Chapter 7
k-Planar Graphs

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Abstract A topological graph is called *k-planar*, for $k \geq 0$, if each edge has at most k crossings; hence, by definition, 0-planar topological graphs are plane. An abstract graph is called *k-planar* if it is isomorphic to a k-planar topological graph, i.e., if it can be drawn on the plane with at most k crossings per edge. While planar and 1-planar graphs have been extensively studied in the literature and their structure has been well understood, this is not the case for k-planar graphs, with $k \geq 2$. These graphs have a more complex structure, which is significantly more difficult to comprehend. As an example, we mention that tight (possibly up to additive constants) bounds on the edge-density of k-planar graphs are only known for small values of k (that is, for $k \in \{0, 1, 2, 3, 4\}$), even though their existence yields corresponding improvements on the leading constant of the lower bound on the number of crossings of a graph, provided by the well-known Crossing Lemma. In this chapter, we focus on k-planar graphs, with $k \geq 2$, and review the known combinatorial and algorithmic results from the literature. We also identify several interesting open problems in the field.

7.1 Introduction

A topological graph is *k-planar*, for $k \geq 0$, if each edge has at most k crossings; for an example, refer to Fig. 7.1a which depicts a 3-planar topological graph that is not 2-planar (to see the latter observe that, e.g., the bold edge is crossed three times). Accordingly, a graph is *k-planar*, if it is isomorphic to the underlying abstract graph of a k-planar topological graph, i.e., if it can be drawn on the plane with at most k crossings per edge. Equivalently, one can define k-planar graphs in terms of the following forbidden configuration: "an edge is crossed by $k + 1$ or more edges"; for example, Fig. 7.1b shows a crossing configuration that is forbidden in a 3-planar topological graph (since the bold-drawn edge is crossed four times).

Observe that, by definition, every 0-planar topological graph is in fact a plane graph. In addition, every k-planar graph is also $(k + 1)$-planar, which naturally

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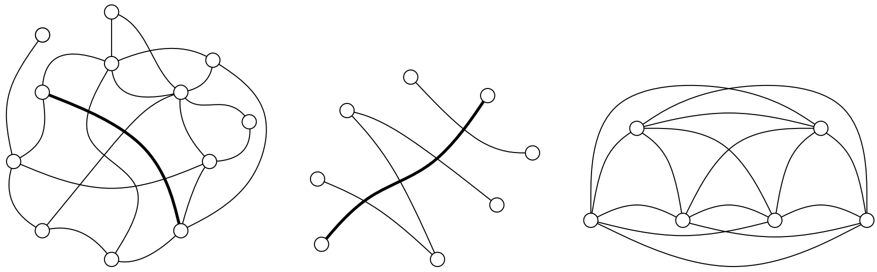


Fig. 7.1 Illustration of: (**a**) a 3-planar topological graph, (**b**) a crossing configuration that is forbidden in a 3-planar topological graph, and of (**c**) the complete graph on six vertices K_6 as 1-planar topological graph

defines a hierarchy in k-planarity. The results in the literature are, in a sense, inversely proportional to this hierarchy. In particular, there is a tremendous amount of results for planar graphs; to mention only some of the most important landmarks, we refer to the characterization of planar graphs in terms of forbidden minors, due to Kuratowski [1], to the existence of linear-time algorithms to test graph planarity [2–4], to the Four-Color Theorem [5, 6], and to the Euler's polyhedron formula (see, e.g., [7]), which can be used to show that n-vertex planar graphs have at most $3n - 6$ edges.

The class of 1-planar graphs, which is the next in the hierarchy, has also been extensively studied in the literature. Early works date back to 1960s [8] and continued over the years; see, e.g., [9–17]. More precisely, the class of 1-planar graphs was initially introduced by Ringel [8], who proved, as a generalization of the Four-Color Theorem, that every 1-planar graph has chromatic number at most 7 and conjectured that this bound could be lowered to 6. Ringel's conjecture was settled by Borodin [10], who showed that indeed the chromatic number of 1-planar graphs is at most 6 and that this bound is tight, as for example, the complete graph on six vertices (which is 1-planar; see, e.g., Fig. 7.1c) requires 6 colors. It is also worth mentioning that, in contrast to the existence of linear-time algorithms to test whether a graph is planar, testing whether a graph is 1-planar is an NP-complete problem [18, 19], and remains NP-complete even if the input graph has bounded bandwidth, pathwidth, or treewidth [20], or it can be obtained from a planar graph by adding a single edge [21]. Efficient recognition algorithms are known only for subclasses of 1-planar graphs; see, e.g., [22, 23]. From a graph drawing perspective, notable is also a result of Thomassen [24], who characterized the 1-planar topological graphs that admit the corresponding 1-planar embedding-preserving straight-line drawings in terms of two forbidden configurations (see also [25]). For a survey on 1-planarity, the reader is referred to [26].

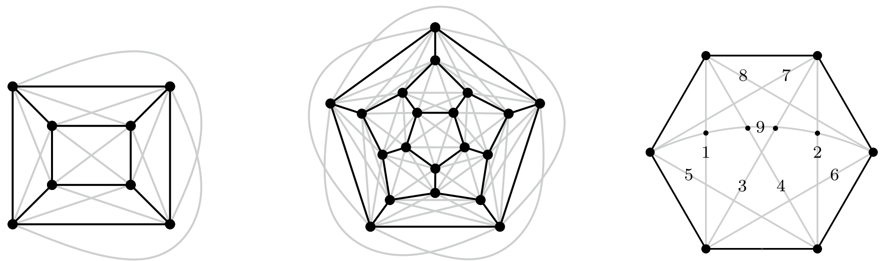
An immediate observation emerging from this short overview is that planar and 1-planar graphs have been extensively studied in the literature and their structure has been well understood. However, this is not the case for k-planar graphs, with

$k \geq 2$, as the results for these graphs are significantly fewer. In the remainder of this chapter, we review the most important combinatorial and algorithmic results from the literature and we also identify several interesting open problems in the field.

7.2 Examples of k-Planar Graphs with High Density

In this section, we present examples of k-planar graphs with high edge-density, for different values of $k \geq 1$. Intuitively, the number of edges of a k-planar graph cannot be quadratic with respect to the number of its vertices, when k is fixed, since the number of crossings along each edge is fixed. For more details on the edge-density of k-planar graphs, refer to Sect. 7.3.

