section of $\pi^*\mathscr{O}(3) \otimes \mathscr{L}^6$. The total space W is smooth if and only if the hypersurfaces in B given by $f = 0$ and $g = 0$ are both smooth and intersect transversally, and moreover, a standard Chern class computation reveals W is Calabi-Yau if and only if $\mathscr{L} = \mathscr{O}(-K_B)$. As such, we will assume from here on that $\mathscr{L} = \mathscr{O}(-K_B)$.

The fiber of $W \to B$ over $p \in B$ is then given by

$$y^2 z = x^3 + f(p) x z^2 + g(p) z^3,$$

which is singular if and only if $p \in \Delta_W$, where $\Delta_W \subset B$ is a hypersurface given by

$$\Delta_W : (4f^3 + 27g^2 = 0) \subset B.$$

Over a generic point of Δ_W the fiber is a nodal cubic, which enhances to a cuspidal cubic over $f = g = 0$. Since the singular fibers of $W \to B$ are all irreducible, it follows that the gauge algebra of W is trivial (it's Mordell-Weil rank r is 0). It has been shown that the Euler characteristic of W may be written in terms of invariants of B [2, 13], namely,

$$\chi(W) = 12c_1(B) \sum_{i=0}^{\dim(B)-1} c_i(B)(-6c_1(B))^{\dim(B)-1-i},$$

where $c_i(B)$ denotes the ith Chern class of B. As such, for B of dimension 2 we have $\chi(W) = -60c_1(B)^2 = -60K_B^2$, so the invariant $\mathfrak{R}_X$ associated with an elliptic 3-fold $X \to B$ (given by (2)) may be re-written as

$$\mathfrak{R}_X = \frac{(-1)^{\dim(X)-1}}{2}(\chi(X) - \chi(W)). \tag{5}$$

Note that the RHS of the above is defined for X of arbitrary dimension, thus we may take the RHS of Eq. (5) as a definition of $\mathfrak{R}_X$ for X of arbitrary dimension (the factor $(-1)^{\dim(X)-1}$ ensures that $\mathfrak{R}_X$ is always non-negative). For $X = W$ we then have $\mathfrak{R}_W = 0$ in all dimensions, which makes sense since the gauge algebra of W is trivial, thus the definition is manifestly relevant to the context at hand.

Now assume $X \to B$ is an elliptic Calabi-Yau with X possibly singular. In the case that X is a 3-fold with $\mathbb{Q}$-factorial singularities, a generalization of $\mathfrak{R}_X$ has been introduced which yields GB-relations as well [10]. In particular, in such a case they define $\mathfrak{R}_X$ as

$$\mathfrak{R}_X = \frac{1}{2}\left(\chi(X) + \sum_{p \in X} \mu(P)\right) + 30K_B^2,$$

where $\mu(p)$ denotes the Milnor number of $p \in X$ (since X has only a finite number of singularities the sum $\sum_{p \in X} \mu(P)$ is finite). As in the smooth case, this can be rewritten as

$$\mathfrak{R}_X = \frac{1}{2}(\chi(X) + \sum_{p \in X} \mu(P) - \chi(W)), \tag{6}$$

where $W \to B$ is a smooth Weierstrass fibration of dimension 3. For X possibly singular of arbitrary dimension, its singular locus may be of positive dimension, thus to adapt (6) to the setting of arbitrary dimension we must write $\sum_{p \in X} \mu(P)$ in a dimension-independent manner. For this, we note that if X has isolated singularities with well-defined Milnor numbers at each of its singular points, then

$$\sum_{p \in X} \mu(P) = (-1)^{\dim(X)} \left(\chi(X) - \int_X c_{\mathrm{F}}(X))\right),$$

where $c_{\mathrm{F}}(X)$ is a characteristic class for singular varieties referred to as the *Chern-Fulton* class [7]. For X a singular 3-fold we then have

$$\chi(X)+\sum_{p\in X}\mu(P)=\int_X c_{\mathrm{F}}(X),$$

thus we can rewrite (6) as

$$\mathfrak{R}_X=\frac{1}{2}\left(\int_X c_{\mathrm{F}}(X)-\chi(W)\right),$$

the RHS of which is manifestly independent of the dimension of X. As such, for $X \to B$ an elliptic Calabi-Yau of arbitrary dimension, we let $\mathfrak{R}_X$ be given by

$$\mathfrak{R}_X=\frac{(-1)^{\dim(X)-1}}{2}\left(\int_X c_{\mathrm{F}}(X)-\chi(W)\right).$$

Remark 1. Suppose $X \to B$ is a *singular* Weierstrass fibration, i.e., X is given by an equation taking the same form as the equation for a *smooth* Weierstrass fibration W (given by (4)), but without the smoothness and transversality assumptions on the coefficients f and g. Then

$$\int_X c_{\mathrm{F}}(X)=\chi(W) \quad\Longrightarrow\quad \mathfrak{R}_X=0.$$

As the singular X considered in [10] are minimal models of singular Weierstrass fibrations, the GB-relations they consider are never applied to such X. As such, the fact that $\mathfrak{R}_X=0$ for singular Weierstrass fibrations may be viewed as a generalization of the fact that $\mathfrak{R}_W=0$ for smooth Weierstrass fibrations.

3 The Invariant $\mathfrak{L}_X$

For an elliptic Calabi-Yau 3-fold $X \to B$, denote the RHS of (3) by $\mathfrak{L}_X$, so that $\mathfrak{L}_X$ is given by

$$\mathfrak{L}_X=(g-1)\dim(\mathrm{adj})_{\mathrm{ch}}+(g'-g)\dim(\rho_0)_{\mathrm{ch}}+\sum_{p\in A}\dim(\rho_p)_{\mathrm{ch}}. \tag{7}$$

The GB-relation for elliptic Calabi-Yau 3-folds then corresponds to the equation $\mathfrak{R}_X=\mathfrak{L}_X$. Now assume B is a compact complex manifold of arbitrary dimension, and let $X \to B$ be an elliptic fibration over B with X Calabi-Yau. We now wish to generalize $\mathfrak{L}_X$ for elliptic Calabi-Yau 3-folds to current setting of arbitrary dimension. As in the case of 3-folds, we assume the discriminant may be written as $\Delta=\Delta_0\cup\Delta_1$, where Δ_0 is the (possibly singular) irreducible component of Δ over which the general fiber is irreducible, and Δ_1 is a smooth irreducible component of Δ, over which the general fiber is of Kodaira type. We note that in all verified cases of the GB-relation, the Mordell-Weil rank r appearing in the definition (1) of the gauge algebra $\mathfrak{g}_X$ is 0. As such, if we assume $r = 0$ in this dimension independent setting, then under our assumptions the gauge algebra $\mathfrak{g}_X$ is precisely the Lie algebra corresponding to the Dynkin diagram of the general fiber over Δ_1.

Now note that equation (7) may be re-written as

$$\mathfrak{L}_X = -\frac{1}{2}\chi(\Delta_1)\mathrm{dim}(\mathrm{adj})_{\mathrm{ch}} - \frac{1}{2}\left(\chi(\Delta_1') - \chi(\Delta_1)\right)\mathrm{dim}(\rho_0)_{\mathrm{ch}} + \sum_{p\in A}\mathrm{dim}(\rho_p)_{\mathrm{ch}},$$

where Δ_1' denotes a certain branched covering of Δ_1. Now take a stratification of Δ such that the generic fiber over each stratum is of Kodaira type, and denote by A_i the set of all codimension i strata. In the 3-fold case we only have one A_i, namely, A_1, thus we can now write the above equation as

$$\mathfrak{L}_X = -\frac{1}{2}\chi(\Delta_1)\mathrm{dim}(\mathrm{adj})_{\mathrm{ch}} - \frac{1}{2}\left(\chi(\Delta_1') - \chi(\Delta_1)\right)\mathrm{dim}(\rho_0)_{\mathrm{ch}} + \sum_{s_1\in A_1}\mathrm{dim}(\rho_{s_1})_{\mathrm{ch}}.$$

With X of arbitrary dimension, we then define

$$\mathfrak{L}_X = \frac{(-1)^{\mathrm{dim}(X)}}{2}\left(\chi(\Delta_1)\mathrm{dim}(\mathrm{adj})_{\mathrm{ch}} + \left(\chi(\Delta_1') - \chi(\Delta_1)\right)\mathrm{dim}(\rho_0)_{\mathrm{ch}}\right) \quad (8)$$
$$+ \sum_{i=1}^{\mathrm{dim}(X)}\sum_{s_i\in A_i}\mathrm{dim}(\rho_{s_i})_{\mathrm{ch}}.$$

Remark 2. It is not necessarily the case that X admits a stratification where the generic fiber over each stratum is of Kodaira type, thus this assumption must be added to the definition of $\mathfrak{L}_X$.

For $X \to B$ an elliptic 3-fold with $\mathbb{Q}$-factorial singularities, in [10] it is shown in a number of examples that

$$\mathfrak{R}_X = -\frac{1}{2}\chi(\Delta_1)\mathrm{dim}(\mathrm{adj})_{\mathrm{ch}} - \frac{1}{2}\left(\chi(\Delta_1') - \chi(\Delta_1)\right)\mathrm{dim}(\rho_0)_{\mathrm{ch}}$$
$$+ \sum_{p\in A}\mathrm{dim}(\rho_p)_{\mathrm{ch}} + \sum_{p\in X}\mu(p),$$

thus the contribution of the singularities of X to the GB-relation is the sum of the Milnor numbers $\sum_{p\in X}\mu(p)$ (if p is in the smooth part of X then $\mu(p) = 0$). For X of arbitrary dimension with isolated singularities, then if the singularities of X all admit well-defined Milnor numbers then

$$\sum_{p\in X} \mu(p) = \int_X \mathcal{M}(X),$$

where $\mathcal{M}(X)$ is a characteristic class supported on the singularities of X referred to as the *Milnor class* of X. The Milnor class however is defined for X with arbitrary singularities, and is given by

$$\mathcal{M}(X) = (-1)^{\dim(X)}(c_{\mathrm{SM}}(X) - c_{\mathrm{F}}(X)),$$

where $c_{\mathrm{SM}}(X)$ is the *Chern-Schwartz-MacPherson class* of X [1, 12], $c_{\mathrm{F}}(X)$ is the Chern-Fulton class (which appears in the definition of $\mathfrak{R}_X$). As such, for X of arbitrary dimension (and arbitrary singularities) we define $\mathfrak{L}_X$ as

$$\mathfrak{L}_X = \widetilde{\mathfrak{L}}_X + \int_X \mathcal{M}(X),$$

where $\widetilde{\mathfrak{L}}_X$ is given by the RHS of (8).

4 GB-relation in Arbitrary Dimension

We now give the following

Definition 1. An elliptic Calabi-Yau $X \to B$ is said to satisfy the *Grothendieck-Brieskorn relation* (or *GB-relation* for short), if and only if $\mathfrak{R}_X = \mathfrak{L}_X$.

The stage is now set to explore GB-relations in higher dimensions. While $\mathfrak{R}_X$ is known for many examples in arbitrary dimension (see [4–6]), $\mathfrak{L}_X$ has yet be explored in dimension greater than 3. Not only is dimension 4 obviously the first direction in which to proceed, it is also the dimension most relevant for physics.

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Control and Composability in Deep Learning — Brief Overview by a Learner

Alexander Ganchev

Abstract A short overview of some applications of optimal control and category theory in deep neural networks is given.

1 Introduction

This is a brief look from an outsider on deep learning. I am addressing it to people like myself coming from mathematical and theoretical physics and knowing close to nothing about machine learning (ML). The topic of ML is huge but there are very few introductions addressing such an audience. Of the vast field of ML I restrict my attention to Deep Neural Networks (DNN) for Supervised Learning and have chosen two aspects I think are of importance: optimal control and compositional formulations.

Learning in DNN can be viewed as a problem of *optimal control* (OC). Calculus of variational is in the foundation of physics while OC, which extends calculus of variations and addresses the question how to adjust a dynamical evolution to achieve a certain goal, is an engineering topic.

In Computer Science (CS) big software systems are unthinkable without the idea and use of *compositions* of smaller parts thus the importance of Category Theory (CT) in CS has grown to the extent that CS has become the main application of CT and one of the main driving force in the development of CT. ML and in particular DNN are considered part of CS.[1] Thus it seems strange that the use of CT in ML is just starting even though for an outside observer it is obvious that learning is compositional.

If we can summarize the key concept that is common to learning in general, DNN in particular, and optimal control is the *bidirectional* transformations involved—the

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[1]In fact ML should be viewed as an interdisciplinary field between CS, Mathematics, Statistics, Information Theory, Cognitive Science, Physics, etc.

V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_44

backpropagation algorithm of DNN and the adjoint-state-method in Pontryagin's Maximum Principle (PMP) of optimal control. The counterpart of these in category theory are 'lenses' (or more generally 'optics') which initially were introduced as a formalism for database updates.[2]

2 Learning and Deep Neural Networks

Machine Learning (ML) is formally a part of Artificial Intelligence (AI) but recently it has grown in size and importance so it is often identified with AI. An important subfield is supervised ML where a machine infers a function mapping inputs to outputs from examples of input-output pairs. An experimental physicist fitting a parametrized function to data points is doing supervised learning but in fitting there are a few parameters while in a typical deep network the number of parameters is huge.

