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Mohammad Ashraf ·
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Gabriel Eduard Vilcu *Editors*

Differential Geometry, Algebra, and Analysis

ICDGAA 2016, New Delhi, India,
November 15–17

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Editors
Mohammad Hasan Shahid
Department of Mathematics
Jamia Millia Islamia
New Delhi, Delhi, India

Mohammad Ashraf
Department of Mathematics
Aligarh Muslim University
Aligarh, India

Falleh Al-Solamy
Department of Mathematics
King Abdulaziz University
Jeddah, Saudi Arabia

Yasunori Kimura
Department of Information Science
Toho University
Chiba, Japan

Gabriel Eduard Vilcu
Department of Mathematics
Petroleum and Gas University of Ploieşti
Ploieşti, Romania

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Preface

This book discusses recent developments and contemporary research in the area of differential geometry, algebra and analysis. It is divided into three parts: the first part contains articles from the discipline of differential geometry; second part presents contributions in algebra and its application; and the third part focuses on topics from analysis.

It is pertinent to mention that some of the included papers were presented at the International Conference on Differential Geometry, Algebra and Analysis (ICDGAA 2016), held at the Department of Mathematics, Jamia Millia Islamia, New Delhi, under the chairmanship of Prof. Abdul Wafi from 15 to 17 November 2016. Participants and speakers from across the country and globe—USA, France, Japan, Poland, Romania, Iran and Morocco—attended the conference and presented their research. The exchange of ideas in different fields formed the link for future collaboration across the world. The focal theme of the conference was to bring together senior and young researchers in the area of differential geometry, algebra and analysis to exchange new ideas and to discuss current challenging problems in mathematics. We are of the opinion that the work presented in this volume will be useful to researchers in these areas. Furthermore, we hope that the research articles given in this book will stimulate the formation of interdisciplinary groups for beneficial collaborative research.

Reviewed by renowned experts, chapters in the book are authored by renowned researchers working in these areas of mathematics. This book covers a wide range of topics such as geometry of submanifolds, geometry of statistical submanifolds, ring theory, module theory, optimization theory, approximation theory, etc., by exhibiting new ideas and methodology for current research in differential geometry, algebra and analysis.

We are thankful to all the contributors, faculty members of the Department of the Mathematics, Jamia Millia Islamia and organizing secretaries Dr. Arshad Khan, Dr. Yahya Abbasi and Dr. Izharuddin for their co-operations. The conference was supported by DRS Department of Mathematics, Jamia Millia Islamia and DST, NBHM, and CSIR.

New Delhi, India — Mohammad Hasan Shahid
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Ploieşti, Romania — Gabriel Eduard Vilcu

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About the Editors

Mohammad Hasan Shahid is Professor at the Department of Mathematics, Jamia Millia Islamia, New Delhi, India. Earlier, he served as Associate Professor at King Abdul Aziz University, Jeddah, Saudi Arabia, from 2001 to 2006. He earned his Ph. D. in Mathematics from Aligarh Muslim University on the topic “On geometry of submanifolds” in 1988, under (Late) Professor Izhar Husain. He was one the recipients of post-doctoral fellowship from the University of Patras, Greece, from October 1997 to April 1998. He has published more than 100 research articles in various national and international journals of repute. Recently, he was awarded the Sultana Nahar Distinguished Teacher award of the Year 2017–2018 for his outstanding contribution in research.

Mohammad Ashraf is Professor at the Department of Mathematics, Aligarh Muslim University, Aligarh, India. He has authored more than 200 research publications in various branches of Mathematics, such as ring theory, coding theory and graph theory. Professor Ashraf received his Ph.D. in Mathematics from Aligarh Muslim University, Aligarh, India, in the year 1986.

Falleh Al-Solamy is Professor of Differential Geometry at King Abdul Aziz University, Jeddah, Saudi Arabia. He studied mathematics at King Abdul Aziz University, Jeddah, Saudi Arabia, and earned his Ph.D. in Mathematics from the University of Wales Swansea, Swansea, United Kingdom, in 1998, under Edwin Beggs. His research interests concern the study of the geometry of submanifolds in Riemannian and semi-Riemannian manifolds, Einstein manifolds, and applications of differential geometry in physics. Co-author of a book, Prof. Al-Solamy’s research papers have been published in journals and conference proceedings of repute. He is a member of the London Mathematical Society, the Institute of Physics, the Saudi Association for Mathematical Sciences, the Tensor Society of India, the Saudi Computer Society and the American Mathematical Society.

Yasunori Kimura is Professor at the Department of Information Science, Faculty of Science, Toho University, Chiba, Japan. He was awarded with a Ph.D. in the year 2000 from the Tokyo Institute of Technology, Japan. He has authored three books and published about 100 research articles in various international and national journals of repute. His area of research is nonlinear analysis and set-valued analysis.

Gabriel Eduard Vilcu is Professor of Mathematics at the Petroleum-Gas University of Ploieşti, Romania. He received his Ph.D. in Mathematics from the University of Bucharest, Romania, in the year 2007. Additionally, he is a senior researcher at the Research Centre in Geometry, Topology and Algebra, Faculty of Mathematics and Computer Science, University of Bucharest, Romania. His main research interest is differential geometry and its applications. Professor Vilcu has authored more than 60 peer-reviewed scientific publications in several renowned international journals and conference proceedings.

Differential Geometry

On Complete Minimal Submanifolds in a Sphere

Shogo Suzuki and Yoshio Matsuyama

Abstract Let M be an n-dimensional complete minimal submanifold in S^{n+p}, $p \geq 2$. If $|\sigma|^2 \leq \dfrac{2}{3}n$, M is either a totally geodesic submanifold or a Veronese surface in S^4.

Keywords Sphere · Minimal submanifold · Parallel second fundamental form

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1 Introduction

Let $S^{n+p}(c)$ be an $(n+p)$-dimensional Euclidean sphere of constant curvature c and M be an n-dimensional minimal submanifold isometrically immersed in $S^{n+p}(c)$. Let A_ξ be the Weingarten endomorphism associated to a normal vector field ξ and T the tensor defined by $T(\xi, \eta) = \mathrm{trace} A_\xi A_\eta$.

Recently, Montiel, Ros and Urbano [5] proved the following: Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Let σ be the second fundamental form of M in $S^{n+p}(c)$. Suppose that M is Einstein and $T = k <,>$. Then it satisfies

$$|\sigma|^2 \geq \frac{np(n+2)}{2(n+p+2)}c$$

and the equality holds if and only if M is isotropic and has the parallel second fundamental form, where $<,>$ is the Riemannian metric.

S. Suzuki · Y. Matsuyama (✉)
Department of Mathematics, Chuo University, 1-13-27, Kasuga, Bunkyo-ku,
Tokyo 112-8551, Japan
e-mail: matuyama@math.chuo-u.ac.jp

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Xia [8] showed: Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then the Ricci curvature satisfies the following:

$$Ric \geq (n-1)c - \frac{p(n+2)}{2(n+p+2)}c \text{ and } T = k <,>$$

if and only if one of the following conditions is satisfied:

(A) $Ric \equiv (n-1)c$ and M is totally geodesic.
(B) $Ric = (n-1)c - \dfrac{p(n+2)}{2(n+p+2)}c$ and M is isotropic and has the parallel second fundamental form.

Using the result of Sakamoto [7], we know that M which is isotropic with parallel second fundamental form is a compact rank one symmetric space. Hence, if the immersion ψ of M into $S^{n+p}(c)$ is full, then ψ is one of the following standard ones: $S^n(c) \to S^n(c)$; $PR^2(\frac{1}{3}c) \to S^4(c)$; $S^2(\frac{1}{3}c) \to S^4(c)$; $CP^2(c) \to S^7(c)$; $QP^2(\frac{3}{4}c) \to S^{13}(c)$; $CP^2(\frac{4}{3}c) \to S^{25}(c)$.

Matsuyama [4] proved the following: Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then

$$|\sigma(v,v)|^2 \leq \frac{p}{n+p+2}c \text{ and } T = k <,>$$

if and only if one of the following conditions is satisfied:

(A) $|\sigma(v,v)|^2 \equiv 0$ and M is totally geodesic.
(B) $|\sigma(v,v)|^2 = \dfrac{p}{n+p+2}c$ and M is isotropic and has parallel second fundamental form.

Yuen and Matsuyama [10] proved the following: Let M be an n-dimensional compact minimal submanifold isometrically immersed in $S^{n+p}(c)$ and ψ the immersion. Then

$$|\sigma|^2 \leq \frac{np(n+2)}{2(n+p+2)}c \text{ and } T = k <,>$$

if and only if one of the following conditions is satisfied:

(A) $|\sigma|^2 \equiv 0$ and M is totally geodesic.
(B) $|\sigma|^2 = \dfrac{np(n+2)}{2(n+p+2)}c$ and M is isotropic and has parallel second fundamental form.

Hence, if ψ is full, then ψ is one of the following standard ones: $S^n(c) \to S^n(c)$; $PR^2(\frac{1}{3}c) \to S^4(c)$; $S^2(\frac{1}{3}c) \to S^4(c)$; $CP^2(c) \to S^7(c)$; $QP^2(\frac{3}{4}c) \to S^{13}$

(c); $CP^2(\frac{4}{3}c) \to S^{25}(c)$.

Moreover, they obtain the result of the case of M being complete: Let M be an n-dimensional complete minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then

$$|\sigma|^2 \leq \frac{np(n+2)}{2(n+p+2)}c \text{ and } T = k <,>$$

if and only if one of the following conditions is satisfied:

(A) $|\sigma|^2 \equiv 0$ and M is totally geodesic.
(B) $|\sigma|^2 = \dfrac{np(n+2)}{2(n+p+2)}c$ and M is isotropic and has parallel second fundamental form.

Related to these results, Li and Li [2] obtained without assumption of $T = k <,>$ as follows: Let M be an n-dimensional compact minimal submanifold isometrically immersed in S^{n+p} of curvature 1. We denote $A_1, A_2, \ldots, A_p$ be symmetric $(n \times n)$-matrices $(p \geq 2)$ and $S_{\alpha\beta} = \mathrm{trace}\,{}^tA_\alpha A_\beta$, $S_\alpha = S_{\alpha\alpha} = N(A_\alpha)$ and $S = S_1 + \cdots + S_p$, respectively. Then we have

$$\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq \frac{3}{2}S^2 \quad (*)$$

and the equality holds if and only if one of the following conditions holds:
(1) $A_1 = A_2 = \ldots = A_p = 0$.
(2) Only two of the matrices $A_1, A_2, \ldots, A_p$ are different from zero. Moreover, assuming $A_1 \neq 0$, $A_2 \neq 0$ and $A_3 = \ldots = A_p = 0$, then $S_1 = S_2$, and there exists an orthogonal $(n \times n)$-matrix T such that

$${}^tTA_1T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{0} \\ 0 & -1 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right),$$

$${}^tTA_2T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{0} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right)$$

Using the result, they proved the following: Let M be an n-dimensional compact minimal submanifold in S^{n+p}, $p \geq 2$. If $|\sigma|^2 \leq \frac{2}{3}n$ everywhere on M, then M is either a totally geodesic submanifold or a Veronese surface in S^4.

In the present paper, we would like to consider the case where M is complete The main results are the following:

Theorem 1 *Let M be an n-dimensional complete minimal submanifold in S^{n+p}, $p \geq 2$. If $|\sigma|^2 \leq \frac{2}{3}n$ everywhere on M, then M is isotropic and either a totally geodesic and isotropic submanifold or a Veronese surface in S^4.*

Theorem 2 *Let M be an n-dimensional complete minimal submanifold in $S^{n+p}(c)$. If* $\operatorname{trace} A_\alpha^2 \leq \frac{n(n+2)}{2(n+p+2)}c$ *for any α, then M is isotropic and either a totally geodesic submanifold or M has parallel second fundamental form. Especially, if $n = 2$, then we see that $S^2(c) \to S^2(c)$; $PR^2(\frac{1}{3}c) \to S^4(c)$; $S^2(\frac{1}{3}c) \to S^4(c)$.*

2 Preliminaries

Let $\tilde{M}$ be a Riemannian manifold. We suppose that M is isometrically immersed in an $(n+p)$-dimensional Riemannian manifold $\tilde{M}$. Let UM be unit tangent bundle of M and UM_x the fibre of UM over a point x of M. We denote by $<,>$ the metric of $\tilde{M}$ as well as that induced on M. Let ∇ and σ be the Riemannian connection and the second fundamental form of the immersion, respectively. When $\nabla^\perp$ is the normal connection, the first and the second covariant derivatives of the normal valued tensor σ are given by

$$(\nabla\sigma)(X, Y, Z) = \nabla_X^\perp(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

and

$$\begin{aligned}(\nabla^2\sigma)(X, Y, Z, W) = &\nabla_X^\perp((\nabla\sigma)(Y, Z, W)) - (\nabla\sigma)(\nabla_X Y, Z, W)\\ &-(\nabla\sigma)(Y, \nabla_X Z, W) - (\nabla\sigma)(Y, Z, \nabla_X W),\end{aligned}$$

respectively, for any vector fields X, Y, Z and W tangent to M. Let R and $R^\perp$ denote the curvature tensor associated with ∇ and $\nabla^\perp$, respectively. Then σ and $\nabla\sigma$ are symmetric and for $\nabla^2\sigma$, we have the Ricci-identity

$$\begin{aligned}&(\nabla^2\sigma)(X, Y, Z, W) - (\nabla^2\sigma)(Y, X, Z, W) \\ &= R^\perp(X, Y)\sigma(Z, W) - \sigma(R(X, Y)Z, W) - \sigma(Z, R(X, Y)W).\end{aligned} \tag{1}$$

Let $S^{n+p}(c)$ be an $(n+p)$-dimensional Euclidean sphere of constant curvature c. We replace $S^{n+p}(c)$ with $\tilde{M}$. If Ric is the Ricci tensor of M, since M is a minimal submanifold in $S^{n+p}(c)$, then from the Gauss equation we have

$$Ric(v, w) = (n-1)c < v, w > - \sum_{i=1}^{n} < A_{\sigma(v,e_i)}e_i, w > . \tag{2}$$

Now let $v \in UM_x$, $x \in M$. If $e_2, \ldots, e_n$ are orthonormal vectors in UM_x orthogonal to v, then we can consider $\{e_2, \ldots, e_n\}$ as an orthonormal basis of $T_v(UM_x)$. We remark that $\{v = e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of T_xM. If we denote the Laplacian of $UM_x \sim S^{n-1}$ by Δ, then $\Delta f = e_2e_2 f + \cdots + e_ne_n f$, where f is a differentiable function on UM_x.

Define a function f_1 on UM_x, $x \in M$, by

$$f_1(v) = \sum_{i=1}^{n} < A_{\sigma(v,e_i)}v,\, A_{\sigma(v,v)}e_i > .$$

Noting that $\nabla_{e_k} v = -e_k$, $\nabla_{e_k} e_\ell = \delta_{k\ell} v$, $k, \ell = 2, \ldots, n$, we have

$$\begin{aligned}(\Delta f_1) &= \sum_{k=2}^{n} (\nabla f_1)(e_k, e_k, v) \\ &= -\sum_{k=2}^{n} \nabla_{e_k} (\sum_{i=1}^{n} < A_{\sigma(e_k,e_i)}v,\, A_{\sigma(v,v)}e_i > \\ &\quad + \sum_{i=1}^{n} < A_{\sigma(v,e_i}e_k,\, A_{\sigma(v,v)}e_i > +2\sum_{i=1}^{n} < A_{\sigma(v,e_i}v,\, A_{\sigma(e_k,v)}e_i > \\ &= -4\sum_{k=2}^{n} f_1(v) + 2\sum_{i=1}^{n}\sum_{k=2}^{n} < A_{\sigma(e_k,e_i}e_k,\, A_{\sigma(v,v)}e_i > \\ &\quad +4\sum_{i=1}^{n}\sum_{k=2}^{n} < A_{\sigma(e_k,e_i}v,\, A_{\sigma(e_k,v)}e_i > +4\sum_{i=1}^{n}\sum_{k=2}^{n} < A_{\sigma(v,e_i}e_k,\, A_{\sigma(e_k,v)}e_i >\end{aligned}$$

Using the minimality of M, we can prove that

$$\begin{aligned}(\Delta f_1)(v) &= -4(n+2)f_1(v) + 2\sum_{i,j=1}^{n} < A_{\sigma(e_j,e_i)}e_j,\, A_{\sigma(v,v)}e_i > \\ &\quad +4\sum_{i,j=1}^{n} < A_{\sigma(e_j,e_i)}v,\, A_{\sigma(e_j,v)}e_i > +4\sum_{i,j=1}^{n} < A_{\sigma(v,e_i)}e_j,\, A_{\sigma(e_j,v)}e_i >\end{aligned} \tag{3}$$

Similarly, define $f_2(v), f_3(v), \cdots, f_{15}(v)$ and $f_{16}(v)$ by

$$f_2(v) = \sum_{i=1}^{n} < A_{\sigma(e_j,e_i)}e_j,\, A_{\sigma(v,v)}e_i >,$$

$$f_3(v) = \sum_{i=1}^{n} < A_{\sigma(v,v)}v,\, A_{\sigma(v,e_i)}e_i >,$$

$$f_4(v) = \sum_{i,j=1}^{n} < A_{\sigma(e_j,e_i)}e_j, A_{\sigma(v,e_i)}v >,$$

$$f_5(v) = \sum_{i,j=1}^{n} < A_{\sigma(e_i,v)}e_i, A_{\sigma(v,e_j)}e_j >,$$

$$f_6(v) = \sum_{i=1}^{n} < A_{\sigma(v,v)}e_i, A_{\sigma(v,v)}e_i >,$$

$$f_7(v) = |\sigma(v,v)|^2,$$

$$f_8(v) = \sum_{i,j=1}^{n} < A_{\sigma(v,e_i)}e_j, A_{\sigma(e_j,v)}e_i >,$$

$$f_9(v) = \sum_{i,j=1}^{n} < A_{\sigma(e_j,v)}e_i, A_{\sigma(e_j,v)}e_i >,$$

$$f_{10}(v) = \sum_{i=1}^{n} < A_{\sigma(v,e_i)}e_i, v >$$

$$f_{11}(v) = |A_{\sigma(v,v)}v|^2,$$

$$f_{12}(v) = \sum_{i=1}^{n} < A_{\sigma(v,e_i)}v, A_{\sigma(v,e_i)}v >,$$

$$f_{13}(v) = |\sigma(v,v)|^4,$$

$$f_{14}(v) = \sum_{i=1}^{n} < A_{\sigma(v,e_i)}e_i, v > |\sigma(v,v)|^2,$$

$$f_{15}(v) = (\sum_{i=1}^{n} < A_{\sigma(v,e_i)}e_i, v >)^2,$$

$$f_{16}(v) = |\sigma|^2|\sigma(v,v)|^2,$$

respectively. Then we obtain

$$(\Delta f_2)(v) = -2nf_2(v), \tag{4}$$

$$(\Delta f_3)(v) = -4(n+2)f_3(v) + 2f_4(v) + 4f_5(v) + 2f_2(v), \tag{5}$$

$$(\Delta f_4)(v) = -2nf_4(v) + 2\sum_{i,j,k=1}^{n} < A_{\sigma(e_j,e_i)}e_j, A_{\sigma(e_k,e_i)}e_k >, \tag{6}$$

$$(\Delta f_5)(v) = -2nf_5(v) + 2\sum_{i,j,k=1}^{n} < A_{\sigma(e_j,e_i)}e_j, A_{\sigma(e_k,e_i)}e_k >, \tag{7}$$

$$(\Delta f_6)(v) = -4(n+2) f_6(v) + 8 f_9(v), \tag{8}$$

$$(\Delta f_7)(v) = -4(n+2) f_7(v) + 8 \sum_{i=1}^{n} < A_{\sigma(v,e_i)} e_i, v >, \tag{9}$$

$$(\Delta f_8)(v) = -2n f_8(v) + 2 \sum_{i,j,k=1}^{n} < A_{\sigma(e_j,e_i)} e_k, A_{\sigma(e_k,e_i)} e_j >, \tag{10}$$

$$(\Delta f_9)(v) = -2n f_9(v) + 2 \sum_{i,j,k=1}^{n} < A_{\sigma(e_j,e_k)} e_j, A_{\sigma(e_j,e_k)} e_i >, \tag{11}$$

$$(\Delta f_{10})(v) = -2n f_{10}(v) + 2|\sigma|^2 \tag{12}$$

$$\begin{aligned}(\Delta f_{11})(v) = {} & -6(n+4) f_{11}(v) + 8 f_3(v) + 2 f_6(v) \\ & +8 f_{12}(v) + 8 \sum_{i=1}^{n} < A_{\sigma(v,v)} e_i, A_{\sigma(v,e_i)} v >\end{aligned} \tag{13}$$

$$\begin{aligned}(\Delta f_{12})(v) = {} & -4(n+2) f_{12}(v) + 4 f_4(v) + 2 f_9(v) \\ & +4 \sum_{i=1}^{n} < A_{\sigma(e_j,e_i)} v, A_{\sigma(v,e_i)} e_j > +4 \sum_{i=1}^{n} < A_{\sigma(e_j,e_i)} v, A_{\sigma(e_j,e_i)} v >\end{aligned} \tag{14}$$

$$\begin{aligned}(\Delta f_{13})(v) &= -8(n+6) f_{13}(v) + 32 f_{11}(v) + 16 f_{14}(v) \\ &\geq -8(n+2) f_{11}(v) + 16 f_{14}(v)\end{aligned} \tag{15}$$

$$(\Delta f_{14})(v) = -6(n+4) f_{14}(v) + 16 f_3(v) + 8 f_{12}(v) + 2 f_{16}(v), \tag{16}$$

$$(\Delta f_{15})(v) = -4(n+2) f_{15}(v) + 8 f_5(v) + 4|\sigma|^2 \sum_{i=1}^{n} < A_{\sigma(v,e_i)} e_i, v >, \tag{17}$$

$$(\Delta f_{16})(v) = -4(n+2) f_{16}(v) + 8|\sigma|^2 \sum_{i=1}^{n} < A_{\sigma(v,e_i)} e_i, v > . \tag{18}$$

The following generalized maximum principle due to Omori [6] and [9] will be used in order to prove our theorems.