We start our discussion with the case $k = 1$, as the corresponding constructions for $k = 2$ and $k = 3$, that we will present, are similar. In the literature, the 1-planar graphs with a fixed number of vertices and maximum edge-density are referred to as *optimal* 1-planar graphs and they have been characterized [16, 27] as the graphs obtained by drawing a pair of crossing edges in the interior of each face of a 3-connected *quadrangulation*, i.e., of a planar graph whose faces are all of length four; see Fig. 7.2a for example. Since by Euler's polyhedron formula, a quadrangulation with n vertices has exactly $2(n - 2)$ edges and $n - 2$ faces, it follows that the graphs obtained by the aforementioned procedure have exactly $2(n - 2) + 2 \cdot (n - 2) = 4n - 8$ edges. In addition, they contain neither parallel edges nor self-loops, since the underlying quadrangulations are 3-connected. The reader is also referred to the work by Brinkmann et al. [28], who describe how one can generate all 3-connected quadrangulations with n vertices by means of two different operations, and to the



(a) (b) (c)

Fig. 7.2 Illustration of: (**a**) a 1-planar topological graph with $n = 8$ vertices and $4n - 8 = 24$ edges obtained by adding a pair of crossing in the interior of each face of the cube graph, (**b**) a 2-planar topological graph with $n = 20$ vertices and $5n - 10 = 90$ edges obtained by adding a pentagram in the interior of each face of the dodecahedral graph, and of (**c**) the fact that if one draws nine edges in the interior of a face of length six, then at least one edge will have inevitably four crossings work by Brandenburg [23], who gives a linear-time recognition algorithm for optimal 1-planar graphs.

The characterization of the optimal 2-planar graphs, i.e., those 2-planar graphs with maximum density, is similar; see [29]. These graphs are obtained by drawing a *pentagram* (that is, five mutually crossing edges) in the interior of each face of a 3-connected *pentagulation*, i.e., of a planar graph whose faces are all of length five; see Fig. 7.2b for an example. Similarly to the case of optimal 1-planar graphs, one can show that a pentagulation with n vertices has exactly $5(n-2)/3$ edges and $2(n-2)/3$ faces. This directly implies that the graphs obtained by drawing a pentagram in the interior of each face of a 3-connected pentagulation with n vertices have exactly $5(n-2)/3 + 5 \cdot 2(n-2)/3 = 5n - 10$ edges. Note that Hasheminezhad et al. [30], describe eight different operations to generate all pentagulations with n vertices (and therefore, all different optimal 2-planar graphs). However, to the best of our knowledge, there is no polynomial time algorithm to recognize optimal 2-planar graphs (note that in general it is NP-complete to decide whether a graph is k-planar [31]). This brings us to the first open problem of this section.

Open Problem 1 *Can optimal* 2*-planar graphs be recognized in polynomial time?*

At this point, we can make an observation. The densest n-vertex 0-planar graphs have $3n-6$ edges (as they are maximal planar). Accordingly, the densest n-vertex 1- and 2-planar graphs (that is, the optimal 1- and 2-planar graphs, respectively) have $4n-8$ and $5n-10$ edges, respectively. So, one would naturally expect that the densest n-vertex 3-planar graphs have $6n-12$ edges. Also, following the corresponding constructive approaches for 1- and 2-planar graphs that we gave above, one would also expect that examples of densest 3-planar graphs can be derived by drawing nine edges in the interior of each face of a 3-connected planar graph whose faces are all of length six. However, neither of the two expected properties hold. Indeed, it is not difficult to see that if one draws nine edges in the interior of a face of length six, then some of the drawn edges will inevitably have four crossings, which is forbidden by 3-planarity (see, e.g., Fig. 7.2c). In addition, there does not exist 3-connected planar graphs whose faces are all of length six, as otherwise the dual[1] of such a graph would be a 6-regular planar graph, which contradicts Euler's polyhedron formula.

In fact, as we will shortly see in Sect. 7.3, an n-vertex 3-planar graph can have at most $5.5n - O(1)$ edges, while the bound of $6n-12$ edges holds for n-vertex 4-planar graphs. However, by appropriately adjusting the constructions that we gave earlier for optimal 1- and 2-planar graphs, we can still derive 3- and 4-planar graphs that have $5.5n - O(1)$ and $6n - O(1)$ edges, respectively. We describe two different constructions for the case of 3-planar graphs; the corresponding ones for 4-planar graphs are analogous. Since there do not exist 3-connected planar graphs whose faces are all of length six, a first idea is to start from a 3-connected planar graph, whose faces are all of length six except for few that have length five.

[1] Recall that the *dual* G^* of a plane graph G is defined as follows: G^* has a vertex for each face of G and for every two vertices of G^* there is an edge connecting them if and only if there corresponding faces of G share an edge.

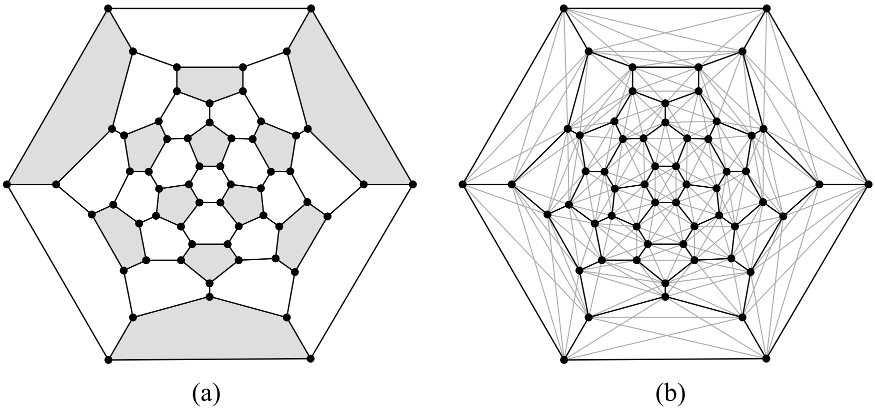


Fig. 7.3 Illustration of: **a** the football graph (also known as truncated icosahedron), which has twenty faces of length six, twelve faces of length five (gray colored), 60 vertices, and 90 edges, and **b** the 3-planar topological graph with $n = 60$ vertices and $5.5n - 20 = 310$ edges obtained by adding eight edges in the interior of each face of length six, and five edges in the interior of each face of length five of the football graph

In fact, it is not difficult to construct 3-connected planar graphs, whose faces are all of length six except for exactly twelve that have length five; these graphs are also known as *fullerene*. Figure 7.3a shows such an example with 60 vertices and 90 edges; this graph is known as *football graph* or *truncated icosahedron*. By Euler's polyhedron formula, an n-vertex fullerene graph has exactly $3n/4$ edges and $n/2 + 2$ faces. It follows that if we add eight edges in the interior of each face of length six, and five edges in the interior of each face of length five, then the resulting graph is 3-planar with exactly $3n/4 + 8 \cdot (n/2 + 2 - 12) + 5 \cdot 12 = 5.5n - 20$ edges. The corresponding bound for 4-planar graphs is derived by adding nine (instead of eight) edges in the interior of each face of length six.