An artificial neuron, a crude model or a real neuron, receives signals x_i, $i = 1, \ldots, n$, from the previous n neurons, each signal x_i is multiplied by a weight w_i and summed. (Assume all signals and weights are real numbers.) The result is fed into an activation function[3] σ which outputs the outgoing signal y. A single neuron can be viewed as a map $(x_1, \ldots, x_n) \mapsto \sigma\left(\sum_{i=1}^n w_i x_i + b\right) = y$ where the weights w_i and the bias b are treated as parameters to be learned. If σ is the step function then the neuron answers the question on which side of an affine hyperplane is the input vector $(x_1, \ldots, x_n)$.

The neurons in the cortex are arranged in layers, e.g., the retina is the input layer of the visual cortex. Mimicking the architecture of the (biological) neurons in the brain a DNN consists of (artificial) neurons arranged in layers with the output of the neurons from layer k being fed in as the input to the neurons of layer $k + 1$, in vector notation

$$\mathbf{x}^{(k)} \mapsto \mathbf{x}^{(k+1)} = \sigma\left(W^{(k)}\mathbf{x}^{(k)} + \mathbf{b}^{(k)}\right) \tag{1}$$

where the activation function σ is applied coordinate wise. As the signal propagates from the input to the output we have compositions of alternating affine transformations and activations function application. For every point $p \in P$ in parameter space (weights and biases) a DNN is a function $F_p \colon X \to Y$ mapping, propagating forward, an input x to an output y which approximates some, in general unknown, function $F \colon X \to Y$. In DNN jargon the *hyperparameters* are things which are not learned by the DNN but are fixed in advance, e.g., the architecture of the net, the choice of activation function, the choice of cost function, etc.

[2]The cited bibliography in this note is far from exhaustive. I try to cite recent sources from which an interested reader can backpropagate to the original papers.

[3]There are many examples of activation functions, e.g. the Heaviside step function, the sigmoid or logistic function which is a smooth version of the step function, the integral of the step function known as ReLU in learning jargon, etc.

3 Optimal Control

A dynamical system (DS) describes the evolution in discrete/continuous time of the state of a system. In the continuous case a DS is given by a first order ordinary differential equation (ODE) $dx(t)/dt = f(x(t), t)$, where $x(t)$ is a trajectory in the state space M (in general a manifold) and f is a vector field on M, i.e., a section of the tangent bundle TM. In a controlled ODE $dx(t)/dt = f(x(t), u(t))$ the field is controlled by a control path $[0, T] \ni t \mapsto u(t) \in U$, where U is the space of control parameters. Fixing a control $u = u(t)$ (for better readability we skip the dependence on t) and an initial condition the (forward propagating) controlled ODE gives a solution $x = x(u) = x(t, u)$. In optimal control (e.g. [18]) we want to choose the control which optimizes a given objective (or cost/loss/error) function. The main approaches are Bellman's Dynamic Programming (BDP) and Pontryagin's Maximum Principle (PMP). Below is a caricature of the second, without the slightest pretense for rigor, trying to indicate how backpropagation appears.

The cost (a functional of the control $u(t)$) is given by $J(u) = G(x(u), u) = h(x(T, u)) + \int_0^T g(x(t, u), u(t))\, dt$ where $x(u)$ is a solution of $\dot{x} = f(x, u)$ and h and g are the terminal and running costs. The goal is to find the variation of the cost as the control varies, and move along the gradient (or steepest) descent to the minimum of J. In the variation of the cost $\delta J = G_u \delta u + G_x \delta x$ we want to express δx in terms of δu. For that reason take a variation of the forward ODE $\partial_t \delta x = f_x \delta x + f_u \delta u$ thus

$$\delta x = [\partial_t - f_x]^{-1} f_u \delta u. \tag{2}$$

Introduce the adjoint/costate (variable and path) λ, living in the cotangent bundle T^*M, as solution of the adjoint (backward propagating) equation

$$[\partial_t - f_x]^* \lambda = G_x. \tag{3}$$

Hence for the second term in δJ we get

$$G_x \delta x = \langle G_x, \delta x \rangle = \langle [\partial_t - f_x]^* \lambda,\ [\partial_t - f_x]^{-1} f_u \delta u \rangle = \langle \lambda,\ f_u \delta u \rangle \tag{4}$$

and finally we obtain

$$\delta J = [G_u + \langle \lambda,\ f_u \rangle] \delta u. \tag{5}$$

A feed-forward deep neural network is a discrete time dynamical system $\mathbf{x}^{(k+1)} = \sigma\left(W^{(k)} \mathbf{x}^{(k)}\right)$, with the layer index k interpreted as time. It is controlled by the weights $W^{(k)}$. Let us do the above once more but in discrete time and with more details. The state, costate, and control paths become sequences $x_\bullet = (x_0, x_1, \ldots, x_n)$, $\lambda_\bullet = (\lambda_0, \lambda_1, \ldots, \lambda_n)$, $u_\bullet = (u_0, u_1, \ldots, u_{n-1})$. The dynamics is

$$x_{k+1} = f(x_k, u_k). \tag{6}$$

The cost functional is $J(u_\bullet) = h(x_n) + \sum_{k=0}^{n-1} g(x_k, u_k)$. The Lagrangian is the cost plus the dynamics multiplied by the costate $\lambda_\bullet$, a Lagrange multiplier:

$$L(x_\bullet, \lambda_\bullet, u_\bullet) = h(x_n) + \sum_{k=0}^{n-1} \left(g(x_k, u_k) + \langle \lambda_{k+1}, (f(x_k, u_k) - x_{k+1})\rangle\right). \quad (7)$$

Introducing the function

$$H(k) \equiv H(x_k, u_k, \lambda_{k+1}) = g(x_k, u_k) + \langle \lambda_{k+1}, f(x_k, u_k)\rangle$$

we can rewrite

$$L = h(x_n) - \langle \lambda_n, x_n\rangle + \langle \lambda_0, x_0\rangle + \sum_{k=0}^{n-1} \left(H(k) - \langle \lambda_k, x_k\rangle\right). \quad (8)$$

Take the variation of L. The variation with respect of the costate λ recovers the dynamics of x. Assuming the dynamics of x holds, for the rest we get

$$\begin{aligned} dL = {} & \langle h_x(x_n) - \lambda_n, dx_n\rangle + \langle \lambda_0, dx_0\rangle \\ & + \sum_{k=0}^{n-1} \left(\langle \partial_x H(k) - \lambda_k, dx_k\rangle + \langle \partial_u H(k), du_k\rangle\right). \end{aligned}$$

The variation with respect to the state x gives the adjoint dynamics of the costate λ:

$$\lambda_k = \partial_x H(k) \equiv \partial_x g(x_k, u_k) + \langle \lambda_{k+1}, \partial_x f(x_k, u_k)\rangle \quad (9)$$

with initial condition $\lambda_n = \partial_x h(x_n)$. Note the backpropagation. The gradient of the total cost is

$$\partial_u H(k) = \partial_u g(x_k, u_k) + \langle \lambda_{k+1}, \partial_u f(x_k, u_k)\rangle. \quad (10)$$

The connections between DNN's backpropagation and optimal control have been emphasized in [16] and more recently [6, 17, 19], etc.

4 Compositionality—Lenses, Learners, Games, Systems, Automatic Differentiation, and Differentiable Programming

All around us there are "processes" that can be composed "in time" and "in space", i.e., all around us there are symmetric monoidal categories.[4]

The construction of [10] gives the first compositional view of supervised deep learning. The goal of supervised learning is to approximate an unknown function $f: A \to B$ having a training collection of input-output pairs $(a, b \approx f(a))$. Learning is described as a composition of functors **NNet** $\to$ **Para** $\to$ **Learn** with the first functor $I^{(\sigma)}$ depending on the hyperparameter "choice of activation function σ", while the second $L^{(E,\epsilon)}$ depends on the hyperparameters "choice of steepest decent step ϵ and choice of error function E". The objects of the category **NNet** give the different architectures of deep networks with morphisms the concatenation of layers. The category **Para** has objects Euclidean spaces, morphisms are (equivalence classes) of differentiable parametrized function (the parameter spaces being also Euclidean), and composition is composition of functions with Cartesian product of the parameter spaces.

The objects of **Learn** are sets $A, B, \ldots$. The description of morphisms is more involved and involves both a 'forward' and 'backward' components. The *implementation* function $I: P \times A \to B$ encodes the 'forward pass' where fixing a parameter $p \in P$ we have an approximation to the unknown function from A to B that is being learned. Next we describe the 'backward' components of a morphism. Given an input-output pair (a, b) the parameter p is updated to $p' = U(p, a, b)$ to get a 'better' approximation $I_{p'}$ of f, hence the *update* function $U: P \times A \times B \to P$. The composition of the implementation (forward) function I is straightforward while to compose updates U we need to back propagate information from B to A. For a parameter p and a pair (a, b) we want a new input $a' = r_p(a, b)$ such that $I_p(a')$ is a 'better' approximation of b than $I_p(a)$. Thus the need for the *request* function $r: P \times A \times B \to A$.

Given morphisms $(P, I, U, r): A \to B$ and $(Q, J, V, s): B \to C$ the composition $A \to B \to C$ is $(Q \times P,\ J \circ I,\ V \circ U,\ s \circ r)$ is defined as follows: $(J \circ I)_{(q,p)} = J_q \circ I_p$, $\quad (V \circ U)(q, p, a, c) = (q',\ p') = (V(q, I_p(a), c),\ U(p, a, s_q(I_p(a), c))$, $(s \circ r)_{(q,p)}(a, c) = r_q(a, s_q(I_p(a), c)$ (the composition of requests goes "backwards" because this back propagates information).

[4]E.g., cooking, electric circuits, Feynman diagrams, computing, learning, etc., etc. Composing processes "in time" (sequentially) gives a category. Composing processes sequentially "in time" and concurrently "in 1-space" gives a monoidal or tensor category (a 2-category with a single 0-arrow/object). Composing "in time" and "in 2-space" gives a braided monoidal category (a 3-category with a single 0-arrow and a single 1-arrow). Composing "in time" and "in 3-space" gives a symmetric monoidal category (a 4-category with a single 0-arrow, 1-arrow, and 2-arrow). Adding more "space" dimensions does not add any more changes (the Baez-Dolan stabilization hypothesis). Thus the ubiquity of symmetric monoidal categories in our 3+1 dimensional space-time.

Next we define the functor $L^{(E,\epsilon)}\colon \textbf{Para} \to \textbf{Learn}$. In **Para** an arrow $(P, I)\colon A \to B$ consists of a parameter space P and differentiable function $I\colon P \times A \to B$. Choose an 'error function', e.g., $E^I_p(a, b) = \|I_p(a) - b\|^2/2$ where $\|.\|$ is the Euclidean norm on B, and a real number $\epsilon > 0$, the "step size". The functor $L^{(E,\epsilon)}$ is identity on objects and maps the arrow (P, I) to (P, I, U^I, r^I) where $U^I(p, a, b) = p - \epsilon\,\nabla_p E^I_p(a, b)$ and $r^I_p(a, b) = a^i - \sum_j (I^j_p(a) - b^j)(\partial I^j/\partial a^i))$ with $i = 1, \ldots, \dim A$ and $j = 1, \ldots, \dim B$.

The functor $I^\sigma\colon \textbf{NNet} \to \textbf{Para}$ is defined by making a choice of the hyperparameter σ, an activating function, assigning a weight for every connection between to neurons (the weights are the parameters) and mapping an arrow in **NNet**, which is described by the connections of neurons in two layers, to the function (1).

The three categories discussed can be equipped in a natural way with monoidal structure. For lack of space we skip this description. Note that the hyperparameters of a learning machine have been moved to the two functors. Thus one may view change of hyperparameters and their eventual 'learning' as natural transformations.

There has been recently a lot of activity of taking a compositional or categorical view on things that before have been studied in isolation, e.g., open (dynamical) systems, open Petri nets, open Markov processes, open games, learning (associated with the names of J. Baez, B. Fong, J. Hedges, E. Lerman, D. Spivak, their collaborators, and others). Some of these (in particular 'learners' and 'games', but probably many others such as dynamical systems) fall in the general notion of categorical lenses. An inspiration for the development of a categorical approach to open systems has been the paper [23] (see also the post of E. Lorch and J. Tan[5]). A faithful monoidal functor from learners to symmetric lenses has been described in [9]. The connection between open games and learners is the subject of [14] while connections between open games and lenses have been made in [12].

Lenses are an example of bidirectional transformations. Initially lenses were used to describe the (covariant or forward propagating) process $\lambda_g\colon A \to B$ of 'getting' or 'viewing' a small piece B of a database A, possibly modifying it, and 'putting' back, or 'updating' it, in the database, $\lambda_p\colon A \times B \to A$ (which is a contravariant or back-propagating process). More generally, a morphism of (bimorphic) lenses is given by $\lambda\colon (A, A') \to (B, B')$ with $\lambda_g\colon A \to B$ and $\lambda_p\colon A \times B' \to A'$. Given a second lens morphism $\mu\colon (B, B') \to (C, C')$ the composition $\mu \circ \lambda$ is defined by $(\mu \circ \lambda)_g = \mu_g \circ \lambda_g$ and $(\mu \circ \lambda)_p(a, c') = \lambda_p(a, \mu_p(\lambda_g(a), c'))$. (Note the contravariance in the composition of the 'put' part.) The literature on bidirectional transformations, lenses, and categories of optics is big. For recent references see [13, 20–22].

