Chapter 1
Beyond Planar Graphs: Introduction

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Abstract Recent research topics in topological graph theory and graph drawing generalize the notion of planarity to sparse non-planar graphs called *beyond planar graphs* with forbidden crossing patterns. In this chapter, we introduce various types of beyond planar graphs and briefly review known results on the edge density, computational complexity, and algorithms for testing beyond planar graphs.

1.1 Beyond Planar Graphs: Edge Density

Recent research topics in topological graph theory and graph drawing generalize the notion of planarity to sparse non-planar graphs, called *beyond planar graphs*, either with forbidden edge crossing patterns or with specific types of edge crossings. Examples include:

- *k-planar graphs*: graphs which can be embedded with at most k crossings per edge [40].
- *k-quasi-planar graphs*: graphs which can be embedded without k mutually crossing edges [2].
- *RAC graphs*: graphs which can be embedded with right angle crossings [19].
- *fan-crossing-free graphs*: graphs which can be embedded without fan-crossings [17].
- *fan-planar graphs*: graphs which can be embedded such that each edge is crossed by a bundle of edges incident to a common vertex [35].
- *k-gap-planar graphs*: graphs which can be embedded such that each crossing is assigned to one of the two involved edges and each edge is assigned at most k of its crossings.

Figure 1.1 shows examples of forbidden crossing patterns for beyond planar graphs.

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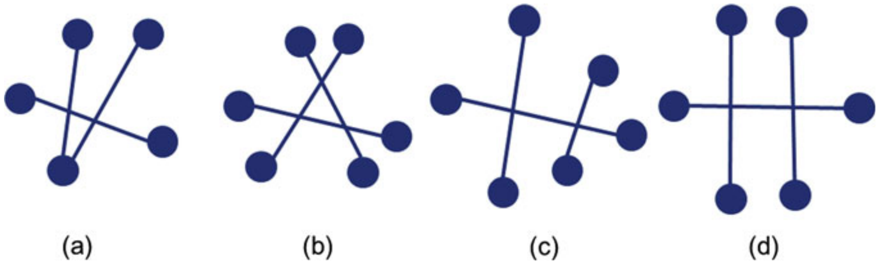

Fig. 1.1 Examples of crossing patterns: **a** fan-crossing (fan-planar and 2-planar graph, but not fan-crossing-free graph); **b** 3 mutually crossing edges (not quasi-planar graph); **c** fan-crossing-free and 2-planar graph (but not fan-planar); **d** RAC and 2-planar graph

Combinatorial aspects of beyond planar graphs are well studied, for example, the maximum number of edges of beyond planar graphs:

- k-planar graphs: Pach and Toth [40] proved that 1-planar graphs with n vertices have at most $4n - 8$ edges.
- k-quasi-planar graphs: Agarwal et al. [2] (respectively, Ackerman [1]) proved that 3 (respectively, 4)-quasi-planar graphs have linear number of edges. Fox et al. [26] showed that k-quasi-planar graphs have at most $O(n \log^{1+o(1)} n)$ edges.
- RAC graphs: Didimo et al. [19] proved that RAC graphs have at most $4n - 10$ edges.
- fan-crossing-free graphs: Cheong et al. [17] proved a tight bound of $4n - 8$ on the maximum number of edges for a 2-fan-crossing-free graph, and an upper bound of $3(k-1)(n-2)$ edges for $k \geq 3$.
- fan-planar graphs: Kaufmann and Ueckerdt [35] showed that fan-planar graphs have at most $5n - 10$ edges.
- k-gap-planar graphs: Bae et al. [7] proved that every k-gap-planar graph has $O(\sqrt{k}n)$ edges (for $k = 1$, an upper bound is $5n - 10$). They also study relationships to other classes of beyond planar graphs.

We now briefly review latest results on beyond planar graphs, mainly focusing on the computational complexity and algorithmic aspects.