Generalized Maximum principle (Omori [6] and Yau [9]). *Let M^n be a complete Riemannian manifold whose Ricci curvature is bounded form below and $f \in C^2(M)$ a function bounded from above on M^n. Then, for any $\epsilon > 0$, there exists a point $p \in M^n$ such that*

$$f(p) \geq \sup f - \epsilon, \; ||\text{grad} f|| < \epsilon, \; \Delta f(p) < \epsilon.$$

3 Lemma

We have the following (See [3, 5]):

Lemma *Let M be an n-dimensional minimal submanifold isometrically immersed in $S^{n+p}(c)$. Then for $v \in UM_x$ we have*

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) &= \sum_{i=1}^{n}|(\nabla\sigma)(e_i, v, v)|^2 + nc|\sigma(v, v)|^2 \\ &+ 2\sum_{i=1}^{n} < A_{\sigma(v,v)}e_i, A_{\sigma(e_i,v)}v > -2\sum_{i=1}^{n} < A_{\sigma(v,e_i)}e_i, A_{\sigma(v,v)}v > \\ &- \sum_{i=1}^{n} < A_{\sigma(v,v)}e_i, A_{\sigma(v,v)}e_i > . \\ &= \sum_{i=1}^{n}|(\nabla\sigma)(e_i, v, v)|^2 + ncf_7(v) + 2f_1(v) - 2f_3(v) - f_6(v)\end{aligned}$$

Proof The second covariant derivatives of $f_7(v)$ is given by

$$\begin{aligned}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) &= 2\sum_{i=1}^{n} < (\nabla^2 h)(e_i, e_i, v, v), h(v, v) > \\ &\quad 2\sum_{i=1}^{n} < (\nabla h)(e_i, v, v), (\nabla h)(e_i, v, v) > .\end{aligned}$$

Since M is minimal, from (1) and the Gauss and Ricci equations, it follows:

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) \qquad (19)\\ &= \sum_{i=1}^{n}|(\nabla\sigma)(e_i, v, v)|^2 + \sum_{i=1}^{n} < (\nabla^2 h)(e_i, e_i, v, v), h(v, v) > \\ &= \sum_{i=1}^{n}|(\nabla\sigma)(e_i, v, v)|^2 + \sum_{i=1}^{n} < (\nabla^2 h)(e_i, v, v, e_i), h(v, v) > \\ &= \sum_{i=1}^{n}|(\nabla\sigma)(e_i, v, v)|^2 + \sum_{i=1}^{n} < (\nabla^2 h)(v, e_i, v, e_i), h(v, v) > \\ &+ \sum_{i=1}^{n} R^{\perp}(e_i, v, \sigma(v, e_i), \sigma(v, v)) - \sum_{i=1}^{n} < \sigma(R(e_i, v)v, e_i), \sigma(v, v) >\end{aligned}$$

$$-\sum_{i=1}^{n} < \sigma(v, R(e_i, v)e_i, \sigma(v, v) >$$

$$= \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + \sum_{i=1}^{n} < R^{\perp}(e_i, v, \sigma(v, e_i), \sigma(v, v) >$$
$$-\sum_{i=1}^{n} R(e_i, v, v, A_{\sigma(v,v)}e_i) + \sum_{i=1}^{n} R(v, e_i, e_i, A_{\sigma(v,v)}v)$$
$$= \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + \sum_{i=1}^{n} < [A_{\sigma(v,e_i)}, A_{\sigma(v,v)}]e_i, v >$$
$$-\sum_{i=1}^{n} R(e_i, v, v, A_{\sigma(v,v)}e_i) - \sum_{i=1}^{n} < \sigma(A_{\sigma(v,v)}e_i, e_i), \sigma(v, v) >$$
$$-\sum_{i=1}^{n} < \sigma(A_{\sigma(v,v)}e_i, \sigma(v, e_i) > +Ric(v, A_{\sigma(v,v)}v) - \sum_{i=1}^{n} < \sigma(A_{\sigma(v,v)}e_i, e_i), \sigma(v, v) >$$
$$= \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + 2\sum_{i=1}^{n} < A_{\sigma(v,v)}e_i, A_{\sigma(v,e_i)}v >$$
$$-\sum_{i=1}^{n} < A_{\sigma(v,e_i)}e_i, A_{\sigma(v,v)}v > - \sum_{i=1}^{n} < A_{\sigma(v,v)}e_i, A_{\sigma(v,v)}e_i > +Ric(v, A_{\sigma(v,v)}v)$$
$$-c\sum_{i=1}^{n}(< e_i, A_{\sigma(v,v)}e_i >< v, v > + < e_i, v >< v, A_{\sigma(v,v)}e_i >).$$

Then, from (2) and using minimality, we have

$$Ric(v, A_{\sigma(v,v)}v) - c\sum_{i=1}^{n}(< e_i, A_{\sigma(v,v)}e_i >< v, v > \tag{20}$$
$$+ < e_i, v >< v, A_{\sigma(v,v)}e_i >)$$
$$= \sum_{i=1}^{n}(n-1)c < v, A_{\sigma(v,v)}v > - \sum_{i=1}^{n} < A_{\sigma(v,e_i)}e_i, A_{\sigma(v,v)}v >$$
$$-c\sum_{i=1}^{n} < \sigma(e_i, e_i), \sigma(v, v) > +c\sum_{i=1}^{n} < e_i, v >< \sigma(v, e_i), \sigma(v, v) >$$
$$= nc|\sigma(v, v)|^2 - \sum_{i=1}^{n} < A_{\sigma(v,e_i)}e_i, A_{\sigma(v,v)}v >$$

Thus, from (19) and (20), we obtain the Lemma.

4 Proof of Theorems

From the assumption of theorem, we have

$$|\sigma|^2 \le \frac{2}{3}c. \tag{21}$$

The following equations hold for $v \in UM_x$, $x \in M$:

$$\sum_{i,j=1}^{n} < A_{\sigma(e_j,e_i)}v, A_{\sigma(e_j,v)}e_i >= \sum_{i,j=1}^{n} < A_{\sigma(v,e_i)}e_j, A_{\sigma(v,e_j)}e_i >, \tag{22}$$

$$\sum_{i,j=1}^{n} < A_{\sigma(e_j,e_i)}v, A_{\sigma(e_j,e_i)}v >= \sum_{i,j=1}^{n} < A_{\sigma(v,e_i)}e_j, A_{\sigma(v,e_i)}e_j > . \tag{23}$$

Hence, we get

$$(\Delta f_1)(v) = -4(n+2)f_1(v) + 2f_2(v) + 8f_8(v). \tag{24}$$

Summing up $\frac{1}{2(n+2)}((\Delta f_1)(v) + \frac{1}{n}(\Delta f_2)(v))$ the both sides of lemma and using (24) and (4), we have

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) + \frac{1}{2(n+2)}((\Delta f_1)(v) + \frac{1}{n}(\Delta f_2)(v)) \\ &= \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + ncf_7(v) - 2f_3(v) - f_6(v) + \frac{4}{n+2}f_8(v) \end{aligned} \tag{25}$$

Secondly, subtracting $\frac{1}{2(n+2)}((\Delta f_3)(v) + \frac{1}{n}(\Delta f_2)(v)) + \frac{1}{4(n+2)}(\Delta f_6)(v)$ from the both sides of (25) and using (4), (5) and (8), we have

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) + \frac{2}{2(n+2)}((\Delta f_1)(v) + \frac{1}{n}(\Delta f_2)(v)) \\ &\quad - \frac{1}{2(n+2)}((\Delta f_3)(v) + \frac{1}{n}(\Delta f_2)(v)) - \frac{1}{4(n+2)}(\Delta f_6)(v) \end{aligned} \tag{26}$$

$$= \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + ncf_7(v) - \frac{2}{n+2} f_4(v) - \frac{2}{n+2} f_5(v)$$
$$+ \frac{4}{n+2} f_8(v) - \frac{2}{n+2} f_9(v).$$

Thirdly, adding $\frac{1}{n(n+2)}(\Delta f_4)(v) - \frac{1}{n(n+2)}(\Delta f_5)(v) + \frac{nc}{4(n+2)}(\Delta f_7)(v)$ in the both sides of (26) and using (6), (7) and (9), we have

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) + \frac{1}{2(n+2)}((\Delta f_1)(v) + \frac{1}{n}(\Delta f_2)(v)) \\
&- \frac{1}{2(n+2)}((\Delta f_3)(v) + \frac{1}{n}(\Delta f_2)(v)) - \frac{1}{4(n+2)}(\Delta f_6)(v) \\
&+ \frac{1}{n(n+2)}(\Delta f_4)(v) - \frac{1}{n(n+2)}(\Delta f_5)(v) + \frac{nc}{4(n+2)}(\Delta f_7)(v) \\
= &\sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + \frac{2nc}{n+2} f_{10}(v) - \frac{4}{n+2} f_4(v) + \frac{4}{n+2} f_8(v) - \frac{2}{n+2} f_9(v).
\end{aligned} \tag{27}$$

Finally, summing up $\frac{c}{n+2}(\Delta f_{10})(v)$ on both sides of (27) and using (12), we have

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) + \frac{2}{2(n+2)}((\Delta f_1)(v) + \frac{1}{n}(\Delta f_2)(v)) \\
&- \frac{1}{2(n+2)}((\Delta f_3)(v) + \frac{1}{n}(\Delta f_2)(v)) - \frac{1}{4(n+2)}(\Delta f_6)(v) \\
&+ \frac{1}{n(n+2)}(\Delta f_4)(v) - \frac{1}{n(n+2)}(\Delta f_5)(v) + \frac{nc}{4(n+2)}(\Delta f_7)(v) \\
&+ \frac{c}{n+2}(\Delta f_{10})(v) \\
= &\sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + \frac{2c}{n+2}|\sigma|^2 - \frac{4}{n+2} f_4(v) + \frac{4}{n+2} f_8(v) - \frac{2}{n+2} f_9(v).
\end{aligned} \tag{28}$$

With respect to $(\Delta f_4)(v)$, $(\Delta f_8)(v)$, $(\Delta f_9)(v)$, we can rewrite

$$\begin{aligned}
(\Delta f_4)(v) &= -2nf_4(v) + 2\sum_{i,j,k=1}^{n} < A_{\sigma(e_j, e_i)}e_j, A_{\sigma(e_k, e_i)}e_k > \\
&= -2nf_4(v) + 2\sum_{\alpha,\beta=1}^{p} \mathrm{trace} A_\alpha^2 A_\beta^2,
\end{aligned}$$

$$
\begin{aligned}
(\Delta f_8)(v) &= -2nf_8(v) + 2\sum_{i,j,k=1}^{n} < A_{\sigma(e_j,e_i)}e_k, A_{\sigma(e_k,e_i)}e_j > \\
&= -2nf_8(v) + 2\sum_{\alpha,\beta=1}^{p} \mathrm{trace}(A_\alpha A_\beta)^2, \\
(\Delta f_9)(v) &= -2nf_9(v) + 2\sum_{i,j,k=1}^{n} < A_{\sigma(e_j,e_k)}e_i, A_{\sigma(e_j,e_k)}e_i > \\
&= -2nf_9(v) + 2\sum_{\alpha,\beta=1}^{p} (\mathrm{trace} A_\alpha A_\beta)^2.
\end{aligned}
$$

Then, from (27) and [2], we have the following:

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) + \frac{2}{2(n+2)}((\Delta f_1)(v) - \frac{1}{2(n+2)}((\Delta f_3)(v) \\
&-\frac{1}{n(n+2)}(\Delta f_4)(v) - \frac{1}{n(n+2)}(\Delta f_5)(v) - \frac{1}{4(n+2)}(\Delta f_6)(v) \\
&+\frac{nc}{4(n+2)}(\Delta f_7)(v) + \frac{2}{n(n+2)}(\Delta f_8)(v) - \frac{1}{n(n+2)}(\Delta f_9)(v) \\
&+\frac{c}{n+2}(\Delta f_{10})(v) \\
= &\sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + \frac{2c}{n+2}|\sigma|^2 - \frac{4}{n(n+2)}\sum \mathrm{trace} A_\alpha^2 A_\beta^2 \\
&+\frac{4}{n(n+2)}\sum \mathrm{trace}(A_\alpha A_\beta)^2 - -\frac{2}{n(n+2)}\sum (\mathrm{trace} A_\alpha A_\beta)^2 \\
&\geq \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + \frac{2}{n(n+2)}\{nc|\sigma|^2 - (N(A_\alpha A_\beta - A_\beta A_\alpha) + S_{\alpha\beta})\} \\
&\geq \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + \frac{2}{n(n+2)}|\sigma|^2(nc - \frac{3}{2}|\sigma|^2).
\end{aligned} \tag{29}
$$

Define a function g on $U_x M$ by the following equation:

$$
\begin{aligned}
g(v) = &\frac{2}{2(n+2)} f_1(v) - \frac{1}{2(n+2)} f_3(v) - \frac{1}{n(n+2)} f_4(v) \\
&-\frac{1}{n(n+2)} f_5(v) - \frac{1}{4(n+2)} f_6(v) + \frac{nc}{4(n+2)} f_7(v) \\
&+\frac{2}{n(n+2)} f_8(v) - \frac{1}{n(n+2)} f_9(v) + \frac{c}{n+2} f_{10}(v).
\end{aligned} \tag{30}
$$

Related to $f_3(v)$, we can rewrite

$$f_3(v) = \sum_{i=1}^{n} < A_{\sigma(v,v)}v, A_{\sigma(v,e_i)}e_i >$$
$$= \sum_{\alpha,\beta=1}^{p} < A_{\xi_\beta}v, v >< A_{\xi_\beta}v, A^2_{\xi_\alpha}v > .$$

Now, since $Ric = (n-1)I - \sum_{\alpha=1}^{p} A^2_{\xi_\alpha}$ is symmetric, we can choose an orthonormal basis $\{v = e_1, e_2, \ldots, e_n\}$ such that the matrix $\sum_{\alpha=1}^{p} A^2_{\xi_\alpha}$ is diagonalized, where $\{\xi_1, \xi_2, \ldots, \xi_p\}$ is any orthonormal normal basis and $1 \leq \alpha \leq p$. Then we obtain

$$f_3(v) = \lambda \sum_{\alpha=1}^{p} < A_{\xi_\alpha}v, v >^2 \geq 0, \tag{31}$$

where λ is an eigenvalue of $\sum_{\alpha=1}^{p} A^2_{\xi_\alpha}$ corresponding to v.
With respect to $f_4(v)$, we have

$$f_4(v) = \sum_{i,j=1}^{n} < A_{\sigma(e_j,e_i)}e_j, A_{\sigma(v,e_i)}v > \tag{32}$$
$$= \sum_{i,j=1}^{n} < A^2_\alpha A^2_\beta v, v > \geq 0.$$

Similarly, we can show

$$f_5(v) \geq 0, \tag{33}$$

$$f_6(v) \geq 0, \tag{34}$$

$$f_7(v) \geq 0. \tag{35}$$

By (30), (31), (32), (33), (34) and (35) we get

$$g(v) \leq \frac{2}{2(n+2)} f_1(v) + \frac{nc}{4(n+2)} f_7(v) + \frac{2}{n(n+2)} f_8(v) + \frac{c}{n+2} f_{10}(v). \tag{36}$$

From the assumption of (21), we see that the Ricci curvature is bounded from below. Taking the contraction with respect to above basis of the both sides of (29) corresponding to Y_1 and Y_4 for $\sum_{i=1}^{n} < (\nabla\sigma)(e_i, Y_1, Y_2), (\nabla\sigma)(e_i, Y_3, Y_4 >$ and noting that (36) by the Generalized Maximum Principle due to Omori [6] and Yau [9], we can prove $\nabla\sigma(e_i, e_j, e_k) = 0$ and $|\sigma|^2 \equiv 0$ or $|\sigma|^2 \equiv \frac{2}{3}nc$. If $|\sigma|^2 \equiv 0$, then M is totally geodesic. Assume that $|\sigma|^2 \equiv \frac{2}{3}nc$. Then we may assume that

$$A_1^* = \frac{\sqrt{\frac{2n}{3}c}}{2} \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline 0 & & 0 \end{array} \right),$$

$$A_2^* = \frac{\sqrt{\frac{2n}{3}c}}{2} \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline 0 & & 0 \end{array} \right)$$

$$A_\alpha^* = 0 \quad \text{for } \alpha \geq n+3$$

by [2]. The same argument as in [1] shows that $\dim M = 2$ and 2-dimensional surface with $|\sigma|^2 = \frac{4}{3}c$. The surface must be a Veronese surface in $S^4(c)$ (See [1]).

On the other hand, by those equations and lemma, we get the following for $v \in UM_x$:

$$\begin{aligned} &\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) - \frac{1}{6}(\Delta f_{11})(v) - \frac{1}{3(n+2)}(\Delta f_{12})(v) \\ &+\frac{1}{6(n+2)}(\Delta f_1)(v) + \frac{1}{3n(n+2)}(\Delta f_2)(v) + \frac{1}{6(n+2)}(\Delta f_3)(v) \\ &-\frac{1}{3n(n+2)}(\Delta f_4)(v) + \frac{1}{3n(n+2)}(\Delta f_5)(v) + \frac{1}{6(n+2)}(\Delta f_6)(v) \\ =& \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + ncf_7(v) + (n+4) f_{11}(v) - 4f_3(v) - 2f_6(v) \end{aligned} \tag{37}$$

In terms of (15) and (37), we have

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) - \frac{1}{6}(\Delta f_{11})(v) - \frac{1}{3(n+2)}(\Delta f_{12})(v) \\
&+\frac{1}{6(n+2)}(\Delta f_1)(v) + \frac{1}{3n(n+2)}(\Delta f_2)(v) + \frac{1}{6(n+2)}(\Delta f_3)(v) \\
&-\frac{1}{3n(n+2)}(\Delta f_4)(v) + \frac{1}{3n(n+2)}(\Delta f_5)(v) + \frac{1}{6(n+2)}(\Delta f_6)(v) \\
&+\frac{n+4}{8(n+2)}(\Delta f_{13})(v) \\
\geq &\sum_{i=1}^{n}|(\nabla\sigma)(e_i, v, v)|^2 + ncf_7(v) + \frac{2(n+4)}{n+2} f_{14}(v) - 4 f_3(v) - 2 f_6(v).
\end{aligned} \tag{38}$$

By (3), (4), (5), (14), (16), (17) and (18) we have

$$\begin{aligned}
&-\frac{1}{3(n+2)}(-\frac{2}{n+2}(\frac{1}{n}(\Delta f_2)(v) + (\Delta f_3)(v) + \frac{2}{n}(\Delta f_4)(v) \\
&-\frac{2}{n}(\Delta f_5)(v)) + (\Delta f_{14})(v) - \frac{1}{n+2}((\Delta f_{16})(v) - 2(\Delta f_{15})(v))) \\
=& \frac{2n}{n+2} f_{14}(v) - \frac{2}{n+2} f_{16}(v).
\end{aligned} \tag{39}$$

Combining (38) with (39), we have

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}(\nabla^2 f_7)(e_i, e_i, v) - \frac{1}{6}(\Delta f_{11})(v) - \frac{1}{3(n+2)}(\Delta f_{12})(v) \\
&+\frac{1}{6(n+2)}(\Delta f_1)(v) + \frac{n+4}{3n(n+2)^2}(\Delta f_2)(v) + \frac{n+6}{6(n+2)^2}(\Delta f_3)(v) \\
&-\frac{n-2}{3n(n+2)^2}(\Delta f_4)(v) + \frac{n-2}{3n(n+2)^2}(\Delta f_5)(v) + \frac{1}{6(n+2)}(\Delta f_6)(v) \\
&+\frac{n+4}{8(n+2)}(\Delta f_{13})(v) - \frac{1}{3(n+2)}(\Delta f_{14}(v) - \frac{2}{3(n+2)^2}(\Delta f_{15})(v) \\
\geq &\sum_{i=1}^{n}|(\nabla\sigma)(e_i, v, v)|^2 + ncf_7(v) - 2 f_6(v) - \frac{2}{n+2} f_{16}(v) + 4 f_{14}(v) - 4 f_3(v).
\end{aligned} \tag{40}$$

Let we assume codimension = p and

$$\text{trace} A_\alpha^2 \leq \frac{n(n+2)}{2(n+p+2)} c \quad \text{for } \forall\ \alpha$$

everywhere on M, then

$$|\sigma|^2 = \sum_{\alpha=1}^{p} \mathrm{trace} A_\alpha^2 \leq \frac{np(n+2)}{2(n+p+2)} c$$

and we can get

$$f_6(v) = \sum_{i=1}^{n} < A_{\sigma(v,v)} e_i, A_{\sigma(v,v)} e_i > \quad (41)$$
$$= \sum_{i=1}^{n} |\sigma(v,v)|^2 < A_{\frac{\sigma(v,v)}{|\sigma(v,v)|}} e_i, A_{\frac{\sigma(v,v)}{|\sigma(v,v)|}} e_i >$$
$$\leq \frac{n(n+2)}{2(n+p+2)} c |\sigma(v,v)|^2$$

From (40) and (41) we obtain

$$\sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + ncf_7(v) - 2f_6(v) - \frac{2}{n+2} f_{16}(v) + +4f_{14}(v) - 4f_3(v) \quad (42)$$
$$\geq \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + ncf_7(v) - 2f_6(v) - \frac{2p}{n+2} f_6(v) + 4f_{14}(v) - 4f_3(v)$$
$$= \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + ncf_7(v) - \frac{2(n+p+2)}{n+2} f_6(v) + 4f_{14}(v) - 4f_3(v)$$
$$\geq \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + (nc - \frac{2(n+p+2)}{n+2} . \frac{n(n+2)}{2(n+p+2)} c) |\sigma(v,v)|^2 + 4f_{14}(v) - 4f_3(v)$$
$$= \sum_{i=1}^{n} |(\nabla\sigma)(e_i, v, v)|^2 + 4f_{14}(v) - 4f_3(v).$$

As in the proof of Theorem 1, we choose an orthonormal basis $\{v = e_1, e_2, \dots, e_n\}$ such that the matrix $\sum_{\alpha=1}^{p} A_{\xi_\alpha}^2$ is diagonalized, where $\{\xi_1, \xi_2, \dots, \xi_p\}$ is any orthonormal normal basis and $1 \leq \alpha \leq p$. Then we have

$$f_3(v) = f_{14}(v) \quad (43)$$

Taking the contraction with respect to above basis of the both sides of (42) corresponding to Y_1 and Y_4 for $\sum_{i=1}^{n} < (\nabla\sigma)(e_i, Y_1, Y_2), (\nabla\sigma)(e_i, Y_3, Y_4 >$, then we obtain

$$\text{trace} A_\alpha^2 = \frac{n(n+2)}{n+p+2}c \quad \text{for } \forall\, \alpha$$

and we can prove

$$|\sigma|^2 \equiv 0 \quad \text{or } |\sigma|^2 \equiv \frac{np(n+2)}{2(n+p+2)}c \tag{44}$$

by the Maximum principle due to Omori [6] and Yau [9]. From the latter of (44), we see that in the case of $n = 2$ $S^2(c) \to S^2(c)$; $PR^2(\frac{1}{3}c) \to S^4(c)$; $S^2(\frac{1}{3}c) \to S^4(c)$.

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The Study of Ricci-Semi-Symmetry of Normal Complex Contact Manifold

Mohamed Belkhelfa and Fatima Z. Kadi

Abstract It is well known that a Sasakian space form is Ricci-semi-symmetric if and only if it is locally isometric to $\mathbb{S}^{2n+1}(1)$. In this paper, we study the Ricci-semi-symmetry of a normal complex contact manifold, in particular complex contact space form.

Keywords Complex contact manifold · Pseudo-symmetry · Semi-symmetry · Ricci-semi-symmetry

1991 Mathematics Subject Classification 53C15 · 53D10

1 Introduction

This paper is a presentation that the author was invited to give a talk in International Conference on Differential Geometry, Algebra and Analysis (ICDGAA-16) during November 15–17, 2016 at the Department of mathematics, JMI, New Delhi, India. Blair and Mihai [5] proved that a locally symmetry normal complex contact space is locally isometric to $\mathbb{C}P^{2n+1}(4)$, with the Fibini-Study metric. In [4], Blair and Martín-Molina proved that normal complex contact metric manifolds that are Bochner flat must have constant holomorphic sectional curvature 4 and be Kahler, and they showed that it is not possible for normal complex contact metric manifolds to be conformally flat.

In Sect. 2, we recall definitions and some properties of complex contact manifold and the expression of the curvature of complex contact space form. Definitions

M. Belkhelfa · F. Z. Kadi
Laboratory L. P. Q. 3M, Mascara University, Mascara, Algeria
e-mail: kadi.fatimazohra@gmail.com

M. Belkhelfa (✉)
Faculty of Sciences and Arts at Al Ula, Department of Mathematics,
Taibah University, Medina, Saudi Arabia
e-mail: mohamed.belkhelfa@gmail.com

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of semi-symmetry and Ricci-semi-symmetry are given in Sect. 3, we investigate in Sect. 4 the Ricci-symmetry properties of normal complex contact space, we show that a normal complex contact manifold is Ricci-symmetric, or Ricci-semi-symmetric if it is an Einstein manifold.

