



A Non-stationary Analysis of Erlang Loss Model

Amit Kumar Singh^{1(✉)}, Dilip Senapati², Sujit Beborntta²,
and Nikhil Kumar Rajput¹

¹ Jawaharlal Nehru University, New Delhi, India
amit.scss@gmail.com

² Ravenshaw University, Cuttack, Odisha, India

Abstract. A complex issue in handling systems with continually changing processing demands is an intractable task. A more current example of these systems can be observed in wireless sensor networks and traffic-intensive IoT networks. Thus, an adaptive framework is desired which can handle the load and can also assist in enhancing the performance of the system. In this paper, our objective is to provide the non-stationary solution of Erlang loss queueing model where s servers can serve at most s jobs at a time. We have employed time-dependent perturbation theory to obtain the probability distribution of $M/M/s/s$ queueing model. The time-dependent arrival and service rates are assumed to be in sinusoidal form. The opted theory gives approximation for probability distribution correct up to first and second order. The result shows that first- and second-order approximations provide better approximation than the existing ones.

Keywords: Erlang loss queueing model · Non-stationary queues · Blocking probability · Time-dependent perturbation theory

1 Introduction

Queueing system is an important tool for designing and analyzing the systems ranging from IOT, blockchain technology to telecommunication systems [1–3]. Suitable queueing models have been designed for different systems. Erlang loss queueing model is applicable in many domains like telecommunication networks, transportation systems, etc. [5, 6]. Such systems are designed better and analyzed on the basis of their performance measures. In order to evaluate performance measures, the systems are assumed to be in a steady state. Based on this assumption, the systems are analyzed. The reason for this assumption is that stationary solutions are easy to obtain. However, in reality, the systems are not in steady

Amit Kumar Singh was a Ph.D. student at JNU, New Delhi.

state and steady-state solutions are far from reality. The arrival and service rates are not constant; rather, they are a function of time. For instance, the pattern of calls or vehicles on road is different in morning and in evening [7, 8]. The non-stationary solutions of the queueing models are difficult to obtain. So, the researchers are trying to obtain approximations [6, 7]. Several approximations have been found in the literature to approximate and analyze the non-stationary queueing systems. Some of these are simple stationary approximation (SSA), pointwise stationary approximation (PSA), modified offer load (MOL), etc [9–13]. The arrival and service rates are assumed to be in sinusoidal form to test these approximations. In this paper, we have opted time-dependent perturbation (TDP) theory, extensively used to solve numerous problems of quantum mechanics [13–15]. TDP is applied on the non-stationary Erlang loss model with time-varying arrival and service rates. The sinusoidal rates are generally used so that we can accurately estimate the performance of the approximation scheme, although this method can also be applied for general perturbed rates.

The rest of the chapter is as follows: After the introduction section, Sect. 2 presents a literature review of non-stationary queues. Section 3 provides Erlang loss model and its CR equations. Time-dependent perturbation theory is briefly discussed in Sect. 4. Section 5 presents numerical results. The last section is the conclusion.

2 Literature Review

Non-stationary queueing models developed for analyzing the systems are complex in nature [13]. Simulation techniques can predict the behavior of the queueing models but these techniques, for example, Monte Carlo simulation, take more computational power as well as time. People choose approximation methods based on steady-state results to predict the behavior of systems. These approximations are much easier and require less computation. Simple stationary approximation (SSA) is one of the techniques based on steady-state solution. The average rate is calculated over a time interval and inserted in steady-state formula. Mathematically, the average arrival rate (similarly average service rate) over a time of length T is given by:

$$\bar{\lambda} = \frac{1}{T} \int_0^T \lambda(t) dt$$

All Markovian queues can be evaluated by using these average arrival rates. However, SSA method ignores non-stationarity behavior and provides stationary solution for the number of jobs in the system. Green et al. [10] found that SSA gives a reasonable approximation if the arrival rate function varies by only 10% about its average arrival rate. Green et al. [10] proposed pointwise stationary approximation (PSA) method used to compute non-stationary behavior. PSA provides time-dependent behavior based on steady-state behavior of a stationary model. It uses arrival and/or service rate that prevails at the time at which we