1.2 Computational Complexity: NP-Hardness

Recently, computational complexity for testing beyond planarity has been studied. More specifically:

- 1-planar graphs: Grigoriev and Bodlaender [29], and Korzhik and Mohar [37] independently proved that testing 1-planarity of a graph is NP-complete. Auer et al. [6]. showed that it remains NP-hard, even if a *rotation system* (i.e., the circular ordering of edges for each vertex) is given.

Furthermore, Cabello and Mohar [15] showed that NP-hardness holds even if the input graph is an *almost planar graph* (i.e., deletion of an edge makes the resulting graph planar). More recently, Bannister et al. [8] studied the fixed parameter complexity of 1-planarity.

- RAC graphs: Argyriou et al. [4] proved that testing whether a given graph admits a straight-line RAC drawing is NP-hard, by presenting an infinite class of graphs with unique RAC embedding.
- fan-planar graphs: Binucci et al. [12] proved that testing fan-planarity of graphs is NP-complete; Bekos et al. [10] showed that it remains NP-hard, even if a rotation system is given.
- gap-planar graphs: Bae et al. [7] proved that testing k-gap-planarity of graphs is NP-complete.

1.3 Polynomial-Time Testing Algorithm

On the positive side, polynomial-time algorithms are available for testing restricted subclasses of beyond planar graphs with additional constraints, as well as computing such an embedding, if it exists.

For example, algorithms for testing special subclasses of 1-planar graphs are well studied:

- *Maximal-1-planar graphs*: Eades et al. [21] showed that the problem of testing the maximal 1-planarity (i.e., addition of an edge destroys 1-planarity) of a graph can be solved in linear time, if a rotation system is given. The embedding is unique, if it exists, and the algorithm also produces the embedding.
- *Outer-1-planar graphs*: Hong et al. [30] and Auer et al. [5] independently presented a linear-time algorithm for testing outer-1-planarity (i.e., 1-planar embedding with each vertex lies on the outer face) of a graph. The algorithm also computes such an embedding, if it exists.
- *Optimal 1-planar graphs*: Optimal-1-planar graph is a special subclass of 1-planar graphs with the maximum of $4n - 8$ edges [41]. A linear-time algorithm was given for testing optimal 1-planarity by Brandenburg [14], using a reduction from optimal 1-planar graphs to irreducible extended wheel graphs.

Figure 1.2 shows examples of maximal 1-planar graphs and outer-1-planar graphs.

For other types of beyond planar graphs, polynomial-time algorithms are also available for testing restricted subclasses of beyond planar graphs with additional constraints. Examples include:

- *Outer-2-planar graphs*: A graph is outer-2-planar, if it admits a drawing where each vertex is placed on the outer boundary and no edge has more than two crossings. A graph is *fully outer-2-planar*, if it admits an outer-2-planar embedding such that no crossing appears along the outer boundary.

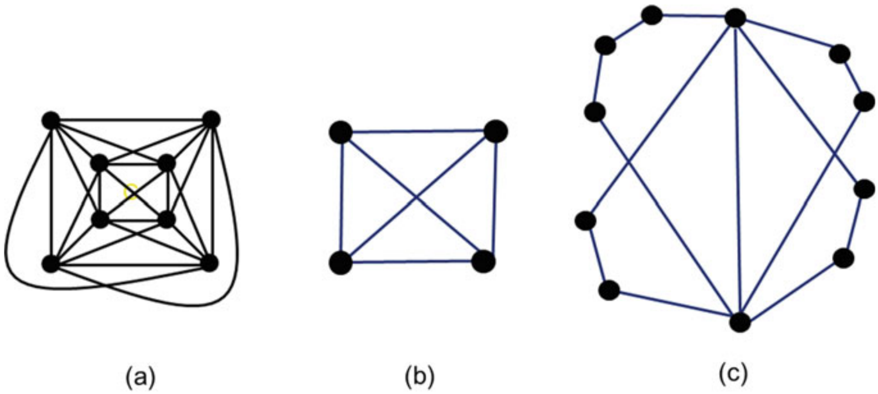

Fig. 1.2 Examples of: **a** maximal 1-planar graphs; **b** triconnected outer-1-planar graphs; **c** biconnected outer-1-planar graphs

Hong and Nagamochi [32] showed that every triconnected full-outer-2-planar graph has a constant number of full-outer-2-planar embeddings. Based on these properties, linear-time algorithms for testing full-outer-2-planarity of a connected, biconnected, and triconnected graph were presented. The algorithms also produce a full-outer-2-planar embedding of a graph, if it exists.