2 Complex Contact Manifold

We recall some definitions and properties of complex contact manifolds [3, 15, 16]

A complex manifold M with $dim_{\mathbb{C}} M = 2n + 1$ and complex structure J is a complex contact manifold if there exists an open covering $\mathcal{U} = \{\mathcal{O}_\alpha\}$ of M, such that

(1) on each $\mathcal{O}_\alpha$, there is a holomorphic 1-form ω_α with $\omega_\alpha \wedge (d\omega_\alpha)^n \neq 0$ everywhere,
and
(2) if $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$, then there is a nonvanishing holomorphic function $\lambda_{\alpha\beta}$ in $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ such that

$$\omega_\alpha = \lambda_{\alpha\beta}\omega_\beta \; in \; \mathcal{O}_\alpha \cap \mathcal{O}_\beta.$$

On each $\mathcal{O}_\alpha$, we define $\mathcal{H}_\alpha = \{X \in T\mathcal{O}_\alpha | \omega_\alpha(X) = 0\}$. Since $\lambda_{\alpha\beta}$'s are nonvanishing, $\mathcal{H}_\alpha = \mathcal{H}_\beta$ on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$. So $\mathcal{H} = \cup\mathcal{H}_\alpha$ is a well-defined, holomorphic, non-integrable subbundle on M called the horizontal subbundle.

From now on, we will suppress the subscripts if $\mathcal{O}_\alpha$ is understood.

A complex contact manifold M admits a complex almost contact metric structure [13], i.e., local real 1-forms u, $v = uJ$, (1, 1)-tensors G, $H = GJ$, unit vector fields U and $V = -JU$ and a Hermitian metric g such that

$$\begin{gathered}
H^2 = G^2 = -Id + u \otimes U + v \otimes V \\
g(GX, Y) = -g(X, GY), \qquad g(U, X) = u(X) \\
GJ = -JG, \qquad GU = 0, \qquad u(U) = 1,
\end{gathered}$$

and on the overlaps, the above tensors transform as

$$\begin{aligned}
u' &= au - bv, & v' &= bu + av \\
G' &= aG - bH, & H' &= bG + aH
\end{aligned}$$

for some functions a, b defined on the overlaps with $a^2 + b^2 = 1$. As a result of the above identities, on a complex almost contact metric manifold M, the following identities also hold:

$$\begin{aligned}
& HG = -GH = J + u \otimes V - v \otimes U, \\
& JH = -HJ = G, \qquad g(HX, Y) = -g(X, HY), \\
& GV = HU = HV = 0, \qquad uG = vG = uH = vH = 0, \\
& JV = U, \qquad g(U, V) = 0.
\end{aligned}$$

If, in addition, $g(X, GY) = du(X, Y)$ and $g(X, HY) = dv(X, Y)$ for all X, Y in $\mathcal{H}$, we say that M has a *complex contact metric structure*. If M is a complex contact manifold, then it has a complex contact metric structure [10].

On a complex contact metric manifold M, we can write $TM = \mathcal{H} \bigoplus \mathcal{V}$, where $\mathcal{V}$ is the vertical subbundle on M, locally spanned by U and $V = -JU$. We will assume that the subbundle $\mathcal{V}$ is integrable.

From now on, we will work with a complex contact metric manifold M with structure tensors (u, v, U, V, G, H, g) and complex structure J. Define 2-forms $\hat{G}$ and $\hat{H}$ on M by $\hat{G}(X, Y) = g(X, GY)$, $\hat{H}(X, Y) = g(X, HY)$. Then

$$\hat{G} = du - \sigma \wedge v, \quad \hat{H} = dv + \sigma \wedge u \tag{2.1}$$

where $\sigma(X) = g(\nabla_X U, V)$

2.1 Normality on Complex Contact Metric Manifolds

The concept of normality introduced by Ishihara and Konishi [12] is related by vanishing of the two tensor fields S and T given by

$$\begin{aligned} S(X, Y) &= [G, G](X, Y) + 2v(Y)HX - 2v(X)HY + 2g(X, GY)U \\ &\quad - 2g(X, HY)V - \sigma(GX)HY + \sigma(GY)HX + \sigma(X)GHY - \sigma(Y)GHX \\ T(X, Y) &= [H, H](X, Y) + 2u(Y)GX - 2u(X)GY + 2g(X, HY)V \\ &\quad - 2g(X, GY)U + \sigma(HX)GY - \sigma(HY)GX - \sigma(X)HGY + \sigma(Y)HGX \end{aligned}$$

where

$$[G, G](X, Y) = (\nabla_{GX} G)Y - (\nabla_{GY} G)X - G(\nabla_X G)Y + G(\nabla_Y G)X$$

is the Nijenhuis torsion of G and similar for $H,\ X, Y \in \mathfrak{X}(M)$.

It implies that the normality is restricted to the Kählarian space and hence it excludes the Heisenberg group, when this space has a normal contact structure in the real case. Here, we adopt the notion of normality given by Korkmaz [15], which is a generalization of the last one.

Definition 2.1 A complex contact metric manifold M is *normal* if

(1) $S(X, Y) = T(X, Y) = 0$ for all X, Y in $\mathcal{H}$, and
(2) $S(U, X) = T(V, X) = 0$ for all X.

A complex contact manifold with a global holomorphic 1-form is called a *complex Sasakian manifold* [11].

On a normal contact metric manifold, we have

$$\nabla_X U = -GX + \sigma(X)V \tag{2.2}$$
$$\nabla_X V = -HX - \sigma(X)U. \tag{2.3}$$

A complex contact manifold is normal if and only if the covariant derivatives of G and H have the following forms:

$$\begin{aligned} g((\nabla_X G)Y, Z) &= \sigma(X)g(HY, Z) + v(X)d\sigma(GZ, GY) - 2v(X)g(HGY, Z) \\ &\quad - u(Y)g(X, Z) - v(Y)g(JX, Z) + u(Z)g(X, Y) - v(Z)g(X, JY) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} g((\nabla_X H)Y, Z) &= -\sigma(X)g(GY, Z) + u(X)d\sigma(HZ, HY) - 2u(X)g(HGY, Z) \\ &\quad + u(Y)g(JX, Z) - v(Y)g(X, Z) + u(Z)g(X, JY) + v(Z)g(X, Y). \end{aligned} \tag{2.5}$$

For underlying Hermitian structure, we have

$$\begin{aligned} g((\nabla_X J)Y, Z) &= u(X)(d\sigma(Z, GY) - 2g(HY, Z)) \\ &\quad + v(X)(d\sigma(Z, HY) + 2g(GY, Z)), \end{aligned} \tag{2.6}$$

$$R(U, V)V = -2d\sigma(U, V)U, \qquad R(V, U)U = -2d\sigma(U, V)V.$$

Then the sectional curvature $R(U, V, V, U) = -2d\sigma(U, V)$. If M is a complex Sasakian manifold then the sectional curvature of the vertical subbundle is flat [11].

For all horizontal vector fields X and Y, we have (see [15] for details)

$$R(X, U)U = X \qquad R(X, V)V = X, \tag{2.7}$$
$$R(X, Y)U = 2(g(X, JY) + 2d\sigma(X, Y))V, \tag{2.8}$$
$$R(X, Y)V = -2(g(X, JY) + 2d\sigma(X, Y))U, \tag{2.9}$$
$$R(X, U)V = \sigma(U)GX + (\nabla_U H)X - JX, \tag{2.10}$$
$$R(X, V)U = -\sigma(V)HX + (\nabla_U G)X + JX, \tag{2.11}$$
$$R(X, U)Y = -g(X, Y)U + g(X, JY)V + d\sigma(HY, HX)V, \tag{2.12}$$
$$R(X, V)Y = -g(X, Y)V + g(JX, Y)U + d\sigma(HX, HY)U. \tag{2.13}$$

2.2 GH-Sectional Curvature

Let M be a normal complex contact metric manifold with structure tensors u, v, U, V, G, H, J, g. For a horizontal vector field X, the plane section generated by X and $Y = aGX + bHX$, $a^2 + b^2 = 1$ is called a GH-section or an $\mathcal{H}$-holomorphic section. We define the GH-sectional curvature $\mathcal{GH}_{a,b}(X)$ as the curvature of a GH-section:

$$\mathcal{GH}_{a,b}(X) = K(X, aGX + bHX),$$

where $K(X, Y)$ is the curvature of the plane section generated by X and Y.

If the GH-sectional curvature is independent of the choice of GH-section at each point, it is constant on the manifold, and we say that M is *a complex contact space form*. The curvature tensor and the following theorems were obtained by Korkmaz [15]; explicitly, the curvature tensor is

$$\begin{aligned}
R(X,Y)Z &= \frac{c+3}{4}(g(Y,Z)X - g(X,Z)Y + g(Z,JY)JX - g(Z,JX)JY + 2g(X,JY)JZ)\\
&+ \frac{c-1}{4}(-(u(Y)u(Z) + v(Y)v(Z))X + (u(X)u(Z) + v(X)v(Z))Y\\
&+ 2u\wedge v(Z,Y)JX - 2u\wedge v(Z,X)JY + 4u\wedge v(X,Y)JZ\\
&+ g(Z,GY)GX - g(Z,GX)GY + 2g(X,GY)GZ\\
&+ g(Z,HY)HX - g(Z,HX)HY + 2g(X,HY)HZ\\
&+ (-u(X)g(Y,Z) + u(Y)g(X,Z) + v(X)g(JY,Z) - v(Y)g(JX,Z) + 2v(Z)g(X,JY))U\\
&+ (-v(X)g(Y,Z) + v(Y)g(X,Z) - u(X)g(JY,Z) + u(Y)g(JX,Z) - 2u(Z)g(X,JY))V)\\
&- \frac{4}{3}(d\sigma(U,V) + c + 1)((v(X)u\wedge v(Z,Y) - v(Y)u\wedge v(Z,X) + 2v(Z)u\wedge v(X,Y))U\\
&- (u(X)u\wedge v(Z,Y) - u(Y)u\wedge v(Z,X) + 2u(Z)u\wedge v(X,Y))V),
\end{aligned}$$

and the Ricci curvature ρ is given by

$$\rho(X,Y) = ((n+2)c + 3n + 2)g(X,Y) - ((n+2)c - n + 2 + 2d\sigma(U,V))(u(X)\otimes u(Y) + v(X)\otimes v(Y)). \tag{2.14}$$

Example 2.2

– The odd-dimensional complex projective space $\mathbb{C}P^{2n+1}$ with the Fubini Study metric g of constant holomorphic curvature 4 admits a normal complex contact metric structure via the Hopf fibering

$$\pi : \mathbb{S}^{4n+3} \to \mathbb{C}P^{2n+1}.$$

Endowed with this structure the complex projective space $\mathbb{C}P^{2n+1}$ is a complex contact space form with $c = 1$ [15], moreover it is an Einstein space and its Ricci curvature is given by

$$\rho = (4n+4)g.$$

– The complex Heisenberg group $H_{\mathbb{C}}$ is the closed subgroup of $GL(3,\mathbb{C})$ given by [1]

$$\left\{ \begin{pmatrix} 1 & z_2 & z_3 \\ 0 & 1 & z_1 \\ 0 & 0 & 1 \end{pmatrix} / z_1, z_2, z_3 \in \mathbb{C} \right\} \simeq \mathbb{C}^3.$$

If L_B denotes left translation by $B \in H_{\mathbb{C}}$, then $L_B^* dz_1 = dz_1$, $L_B^* dz_2 = dz_2$, $L_B^*(dz_3 - z_2 dz_1) = dz_3 - z_2 dz_1$. The vector fields $\frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}$ are dual to the 1-forms dz_1, dz_2, and $dz_3 - z_2 dz_1$ are left-invariant vector fields. Moreover, relative to the coordinates $(z_1, z_2, z_3, \overline{z}_1, \overline{z}_2, \overline{z}_3)$ the Hermitian metric

$$g = \frac{1}{8} \left(\begin{array}{ccc|ccc} & & & 1+|z_2|^2 & 0 & -z_2 \\ & 0 & & 0 & 1 & 0 \\ & & & -\overline{z}_2 & 0 & 1 \\ \hline 1+|z_2|^2 & 0 & -\overline{z}_2 & & & \\ 0 & 1 & 0 & & 0 & \\ -z_2 & 0 & 1 & & & \end{array} \right)$$

is a left-invariant metric on $H_{\mathbb{C}}$, but is not Kähler metric, the form

$$\theta = \frac{1}{2}(dz_3 - z_2 dz_1) = u - iv$$

is a complex contact structure on $H_{\mathbb{C}}$. Moreover, the tensors G and H and the covariant derivatives of G and H given by

$$G = \left(\begin{array}{ccc|ccc} & & & 0 & 1 & 0 \\ & 0 & & -1 & 0 & 0 \\ & & & 0 & z_2 & 0 \\ \hline 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & 0 & \\ 0 & \overline{z}_2 & 1 & & & \end{array} \right), \quad H = \left(\begin{array}{ccc|ccc} & & & 0 & -i & 0 \\ & 0 & & i & 0 & 0 \\ & & & 0 & -iz_2 & 0 \\ \hline 0 & i & 0 & & & \\ -i & 0 & 0 & & 0 & \\ 0 & i\overline{z}_2 & 1 & & & \end{array} \right)$$

$$(\nabla_X G)Y = g(X,Y)U - u(Y)X - g(X,JY)V - v(Y)JX + 2v(X)GHY,$$
$$(\nabla_X H)Y = g(X,Y)V - v(Y)X - g(X,JY)U + u(Y)JX - 2u(X)GHY.$$

and

$$g(\nabla_X U, V) = \sigma(X) = 0, \qquad \nabla_X U = -GX, \qquad \nabla_X V = -HX.$$

Therefore this space has constant GH-sectional curvature (-3) and a holomorphic curvature 0, moreover its Ricci curvature is given by

$$\rho = -4g + (4n+4)(u \otimes u + v \otimes v).$$

2.3 H-Homothetic Deformation

Let (M, u, v, U, V, G, H, g) be a complex contact metric manifold. For a positive constant α, we define new tensors by

$$\widetilde{u} = \alpha u, \qquad \widetilde{v} = \alpha v, \qquad \widetilde{U} = \frac{1}{\alpha}U, \qquad \widetilde{V} = \frac{1}{\alpha}V, \qquad \widetilde{G} = G, \qquad \widetilde{H} = H,$$
$$\widetilde{g} = \alpha g + \alpha(\alpha - 1)(u \otimes u + v \otimes v).$$

This change of structure is called an H-homothetic deformation. The new structure $(\widetilde{u}, \widetilde{v}, \widetilde{U}, \widetilde{V}, \widetilde{G}, \widetilde{H}, \widetilde{g})$ is also a complex contact metric structure on (M, J), if the given structure is normal (has a constant GH-sectional curvature c), so the new structure is normal (has a constant sectional curvature $\widetilde{c} = \frac{c+3}{\alpha} - 3,$), respectively. Under an H-homothetic deformation, the 1-form σ (and hence the 2-form $\Omega = d\sigma$) does not change moreover $\Omega(U, V) = \alpha^2 \widetilde{\Omega}(\widetilde{U}, \widetilde{V})$ [15].

3 Semi-Symmetry

Let (M^n, g) be connected n-dimensional $(n \geq 3)$ semi-Riemannian manifold of class C^∞, we denote by ∇, R, and ρ the Levi–Civita connection, the Riemannian curvature and the Ricci curvature, respectively,

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

and we define an endomorphism $X \wedge Y$ by

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y.$$

For all vector fields, X, Y and Z on M.

$R(X, Y)$ and $(X \wedge Y)$ are a (1,1)-tensor fields, which can be uniquely extended to a derivation of the tensor algebra $\mathfrak{T}(M)$ over M commute with contraction of the tensor algebra and it is zero on $\mathfrak{F}(M)$ [14], then for every tensor field T of type (0,k) we can define (0,k + 2)-tensor fields $(R.T)$ and $Q(g, T)$ by

$$\begin{aligned}
(R.T)(X_1, X_2, \ldots, X_k; X, Y) &= (R(X, Y).T)(X_1, X_2, \ldots, X_k) \\
&= -\sum_{i=1}^{k} T(X_1, \ldots, X_{i-1}, R(X, Y)X_i, \ldots, X_k) \\
Q(g, T)(X_1, X_2, \ldots, X_k; X, Y) &= ((X \wedge Y).T)(X_1, X_2, \ldots, X_k) \\
&= -\sum_{i=1}^{k} T(X_1, \ldots, X_{i-1}, (X \wedge Y)X_i, \ldots, X_k)
\end{aligned}$$

for all vector fields $X_1, X_2, \ldots, X_k$ on M.

M is called *semi-symmetric* if $R.R = 0$, [17] as a generalization of this notion Deszcz [6, 7], [18] introduced the notion of pseudo-symmetry, M is called *pseudo-symmetric* if $R.R$ and the Tachibana tensor $Q(g, R)$ are linearly dependent, i.e., there exists a function L_R on M such that

$$R.R = L_R Q(g, R)$$

holds on $\mathcal{U}_R = \{x \in M/R - \frac{s}{n(n-1)}\widetilde{G} \neq 0 \text{ at } x\}$, s is a scalar curvature and G is the (0,4) tensor defined by $\widetilde{G}(X_1, X_2, X_3, X_4) = g((X_1 \wedge X_2)X_3, X_4)$.

3.1 Ricci-Semi-Symmetry

M is called *Ricci-semi-symmetric* if $R.\rho = 0$ and it is called *Ricci-pseudo-symmetric* if $R.\rho$ and $Q(g, \rho)$ are linearly dependent, i.e., there exists a function L_ρ on M such that

$$R.\rho = L_\rho Q(g, \rho)$$

holds on $\mathcal{U}_\rho = \{x \in M/\rho - \frac{s}{n}g \neq 0 \text{ at } x\}$. For more details see [6–9, 18]

4 Symmetry Property

Lemma 4.1 *Let (M, u, v, U, V, G, H, g) be a normal complex contact manifold with $dim_{\mathbb{C}}M = 2n + 1$. Then $QU = (4n - 2d\sigma(U, V))U$ and $QV = 4n - 2d\sigma(U, V)V$ where Q is the Ricci operator.*

Proof We choose a local orthonormal basis of the form $\{X_i, GX_i, HX_i, JX_i, U, V\ /1 \le i \le n\}$. We have

$$QU = \sum_{i=1}^{n}[R(U, X_i)X_i + R(U, GX_i)GX_i + R(U, HX_i)HX_i \tag{4.1}$$
$$+ R(U, JX_i)JX_i] + R(U, V)V. \tag{4.2}$$

We use formula (2.12) and get

$$\begin{aligned} QU &= \sum_{i=1}^{n}[g(X_i, X_i) + g(GX_i, GX_i) + g(HX_i, HX_i) \\ &\quad + g(JX_i, JX_i)]U - 2d\sigma(U, V)U \\ &= (4n - 2d\sigma(U, V))U. \end{aligned} \tag{4.3}$$

To compute QV, we use formula (2.13), then we obtian

$$
\begin{aligned}
QV &= \sum_{i=1}^{n}[R(V, X_i)X_i + R(V, GX_i)GX_i + R(V, HX_i)HX_i \\
&+ R(V, JX_i)JX_i] + R(V, U)U \\
&= \sum_{i=1}^{n}[g(X_i, X_i) + g(GX_i, GX_i) + g(HX_i, HX_i) \\
&+ g(JX_i, JX_i)]V - 2d\sigma(U, V)V \\
&= (4n - 2d\sigma(U, V))V. \qquad (4.4)
\end{aligned}
$$

□

If M is a complex Sasakian manifold then $QU = 4nU$ and $QV = 4nV$.

Theorem 4.2 *A normal complex contact manifold M is Ricci-semi-symmetric if and only if it is an Einstein manifold.*

Corollary 4.3 *A complex contact space form is Ricci-semi-symmetric if and only if an Einstein manifold.*

Proof It is clear that every Einstein manifold is Ricci-symmetric then it is Ricci-semi-symmetric.

If M is Ricci-semi-symmetric, then for all vector fields X, Y, Z, and W on M we have

$$
\begin{aligned}
R(X, Y)\rho(Z, W) &= -\rho(R(X, Y)Z, W) - \rho(Z, R(X, Y)W) \\
&= 0 \qquad (4.5)
\end{aligned}
$$

We use formula (2.14), Lemma 4.1 and replacing Y and Z by U, we obtain for a horizontal vector field X

$$
\begin{aligned}
\rho(R(X, U)U, W) + \rho(U, R(X, U)W) &= \rho(X, W) - (4n - 2d\sigma(U, V))g(X, W) \\
&= 0
\end{aligned}
$$

this implies

$$
\rho(X, W) = (4n - d\sigma(U, V))g(X, W). \qquad (4.6)
$$

On the other hand, for an arbitrary vector field $X = X_0 + u(X)U + v(X)V$ on M we have

$$
\begin{aligned}
\rho(X, W) &= \rho(X_0, W) + (4n - d\sigma(U, V))(u(X)u(W) + v(X)v(W)) \\
&= (4n - d\sigma(U, V))[g(X_0, W) + u(X)u(W) + v(X)v(W)] \\
&= (4n - d\sigma(U, V))g(X, W) \qquad (4.7)
\end{aligned}
$$

then M is an Einstein manifold. □

Corollary 4.4 *A normal complex contact manifold M is Ricci-symmetric* ($\nabla \rho = 0$) *if and only if M is an Einstein manifold.*

Corollary 4.5 *A complex Heisenberg group $H_{\mathbb{C}}(-3)$ group is not Ricci-semi-symmetric.*

Remark 4.6

- Under H-homothetic deformation ($\alpha \neq 1$), the curvature of the odd-dimensional complex projective space $\mathbb{C}P^{2n+1}(4)$ with the Fubini-Study metric will be changed then the new structure is not Ricci-semi-symmetric.
- As we mentioned earlier, the complex Heinseberg group is not Ricci-semi-symmetric, moreover its curvature does not change under H-homothetic deformation. Then the complex Heisenberg group $H_{\mathbb{C}}(-3)$ is not H-homothetic to Ricci-semi-symmetric complex contact space form.

For more details on Ricci-pseudo-symmetry and pseudo-symmetry see [2].

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Warped Product Slant Lightlike Submanifolds of Indefinite Kaehler Manifolds

Rashmi Sachdeva, Rachna Rani, Rakesh Kumar, and Satvinder Singh Bhatia

Abstract We study warped product slant lightlike submanifolds of indefinite Kaehler manifolds. We obtain some characterization theorems for the nonexistence of warped product slant lightlike submanifolds of indefinite Kaehler manifolds.

Keywords Indefinite Kaehler manifolds · Slant lightlike submanifolds · Warped product slant lightlike submanifolds

1 Introduction

Warped product manifolds are known to have applications in physics as they provide an excellent setting to model space time. Bishop and O'Neill [2] introduced the notion of warped product manifolds in order to construct a large variety of manifolds of negative curvature. From a geometric point of view, this study got momentum, when the study of warped product of CR-submanifolds of Kaehler manifolds were introduced by Chen [5, 6]. Following this field, many geometers started working along this line and presented numerous results. Recently, Sahin [18] obtained some results on warped product submanifolds of Kaehler manifolds with a slant factor. Although, there are significant applications of warped product submanifolds in the general theory of relativity, a very limited specific information is available on its

R. Sachdeva · R. Kumar (✉)
Department of Basic and Applied Sciences, Punjabi University, Patiala, India
e-mail: dr_rk37c@yahoo.co.in

R. Sachdeva
e-mail: rashmi.sachdeva86@gmail.com

R. Rani
Department of Mathematics, University College, Ghanaur, Patiala, Punjab, India
e-mail: rachna@pbi.ac.in

S. S. Bhatia
School of Mathematics, Thapar Institute of Engineering and Technology, Patiala, India
e-mail: ssbhatia@thapar.edu

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lightlike case. This motivated the geometers to carry out work on the geometry of warped product lightlike submanifolds.

