Propagation of Viscoelastic Waves in a Single Layered Media with a Free Surface

Pankaj Kumar, Anirvan DasGupta, and Ranjan Bhattacharyya

Abstract In this work, a wave propagation formulation in a layered half space is presented. A viscoelastic layer is considered to be fixed to an elastic half space. The inhomogeneity in various reflected and refracted waves is caused due to the viscoelastic layer. In the analysis, the low-loss approximation is considered for the viscoelastic layer. Using the matrix formulation of Thomson [1] and Haskell [2], the free surface displacement functions are derived by considering boundary conditions. The effect of medium parameters and wave types on the wave propagation behavior is studied numerically and conclusions are drawn.

Keywords Bulk waves · Layered media · Viscoelasticity

Nomenculture

ψ_{ijkl}	Relaxation tensor
Φ	Scalar potential
$\mathbf{\Psi}$	Vector potential
λ,μ	Lamé parameters
Q	Quality factor
$\mathbf{p_L}$	Propagation vector
$\mathbf{a_L}$	Attenuation vector
c_L, c_T	Wave speeds

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S. Dutta et al. (eds.), *Advances in Structural Vibration*, Lecture Notes in Mechanical Engineering, https://doi.org/10.1007/978-981-15-5862-7_1

1 Introduction

The study of bulk waves in viscoelastic media is important in many aspects. In materials characterization and non-destructive testing, the understanding of the wave-medium interaction plays a significant role. Also, due to propagating bulk waves, the free surface displacement can develop a driving force along the propagation, which can be used in applications like positioning, segregation, transportation, etc. This motivates the present analysis.

Anelasticity introduces inhomogeneity in various waves associated with viscoelastic layer. The inhomogeneity can be described by the general theory of viscoelasticity. In addition, the theory gives the basis to explain the attenuation in bulk waves. In viscoelastic medium, the bulk wave behavior has been evaluated by many researchers. Some of the earlier studies on bulk waves in viscoelastic media dealt with its energy relation and physical characteristics [3–6]. Buchen [3] and Borcherdt [4] considered harmonic P and SV wave, and details are provided on associated energy with these waves. Cooper [5] and Borcherdt [6] investigated the propagation of bulk waves at a planner interface composed of two viscoelastic half spaces. In some works [7, 8], reflection and refraction of waves and their frequency relation in a composite medium are considered. A composite consists of alternating layers of both elastic and viscoelastic materials, which is simplified to a homogeneous composite. Recently, problems on wave propagation in layered media have been studied in the context of welded interface in Kaur et al. [9] and of imperfect interface in Liu et al. [10].

In this paper, we are interested in the study of free surface displacement. The excitation is caused by an incident harmonic wave in the half space. The extended matrix formulation of Thomson [1] and Haskell [2] is presented and discussed to analyze attenuation in viscoelastic waves. To keep the problem simple, we consider a viscoelastic layer on an elastic half space. The analysis is carried out considering an incident P and SV waves.

2 Problem Formulation

In a linear viscoelastic medium, the time dependent stress σ_{ij} and ε_{ij} can be related by the following constitutive equation

$$\sigma_{ij} = \psi_{ijkl} * \dot{\varepsilon}_{kl}, \tag{1}$$

where the most general relaxation tensor ψ_{ijkl} can be expressed in terms of the bulk relaxation ψ_1, and the shear relaxation ψ_2 as

$$\psi_{ijkl} = (\psi_1 - \tfrac{2}{3}\psi_2)\delta_{ij}\delta_{kl} + \psi_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2}$$

Substituting (2) in (1), and using the relation $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ for an infinitesimal displacement u_i, the stress-divergence is then obtained as

$$\sigma_{ij,j} = \left(\psi_1 - \tfrac{2}{3}\psi_2\right)_{,t} * (\nabla \cdot \mathbf{u})_{,i} + (\psi_2)_{,t} * \left[\nabla^2 u_i + (\nabla \cdot \mathbf{u})_{,i}\right]. \tag{3}$$