- *Outer k-planar graphs*: Chaplick et al. [16] showed that every outer k-planar graph has a small balanced separator of size at most $2k + 3$, which allow testing outer k-planarity in quasi-polynomial time.
  It was also shown that *closed outer k-planarity* (i.e., the vertex sequence on the boundary is a cycle in the graph) is linear time testable, since outer k-planar graphs have bounded treewidth.
- *Circular-RAC graphs*: Circular-RAC drawing is a circular layout where each vertex lies on the circle and all crossings are with right angles. Dehkordi et al. [18] presented a characterization for circular-RAC graphs, and a linear-time algorithm for testing and constructing such a drawing, if it exists.
- *2-layer RAC graphs*: A 2-layer RAC drawing of a bipartite graph is a straight-line drawing, where each vertex is placed on one of two parallel lines such that no two vertices on the same line are adjacent, and each crossing angle is a right angle. Di Giacomo et al. [27] characterized 2-layer RAC graphs, and presented linear-time testing and embedding algorithms.
- *Maximal outer-fan-planar graphs*: A graph is maximal outer-fan-planar if it has a fan-planar embedding, where every vertex is on the outer face, and insertion of an edge destroys its outer-fan-planarity. Bekos et al. [10] presented a linear-time algorithm for testing whether a graph is maximal outer-fan-planar. The algorithm also computes such an embedding, if it exists.

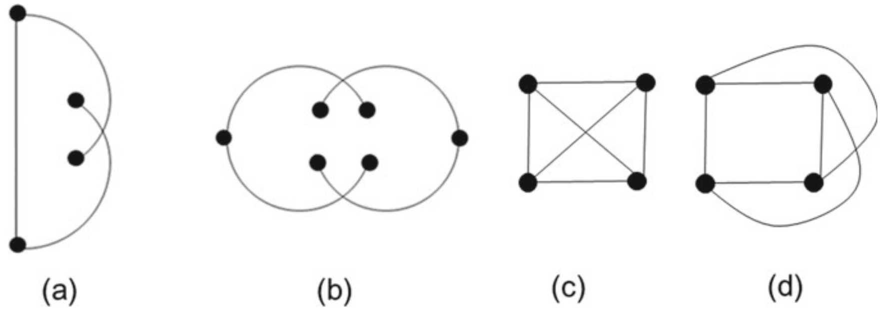

Fig. 1.3 Examples of: **a** B graph; **b** W graph; **c** straight-line 1-planar drawing of K_4; **d** 1-planar embedding of K_4 containing the B subgraph

1.4 Straight-Line Drawing

The classical *Fáry's Theorem* [25] showed that every *plane graph* (i.e., a planar graph with a given planar embedding) admits a planar straight-line drawing. Indeed, planar straight-line drawing is one of the most popular drawing conventions in Graph Drawing; consequently many straight-line drawing algorithms are available for planar graphs [9, 39].

On the other hand, Thomassen [42] showed that 1-plane graphs (i.e., 1-planar graphs with a given 1-planar embedding) have two forbidden subgraphs, called B graph and W graph, to admit a straight-line drawing. Figure 1.3 shows two forbidden subgraphs of 1-planar graphs.