A slightly improved lower bound construction is due to Pach et al. [32]. The idea here is to relax the 3-connectivity constraint in the underlying planar graph. On one hand, this implies that we will not be able to add all edges in the faces of the underlying planar graph, since parallel edges will be inevitably introduced. On the other hand, by relaxing the 3-connectivity constraint, the construction of a planar graph whose faces are all of length six is possible. The construction is illustrated in Fig. 7.4. By identifying the topmost vertices with their corresponding bottommost ones (that is, by wrapping the construction around a cylinder), the faces of the underlying planar graph (drawn in black in Fig. 7.4) are all of length six. However, in order to avoid introducing parallel edges, in each of the two faces corresponding to the bases of the cylinder, only six (instead of eight) edges can be drawn. Hence, the derived graph has $5.5n - 11 - 4 = 5.5n - 15$ edges. The corresponding bound for 4-planar graphs is $6n - 18$; see also [33]. In view of these two results, we state the following open problem.

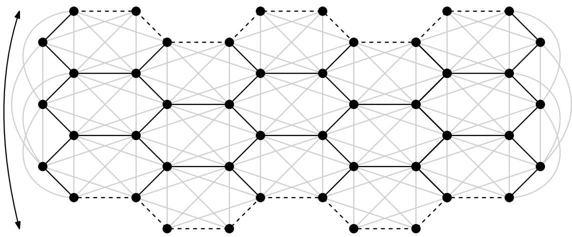


Fig. 7.4 Illustration of a 3-planar topological graph with $n = 36$ vertices and $5.5n - 15 = 183$ edges. The construction is wrapped around a cylinder by identifying the topmost vertices with their corresponding bottommost ones. To avoid introducing parallel edges, in each of the two faces corresponding to the bases of the cylinder only six edges can be drawn

Open Problem 2 *Is there a* 3*-planar (a* 4*-planar) graph with n vertices and more than* $5.5n - 15$ *(*$6n - 18$*, respectively) edges?*

For values of k greater than 3, Pach and Tóth [34] suggest a construction, which yields k-planar graphs with n vertices and approximately $1.92\sqrt{k}n$ edges; however, their construction is only limited to high values of k. Without entering all the details of the construction, we mention that the vertices of the constructed graphs are arranged on a $\sqrt{n} \times \sqrt{n}$ grid, whose points have been slightly perturbed so to be in general position (in order to avoid edge overlaps). Then, two vertices are connected by an edge if and only if their distance is at most d, where d is selected such that no edge is crossed more than k times. Indeed, the authors prove that if d is set to $\sqrt[4]{3k/2}(1 - o(1))$, then no edge is crossed more than k times and the graph has approximately $\frac{d^2\pi}{2}\ n = \sqrt{3k/8}\pi\ n \approx 1.92\sqrt{k}n$ edges.

7.3 Density Results and Their Implications to the Crossing Lemma

There exists several results for the edge-density of k-planar graphs for different values of k; for an overview refer to Table 7.1. As we will shortly see, the bounds for 1-, 2-, 3-, and 4-planar graphs have led to successive improvements on the upper bound on the edge-density of general k-planar graphs, from $4.108\sqrt{k}n$ [34], to $3.95\sqrt{k}n$ [32] and to $3.81\sqrt{k}n$ [33], and on the leading constant of the lower bound on the number of crossings of a graph, provided by the well-known Crossing Lemma, from $\frac{1}{100} = 0.01$ [35, 36] to $\frac{1}{64} \approx 0.0156$ [7], to $\frac{1}{33.75} \approx 0.0296$ [34], to $\frac{1}{31.1} \approx 0.0322$ [32], to $\frac{1}{29} \approx 0.0345$ [33]. In the following, we present three different techniques for obtaining the upper bounds on the number of edges of 1-, 2-, 3-, and 4-planar graphs.

In Sect. 7.2, we have already explained how to construct 1- and 2-planar graphs with $4n - 8$ and $5n - 10$ edges, respectively. We now show that these two bounds are also upper bounds on the number of edges of 1- and 2-planar graphs, respectively;

Table 7.1 Bounds on the number of edges of k-planar graphs with n vertices for different values of k; the ones marked with an asterisk $(*)$ are asymptotically tight; the remaining ones are either tight or tight up to small additive constants (recall the constructions from Sect. 7.2)

Model	General		Bipartite	
	Bound	Ref.	Bound	Ref.
1-planar:	$4n - 8$	[34]	$3n - 8$	[37]
2-planar:	$5n - 10$	[34]	$3.5n - 12$	[38]
3-planar:	$5.5n - 10.5$	[29, 32]	–	–
4-planar:	$6n - 12$	[33]	–	–
k-planar:	$3.81\sqrt{k}n$ *	[33]	$3.005\sqrt{k}n$ *	[38]

we sketch the proof given by Pach and Tóth [34], which holds for k-planar graphs with $k \in \{1, 2, 3, 4\}$. However, while for $k = 1$ and $k = 2$ the obtained bounds are tight, for the corresponding bounds for $k = 3$ and $k = 4$ there exist improvements, as we will see later in this section.

Consider a k-planar n-vertex graph G with $k \in \{1, 2, 3, 4\}$ and denote by G_p a spanning subgraph of G with the largest number of edges, called *maximal-planar substructure*, such that in the drawing of G_p (that is inherited from the one of G) no two edges cross each other. With a slight abuse of notation, let $G - G_p$ be the graph obtained from G by removing only the edges of G_p and let e be an edge of $G - G_p$. Since G_p is maximal, edge e must cross at least one edge of G_p. The part of edge e between a vertex of e and the nearest crossing with an edge of G_p is referred to as *half-edge*; see Fig. 7.5 for an illustration. Clearly, the number of edges of $G - G_p$ equals half of the number of half-edges, since each edge of $G - G_p$ contains two half-edges. By k-planarity, each half-edge *is contained in* a face f of G_p and crosses at most $k - 1$ other half-edges (and a boundary edge of f).