In absence of body forces, using (3) in equation $\rho \ddot{u}_{,i} - \sigma_{ij,j} = 0$, one obtains the equation of motion as

$$\rho \ddot{\mathbf{u}} - \left(\psi_1 - \tfrac{1}{3}\psi_2\right)_{,t} * (\nabla \cdot \mathbf{u}) - (\psi_2)_{,t} * \nabla^2 \mathbf{u} = 0, \tag{4}$$

using Helmholtz theorem, the displacement field $\mathbf{u}$ in (4) can be rewritten as

$$\mathbf{u} = \nabla \Phi + \nabla \times \boldsymbol{\Psi}, \tag{5}$$

where Φ and $\boldsymbol{\Psi}$ are, respectively, the scalar and vector potentials with $\nabla \cdot \boldsymbol{\Psi} = 0$. Now, substituting displacement (5) into equation of motion (4) yields on simplification

$$\nabla \cdot [\rho \ddot{\Phi} - (\psi_1 - \tfrac{4}{3}\psi_2)_{,t} * \nabla^2 \Phi] + \nabla \times [\rho \ddot{\boldsymbol{\Psi}} - (\psi_2)_{,t} * \nabla^2 \boldsymbol{\Psi}] = 0, \tag{6}$$

Hence, the equation of motion (4) is satisfied by the displacement field (5) only if the particular vector quantities in (6) vanish. With these, Eq. (6) may be rewritten in the standard form of wave equation as

$$\ddot{\Phi} - c_L^2 \nabla^2 \Phi = 0, \quad \text{and} \quad \ddot{\boldsymbol{\Psi}} - c_T^2 \nabla^2 \boldsymbol{\Psi} = 0, \tag{7}$$

here the complex speeds $c_L^2 = \mathcal{F}(\dot{\psi}_L)/\rho = (\lambda + 2\mu)/\rho$ and $c_T^2 = \mathcal{F}(\dot{\psi}_T)/\rho = (\mu)/\rho$ are in general frequency dependent. The constants λ and μ are known as Lamé parameters. The operator $\mathcal{F}(\cdot)$ denotes the Fourier transform, and the quantities $\psi_L = \psi_1 - \frac{4}{3}\psi_2$ and $\psi_T = \psi_2$. Hence, one can also define the viscoelastic moduli for P and SV waves using the velocity relations.

Now, consider the general harmonic viscoelastic P wave as

$$\Phi = \Phi_0(\omega) \exp(-i \mathbf{k_L} \cdot \mathbf{e}), \tag{8}$$

where $\mathbf{k_L} = \mathbf{p_L} - i\mathbf{a_L}$, with $\mathbf{p_L}$ and $\mathbf{a_L}$ as the propagation and attenuation vectors, respectively. In the expression of complex wave number $(k_L - \omega/c_L)$, the complex velocity c_L is defined earlier in (7). Substituting (8) in $(7)_1$ and using velocity relation yields on simplification

$$k_L^2 = \mathbf{k_L} \cdot \mathbf{k_L} = \Re[k_L^2] \left(1 + i \frac{\Im[k_L^2]}{\Re[k_L^2]}\right) = \Re[k_L^2] \left(1 - i Q_L^{-1}\right), \tag{9}$$

where $\Re[\cdot]$ and $\Im[\cdot]$ denote real and imaginary parts, respectively, and the quality factor is $Q_L = -(\Re[k_L^2]/\Im[k_L^2])$. In a viscoelastic medium, the quality factor also depends upon the angle γ_L between the vectors $\mathbf{p_L}$ and $\mathbf{a_L}$. Next, extracting the real and imaginary parts from (9), and after solving and simplification for p_L and a_L, leads to

$$\begin{aligned} 2p_L^2 &= \Re[k_L^2]\left(1+\sqrt{1+Q_L^{-2}\sec^2\gamma_L}\right), \\ 2a_L^2 &= \Re[k_L^2]\left(-1+\sqrt{1+Q_L^{-2}\sec^2\gamma_L}\right). \end{aligned} \tag{10}$$

Similarly, using SV wave solution, the expressions for SV wave can be derived. The present study considers the reflection and refraction of bulk waves in a layered viscoelastic medium, and hence the direction of vectors **p** and **a** for each propagating waves must be defined. The Snell's law along with the boundary conditions will give these vectors separately. Consider next the case of elastic half space ($Q \approx \infty$), attenuation vector will be zero, and all other fields are modified accordingly.