As such, it opened the way for the investigation for straight-line drawings of beyond planar graphs:

- *1-plane graphs*: Based on the forbidden subgraph characterization by Thomassen [42], Hong et al. [33] presented a linear-time testing and drawing algorithm to construct a straight-line drawing of 1-plane graphs, if it exists. It was also shown that some 1-planar graphs require exponential area for any straight-line drawing.
- *Re-embedding 1-plane graphs*: Re-embedding a 1-plane graph is to change the rotation system or the outer face of the given 1-planar embedding of the 1-plane graph, while preserving the same set of pairs of crossing edges.

  Hong and Nagamochi [31] considered the problem of re-embedding a 1-plane graph, which contains the forbidden subgraphs (i.e., B graph or W graph), to a new 1-planar embedding which admits a straight-line drawing (i.e., 1-planar embedding without B graph or W graph). They presented a characterization of forbidden configuration.

  Based on the characterization, a linear-time algorithm for finding a straight-line drawable 1-planar embedding or the forbidden configuration was presented.

- *Almost planar graphs*: Almost planar graph consists of a planar graph plus one edge, also called graphs with *1-skewness* (i.e., removal of an edge makes the graph planar).
  Eades et al. [22] presented a characterization of almost planar topological graphs that admit a straight-line drawing. Based on the characterization, linear-time algorithms were presented for testing whether an almost planar graph admits a straight-line drawing, and for constructing such a drawing if it exists. It was also shown that some almost planar graphs require exponential area for any straight-line drawing.
- *General embedded non-planar graphs*: Nagamochi [38] investigated the stretchability problem (i.e., straight-line drawings) of general embedded graphs. It was shown that there is a 3-planar embedding and quasi-planar embedding that admits no straight-line drawing, which cannot be characterized by forbidden configuration.
  He also considered a problem of whether a given embedded graph G admits a straight-line drawing under the same *frame*, which is defined by a fixed biconnected planar spanning subgraph of G, and presented forbidden configurations (i.e., a given embedding admits a straight-line drawing under the same frame if and only if it contains no forbidden configuration).
  If a given embedding is quasi-planar (i.e. no pairwise crossing edges) and its crossing-free edges induce a biconnected spanning subgraph, then the stretchability can be tested in polynomial time using forbidden configurations.

1.5 Outlook and Open Problem

This chapter introduces beyond planar graphs and briefly reviews known results on the edge density, computational complexity and algorithmic results on testing and drawing beyond planar graphs.

Many combinatorial results are also studied for beyond planar graphs, including structural properties, various geometric representations, as well as the relationships between beyond planar graphs. Examples include:

- *Structural properties*: Structures of 1-planar graphs are well studied.
  For example, Borodin [13] studied the coloring problem of 1-planar graphs. Fabrici and Madaras [24] presented structural results on 1-planar graphs, while Hudak et al. [34] studied structural properties of maximal 1-planar graphs. Suzuki [41] investigated structural properties of optimal 1-planar graphs.
- *Geometric representation*: Various geometric representations of beyond planar graphs, such as orthogonal drawings, polyline drawings, visibility representations, and book embeddings are also studied.

For example, Biedl et al. [11] studied RVR (Rectangle Visibility Representation) of embedded graphs. Di Giacomo et al. [28] studied polyline drawings of topological graphs with few bends per edge.

- *Relationships between beyond planar graphs*: Relationships between k-planar graphs, RAC graphs, k-quasi-planar graphs, fan-planar graphs and gap-planar graphs are well studied.
  For example, Eades and Liotta [23] studied the relationship between RAC and 1-planar graphs. Angelini et al. [3] showed that every simple k-planar topological graph can be transformed into a simple k-quasi-planar topological graph.

For more details, we refer to corresponding chapters in this book and a recent survey on 1-planar graphs [36] and beyond planar graphs [20].

Finally, we conclude with open problems related to the topics covered in this chapter.