want to describe the performance. For example, the blocking probability $P_B(t)$ which is the probability that no server is available is calculated approximately for $M(t)/M(t)/s/s$ queue as:

$$P_B(t) \approx \frac{\rho(t)^s / s!}{\sum_{i=0}^s \rho(t)^i / i!}$$

where $\rho(t) = \lambda(t) / \mu(t)$

The expected number of busy servers $E[S(t)]$ can be given approximately as

$$E[S(t)] \approx (1 - P_B(t))\rho(t)$$

Massey and Whitt [12] showed that PSA is asymptotically correct as arrival rate changes less rapidly. Modified offered load (MOL) model was introduced to approximate telephone traffic [12]. MOL is applicable to only Erlang loss models and works effectively if the blocking probability is small.

3 Erlang Loss Model

Consider a queueing system with s number of servers and with zero buffer size. That is, there can be at most s jobs in the system that are under service. The arrival pattern follows Poisson process with a non-negative time-varying rate $\lambda(t)$. The service of each job follows exponential distribution with rate $\mu(t)$. Let $p(n, t)$ be the probability of n jobs in the system at time t . The Chapman–Kolmogorov (CR) equations for $M(t)/M(t)/s/s$ queueing system [5] are given:

$$\begin{aligned} \frac{d}{dt}p(0, t) &= -\lambda(t)p(0, t) + \mu(t)p(1, t) \\ \frac{d}{dt}p(n, t) &= \lambda(t)p(n - 1, t) - (\lambda(t) + n\mu(t))p(n, t) + (n + 1)\mu(t)p(n + 1, t) \\ \frac{d}{dt}p(s, t) &= \lambda(t)p(s - 1, t) - s\mu(t)p(s, t) \end{aligned} \tag{1}$$

$0 < n \leq s$

The non-autonomous linear differential equation in matrix form of above set of CR equations can be written as:

$$\frac{dP(t)}{dt} = M(t)P(t) \tag{2}$$

where

$$P(t) = (p(0, t), p(1, t), \dots, p(s, t))^T$$

and $M(t)$ be a matrix obtained from the set of differential equations. The solution of Eq. (2), if $M(t)$ is constant ($M(t) = M_0$), can easily be obtained by matrix

differential equation with constant coefficients. If the initial condition $p(n, 0)$ is known, then the solution is given by

$$P(t) = \sum_{i=1}^{s+1} a_i e^{-b_i t} m_i \tag{3}$$

where b_i is the eigenvalue and m_i is the corresponding eigenvector of M_0 and coefficients a_i can be easily obtained by the initial probability values. However, if the entries of matrix $M(t)$ in Eq. (2) are time-dependent, then it is complex to obtain the exact solution. For solving the differential Eq. (2), researchers proposed a different method. For the support of their method, researchers assumed that the time-varying rates are periodic in sinusoidal form [12]. But, these methods can also be applied on general rates. In the light of this assumption [13], we assume that the time-varying rates can be split into two parts: a constant and some perturbed part. The perturbation part of the rates is in sinusoidal form. Formally, the rates can be written as

$$\begin{aligned} \lambda(t) &= \lambda + \eta_\lambda \sin(\tau_\lambda t + \phi_\lambda) \\ \mu(t) &= \mu + \eta_\mu \sin(\tau_\mu t + \phi_\mu) \end{aligned}$$

The matrix $M(t)$ thus can be seen as a sum of two matrices M_0 and $X(t)$ where M_0 is constant and $X(t)$ is sinusoidal form.