Differentiation is a key ingredient in the algorithms of both DNN and scientific computing and the method of choice for both is automatic (or algorithmic) differentiation (AD), e.g., see [1] for a recent review. AD is neither numerical nor symbolic differentiation. As symbolic differentiation AD does not use approximations (like a finite difference) while as numerical differentiation it manipulates concrete numbers. AD differentiation comes in two modes direct and reverse. The automatic differentiation of a function from a high dimensional space to a one dimensional space

[5] https://golem.ph.utexas.edu/category/2018/06/the_behavioral_approach_to_sys.html.

(such as the cost/loss/error function) is far more efficient in reverse mode, i.e., as backpropagation. A categorical formulation of AD is given in [8]. The more general program for differential lambda calculus started in [7] and a categorical semantics for it has been developed with the most recent paper [3] considering reverse mode AD. The paper [2] which proposes to take the optimal control view seriously and even to substitute DNN with Neural ODE is one of the signs of a rebranding of DNN as *differentiable programming*[6] thus the importance of papers such as [8] and [3] is bound to increase.

5 Discussion

We have seen that supervised learning in DNN is an instance of optimal control (in the approach of PMP) and how DNN learning can be formulated categorically. The common tread in both is their bidirectional character which for the formulation of [10] results in the interpretation of learners as lenses. A categorical interpretation of optimal control remains to be done. The approach of Willems [23], which has been the main inspiration for the recent developments in open dynamical systems, departs from the older input/controlled-system/output approach and merges together the inputs and the controls. In some cases, in particular in DNN, the older approach is more natural. Such a three-way (input/control/output) proposal is contained in [4] and probably it could be a basis for developing a compositional view on PMP optimal control.

Reinforcement Learning (RL) is another big topic in ML. In RL learning there is no supervisor but the learner interacts with the environment and as feedback receives rewards or punishments. RL is closely connected with the BDP approach to optimal control and with game theory. RL is waiting a compositional approach.

This brief note is the second one in which I look at DNN from outside. My first note [11] concentrated on connections between DNN and renormalization group flow and the bulk/boundary duality. It is curious to note that Tambara modules play an important role in lenses [20, 21] as well as in the Drinfeld center of a monoidal category while the center construction plays a role in bulk/boundary duality [5, 15]. Is this a coincidence or there is something deeper?

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Lie Theory and Infinitesimal Extensions in Algebraic Geometry

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Abstract We apply the Lie theoretic formalism of Calaque-Caldararu-Tu, to some extension problems of vector bundles to the first infinitesimal neighbourhood of a subvariety in the complex projective space.

1 Introduction

In recent years, Lie algebraic type constructions received many applications going from low dimensional topology to pure algebraic geometry.

We want to focus on one aspect of these applications, namely a result of Arinkin-Caldararu [1], and its connection to the classical Serre's construction in the world of vector bundles on projective varieties.

In the first part we shall introduce the relevant definitions concerning Lie-type phenomena in the algebraic geometric world together with their original Lie-algebraic counterpart, following mainly [3] and [7].

The second part is devoted to Serre's construction, and to the main result of this paper. The last section contains a question which could be considered for future research.

2 Lie Algebras and Duflo's Conjecture

In what follows, we fix a commutative field k of characteristic 0, and a Lie algebra g over k. The following fact is well known:

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_45

Theorem 1. *The natural map*

$$S(g) \to U(g) \tag{1}$$

is an isomorphism of vector spaces (but not of algebras).

It is the PBW-isomorphism. Also, the restriction of the PBW-map to the g-invariant part is also a linear isomorphism:

$$S(g)^g \to U(g)^g. \tag{2}$$

Neither this restriction is an algebra isomorphism, but the deep insight of Duflo was that a certain modification becomes an isomorphism of algebras.

More precisely, let's consider the following "magic" element in $(\hat{S})(g^\star)^g$:

$$J = det(\frac{1 - e^{-ad}}{ad}). \tag{3}$$

Formally, it is expressed as a series in the elements $tr(ad^k)$. An important fact is that the above elements are g-invariant w.r.t. the co-adjoint action extended by derivations. Moreover, $g^\star$ acts on $S(g)$ by derivations, and this action preserves the g-invariant part.

Putting all the above together, Duflo's theorem can be stated as follows:

Theorem 2. *The composition*

$$PBW \circ J^{\frac{1}{2}} : S(g)^g \to U(g)^g \tag{4}$$

is an isomorphism of algebras.

The above result can be extended at least at a conjectural level as follows. Let $h \subseteq g$ be an inclusion of Lie algebras, such that $g = h \oplus m$ with $[h, m] \subseteq m$. In this setting, Duflo conjectured the following:

Duflo's Conjecture 1. *The Poisson center of $S(m)^h$, and the center of $(U(g)/hU(g))^h$ are isomorphic as algebras.*

Despite many efforts, this conjecture is widely open even today.

In the same spirit, Calaque posed the following weaker question:

Calaque's Question 1. *Under which assumptions $S(g/h)$ and $(U(g)/hU(g))$ are isomorphic as h-modules?*

The answer obtained by Calaque-Caldararu-Tu [4, 5] concerns the splitting of the natural filtration on $(U(g)/hU(g))$, a question which is also relevant in some deformation quantization problems. In the particular case of the diagonal embedding $h \subset h \oplus h$ their result recovers the usual PBW isomorphism.

The interesting part in the C-C-T solution to the above question is the fact that the ideas are inspired by an analogous algebraic-geometric problem through a surprising dictionary:

let's fix two algebraic varieties X and Y, and an embedding $X \hookrightarrow Y$;

by $T_X[-1]$ and $T_Y[-1]$ we denote the shifted tangent bundles of X and Y, as objects in the derived categories of coherent sheaves $D(X)$ and $D(Y)$;

the Lie algebras h and g are the counterpart of the Lie algebra objects $T_X[-1]$ and $T_Y[-1]$.

In what follows we define the relevant notions needed to explain the algebraic-geometric analog of the C-C-T problem.

3 The Derived Category of Coherent Sheaves and Lie Algebra Objects

The path to the derived category of coherent sheaves goes through two well known procedures in category theory: in the first place, we construct a category of fractions, and secondly, we perform a localization of that category.

Let's fix X a smooth projective algebraic variety over the field of complex numbers. We denote by $Coh(X)$ the category of coherent sheaves on X. (If unfamiliar with the subject one can think simply of holomorphic vector bundles over X.) $Coh(X)$ is an abelian category, and one can take the usual category of complexes $Cpx(X)$: here the objects are complexes of coherent sheaves

$$... \to F_{i-1} \to F_i \to F_{i+1} \to ... \tag{5}$$

and the morphisms are morphisms of complexes.

The first operation is to take $Hot(X)$, the homotopy category of $Cpx(X)$: the objects are still complexes, but the morphisms are modified by taking the quotient w.r.t. homotopy equivalence of morphisms between complexes. The second operation, which produces finally $D(X)$ from $Hot(X)$, is the localisation w.r.t. the class of quasi-isomorphisms. For this step recall that a quasi-isomorphism is a map between two complexes which induces isomorphisms at the level of homology groups. Morphisms in $D(X)$ are more complicated than the usual maps between complexes; in fact they can be represented by triangles $F^\bullet \leftarrow G^\bullet \to E^\bullet$ of usual morphisms, such that the left hand side morphism is a quasi-isomorphism. Also, $D(X)$ inherits from $Cpx(X)$ the translation functor [1], which acts as follows: $F[1]^i = F^{i+1}$.

3.1 The Atiyah Class and the Lie Algebra Object $T_X[-1]$

The Atiyah class can be constructed for an arbitrary object in $D(X)$. It can be interpreted in various ways, as an extension class, as a morphism in the derived category, or in the simplest case of a vector bundle, as the obstruction to have a holomorphic connection. Let's consider the simplest situation: take E a holomorphic vector bundle over the smooth projective variety X. The bundle of 1-jets JE is the sheaf $E \oplus (E \otimes \Omega^1)$ with the following $\mathcal{O}_X$-module structure:

$$f(s, t \otimes \theta) = (f \cdot s, f \cdot t \otimes \theta + t \otimes df) \tag{6}$$

With this structure, JE is described as the following extension:

$$0 \to E \otimes \Omega^1 \to JE \to E \to 0 \tag{7}$$

From the sequence above, the extension class $[JE]$ can be seen in either of the following groups:

$$Ext^1(E, E \otimes \Omega^1) \simeq Ext^1(E \otimes T, E)$$

The Atiyah class defines a structure of a Lie algebra object on $T[-1]$ through the following result [7]:

Proposition 1.

$$Hom_{D(X)}(E, F[i]) \simeq Ext^i(E, F).$$

The above Proposition expresses the Atiyah class $[JE]$ as a morphism in the derived category $D(X)$:

$$E \otimes T[-1] \to E. \tag{8}$$

In the particular case $E = T[-1]$, the above morphism in $D(X)$, endows $T[-1]$ with a “bracket”, such that we have the following [7]:

Proposition 2. *The bracket induced on $T[-1]$ by the Atiyah class verifies the axioms of a Lie algebra object in $D(X)$.*

Moreover, the above morphism induced by the Atiyah class for an arbitrary object E in $D(X)$, endows E with a $T[-1]$-action, such that we have [7]:

Proposition 3. *The $T[-1]$-action on E, induced by the Atiyah class*

$$E \otimes T[-1] \to E,$$

verifies the axioms of a Lie algebra action on E.

In view of the above proposition, one can ask the following:

Question 1. *Is an arbitrary morphism $E \to F$ in $D(X)$ a morphism of $T[-1]$-modules ?*

The result below [7] gives a positive answer, and expresses the fact that the derived category is the representation category of the Lie algebra object $T[-1]$.

Proposition 4. *Any morphism $E \to F$ in $D(X)$ is a morphism of $T[-1]$-modules.*

4 The Arinkin's-Caldararu Extendability Criterion

In the sequel, we will translate the above Lie algebraic facts into their algebraic-geometric counterparts using a formal dictionary between Lie algebras and Lie algebra objects $T[-1]$ in the derived categories $D(X)$. Concerning the Duflo's isomorphism, we remark firstly that on the algebraic-geometric side one has [7]:

Proposition 5. *The symmetric and the universal enveloping algebras of* $T[-1]$, *are:*
$S = \oplus(\Lambda^k T)[-k]$ *and*
$U = p_\star(R\mathcal{H}om(\mathcal{O}_\Delta, \mathcal{O}_\Delta))$.

The invariant parts must be defined in a categorical setting as $Hom(\mathcal{O}_X, -)$.

For S, respectively U, the invariant parts are: $\oplus H^\star(\Lambda^\star T)$, respectively $Ext^\star_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$. The isomorphism between these is the usual Hochschild-Kostant-Rosenberg isomorphism. It is only an isomorphism of vector spaces. The analogue of Duflo's map for the HKR isomorphism was obtained by Kontsevich by contracting with a similar "magic" element [6].

In the algebraic-geometric setting, the Lie algebras h and g are replaced by the shifted tangent bundles $T_X[-1]$ and $T_Y[-1]$ for an embedding of algebraic varieties $i : X \hookrightarrow Y$. The invariant part $S(m)^h$ is replaced by $\oplus\Lambda^k(N_{X,Y})[-k]$, where N is the normal bundle of the embedding. On the other hand, the invariant part $(U(g)/hU(g))^h$ is replaced by $R\mathcal{H}om_X(i^\star i_\star\mathcal{O}_X, \mathcal{O}_X)$. In this setting, Duflo's Conjecture translates into the question of finding conditions that ensure the above invariant parts are isomorphic.

The answer is given in the theorem below in terms of the composition of the extension

$$0 \to T_X \to T_{Y|X} \to N_{X,Y} \to 0, \tag{9}$$

viewed as a morphism $N_{X,Y}^{\otimes 2} \to N_{X,Y} \otimes T_X[1]$ in $D(X)$, and the Atiyah class of $N_{X,Y}$. Let's denote by $a \in Ext^2(N_{X,Y}^{\otimes 2}, N_{X,Y})$ the resulting extension class.

With the above notation, the following result was proved by Arinkin and Caldararu [1]:

Theorem (Arinkin-Caldararu) 1. *The following are equivalent:*
1. *The extension class* $a = 0$.
2. $N_{X,Y}$ *admits an extension to the first infinitesimal neighbourhood of* X *in* Y.
3. *The algebraic geometric Duflo's Conjecture holds true in this situation.*

5 Serre's Construction and Applications

We can interpret the above theorem as a criterion for the algebraic-geometric Duflo's theorem to hold, in terms of the extendability of the normal bundle, at least to the first infinitesimal neighbourhood. Fortunately, this geometric condition is known to be true in a variety of special cases. Among these, we wish to restrict to the case of co-dimension 2 smooth sub-manifolds $X \subset Y$. In this case, Serre's construction asserts the following:

Theorem (Serre's Construction) 1. *Let I_X be the ideal sheaf of X, $L = (det\ N)^\vee$. Then $\mathcal{EXT}_{\mathcal{O}_X}(I_X, L)$ is a trivial line bundle, and its generator defines an extension*

$$0 \to L \to E \to I_X \to 0.$$

Moreover, the sheaf E is locally free, and $E^\vee$ extends N to all of Y.