Figure 1 shows the geometry of a viscoelastic layer of thickness h, with lower boundary fixed with an elastic half space. We assumed that a harmonic wave is incident on the interface $x_3 = h$ at any arbitrary angle θ. Let ρ, c_L, and c_T denote density, and wave speeds for the viscoelastic layer, and ω is incidence frequency. The corresponding primed variables stand for the elastic half space. In the layer, one can write the total displacement potential as

$$\begin{aligned} \Phi &= [\Phi_1 \exp(i\alpha x_3) + \Phi_2 \exp(-i\alpha x_3)]\exp(-ikx_1), \\ \Psi &= \Phi_{x_2} = [\Psi_1 \exp(i\beta x_2) + \Psi_2 \exp(-i\beta x_2)]\exp(-ikx_1). \end{aligned} \tag{11}$$

where $\alpha = \sqrt{k_L^2 - k^2}$ and $\beta = \sqrt{k_T^2 - k^2}$, and the amplitudes Φ_i and Ψ_i $(i = i, 2)$ are in general complex, and k is horizontal wave number. The propagator term in (11) is dropped since it is common in all fields. Since the wave numbers are complex in the viscoelastic medium, the quantities α and β are given by the corresponding principal values.

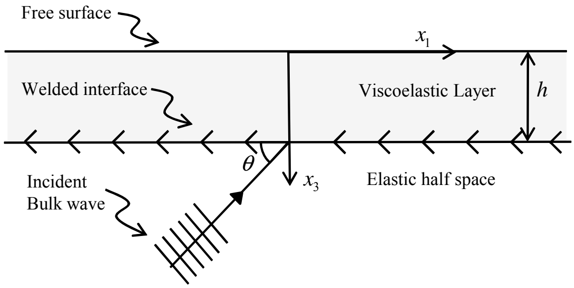


Fig. 1 Wave incident on the interface of layered half space

Let us write the various fields in the layer using (3), (5), and (11) in matrix form as

$$\begin{bmatrix} u_1 \\ u_3 \\ \sigma_{33} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} -ik\cos\alpha x_3 & \beta\sin\beta x_3 & k\sin\alpha x_3 & -i\beta\cos\beta x_3 \\ -\alpha\sin\alpha x_3 & -ik\cos\beta x_3 & i\alpha\cos\alpha x_3 & k\sin\beta x_3 \\ -\Omega\cos\alpha x_3 & 2i\beta k\mu\sin\beta x_3 & -i\Omega\sin\alpha x_3 & 2i\beta k\mu\cos\beta x_3 \\ 2ik\alpha\mu\sin\alpha x_3 & \mu\Theta\cos\beta x_3 & 2k\alpha\mu\sin\alpha x_3 & i\mu\Theta\sin\beta x_3 \end{bmatrix} \begin{bmatrix} \Phi_1+\Phi_2 \\ \Psi_1+\Psi_2 \\ \Phi_1-\Phi_2 \\ \Psi_1-\Psi_2 \end{bmatrix} \tag{12}$$

where

$$\Omega = (k^2\lambda + \alpha^2(\lambda+2\mu)), \quad \text{and} \quad \Theta = (\beta^2 - k^2).$$

For simplicity, (12) in a compact form is represented as

$$\mathbf{X}(x_3) = \mathbf{C}(x_3)\mathbf{A}. \tag{13}$$

Now, one can relate easily the fields at $x_3 = 0$ and $x_3 = h$, by using (13). Thus, eliminating the amplitude vector yields a linear relationship between the fields at $x_3 = 0$ and $x_3 = h$ as

$$\mathbf{X}(h) = \mathbf{C}(h)\mathbf{C}(0)^{-1}\mathbf{X}(0). \tag{14}$$

Next, using the fixed boundary conditions at the interface $x_3 = h$, we obtain the amplitude relation for the half space as

$$\mathbf{A}' = \mathbf{C}'(0)^{-1}\mathbf{C}(h)\mathbf{C}(0)^{-1}\mathbf{X}(0), \tag{15}$$

where $\mathbf{A}' = [\Phi'_1 + \Phi'_2 \quad \Psi'_1 + \Psi'_2 \quad \Phi'_1 - \Phi'_2 \quad \Psi'_1 - \Psi'_2]$ is the amplitude vector in the half space. In elastic half space, the matrix $\mathbf{C}'$ is calculated similarly as calculated in (12) for viscoelastic layer. In (15), the field vector $\mathbf{X}(0)$ is $[u_1 \quad u_3 \quad 0 \quad 0]^T$ (since the stresses are zero at free surface, $x_3 = 0$).