The crucial part in the proof by Pach and Tóth [34], is the following upper bound on the number of half-edges, denoted by $h(f)$, contained in a face f of G_p, whose boundary is connected and consists of $|f| \geq 3$ edges

$$h(f) \leq (|f| - 2)(k + 1) - 1. \tag{7.1}$$

Using this upper bound, the number of half-edges of a general face f (i.e., whose boundary is not necessarily connected) can be related to the number of triangles in a triangulation of f, denoted by $t(f)$, and to the number of boundary edges of f, denoted by $|f|$, as follows:

$$h(f) \leq t(f)k + |f| - 3. \tag{7.2}$$

In fact, if the boundary of f is connected, then the number of triangles in a triangulation of f is $|f| - 2$, that is, $t(f) = |f| - 2$. Hence, Eq. 7.2, directly follows from Eq. 7.1. On the other hand, if the boundary of f is not connected, then $t(f) \geq |f|$ holds. In this case, the bound follows by the observation that the number of half-edges

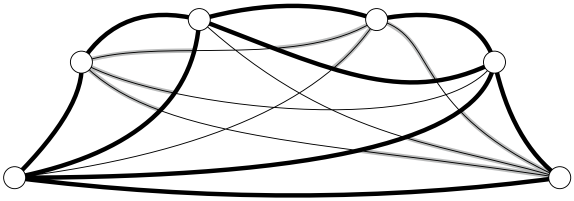


Fig. 7.5 Illustration of a topological 4-planar graph G. Its maximal-planar substructure G_p is drawn bold. The half-edges contained in the three faces of G_p incident to its unbounded face are highlighted in gray

contained in f cannot be more than $k|f|$, because, by k-planarity, every boundary edge of f has at most k crossings. Hence, $h(f) \leq |f|k \leq t(f)k + |f| - 3$.

Since each edge of $G - G_p$ contains two half-edges, it follows by Eq. 7.2, that the number of edges of $G - G_p$ is at most

$$\frac{1}{2} \sum_{f \in F_p} (t(f)k + |f| - 3), \tag{7.3}$$

where F_p denotes the set of faces of G_p. On the other hand, since in order to triangulate a face f of G_p, one needs at least $|f| - 3$ edges, it follows that the number of edges of G_p is at most

$$3n - 6 - \sum_{f \in F_p} (|f| - 3). \tag{7.4}$$

Combining Eqs. 7.3 and 7.4, with the fact in a triangulation of G_p there exist in total $2n - 4$ triangles (that is, $\sum_{f \in F_p} |f| = 2n - 4$), it follows that the total number of edges of G is at most

$$3n - 6 + \frac{1}{2} \sum_{f \in F_p} (t(f)k - (|f| - 3)) \leq 3n - 6 + (n-2)k = (k+3)(n-2).$$

We summarize this bound in the following theorem.

Theorem 1 (Pach and Tóth [34]) *For* $k \in \{1, 2, 3, 4\}$*, a* k*-planar graph with* n *vertices has at most* $(k+3)(n-2)$ *edges.*

To derive the bound of $5.5n - 11$ edges on the edge-density of 3-planar graphs with n vertices, Pach et al. [32], propose a similar proof as the one we described above, which is, however, more technical. Here, we describe an alternative proof suggested by Bekos et al. [39], that is based on structural properties of these graphs and also holds for multigraphs containing neither homotopic parallel edges nor homotopic self-loops. Note that a similar proof has been proposed by Bae et al. [40], to derive the upper bound of $5n - 10$ edges on the edge-density of gap-planar graphs.

Among all possible 3-planar graphs with n vertices and maximum density, Bekos et al. [39], choose one, called *crossing-minimal*, with the following two properties:

(i) its maximal-planar substructure has maximum number of edges among all possible maximal-planar substructures of all 3-planar graphs with n vertices and maximum density, and
(ii) the number of crossings is minimum over all corresponding 3-planar such graphs subject to (i).

With a slightly technical proof, it can be proved that the maximal-planar substructure G_p of a crossing-minimal 3-planar graph G with n vertices and maximum density is a triangulation. Hence, the number of edges of G_p is exactly $3n-6$. It follows that in order to count the number of edges of $G - G_p$, it suffices to count the total number of half-edges contained in the faces of G_p.

By 3-planarity, at most three half-edges are contained in each (triangular) face of G_p. Now, observe that if each face of G_p contained exactly three half-edges, then $G - G_p$ would have $3/2(2n-4) = 3n-6$ edges, since G_p has exactly $2n-4$ triangular faces. This would imply that the total number of edges of G is $6n-12$, which, however, is an overestimation. To adjust the bound, Bekos et al. [39], show that each face of G_p containing exactly three half-edges can be uniquely associated to a neighboring face of G_p containing at most two half-edges.

To see this, consider a face $\langle v_1, v_2, v_3 \rangle$ of G_p. We say that this face is of *type* (τ_1, τ_2, τ_3) if and only if for each $i = 1, 2, 3$ vertex v_i is an endvertex of τ_i half-edges contained in it. Without loss of generality, we may further assume that $\tau_1 \geq \tau_2 \geq \tau_3$. Here, we only describe how the association is performed when $\langle v_1, v_2, v_3 \rangle$ is of type $(2, 1, 0)$; the $(3, 0, 0)$ case is slightly more technical, while the $(1, 1, 1)$ case cannot occur due to the fact that G is of maximum density (for more details refer to [39]). Since v_2 is the vertex of one half-edge contained in $\langle v_1, v_2, v_3 \rangle$, edge (v_1, v_3) is crossed at least once. This allows the association of $\langle v_1, v_2, v_3 \rangle$ with the triangular face T of G_p neighboring $\langle v_1, v_2, v_3 \rangle$ along (v_1, v_3). More precisely, since the half-edge contained in $\langle v_1, v_2, v_3 \rangle$ that is incident to v_2 has three crossings in $\langle v_1, v_2, v_3 \rangle$, it is clear that there exist no half-edge contained in T having as endpoint either v_1 or v_3. In particular, T may contain at most one additional half-edge incident to the vertex of T that is different from v_1 and v_3, which implies that T contains at most two half-edges, as desired.

From the above association, it follows that if we denote by t_i the number of triangular faces of G_p containing exactly i half-edges, $0 \leq i \leq 3$, then $t_3 \leq t_0 + t_1 + t_2$. This implies that $t_3 \leq (2n-4)/2 = n-2$, since the number of faces of G_p is $2n-4$. Hence, the number of edges of $G - G_p$ are

$$
\begin{aligned}
(t_1 + 2t_2 + 3t_3)/2 &= (t_1 + t_2 + t_3) + (t_3 - t_1)/2 \\
&\leq (2n - 4 - t_0) + t_3/2 \\
&\leq 2n - 4 + (n-2)/2 \\
&\leq 5/4(2n-4).
\end{aligned}
$$

So, the total number of edges of G are

$$3n - 6 + 5(2n - 4)/4 = 11n/2 - 11.$$

Note that in [29], it is shown that the aforementioned upper bound can be achieved only if G is a multi-graph containing parallel edges or self-loops. We summarize the discussion above in the following theorem.