On the other hand, the notion of slant submanifolds was initiated by Chen, as a generalization of both holomorphic and totally real submanifolds in complex geometry [3, 4]. Since then such submanifolds have been studied by many authors and they all studied the geometry of slant submanifolds with positive definite metrics. Therefore, this geometry may not be applicable to the other branches of mathematics and physics, where the metric is not necessarily positive definite. Thus, the geometry of slant submanifolds with indefinite metric became a topic of interest and Sahin [17] played a very crucial role in this study by introducing the notion of slant lightlike submanifolds of indefinite Hermitian manifolds. In [12, 13], we have studied slant lightlike submanifolds of indefinite contact manifolds and discussed the nonexistence of totally contact umbilical slant lightlike submanifolds. Recently in [14], we have explored warped product slant lightlike submanifolds of indefinite Sasakian manifolds and obtained some characterization theorems for the nonexistence of warped product slant lightlike submanifolds of indefinite Sasakian manifolds. In the present chapter, the theory of warped product submanifolds has been clubbed with slant lightlike submanifolds. In particular, we study warped product slant lightlike submanifolds of indefinite Kaehler manifolds and explore the nonexistence of warped product slant lightlike submanifolds of indefinite Kaehler manifolds.

2 Preliminaries

A $2k$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g}, J)$ of constant index q, $0 < q < 2k$, is called an indefinite almost Hermitian manifold if there exists a tensor field J of type $(1, 1)$ on $\bar{M}$ such that

$$J^2 = -I, \quad \text{and} \quad \bar{g}(X, Y) = \bar{g}(JX, JY), \quad \forall X, Y \in \Gamma(T\bar{M}), \tag{1}$$

where I denotes the identity transformation of $T_p\bar{M}$. Moreover, $\bar{M}$ is called an indefinite Kaehler manifold [1] if J is parallel with respect to $\bar{\nabla}$, that is,

$$(\bar{\nabla}_X J)Y = 0, \quad \forall X, Y \in \Gamma(T\bar{M}), \tag{2}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $Rad(TM)$ is called the radical distribution on M.

If $S(TM)$ be a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, called the screen distribution, then

$$TM = Rad(TM) \perp S(TM),$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{3}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{4}$$

Let $\mathcal{U}$ be a local coordinate neighborhood of M and $\{\xi_1, ..., \xi_r, W_{r+1}, ..., W_n, N_1, ..., N_r, X_{r+1}, ..., X_m\}$ be the local quasi-orthonormal fields of frames of $\bar{M}$, where $\{\xi_1, ..., \xi_r\}, \{N_1, ..., N_r\}$ are local lightlike bases of $\Gamma(Rad(TM)|_{\mathcal{U}})$, $\Gamma(ltr(TM)\mid_{\mathcal{U}})$ and $\{W_{r+1}, ..., W_n\}, \{X_{r+1}, ..., X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})\mid_{\mathcal{U}})$ and $\Gamma(S(TM)\mid_{\mathcal{U}})$, respectively. For this quasi-orthonormal fields of frames, we have

Theorem 2.1 ([7]) *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_{\mathcal{U}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_{\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad \textit{for any} \quad i, j \in \{1, 2, .., r\}, \tag{5}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (4), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{6}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is linear a operator on M, known as shape operator.

According to (3), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, then (6) gives

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{7}$$

where we put $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$. As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued, respectively, and therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular,

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{8}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{9}$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. By using (3)–(4) and (7)–(9), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{10}$$

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0. \tag{11}$$

3 Slant Lightlike Submanifolds

A lightlike submanifold has two distributions, namely, the radical distribution and the screen distribution. The radical distribution is totally lightlike and it is not possible to define the angle between two vector fields of radical distribution. Furthermore, the screen distribution is nondegenerate. There are some definitions for angle between two vector fields in Lorentzian setup [11], but not appropriate for our goal because a manifold with Lorentzian metric cannot admit an almost Hermitian structure. Therefore to introduce the notion of slant lightlike submanifolds (one needs a Riemannian distribution), Sahin [17] proved the following lemma.

Lemma 3.1 *Let M be an r-lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2r$. Suppose that $JRad(TM)$ is a distribution on M such that $Rad(TM) \cap JRad(TM) = \{0\}$. Then any complementary distribution to $JRad(TM) \oplus Jltr(TM)$ in $S(TM)$ is Riemannian.*

In the light of above lemma, Sahin [17] defines slant lightlike submanifolds as

Definition 3.2 Let M be an r-lightlike submanifold of an indefinite almost Hermitian manifold $\bar{M}$ of index $2r$. Then M is a slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(A) $Rad(TM)$ is a distribution on M such that $JRad(TM) \cap Rad(TM) = \{0\}$.
(B) For each nonzero vector field X tangent to D^{θ} at $p \in U \subset M$, the angle $\theta(X)$ between JX and the vector space D^{θ}_p is constant, that is, it is independent of the choice of $p \in U \subset M$ and $X \in D^{\theta}_p$, where D^{θ} is complementary distribution to $JRad(TM) \oplus Jltr(TM)$ in the screen distribution $S(TM)$.

This constant angle $\theta(X)$ is called slant angle of the distribution D^{θ}. A slant lightlike submanifold is said to be proper if $D^{\theta} \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

Then the tangent bundle TM of a slant lightlike submanifold M is decomposed as

$$TM = Rad(TM) \perp S(TM) = Rad(TM) \perp (JRad(TM) \oplus Jltr(TM)) \perp D^{\theta}. \tag{12}$$

Therefore, for any $X \in \Gamma(TM)$, we write

$$JX = TX + FX, \tag{13}$$

where TX and FX are the tangential component and the transversal component of JX, respectively. Similarly for any $V \in \Gamma(tr(TM))$ we write

$$JV = BV + CV, \tag{14}$$

where BV and CV are the tangential component and the transversal component of JV, respectively. Using the decomposition in (12), denote by P_1, P_2, Q_1 and Q_2 the projections on the distributions $Rad(TM)$, $JRad(TM)$, $Jltr(TM)$ and D^{θ} respectively. Then for any $X \in \Gamma(TM)$, we can write

$$X = P_1X + P_2X + Q_1X + Q_2X. \tag{15}$$

Applying J to (15), we obtain

$$JX = JP_1X + JP_2X + FQ_1X + TQ_2X + FQ_2X. \tag{16}$$

Then using (15) and (16), we get

$$JP_1X = TP_1X \in \Gamma(JRad(TM)), \quad JP_2X = TP_2X \in \Gamma(Rad(TM)),$$

$$FP_1X = FP_2X = 0, \quad TQ_2X \in \Gamma(D^{\theta}), \quad FQ_1X \in \Gamma(ltr(TM)).$$

In [17], Sahin gave characterization theorem for slant lightlike submanifolds as

Theorem 3.3 *Let M be an r-lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2r$. Then M is slant lightlike submanifold if and only if the following conditions are satisfied:*

(i) $Jltr(TM)$ is a distribution on M.
(ii) There exists a constant $\lambda \in [-1, 0]$ such that

$$(Q_2T)^2X = \lambda X, \forall X \in \Gamma(TM). \tag{17}$$

Moreover, in such case, $\lambda = -cos^2\theta$.

Theorem 3.4 *Let M be an r-lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$ of index $2r$. Then M is slant lightlike submanifold if and only if the following conditions are satisfied:*

(i) $Jltr(TM)$ is a distribution on M.
(ii) There exists a constant $\mu \in [-1, 0]$ such that

$$BFQ_2X = \mu Q_2X, \forall X \in \Gamma(TM). \tag{18}$$

In this case, $\mu = -sin^2\theta$, where θ is the slant angle of M.

Corollary 3.5 *Let M be a slant lightlike submanifold of an indefinite Hermitian manifold $\bar{M}$. Then for any $X, Y \in \Gamma(D^\theta)$, we have*

$$g(TQ_2X, TQ_2Y) = cos^2\theta g(Q_2X, Q_2Y), \quad g(FQ_2X, FQ_2Y) = sin^2\theta g(Q_2X, Q_2Y). \tag{19}$$

Lemma 3.6 *Let M be a slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ then $FQ_2X \in \Gamma(S(TM^\perp))$, for any $X \in \Gamma(TM)$.*

Proof Using (3) and (5), it is clear that $FQ_2X \in \Gamma(S(TM^\perp))$ if $g(FQ_2X, \xi) = 0$. Therefore, $g(FQ_2X, \xi) = g(JQ_2X - TQ_2X, \xi) = g(JQ_2X, \xi) = -g(Q_2X, J\xi) = 0$. Hence, the result follows. □

Thus from the Lemma (3.6) it follows that $F(D^\theta)$ is a subspace of $S(TM^\perp)$. Therefore, there exists an invariant subspace μ of $T\bar{M}$ such that $S(T_pM^\perp) = F(D_p^\theta) \perp \mu_p$ and

$$T_p\bar{M} = S(T_pM) \perp \{Rad(T_pM) \oplus ltr(T_pM)\} \perp \{F(D_p^\theta) \perp \mu_p\}.$$

The main results of this chapter has been presented followed by some definitions, which will be used for proving them.

Definition 3.7 ([15]) Let M be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is said to be a transversal lightlike submanifold if the following conditions are satisfied:

(i) $Rad(TM)$ is transversal with respect to J, that is, $J(Rad(TM)) = ltr(TM)$.
(ii) $S(TM)$ is transversal with respect to J, that is, $J(S(TM)) \subseteq S(TM^\perp)$.

Definition 3.8 ([8]) Let M be a lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g})$. Then M is said to be a Screen Cauchy Riemann (SCR) lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:

(i) $Rad(TM)$ is invariant with respect to J.
(ii) There exist real non-null distributions $D \subset S(TM)$ such that

$$S(TM) = D \oplus D^{\perp}, \quad JD^{\perp} \subset S(TM^{\perp}), \quad D \cap D^{\perp} = \{0\},$$

where $D^{\perp}$ is orthogonal complementary to D in $S(TM)$.

Theorem 3.9 ([8]) *A SCR-lightlike submanifold M, of an indefinite Kaehler manifold $\bar{M}$ is a holomorphic or complex (resp. screen real) lightlike submanifold if and only if $D^{\perp} = \{0\}$ (resp. $D = \{0\}$).*

4 Nonexistence of Warped Product Slant Lightlike Submanifolds

Assume that (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and $f > 0$ be a differentiable function on M_1. Let $\pi : M_1 \times M_2 \to M_1$ and $\phi : M_1 \times M_2 \to M_2$ be the projection morphisms from the product manifold $M_1 \times M_2$. Then product manifold $M_1 \times M_2$ equipped with Riemannian metric g given by

$$g = g_1 + f^2 g_2, \tag{20}$$

is called warped product manifold and denoted by $M_1 \times_f M_2$, where function f is known as the warping function of the warped product manifold $M_1 \times_f M_2$. Then using (20), for any tangent vector X to $M_1 \times_f M_2$ at (p, q), we have

$$\|X\|^2 = \|\pi_* X\|^2 + f^2(\pi(X))\|\phi_* X\|^2.$$

The gradient ∇f of f on the warped product manifold $M_1 \times_f M_2$ is given by $g(\nabla f, X) = Xf$, for all $X \in \Gamma(TM)$.

Lemma 4.1 ([2]) *Let $M_1 \times_f M_2$ be a warped product manifold. Then for any $X, Y \in T(M_1)$ and $U, Z \in T(M_2)$, we have*

$$\nabla_X Y \in T(M_1) \quad \nabla_U Z = -\frac{g(U, Z)}{f}\nabla f.$$

$$\nabla_X U = \nabla_U X = \frac{Xf}{f}U = X(ln\ f)U. \tag{21}$$

Remark 1 It should be noted that M_1 and M_2 are totally geodesic and totally umbilical submanifolds for warped product manifold $M_1 \times_f M_2$, respectively.

Theorem 4.2 *There does not exist warped product submanifold $M = M_\theta \times_f M_T$ of indefinite Kaehler manifold $\bar{M}$, where M_θ and M_T are proper slant lightlike submanifold and holomorphic SCR-lightlike submanifold of $\bar{M}$, respectively.*

Proof Assume that $Z \in \Gamma(D^\theta)$ of M_θ and $X \in \Gamma(S(TM))$ of M_T then using (21), we derive

$$g(\nabla_{JX}Z, X) = Z(ln\ f)g(JX, X) = 0.$$

Then on using (1), (2), (7) and (13), it follows that

$$\begin{aligned} 0 &= \bar{g}(\bar{\nabla}_{JX}Z, X) = -\bar{g}(JZ, \bar{\nabla}_{JX}JX) = \bar{g}(\bar{\nabla}_{JX}TZ, JX) - \bar{g}(FZ, \bar{\nabla}_{JX}JX) \\ &= \bar{g}(\nabla_{JX}TZ, JX) - \bar{g}(FZ, h^s(JX, JX)), \end{aligned}$$

by virtue of (21), it further implies that

$$TZ(ln\ f)g(X, X) = \bar{g}(h^s(JX, JX), FZ).$$

On using the polarization identity, we obtain

$$TZ(ln\ f)g(X, Y) = \bar{g}(h^s(JX, JY), FZ), \tag{22}$$

for any $X, Y \in \Gamma(S(TM))$ of M_T and $Z \in \Gamma(D^\theta)$ of M_θ. On the other hand, using (9) and (21), we have

$$\begin{aligned} g(A_{FZ}JX, JY) &= -g(\bar{\nabla}_{JX}FZ, JY) = \bar{g}(Z, \bar{\nabla}_{JX}Y) - \bar{g}(TZ, \bar{\nabla}_{JX}JY) \\ &= -\bar{g}(\bar{\nabla}_{JX}Z, Y) + \bar{g}(\bar{\nabla}_{JX}TZ, JY) \\ &= -Z(ln\ f)g(JX, Y) + TZ(ln\ f)g(X, Y). \end{aligned}$$

Now on using (10), we get $\bar{g}(h^s(JX, JY), FZ) = g(A_{FZ}JX, JY)$, then we derive

$$\bar{g}(h^s(JX, JY), FZ) = -Z(ln\ f)g(JX, Y) + TZ(ln\ f)g(X, Y). \tag{23}$$

Hence from (22) and (23), we obtain $Z(ln\ f)g(JX, Y) = 0$, for any $X, Y \in \Gamma(S(TM))$ of M_T and $Z \in \Gamma(D^\theta)$ of M_θ. Since $M_T \neq \{0\}$ is Riemannian and invariant and therefore

$$Zln\ f = 0,$$

this shows that f is constant and hence the proof is complete. □

Theorem 4.3 *There does not exist warped product submanifold $M = M_T \times_f M_\theta$ of an indefinite Kaehler manifold $\bar{M}$, where M_θ and M_T are as in Theorem 4.2.*

Proof Assume that $Z \in \Gamma(D^\theta)$ of M_θ and $X \in \Gamma(S(TM))$ of M_T then using (21), we obtain $g(\nabla_{TZ}X, Z) = X(ln\ f)g(TZ, Z) = 0$, this further using with (9), (10) and (19) implies that

$$\begin{aligned} 0 &= \bar{g}(\bar{\nabla}_{TZ}X, Z) = -\bar{g}(JX, \bar{\nabla}_{TZ}TZ) - \bar{g}(JX, \bar{\nabla}_{TZ}FZ) \\ &= g(\nabla_{TZ}JX, TZ) + g(JX, A_{FZ}TZ) \\ &= g(\nabla_{TZ}JX, TZ) + g(h^s(JX, TZ), FZ) \\ &= JX(ln\ f)g(TZ, TZ) + \bar{g}(h^s(JX, TZ), FZ) \\ &= JX(ln\ f).cos^2\theta g(Z, Z) + \bar{g}(h^s(JX, TZ), FZ). \end{aligned}$$

On replacing X by JX in the last expression, we get

$$X(ln\ f).cos^2\theta g(Z, Z) + \bar{g}(h^s(X, TZ), FZ) = 0, \tag{24}$$

and after replacing Z by TZ and then using (17), (19), we obtain

$$\bar{g}(h^s(X, Z), FTZ) = X(ln\ f).cos^2\theta g(Z, Z). \tag{25}$$

On the other hand using (7), (13) (17), (19) and (21), for any $X \in \Gamma(S(TM))$ of M_T and $Y, Z \in \Gamma(D^\theta)$ of M_θ, we derive

$$\begin{aligned} \bar{g}(h^s(TZ, X), FY) &= -\bar{g}(TZ, \bar{\nabla}_X JY) + \bar{g}(TZ, \bar{\nabla}_X TY) \\ &= \bar{g}(T^2Z, \bar{\nabla}_X Y) + \bar{g}(FTZ, \bar{\nabla}_X Y) + \bar{g}(TZ, \nabla_X TY) \\ &= -cos^2\theta X(ln\ f)g(Z, Y) + \bar{g}(FTZ, h^s(X, Y)) + X(ln\ f)g(TZ, TY) \\ &= \bar{g}(FTZ, h^s(X, Y)). \end{aligned}$$

On putting $Y = Z$, we get

$$\bar{g}(h^s(TZ, X), FZ) = \bar{g}(FTZ, h^s(X, Z)). \tag{26}$$

Thus from (24)–(26), it follows that

$$X(ln\ f)cos^2\theta g(Z, Z) = 0.$$

Since D^θ is a proper slant and Z is non-null, we obtain $X(ln\ f) = 0$, which proves our assertion. □

Theorem 4.4 *There does not exist warped product submanifold $M = M_\perp \times_f M_\theta$ of an indefinite Kaehler manifold $\bar{M}$, where $M_\perp$ and M_θ are transversal lightlike submanifold and proper slant lightlike submanifold of $\bar{M}$, respectively.*

Proof Let $Z \in \Gamma(D^\theta)$ of M_θ and $X \in \Gamma(S(TM))$ of $M_\perp$ then using (1), (2), (9), (13), (19), and (21), we derive

$$\begin{aligned} g(A_{JX}TZ, Z) &= \bar{g}(\bar{\nabla}_{TZ}X, JZ) = g(\nabla_{TZ}X, TZ) + \bar{g}(h^s(TZ, X), FZ) \\ &= X(ln\ f)g(TZ, TZ) + \bar{g}(h^s(TZ, X), FZ) \\ &= X(ln\ f)cos^2\theta g(Z, Z) + \bar{g}(h^s(TZ, X), FZ). \end{aligned}$$

Further, on using (10) in the last expression equation, we obtain

$$\bar{g}(h^s(TZ, Z), JX) = X(ln\ f)cos^2\theta g(Z, Z) + \bar{g}(h^s(TZ, X), FZ). \tag{27}$$

On Replacing Z by TZ in (27) and then using (17) and (19), it follows that

$$\bar{g}(h^s(Z, TZ), JX) = -X(ln\ f)cos^2\theta g(Z, Z) + \bar{g}(h^s(Z, X), FTZ). \tag{28}$$

On the other hand, using (1), (2), (9), (13), (17), and (21), we derive

$$\begin{aligned} g(A_{FZ}Y, TZ) &= -\bar{g}(\bar{\nabla}_X FZ, TZ) = \bar{g}(\bar{\nabla}_X Z, JTZ) + \bar{g}(\bar{\nabla}_X TZ, TZ) \\ &= \bar{g}(\bar{\nabla}_X Z, T^2Z) + \bar{g}(\bar{\nabla}_X Z, FTZ) + g(\nabla_X TZ, TZ) \\ &= -cos^2\theta g(\nabla_X Z, Z) + \bar{g}(h^s(X, Z), FTZ) + X(ln\ f)g(TZ, TZ) \\ &= -cos^2\theta X(ln\ f)g(Z, Z) + \bar{g}(h^s(X, Z), FTZ) + X(ln\ f)cos^2\theta g(Z, Z) \\ &= \bar{g}(h^s(X, Z), FTZ), \end{aligned}$$

hence on using (10), we obtain

$$\bar{g}(h^s(TZ, X), FZ) = \bar{g}(h^s(X, Z), FTZ). \tag{29}$$

Thus on using (27)–(29), we get

$$2X(ln\ f)cos^2\theta g(Z, Z) = 0.$$

Since M_θ is proper slant lightlike submanifold and D^θ is Riemannian then $X(ln\ f) = 0$, this proves the assertion. □

Remark 2 From the Theorems 4.2, 4.3, and 4.4, it is clear that there do not exist warped product lightlike submanifolds of the following forms:

- $M = M_\theta \times_f M_T$
- $M = M_T \times_f M_\theta$
- $M = M_\perp \times_f M_\theta$

Hence, now onwards, we call $M = M_\theta \times_f M_\perp$ as a warped product slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$.

Lemma 4.5 *Let $M = M_\theta \times_f M_\perp$ be a warped product slant lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, where $M_\perp$ and M_θ are transversal lightlike submanifold and proper slant lightlike submanifold of $\bar{M}$, respectively. Then*

$$g(h^s(X, Y), JZ) = -TX(ln\ f)g(Y, Z),$$

for any $X \in \Gamma(D^\theta)$ *of* M_θ *and* $Y, Z \in \Gamma(S(TM))$ *of* $M_\perp$.

Proof Let $X \in \Gamma(D^\theta)$ of M_θ and $Y, Z \in \Gamma(S(TM))$ of $M_\perp$, then using (1), (2), (9) and (13), we derive

$$g(h^s(TX, Y), JZ) = g(\bar{\nabla}_Y TX, JZ) = g(\nabla_Y X, Z) + g(\bar{\nabla}_Y JFX, Z).$$

Since $F(D^\theta) \subset S(TM^\perp)$ and μ is invariant therefore using (14), we have $JFX = BFX$ and $CFX = 0$. Hence using (18) and (21), we obtain

$$g(h^s(TX, Y), JZ) = X(ln\ f)g(Y, Z) - sin^2\theta g(\bar{\nabla}_Y X, Z).$$

Again using (21), we derive

$$g(h^s(TX, Y), JZ) = (1 - sin^2\theta)X(ln\ f)g(Y, Z) = cos^2\theta X(ln\ f)g(Y, Z).$$

On replacing X by TX in the last expression and then using (17), our assertion follows. □

Remark 3 It is natural to extend the results obtained in this work in the case of other ambient spaces, like indefinite cosymplectic manifolds, indefinite quaternionic manifolds or paraquaternionic manifolds, whose metrics are always semi-Riemannian. Note that lightlike submanifolds in these spaces were investigated by many authors (see [9, 10, 16]), but the existence of warped product lightlike submanifolds of such ambient spaces is an open problem.

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Concircular Curvature Tensor's Properties on Lorentzian Para-Sasakian Manifolds

Shashikant Pandey, Rajendra Prasad, and Sandeep Kumar Verma

Abstract The objective of the present paper is to study and investigate the geometric properties of Concircular curvature tensor on Lorentzian type para-Sasakian manifold endowed with the quarter-symmetric nonmetric connection. At last, we provide an example which satisfies the conditions of ξ_*-Concircularly flat and ϕ_*-Concircularly flat Lorentzian type para-Sasakian manifold endowed with the quarter-symmetric nonmetric connection.

Keywords Lorentzian type para-Sasakian manifolds · Quarter-symmetric nonmetric connection · Concircular curvature tensor · η_*-Einstein manifold

2010 Mathematics Subject Classification 53C15 · 53C25

1 Introduction

In 1989, K. Matsumoto [11] introduced the notion of Lorentzian type para-Sasakian manifolds, and an almost para-contact manifold was introduced by [18]. I. Mihai and R. Rosca [13] introduced the same notion independently and obtained several results. The Lorentzian type para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [12], U. C. De and A. A. Shaikh [14] and many others such as [14, 16, 19, 20]. In 1924, Friedmann and Schouten [7] introduced the idea of semi-symmetric connection on a differentiable manifold. Some interesting results were obtained for conformally recurrent and conformally symmetric P-Sasakian

S. Pandey · R. Prasad (✉) · S. K. Verma
Department of Mathematics & Astronomy, University of Lucknow,
Lucknow 226007, U.P., India
e-mail: rp.manpur@rediffmail.com

S. Pandey
e-mail: shashi.royal.lko@gmail.com

S. K. Verma
e-mail: skverma1208@gmail.com

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manifold in [1]. A linear connection $\bar{\nabla}$ on a differentiable manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection satisfies

$$T(X, Y) = \eta_*(Y)X - \eta_*(X)Y,$$

where η_* is a 1-form and ξ_* is a vector field defined by $\eta_*(X) = g(X, \xi_*)$, for all vector fields X on $\Gamma(TM)$, $\Gamma(TM)$ is the set of all differentiable vector fields on M. Semi-symmetric metric and nonmetric connection on para-Sasakian manifold was studied by [2, 3]. In 1975, Golab [8] defined and studied quarter-symmetric connection in differentiable manifolds with affine connections.