4 Time-Dependent Perturbation (TDP) Theory

The matrix differential equations with time-dependent coefficients are generally found in quantum mechanics, for instance, the Schrodingers wave equation with time-dependent Hamiltonian [14–16]. TDP theory has been very successful in computing the solution of such problems. Equation (2) can be rewritten where the coefficient matrix is the sum of constant and perturbed part, i.e.,

$$\frac{dP(t)}{dt} = (M_0 + X(t))P(t), \quad P(0) = p_0. \tag{4}$$

In case of $M(t)$ as constant, the solution of Eq. (4) as discussed in the previous section is given by

$$P(t) = \sum_{i=1}^{s+1} a_n e^{-b_i t} m_i \tag{5}$$

In a similar fashion, if the coefficient matrix as given in Eq. (4) is time-dependent, the unknown coefficients a_n must be $a(n, t)$. The general solution of Eq. (4) can be written as

$$P(t) = \sum_{i=1}^{s+1} a(i, t) e^{-b_i t} m_i \tag{6}$$

Substituting the value of $P(t)$ given in Eq. (6) into (4) and simplifying it, we have

$$\sum_i \frac{da(i,t)}{dt} e^{-b_i t} m_i = c \sum_i a(i,t) e^{-b_i t} m_i \tag{7}$$

If m_j' is left eigenvector corresponding to eigenvalue b_j and m_i is right eigenvector corresponding to b_i , then $m_j' m_i \neq 0$, if $i = j$, otherwise 0 [17]. So, we multiply reduced Eq. (7) by m_j' from left-hand side

$$\begin{aligned} \frac{da(j,t)}{dt} &= c \sum_n m_j' X(t) m_i e^{(b_i - b_j)t} a(i,t) \\ &= c \sum_n a(i,t) X_{ij} e^{(b_i - b_j)t} \end{aligned} \tag{8}$$

where $X_{ij} = m_j' X(t) m_i$.

The term $a(i,t)$ can be expanded in the power of c as a perturbation approximation

$$a(n,t) = a(n,t)^{(0)} + c^1 a(n,t)^{(1)} + c^2 a(n,t)^{(2)} + \dots \tag{9}$$

Using this expression into Eq. (8) and on equating the coefficients of equal power of c , one can obtain

$$\frac{da_j^{(0)}(t)}{dt} = 0 \tag{10}$$

The first-order coefficients are evaluated by

$$\frac{da_j^{(1)}(t)}{dt} = \sum_i X_{ij} e^{(b_i - b_j)t} m_i a_i^{(0)} \tag{11}$$

Similarly, higher-order coefficients can be calculated using

$$\frac{da_j^{(r+1)}(t)}{dt} = \sum_i X_{ij} e^{(b_i - b_j)t} m_i a_i^{(r)}(t) \tag{12}$$

The zeroth-order coefficients $a(n,0)$ are obtained from initial condition

$$\zeta(0) = \sum_i a_i^{(0)} m_i$$

which is given in the problem. The first-order coefficients can be obtained by using zeroth-order coefficients. The higher-order coefficients can be obtained in the same manner [17–19].

5 Numerical Results

In this section, we implemented TDP method on Erlang loss model. The results obtained from TDP(correct up to first and second order), MOL and PSA are compared with the solution obtained by the ode45 function available in MATLAB ODE suit. The relative error RE of the existing and TDP methods is computed to get insights by

$$RE = \frac{\text{Value_obtained_by_ode45} - \text{Approx_value}}{\text{Value_obtained_by_ode45}}$$

We take $M(t)/M(t)/5/5$ queue as an instance to gain insight on how the solution based on TDP correct up to first and second order performs. We take arrival rate

$$\lambda(t) = 3 + 0.2 \sin(2t)$$

and service rate $\mu(t) = 4$. In Fig. 1, we can easily see that TDP (second-order approx.) performs better than the other approximations (PSA, MOL, TDP (first-order approx.)). Figure 2 shows the relative error of PSA, MOL, TDP (first-order and second-order approx). For better clarity, Fig. 3 shows only the relative error of only MOL and TDP (second-order approx). From Figs. 2 and 3, it can be easily seen that the TDP method outperforms the other existing methods.