- *Computational complexity*: For most beyond planar graphs, testing problem is known to be NP-complete. However, it is still open for some classes of beyond planar graphs.
  - **Open Problem 1**: Is it NP-complete to test quasi-planarity?
  - **Open Problem 2**: Is it NP-complete to test whether a given graph is a fan-crossing-free graph?
- *Testing algorithm*: Polynomial-time algorithms are available for testing restricted subclass of beyond planar graphs. For example, testing problem becomes tractable when further restrictions such as a rotation system, maximality/optimality, or outer-beyond planarity are assumed.
  - **Open Problem 3**: Is it polynomial time solvable to test maximal quasi-planarity?
  - **Open Problem 4**: Is it polynomial time solvable to test whether a given graph is a maximal fan-crossing-free graph?
- *Straight-line drawability*: Forbidden subgraph characterization to admit a straight-line drawing and linear-time algorithm to construct straight-line drawing if it exists are known for 1-planar graphs and almost planar graphs. For other beyond planar graphs, straight-line drawability problem need further investigation.
  - **Open Problem 5**: Characterize forbidden configuration of RAC graphs to admit a straight-line drawing. Is there an efficient algorithm to construct a straight-line drawing of a RAC graph?
  - **Open Problem 6**: Characterize forbidden configuration of 2-skewness graphs (i.e., removal of two edges makes the resulting graph planar) to admit a straight-line drawing. Is there an efficient algorithm to construct a straight-line drawing of a 2-skewness graph?

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References

1. Ackerman, E.: On the maximum number of edges in topological graphs with no four pairwise crossing edges. Discret. Comput. Geom. **41**(3), 365–375 (2009). https://doi.org/10.1007/s00454-009-9143-9
2. Agarwal, P.K., Aronov, B., Pach, J., Pollack, R., Sharir, M.: Quasi-planar graphs have a linear number of edges. Combinatorica **17**(1), 1–9 (1997). https://doi.org/10.1007/BF01196127
3. Angelini, P., Bekos, M.A., Brandenburg, F.J., Da Lozzo, G., Di Battista, G., Didimo, W., Hoffmann, M., Liotta, G., Montecchiani, F., Rutter, I., Tóth, C.D.: Simple k-planar graphs are simple (k839+82391)-quasiplanar. J. Comb. Theory, Ser. B **142**, 1–35 (2020). https://doi.org/10.1016/j.jctb.2019.08.006
4. Argyriou, E.N., Bekos, M.A., Symvonis, A.: The straight-line RAC drawing problem is np-hard. J. Graph Algorithms Appl. **16**(2), 569–597 (2012). https://doi.org/10.7155/jgaa.00274
5. Auer, C., Bachmaier, C., Brandenburg, F.J., Gleißner, A., Hanauer, K., Neuwirth, D., Reislhuber, J.: Outer 1-planar graphs. Algorithmica **74**(4), 1293–1320 (2016). https://doi.org/10.1007/s00453-015-0002-1
6. Auer, C., Brandenburg, F.J., Gleißner, A., Reislhuber, J.: 1-planarity of graphs with a rotation system. J. Graph Algorithms Appl. **19**(1), 67–86 (2015). https://doi.org/10.7155/jgaa.00347