A linear connection $\bar{\nabla}$ on an n-dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection [8], if its torsion tensor T satisfies

$$T(X, Y) = \eta_*(Y)\phi_*X - \eta_*(X)\phi_*Y,$$

where ϕ_* is a (1,1) tensor field.

In particular, if $\phi_*X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [7].

Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y, Z) \neq 0, \tag{1.1}$$

for some X, Y, Z on $\Gamma(TM)$, then $\bar{\nabla}$ is said to be a quarter-symmetric nonmetric connection. A relation between the quarter-symmetric nonmetric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in an n-dimensional Lorentzian type para-Sasakian manifold M is given by [6]

$$\bar{\nabla}_X Y = \nabla_X Y - \eta_*(X)\phi_*Y. \tag{1.2}$$

The 1-form η_* is defined by $\eta_*(X) = g(X, \xi_*)$ and ξ_* is the corresponding vector field. Venkatesha and C.S. Bagewadi [23] studied concircular ϕ_*-recurrent Lorentzian type para-Sasakian manifolds which generalize the notion of locally concircular ϕ_*-symmetric Lorentzian type para-Sasakian manifolds and obtained some interesting results.

Let $\bar{R}$ and R be the curvature tensors with respect to the quarter-symmetric nonmetric connection $\bar{\nabla}$ and the Levi-Civita connection ∇, respectively. Then, we have from [6]

$$\begin{aligned}\bar{R}(X, Y)Z = R(X, Y)Z &+ (\eta_*(X)Y - \eta_*(Y)X)\eta_*(Z) \\ &+ g(Y, Z)\eta_*(X)\xi_* - g(X, Z)\eta_*(Y)\xi_*,\end{aligned} \tag{1.3}$$

$$\bar{R}(\xi_*, Y)Z = -\bar{R}(Y, \xi_*)Z = -2\eta_*(Z)Y - 2\eta_*(Y)\eta_*(Z)\xi_*, \tag{1.4}$$

$$\bar{S}(Y, Z) = S(Y, Z) - g(Y, Z) - n\eta_*(Y)\eta_*(Z), \tag{1.5}$$

$$\bar{S}(Y, \xi_*) = 2(n-1)\eta_*(Y), \;\; S(\xi_*, \xi_*) = -2(n-1), \tag{1.6}$$

$$\bar{S}(\phi_* Y, \phi_* Z) = \bar{S}(Y, Z) - g(Y, Z) - (n-2)\eta_*(Y)\eta_*(Z), \tag{1.7}$$

for all vector fields $X, Y, Z \in \Gamma(TM)$, where $\bar{S}$ and S are the Ricci tensors with respect to the quarter-symmetric nonmetric connection $\bar{\nabla}$ and the Levi-Civita connection ∇, respectively.

A Riemannian manifold M is locally symmetric if its curvature tensor R satisfies $\nabla R = 0$. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be semi-symmetric if its curvature tensor R satisfies $R(X, Y) \cdot R = 0$, where $R(X, Y)$ acts on R as a derivation. A Riemannian manifold M is said to be Ricci-semi-symmetric manifold if the relation $\bar{R}(X, Y) \cdot \bar{S} = 0$. holds, where $\bar{R}(X, Y)$ the curvature operator.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [9, 21]. A concircular transformation is always a conformal transformation [9]. Here, geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [5]). An interesting invariant of a concircular transformation is the concircular curvature tensor $\bar{C}$. It is defined by [21, 22]

$$\bar{C}(X, Y)Z = \bar{R}(X, Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \tag{1.8}$$

Using (1.8), we obtain

$$\tilde{\bar{C}}(X, Y, Z, U) = \tilde{\bar{R}}(X, Y, Z, U) - \frac{\bar{r}}{n(n-1)}[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)], \tag{1.9}$$

and $\tilde{\bar{C}}(X, Y, Z, U) = g(\bar{C}(X, Y)Z, U)$, $\tilde{\bar{R}}(X, Y, Z, U) = g(\bar{R}(X, Y)Z, U)$, where $X, Y, Z, U \in \Gamma(TM)$ and $\bar{C}$ is the concircular curvature tensor and $\bar{r}$ is the scalar curvature with respect to the quarter-symmetric nonmetric connection, respectively. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

Putting $X = \xi_*$ in (1.8) and using (1.4), we obtain

$$\bar{C}(\xi_*, Y)Z = (-\frac{\bar{r}}{n(n-1)})g(Y, Z)\xi_* + (\frac{\bar{r}}{n(n-1)} - 2)\eta_*(Z)Y - 2\eta_*(Y)\eta_*(Z)\xi_*. \tag{1.10}$$

In this paper, we study quarter-symmetric nonmetric connection on Lorentzian type para-Sasakian manifolds.

The paper is organized as follows: After the introduction, section two is equipped with some prerequisites of a Lorentzian type para-Sasakian manifold. Section three is devoted to studying ξ_*-Concircularly flat in a Lorentzian type para-Sasakian manifold with respect to the quarter-symmetric nonmetric connection. ϕ_*-Concircularly flat Lorentzian type para-Sasakian manifolds with respect to the quarter-symmetric nonmetric connection have been studied in section four. In the next section, we investigate Ricci-semi-symmetric manifolds with respect to the quarter-symmetric nonmetric connection of a Lorentzian type para-Sasakian manifold. At last, we construct an example of 5-dimensional Lorentzian type para-Sasakian manifolds endowed with the quarter-symmetric nonmetric connections which verify the results of section three and section four.

2 Preliminaries

An n-dimensional differentiable manifold M is said to be an almost para-contact manifold, if it admits an almost para-contact structure $(\phi_*, \xi_*, \eta_*, g)$ consisting of a $(1, 1)$ tensor field ϕ_*, vector field ξ_*, 1-form η_* and Lorentzian metric g satisfying

$$\phi_* \circ \xi_* = 0, \eta_* \circ \phi_* = 0, \eta_*(\xi_*) = -1, g(X, \xi_*) = \eta_*(X), \tag{2.1}$$

$$\phi_*^2 X = X + \eta_*(X)\xi_*, \tag{2.2}$$

$$g(\phi_* X, \phi_* Y) = g(X, Y) + \eta_*(X)\eta_*(Y), \tag{2.3}$$

$$(\nabla_X \eta_*)\, Y = g(X, \phi_* Y) = (\nabla_Y \eta_*)\, X, \tag{2.4}$$

for any vector field X, Y on M. Such a manifold is termed as Lorentzian para-contact manifold and the structure $(\phi_*, \xi_*, \eta_*, g)$ a Lorentzian para-contact structure [11].

If moreover $(\phi_*, \xi_*, \eta_*, g)$ satisfies the conditions

$$d\eta_* = 0, \quad \nabla_X \xi_* = \phi_* X, \tag{2.5}$$

$$(\nabla_X \phi_*)\, Y = g\,(X, Y)\, \xi_* + \eta_*\,(Y)\, X + 2\eta_*\,(X)\, \eta_*\,(Y)\, \xi_*, \tag{2.6}$$

for X, Y tangent to M, then M is called a Lorentzian type para-Sasakian manifold or briefly LP type Sasakian manifold, where ∇ denotes the covariant differentiation with respect to Lorentzian metric g.

Moreover, for the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a Lorentzian type para-Sasakian manifold M with respect to the Levi-Civita connection, the following relation holds [17]

$$\eta_*\,(R\,(X, Y)\, Z) = g\,(Y, Z)\, \eta_*\,(X) - g\,(X, Z)\, \eta_*\,(Y)\,, \tag{2.7}$$

$$R\,(\xi_*, X)\, Y = g\,(X, Y)\, \xi_* - \eta_*\,(Y)\, X, \tag{2.8}$$

$$R(\xi_*, X)\xi_* = -R(X, \xi_*)\xi_* = X + \eta_*(X)\xi_*, \tag{2.9}$$

$$R(X, Y)\xi_* = \eta_*(Y)X - \eta_*(X)Y, \tag{2.10}$$

$$S(X, \xi_*) = (n-1)\eta_*(X), \; Q\xi_* = (n-1)\xi_* \tag{2.11}$$

$$S(\phi_* X, \phi_* Y) = S(X, Y) + (n-1)\eta_*(X)\eta_*(Y), \tag{2.12}$$

for all vector fields $X, Y \in \Gamma(TM)$.

3 ξ_*-Concircularly Flat Lorentzian Type Para-Sasakian Manifolds with Respect to the Quarter-Symmetric Nonmetric Connection

Definition 3.1 A Lorentzian type para-Sasakian manifold is said to be ξ_*-Concircularly flat [4] with respect to the quarter-symmetric nonmetric connection if

$$\bar{C}(X, Y)\xi_* = 0,$$

where $X, Y \in \Gamma(TM)$.

Theorem 3.2 *A Lorentzian type para-Sasakian manifold admitting a quarter-symmetric nonmetric connection is ξ_*-Concircularly flat if and only if the scalar curvature $\bar{r}$ with respect to the quarter-symmetric nonmetric connection is equal to $2n(n-1)$.*

Proof Combining (1.3) and (1.8), it follows that

$$\bar{C}(X, Y)Z = R(X, Y)Z + \eta_*(Z)[\eta_*(X)Y - \eta_*(Y)X] - [\eta_*(X)g(Y, Z) - \eta_*(Y)g(X, Z)]\xi_* - \frac{\bar{r}}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \tag{3.1}$$

Putting $Z = \xi_*$ in (3.1) and using (2.1), we have

$$\bar{C}(X, Y)\xi_* = R(X, Y)\xi_* + [-\eta_*(X)Y + \eta_*(Y)X] + \frac{\bar{r}}{n(n-1)}[-\eta_*(X)Y + \eta_*(Y)X]. \tag{3.2}$$

Using (2.10) and (3.2), we get

$$\bar{C}(X, Y)\xi_* = [2 - \frac{\bar{r}}{n(n-1)}]R(X, Y)\xi_*. \tag{3.3}$$

If $\bar{C}(X, Y)\xi_* = 0$, then $\bar{r} = 2n(n-1)$ or $R(X, Y)\xi_* = \eta_*(Y)X - \eta_*(X)Y = 0$ implies that $\eta_*(X) = 0$ which is not possible.

Conversely, if $\bar{r} = 2n(n-1)$, then from (3.3), it follows that $\bar{C}(X, Y)\xi_* = 0$.

This completes the proof. □

Theorem 3.3 *If Lorentzian type para-Sasakian manifolds satisfy $\bar{R}(\xi_*, Y) \cdot \bar{C} = 0$ with respect to a quarter-symmetric nonmetric connection, then the manifold is an η_*-Einstein manifold with respect to the quarter-symmetric nonmetric connection, and the scalar curvature $\bar{r}$ with respect to the quarter-symmetric nonmetric connection is a positive constant.*

Proof For Lorentzian type para-Sasakian manifolds with respect to the quarter-symmetric nonmetric connection the curvature tensor satisfies the condition

$$\bar{R}(X, Y) \cdot \bar{C} = \bar{R}(X, Y) \cdot \bar{R}$$

$$(\bar{R}(\xi_*, Y) \cdot \bar{R})(U, V)W = 0, \tag{3.4}$$

From (3.4), we have

$$\bar{R}(\xi_*, Y)\bar{R}(U, V)W + \bar{R}(\bar{R}(\xi_*, Y)U, V)W$$
$$-\bar{R}(U, \bar{R}(\xi_*, Y)V)W - \bar{R}(U, V)\bar{R}(\xi_*, Y)W = 0$$

Now, using (2.8) and (3.4), we have

$$-\eta_*(\bar{R}(U, V)W)Y - \eta_*(Y)\eta_*(\bar{R}(U, V)W)\xi_* + \eta_*(U)\bar{R}(Y, V)W \tag{3.5}$$
$$+\eta_*(V)\bar{R}(U, Y)W + \eta_*(W)\bar{R}(U, V)Y + \eta_*(Y)\eta_*(W)\bar{R}(U, V)\xi = 0.$$

Putting $U = \xi_*$ in (3.5), we have

$$\bar{R}(Y, V)W = -2\eta_*(V)\eta_*(W)Y + 2\eta_*(V)\eta_*(W)\eta_*(Y)\xi_* \tag{3.6}$$

Now contracting (3.6) with respect to Y, we obtain

$$\bar{S}(V, W) = -2(n-1)\eta_*(V)\eta_*(W). \tag{3.7}$$

We know that scaler curvature $\bar{r} = \sum_{i=1}^{n} \epsilon_i \bar{S}(e_i, e_i)$ so using (3.7), we get

$$\bar{r} = 2(n-1).$$

Which shows that the scalar curvature $\bar{r}$ with respect to the quarter-symmetric nonmetric connection is a positive constant.

This completes the proof of Theorem 3.3.

4 ϕ_*-Concircularly Flat Lorentzian Type Para-Sasakian Manifold with Respect to the Quarter-Symmetric Nonmetric Connection

Definition 4.1 A Lorentzian type para-Sasakian manifold is said to be ϕ_*-Concircularly flat with respect to the quarter-symmetric nonmetric connection if

$$\tilde{\bar{C}}(\phi_* X, \phi_* Y, \phi_* Z, \phi_* U) = 0,$$

where $X, Y, Z, U \in \Gamma(TM)$.

Definition 4.2 A Lorentzian type para-Sasakian manifold is said to be an η_*-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form

$$S(X, Y) = ag(X, Y) + b\eta_*(X)\eta_*(Y),$$

where a and b are smooth functions on the manifold.

Theorem 4.3 *If a Lorentzian type para-Sasakian manifold admitting a quarter-symmetric nonmetric connection is ϕ_*-Concircularly flat, then the manifold with respect to the quarter-symmetric nonmetric connection is an η_*-Einstein manifold.*

Proof Using (1.9)we obtain

$$\tilde{\bar{C}}(X, Y, Z, U) = \tilde{\bar{R}}(X, Y, Z, U) - \frac{\bar{r}}{n(n-1)}[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)], \tag{4.1}$$

where $\tilde{\bar{C}}(X, Y, Z, U) = g(\bar{C}(X, Y)Z, U)$ and $\tilde{\bar{R}}(X, Y, Z, U) = g(\bar{R}(X, Y)Z, U)$.

Now putting $X = \phi_* X,\ Y = \phi_* Y,\ Z = \phi_* Z,\ U = \phi_* U$ in (4.1) and using (2.1) and (2.2), we get

$$\tilde{\bar{C}}(\phi_* X, \phi_* Y, \phi_* Z, \phi_* U) = \tilde{\bar{R}}(\phi_* X, \phi_* Y, \phi_* Z, \phi_* U) - \frac{\bar{r}}{n(n-1)}[g(\phi_* Y, \phi_* Z)g(\phi_* X, \phi_* U) - g(\phi_* X, \phi_* Z)g(\phi_* Y, \phi_* U)]. \tag{4.2}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi_*\}$ be a local orthonormal basis of vector fields in M, then $\{\phi_* e_1, \phi_* e_2, ..., \phi_* e_{n-1}, \xi_*\}$ is also a local orthonormal basis. Putting $X = U = e_i$ in (4.2) and summing over $i = 1$ to $n-1$, we obtain

$$\sum_{n=1}^{n-1} \tilde{\bar{C}}(\phi_* e_i, \phi_* Y, \phi_* Z, \phi_* e_i) = \bar{S}(\phi_* Y, \phi_* Z) - \frac{(n-2)\bar{r}}{n(n-1)} g(\phi_* Y, \phi_* Z). \tag{4.3}$$

Using (2.3) and (2.12) in (4.3), we have

$$\sum_{n=1}^{n-1} \tilde{\bar{C}}(\phi_* e_i, \phi_* Y, \phi_* Z, \phi_* e_i) = \bar{S}(Y, Z) - g(Y, Z) - (n-2)\,\eta_*(Y)\,\eta_*(Z) - \frac{(n-2)\,\bar{r}}{n(n-1)}\left[g(Y, Z) + \eta_*(Y)\,\eta_*(Z)\right]. \quad (4.4)$$

By virtue of (1.5) and (4.4), we have

$$\sum_{n=1}^{n-1} \tilde{\bar{C}}(\phi_* e_i, \phi_* Y, \phi_* Z, \phi_* e_i) = \bar{S}(Y, Z) - \left[1 + \frac{(n-2)\,\bar{r}}{n(n-1)}\right] g(Y, Z) - \left[(n-2) + \frac{(n-2)\,\bar{r}}{n(n-1)}\right] \eta_*(Y)\,\eta_*(Z). \quad (4.5)$$

If $\sum_{n=1}^{n-1} \tilde{\bar{C}}(\phi_* e_i, \phi_* Y, \phi_* Z, \phi_* e_i) = 0$, then

$$\bar{S}(Y, Z) = \left[1 + \frac{(n-2)\,\bar{r}}{n(n-1)}\right] g(Y, Z) + \left[(n-2) + \frac{(n-2)\,\bar{r}}{n(n-1)}\right] \eta_*(Y)\,\eta_*(Z),$$

or

$$\bar{S}(Y, Z) = a g(Y, Z) + b\eta_*(Y)\,\eta_*(Z),$$

where $a = \left[1 + \frac{(n-2)\bar{r}}{n(n-1)}\right]$ and $b = \left[(n-2) + \frac{(n-2)\bar{r}}{n(n-1)}\right]$.

From which it follows that the manifold is an η_*-Einstein manifold with respect to the quarter-symmetric nonmetric connection.

Hence, proof of the Theorem 4.3 is completed. □

5 Lorentzian type para-Sasakian manifold satisfying $\bar{C} \cdot \bar{S} = 0$ with respect to a quarter-symmetric nonmetric connection

Theorem 5.1 *If Lorentzian type para-Sasakian manifolds satisfy $\bar{C} \cdot \bar{S} = 0$ with respect to a quarter-symmetric nonmetric connection, then the manifold is an η_*-Einstein manifold with respect to a quarter-symmetric nonmetric connection.*

Proof We consider Lorentzian type para-Sasakian manifolds with respect to a quarter-symmetric nonmetric connection $\bar{\nabla}$ satisfying the curvature condition $\bar{C} \cdot \bar{S} = 0$. Then

$$\left(\bar{C}(X, Y) \cdot \bar{S}\right)(U, V) = 0.$$

So,

$$\bar{S}(\bar{C}\,(X,Y)\,U,V) + \bar{S}\left(U, \bar{C}\,(X,Y)\,V\right) = 0. \tag{5.1}$$

Putting $X = \xi_*$ in (5.1) and using (1.10), we get

$$\left(-\frac{\bar{r}}{n(n-1)}\right)[g(Y,U)\bar{S}(\xi_*,V) + g(Y,V)\bar{S}(\xi_*,U)] + \left(\frac{\bar{r}}{n(n-1)} - 2\right)[\eta_*(U)\bar{S}(Y,V) + \eta_*(V)\bar{S}(Y,U)] - 2\eta_*(Y)\eta_*(U)\bar{S}(\xi_*,V) - 2\eta_*(Y)\eta_*(V)\bar{S}(\xi_*,U) = 0. \tag{5.2}$$

Again putting $U = \xi_*$ in (5.2) implies that

$$\left(-\frac{\bar{r}}{n(n-1)}\right)[2(n-1)\eta_*(Y)\eta_*(V) - 2(n-1)g(Y,V)] + \left(\frac{\bar{r}}{n(n-1)} - 2\right)[-\bar{S}(Y,V) + 2(n-1)\eta_*(Y)\eta_*(V)] + 8(n-1)\eta_*(Y)\eta_*(V) = 0.$$

$$\bar{S}(Y,V) = \left[\frac{2(n-1)\bar{r}}{\bar{r} - 2n(n-1)}\right] g(Y,V) + \left[\frac{4n(n-1)^2}{\bar{r} - 2n(n-1)}\right]\eta_*(Y)\eta_*(V).$$

Therefore, $\bar{S}(Y,V) = ag(Y,V) + b\eta_*(Y)\eta_*(V)$ where $a = \frac{2(n-1)\bar{r}}{\bar{r}-2n(n-1)}$ and $b = \frac{4n(n-1)^2}{\bar{r}-2n(n-1)}$.

This means that the manifold is an η_*-Einstein manifold with respect to the quarter-symmetric nonmetric connection.

This completes the proof. □

6 Example

In this section, we construct an example on Lorentzian type para-Sasakian manifold with respect to the quarter-symmetric nonmetric connection $\bar{\nabla}$, which verifies the results of section three and section four. We consider the 5-dimensional manifold $(x, y, z, u, v) \in \mathbb{R}^5$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$e_1 = -\frac{\partial}{\partial x},\; e_2 = e^{-x}\frac{\partial}{\partial y},\; e_3 = e^{-x}\frac{\partial}{\partial z},\; e_4 = e^{-x}\frac{\partial}{\partial u},\; e_5 = e^{-x}\frac{\partial}{\partial v},$$

which are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$g(e_1,e_1) = -1,\; g(e_2,e_2) = 1,\; g(e_3,e_3) = 1,\; g(e_4,e_4) = 1,\; g(e_5,e_5) = 1,$$

and $g(e_i, e_j) = 0$ if $i \neq j$.

Let ϕ_* be the (1, 1)-tensor field and $\eta_*(X) = g(X, e_1 = \xi_*)$ be a 1-form, defined by

$$\phi_* e_1 = 0,\ \phi_* e_2 = -e_2,\ \phi_* e_3 = -e_3,\ \phi_* e_4 = -e_4,\ \phi_* e_5 = -e_5.$$

Using the linearity of ϕ_* and g, we obtain

$$\phi_*^2 X = X + \eta_*(X)\xi_*,\ \eta_*(\xi_*) = -1,\ \eta_*(X) = g(X, \xi_*)$$

and

$$g(\phi_* X, \phi_* Y) = g(X, Y) + \eta_*(X)\eta_*(Y),$$

for any vector fields $X, Y \in \Gamma(TM)$. Thus for $e_1 = \xi_*$, the structure $(\phi_*, \xi_*, \eta_*, g)$ defines an almost para-contact metric structure on M. Then, we have

$$[e_1, e_2] = e_2,\ [e_1, e_3] = e_3,\ [e_1, e_4] = e_4,\ [e_1, e_5] = e_5,$$

$$[e_2, e_3] = [e_2, e_4] = 0,\ [e_2, e_5] = [e_3, e_4] = [e_3, e_5] = [e_4, e_5] = 0.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul's formula, we obtain the following:

$$\begin{aligned}
\nabla_{e_1} e_1 &= 0,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = 0,\ \nabla_{e_1} e_4 = 0,\ \nabla_{e_1} e_5 = 0,\\
\nabla_{e_2} e_1 &= -e_2,\ \nabla_{e_2} e_2 = e_1,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0,\\
\nabla_{e_3} e_1 &= -e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = e_1,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = 0,\\
\nabla_{e_4} e_1 &= -e_4,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = e_1,\ \nabla_{e_4} e_5 = 0,\\
\nabla_{e_5} e_1 &= -e_5,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = e_1.
\end{aligned}$$

In view of the above relations, we see that

$$(\nabla_X \phi_*)Y = g(X, Y)\xi_* + \eta_*(Y)X + 2\eta_*(X)\eta_*(Y)\xi_*,$$

$$\nabla_X \xi_* = \phi_* X,$$

for all $X, Y \in \Gamma(TM)$ and $\xi_* = e_1$.