Putting together the Arinkin-Caldararu theorem and Serre's construction, we arrive at our main result:

Theorem 3. *For a co-dimension 2 smooth sub-manifold, the algebraic-geometric Duflo's Conjecture holds true.*

6 Conclusions and Future Directions

We combined Serre's construction and the Arinkin-Caldararu theorem, to argue that in the co-dimension 2 case, the algebraic-geometric Duflo's Conjecture holds true. The question of the extendability of vector bundles over a sub-manifold X up to a certain infinitesimal neighbourhood is a very old one. In recent years, for example Badescu [2] obtained new results in this direction in the situation of small co-dimension and a vector bundle of rank 2 on X. A natural question could then be the following:

Question 2. *Find co-homological obstructions of the type of the class a, as in the Arinkin-Caldararu theorem, that prevent the extension of an arbitrary rank 2 vector bundle on X to its first infinitesimal neighbourhood.*

Also, another natural question is the following: is there a canonical "associative algebra object" in the derived category D(X), that plays a role analogous to the one that deformation quantization has for Poisson (Lie) structures? Such an object would have topological relevance for construction of weight systems producing not only knot invariants (see [7]), but also 3-manifold invariants.

Acknowledgements The first author is grateful to the organisers of the XIIIth International Workshop Lie Theory and Its Applications in Physics, for partial financial support.

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The Low-Dimensional Algebraic Cohomology of the Witt and the Virasoro Algebra with Values in General Tensor Densities Modules

Jill Ecker

Abstract In this contribution we give results concerning the low-dimensional algebraic cohomology of the Witt and the Virasoro algebra with values in the general tensor densities modules $\mathcal{F}^\lambda$. More precisely, we provide the first and the second algebraic cohomology of the Witt and the Virasoro algebra with values in $\mathcal{F}^\lambda$ for every $\lambda \in \mathbb{C}$, and we give results for the third algebraic cohomology for certain values of λ in the case of the Witt algebra. The low-dimensional algebraic cohomology vanishes except for certain values of λ for which the dimension of the cohomology is one or two, depending on λ. We consider algebraic cohomology, i.e. we do not put any continuity restrictions on our cochains and cocycles. Moreover, the given results are independent of any concrete realization of the Lie algebras under consideration, and thus, independent of any choice of topology.

1 Introduction

The Witt algebra and the Virasoro algebra are infinite-dimensional Lie algebras widely used in mathematics and in physics. In physics, the Virasoro algebra is omnipresent in String Theory and two-dimensional conformal field theory. The low-dimensional Lie algebra cohomology is also of outermost importance both in mathematics and in physics. In fact, contrary to higher dimensional Lie algebra cohomology, the low-dimensional cohomology has an easy interpretation in terms of known objects such as derivations, extensions, deformations, obstructions or crossed modules, see Gerstenhaber [12–14], and also Wagemann [26, 27] for the crossed modules. The study of these objects provides new insights into the Lie algebra under consideration and hence a better understanding of it. Besides, new Lie algebras can be constructed by using extensions or deformations, and crossed modules can be used to classify Lie-2-algebras, see Baez and Crans [1]. However, also in physics, the low-dimensional cohomology is important. For example, central extensions of a Lie

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V. Dobrev (ed.), *Lie Theory and Its Applications in Physics*,
Springer Proceedings in Mathematics & Statistics 335,
https://doi.org/10.1007/978-981-15-7775-8_46

algebra can be used in order to work with linear representations instead of projective ones, see e.g. Tuynman and Wiegerinck [23]. Also, the low-dimensional cohomology appears in the study of anomalies, see e.g. Roger [19].

The main aim of this contribution is to provide results concerning the first, the second and the third algebraic cohomology of the Witt and the Virasoro algebra with values in general tensor densities modules $\mathcal{F}^\lambda$ with $\lambda \in \mathbb{C}$. The proofs will be given elsewhere [3].

We consider here algebraic cohomology and not continuous cohomology, i.e. we deal with arbitrary cochains and not only continuous ones. In the case of continuous cohomology with general coefficients, much work was done already decades ago by Gelfand and Fuks [11] and Tsujishita [22], see also the book by Fuks [10]. Explicit expressions for the generating cocycles of $\mathrm{H}^2_{\mathrm{cont}}(\mathcal{W}, \mathcal{F}^\lambda)$ were derived by Ovsienko and Roger [17]. Using results of Goncharova [15], Reshetnikov [18] and Tsujishita [22], the entire continuous cohomology of the Witt algebra with coefficients in general tensor modules was computed by Fialowski and Schlichenmaier in [9].

Less is known in the case of algebraic cohomology, which is an active, on-going, research area. Algebraic cohomology is in general much harder to compute than the continuous one as there is no underlying topology, hence no results from topology or geometry can be used. The second algebraic cohomology of the Witt algebra with values in the trivial module has been known for a very long time. An algebraic computation of it can be found e.g. in [16]. The second algebraic cohomology of the Witt and the Virasoro algebra with values in the adjoint module was computed by Schlichenmaier in [20, 21], see also Fialowski [8]. In case of the Witt algebra, the result was announced already in [7] by Fialowski, without proof. In [21], Schlichenmaier also computed the second algebraic cohomology with values in the trivial module of the Virasoro algebra. In [24, 25], Van den Hijligenberg and Kotchetkov computed the second algebraic cohomology with values in the adjoint module of the super-algebraic versions of the Witt and the Virasoro algebra. The third algebraic cohomology is already much harder to compute than the second. Results have been obtained for the Witt and the Virasoro algebra in the case of the adjoint and the trivial module by Ecker and Schlichenmaier in [4–6]. The first algebraic cohomology is rather easy to compute; it has been computed for the Witt and the Virasoro algebra in the case of the adjoint and trivial modules in [4].

2 The Witt and the Virasoro Algebra

The Witt algebra $\mathcal{W}$ is an infinite-dimensional $\mathbb{Z}$-graded Lie algebra. As a vector space, it is generated over a base field $\mathbb{K}$ with characteristic zero by basis elements $\{e_n \mid n \in \mathbb{Z}\}$ that fulfill the following Lie structure equation:

$$[e_n, e_m] = (m-n)e_{n+m}, \qquad n, m \in \mathbb{Z}. \tag{1}$$

The $\mathbb{Z}$-grading of the Witt algebra is obtained by defining the degree of an element e_n by $deg(e_n) := n$. From the relation $[e_0, e_m] = me_m = deg(e_m)e_m$, one sees that the grading is given by one of the elements of the Witt algebra itself, namely the element e_0. Hence, the grading of the Witt algebra is an internal grading.

The Witt algebra has three popular concrete realizations. An algebraic realization of the Witt algebra is given by the Lie algebra of derivations of the infinite-dimensional associative $\mathbb{K}$-algebra of Laurent polynomials $\mathbb{K}[Z^{-1}, Z]$. By considering $\mathbb{K} = \mathbb{C}$, a geometrical realization of the Witt algebra is obtained, corresponding to the algebra of meromorphic vector fields on the Riemann sphere $\mathbb{CP}^1$ that are holomorphic outside of 0 and ∞. In this realization, the basis elements of the Witt algebra can be written as $e_n = z^{n+1}\frac{d}{dz}$, where z corresponds to the quasi-global complex coordinate. Finally, the Witt algebra can be realized as the complexified Lie algebra of polynomial vector fields on the circle S^1, in which case the generators are given by $e_n = e^{in\varphi}\frac{d}{d\varphi}$, where φ is the coordinate along S^1.

In this article, we consider general tensor densities $\mathcal{F}^\lambda$ as $\mathcal{W}$-modules.

The action of the Witt algebra on these modules is given by:

$$e_n \cdot f_m^\lambda = (m + \lambda n) f_{n+m}^\lambda, \qquad n, m \in \mathbb{Z}\,, \tag{2}$$

where $e_n \in \mathcal{W}$, $f_m^\lambda \in \mathcal{F}^\lambda$ and $\lambda \in \mathbb{C}$.

In the geometric picture on $\mathbb{CP}^1$ and for integer λ, the elements f_m^λ can be interpreted as meromorphic forms of weight λ and written as $f_m^\lambda = z^{m-\lambda}(dz)^\lambda$. For example, $\lambda = 0$ corresponds to meromorphic functions, the adjoint module $\lambda = -1$ corresponds to meromorphic vector fields and $\lambda = 1$ corresponds to meromorphic differential 1-forms. However, in this article we consider $\lambda \in \mathbb{C}$ as we are working on a purely algebraic level.

The Witt algebra comes with a non-trivial central extension $\mathcal{V}$, which is unique up to equivalence and rescaling, and which corresponds actually to a universal central extension:

$$0 \longrightarrow \mathbb{K} \xrightarrow{\;i\;} \mathcal{V} \xrightarrow{\;\pi\;} \mathcal{W} \longrightarrow 0\,, \tag{3}$$

where $\mathbb{K}$ is in the center of $\mathcal{V}$. This extension $\mathcal{V}$ is called the Virasoro algebra.

The Virasoro algebra is given as a vector space by the direct sum $\mathcal{V} = \mathbb{K} \oplus \mathcal{W}$. The generators are $\hat{e}_n := (0, e_n)$ and $t := (1, 0)$, which satisfy the following Lie structure equation:

$$[\hat{e}_n, \hat{e}_m] = (m - n)\hat{e}_{n+m} + \alpha(e_n, e_m) \cdot t \qquad n, m \in \mathbb{Z}\,, \tag{4}$$

$$[\hat{e}_n, t] = [t, t] = 0\,, \tag{5}$$

where α is the so-called Virasoro 2-cocycle, given by:

$$\alpha(e_n, e_m) = -\frac{1}{12}(n^3 - n)\delta_{n+m,0}\,. \tag{6}$$

The cubic term n^3 is the most important term, the linear term n corresponding to a coboundary term only.[1]
The Virasoro algebra is also an internally $\mathbb{Z}$-graded Lie algebra, with $deg(\hat{e}_n) := deg(e_n) = n$ and $deg(t) := 0$.

3 The Chevalley-Eilenberg Cohomology

Let us briefly recall some facts about the Lie algebra cohomology, for the convenience of the reader.
Let $\mathcal{L}$ be a Lie algebra and M a $\mathcal{L}$-module. We denote by $C^q(\mathcal{L}, M)$ the space of q-multilinear maps on $\mathcal{L}$ with values in M, $C^q(\mathcal{L}, M) := \mathrm{Hom}_{\mathbb{K}}(\wedge^q \mathcal{L}, M)$. The elements of $C^q(\mathcal{L}, M)$ are called q-cochains. By convention, $C^0(\mathcal{L}, M) := M$. The coboundary operators δ_q are given by:

$$\forall q \in \mathbb{N},\ \delta_q : C^q(\mathcal{L}, M) \to C^{q+1}(\mathcal{L}, M) : \psi \mapsto \delta_q \psi\,,$$

$$(\delta_q \psi)(x_1, \ldots x_{q+1}) := \textstyle\sum_{1\le i<j\le q+1} (-1)^{i+j+1}\ \psi([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1})$$

$$+ \textstyle\sum_{i=1}^{q+1} (-1)^i\ x_i \cdot \psi(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1})\,,$$

with $x_1, \ldots, x_{q+1} \in \mathcal{L}$, $\hat{x}_i$ means that the entry x_i is omitted and the dot $\cdot$ stands for the module structure. The coboundary operators square to zero, $\delta_{q+1} \circ \delta_q = 0\ \forall\ q \in \mathbb{N}$, giving rise to the Chevalley-Eilenberg complex $(C^*(\mathcal{L}, M), \delta)$, and the Chevalley-Eilenberg cohomology $\mathrm{H}^q(\mathcal{L}, M) := Z^q(\mathcal{L}, M)/B^q(\mathcal{L}, M)$. Elements in $Z^q(\mathcal{L}, M) :=\ \ker \delta_q$ are called q-cocycles and elements in $B^q(\mathcal{L}, M) :=\ \mathrm{im}\ \delta_{q-1}$ are called q-coboundaries. The original literature is given by Chevalley and Eilenberg [2].

4 Results

Theorem 1. *The first algebraic cohomology of the Witt $\mathcal{W}$ and the Virasoro $\mathcal{V}$ algebra over a field $\mathbb{K}$ with char$(\mathbb{K}) = 0$ and values in the tensor densities modules $\mathcal{F}^\lambda$ is zero for all $\lambda \in \mathbb{C}$ except for $\lambda = 0, 1, 2$, i.e.*

$$\dim\ \mathrm{H}^1(\mathcal{L}, \mathcal{F}^\lambda) = \begin{cases} 2 \text{ if } \lambda = 0 \\ 1 \text{ if } \lambda = 1, 2 \\ 0 \text{ else} \end{cases}$$

where $\mathcal{L}$ stands for $\mathcal{W}$ or $\mathcal{V}$.

[1] The symbol $\delta_{i,j}$ is the Kronecker Delta, equaling one if $i = j$ and zero otherwise.