2.1 Incident P **wave**: Let us consider an incident longitudinal wave in the half space. This obtained by setting $\Psi'_1 = 0$ in (15). Solving for free surface displacements u_{10} and u_{20} in terms of incident amplitude from (15) yields

$$\begin{aligned} u_{10} &= -(D_{12}D_{31} - D_{11}D_{32})^{-1}D_{32}\Phi'_1, \\ u_{20} &= (D_{12}D_{31} - D_{11}D_{32})^{-1}D_{31}\Phi'_1. \end{aligned} \tag{16}$$

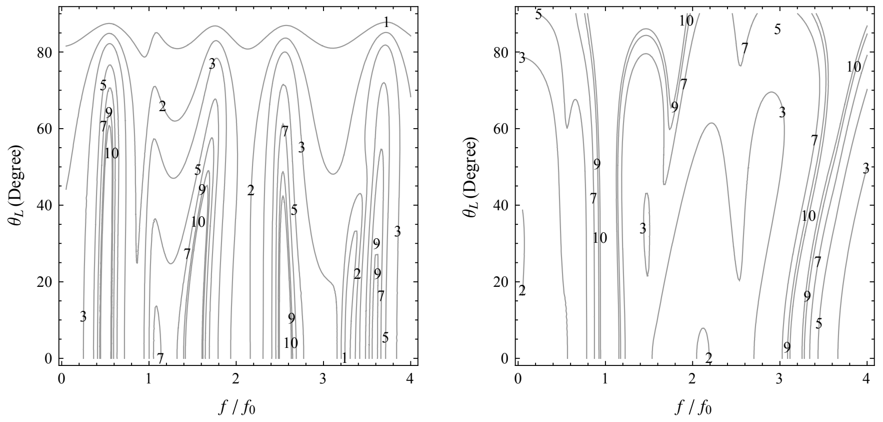
2.2 Incident SV **wave**: The expression for this case is obtained by setting $\Phi'_1 = 0$ in (15). The resulting displacements are calculated in terms of Ψ'_1 as

$$\begin{aligned} u_{10} &= (D_{12}D_{31} - D_{11}D_{32})^{-1}D_{12}\Psi'_1, \\ u_{20} &= -(D_{12}D_{31} - D_{11}D_{32})^{-1}D_{11}\Psi'_1. \end{aligned} \tag{17}$$

where matrix $\mathbf{D} = \mathbf{C}'(0)^{-1}\mathbf{C}(h)\mathbf{C}(0)^{-1}$ depends upon both medium and incident wave parameters.

Table 1 Elastic and viscoelastic material parameters

Material	$\rho\,(\mathrm{kg/m^3})$	$c_L\,(\mathrm{m/s})$	$c_T\,(\mathrm{m/s})$	Q_L	Q_T
Layer (E&C Epoxy)	1600	2960	1450	36	50
Half space (Steel)	7800	6020	3218	–	–