Therefore, the manifold is a Lorentzian type para-Sasakian manifold with the structure $(\phi_*, \xi_*, \eta_*, g)$. Using (1.1) in the above equations, we obtain

$$
\begin{aligned}
\bar{\nabla}_{e_1}e_1 &= 0,\ \bar{\nabla}_{e_1}e_2 = 0,\ \bar{\nabla}_{e_1}e_3 = 0,\ \bar{\nabla}_{e_1}e_4 = 0,\ \bar{\nabla}_{e_1}e_5 = 0,\\
\bar{\nabla}_{e_2}e_1 &= -2e_2,\ \bar{\nabla}_{e_2}e_2 = 2e_1,\ \bar{\nabla}_{e_2}e_3 = 0,\ \bar{\nabla}_{e_2}e_4 = 0,\ \bar{\nabla}_{e_2}e_5 = 0,\\
\bar{\nabla}_{e_3}e_1 &= -2e_3,\ \bar{\nabla}_{e_3}e_2 = 0,\ \bar{\nabla}_{e_3}e_3 = 2e_1,\ \bar{\nabla}_{e_3}e_4 = 0,\ \bar{\nabla}_{e_3}e_5 = 0,\\
\bar{\nabla}_{e_4}e_1 &= -2e_4,\ \bar{\nabla}_{e_4}e_2 = 0,\ \bar{\nabla}_{e_4}e_3 = 0,\ \bar{\nabla}_{e_4}e_4 = 2e_1,\ \bar{\nabla}_{e_4}e_5 = 0,\\
\bar{\nabla}_{e_5}e_1 &= -2e_5,\ \bar{\nabla}_{e_5}e_2 = 0,\ \bar{\nabla}_{e_5}e_3 = 0,\ \bar{\nabla}_{e_5}e_4 = 0,\ \bar{\nabla}_{e_5}e_5 = 2e_1.
\end{aligned}
$$

Now, we can easily obtain the nonzero components of the curvature tensors R as follows:

$$
\begin{aligned}
&R(e_1,e_2)e_1 = -e_2,\ R(e_1,e_2)e_2 = e_1,\ R(e_1,e_3)e_1 = -e_3,\ R(e_1,e_3)e_3 = e_1,\\
&R(e_1,e_4)e_1 = -e_4,\ R(e_1,e_4)e_2 = e_1,\ R(e_1,e_5)e_1 = -e_5,\ R(e_1,e_5)e_3 = e_1,\\
&R(e_2,e_3)e_1 = -e_3,\ R(e_2,e_3)e_2 = e_2,\ R(e_2,e_4)e_1 = -e_4,\ R(e_2,e_4)e_3 = e_2,\\
&R(e_2,e_5)e_1 = -e_5,\ R(e_2,e_5)e_2 = e_2,\ R(e_3,e_4)e_1 = -e_4,\ R(e_3,e_4)e_3 = e_3,\\
&R(e_3,e_5)e_1 = -e_5,\ R(e_3,e_5)e_2 = e_3,\ R(e_4,e_5)e_1 = -e_5,\ R(e_4,e_5)e_3 = e_4,
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{R}(e_1,e_2)e_1 = -2e_2,\ \bar{R}(e_1,e_2)e_2 = 2e_1,\ \bar{R}(e_1,e_3)e_1 = -2e_3,\ \bar{R}(e_1,e_3)e_3 = 2e_1,\\
&\bar{R}(e_1,e_4)e_1 = -2e_4,\ \bar{R}(e_1,e_4)e_2 = 2e_1,\ \bar{R}(e_1,e_5)e_1 = -2e_5,\ \bar{R}(e_1,e_5)e_3 = 2e_1,\\
&\bar{R}(e_2,e_3)e_1 = -2e_3,\ \bar{R}(e_2,e_3)e_2 = 2e_2,\ \bar{R}(e_2,e_4)e_1 = -2e_4,\ \bar{R}(e_2,e_4)e_3 = 2e_2,\\
&\bar{R}(e_2,e_5)e_1 = -2e_5,\ \bar{R}(e_2,e_5)e_2 = 2e_2,\ \bar{R}(e_3,e_4)e_1 = -2e_4,\ \bar{R}(e_3,e_4)e_3 = 2e_3,\\
&\bar{R}(e_3,e_5)e_1 = -2e_5,\ \bar{R}(e_3,e_5)e_2 = 2e_3,\ \bar{R}(e_4,e_5)e_1 = -2e_5,\ \bar{R}(e_4,e_5)e_3 = 2e_4.
\end{aligned}
$$

With the help of the above curvature tensors with respect to the quarter-symmetric nonmetric connection, we find the Ricci tensors $\bar{S}$ as follows:

$$\bar{S}(e_1,e_1) = \bar{S}(e_2,e_2) = \bar{S}(e_3,e_3) = \bar{S}(e_4,e_4) = \bar{S}(e_5,e_5) = 8.$$

Also, it follows that the scalar curvature tensor with respect to the quarter-symmetric nonmetric connection is $\bar{r} = 40$.

Let X, Y, Z and U be any four vector fields given by

$$
\begin{aligned}
X &= a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5,\ Y = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5\\
Z &= c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5,\ W = d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5
\end{aligned}
$$

where a_i, b_i, c_i, d_i , for all $i = 1, 2, 3, 4, 5$ are all nonzero real numbers. Using the above curvature tensors and the scalar curvature tensors of the quarter-symmetric nonmetric connection, we have

$$\begin{aligned}\bar{C}(X,Y)\xi_* &= a_1b_1(2e_2-2e_2)+a_1b_5(2e_5-2e_5)+a_1b_4(2e_4-2e_4)\\&\quad+a_1b_3(2e_3-2e_3)=0,\end{aligned}$$

which verifies the result of section three.

Now, we see that the ϕ_*-Concircularly flat with respect to the quarter-symmetric nonmetric connection from the above relations is as follows:

$$\begin{aligned}\bar{C}(\phi_*X,\phi_*Y,\phi_*Z,\phi_*U) &= 2a_2b_3(c_2d_3-c_3d_2)+2a_2b_5(c_2d_5-c_5d_2)+2a_3b_4(c_3d_4-c_4d_3)\\&\quad+2a_4b_5(c_4d_5-c_5d_4)+2a_2b_4(c_2d_4-c_4d_2)+2a_3b_5(c_3d_5-c_5d_3)\\&=0\end{aligned}$$

Hence Lorentzian type para-Sasakian manifold will be ϕ_*-Concircularly flat with respect to the quarter-symmetric nonmetric connection if $\frac{c_2}{d_2}=\frac{c_3}{d_3}=\frac{c_4}{d_4}=\frac{c_5}{d_5}$. The above arguments tell us that the 5-dimensional Lorentzian type para-Sasakian manifold with respect to the quarter-symmetric nonmetric connection under consideration agrees with section four.

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Inequalities for Statistical Submanifolds in Sasakian Statistical Manifolds

Mohd. Aquib, Falleh R. Al-Solamy, Mohammed Jamali, Maryam T. Aldossary, and Michel Nguiffo Boyom

Abstract In this paper, we obtain the bounds for the normalized δ-Casorati curvatures for statistical submanifolds in Sasakian statistical manifolds with constant curvature using T. Oprea optimization technique. We also obtain the bounds in the general setting of a semi-symmetric metric connection.

Keywords Casorati curvatures · Semi-symmetric metric connection · Statistical manifolds · Sasakian statistical manifolds

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Mohd. Aquib (✉)
Faculty of Natural Sciences, Department of Mathematics, Jamia Millia Islamia, New Delhi 110025, India
e-mail: aquib80@gmail.com

F. R. Al-Solamy
Faculty of Science, Department of Mathematics, Kings Abdul Aziz University, Jeddah 21589, Saudi Arabia
e-mail: fsolamy@kau.adu.sa

M. Jamali
Department of Mathematics, Al-Falah University, Haryana 121004, India
e-mail: jamali_dbd@yahoo.co.in

M. T. Aldossary
Department of Mathematics, Science College, Imam Abdulrahman Bin Faisal University, Dammam, Saudi Arabia
e-mail: maaldosari@iau.edu.sa

M. N. Boyom
IMAG: Alexander Grothendieck Research Institute, University of Montpellier, Montpellier, France
e-mail: nguiffo.boyom@gmail.com

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1 Introduction

Vos [22] introduced the notion of statistical submanifold in 1989. The statistical submanifold theory originates in Information geometry and is now deepening through pure differential geometry, though due to the hardness to find classical differential geometric approaches for study of statistical submanifolds, it has made very little progress. The geometry of statistical manifolds is developing as deformation of Riemannian geometry, related to Information geometry, Hessian geometry, the differential geometry of affine hypersurfaces and many other fields [2, 4, 5, 9, 10, 19].

Casorati [6] introduced and studied Casorati curvature in 1999. The geometrical importance of the Casorati curvature have been discussed in [7, 13, 20]. A number of geometric results has been obtained for the Casorati curvatures using different optimization techniques [1, 3, 11, 14–17, 21].

In the present article, in Sect. 2, we review the statistical manifold theory briefly. In Sect. 3, we obtain bounds for normalized $\delta-$Casorati curvature for statistical submanifolds in Sasakian statistical manifolds of constant ϕ-sectional curvature. In Sect. 4, we derive normalized $\delta-$Casorati curvature for statistical submanifolds in Sasakian statistical manifolds of constant ϕ-sectional curvature with semi-symmetric metric connection and provide the condition under which the equality holds.

2 Preliminaries

In this section, we review the statistical manifolds and their submanifolds in brief.

Definition 2.1 Let $(\overline{\mathtt{M}}, \mathtt{g})$ be a Riemannian manifold and $\overline{\nabla}$ and $\overline{\nabla}^*$ be torsion-free affine connections on $\overline{\mathtt{M}}$ such that

$$\mathtt{G}\mathtt{g}(\mathtt{E}, \mathtt{F}) = \mathtt{g}(\overline{\nabla}_{\mathtt{G}}\mathtt{E}, \mathtt{F}) + \mathtt{g}(\mathtt{E}, \overline{\nabla}^*_{\mathtt{G}}\mathtt{F}), \tag{2.1}$$

for $\mathtt{E}, \mathtt{F}, \mathtt{G} \in \Gamma(T\overline{\mathtt{M}})$. Then Riemannian manifold $(\overline{\mathtt{M}}, \mathtt{g})$ is called a statistical manifold. It is denoted by $(\overline{\mathtt{M}}, \mathtt{g}, \overline{\nabla}, \overline{\nabla}^*)$.

Remark 2.2 Here,

1. the connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are called dual connections. The pair $(\overline{\nabla}, \mathtt{g})$ is said to be a statistical structure.
2. if $(\overline{\nabla}, \mathtt{g})$ is a statistical structure on $\overline{\mathtt{M}}$, then $(\overline{\nabla}^*, \mathtt{g})$ is also a statistical structure on $\overline{\mathtt{M}}$.
3. we have

$$\overline{\nabla}^{\circ} = \frac{1}{2}(\overline{\nabla} + \overline{\nabla}^*), \tag{2.2}$$

where $\overline{\nabla}^{\circ}$ is Levi-Civita connection for $\overline{\mathtt{M}}$.

Let $\overline{\mathtt{M}}$ be a $(2m+1)$-dimensional manifold and let $\mathtt{M}$ be an n-dimensional submanifolds of $\overline{\mathtt{M}}$. Then, the Gauss formulae are [22]

$$\begin{cases} \overline{\nabla}_{\mathtt{E}}\mathtt{F} = \nabla_{\mathtt{E}}\mathtt{F} + \sigma(\mathtt{E}, \mathtt{F}), \\ \overline{\nabla}^{*}_{\mathtt{E}}\mathtt{F} = \nabla^{*}_{\mathtt{E}}\mathtt{F} + \sigma^{*}(\mathtt{E}, \mathtt{F}), \end{cases} \tag{2.3}$$

where σ and σ^{*} are symmetric, bilinear, imbedding curvature tensors of $\mathtt{M}$ in $\overline{\mathtt{M}}$ for $\overline{\nabla}$ and $\overline{\nabla}^{*}$, respectively.

The curvature tensor fields $\overline{\mathtt{R}}$ and $\overline{\mathtt{R}}^{*}$ of $\overline{\nabla}$ and $\overline{\nabla}^{*}$, respectively, satisfy

$$\mathtt{g}(\overline{\mathtt{R}}^{*}(\mathtt{E}, \mathtt{F})\mathtt{G}, \mathtt{W}) = -\mathtt{g}(\mathtt{G}, \overline{\mathtt{R}}(\mathtt{E}, \mathtt{F})\mathtt{W}), \tag{2.4}$$

where $\overline{\mathtt{R}}$ and $\overline{\mathtt{R}}^{*}$ are given by [22]

$$\begin{aligned} \mathtt{g}(\overline{\mathtt{R}}(\mathtt{E}, \mathtt{F})\mathtt{G}, \mathtt{W}) &= \mathtt{g}(\mathtt{R}(\mathtt{E}, \mathtt{F})\mathtt{G}, \mathtt{W}) + \mathtt{g}(\sigma(\mathtt{E}, \mathtt{G}), \sigma^{*}(\mathtt{F}, \mathtt{W})) \\ &\quad - \mathtt{g}(\sigma^{*}(\mathtt{E}, \mathtt{W}), \sigma(\mathtt{F}, \mathtt{G})), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \mathtt{g}(\overline{\mathtt{R}}^{*}(\mathtt{E}, \mathtt{F})\mathtt{G}, \mathtt{W}) &= \mathtt{g}(\mathtt{R}^{*}(\mathtt{E}, \mathtt{F})\mathtt{G}, \mathtt{W}) + \mathtt{g}(\sigma^{*}(\mathtt{E}, \mathtt{G}), \sigma(\mathtt{F}, \mathtt{W})) \\ &\quad - \mathtt{g}(\sigma(\mathtt{E}, \mathtt{W}), \sigma^{*}(\mathtt{F}, \mathtt{G})). \end{aligned} \tag{2.6}$$

Let us denote the normal bundle of $\mathtt{M}$ by $T\mathtt{M}^{\perp}$. The linear transformations A_N and A^{*}_N are defined by

$$\begin{cases} \mathtt{g}(A_N\mathtt{E}, \mathtt{F}) = \mathtt{g}(\sigma(\mathtt{E}, \mathtt{F}), N), \\ \mathtt{g}(A^{*}_N\mathtt{E}, \mathtt{F}) = \mathtt{g}(\sigma^{*}(\mathtt{E}, \mathtt{F}), N), \end{cases} \tag{2.7}$$

for any $N \in T\mathtt{M}^{\perp}$ and $\mathtt{E}, \mathtt{F} \in T\mathtt{M}$. The corresponding Weingarten formulae are [22]

$$\begin{cases} \overline{\nabla}_{\mathtt{E}}N = -A^{*}_N\mathtt{E} + \nabla^{\perp}_{\mathtt{E}}N, \\ \overline{\nabla}^{*}_{\mathtt{E}}N = -A_N\mathtt{E} + \nabla^{*\perp}_{\mathtt{E}}N, \end{cases} \tag{2.8}$$

where $N \in T\mathtt{M}^{\perp}$, $\mathtt{E} \in T\mathtt{M}$ and $\nabla^{\perp}_{\mathtt{E}}$ and $\nabla^{*\perp}_{\mathtt{E}}$ are Riemannian dual connections with respect to the induced metric on $T\mathtt{M}^{\perp}$.

Let $\overline{\mathtt{M}}$ be an odd-dimensional manifold and ϕ be a tensor field of type $(1, 1)$, ξ a vector field, and a 1-form η on $\overline{\mathtt{M}}$ satisfying the conditions

$$\eta(\xi) = 1,$$
$$\phi^{2}\mathtt{E} = -\mathtt{E} + \eta(\mathtt{E})\xi,$$

for any vector field $\mathtt{E}$ on $\overline{\mathtt{M}}$. Then $\overline{\mathtt{M}}$ is said to have an almost contact structure (ϕ, ξ, η).

Definition 2.3 ([9]) An almost contact structure $(\phi, \xi, \mathtt{g})$ on $\overline{\mathtt{M}}$ is said to be a Sasakian structure if

$$(\overline{\nabla}^{\circ}_{\mathtt{E}}\phi)\mathtt{F} = \mathtt{g}(\mathtt{F}, \xi)\mathtt{E} - \mathtt{g}(\mathtt{F}, \mathtt{E})\xi$$

holds for any $\mathtt{E}, \mathtt{F} \in T\overline{\mathtt{M}}$.

Definition 2.4 ([9]) A quadruple $(\overline{\nabla}, \mathtt{g}, \phi, \xi)$ is called a Sasakian statistical structure on $\overline{\mathtt{M}}$ if $(\overline{\nabla}, \mathtt{g})$ is a statistical structure, $(\mathtt{g}, \phi, \xi)$ is a Sasakian structure on $\overline{\mathtt{M}}$ and the formula

$$\mathtt{K}_{\mathtt{E}}\phi\mathtt{F} + \phi\mathtt{K}_{\mathtt{E}}\mathtt{F} = 0$$

holds for any $\mathtt{E}, \mathtt{F} \in T\overline{\mathtt{M}}$, where $\mathtt{K}_{\mathtt{E}}\mathtt{F} = \overline{\nabla}_{\mathtt{E}}\mathtt{F} - \overline{\nabla}^{\circ}_{\mathtt{E}}\mathtt{F}$.

Definition 2.5 ([9]) Let $(\overline{\mathtt{M}}, \overline{\nabla}, \mathtt{g}, \phi, \xi)$ be a Sasakian statistical manifold and $c \in \mathtt{R}$. The Sasakian statistical structure is said to be of constant ϕ-sectional curvature c if the curvature tensor $\overline{S}$ is given by

$$\begin{aligned}\overline{S}(\mathtt{E}, \mathtt{F})\mathtt{G} = &\frac{c+3}{4}\{\mathtt{g}(\mathtt{F}, \mathtt{G})\mathtt{E} - \mathtt{g}(\mathtt{E}, \mathtt{G})\mathtt{F}\} + \frac{c-1}{4}\{\mathtt{g}(\phi\mathtt{F}, \mathtt{G})\phi\mathtt{E} \\ &- \mathtt{g}(\phi\mathtt{E}, \mathtt{G})\phi\mathtt{F} - 2\mathtt{g}(\phi\mathtt{E}, \mathtt{F})\phi\mathtt{G} - \mathtt{g}(\mathtt{F}, \xi)\mathtt{g}(\mathtt{G}, \xi)\mathtt{E} + \mathtt{g}(\mathtt{E}, \xi)\mathtt{g}(\mathtt{G}, \xi)\mathtt{F} \\ &+ \mathtt{g}(\mathtt{F}, \xi)\mathtt{g}(\mathtt{G}, \mathtt{E})\xi - \mathtt{g}(\mathtt{E}, \xi)\mathtt{g}(\mathtt{G}, \mathtt{F})\xi\},\end{aligned} \tag{2.9}$$

where $\mathtt{E}, \mathtt{F}, \mathtt{G} \in T\overline{\mathtt{M}}$ and

$$2\overline{S}(\mathtt{E}, \mathtt{F})\mathtt{G} = \overline{\mathtt{R}}(\mathtt{E}, \mathtt{F})\mathtt{G} + \overline{\mathtt{R}}^*(\mathtt{E}, \mathtt{F})\mathtt{G}. \tag{2.10}$$

We denote a Sasakian statistical manifold with constant ϕ-sectional curvature c by $\overline{\mathtt{M}}(c)$.

Let $\{e_1, \ldots, e_n\}$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ be tangent orthonormal frame and normal orthonormal frame, respectively, on $\mathtt{M}$. The mean curvature vector fields with respect to $\overline{\nabla}$, $\overline{\nabla}^*$ and $\overline{\nabla}^{\circ}$, respectively, are given by

$$\begin{cases} \mathtt{H} = \frac{1}{n}\sum_{i=1}^{n}\sigma(e_i, e_i), \\ \mathtt{H}^* = \frac{1}{n}\sum_{i=1}^{n}\sigma^*(e_i, e_i), \\ \mathtt{H}^{\circ} = \frac{1}{n}\sum_{i=1}^{n}\sigma^{\circ}(e_i, e_i). \end{cases} \tag{2.11}$$

We also set

$$\begin{cases} \|\sigma\|^2 = \sum_{i,j=1}^{n} g(\sigma(e_i, e_j), \sigma(e_i, e_j)), \\ \|\sigma^*\|^2 = \sum_{i,j=1}^{n} g(\sigma^*(e_i, e_j), \sigma^*(e_i, e_j)), \\ \|\sigma^\circ\|^2 = \sum_{i,j=1}^{n} g(\sigma^\circ(e_i, e_j), \sigma^\circ(e_i, e_j)). \end{cases} \tag{2.12}$$

The second fundamental form σ°(resp. σ, or σ^*) has several geometric properties due to which we got the following different classes of the submanifolds.

- A submanifold is said to be totally a geodesic submanifold with respect to $\overline{\nabla}^\circ$ (resp. $\overline{\nabla}$, or $\overline{\nabla}^*$), if the second fundamental form σ° (resp. σ, or σ^*) vanishes identically, that is $\sigma^\circ = 0$ (resp. $\sigma = 0$, or $\sigma^* = 0$).
- A submanifold is said to be a minimal submanifold with respect to $\overline{\nabla}^\circ$ (resp.$\overline{\nabla}$, or $\overline{\nabla}^*$), if the mean curvature vector H° (resp.H, or H^*) vanishes identically, that is $H^\circ = 0$ (resp. $H = 0$, or $H^* = 0$).

Let $K(\pi)$ denote the sectional curvature of a Riemannian manifold M of the plane section $\pi \subset T_pM$ at a point $p \in M$. If $\{e_1, \dots, e_n\}$ is an orthonormal basis of T_pM and $\{e_{n+1}, \dots, e_{2m+1}\}$ is an orthonormal basis of $T_p^\perp M$ at any $p \in M$, then

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j), \tag{2.13}$$

where τ is the scalar curvature. The normalized scalar curvature ρ is defined as

$$\rho = \frac{2\tau}{n(n-1)}. \tag{2.14}$$

We also put

$$\sigma_{ij}^{\gamma} = g(\sigma(e_i, e_j), e_\gamma), \qquad \sigma_{ij}^{*\gamma} = g(\sigma^*(e_i, e_j), e_\gamma),$$

$i, j \in 1, 2, \dots, n, \gamma \in \{n+1, n+2, \dots, 2m+1\}$. The squared norms of the second fundamental forms σ and σ^* are denoted by C and C^*, respectively, and are given as

$$C = \frac{1}{n} \sum_{\gamma=n+1}^{2m+1} \sum_{i,j=1}^{n} \left(\sigma_{ij}^{\gamma}\right)^2 \quad \text{and} \quad C^* = \frac{1}{n} \sum_{\gamma=n+1}^{2m+1} \sum_{i,j=1}^{n} \left(\sigma_{ij}^{*\gamma}\right)^2, \tag{2.15}$$

called Casorati curvatures of the submanifold.

Let L be an r-dimensional subspace of TM, $r \ge 2$, and $\{e_1, e_2, \dots, e_r\}$ is an orthonormal basis of L. Then

$$\tau(L) = \sum_{1 \le \gamma < \beta \le r} K(e_\gamma \wedge e_\beta)$$

is called the scalar curvature of the r-plane section. The Casorati curvatures $\mathtt{C}$ and $\mathtt{C}^*$ of that r-plane section L are

$$\mathtt{C}(\mathtt{L}) = \frac{1}{r}\sum_{\gamma=n+1}^{2m+1}\sum_{i,j=1}^{r}\left(\sigma_{ij}^{\gamma}\right)^2 \quad \text{and} \quad \mathtt{C}^*(\mathtt{L}) = \frac{1}{r}\sum_{\gamma=n+1}^{2m+1}\sum_{i,j=1}^{r}\left(\sigma_{ij}^{*\gamma}\right)^2. \tag{2.16}$$

The normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\widehat{\delta}_c(n-1)$ are defined as

$$[\delta_c(n-1)]_p = \frac{1}{2}\mathtt{C}_p + \frac{n+1}{2n} inf\{\mathtt{C}(\mathtt{L})|\mathtt{L} : \text{a hyperplane of } T_p\mathtt{M}\} \tag{2.17}$$

and

$$[\widehat{\delta}_c(n-1)]_p = 2\mathtt{C}_p + \frac{2n-1}{2n} sup\{\mathtt{C}(\mathtt{L})|\mathtt{L} : \text{a hyperplane of } T_p\mathtt{M}\}. \tag{2.18}$$

Moreover, the dual normalized δ^*-Casorati curvatures $\delta_c^*(n-1)$ and $\widehat{\delta}_c^*(n-1)$ are defined as

$$[\delta_c^*(n-1)]_p = \frac{1}{2}\mathtt{C}_p^* + \frac{n+1}{2n} inf\{\mathtt{C}^*(\mathtt{L})|\mathtt{L} : \text{a hyperplane of } T_p\mathtt{M}\} \tag{2.19}$$

and

$$[\widehat{\delta}_c^*(n-1)]_p = 2\mathtt{C^*}_p + \frac{2n-1}{2n} sup\{\mathtt{C}^*(\mathtt{L})|\mathtt{L} : \text{a hyperplane of } T_p\mathtt{M}\}. \tag{2.20}$$

Let $\overline{\mathtt{M}}$ be a Riemannian manifold and $\mathtt{M}$ be a submanifold of $\overline{\mathtt{M}}$ and $f : \overline{\mathtt{M}} \to \mathbb{R}$ be a differentiable function. If we assume an optimal problem

$$min_{x\in\mathtt{M}} f(x), \tag{2.21}$$

then we have the following lemma for later use.