Theorem 2. *The second algebraic cohomology of the Witt $\mathcal{W}$ and the Virasoro $\mathcal{V}$ algebra over a field $\mathbb{K}$ with char$(\mathbb{K}) = 0$ and values in the tensor densities modules $\mathcal{F}^\lambda$ is zero for all $\lambda \in \mathbb{C}$ except for $\lambda = 0, 1, 2, 5, 7$, i.e.*

$$\dim\ \mathrm{H}^2(\mathcal{W}, \mathcal{F}^\lambda) = \begin{cases} 2 \text{ if } \lambda = 0, 1, 2 \\ 1 \text{ if } \lambda = 5, 7 \\ 0 \text{ else} \end{cases} \qquad \dim\ \mathrm{H}^2(\mathcal{V}, \mathcal{F}^\lambda) = \begin{cases} 2 \text{ if } \lambda = 1, 2 \\ 1 \text{ if } \lambda = 0, 5, 7 \\ 0 \text{ else} \end{cases}.$$

Theorem 3. *The third algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with char$(\mathbb{K}) = 0$ and values in the tensor densities modules $\mathcal{F}^\lambda$ is zero for $\lambda \in I = [-100, \ldots, -1] \cup [6, 8, 10, 14, 16, 18, 20, 22, 24, 26]$, i.e.*

$$\mathrm{H}^3(\mathcal{W}, \mathcal{F}^\lambda) = \{0\} \text{ if } \lambda \in I\ .$$

Acknowledgements The author would like to thank Martin Schlichenmaier for constructive discussions and valuable remarks. Partial support by the Internal Research Project GEOMQ15, University of Luxembourg, and by the OPEN programme of the Fonds National de la Recherche (FNR), Luxembourg, project QUANTMOD O13/570706 is gratefully acknowledged.

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Derivations and Automorphisms of Free Nilpotent Lie Algebras and Their Quotients

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Abstract Let $\mathfrak{n}_{d,t}$ be the free nilpotent Lie algebra of type d and nilindex t. Starting out with the derivation algebra and the automorphism group of $\mathfrak{n}_{d,t}$, we get a natural description of derivations and automorphisms of any generic nilpotent Lie algebra of the same type and nilindex. Moreover, along the paper we discuss several examples to illustrate the obtained results.

1 Introduction

In the middle of the 20th century, the study of derivations and automorphisms of algebras was a central topic of research. It is well known that many linear algebraic Lie groups and their Lie algebras arise from the automorphism groups and the derivation algebras of certain nonassociative algebras. In fact, for a given finite-dimensional real nonassociative algebra A, the automorphism group Aut$A$ is a closed Lie subgroup of the lineal group $\mathtt{GL}(A)$ and the derivation algebra Der$A$ is the Lie algebra of Aut$A$ (see [20, Proposition 7.1 and 7.3, Chap. 7]).

Paying attention to Lie algebras, a lot of research papers on this topic are devoted to the study of the interplay between the structures of their derivation algebras, their groups of automorphisms and Lie algebras themselves (see [24] and references therein). We point out two simple but elegant results on this direction. According to [1], any Lie algebra that has an automorphism of prime period without nonzero fixed points is nilpotent. The same result is valid in the case of Lie algebra has a nonsingular derivation (see [11, Theorem 2]). So, automorphisms and derivations and the nature of their elements are interesting tools in the study of structural properties of algebras.

The main motif of this paper is to describe the group of automorphisms and the algebra of derivations of any finite-dimensional t-step nilpotent Lie algebra $\mathfrak{n}$ (this means that $\mathfrak{n}^t \neq 0 = \mathfrak{n}^{t+1}$) generated by a set U of d elements. The description will

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https://doi.org/10.1007/978-981-15-7775-8_47

be given through the derivation algebra and the automorphism group of the free t-step nilpotent Lie algebra $\mathfrak{n}_{d,t}$ generated by U. Denoting by $\mathfrak{u} = \mathrm{span}\langle U \rangle$, the elements of the derivation algebra, $\mathrm{Der}\mathfrak{n}_{d,t}$, arise by extending and combining, in a natural way, linear maps from $\mathfrak{u}$ into $\mathfrak{u}$ and from $\mathfrak{u}$ to $\mathfrak{n}^2_{d,t}$. The group of automorphisms, $\mathrm{Aut}\mathfrak{n}_{d,t}$, is described through automorphism induced by elements of the general linear group $\mathtt{GL}(\mathfrak{u})$ and automorphisms provide by linear maps from $\mathfrak{u}$ to $\mathfrak{n}_{d,t}$ which induce the identity mapping on $\frac{\mathfrak{n}_{d,t}}{\mathfrak{n}^2_{d,t}}$.

The paper splits into three sections starting from number 2. Section 2 collects some known results about derivations and automorphisms of free nilpotent Lie algebras. This information let establish the structure of derivations and automorphisms of any nilpotent Lie algebra in Theorems 1 and 2. Section 3 contains examples which show the way to compute automorphism groups and derivation algebras. This last section may also be consider as a short illustration of techniques which may be used in this regard.

Along the paper, vector spaces are of finite dimension over a field $\mathbb{F}$ of characteristic zero. All unexplained definitions may be found in [12] or [10].

2 Theoretical Results

We begin by recalling some basics facts and notations about Lie algebras. Let $\mathfrak{n}$ be a Lie algebra with bilinear product $[x, y]$, and $\mathrm{Aut}\mathfrak{n}$ denote the automorphism group of $\mathfrak{n}$, that is, the set of linear bijective maps $\varphi\colon \mathfrak{n} \to \mathfrak{n}$ such that $\varphi[x, y] = [\varphi(x), \varphi(y)]$. Also, let $\mathrm{Der}\mathfrak{n}$ denote the set of derivations of $\mathfrak{n}$, so, the set of linear maps $d\colon A \to A$ such that $d[x, y] = [d(x), y] + [x, d(y)]$. If V, W are subspaces of $\mathfrak{n}$, $[V, W]$ denotes the space spanned by all products $[v, w]$, $v \in V$, $w \in W$. The terms of the lower central series of $\mathfrak{n}$ are defined by $\mathfrak{n}^1 = \mathfrak{n}$, and $\mathfrak{n}^{i+1} = [\mathfrak{n}, \mathfrak{n}^i]$ for $i \geq 2$. If $\mathfrak{n}^t \neq 0$ and $\mathfrak{n}^{t+1} = 0$, then $\mathfrak{n}$ is said nilpotent of nilpotent index or nilindex t. We refer to them also as t-step nilpotent. A nilpotent algebra $\mathfrak{n}$ is generated by any set $\{y_1, \ldots, y_d\}$ of $\mathfrak{n}$ such that $\{y_i + \mathfrak{n}^2 : i = 1, \ldots, d\}$ is a basis of $\frac{\mathfrak{n}}{\mathfrak{n}^2}$ (see [8, Corollary 1.3]). The dimension of this space is called the type of $\mathfrak{n}$ and $\{y_1, \ldots, y_d\}$ is said *minimal set of generators* (m.s.g.). So, the type is just the dimension of any subspace $\mathfrak{u}$ such that $\mathfrak{n} = \mathfrak{u} \oplus [\mathfrak{n}, \mathfrak{n}]$.

The free t-step nilpotent Lie algebra on the set $U = \{x_1, \ldots, x_d\}$ (where $d \geq 2$) is the quotient algebra $\mathfrak{n}_{d,t} = \mathfrak{FL}(U)/\mathfrak{FL}(U)^{t+1}$, where $\mathfrak{FL}(U)$ is the free Lie algebra generated by U (see [12, Section 4, Chapter V]). The elements of $\mathfrak{FL}(U)$ are linear combinations of monomials $[x_{i_1}, \ldots, x_{i_s}] = [\ldots[[x_{i_1}, x_{i_2}], x_{i_3}], \ldots, x_{i_s}]$, $s \geq 1$ and $x_{i_j} \in U$. So, the free nilpotent algebra $\mathfrak{n}_{d,t}$ is generated as vector space by s-monomials $[x_{i_1}, \ldots, x_{i_s}]$, for $1 \leq s \leq t$.

Again, if we set $\mathfrak{u} = \mathrm{span}\langle U \rangle$, the subspace $\mathfrak{u}^s = [\mathfrak{u}^{s-1}, \mathfrak{u}]$ is the linear span of the s-monomials. Thus $\mathfrak{n}_{d,t}$ is an $\mathbb{N}$-graded algebra whose s-th homogeneous component is $\mathfrak{u}^s$. The dimension of any subspace $\mathfrak{u}^s$, $1 \leq s \leq t$ is:

$$\frac{1}{s}\sum_{a|s}\mu(a)d^{s/a},$$

where μ is the Möebius function.

The algebra $\mathfrak{n}_{d,t}$ enjoys the following *Universal Mapping Property* (see [8, Proposition 1.4] and [21, Proposition 4]): for any k-step nilpotent Lie algebra $\mathfrak{n}$ with $k \le t$ of type d, and any d-elements $y_1, \dots, y_d$ of $\mathfrak{n}$, the correspondence $x_i \mapsto y_i$ extends to a unique algebra homomorphism $\mathfrak{n}_{d,t} \to \mathfrak{n}$. In the particular case that $\{y_1, \dots, y_d\}$ is a m.s.g., the image contains a set of generators, so the map is surjective. Therefore, any t-step nilpotent Lie algebra of type d is an homomorphic image $\frac{\mathfrak{n}_{d,t}}{\mathfrak{t}}$ where $\mathfrak{t}$ is an ideal such that $\mathfrak{t} \subseteq \mathfrak{n}^2_{d,t}$ and $\mathfrak{n}^t_{d,t} \nsubseteq \mathfrak{t}$.

Derivations and automorphisms of $\mathfrak{n}_{d,t}$ are completely determine by their effect on $\mathfrak{u}$. Conversely, any linear map from $\mathfrak{u}$ into $\mathfrak{n}_{d,t}$ (bijection from a basis of $\mathfrak{u}$ to any m.s.g.) determines a unique derivation (automorphism) of $\mathfrak{n}_{d,t}$. This assertion is covered by the next result and its corollary. A detailed proof can be found in [21, Propositions 2 and 3].

Proposition 1. *Let φ denote any linear map from the vector space $\mathfrak{u} = span\langle x_1, \dots, x_d\rangle$ into $\mathfrak{n}_{d,t}$, where $\{x_1, \dots, x_d\}$ is a m.s.g. of $\mathfrak{n}_{d,t}$. Then:*

a) φ extends to a derivation of $\mathfrak{n}_{d,t}$ by declaring

$$d_\varphi([x_{\alpha_1}, \dots, x_{\alpha_r}]) = \sum_{1\le i\le r}[x_{\alpha_1}, \dots, \varphi(x_{\alpha_i}), \dots, x_{\alpha_r}].$$

b) φ extends to an algebra homomorphism of $\mathfrak{n}_{d,t}$ by declaring

$$\Phi_\varphi([x_{\alpha_1}, \dots, x_{\alpha_r}]) = [\varphi(x_{\alpha_1}), \dots, \varphi(x_{\alpha_r})].$$

Moreover if $p_{\mathfrak{u}}$ stands for the projection map from $\mathfrak{n}_{d,t}$ into $\mathfrak{u}$, then Φ_φ is an automorphism iff $\{p_{\mathfrak{u}}(\varphi(x_1)), \dots, p_{\mathfrak{u}}(\varphi(x_n))\}$ is a linearly independent set.

Corollary 1. *Let $\mathfrak{n}_{d,t}$ be the free t-nilpotent Lie algebra on d-generators $x_1, \dots, x_d$ and $\mathfrak{u} = span\langle x_i\rangle$. The derivation algebra and the automorphism group of $\mathfrak{n}_{d,t}$ are described as $Der\mathfrak{n}_{d,t} = \{d_\varphi : \varphi \in Hom(\mathfrak{u}, \mathfrak{n}_{d,t})\}$ and $Aut\mathfrak{n}_{d,t} = \{\Phi_\varphi : \varphi \in Hom(\mathfrak{u}, \mathfrak{n}_{d,t})$ and $\{p_{\mathfrak{u}}(\varphi(x_1)), \dots, p_{\mathfrak{u}}(\varphi(x_d))\}$ m.s.g.$\}$.*

Remark 1. The Levi factor $\mathfrak{S}_{d,t}$ of $\mathrm{Der}\mathfrak{n}_{d,t}$ is given by the maps d_φ for $\varphi \in \mathfrak{sl}(\mathfrak{u})$. Clearly, $\mathfrak{S}_{d,t}$ is isomorphic to the special Lie algebra $\mathfrak{sl}_d(\mathbb{F})$. The elements of the nilpotent radical $\mathfrak{N}_{d,t}$ are the linear maps d_φ where $\varphi \in \mathrm{Hom}(\mathfrak{u}, \mathfrak{n}^2_{d,t})$. And the solvable radical is just $\mathfrak{R}_{d,t} = k \cdot id_{d,t} \oplus \mathfrak{N}_{d,t}$ where $id_{d,t}(a_k) = k \cdot a_k$ for any $a_k \in \mathfrak{u}^k$ (see [3, Proposition 2.4]).

Remark 2. The group $\mathrm{Aut}\mathfrak{n}_{d,t}$ is the semidirect product of the general linear group $\mathtt{GL}(d, t)$, obtained from the automorphisms Φ_φ where $\varphi \in GL(\mathfrak{u})$, and the nilpotent group $\mathtt{NL}(d, t)$, whose elements are Φ_σ and $\sigma = \mathrm{Id}_{\mathfrak{u}} + \delta$ and $\delta \in \mathrm{End}(\mathfrak{u}, n^2_{d,t})$ (see [4, Proposition 3.1]).