Theorem 2 (Bekos et al. [29, 39]) *A* 3*-planar graph with n vertices has at most* $5.5n - 10.5$ *edges.*

We conclude this section by briefly discussing the case $k = 4$. In this case, the best-known upper bound is due to Ackerman [33], who employed a charging technique to show that a 4-planar graph $G = (V, E)$ with n vertices cannot have more than $6n - 12$ edges. According to this technique, graph G is first *planarized*, that is, it is transformed into a plane graph $G' = (V', E')$ by replacing each crossing of 4-planar drawing of G with a dummy vertex. Denote by F' the set of faces of G' and for a face $f \in F'$ let $V(f)$ be the set of non-dummy vertices on the boundary of f. Recall that by Euler's polyhedron formula, $|V'| + |F'| - |E'| = 2$ holds. Initially, each face $f \in F'$ is assigned a charge equal to $|f| + |V(f)| - 4$. Therefore, the sum of the charges over all faces of G' is

$$\begin{aligned}
\sum_{f \in F'} (|f| + |V(f)| - 4) &= 2|E'| + \sum_{u \in V} \deg(u) - 4|F'| \\
&= 2|E'| + \sum_{u \in V'} \deg(u) - \sum_{u \in V' - V} \deg(u) - 4|F'| \\
&= 2|E'| + 2|E'| - 4(|V'| - n) - 4|F'| \\
&= 4n + 4(|E'| - |V'| - |F'|) \\
&= 4n - 8.
\end{aligned}$$

In subsequent steps, the charge is redistributed such that eventually the charge of each face of G' is nonnegative and the charge of each non-dummy vertex $u \in V$ is $\deg(u)/3$. Then, the upper bound on the number of edges of G is derived as follows:

$$\frac{2}{3}|E| = \sum_{u \in V} \deg(u)/3 \leq 4n - 8 \Rightarrow |E| \leq 6n - 12.$$

We summarize this bound in the following theorem.

Theorem 3 (Ackerman [33]) *A* 4*-planar graph with n vertices has at most* $6n - 12$ *edges.*

In view of the above results, we state the following two open problems.

Open Problem 3 *What is the maximum number of edges of a* 5*-planar graph with* n *vertices? In particular, does there exist a* 5*-planar graph with* n *vertices and more than* $6.33n - O(1)$ *edges?*

Open Problem 4 *What is the maximum number of edges of a bipartite* 3*-planar graph with* n *vertices?*

Note that an answer to Open Problem 3 may yield to a further improvement on the leading constant of the lower bound on the number of crossings of a graph, provided by the Crossing Lemma, as will see in the next section. On the other hand, an answer to Open Problem 4 may yield to an improvement on the leading constant of the corresponding lower bound for bipartite graphs.

7.3.1 Two Important Implications

In the following, we describe two important implications of the currently best-known upper bound on the edge-density of 4-planar graphs. The first one is on the well-known Crossing Lemma, which provides a lower bound on the number of crossings of a graph G, denoted by $\mathrm{cr}(G)$. The roots of this result date back to 1973, when Erdős and Guy conjectured that there exists a positive constant c such that, if G has at least a certain number of edges, then

$$\mathrm{cr}(G) \geq c \cdot \frac{m^3}{n^2}.$$

The first proofs were by Leighton [36] and by Ajtai et al. [35], who independently answered in affirmative the conjecture, when the leading constant c is 0.01. An improvement on the leading constant from 0.01 to $\frac{1}{64} \approx 0.0156$ was presented in [7]. The main ingredient in the proof is a simple probabilistic argument, which later was reused by Pach and Tóth [34], by Pach et al. [32] and by Ackerman [33], to progressively further improve the leading constant to $\frac{1}{33.75} \approx 0.0296$, to $\frac{1}{31.1} \approx 0.0322$ and to $\frac{1}{29} \approx 0.0345$, respectively. The technique is summarized in the proof of the following theorem, which is a slightly weaker version of the corresponding one of [33], in order to avoid a rather technical part in the proof.

Theorem 4 (Ackerman [33]) *Let* G *be a graph with* $n \geq 3$ *vertices and* m *edges, such that* $m \geq \frac{141}{20}n$*. Then*

$$\mathrm{cr}(G) \geq \frac{2000}{59643} \cdot \frac{m^3}{n^2} \approx \frac{1}{29} \cdot \frac{m^3}{n^2}.$$

Proof The proof consists of two main steps. In the first step, a weaker lower bound on the number of crossings $\mathrm{cr}(G)$ of G is guaranteed by exploiting the bounds on the edge-density of 0-, 1-, 2-, 3-, and 4-planar graphs. Note that we assume that $m > 3n - 6$, as otherwise there is nothing to prove. Intuitively, the idea is the following. By Euler's polyhedron formula, it follows that if $m > 3n - 6$, then G has an edge crossed by at least one other edge. It follows from [32] that if $m > 4n - 8$ (if $m > 5n - 10$), then G has an edge crossed by at least two (by at least three, respectively) other edges. Also, by [32], we know that if $m > 5.5n - 11$, then G has an edge crossed by at least four other edges. Finally, it follows from [33], that if $m > 6n - 12$ then G has an edge crossed by at least five other edges. We obtain by induction on the number of edges of G that $\mathrm{cr}(G)$ is at least

$$m - (3n - 6) + m - (4n - 8) + m - (5n - 10) + m - (5.5n - 11) + m - (6n - 12).$$

Hence

$$\mathrm{cr}(G) \geq 5m - \frac{47}{2}(n - 2). \tag{7.5}$$

In the second step, the aforementioned lower bound is used in a probabilistic argument. Consider a drawing of G with $\mathrm{cr}(G)$ crossings and let $p = \frac{141n}{20m} \leq 1$. Next, construct a random subgraph H_p of G as follows. Choose independently every vertex of G with probability p, and denote by H_p the subgraph of G induced by the chosen vertices. Let also n_p, m_p, and c_p be the random variables corresponding to the number of vertices, of edges and of crossings of H_p. Then, it is not difficult to see that the expected values of these variables are as follows:

$$E(n_p) = p \cdot n \qquad E(m_p) = p^2 \cdot m \qquad E(c_p) = p^4 \cdot \mathrm{cr}(G).$$

By Eq. 7.5, it follows that

$$\mathrm{cr}(H_p) \geq 5m_p - \frac{47}{2}(n_p - 2). \tag{7.6}$$

By taking expectations in Eq. 7.6, and by the linearity of expectations, we have

$$p^4\mathrm{cr}(G) \geq 5p^2 m - \frac{47}{2}pn \qquad \Rightarrow \qquad \mathrm{cr}(G) \geq \frac{5m}{p^2} - \frac{47n}{2p^3}.$$

The proof follows by plugging $p = \frac{141n}{20m}$ (which is at most 1 by our assumption) to the last inequality, that is

$$\mathrm{cr}(G) \geq \frac{2000}{59643} \cdot \frac{m^3}{n^2}.$$

This concludes the proof. □

Note that, by exploiting properties of the crossing-free structure of G, Ackerman [33], presents a slightly improved leading constant that is exactly $\frac{1}{29}$ and holds when $m \geq 6.95n$; for details the interested reader is referred to [32], where the technique above has been used for the first time. We also note that, if one establishes an upper bound on the edge-density of 5-planar graphs (see Open Problem 3), then the second step of the proof of Theorem 4, will start with a lower bound on $\mathrm{cr}(G)$ that is different from the one of Eq. 7.5. This is expected to lead to a further improvement of the leading constant. Finally, we note that following the two steps presented in the proof of Theorem 4, and based on the corresponding upper bounds on the edge-density of bipartite 1-planar [37] and 2-planar graphs [38] (see also Table 7.1), Bekos et al. [38], have derived a different leading constant for the lower bound on the number of crossings for bipartite graphs, which is given in the following theorem.