Lemma 2.6 ([18]) *If $x_\circ \in \mathtt{M}$ is a solution of the problem (2.21), then*

- $(grad(f))(x_\circ) \in T_{x_\circ}^{\perp}\mathtt{M}$
- *the bilinear form*

$$\Lambda : T_{x_\circ}\mathtt{M} \times T_{x_\circ}\mathtt{M} \to \mathbb{R},$$

$$\Lambda(\mathtt{E},\mathtt{F}) = Hessf(\mathtt{E},\mathtt{F}) + g(\sigma(\mathtt{E},\mathtt{F}), (gradf)(x))$$

is positive semi-definite, where σ is the second fundamental form of $\mathtt{M}$ in $\overline{\mathtt{M}}$ and $grad(f)$ is the gradient of f.

3 Normalized δ-Casorati Curvature

In this section, we mainly find the bounds for normalized scalar curvature in terms of the normalized $\delta-$Casorati curvatures for statistical submanifolds of Sasakian statistical manifold with constant ϕ-sectional curvature.

Theorem 3.1 *Let* $\mathtt{M}$ *be a statistical submanifold in a Sasakian statistical manifold* $\overline{\mathtt{M}}(c)$ *such that* $\mathtt{M}$ *is tangent to the structure vector field* ξ *of* $\overline{\mathtt{M}}(c)$*. Then, the normalized* δ*-Casorati curvatures* $\delta_c(n-1)$ *and* $\delta_c^*(n-1)$ *satisfy*

$$\rho \leq 2\delta_c^\circ(n-1) - \frac{4n^2(n-1)(n^3+4n^2-3n-20)}{(n^2+4n+1)^2}\|\mathtt{H}^\circ\|^2 + \frac{c-1}{2n(n-1)}\left\{2(n-1)+3\|P\|^2\right\} + \frac{c+3}{2} + \frac{2}{(n-1)}\mathtt{C}^\circ + \frac{n}{n-1}(\|\mathtt{H}\|^2, \|\mathtt{H}^*\|^2), \tag{3.1}$$

for real t*,* $0 < t < n(n-1)$*, where* $2\delta_c^\circ(n-1) = \delta_c(n-1) + \delta_c^*(n-1)$ *and* $2\mathtt{C}^\circ = \mathtt{C} + \mathtt{C}^*$.

Proof From (2.5), (2.6), (2.9) and (2.10), we have

$$\begin{aligned} &\mathtt{g}(\mathtt{R}(\mathtt{E},\mathtt{F})\mathtt{G},\mathtt{W}) + \mathtt{g}(\mathtt{R}^*(\mathtt{E},\mathtt{F})\mathtt{G},\mathtt{W}) \\ &= \frac{c+3}{2}\{\mathtt{g}(\mathtt{F},\mathtt{G})\mathtt{g}(\mathtt{E},\mathtt{W}) - \mathtt{g}(\mathtt{E},\mathtt{G})\mathtt{g}(\mathtt{F},\mathtt{W}\} \\ &+ \frac{c-1}{2}\{\mathtt{g}(\phi\mathtt{F},\mathtt{G})\mathtt{g}(\phi\mathtt{E},\mathtt{W}) - \mathtt{g}(\phi\mathtt{E},\mathtt{G})\mathtt{g}(\phi\mathtt{F},\mathtt{W}) \\ &- 2\mathtt{g}(\phi\mathtt{E},\mathtt{F})\mathtt{g}(\phi\mathtt{G},\mathtt{W}) - \mathtt{g}(\mathtt{F},\xi)\mathtt{g}(\mathtt{G},\xi)\mathtt{g}(\mathtt{E},\mathtt{W}) \\ &+ \mathtt{g}(\mathtt{E},\xi)\mathtt{g}(\mathtt{G},\xi)\mathtt{g}(\mathtt{F},\mathtt{W}) + \mathtt{g}(\mathtt{F},\xi)\mathtt{g}(\mathtt{G},\mathtt{E})\mathtt{g}(\xi,\mathtt{W}) \\ &- \mathtt{g}(\mathtt{E},\xi)\mathtt{g}(\mathtt{G},\mathtt{F})\mathtt{g}(\xi,\mathtt{W})\} - \mathtt{g}(\sigma(\mathtt{E},\mathtt{G}),\sigma^*(\mathtt{F},\mathtt{W})) \\ &+ \mathtt{g}(\sigma^*(\mathtt{E},\mathtt{W}),\sigma(\mathtt{F},\mathtt{G})) - \mathtt{g}(\sigma^*(\mathtt{E},\mathtt{G}),\sigma(\mathtt{F},\mathtt{W})) \\ &+ \mathtt{g}(\sigma(\mathtt{E},\mathtt{W}),\sigma^*(\mathtt{F},\mathtt{G})). \end{aligned} \tag{3.2}$$

Putting $\mathtt{F} = \mathtt{W} = e_i$ and $\mathtt{E} = \mathtt{G} = e_j$, in (3.2), we get

$$
\begin{aligned}
&\mathtt{g}(\mathtt{R}(e_i, e_j)e_j, e_i) + \mathtt{g}(\mathtt{R}^*(e_i, e_j)e_j, e_i) \\
&\quad = \frac{c+3}{2}\{\mathtt{g}(e_j, e_j)\mathtt{g}(e_i, e_i) - \mathtt{g}(e_i, e_j)\mathtt{g}(e_j, e_i\} \\
&\quad + \frac{c-1}{2}\{\mathtt{g}(\phi e_j, e_j)\mathtt{g}(\phi e_i, e_i) - \mathtt{g}(\phi e_i, e_j)\mathtt{g}(\phi e_j, e_i) \\
&\quad - 2\mathtt{g}(\phi e_i, e_j)\mathtt{g}(\phi e_j, e_i) - \mathtt{g}(e_j, \xi)\mathtt{g}(e_j, \xi)\mathtt{g}(e_i, e_i) \\
&\quad + \mathtt{g}(e_i, \xi)\mathtt{g}(e_j, \xi)\mathtt{g}(e_j, e_i) + \mathtt{g}(e_j, \xi)\mathtt{g}(e_j, e_i)\mathtt{g}(\xi, e_i) \\
&\quad - \mathtt{g}(e_i, \xi)\mathtt{g}(e_j, e_j)\mathtt{g}(\xi, e_i)\} - \mathtt{g}(\sigma(e_i, e_j), \sigma^*(e_j, e_i)) \\
&\quad + \mathtt{g}(\sigma^*(e_i, e_i), \sigma(e_j, e_j)) - \mathtt{g}(\sigma^*(e_i, e_j), \sigma(e_j, e_i)) \\
&\quad + \mathtt{g}(\sigma(e_i, e_i), \sigma^*(e_j, e_j)). \qquad (3.3)
\end{aligned}
$$

Applying summation $1 \leq i, j \leq n$ and using (2.11) and (2.12) in (3.3), we obtain

$$
\begin{aligned}
&\sum_{1\leq i,j\leq n} [\mathtt{g}(\mathtt{R}(e_i, e_j)e_j, e_i) + \mathtt{g}(\mathtt{R}^*(e_i, e_j)e_j, e_i)] \\
&\quad = \frac{c+3}{2}n(n-1) + 2n^2\mathtt{g}(\mathtt{H}, \mathtt{H}^*) \\
&\quad + \frac{c-1}{2}\left\{2(1-n) + 3\sum_{1\leq i,j\leq n}\mathtt{g}^2(\phi e_i, e_j)\right\} \\
&\quad - \sum_{1\leq i,j\leq n}\mathtt{g}(\sigma(e_i, e_j), \sigma^*(e_j, e_i)) - \sum_{1\leq i,j\leq n}\mathtt{g}(\sigma^*(e_i, e_j), \sigma(e_j, e_i)) \\
&\quad = \frac{c+3}{2}n(n-1) \\
&\quad + n^2\{\mathtt{g}(\mathtt{H}^* + \mathtt{H}, \mathtt{H}^* + \mathtt{H}) - \mathtt{g}(\mathtt{H}, \mathtt{H}) - \mathtt{g}(\mathtt{H}^*, \mathtt{H}^*)\} \\
&\quad + \frac{c-1}{2}\{2(1-n) + 3\sum_{1\leq i,j\leq n}\mathtt{g}^2(\phi e_i, e_j)\} \\
&\quad - \sum_{1\leq i,j\leq n}\{\mathtt{g}(\sigma(e_i, e_j) + \sigma^*(e_j, e_i), \sigma^*(e_i, e_j) + \sigma(e_j, e_i)) \\
&\quad - \mathtt{g}(\sigma(e_i, e_j), \sigma(e_i, e_j)) - \mathtt{g}(\sigma^*(e_j, e_i), \sigma^*(e_j, e_i))\}. \qquad (3.4)
\end{aligned}
$$

Since from Eq. (2.2) $2\mathtt{H}^\circ = \mathtt{H} + \mathtt{H}^*$, it follows from the above equation and Eq. (2.15) that

$$
\begin{aligned}
2\tau &= \frac{c+3}{2}n(n-1) + \frac{c-1}{2}\{2(1-n) + 3\|P\|^2\} \\
&+ 4n^2\|\mathtt{H}^\circ\|^2 - n^2(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2) - 4n\mathtt{C}^\circ + n(\mathtt{C} + \mathtt{C}^*). \qquad (3.5)
\end{aligned}
$$

Define the following function, denoted by $\mathtt{Q}$, a quadratic polynomial in the components of the second fundamental form

$$
\begin{aligned}
\mathtt{Q} &= n(n-1)\mathtt{C}^{\circ} + (n-1)(n+1)\mathtt{C}^{\circ}(\mathtt{L}) + \frac{c+3}{2}n(n-1) \\
&+ \frac{c-1}{2}\left\{2(1-n) + 3\|P\|^2\right\} - 2\tau + n(\mathtt{C} + \mathtt{C}^*) \\
&- n^2(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2),
\end{aligned} \tag{3.6}
$$

where $\mathtt{L}$ is the hyperplane of $T_p\mathtt{M}$. Without loss of generality, let us assume that $\mathtt{L}$ is spanned by $e_1, \ldots, e_{n-1}$, then from (3.6)

$$
\begin{aligned}
\mathtt{Q} &= 2n\sum_{\gamma=n+1}^{2m+1}\sum_{i=1}^{n-1}(\sigma_{ii}^{\circ\gamma})^2 + 4(n+2)\sum_{\gamma=n+1}^{2m+1}\sum_{i<j=1}^{n-1}(\sigma_{ij}^{\circ\gamma})^2 \\
&+ (n-1)\sum_{\gamma=n+1}^{2m+1}(\sigma_{nn}^{\circ\gamma})^2 + 2(n+3)\sum_{\gamma=n+1}^{2m+1}\sum_{i=1}^{n-1}(\sigma_{in}^{\circ\gamma})^2 \\
&- 8\sum_{\gamma=n+1}^{2m+1}\sum_{i<j=1}^{n}\sigma_{ii}^{\circ\gamma}\sigma_{jj}^{\circ\gamma},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\frac{1}{4}\mathtt{Q} &= \sum_{\gamma=n+1}^{2m+1}\sum_{i=1}^{n-1}\left[\frac{n}{2}(\sigma_{ii}^{\circ\gamma})^2 + \frac{n+3}{2}\sum_{i=1}^{n-1}(\sigma_{in}^{\circ\gamma})^2\right] \\
&+ \sum_{\gamma=n+1}^{2m+1}\left[(n+2)\sum_{i<j=1}^{n-1}(\sigma_{ij}^{\circ\gamma})^2 + \frac{n-1}{4}\sum_{\gamma=n+1}^{2m+1}(\sigma_{nn}^{\circ\gamma})^2\right. \\
&\left. - 2\sum_{\gamma=n+1}^{2m+1}\sum_{i<j=1}^{n}\sigma_{ii}^{\circ\gamma}\sigma_{jj}^{\circ\gamma}\right] \\
&\geq \sum_{\gamma=n+1}^{2m+1}\left[\frac{n}{2}(\sigma_{ii}^{\circ\gamma})^2 + \frac{n-1}{4}(\sigma_{nn}^{\circ\gamma})^2 - 2\sum_{\gamma=n+1}^{2m+1}\sum_{i<j=1}^{n}\sigma_{ii}^{\circ\gamma}\sigma_{jj}^{\circ\gamma}\right].
\end{aligned} \tag{3.7}
$$

Now, we consider the quadratic forms $f_\gamma : \mathbb{R}^n \to \mathbb{R}$ such that

$$
\begin{aligned}
&f_\gamma(\sigma_{11}^{\circ\gamma}, \sigma_{22}^{\circ\gamma}, \cdots, \sigma_{nn}^{\circ\gamma}) \\
&\quad = \sum_{\gamma=n+1}^{2m+1}\left[\frac{n}{2}(\sigma_{ii}^{\circ\gamma})^2 + \frac{n-1}{4}(\sigma_{nn}^{\circ\gamma})^2 - 2\sum_{\gamma=n+1}^{2m+1}\sum_{i<j=1}^{n}\sigma_{ii}^{\circ\gamma}\sigma_{jj}^{\circ\gamma}\right].
\end{aligned} \tag{3.8}
$$

Then from (3.7), we deduce that

$$
\mathtt{Q} \geq 4\sum_{\gamma} f_\gamma \tag{3.9}
$$

Next, we start with the problem

$$minf_\gamma, \quad subject \quad to \quad \Gamma : \sigma_{11}^{\circ\gamma} + \sigma_{22}^{\circ\gamma} + \cdots + \sigma_{nn}^{\circ\gamma} = k^\gamma,$$

where k^γ is a real constant.

The partial derivatives of the function f_γ are

$$\begin{cases} \frac{\partial f_\gamma}{\partial \sigma_{11}^{\circ\gamma}} = n\sigma_{11}^{\circ\gamma} - 2\sum_{i\neq 1}\sigma_{ii}^{\circ\gamma} \\ \frac{\partial f_\gamma}{\partial \sigma_{22}^{\circ\gamma}} = n\sigma_{22}^{\circ\gamma} - 2\sum_{i\neq 2}\sigma_{ii}^{\circ\gamma} \\ \vdots \\ \frac{\partial f_\gamma}{\partial \sigma_{n-1n-1}^{\circ\gamma}} = n\sigma_{n-1n-1}^{\circ\gamma} - 2\sum_{i\neq n-1}\sigma_{ii}^{\circ\gamma} \\ \frac{\partial f_{\circ\gamma}}{\partial \sigma_{nn}^{\circ\gamma}} = \frac{n-1}{2}\sigma_{nn}^{\circ\gamma} - 2\sum_{i\neq n}\sigma_{ii}^{\circ\gamma}. \end{cases} \tag{3.10}$$

For an optimal solution $(\sigma_{11}^{\circ\gamma}, \sigma_{22}^{\circ\gamma}, \ldots, \sigma_{nn}^{\circ\gamma})$, the vector $grad(f_\gamma)$ is normal at Γ. That is, it is collinear with the vector $(1, 1, \ldots, 1)$.

From Eq. (3.10) and Lemma 2.6 a critical point of the problem has the following form

$$\begin{cases} \sigma_{11}^{\circ\gamma} = \sigma_{22}^{\circ\gamma} = \cdots = \sigma_{n-1n-1}^{\circ\gamma} = \frac{n+3}{n^2+4n+1}k^\gamma \\ \sigma_{nn}^{\circ\gamma} = \frac{2(n+2)}{n^2+4n+1}k^\gamma. \end{cases} \tag{3.11}$$

We fix an arbitrary point $x \in \Gamma$. The 2-form

$$\Lambda : T_x\Gamma \times T_x\Gamma \to \mathbb{R}$$

has the expression

$$\Lambda(\mathtt{E}, \mathtt{F}) = Hessf_\gamma(\mathtt{E}, \mathtt{F}) + \langle \sigma'(\mathtt{E}, \mathtt{F}), (gradf_\gamma)(x)\rangle,$$

where σ' is the second fundamental form of Γ in $\mathbb{R}^n$ and $\langle ., .\rangle$ is the standard inner-product of $\mathbb{R}^n$.

From (3.10), we get

$$\begin{cases} \frac{\partial^2 f_\gamma}{(\partial \sigma_{ii}^{\circ\gamma})^2} = n, \quad i \in \{1, \ldots, n-1\} \\ \frac{\partial^2 f_\gamma}{(\partial \sigma_{nn}^{\circ\gamma})^2} = \frac{n-1}{2} \\ \frac{\partial^2 f_\gamma}{\partial \sigma_{ii}^{\circ\gamma}\partial \sigma_{jj}^{\circ\gamma}} = -2, \quad i, j \in \{1, \ldots, n\}, \quad i \neq j. \end{cases} \tag{3.12}$$

Thus, the Hessian matrix of f_γ is

$$Hess f_\gamma = \begin{pmatrix} n & -2 & \dots & -2 & -2 \\ -2 & n & \dots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \dots & n & -2 \\ -2 & -2 & \dots & -2 & \frac{n-1}{2} \end{pmatrix}.$$

As Γ is totally geodesic in $\mathbb{R}^n$, considering a vector $\mathtt{E}$ tangent to Γ at the arbitrary point x on Γ, that is, verifying the relation $\sum_{i=1}^{n} \mathtt{E} = 0$, we obtain

$$\begin{aligned} \Lambda(\mathtt{E}, \mathtt{E}) &= (n+2) \sum_{i=1}^{n-1} \mathtt{E}_i^2 \\ &+ \frac{n+3}{2} \mathtt{E}_n^2 - 2(\mathtt{E}_1 + \mathtt{E}_2 + \cdots + \mathtt{E}_n)^2 \\ &= (n+2) \sum_{i=1}^{n-1} \mathtt{E}_i^2 + \frac{n+3}{2} \mathtt{E}_n^2 \\ &\geq 0. \end{aligned}$$

Hence from (3.11), the point $(\sigma_{11}^{\circ\gamma}, \sigma_{22}^{\circ\gamma}, \dots, \sigma_{nn}^{\circ\gamma})$ is a global minimum point. Therefore using (3.11) in f_γ, a lengthy but straight forward computation yields

$$f_\gamma \geq \frac{(1-n)(n^3+4n^2-3n-20)}{2(n^2+4n+1)^2} (k^\gamma)^2. \tag{3.13}$$

Taking into account (3.9) and (3.13), we obtain

$$\begin{aligned} \mathtt{Q} &\geq \sum_\gamma \frac{2(1-n)(n^3+4n^2-3n-20)}{(n^2+4n+1)^2} (k^\gamma)^2 \\ &= \frac{2(1-n)(n^3+4n^2-3n-20)}{(n^2+4n+1)^2} \sum_\gamma (k^\gamma)^2 \\ &= \frac{2n^2(1-n)(n^3+4n^2-3n-20)}{(n^2+4n+1)^2} \|\mathtt{H}^\circ\|^2. \end{aligned} \tag{3.14}$$

Combining (3.6) and (3.14) we have our assertion. □

The following result flows from Theorem 3.1.

Corollary 3.2 *Let* $\mathtt{M}$ *be a minimal statistical submanifold in a Sasakian statistical manifold* $\overline{\mathtt{M}}(c)$*, that is* $\mathtt{H}^\circ = 0$*, such that* $\mathtt{M}$ *is tangent to the structure vector field* ξ *of* $\overline{\mathtt{M}}(c)$*. Then, the normalized* δ*-Casorati curvatures* $\delta_c(n-1)$ *and* $\delta_c^*(n-1)$ *satisfy*

$$\rho \leq 2\delta_c^\circ(n-1) - \frac{n}{n-1}\mathtt{g}(\mathtt{H}, \mathtt{H}^*) + \frac{c+3}{2} + \frac{c-1}{2n(n-1)}\left\{2(n-1) + 3\|P\|^2\right\} + \frac{2}{(n-1)}\mathtt{C}^\circ. \tag{3.15}$$

Remark 3.3 Proof of the above corollary follows from Theorem 3.1 by using the definition of minimal submanifolds.

We state the similar results by assuming that $\mathtt{M}$ is normal to the structure vector field ξ of $\overline{\mathtt{M}}(c)$.

Theorem 3.4 *Let* $\mathtt{M}$ *be a statistical submanifold in a Sasakian statistical manifold* $\overline{\mathtt{M}}(c)$ *such that* $\mathtt{M}$ *is normal to the structure vector field* ξ *of* $\overline{\mathtt{M}}(c)$*. Then, the normalized* δ*-Casorati curvatures* $\delta_c(n-1)$ *and* $\delta_c^*(n-1)$ *satisfy*

$$\rho \leq 2\delta_c^\circ(n-1) - \frac{4n^2(n-1)(n^3+4n^2-3n-20)}{(n^2+4n+1)^2}\|\mathtt{H}^\circ\|^2 + \frac{3(c-1)}{2n(n-1)}\|P\|^2 + \frac{c+3}{2} + \frac{2}{(n-1)}\mathtt{C}^\circ + \frac{n}{n-1}(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2), \tag{3.16}$$

where $2\delta_c^\circ(n-1) = \delta_c(n-1) + \delta_c^*(n-1)$ *and* $2\mathtt{C}^\circ = \mathtt{C} + \mathtt{C}^*$.

Remark 3.5 This result can be proved with the same techniques as we used in the case of Theorem 3.1 and just keeping in mind that $\mathtt{M}$ is normal to the structure vector field ξ of $\overline{\mathtt{M}}(c)$.

As a consequence of the above theorem, we have following result.