For any ideal $\mathfrak{t}$ of $\mathfrak{n}_{d,t}$ such that $\mathfrak{n}^t_{d,t} \not\subseteq \mathfrak{t} \subseteq \mathfrak{n}^2_{d,t}$, let denote by $\mathrm{Der}_{\mathfrak{t}}\, \mathfrak{n}_{d,t}$ and $\mathrm{Der}_{\mathfrak{n}_{d,t},\mathfrak{t}}\, \mathfrak{n}_{d,t}$ the subset of derivations which map $\mathfrak{t}$ into itself, and $\mathfrak{n}_{d,t}$ into $\mathfrak{t}$ respectively. Both sets are subalgebras of $\mathrm{Der}\mathfrak{n}_{d,t}$, even more, $\mathrm{Der}_{\mathfrak{n}_{d,t},\mathfrak{t}}\, \mathfrak{n}_{d,t}$ is an ideal inside $\mathrm{Der}_{\mathfrak{t}}\, \mathfrak{n}_{d,t}$, and the following result follows [21, Proposition 5]:

Theorem 1. *Let $\mathfrak{t}$ be an ideal of $\mathfrak{n}_{d,t}$ such that $\mathfrak{n}^t_{d,t} \not\subseteq \mathfrak{t} \subseteq \mathfrak{n}^2_{d,t}$, the algebra of derivations of $\dfrac{\mathfrak{n}_{d,t}}{\mathfrak{t}}$ is isomorphic to $\dfrac{Der_{\mathfrak{t}}\, \mathfrak{n}_{d,t}}{Der_{\mathfrak{n}_{d,t},\mathfrak{t}}\, \mathfrak{n}_{d,t}}$, where $Der_{\mathfrak{t}}\, \mathfrak{n}_{d,t}$ and $Der_{\mathfrak{n}_{d,t},\mathfrak{t}}\, \mathfrak{n}_{d,t}$ maps $\mathfrak{t}$ and $\mathfrak{n}_{d,t}$ into $\mathfrak{t}$ respectively.*

In a similar vein to the previous theorem, it is possible to arrive at a structural description of automorphisms of homomorphic images of free nilpotent algebras. For any ideal $\mathfrak{t}$ of $\mathfrak{n}_{d,t}$, $\mathfrak{n}^t_{d,t} \not\subseteq \mathfrak{t} \subseteq \mathfrak{n}^2_{d,t}$, let denote by $\mathrm{Aut}_{\mathfrak{t}}\, \mathfrak{n}_{d,t}$ the subset of automorphisms which map $\mathfrak{t}$ into itself. It is easily checked that $\mathrm{Aut}_{\mathfrak{t}}\, \mathfrak{n}_{d,t}$ is a subgroup of $\mathrm{Aut}\mathfrak{n}_{d,t}$. Consider now the map

$$\theta\colon \mathrm{Aut}_{\mathfrak{t}}\, \mathfrak{n}_{d,t} \to \mathrm{Aut}\frac{\mathfrak{n}_{d,t}}{\mathfrak{t}}, \quad \theta(\Phi)(x + \mathfrak{t}) = \Phi(x) + \mathfrak{t}.$$

By using $\Phi(\mathfrak{t}) = \mathfrak{t}$ and Φ homomorphism, we can easily check that θ is well defined. Now, a straightforward computation shows that θ is a group homomorphism with kernel,

$$\mathrm{Ker}\,\theta = \{\Phi \in \mathrm{Aut}\mathfrak{n}_{d,t} : \mathrm{Im}\,(\Phi - Id) \subseteq \mathfrak{t}\}.$$

Then, we have the following result:

Theorem 2. *For any ideal $\mathfrak{t}$ of $\mathfrak{n}_{d,t}$ such that $\mathfrak{n}^t_{d,t} \not\subseteq \mathfrak{t} \subseteq \mathfrak{n}^2_{d,t}$, the set $Aut^{\circ}_{\mathfrak{t}}\, \mathfrak{n}_{d,t} = \{\Phi \in Aut\mathfrak{n}_{d,t} : Im\,(\Phi - Id) \subseteq \mathfrak{t}\}$ is a normal subgroup of the group of automorphisms of $\dfrac{\mathfrak{n}_{d,t}}{\mathfrak{t}}$. Moreover $Aut\dfrac{\mathfrak{n}_{d,t}}{\mathfrak{t}}$ is isomorphic to $\dfrac{Aut_{\mathfrak{t}}\, \mathfrak{n}_{d,t}}{Aut^{\circ}_{\mathfrak{t}}\, \mathfrak{n}_{d,t}}$, where $Aut_{\mathfrak{t}}\, \mathfrak{n}_{d,t}$ maps $\mathfrak{t}$ into $\mathfrak{t}$.*

Proof. From previous comments we only need to proof that the map θ is onto. Let $\rho_{\mathfrak{t}} : \mathfrak{n}_{d,t} \to \frac{\mathfrak{n}_{d,t}}{\mathfrak{t}}$ be the canonical projection and let $\{f_1 + \mathfrak{t}, \dots, f_k + \mathfrak{t}\}$ be a basis of $\frac{\mathfrak{n}_{d,t}}{\mathfrak{t}}$ and $\{e_1 + \mathfrak{t}, \dots, e_d + \mathfrak{t}\}$ a m.s.g. of $\frac{\mathfrak{n}_{d,t}}{\mathfrak{t}}$. Then $\{e_1, \dots, e_d\}$ is also a m.s.g. of $\mathfrak{n}_{d,t}$. If we take a generic automorphism $\hat{A} \in \mathrm{Aut}\dfrac{\mathfrak{n}_{d,t}}{\mathfrak{t}}$,

$$\hat{A}(e_i + \mathfrak{t}) = \sum_{j=1}^{k} \alpha_{ij} f_j + \mathfrak{t}, \text{ and declare } A(e_i) = \sum_{j=1}^{k} \alpha_{ij} f_j,$$

A extends to a linear homomorphism, $A\colon \mathfrak{e} \to \mathfrak{n}_{d,t}$, where $\mathfrak{e} = \mathrm{span}\langle e_1, \dots, e_d\rangle$. Let Φ_A be the homomorphism given by Proposition 1. We check that $\theta(\Phi_A) = \hat{A}$ noting that, for a generic element $[[\dots[a_1, a_2], \dots, a_l]$ where $a_i \in \mathfrak{e}$, up to linear combinations, we have that

$$\begin{aligned}\rho_{\mathfrak{t}} \circ \Phi_A[[\dots[a_1,a_2],\dots,a_l] &= [[\dots[\rho_{\mathfrak{t}} \circ A(a_1), \rho_{\mathfrak{t}} \circ A(a_2)],\dots,\rho_{\mathfrak{t}} \circ A(a_l)]\\ &= [[\dots[\hat{A}\circ\rho(a_1), \hat{A}\circ\rho_{\mathfrak{t}}(a_2)],\dots,\hat{A}\circ\rho_{\mathfrak{t}}(a_l)]\\ &= \hat{A}\circ\rho_{\mathfrak{t}}[[\dots[a_1,a_2],\dots,a_l].\end{aligned}$$

The second equality follows because for every $a_i = \sum_{j=1}^{d} \beta_{ji} e_j$,

$$\begin{aligned}\rho_{\mathfrak{t}} \circ A(a_i) = \rho_{\mathfrak{t}} \circ A\left(\sum_{j=1}^{d}\beta_{ji}e_j\right) = \sum_{j=1}^{d}\beta_{ji}\rho_{\mathfrak{t}} \circ A(e_j) = \sum_{j=1}^{d}\beta_{ji}\rho_{\mathfrak{t}}\left(\sum_{l=1}^{k}\alpha_{jl}f_l\right)\\ = \sum_{j=1}^{d}\beta_{ji}\sum_{l=1}^{k}(\alpha_{jl}f_l+\mathfrak{t}) = \sum_{j=1}^{d}\beta_{ji}\hat{A}(e_i+\mathfrak{t}) = \sum_{j=1}^{d}\beta_{ji}\hat{A}\circ\rho_{\mathfrak{t}}(e_i) = \hat{A}\circ\rho_{\mathfrak{t}}(a_i).\end{aligned}$$

Now Ker $\rho_{\mathfrak{t}} = \mathfrak{t}$ and $\rho_{\mathfrak{t}} \circ \Phi_A = \hat{A}\circ\rho_{\mathfrak{t}}$ implies $\Phi_A(\mathfrak{t}) = \mathfrak{t}$ and, $\Phi_A(\mathfrak{t})$ automorphism, follows by using the equivalence given in Proposition 1 and the fact that $\hat{A}$ is an automorphism.

3 Examples, Techniques and Patterns

From the generator set $U = \{x_1,\dots,x_d\}$, we easily get the standard monomials $[x_{i_1},\dots,x_{i_r}]$ that (linearly) generate the Lie algebra $\mathfrak{n}_{d,t}$. However, the anticommutativity law ($[x_i,x_j]+[x_j,x_i]=0$) and the Jacobi identity ($\sum_{\text{cyclic}}[[x_i,x_j],x_k]=0$), both set linear dependence relations. This makes it difficult to find a basis formed by monomials. The problem was solved by M. Hall in 1950. Focusing on the behavior of algorithms, the most natural basis to work on free nilpotent Lie algebras, is the *Hall basis* (see [9] for definition, and [23, Chapter IV, Section 5] for a detailed construction).

Starting with the total order $x_d < x_{d-1} < \cdots < x_1$, the definition of Hall basis states recursively if a given standard monomial depends on the previous ones. The recursive algorithm is covered by the pseudocode given in Table 1 and provides a Hall basis that we will denote as $H_{d,t}(U_<)$ or $H_{d,t}$ if the total order in U is clear. This algorithm checks if an element v belongs to the Hall basis once we have defined a monomial order. For some small d and t values, the output of Hall basis algorithm is given in Table 2.
Now we introduce several examples which illustrate (among other things):

1. The way to describe a generic d-generated t-nilpotent Lie algebra as an homomorphic image of $\mathfrak{n}_{d,t}$.
2. The way to compute automorphisms and derivations regarding Proposition 1 and Theorems 1 and 2.
3. The recognition of some structural patterns of nilpotent algebras depending on the nature of their derivations and automorphisms.

Table 1 Hall basis algorithm

```
isCanonical(v):
    if deg v == 1 then true;
    else if (not isCanonical(v_1) or not isCanonical(v_2) or v_2 > v_1) then false;
    else if deg v_1 > 1 then (isCanonical(v_{1,1}) or isCanonical(v_{1,2}) or v_2 ≥ v_{1,2});
    else true;
```

[a] Note that this is a recursive algorithm. Here $v = [v_1, v_2]$. In order to generate Hall Basis elements of degree n we can combine v_1 and v_2 in level $n - k$ and k respectively, where $k = 1, \dots, n/2$.

Table 2 Hall basis of $\mathfrak{n}_{d,t}$

(d,t)	$H_{d,t}$
$(2,6)$	$x_2, x_1, [x_1,x_2], [[x_1,x_2],x_2], [[x_1,x_2],x_1], [[[x_1,x_2],x_2],x_2], [[[x_1,x_2],x_2],x_1],$ $[[[x_1,x_2],x_1],x_1], [[[[x_1,x_2],x_2],x_2],x_2], [[[[x_1,x_2],x_2],x_2],x_1],$ $[[[x_1,x_2],x_2],[x_1,x_2]], [[[[x_1,x_2],x_2],x_1],x_1], [[[x_1,x_2],x_1],[x_1,x_2]],$ $[[[[x_1,x_2],x_1],x_1],x_1], [[[[[x_1,x_2],x_2],x_2],x_2],x_2], [[[[[x_1,x_2],x_2],x_2],x_2],x_1],$ $[[[[x_1,x_2],x_2],x_2],[x_1,x_2]], [[[[[x_1,x_2],x_2],x_2],x_1],x_1], [[[[x_1,x_2],x_2],x_1],[x_1,x_2]],$ $[[[[[x_1,x_2],x_2],x_1],x_1],x_1], [[[x_1,x_2],x_1],[[x_1,x_2],x_2]], [[[[x_1,x_2],x_1],x_1],[x_1,x_2]],$ $[[[[[x_1,x_2],x_1],x_1],x_1],x_1]$
$(4,3)$	$x_4, x_3, x_2, x_1, [x_3,x_4], [x_2,x_4], [x_2,x_3], [x_1,x_4], [x_1,x_3], [x_1,x_2], [[x_3,x_4],x_4],$ $[[x_3,x_4],x_3], [[x_3,x_4],x_2], [[x_3,x_4],x_1], [[x_2,x_4],x_4], [[x_2,x_4],x_3], [[x_2,x_4],x_2],$ $[[x_2,x_4],x_1], [[x_2,x_3],x_3], [[x_2,x_3],x_2], [[x_2,x_3],x_1], [[x_1,x_4],x_4], [[x_1,x_4],x_3],$ $[[x_1,x_4],x_2], [[x_1,x_4],x_1], [[x_1,x_3],x_3], [[x_1,x_3],x_2], [[x_1,x_3],x_1], [[x_1,x_2],x_2],$ $[[x_1,x_2],x_1]$
$(6,2)$	$x_6, x_5, x_4, x_3, x_2, x_1, [x_5,x_6], [x_4,x_6], [x_4,x_5], [x_3,x_6], [x_3,x_5], [x_3,x_4], [x_2,x_6],$ $[x_2,x_5], [x_2,x_4], [x_2,x_3], [x_1,x_6], [x_1,x_5], [x_1,x_4], [x_1,x_3], [x_1,x_2]$

[b] We point out that from expanded basis $H_{4,3}$ and $H_{2,6}$ we can recover Hall basis of $\mathfrak{n}_{4,2}$ and $\mathfrak{n}_{2,t}$ for $t = 2, 3, 4, 5$.

In the sequel, if a map φ is given in a matrix form $A = (a_{ij})$ attached to a basis $\mathcal{B} = \{v_1, \dots, v_n\}$, then $\varphi(v_i) = \sum_{j=1}^{n} a_{ji} v_j$.