7. Bae, S.W., Baffier, J., Chun, J., Eades, P., Eickmeyer, K., Grilli, L., Hong, S., Korman, M., Montecchiani, F., Rutter, I., Tóth, C.D.: Gap-planar graphs. Theor. Comput. Sci. **745**, 36–52 (2018). https://doi.org/10.1016/j.tcs.2018.05.029
8. Bannister, M.J., Cabello, S., Eppstein, D.: Parameterized complexity of 1-planarity. J. Graph Algorithms Appl. **22**(1), 23–49 (2018). https://doi.org/10.7155/jgaa.00457
9. Battista, G.D., Eades, P., Tamassia, R., Tollis, I.G.: Graph Drawing: Algorithms for the Visualization of Graphs. Prentice-Hall (1999)
10. Bekos, M.A., Cornelsen, S., Grilli, L., Hong, S., Kaufmann, M.: On the recognition of fan-planar and maximal outer-fan-planar graphs. Algorithmica **79**(2), 401–427 (2017). https://doi.org/10.1007/s00453-016-0200-5
11. Biedl, T.C., Liotta, G., Montecchiani, F.: Embedding-preserving rectangle visibility representations of nonplanar graphs. Discret. Comput. Geom. **60**(2), 345–380 (2018). https://doi.org/10.1007/s00454-017-9939-y
12. Binucci, C., Giacomo, E.D., Didimo, W., Montecchiani, F., Patrignani, M., Symvonis, A., Tollis, I.G.: Fan-planarity: properties and complexity. Theor. Comput. Sci. **589**, 76–86 (2015). https://doi.org/10.1016/j.tcs.2015.04.020
13. Borodin, O.V.: Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs. Metody Diskret. Analiz. **41**, 12–26, 108 (1984)
14. Brandenburg, F.J.: Recognizing optimal 1-planar graphs in linear time. Algorithmica **80**(1), 1–28 (2018). https://doi.org/10.1007/s00453-016-0226-8
15. Cabello, S., Mohar, B.: Adding one edge to planar graphs makes crossing number and 1-planarity hard. SIAM J. Comput. **42**(5), 1803–1829 (2013). https://doi.org/10.1137/120872310
16. Chaplick, S., Kryven, M., Liotta, G., Löffler, A., Wolff, A.: Beyond outerplanarity. In: F. Frati, K. Ma (eds.) Graph Drawing and Network Visualization - 25th International Symposium, GD 2017, Boston, MA, USA, September 25-27, 2017, Revised Selected Papers, Lecture Notes in Computer Science, vol. 10692, pp. 546–559. Springer (2017). https://doi.org/10.1007/978-3-319-73915-1_42
17. Cheong, O., Har-Peled, S., Kim, H., Kim, H.: On the number of edges of fan-crossing free graphs. Algorithmica **73**(4), 673–695 (2015). https://doi.org/10.1007/s00453-014-9935-z
18. Dehkordi, H.R., Eades, P., Hong, S., Nguyen, Q.H.: Circular right-angle crossing drawings in linear time. Theor. Comput. Sci. **639**, 26–41 (2016). https://doi.org/10.1016/j.tcs.2016.05.017
19. Didimo, W., Eades, P., Liotta, G.: Drawing graphs with right angle crossings. Theor. Comput. Sci. **412**(39), 5156–5166 (2011). https://doi.org/10.1016/j.tcs.2011.05.025
20. Didimo, W., Liotta, G., Montecchiani, F.: A survey on graph drawing beyond planarity. ACM Comput. Surv. **52**(1), 4:1–4:37 (2019). https://doi.org/10.1145/3301281