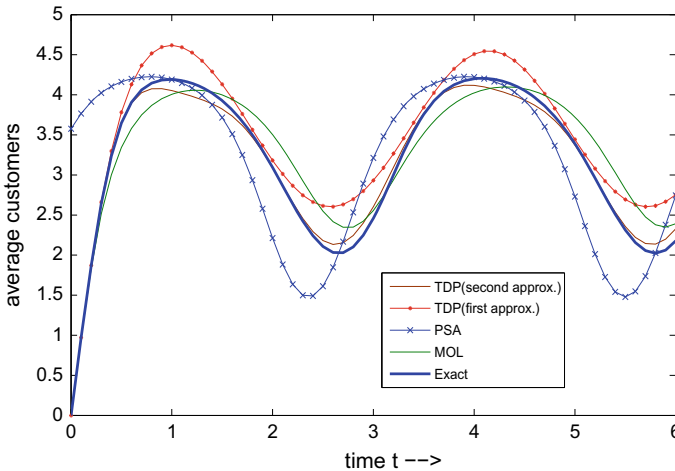


Fig. 1. Average number of customers in $M(t)/M/5/5$. ($\lambda = 10 + 7\sin(2t)$, $\mu = 2$)

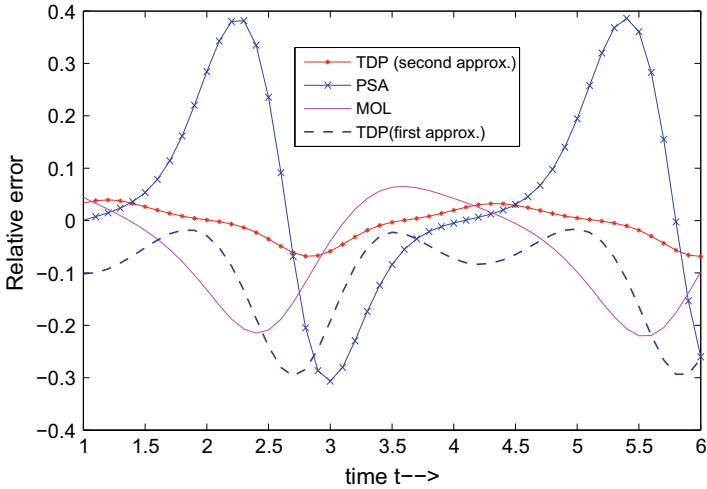


Fig. 2. Relative error of TDP(second approx.) ($\lambda = 10 + 7\sin(2t), \mu = 2$)

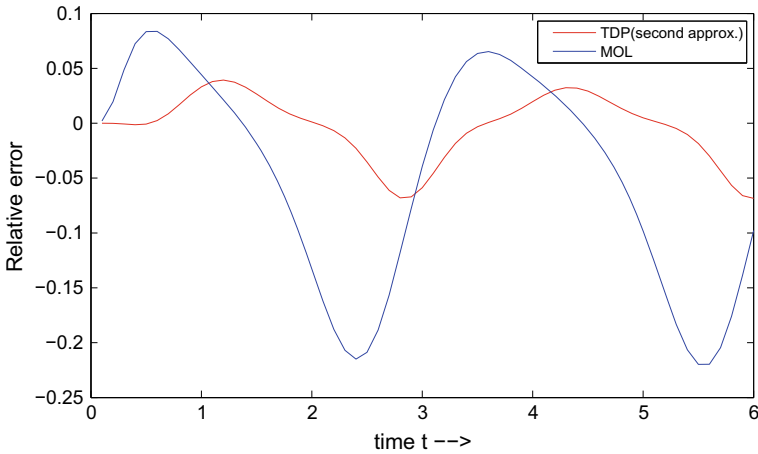


Fig. 3. Relative error of TDP(second approx.)($\lambda = 10 + 7\sin(2t), \mu = 2$)

6 Conclusion

In this chapter a time-dependent perturbation theory is used for computing probability distribution and blocking probability for non-stationary Erlang loss model. The scope of this study covers a plethora of systems with dynamic organization. The numerical results show that the opted theory outperforms the

existing approximations. The theory can be applied extensively to obtain approximations for various non-stationary queueing models.