The Universal Mapping Property lets us describe any t-nilpotent Lie algebra $\mathfrak{n}$ of type d as a homomorphic image of $\mathfrak{n}_{d,t}$ in a easy way. From any m.s.g. $\{e_1, \dots, e_d\}$ of $\mathfrak{n}$, the correspondence $x_i \mapsto e_i$ for $i = 1, \dots, d$ extends uniquely to a surjective algebra homomorphism $\theta_{\mathfrak{n}} \colon \mathfrak{n}_{d,t} \to \mathfrak{n}$ and $\mathfrak{n} \cong \dfrac{\mathfrak{n}_{d,t}}{\operatorname{Ker} \theta_{\mathfrak{n}}}$. We will compute ideals of this type in our following example:

Example 1. Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be the 8-dimensional and 5-dimensional Lie algebras described through the basis $\{e_1, \dots, e_8\}$ and $\{u_1, \dots, u_5\}$ by the following multiplication table ($[a, b] = -[b, a]$ and $[a, b] = 0$ is not in the table):

$$\begin{array}{lll} [e_1, e_2] = e_5, & [e_2, e_3] = e_8, & [e_3, e_5] = -e_7, \\ [e_1, e_3] = e_6, & [e_2, e_4] = e_6, & [e_4, e_6] = -e_8, \\ [e_1, e_4] = e_7, & [e_2, e_6] = -e_7, & [u_1, u_3] = u_5, \\ [e_1, e_5] = -e_8, & [e_3, e_4] = -e_5, & [u_2, u_4] = u_5, \end{array}$$

The lower central series of these algebras are:

$$\mathfrak{n}_1^2 = \operatorname{span}\langle e_5, e_6, e_7, e_8\rangle, \quad \mathfrak{n}_1^3 = \operatorname{span}\langle e_7, e_8\rangle, \quad \mathfrak{n}_1^4 = 0,$$

and

$$\mathfrak{n}_2^2 = \operatorname{span}\langle u_5\rangle, \quad \mathfrak{n}_2^3 = 0.$$

Consider now the maps $\theta_{\mathfrak{n}_1} \colon x_i \to e_i$ for $i = 1, \dots, 4$ from $\mathfrak{n}_{4,3}$ onto $\mathfrak{n}_1$ and $\theta_{\mathfrak{n}_2} \colon x_i \to u_i$ for $i = 1, \dots, 4$ from $\mathfrak{n}_{4,2}$ onto $\mathfrak{n}_2$. Both correspondences extend to homomorphisms of algebras by following the proof in [21, Proposition 4] ($\theta[x_{\alpha_1} \dots x_{\alpha_s}] = [\theta(x_{\alpha_1}) \dots \theta(x_{\alpha_s})]$). It is not hard to see that:

$$\begin{aligned} \operatorname{Ker} \theta_{\mathfrak{n}_1} = \operatorname{span}\langle & [x_3, x_4] + [x_1, x_2], [x_2, x_4] - [x_1, x_3], [x_2, x_3] - [[x_1, x_3], x_1], \\ & [x_1, x_4] + [[x_1, x_2], x_2], [[x_3, x_4], x_4] - [[x_1, x_3], x_1], [[x_3, x_4], x_3], \\ & [[x_3, x_4], x_2] + [[x_1, x_2], x_2], [[x_3, x_4], x_1], [[x_2, x_4], x_4], [[x_1, x_4], x_2], \\ & [[x_2, x_4], x_3] + [[x_1, x_2], x_2], [[x_2, x_4], x_2], [[x_2, x_4], x_1] - [[x_1, x_3], x_1], \\ & [[x_2, x_3], x_3], [[x_2, x_3], x_2], [[x_2, x_3], x_1], [[x_1, x_4], x_4], [[x_1, x_4], x_3], \\ & [[x_1, x_4], x_1], [[x_1, x_3], x_3] + [[x_1, x_2], x_2], [[x_1, x_3], x_2], [[x_1, x_2], x_1]\rangle, \end{aligned}$$

and

$$\operatorname{Ker} \theta_{\mathfrak{n}_2} = \operatorname{span}\langle [x_3, x_4], [x_2, x_3], [x_1, x_4], [x_1, x_2], [x_1, x_3] - [x_2, x_4]\rangle.$$

We point out that $\operatorname{Ker} \theta_{\mathfrak{n}_2}$ is an homogeneous ideal in the $\mathbb{N}$-graded structure of $\mathfrak{n}_{4,2}$ and $\operatorname{Ker} \theta_{\mathfrak{n}_1}$ is not an homogeneous ideal of $\mathfrak{n}_{4,3}$. Therefore, $\mathfrak{n}_2$ inherits the grading of $\mathfrak{n}_{4,2}$, but $\mathfrak{n}_1$ does not inherit that of $\mathfrak{n}_{4,3}$.

Example 2. According to Proposition 1, derivations and automorphisms of $\mathfrak{n}_{2,4}$ in Hall basis $H_{2,4}$ can be easily obtained by iterating the Leibniz rule $\varphi([a, b]) = [\varphi(a), b])] + [a, \varphi(b)]$ and the law $\varphi([a, b]) = [\varphi(a), \varphi(b)]$. The matrices representing the elements of $\operatorname{Aut}\mathfrak{n}_{2,4} = \mathtt{GL}(2, 4) \rtimes \mathtt{NL}(2, 4)$ are product of matrices of the following shapes:

$$\left(\begin{array}{c|c|c|c} \begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix} & 0 & 0 & 0 \\ \hline 0 & \epsilon & 0 & 0 \\ \hline 0 & 0 & \begin{smallmatrix} \epsilon a_1 & \epsilon a_2 \\ \epsilon a_3 & \epsilon a_4 \end{smallmatrix} & 0 \\ \hline 0 & 0 & 0 & \epsilon \cdot A' \end{array}\right) \in \mathtt{GL}(2,4), \quad \left(\begin{array}{c|c|c|c} I_2 & 0 & 0 & 0 \\ \hline b_1 \;\; b_2 & 1 & 0 & 0 \\ \hline \begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix} & \begin{smallmatrix} b_2 \\ -b_1 \end{smallmatrix} & I_2 & 0 \\ \hline \begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4 \\ d_5 & d_6 \end{smallmatrix} & \begin{smallmatrix} c_2 \\ c_4 - c_1 \\ -c_3 \end{smallmatrix} & \begin{smallmatrix} b_2 & 0 \\ -b_1 & b_2 \\ 0 & -b_1 \end{smallmatrix} & I_3 \end{array}\right) \in \mathtt{NL}(2,4);$$

here I_k denotes the $k \times k$ identity matrix, $\epsilon = a_1 a_4 - a_2 a_3 \neq 0$ and

$$A' = \begin{pmatrix} a_1^2 & a_1 a_2 & a_2^2 \\ 2a_1 a_3 & a_1 a_4 + a_2 a_3 & 2a_2 a_4 \\ a_3^2 & a_3 a_4 & a_4^2 \end{pmatrix}.$$

From the decomposition $\mathrm{Der}\mathfrak{n}_{2,4} = \mathfrak{S}_{2,4} \oplus \mathbb{F} \cdot id_{2,4} \oplus \mathfrak{N}_{2,4}$, the matrices that represent derivations of $\mathfrak{n}_{2,4}$ are sum of matrices of three different types:

$$\left(\begin{array}{c|c|c|c} \begin{smallmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{smallmatrix} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \begin{smallmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{smallmatrix} & 0 \\ \hline 0 & 0 & 0 & \begin{smallmatrix} 2a_1 & a_2 & 0 \\ 2a_3 & 0 & 2a_2 \\ 0 & a_3 & -2a_1 \end{smallmatrix} \end{array}\right) \in \mathfrak{S}_{2,4}, \ \lambda id_{2,4} = \left(\begin{array}{c|c|c|c} \begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix} & 0 & 0 & 0 \\ \hline 0 & 2\lambda & 0 & 0 \\ \hline 0 & 0 & \begin{smallmatrix} 3\lambda & 0 \\ 0 & 3\lambda \end{smallmatrix} & 0 \\ \hline 0 & 0 & 0 & \begin{smallmatrix} 4\lambda & 0 & 0 \\ 0 & 4\lambda & 0 \\ 0 & 0 & 4\lambda \end{smallmatrix} \end{array}\right)$$

and

$$\left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline b_1 \;\; b_2 & 0 & 0 & 0 \\ \hline \begin{smallmatrix} c_1 & c_2 \\ c_3 & c_4 \end{smallmatrix} & \begin{smallmatrix} b_2 \\ -b_1 \end{smallmatrix} & 0 & 0 \\ \hline \begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4 \\ d_5 & d_6 \end{smallmatrix} & \begin{smallmatrix} c_2 \\ c_4 - c_1 \\ -c_3 \end{smallmatrix} & \begin{smallmatrix} b_2 & 0 \\ -b_1 & b_2 \\ 0 & -b_1 \end{smallmatrix} & 0 \end{array}\right) \in \mathfrak{N}_{2,4}.$$

For any $0 \neq \lambda \in \mathbb{F}$, the linear map $\varphi_\lambda(x_i) = \lambda x_i$ provides the (semisimple) automorphism $\Phi_{\varphi_\lambda}([x_{\alpha_1} \dots x_{\alpha_r}]) = \lambda^r [x_{\alpha_1} \dots x_{\alpha_r}]$ and the (semisimple) derivation $d_{\varphi_\lambda}([x_{\alpha_1} \dots x_{\alpha_r}]) = r\lambda [x_{\alpha_1} \dots x_{\alpha_r}]$.

Consider now the 5-dimensional Lie algebra $\mathfrak{n}_3$ with basis $\{z_1, \dots, z_5\}$ and nonzero products:

$$[z_1, z_2] = z_3, \quad [z_1, z_3] = z_4, \quad [z_1, z_4] = [z_2, z_3] = z_5.$$

The lower central series is $\mathfrak{n}_3^2 = \mathrm{span}\langle z_3, z_4, z_5 \rangle$, $\mathfrak{n}_3^3 = \mathrm{span}\langle z_4, z_5 \rangle$, $\mathfrak{n}_3^4 = \mathrm{span}\langle z_5 \rangle$ and $\mathfrak{n}_3^5 = 0$. So, the correspondence $x_i \mapsto z_i$ for $i = 1, 2$ extends to a surjective algebra homomorphism $\theta_{\mathfrak{n}_3} : \mathfrak{n}_{2,4} \to \mathfrak{n}_3$ and $\mathfrak{n}_3 \cong \dfrac{\mathfrak{n}_{2,4}}{\mathrm{Ker}\, \theta_{\mathfrak{n}_3}}$. In this case, the kernel is the 3-dimensional ideal:

$$\begin{aligned} \mathrm{Ker}\, \theta_{\mathfrak{n}_3} = \mathrm{span}\langle & [[[x_1, x_2], x_2], x_2], [[[x_1, x_2], x_2], x_1], \\ & [[x_1, x_2], x_2] + [[[x_1, x_2], x_1], x_1] \rangle. \end{aligned}$$

Example 3. Let denote $\mathfrak{t} = \operatorname{Ker}\theta_{\mathfrak{n}_3}$. According to Theorems 1 and 2, derivations (automorphisms) of $\mathfrak{n}_3$ are a quotient of the set of derivations (automorphisms) of $\mathfrak{n}_{2,4}$ that leave $\mathfrak{t}$ invariant. These sets are:

$$\operatorname{Der}_{\mathfrak{t}}\mathfrak{n}_{2,4} : \left(\begin{array}{cc|c|cc|ccc} a_1 & a_2 & 0 & 0 & & 0 & & \\ 0 & \frac{1}{2}a_1 & & & & & & \\ \hline b_1 & b_2 & \frac{3}{2}a_1 & 0 & & 0 & & \\ \hline c_1 & c_2 & b_2 & \frac{5}{2}a_1 & a_2 & 0 & & \\ c_3 & c_4 & -b_1 & 0 & 2a_1 & & & \\ \hline d_1 & d_2 & c_2 & b_2 & 0 & \frac{7}{2}a_1 & a_2 & 0 \\ d_3 & d_4 & c_4 - c_1 & -b_1 & b_2 & 0 & 3a_1 & 2a_2 \\ d_5 & d_6 & -c_3 & 0 & -b_1 & 0 & 0 & \frac{5}{2}a_1 \end{array}\right),$$

and, for $a_4 \neq 0$,

$$\operatorname{Aut}_{\mathfrak{t}}\mathfrak{n}_{2,4} : \left(\begin{array}{cc|c|cc|ccc} a_4^2 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline b_1 & b_2 & a_4^3 & 0 & 0 & 0 & 0 & 0 \\ \hline c_1 & c_2 & a_4^2 b_2 - a_2 b_1 & a_4^5 & a_2 a_4^3 & 0 & 0 & 0 \\ c_3 & c_4 & -a_4 b_1 & 0 & a_4^4 & 0 & 0 & 0 \\ \hline d_1 & d_2 & a_4^2 c_2 - a_2 c_1 & a_4^4 b_2 - a_2 a_4^2 b_1 & a_2\left(a_4^2 b_2 - a_2 b_1\right) & a_4^7 & a_2 a_4^5 & a_2^2 a_4^3 \\ d_3 & d_4 & c_4 a_4^2 - a_4 c_1 - a_2 c_3 & -a_4^3 b_1 & a_4^3 b_2 - 2a_2 a_4 b_1 & 0 & a_4^6 & 2a_2 a_4^4 \\ d_5 & d_6 & -a_4 c_3 & 0 & -a_4^2 b_1 & 0 & 0 & a_4^5 \end{array}\right).$$