21. Eades, P., Hong, S., Katoh, N., Liotta, G., Schweitzer, P., Suzuki, Y.: A linear time algorithm for testing maximal 1-planarity of graphs with a rotation system. Theor. Comput. Sci. **513**, 65–76 (2013). https://doi.org/10.1016/j.tcs.2013.09.029
22. Eades, P., Hong, S., Liotta, G., Katoh, N., Poon, S.: Straight-line drawability of a planar graph plus an edge. In: F. Dehne, J. Sack, U. Stege (eds.) Algorithms and Data Structures - 14th International Symposium, WADS 2015, Victoria, BC, Canada, August 5-7, 2015. Proceedings, Lecture Notes in Computer Science, vol. 9214, pp. 301–313. Springer (2015). https://doi.org/10.1007/978-3-319-21840-3_25
23. Eades, P., Liotta, G.: Right angle crossing graphs and 1-planarity. Discret. Appl. Math. **161**(7–8), 961–969 (2013). https://doi.org/10.1016/j.dam.2012.11.019
24. Fabrici, I., Madaras, T.: The structure of 1-planar graphs. Discrete Mathematics **307**(7–8), 854–865 (2007)
25. Fáry, I.: On straight line representations of planar graphs. Acta Sci. Math. Szeged **11**, 229–233 (1948)
26. Fox, J., Pach, J., Suk, A.: The number of edges in k-quasi-planar graphs. SIAM J. Discrete Math. **27**(1), 550–561 (2013). https://doi.org/10.1137/110858586
27. Giacomo, E.D., Didimo, W., Eades, P., Liotta, G.: 2-layer right angle crossing drawings. Algorithmica **68**(4), 954–997 (2014). https://doi.org/10.1007/s00453-012-9706-7
28. Giacomo, E.D., Eades, P., Liotta, G., Meijer, H., Montecchiani, F.: Polyline drawings with topological constraints. Theor. Comput. Sci. **809**, 250–264 (2020). https://doi.org/10.1016/j.tcs.2019.12.016
29. Grigoriev, A., Bodlaender, H.L.: Algorithms for graphs embeddable with few crossings per edge. Algorithmica **49**(1), 1–11 (2007). https://doi.org/10.1007/s00453-007-0010-x
30. Hong, S., Eades, P., Katoh, N., Liotta, G., Schweitzer, P., Suzuki, Y.: A linear-time algorithm for testing outer-1-planarity. Algorithmica **72**(4), 1033–1054 (2015). https://doi.org/10.1007/s00453-014-9890-8
31. Hong, S., Nagamochi, H.: Re-embedding a 1-plane graph into a straight-line drawing in linear time. In: Y. Hu, M. Nöllenburg (eds.) Graph Drawing and Network Visualization - 24th International Symposium, GD 2016, Athens, Greece, September 19-21, 2016, Revised Selected Papers, Lecture Notes in Computer Science, vol. 9801, pp. 321–334. Springer (2016). https://doi.org/10.1007/978-3-319-50106-2_25
32. Hong, S., Nagamochi, H.: A linear-time algorithm for testing full outer-2-planarity. Discret. Appl. Math. **255**, 234–257 (2019). https://doi.org/10.1016/j.dam.2018.08.018
33. Hong, S.H., Eades, P., Liotta, G., Poon, S.H.: Fáry's theorem for 1-planar graphs. In: J. Gudmundsson, J. Mestre, T. Viglas (eds.) Proceedings of COCOON 2012, Lecture Notes in Computer Science, vol. 7434, pp. 335–346. Springer (2012)
34. Hudák, D., Madaras, T., Suzuki, Y.: On properties of maximal 1-planar graphs. Discussiones Mathematicae Graph Theory **32**(4), 737–747 (2012). https://doi.org/10.7151/dmgt.1639
35. Kaufmann, M., Ueckerdt, T.: The density of fan-planar graphs. CoRR (2014). http://arxiv.org/abs/1403.6184
36. Kobourov, S.G., Liotta, G., Montecchiani, F.: An annotated bibliography on 1-planarity. Comput. Sci. Rev. **25**, 49–67 (2017). https://doi.org/10.1016/j.cosrev.2017.06.002
37. Korzhik, V.P., Mohar, B.: Minimal obstructions for 1-immersions and hardness of 1-planarity testing. J. Graph Theory **72**(1), 30–71 (2013)
38. Nagamochi, H.: Straight-line drawability of embedded graph. Technical Report 2013–005, Department of Applied Mathematics and Physics, Kyoto University, Japan (2013)
39. Nishizeki, T., Rahman, M.S.: Planar Graph Drawing, *Lecture Notes Series on Computing*, vol. 12. World Scientific (2004). https://doi.org/10.1142/5648
40. Pach, J., Tóth, G.: Graphs drawn with few crossings per edge. Combinatorica **17**(3), 427–439 (1997)
41. Suzuki, Y.: Optimal 1-planar graphs which triangulate other surfaces. Discret. Math. **310**(1), 6–11 (2010). https://doi.org/10.1016/j.disc.2009.07.016
42. Thomassen, C.: Rectilinear drawings of graphs. J. Graph Theory **12**(3), 335–341 (1988)