References

1. Rivera D, Cruz-Piris L, Lopez-Civera G, de la Hoz E, Marsa-Maestre I (2015) Applying an unified access control for IoT-based intelligent agent systems. In: 2015 IEEE 8th international conference on service-oriented computing and applications (SOCA), pp 247–251
2. Said D, Cherkaoui S, Khoukhi L (2015) Multi-priority queuing for electric vehicles charging at public supply stations with price variation. *Wirel Commun Mobile Comput* 15(6):1049–1065
3. Yan H, Zhang Y, Pang Z, Da Xu L (2014) Superframe planning and access latency of slotted MAC for industrial WSN in IoT environment. *IEEE Trans Ind Inform* 10(2):1242–1251; Strielkina A, Uzun D, Kharchenko V (2017) Modelling of health-care IoT using the queueing theory. In: 2017 9th IEEE international conference on intelligent data acquisition and advanced computing systems: technology and applications (IDAACS), vol 2
4. Ozmen M, Gursoy MC (2015) Wireless throughput and energy efficiency with random arrivals and statistical queueing constraints. *IEEE Trans Inf Theory* 62(3):1375–1395
5. Shortle JF, Thompson JM, Gross D, Harris (2008) *Fundamentals of queueing theory*. Wiley, C. M
6. Abdalla N, Boucherie RJ (2002) Blocking probabilities in mobile communications networks with time-varying rates and redialing subscribers. *Ann Oper Res* 112:15–34
7. Worthington DJ, Wall AD (2007) Time-dependent analysis of virtual waiting time behaviour in discrete time queues. *Euro J Oper Res* 178(2):482–499
8. Pender J (2015) Nonstationary loss queues via cumulant moment approximations. *Prob Eng Inform Sci* 29(1):27–49
9. Alnowibet KA, Perros HG (2006) The nonstationary queue: a survey. In: Barria J (ed) *Communications and computer systems: a tribute to Professor Erol Gelenbe*, World Scientific
10. Green L, Kolesar P (1991) The pointwise stationary approximation for queues with nonstationary arrivals. *Manage Sci* 37(1):84–97
11. Alnowibet, Perros (2009) Nonstationary analysis of the loss queue and of queueing networks of loss queues. *Euro J Oper Res* 196:1015–1030
12. Massey WA, Whitt W (1997) An analysis of the modified offered-load approximation for the nonstationary loss model. *Ann Appl Prob* 4:1145–1160
13. Massey WA (2002) The analysis of queues with time-varying rates for telecommunication models. *Telecommun Syst* 21:173–204
14. Gottfried K, Yan TM (2013) *Quantum mechanics: fundamentals*. Springer Science & Business Media
15. Griffiths DJ (2016) *Introduction to quantum mechanics*. Cambridge University Press
16. Bransden BH, Joachain (2006) *Quantum mechanics*. Pearson Education Ltd., C. J

17. Singh AK(2015) Performance modeling of communication networks: power law behavior in stationary and non-stationary environments, thesis. Thesis submitted to JNU, New Delhi
18. Dilip S, Karmeshu (2016) Generation of cubic power-law for high frequency intraday returns: maximum Tsallis entropy framework. *Digital Signal Process* 48:276–284
19. Tanmay M, Singh AK, Senapati D (2019) Performance evaluation of wireless communication systems over weibull/ q -lognormal shadowed fading using Tsallis' entropy framework. *Wirel Pers Commun* 106(2):789–803