Theorem 5 (Bekos et al. [38]) *Let G be a bipartite graph with $n \geq 3$ vertices and m edges, such that $m \geq \frac{17}{4}n$. Then*

$$\mathrm{cr}(G) \geq \frac{16}{289} \cdot \frac{m^3}{n^2} \approx \frac{1}{18.1} \cdot \frac{m^3}{n^2}.$$

The second implication that we will present in this section is on the edge-density of k-planar graphs, for general values of $k \geq 5$. As already stated, the bounds for 1-, 2-, 3-, and 4-planar graphs have led to successive improvements on the upper bound on the number of edges of general k-planar graphs, from $4.108\sqrt{k}n$ [34], to $3.95\sqrt{k}n$ [32] and to $3.81\sqrt{k}n$ [33].

Theorem 6 (Ackerman [33]) *Let G be a k-planar graph with $n \geq 3$ vertices and m edges, for some $k \geq 1$. Then*

$$m \leq \sqrt{\frac{29}{2}k}\, n \leq 3.81\sqrt{k}n.$$

Proof For $k \in \{1, 2, 3, 4\}$, the bounds of this theorem are weaker than the corresponding ones by Pach and Tóth [34], by Pach et al. [32] and by Ackerman [33] (see also Table 7.1). So, without loss of generality we can assume that $k > 4$. We can further assume that $m \geq 6.95n$, as otherwise there is nothing to prove. By [33], it follows that

$$\mathrm{cr}(G) \geq \frac{1}{29} \cdot \frac{m^3}{n^2}. \tag{7.7}$$

On the other hand, the fact that G is k-planar trivially implies that

$$\mathrm{cr}(G) \leq \frac{mk}{2}. \tag{7.8}$$

Combining Eqs. 7.7 and 7.8, we obtain that

$$\frac{1}{29} \cdot \frac{m^3}{n^2} \leq \frac{mk}{2} \quad \Rightarrow \quad m \leq \sqrt{\frac{29}{2}k}\, n \leq 3.81\sqrt{k}n.$$

This concludes the proof. □

With similar arguments, Bekos et al. [38], proved a slightly improved upper bound on the number of edges of bipartite k-planar graphs.

Theorem 7 (Bekos et al. [38]) *Let G be a k-planar bipartite graph with $n \geq 3$ vertices and m edges, for some $k \geq 1$. Then*

$$m \leq \frac{17}{8}\sqrt{2k}\, n \leq 3.006\sqrt{k}n.$$

7.4 Interesting Subclasses

In this section, we present results for some important subclasses of k-planar graphs, with $k \geq 2$. We start with the class of 2- and 3-planar graphs with n vertices and maximum density. These graphs are called *optimal* in the literature. The characterizations of optimal 2- and 3-planar graphs extend the corresponding one for optimal 1-planar graphs. Recall that optimal 1-planar graphs with n vertices have exactly $4n - 8$ edges and can be obtained by adding a pair of crossing edges in the interior of each face of an n-vertex quandragulation (see also Sect. 7.2). Analogously, optimal 2- and 3-planar graphs with n vertices have exactly $5n - 10$ and $5.5n - 11$ edges, respectively. The corresponding characterization for optimal 2-planar graphs, which also hold for multigraphs containing neither homotopic parallel edges nor homotopic self-loops, is as follows:

Theorem 8 (Bekos et al. [29]) *A graph G is optimal 2-planar if and only if G is isomorphic to the underlying abstract graph of a 2-planar topological graph H containing neither homotopic parallel edges nor homotopic self-loops, such that the graph induced by the uncrossed edges of H spans all vertices of H, and each of its faces has length five containing five mutually crossing edges in its interior in H.*

To obtain the aforementioned characterization, Bekos et al. [29], exploit several structural properties of a 2-planar topological graph H with n vertices that is isomorphic to G chosen as follows:

(i) the uncrossed edges of H are maximized overall 2-planar topological graphs with n vertices that are isomorhic to G, and
(ii) the number of crossings of H is minimized overall 2-planar topological graphs with n vertices that are isomorhic to G subject to (i).

Graph H has several interesting properties. Since H is 2-planar optimal, any edge that is crossed twice in H lies in the interior of a 5-cycle consisting explicitly of uncrossed edges. On the other hand, H cannot contain edges that are crossed exactly once, otherwise, the fact that H is optimal is led to a contradiction. Next, it can be shown that the graph induced by the uncrossed edges of H is connected. To see this, assume to the contrary that this graph has at least two connected components, say c_1 and c_2. Since H is connected, there exist edges connecting c_1 and c_2, which cannot be uncrossed. However, since these edges are crossed, they belong in the interior of a 5-cycle consisting explicitly of uncrossed edges, which implies that c_1 and c_2 cannot be distinct; a contradiction. To complete the characterization, it suffices to show that each face of the graph induced by the uncrossed edges of H has length five. In fact, faces of length one or two imply the presence of homotopic self-loops and homotopic parallel edges, respectively, which is not possible. A face of length four would contradict the fact that H is optimal, since, e.g., the edge that triangulates this face can be safely added to H without deviating its 2-planarity and without introducing homotopic parallel edges. With slightly more complicated arguments a face of length three can also be excluded, yielding the characterization.

The most intriguing open problem raised by the aforementioned characterization, besides the corresponding recognition question posed in Open Problem 1, is the following one, which is motivated by the fact that the proof of Theorem 8, depends on the choice of the initial topological graph H. So, it is natural to ask whether this dependency can be eliminated.

Open Problem 5 *Given an optimal 2-planar graph G, does there exist a 2-planar topological graph H whose underlying abstract graph is isomorphic to G, such that H does not have the structural properties of the characterization of Theorem 8?*

Note that the corresponding characterization of optimal 3-planar graphs is analogous to the one of Theorem 8 and its proof uses similar arguments.