(a) Vertical displacement (b) horizontal displacement

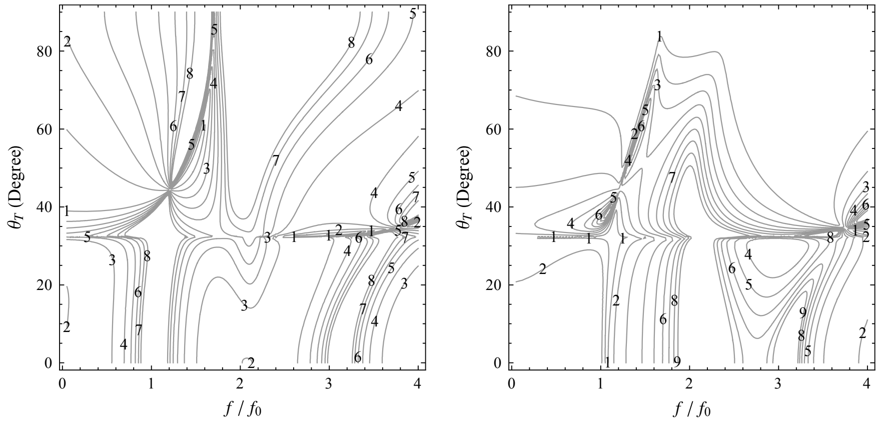
Fig. 2 Normalized absolute displacement variation for incident P wave

3 Results and Discussions

We have analyzed the free surface displacement characteristics for a viscoelastic layered half space. Numerical results are calculated from (16) and (17) and presented to show the behavior of surface displacement. The material parameters for the layered medium are listed in Table 1.

In Fig. 2, the displacement variations are shown as a contour map for an incident P wave. The displacement components are plotted as separate map against incidence angle and normalized incident frequency f/f_0, where f is incidence frequency in Hz, and $f_0 = c_L/4h$ and $c_T/4h$, for an incident P and SV wave, respectively. It is observed that the vertical displacements curves are nearly zero valued at grazing incidence.

For an incident SV wave, the variation of the vertical and horizontal parts of displacement is shown in Fig. 3. For grazing incidence, the horizontal displacement components are zero valued. There are some interesting observations to note in this case. Due to the higher quality factor and travel time, the effect of anelasticity becomes more pronounced here. The displacements fluctuate notably when the incident angle reaches its critical value. Here, the critical angle in this case is calculated as $\sin^{-1}(c'_T/c'_L) = 32.3°$. The results are provided for this incident angle in Fig. 4. Also, Rayleigh wave mode will appear thereafter this critical angle. In the layered



(a) Vertical displacement (b) horizontal displacement

Fig. 3 Normalized absolute displacement variation for incident SV wave

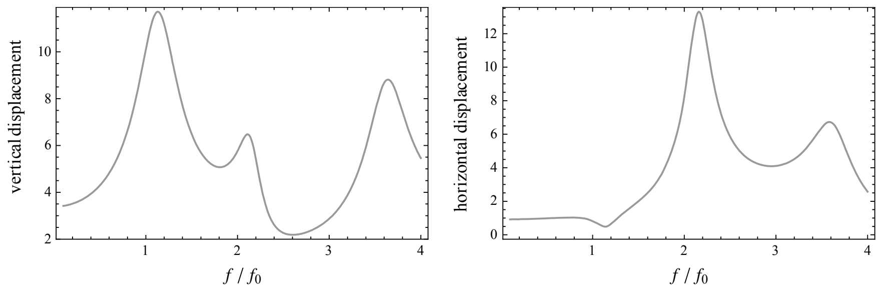


Fig. 4 Variation of normalized displacement for incident SV wave at $\theta_T = 32.3°$

medium, the motions of the material points on the free surface are typically elliptical; this is also due to the effect of anelasticity. In a similar way, this elliptical motion can be analyzed by plotting the phase difference between horizontal and vertical components of displacement.

4 Conclusions

In the present work, the matrix method is expanded to study the bulk wave characteristics in a viscoelastic layer that is fixed from below by an elastic half space. The free surface displacement distributions are derived analytically in closed form. The results for the normalized horizontal and vertical displacements are plotted and analyzed. The effective anelastic constants along with wave constants in the viscoelastic layer are derived. When the incidence angle reaches close to 90°, the vertical displacement components are zero for an incident P wave, and the horizontal displacement components are zero for an incident SV wave, respectively. For a critical incident SV wave, the displacement curves fluctuate suddenly near that angle, for which the results are also provided. These analyses have potential applications in non-destructive testing, material evaluation, ultrasonic assessment, and others. Also, wave assisted motion and handling can be improved based on our understanding of the free surface motion characteristics. The analysis of the interfacial weakness, interfacial bonding, and friction is an important aspect for future applications.

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