Note that the isomorphism $\frac{\mathfrak{n}_{2,4}}{\mathfrak{t}} \to \mathfrak{n}_3$ is provided by the correspondence $z_i' \mapsto z_i$ by taking $z_1' = x_1 + \mathfrak{t}$, $z_2' = x_2 + \mathfrak{t}$, $z_3' = [x_1, x_2] + \mathfrak{t}$, $z_4' = [x_1, [x_1, x_2]] + \mathfrak{t}$, $z_5' = [x_2, [x_1, x_2]] + \mathfrak{t}$. So $\mathcal{B}' = \{z_1', z_2', z_3', z_4', z_5'\}$ is a basis. Now, by using the isomorphisms in Theorem 1 and Theorem 2 and a minor change of basis, we get a complete description of derivations and automorphisms of $\mathfrak{n}_3 \cong \dfrac{\mathfrak{n}_{2,4}}{\mathfrak{t}}$. Relative to the basis $\{z_1, z_2, z_3, z_4, z_5\}$:

$$\operatorname{Der}\frac{\mathfrak{n}_{2,4}}{\mathfrak{t}} : \left(\begin{array}{cc|c|c|c} a_1 & 0 & 0 & 0 & 0 \\ a_3 & 2a_1 & & & \\ \hline b_1 & b_2 & 3a_1 & 0 & 0 \\ \hline c_1 & c_2 & b_2 & 4a_1 & 0 \\ \hline d_1 & d_2 & c_2 - b_1 & a_3 + b_2 & 5a_1 \end{array}\right),$$

and, for $a_4 \neq 0$,

$$\operatorname{Aut}\frac{\mathfrak{n}_{2,4}}{\mathfrak{t}} : \left(\begin{array}{cc|c|c|c} a_4 & 0 & 0 & 0 & 0 \\ a_2 & a_4^2 & 0 & 0 & 0 \\ \hline b_2 & b_1 & a_4^3 & 0 & 0 \\ \hline -c_4 & -c_3 & a_4 b_1 & a_4^4 & 0 \\ \hline d_6 - c_2 & d_5 - c_1 & a_2 b_1 - a_4(a_4 b_2 + c_3) & a_4^2(a_2 a_4 + b_1) & a_4^5 \end{array}\right).$$

From previous descriptions, it is clear that the map $\varphi_\lambda : x_i \to \lambda x_i$, for $i = 1, 2$, extends to a derivation iff $\lambda = 0$ and φ_λ extends to an automorphism iff $\lambda = 1$. We also remark that, $\mathfrak{t}$ is not an homogeneous ideal, so $\mathfrak{n}_3$ does not inherit the natural $\mathbb{N}$-grading of $\mathfrak{n}_{2,4}$. However $\Phi_\lambda : x_i \to \lambda^i x_i$ is an automorphism for all $0 \neq \lambda \in \mathbb{F}$

with eigenvalues λ^i for $1 \le i \le 5$. In the case of $\mathbb{F}$ be the reals and $\lambda > 1$, Φ_λ is an (expanding) automorphism that provides the $\mathbb{N}$-grading $\mathfrak{n}_3 = \oplus_{i=1}^5 S(\lambda^i)$ where $S(\lambda^i) = \{v \in \mathfrak{n}_3 : \Phi_\lambda(v) = \lambda^i v\}$.

As in the previous example, in the final one we get the conditions that determine derivations and automorphisms of $\mathfrak{n}_2$ by using $\mathrm{Der}\mathfrak{n}_{4,2}$ and $\mathrm{Aut}_{4,2}$.

Example 4. Derivations and automorphisms of $\mathfrak{n}_{4,2}$ in Hall basis $H_{4,2}$ are (here $\Delta_{i,j}^{k,l} = a_i a_j - a_k a_l$):

$$\mathrm{Der}\mathfrak{n}_{4,2} : d_A = \left(\begin{array}{cccc|cccccc}
a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_5 & a_6 & a_7 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_9 & a_{10} & a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{13} & a_{14} & a_{15} & a_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
b_1 & b_2 & b_3 & b_4 & a_1+a_6 & a_7 & -a_3 & a_8 & -a_4 & 0 \\
b_5 & b_6 & b_7 & b_8 & a_{10} & a_1+a_{11} & a_2 & a_{12} & 0 & -a_4 \\
b_9 & b_{10} & b_{11} & b_{12} & -a_9 & a_5 & a_6+a_{11} & 0 & a_{12} & -a_8 \\
b_{13} & b_{14} & b_{15} & b_{16} & a_{14} & a_{15} & 0 & a_1+a_{16} & a_2 & a_3 \\
b_{17} & b_{18} & b_{19} & b_{20} & -a_{13} & 0 & a_{15} & a_5 & a_6+a_{16} & a_7 \\
b_{21} & b_{22} & b_{23} & b_{24} & 0 & -a_{13} & -a_{14} & a_9 & a_{10} & a_{11}+a_{16}
\end{array}\right),$$

and, for nonsingular matrices with entries a_i,

$$\mathrm{Aut}\mathfrak{n}_{4,2} : \Phi_A = \left(\begin{array}{cccc|cccccc}
a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_5 & a_6 & a_7 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_9 & a_{10} & a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{13} & a_{14} & a_{15} & a_{16} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
b_1 & b_2 & b_3 & b_4 & \Delta_{1,6}^{2,5} & \Delta_{1,7}^{3,5} & \Delta_{2,7}^{3,6} & \Delta_{1,8}^{4,5} & \Delta_{2,8}^{4,6} & \Delta_{3,8}^{4,7} \\
b_5 & b_6 & b_7 & b_8 & \Delta_{1,10}^{2,9} & \Delta_{1,11}^{3,9} & \Delta_{2,11}^{3,10} & \Delta_{1,12}^{4,9} & \Delta_{2,12}^{4,10} & \Delta_{3,12}^{4,11} \\
b_9 & b_{10} & b_{11} & b_{12} & \Delta_{5,10}^{6,9} & \Delta_{5,11}^{7,9} & \Delta_{6,11}^{7,10} & \Delta_{5,12}^{8,9} & \Delta_{6,12}^{8,10} & \Delta_{7,12}^{8,11} \\
b_{13} & b_{14} & b_{15} & b_{16} & \Delta_{1,14}^{2,13} & \Delta_{1,15}^{3,13} & \Delta_{2,15}^{3,14} & \Delta_{1,16}^{4,13} & \Delta_{2,16}^{4,14} & \Delta_{3,16}^{4,15} \\
b_{17} & b_{18} & b_{19} & b_{20} & \Delta_{5,14}^{6,13} & \Delta_{5,15}^{7,13} & \Delta_{6,15}^{7,14} & \Delta_{5,16}^{8,13} & \Delta_{6,16}^{8,14} & \Delta_{7,16}^{8,15} \\
b_{21} & b_{22} & b_{23} & b_{24} & \Delta_{9,14}^{10,13} & \Delta_{9,15}^{11,13} & \Delta_{10,15}^{11,14} & \Delta_{9,16}^{12,13} & \Delta_{10,16}^{12,14} & \Delta_{11,16}^{12,15}
\end{array}\right).$$

Let $d_A \in \mathrm{Der}\mathfrak{n}_{2,4}$ be and $\Phi_A \in \mathrm{Aut}\mathfrak{n}_{2,4}$. An easy computation shows that

$$d_A \in \mathrm{Der}_{\mathfrak{t}}\mathfrak{n}_{2,4} \text{ iff } \begin{cases} a_{12} = -a_2,\ a_{14} = a_9,\ a_8 = a_3, \\ a_{15} = -a_5,\ a_{16} = -a_1 + a_6 + a_{11}, \end{cases}$$

and

$$d_A \in \mathrm{Aut}_{\mathfrak{t}}\mathfrak{n}_{2,4} \text{ iff } \begin{cases} \Delta_{5,10}^{6,9} + \Delta_{1,14}^{2,13} = \Delta_{7,12}^{8,11} + \Delta_{3,16}^{4,15} = 0, \\ \Delta_{5,11}^{7,9} + \Delta_{1,15}^{3,13} = \Delta_{6,12}^{8,10} + \Delta_{2,16}^{4,14} = 0, \\ \Delta_{7,10}^{8,9} + \Delta_{5,12}^{6,11} + \Delta_{3,14}^{4,13} + \Delta_{1,16}^{2,15} = 0. \end{cases}$$

Therefore, the correspondence $u_i \mapsto \lambda u_i$ for $i = 1, \dots, 4$, and $u_5 \mapsto 2\lambda u_5$ extends by linearity to a derivation of $\mathfrak{n}_2$ for all λ. The correspondence $u_i \mapsto \lambda u_i$ for $i = 1, \dots, 4$, and $u_5 \mapsto \lambda^2 u_5$ extends to an automorphism if $\lambda \neq 0$.

Epilogue

In 1955, N. Jacobson proved in [11, Theorem 3] that any Lie algebra of characteristic zero with a nonsingular derivation is nilpotent. The author also noted that the validity of the converse was an open question. Two years later, J. Dixmier and W.G. Lister supplied in [5] a negative answer to the question by means of the algebra $\mathfrak{n}_1$ that we have revisited in Example 1. Every derivation of $\mathfrak{n}_1$ is nilpotent, so the elements of $\operatorname{Der}\mathfrak{n}_1$ are nilpotent maps, and therefore, $\operatorname{Der}\mathfrak{n}_1$ is a nilpotent Lie algebra. It can be also proved that $\operatorname{Aut}\mathfrak{n}_1$ is not a nilpotent group (see [15]). The existence of $\mathfrak{n}_1$ is the starting point of the study of the so called *characteristically nilpotent Lie algebras*, that is, Lie algebras in which any derivation is nilpotent. Over fields of characteristic zero, this class of algebras matches to the class of algebras in which every semisimple automorphism is of finite order (see [16, Theorem 3]) or the class of algebras in which the algebra of derivations is nilpotent (see [16, Theorem 1]).

Quasi-cyclic Lie algebras were introduced at [17] by G. Leger in 1963. A nilpotent Lie algebra $\mathfrak{n}$ is called quasi-cyclic (also known in the literature as homogenous) if $\mathfrak{n}$ has a subspace $\mathfrak{u}$ such that $\mathfrak{n}$ decomposes as the direct sum of subspaces $\mathfrak{u}^k = [\mathfrak{u}^k, \mathfrak{u}]$; in particular, quasi-cyclic algebras are $\mathbb{N}$-graded. Free nilpotent Lie algebras $\mathfrak{n}_{d,t}$ are examples of this type of algebras. It is not hard to see that a nilpotent Lie algebra $\mathfrak{n} \cong \frac{\mathfrak{n}_{d,t}}{\mathfrak{t}}$ is quasi-cyclic iff $\mathfrak{t}$ is a homogeneous ideal of $\mathfrak{n}_{d,t}$. In fact, quasi-cyclic Lie algebras are the class of nilpotent Lie algebras that contain a minimal set of generators $\{e_1, \dots, e_d\}$, so d is the type of $\mathfrak{n}$, such that the correspondence $e_i \mapsto e_i$ extends to a derivation of $\mathfrak{n}$ according to [13, Corollary 1]. By reviewing $\operatorname{Ker}\theta_{\mathfrak{n}_i}$, we conclude that $\mathfrak{n}_2$ is quasi-cyclic, but $\mathfrak{n}_1$ and $\mathfrak{n}_3$ are not.

An automorphism of a real Lie algebra is called *expanding automorphism* if it is a semisimple automorphism whose eigenvalues are all greater than 1 in absolute value. In 1970, J. L. Dyer states in [6] that quasi-cyclic Lie algebras admits expanding automorphisms, the converse is false. The Lie algebra $\mathfrak{n}_3$ provides a counterexample: according to Example 3, $\mathfrak{n}_3$ admits expanding automorphisms, but it is not quasi-cyclic. The algebra $\mathfrak{n}_3$ has been introduced in [17] and [13] to illustrate the results therein. The latter paper includes the characterization of (real) quasi-cyclic Lie algebras as those algebras that admit *grading automorphisms*.

As it is noted by J. Scheuneman in [22, Section 1], "*(...) the Lie algebra of a simply transitive group of affine motions of* $\mathbb{R}^n$ *has an affine structure (...)"*. The main result in this paper is that each 3-step nilpotent Lie algebra has an complete (also known as transitive) *affine structure*. Therefore, any homomorphic image $\frac{\mathfrak{n}_{d,3}}{\mathfrak{t}}$ can be endowed with a such structure. The notion of (complete or transitive) *affine structure* on Lie algebras is equivalent to that of (complete) *left-symmetric structure* (see [7] and references therein). In fact any positively $\mathbb{Z}$-graded real Lie algebra admits a complete left symmetric structure according to [2, Theorem 3.1]. Therefore, any quasi-cyclic nilpotent Lie algebra admits a complete left symmetric structure. There is an interesting interplay among gradings, expanding and hyperbolic automorphisms and affine structures.

It is not difficult to find recent research on derivations and automorphisms algebras and their applications. For nilpotent Lie algebras we point out [14, 18, 19]. So, this research area deserves to be considered.

Acknowledgements The authors are partially funded by grant MTM2017-83506-C2-1-P of Ministerio de Economía, Industria y Competititividad (Spain). The second-named author is supported by a predoctoral research grant of Universidad de La Rioja.

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