Theorem 9 (Bekos et al. [29]) *A graph G is optimal 3-planar if and only if G is isomorphic to the underlying abstract graph of a 3-planar topological graph H containing neither homotopic parallel edges nor homotopic self-loops, such that the graph induced by the uncrossed edges of H spans all vertices of H, and each of its faces has length six containing eight crossing edges in its interior in H.*

Auer et al. [41] studied how sparse a *maximal* 2-planar graph can be, that is, what is the (least) number of edges that a 2-planar graph can have when the addition of any edge (which is not present in the graph) would deviate 2-planarity. Interestingly enough, they prove that such a graph can be considerably sparser than a planar graph. To this end, their main result is the existence of graphs with n vertices and $\frac{387}{147}n + O(1)$ edges, for infinitely many values of n.

The key observation in their construction is that the average degree of a vertex of a 2-planar graph with n vertices and maximum density (that is, with $5n - 10$ edges) is slightly less than 10. Hence, by lowering the average vertex-degree, the edge- density decreases. To achieve this, Auer et al. [41] introduced *hermits*, which are vertices of degree 2 that are “enclosed” by crossing edges preventing their connections to other vertices. The first member in the suggested family of graphs is obtained from the football graph, illustrated in Fig. 7.3a, by

(i) completing all faces of length five to K_5s,
(ii) drawing in each face of length six its three diagonals, and
(iii) by attaching degree-2 hermits connecting the vertices of each edge of the football graph;

note that the maximality of this graph follows from the fact that the graph without the hermits has a unique 2-planar embedding. For $\kappa > 1$, the κ-th member in the family is obtained by taking κ copies of the first member, and by identifying the same two vertices from each copy, which are chosen to be adjacent along a face of length five in the underlying football graph. The result is summarized in the following theorem.

Theorem 10 (Auer et al. [41]) *For infinitely many values of n, there exist maximal 2-planar graphs with n vertices and* $\frac{387}{147}n + O(1)$ *edges.*

The following open problem follows naturally from the result by Auer et al. [41].

Open Problem 6 *How sparse can a* 3*-planar graph with n vertices be?*

Chaplick et al. [42], and Hong and Nagamochi [43], study the problem of testing whether a graph G is *outer k-planar*, that is, whether G can be drawn as a k-planar topological graph, whose vertices lie on the outer boundary (*outer constraint*). The work by Hong and Nagamochi [43], focuses on the special case $k = 2$, under the additional constraint that no edge-crossings appear along the outer boundary (*full constraint*); these graphs are refereed to as *fully-outer* 2*-planar graphs*. Note that, in contrast to outer 1-planar graphs, which are in fact planar [23], (fully-)outer 2-planar graphs are not necessarily planar (e.g., the complete graph on five vertices K_5 and the complete bipartite graph $K_{3,3}$ are both fully-outer 2-planar graphs; see Fig. 7.6a and b).

An algorithm by Hong and Nagamochi [43], mainly focuses on 3-connected graphs, due to the following two properties:

(i) a graph is fully-outer 2-planar if and only if its biconnected components are fully-outer 2-planar, while
(ii) a biconnected graph is fully-outer 2-planar if and only if in its SPQR-tree (see, e.g., [44, 45]), every P-node has at most two virtual edges, and the skeleton of each R-node is fully-outer 2-planar.

Hence, the main difficulty in the problem of testing whether a graph is fully-outer 2-planar lies in testing whether a 3-connected component is fully-outer 2-planar.

To cope with 3-connected input graphs, Hong and Nagamochi [43] exploit several structural properties. In particular, they prove that a fully-outer 2-planar graph G, which is 3-connected,

(P.1) does not contain three mutually crossing edges, unless G is the complete bipartite graph $K_{3,3}$;
(P.2) the maximum degree is at most four;
(P.3) cannot contain the compete graph K_4 as a subgraph, unless G has less than seven vertices.

Based on Properties (P.1)–(P.3), they further show that a fully-outer 2-planar 3-connected graph G must contain either a (3, 3)-rim (see Fig. 7.6c), or a (3, 4)-rim (see Fig. 7.6d) or a 4-rim (see Fig. 7.6e), each of which is defined on a set B containing either three vertices (as in the case of a (3, 3)-rim or a (3, 4)-rim; see Fig. 7.6c and d) or

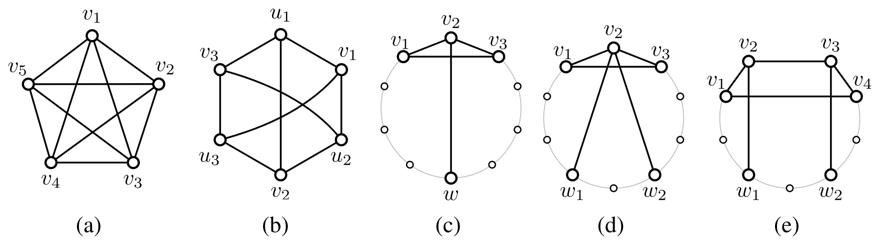


Fig. 7.6 Illustration of: (**a**) the complete graph K_5 on five vertices $\{v_1, \ldots, v_5\}$ as fully-outer 2-planar, (**b**) the complete bipartite graph $K_{3,3} = \{u_1, u_2, u_3\} \times \{v_1, v_2, v_3\}$ as fully-outer 2-planar, (**c**) (3, 3)-rim defined on $B = \{v_1, v_2, v_3\}$, in which vertex v_2 is of degree three, (**d**) a (3, 4)-rim defined on $B = \{v_1, v_2, v_3\}$, in which vertex v_2 is of degree four, and (**e**) a 4-rim defined on $B = \{v_1, v_2, v_3, v_4\}$, in which vertices v_2 and v_3 are of degree three

four vertices (as in the case of a 4-rim; see Fig. 7.6e). If no such rim can be identified in G, then the instance is reported as negative. Otherwise, it is used to transform instance (G, B) into a smaller instance (G', B'), which is solved recursively, where B' is a different rim. The base of the recursion, corresponds to a graph with at most nine vertices, whose fully-outer 2-planarity can be tested (e.g., by a brute-force method) in constant time. Since each transformation requires constant time, the overall time complexity of the algorithm is linear.

Theorem 11 (Hong and Nagamochi [43]) *There is a linear-time algorithm that tests whether a given graph is fully-outer* 2*-planar, and computes a fully-outer* 2*-planar embedding of the graph, if it exists.*

Note that for values of k greater than 2, Chaplick et al. [42], exploit several interesting properties of outer k-planar graphs. In particular, they prove that an outer k-planar graph

(i) has a balanced separator of size at most $2k + 3$,
(ii) is $(\lfloor\sqrt{4k+1}\rfloor + 1)$-degenerate, and therefore
(iii) is $(\lfloor\sqrt{4k+1}\rfloor + 2)$-colorable.

For every constant k, the small balanced separators guaranteed by (i) allow for testing outer k-planarity in quasi-polynomial time [42]. We summarize this result in the following theorem.

Theorem 12 (Chaplick et al. [42]) *For every constant k, there is a quasi-polynomial time algorithm that tests whether a given graph is outer k-planar.*

The following open problem follows naturally from Theorems 11 and 12.

Open Problem 7 *Is it possible to decide in polynomial time whether a graph is (fully-)outer* 3*-planar?*

Since Chaplick et al. [42] give some partial results on the edge-density of outer k-planar graphs (in particular, on the size of the largest outer k-planar clique), it is interesting to ask for a closed formula on the maximum edge-density of a (fully-)outer k-planar graph (that is an improvement of the one for general k-planar graphs).

Open Problem 8 *What is the maximum number of edges of a (fully-)outer k-planar graph with n vertices?*

As a first answer to Open Problem 8, note that for $k = 1$, Auer et al. [46], have shown that an outer 1-planar graph with n vertices has at most $2.5n - 4$ edges. Combining this result with the fact that an n-vertex outerplanar graph has at most $2n - 3$ edges, yields the following lower bound on the number of crossings $\mathrm{cr}(G)$ of an outer k-planar graph G

$$\mathrm{cr}(G) \geq 2m - 4.5n - 7.$$

Plugging this trivial lower bound to the second step in the probabilistic proof of Theorem 6 (with $p = \frac{27n}{8m}$), yields that the crossing number of an outer k-planar graph G with n vertices and $m \geq \frac{27}{8}n$ edges satisfies

$$\mathrm{cr}(G) \geq \frac{128}{2187} \cdot \frac{m^3}{n^2}. \tag{7.9}$$

Thus, by combining Eq. 7.9 with Eq. 7.8, we obtain that an outer k-planar graph with n vertices has at most $\frac{27\sqrt{3k}}{16}\ n \leq 2.93\sqrt{k}\ n$ edges, which is only a slight improvement on the upper bound for general k-planar graphs by Ackerman [33].

7.5 Relationship with k-quasi-planarity

We conclude this chapter by mentioning an interesting relationship with the class of *k-quasi-planar graphs*, which are topological graphs in which no k edges pairwise cross (note that 3-quasi-planar graphs are also called *quasi-planar* in the literature). It can be easily observed that, for $k \geq 1$, every k-planar graph is $(k+2)$-quasi-planar. Indeed, if a k-planar graph G were not $(k+2)$-quasi-planar, then any topological graph isomorphic to G would contain $k+2$ pairwise crossing edges. However, this would imply that any of these edges is crossed at least $k+1$ times, thus contradicting the fact that G is k-planar. This simple relationship was further strengthened by Angelini et al. [47], and Hoffmann and Tóth [48], who showed that for $k \geq 2$, every k-planar graph is $(k+1)$-quasi-planar. Note that this result cannot be extended to the case $k = 1$, as a 2-quasi-planar graph is by definition planar but not every quasi-planar is planar.

Theorem 13 (Angelini et al. [47], Hoffmann and Tóth [48]) *For $k \geq 2$, every k-planar graph is $(k+1)$-quasi-planar.*

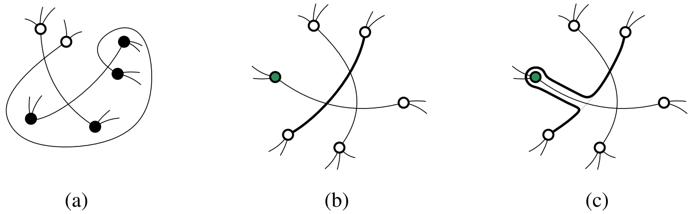


Fig. 7.7 Illustration of: (**a**) an tangled 3-crossing, (**b**) an untangled 3-crossing, and (**c**) the rerouting of the bold-drawn edge $g(X)$ of Fig. 7.7b around the green-colored vertex $f(X)$ of the 3-crossing X of Fig. 7.7b

The core of both approaches by Angelini et al. [47] and by Hoffmann and Tóth [48], is the elimination of all sets of $k+1$ pairwise crossing edges (called $(k+1)$-*crossings* for short, in the following) by appropriately redrawing an edge of each of them; note that the $(k+1)$-crossings are pairwise disjoint edge sets. To achieve this, each $(k+1)$-crossing is first *untangled*, that is, all its $2k+2$ endvertices become incident to a common face; see Fig. 7.7a and b for an illustration of a tangled and of an untangled 3-crossing, respectively. Once all $(k+1)$-crossings have been untangled, the idea is to appropriately define two injective functions f and g, which associate every $(k+1)$-crossing X of the graph with a vertex $f(X)$ and with an edge $g(X)$, respectively, such that an endvertex of edge $g(X)$ and vertex $f(X)$ are consecutive along the face of X containing all the $2k+2$ vertices of X. Then, for each $(k+1)$-crossing X, edge $g(X)$ is *rerouted around* vertex $f(X)$, that is, edge $g(X)$ is redrawn so to pass close to vertex $f(X)$, in such a way that $g(X)$ crosses the all edges incident to $f(X)$ except for the ones in $E[X]$; see Fig. 7.7c, for an illustration. This operation is called *global rerouting*.

In a high-level description, the existence of function f is guaranteed by Hall's theorem applied to an auxiliary bipartite graph whose bipartite sets are the $(k+1)$-crossings, and the vertices of the graph. Angelini et al. [47] then prove that if $k>2$, then the topological graph obtained after the global rerouting is $(k+1)$-quasi-planar. However, in the case $k=2$, the global rerouting may yield new 3-crossings. Hoffmann and Tóth [48] describe a more complicated technique to eliminate all 3-crossings that is still based on the rerouting idea, but it also takes advantage of a specific crossing pattern in the graph, which allows one to eliminate possible new 3-crossings that may appear using recursion. It is worth noting that with their approach the topological graph obtained this way is also *simple*, in the sense that any two edges intersect in at most one point, which is either a common endpoint or a proper crossing (assuming, of course, that the initial 2-planar topological graph was simple). The approach by Angelini et al. [47], on the other hand needs one additional post-processing step to guarantee this property. We conclude this section with the following open problem.

Open Problem 9 *For $k \geq 3$, is every k-planar graph k-quasi-planar?*

Note that for $k = 2$ the answer is trivially negative, as 2-quasi-planar graphs are planar but there are nonplanar quasi-planar graphs.

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