Corollary 3.6 *Let* $\mathtt{M}$ *be a minimal statistical submanifold in a Sasakian statistical manifold* $\overline{\mathtt{M}}(c)$*, that is* $\mathtt{H}^\circ = 0$*, such that* $\mathtt{M}$ *is normal to the structure vector field* ξ *of* $\overline{\mathtt{M}}(c)$*. Then, the normalized* δ*-Casorati curvatures* $\delta_c(n-1)$ *and* $\delta_c^*(n-1)$ *satisfy*

$$\rho \leq 2\delta_c^\circ(n-1) - \frac{n}{n-1}\mathtt{g}(\mathtt{H}, \mathtt{H}^*) + \frac{c+3}{2} + \frac{3(c-1)}{2n(n-1)}\|P\|^2 + \frac{2}{(n-1)}\mathtt{C}^\circ. \tag{3.17}$$

4 Inequalities for Statistical Submanifolds in Sasakian Statistical Manifold with Semi-symmetric Metric Connection

Friedmann and Schouten [8] introduced the notion of semi-symmetric linear connection on a differentiable manifold. Any linear connection is called semi-symmetric connection if its torsion tensor T satisfies

$$T(X, Y) = \omega(y)X - \omega(X)y$$

where ω is the 1-form given by $\omega(X) = g(X, U)$ for any vector fields $X, Y, U \in TM$. Any semi-symmetric connection is said to be a semi-symmetric metric connection $\tilde{\nabla}$ if it further satisfies

$$\tilde{\nabla} g = 0.$$

Theorem 4.1 *Let N be a statistical submanifold in a Sasakian statistical manifold $\overline{N}$ such that N is tangent to the structure vector field ξ. Then the normalized δ-Casorati curvatures $\delta_c(n-1)$ and $\delta_c^*(n-1)$ satisfy the following:*

$$\begin{aligned} 2\rho \leq\, & \delta_c(n-1) + \delta_c^*(n-1) + 4\tilde{\rho}^N + \frac{4n^2 + 2n - 6 + 2\lambda}{n(n-1)} \\ & - \frac{8n}{n-1}\|H^\circ\|^2 + \frac{n}{n-1}\{\|H\|^2 + \|H^*\|^2\} \\ & + \frac{1}{n-1}\{\mathsf{C} + \mathsf{C}^*\} + \frac{4}{n(n-1)} \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} \sigma_{ij}^r \sigma_{ij}^{*r}, \end{aligned}$$

where $\tilde{\rho}^N$ is the normalized scalar curvature with respect to the semi-symmetric metric connection over the submanifold N and λ is given by

$$\begin{aligned} \lambda =\, & \sum_{i,j=1} \{g((\nabla_{e_i} k)(e_j, e_i), e_j) + g((\nabla^*_{e_i} k)(e_j, e_i), e_j)\} \\ & - \sum_{i,j} \{g((\nabla_{e_j} k)(e_i, e_i), e_j) + g((\nabla^*_{e_j} k)(e_i, e_i), e_j)\} \\ & -2 \sum_{i,j} \{\|k_{ij}\|^2 - g(k(e_i, e_i), k(e_j, e_j))\}. \end{aligned}$$

The equality case of the above inequality holds if and only if

$$\sigma_{11}^r = \sigma_{22}^r = \sigma_{33}^r = \cdots \sigma_{n-1n-1}^r = \frac{1}{2}\sigma_{nn}^r = \frac{nH}{n+1}$$

and

$$\sigma_{11}^{*r} = \sigma_{22}^{*r} = \sigma_{33}^{*r} = \cdots \sigma_{n-1n-1}^{*r} = \frac{1}{2}\sigma_{nn}^{*r} = \frac{nH^*}{n+1}.$$

Proof Let N be a statistical submanifold in a Sasakian statistical manifold $\overline{N}(c)$ with semi-symmetric metric connection. It is known from [12] that

$$
\begin{aligned}
\overline{\mathrm{R}}(\mathrm{E},\mathrm{F})\mathrm{G} &= \overline{\mathrm{R}}(\mathrm{E},\mathrm{F})\mathrm{G} + \{\phi^2\mathrm{E}-\phi\mathrm{E}\}g(\mathrm{F},\mathrm{G}) - \{\phi^2\mathrm{F}-\phi\mathrm{F}\}g(\mathrm{E},\mathrm{G})\\
&\quad + g(\phi\mathrm{E},\mathrm{G})\mathrm{F} - g(\phi\mathrm{F},\mathrm{G})\mathrm{E} - \eta(\mathrm{E})\eta(\mathrm{G})\mathrm{F} + \eta(\mathrm{F})\eta(\mathrm{G})\mathrm{E}\\
&\quad - (\nabla_{\mathrm{E}}k)(\mathrm{F},\mathrm{G}) + (\nabla_{\mathrm{F}}k)(\mathrm{E},\mathrm{G}) + k(\mathrm{E},k(\mathrm{F},\mathrm{G})) - k(\mathrm{F},k(\mathrm{E},\mathrm{G}))\\
&= \overline{\mathrm{R}}^*(\mathrm{E},\mathrm{F})\mathrm{G} + \{\phi^2\mathrm{E}-\phi\mathrm{E}\}g(\mathrm{F},\mathrm{G}) - \{\phi^2\mathrm{F}-\phi\mathrm{F}\}g(\mathrm{E},\mathrm{G})\\
&\quad + g(\phi\mathrm{E},\mathrm{G})\mathrm{F} - g(\phi\mathrm{F},\mathrm{G})\mathrm{E} - \eta(\mathrm{E})\eta(\mathrm{G})\mathrm{F} + \eta(\mathrm{F})\eta(\mathrm{G})\mathrm{E}\\
&\quad - (\nabla^*_{\mathrm{E}}k)(\mathrm{F},\mathrm{G}) + (\nabla^*_{\mathrm{F}}k)(\mathrm{E},\mathrm{G}) + k(\mathrm{E},k(\mathrm{F},\mathrm{G})) - k(\mathrm{F},k(\mathrm{E},\mathrm{G})),
\end{aligned} \tag{4.1}
$$

where $\overline{\mathrm{R}}$ is the curvature tensor with respect to the semi-symmetric metric connection $\overline{\nabla}$.

Equation (4.1) implies

$$
\begin{aligned}
2\overline{\mathrm{R}}(\mathrm{E},\mathrm{F})\mathrm{G} &= \overline{\mathrm{R}}(\mathrm{E},\mathrm{F})\mathrm{G} + \overline{\mathrm{R}}^*(\mathrm{E},\mathrm{F})\mathrm{G} + 2\{\phi^2\mathrm{E}-\phi\mathrm{E}\}g(\mathrm{F},\mathrm{G})\\
&\quad - 2\{\phi^2\mathrm{F}-\phi\mathrm{F}\}g(\mathrm{E},\mathrm{G}) + 2g(\phi\mathrm{E},\mathrm{G})\mathrm{F} - 2g(\phi\mathrm{F},\mathrm{G})\mathrm{E}\\
&\quad - 2\eta(\mathrm{E})\eta(\mathrm{G})\mathrm{F} + 2\eta(\mathrm{F})\eta(\mathrm{G})\mathrm{E} - (\nabla_{\mathrm{E}}k)(\mathrm{F},\mathrm{G})\\
&\quad - (\nabla^*_{\mathrm{E}}k)(\mathrm{F},\mathrm{G}) + (\nabla_{\mathrm{F}}k)(\mathrm{E},\mathrm{G}) + (\nabla^*_{\mathrm{F}}k)(\mathrm{E},\mathrm{G})\\
&\quad + 2k(\mathrm{E},k(\mathrm{F},\mathrm{G})) - 2k(\mathrm{F},k(\mathrm{E},\mathrm{G})).
\end{aligned} \tag{4.2}
$$

From (2.5) and (2.6), we write

$$
\begin{aligned}
& g(\overline{\mathrm{R}}(\mathrm{E},\mathrm{F})\mathrm{G},\mathrm{W}) + g(\overline{\mathrm{R}^*}(\mathrm{E},\mathrm{F})\mathrm{G},\mathrm{W})\\
&\qquad = g(\mathrm{R}(\mathrm{E},\mathrm{F})\mathrm{G},\mathrm{W}) + g(\mathrm{R}^*(\mathrm{E},\mathrm{F})\mathrm{G},\mathrm{W})\\
&\qquad + g(\sigma(\mathrm{E},\mathrm{G}),\sigma^*(\mathrm{F},\mathrm{W})) - g(\sigma^*(\mathrm{E},\mathrm{W}),\sigma(\mathrm{F},\mathrm{G}))\\
&\qquad + g(\sigma^*(\mathrm{E},\mathrm{G}),\sigma(\mathrm{F},\mathrm{W})) - g(\sigma(\mathrm{E},\mathrm{W}),\sigma^*(\mathrm{F},\mathrm{G})).
\end{aligned} \tag{4.3}
$$

From (4.2) and (4.3), we find

$$
\begin{aligned}
& g(\overline{\mathrm{R}}(\mathrm{E},\mathrm{F})\mathrm{G},\mathrm{W}) + g(\overline{\mathrm{R}}^*(\mathrm{E},\mathrm{F})\mathrm{G},\mathrm{W})\\
&\qquad = 2g(\tilde{\mathrm{R}}(\mathrm{E},\mathrm{F})\mathrm{G},\mathrm{W}) - 2\{\phi^2\mathrm{E}-\phi\mathrm{E}\}g(\mathrm{F},\mathrm{G})\\
&\qquad + 2\{\phi^2\mathrm{F}-\phi\mathrm{F}\}g(\mathrm{E},\mathrm{G}) + 2g(\phi\mathrm{E},\mathrm{G})\mathrm{F} - 2g(\phi\mathrm{E},\mathrm{G})\mathrm{F}\\
&\qquad + 2g(\phi\mathrm{F},\mathrm{G})\mathrm{E} + 2\eta(\mathrm{E})\eta(\mathrm{G})\mathrm{F} - 2\eta(\mathrm{F})\eta(\mathrm{G})\mathrm{E}\\
&\qquad + [(\nabla_{\mathrm{E}}k)(\mathrm{F},\mathrm{G}) + (\nabla^*_{\mathrm{E}}k)(\mathrm{F},\mathrm{G})] - [(\nabla_{\mathrm{F}}k)(\mathrm{E},\mathrm{G}) + (\nabla^*_{\mathrm{F}}k)(\mathrm{E},\mathrm{G})]\\
&\qquad - 2k(\mathrm{E},k(\mathrm{F},\mathrm{G})) + 2k(\mathrm{F},k(\mathrm{E},\mathrm{G}))\\
&\qquad - g(\sigma(\mathrm{E},\mathrm{G}),\sigma^*(\mathrm{F},\mathrm{W}))) + g(\sigma^*(\mathrm{E},\mathrm{W}),\sigma(\mathrm{F},\mathrm{G}))\\
&\qquad - g(\sigma^*(\mathrm{E},\mathrm{G}),\sigma(\mathrm{F},\mathrm{W})) + g(\sigma) + g(\sigma(\mathrm{E},\mathrm{W}),\sigma^*(\mathrm{F},\mathrm{G})).
\end{aligned} \tag{4.4}
$$

Simplifying (4.4), we arrive at

$$2\tau = 4\tilde{\tau}^N + (n-1) + 2n(n-1) + 2n - 2 + \lambda - n^2 g(\mathtt{H}^*, \mathtt{H})$$
$$- n^2 g(\mathtt{H}, \mathtt{H}^*) + \sum_{i,j} g(\sigma(e_i, e_j), \sigma^*(e_i, e_j)) + \sum_{i,j} g(\sigma^*(e_i, e_j), \sigma(e_i, e_j)),$$

where

$$\lambda = \sum_{i,j}\{g((\nabla_{e_i} k)(e_j, e_i), e_j) + g((\nabla^*_{e_i} k)(e_j, e_i), e_j)\}$$
$$- \sum_{i,j} - \sum_{i,j}\{g((\nabla_{e_j} k)(e_i, e_i), e_j) + g((\nabla^*_{e_j} k)(e_i, e_i), e_j)\}$$
$$- 2\sum_{i,j}\{\|k_{ij}\|^2 - g(k(e_i, e_i), k(e_j, e_j))\}.$$

This gives

$$2\tau = 4\tilde{\tau}^N + 2n^2 + n - 3 + \lambda - n^2 g(\mathtt{H}^*, \mathtt{H}) - n^2 g(\mathtt{H}, \mathtt{H}^*)$$
$$+ \sum_{i,j} g(\sigma(e_i, e_j) + \sigma^*(e_i, e_j), \sigma^*(e_i, e_j) + \sigma(e_i, e_j))$$
$$- \sum_{i,j} g(\sigma(e_i, e_j), \sigma(e_i, e_j)) - \sum_{i,j} g(\sigma^*(e_i, e_j), \sigma^*(e_i, e_j))$$
$$= 4\tilde{\tau}^N + 2n^2 - n^2\{g(\mathtt{H} + \mathtt{H}^*, \mathtt{H} + \mathtt{H}^*) - g(\mathtt{H}, \mathtt{H}) - g(\mathtt{H}^*, \mathtt{H}^*)\}$$
$$- n\mathtt{C} - n\mathtt{C}^* + \sum_{i,j} \|\sigma(e_i, e_j) + \sigma^*(e_i, e_j)\|^2 + n - 3 + \lambda$$
$$= 4\tilde{\tau}^N + 2n^2 + n - 3 + \lambda - 4n^2\|\mathtt{H}^\circ\|^2 + n^2\|\mathtt{H}\|^2 + n^2\|\mathtt{H}^*\|^2$$
$$- n(\mathtt{C} + \mathtt{C}^*) + \sum_{i,j} \|\sigma(e_i, e_j) + \sigma^*(e_i, e_j)\|^2.$$

Now consider the following function Q which is a quadratic polynomial in the components of the second fundamental form.

$$Q = \frac{1}{2}n(n-1)\mathtt{C} + \frac{(n-1)(n+1)}{2}\mathtt{C}(L) - 2\tau + 4\tilde{\tau}^N + 2n^2$$
$$+ n - 3 + \lambda - 4n^2\|\mathtt{H}^\circ\|^2 + n^2\|\mathtt{H}\|^2 + n^2\|\mathtt{H}^*\|^2$$
$$- n(\mathtt{C} + \mathtt{C}^*) + \sum_{i,j} \|\sigma(e_i, e_j) + \sigma^*(e_i, e_j)\|^2. \tag{4.5}$$

Assuming that L is spaned by $e_1, e_2, \ldots e_{n-1}$ which from [23] gives

$$Q \geq \sum_{r=n+1}^{2m+1} \left\{ n \sum_{j=1}^{n-1} (\sigma^r_{ii})^2 + \frac{n-1}{2}(\sigma^r_{nn})^2 - 2 \sum_{1 \leq i < j \leq n} \sigma^r_{ii}\sigma^r_{jj} \right\}$$

Now we set

$$\sigma^r_{11} + \sigma^r_{22} + \cdots + \sigma^r_{nn} = k^r,$$

where k^r is a real constant. □

Then by [12, 23]

$$Q \geq 0 \tag{4.6}$$

with the equality case if and only if

$$\sigma^r_{11} = \sigma^r_{22} = \sigma^r_{33} = \cdots \sigma^r_{n-1n-1} = \frac{1}{2}\sigma^r_{nn}.$$

Combining Eqs. (4.5) and (4.6), we get

$$\begin{aligned}2\tau \leq{}& \frac{n(n-1)}{2}\mathrm{C} + \frac{(n-1)(n+1)}{2}\mathrm{C}(l) + 4\tilde{\tau}^N + 2n^2 + n - 3 + \lambda \\ &- 4n^2\|\mathrm{H}^\circ\|^2 + n^2\|\mathrm{H}^*\|^2 - nC^* \\ &+ \sum_{i,j}\|\sigma(e_i, e_j) + \sigma^*(e_i, e_j)\|^2.\end{aligned} \tag{4.7}$$

Similarly, we get

$$\begin{aligned}2\tau \leq{}& \frac{n(n-1)}{2}\mathrm{C}^* + \frac{(n-1)(n+1)}{2}\mathrm{C}^*(L) + 4\tilde{\tau}^N + 2n^2 + n - 3 + \lambda \\ &- 4n^2\|\mathrm{H}^\circ\|^2 + n^2\|\mathrm{H}\|^2 - n\mathrm{C} \\ &+ \sum_{i,j}\|\sigma(e_i, e_j) + \sigma^*(e_i, e_j)\|^2.\end{aligned} \tag{4.8}$$

Equations (4.7) and (4.8) imply

$$\begin{aligned}4\tau \leq{}& \frac{n(n-1)}{2}[\mathrm{C} + \mathrm{C}^*] + \frac{(n-1)(n+1)}{2}[\mathrm{C}(L) + \mathrm{C}^*(L)] + 8\tilde{\tau}^N \\ &+ 2[2n^2 + n - 3 + \lambda] - 8n^2\|\mathrm{H}^\circ\|^2 + n^2[\|\mathrm{H}^*\|^2 + \|\mathrm{H}\|^2] \\ &- n[\mathrm{C} + \mathrm{C}^*] + 2\sum_{i,j}\|\sigma(e_i, e_j) + \sigma^*(e_i, e_j)\|^2.\end{aligned}$$

Since $\rho = \frac{2\tau}{n(n-1)}$, from the above equation we get

$$2\tau \leq \delta_c(n-1) + \delta_c^*(n-1) + 4\tilde{\rho}^N + \frac{4n^2+2n-6+2\lambda}{n(n-1)}$$
$$- \frac{8n}{n-1}\|\mathrm{H}^\circ\|^2 + \frac{n}{n-1}\{\|\mathrm{H}^*\|^2 + \|\mathrm{H}\|^2\}$$
$$+ \frac{1}{n-1}\{\mathrm{C}+\mathrm{C}^*\} + \frac{4}{n(n-1)}\sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}\sigma_{ij}^r\sigma_{ij}^{*r},$$

where $\tilde{\rho}^N$ is the normalized scalar curvature ρ with respect to the semi-symmetric metric connection over the submanifold N.

Corollary 4.2 *Let N be a statistical submanifold in a Sasakian statistical manifold such that it is tangent to the structure vector field ξ. Then the normalized Casorati curvatures $\delta_c(n-1)$ and $\delta_c^*(n-1)$ satisfy the following:*

$$2\rho \leq \delta_c(n-1) + \delta_c^*(n-1) + 4\tilde{\rho}^N + \frac{4n^2+2n-6+2\lambda}{n(n-1)}$$
$$+ \frac{1}{n-1}\{\mathrm{C}+\mathrm{C}^*\} + \frac{4}{n(n-1)}\sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n}\sigma_{ij}^r\sigma_{ij}^{*r}.$$

Remark 4.3 The equality holds in the corollary $(\mathrm{H} = \mathrm{H}^* = 0)$ if and only if

$$\sigma_{11}^r = \sigma_{22}^r = \sigma_{33}^r = \cdots \sigma_{n-1n-1}^r = \frac{1}{2}\sigma_{nn}^r = 0,$$

$$\sigma_{11}^{*r} = \sigma_{22}^{*r} = \sigma_{33}^{*r} = \cdots \sigma_{n-1n-1}^{*r} = \frac{1}{2}\sigma_{nn}^{*r} = 0$$

and

$$\sigma_{ij}^r = 0, \sigma_{ij}^{*r} = 0, \quad i \neq j.$$

This implies $\sigma = 0$, $\sigma^* = 0$, i.e., equality holds if and only if N is totally geodesic.

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Certain Classes of Warped Product Submanifolds of Sasakian Manifolds

Joydeb Roy, Laurian-Ioan Pişcoran, and Shyamal Kumar Hui

Abstract In Chen and Garay (Turk J Math 36:630–640, 2012) studied pointwise-slant submanifolds of almost Hermitian manifolds. They have obtained many fundamental results, in particular, a characterization of these submanifolds. Later, Park (Pointwise slant and pointwise semi slant submanifolds almost contact metric manifold, 2020) has extended the study for almost contact metric manifolds. In the present article, we have studied warped product submanifold of Sasakian manifolds $\bar{M}$. We have considered the warped product submanifold of the form $M = M_5 \times_f M_{\theta_3}$ where $M_5 = M_{\theta_1} \times M_{\theta_2}$ and M_{θ_1}, M_{θ_2} and M_{θ_3} are pointwise-slant submanifolds of $\bar{M}$. Here, we have obtained a characterization theorem of this class of warped product submanifold.

Keywords Sasakian manifold · Pointwise-slant submanifolds · Warped product submanifolds · Bi-warped product submanifolds

2010 Mathematics Subject Classification 53C15 · 53C40

1 Introduction

The warped product [5] between two Riemannian manifolds (N_1, g_1) and (N_2, g_2) is the Riemannian manifold $N_1 \times_f N_2 = (N_1 \times N_2, g)$, where

J. Roy · S. K. Hui (✉)
Department of Mathematics, The University of Burdwan, Burdwan 713104, West Bengal, India
e-mail: skhui@math.buruniv.ac.in

J. Roy
e-mail: joydeb.roy8@gmail.com

L.-I. Pişcoran
Department of Mathematics and Computer Science, North University Center of Baia Mare, Technical University of Cluj Napoca, Victoriei 76, 430122 Baia Mare, Romania
e-mail: plaurian@yahoo.com

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$$g = g_1 + f^2 g_2 \tag{1.1}$$

for $f : N_1 \to \mathbb{R}^+$ to be a smooth function.

A warped product manifold $N_1 \times_f N_2$ is said to be trivial if f is constant. For $M = N_1 \times_f N_2$, we have [5]

$$\nabla_U X = \nabla_X U = (X \ln f)U \tag{1.2}$$

for any $X \in \Gamma(TN_1)$ and $U \in \Gamma(TN_2)$.

The study of warped product submanifold theory was initiated in [8–10]. After that, many authors have studied the theory of warped product submanifolds of different ambient manifolds, see [17–19, 21]. Some important research for the study of warped product submanifolds of Kenmotsu manifolds were done in a series of works [1–3, 23, 24, 27, 28, 32–41]. Warped product submanifolds of Sasakian manifold have been studied by Hasegawa and Mihai in [15]. Multiply warped products (see [11, 13, 40]) are generalizations of warped product and Riemannian product manifolds, and bi-warped products are special classes of multiply warped products. Bi-warped product submanifolds of different ambient manifolds are studied in [33, 34, 36].

In the present paper, we have studied the warped product submanifolds of Sasakian manifold $\bar{M}$ of the form $M = M_5 \times_f M_{\theta_3}$ such that $\xi \in \Gamma(TM_{\theta_1})$, where $M_5 = M_{\theta_1} \times M_{\theta_2}$ and M_{θ_1}, M_{θ_2} are proper slant submanifolds of $\bar{M}$ and M_{θ_3} represents a proper pointwise-slant submanifold of $\bar{M}$. Next we have studied bi-warped product submanifolds of $\bar{M}$ of the form $M_{\theta_1} \times_{f_1} M_{\theta_2} \times_{f_2} M_{\theta_3}$, where M_{θ_1}, M_{θ_2}, M_{θ_3} are pointwise slant submanifolds of $\bar{M}$ of distinct slant functions θ_1, θ_2 and θ_3, respectively.

The paper is organized as follows: Sect. 2 deals with some preliminary useful results for construction of the paper; Sect. 3 is concerned with the study of a class of submanifold M of $\bar{M}$ such that $TM = \mathcal{D}_1^\theta \oplus \mathcal{D}_2^\theta \oplus \mathcal{D}_3^\theta \oplus \langle \xi \rangle$, where $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ are slant distributions and $\mathcal{D}^{\theta_3}$ is pointwise slant distribution; in Sect. 4 we have studied warped product submanifolds of the form $M = M_5 \times_f M_{\theta_3}$ of $\bar{M}$ where $M_5 = M_{\theta_1} \times M_{\theta_2}$ such that ξ is orthogonal to M_{θ_3} with a supporting example. In Sect. 5, a characterization theorem of the mentioned class has been obtained. Some applications in terms of generalization are also given as corollary.

2 Preliminaries

An odd-dimensional smooth manifold $\bar{M}^{2m+1}$ is said to be an almost contact metric manifold [4] if it admits a $(1, 1)$ tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on $\bar{M}^{2m+1}$.

An almost contact metric manifold $\bar{M}^{2m+1}(\phi, \xi, \eta, g)$ is said to be a Sasakian manifold if the following conditions hold [22]:

$$\bar{\nabla}_X \xi = -\phi X, \tag{2.4}$$

$$(\bar{\nabla}_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.5}$$

where $\bar{\nabla}$ denotes the Riemannian connection of g.

Let M be an n-dimensional submanifold of a Sasakian manifold $\bar{M}$. Throughout the paper, we assume that the submanifold M of $\bar{M}$ is tangent to the structure vector field ξ.

Let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.6}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.7}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h and A_V are second fundamental form and the shape operator (corresponding to the normal vector field V), respectively, for the immersion of M into $\bar{M}$. The second fundamental form h and the shape operator A_V are related by $g(h(X, Y), V) = g(A_V X, Y)$ for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where g is the Riemannian metric on $\bar{M}$ as well as on M.

The mean curvature H of M is given by $H = \frac{1}{n}\text{trace } h$. A submanifold of a Sasakian manifold $\bar{M}$ is said to be totally umbilical, if $h(X, Y) = g(X, Y)H$ for any $X, Y \in \Gamma(TM)$. If $h(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$, then M is totally geodesic and if $H = 0$ then M is minimal in $\bar{M}$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent bundle TM and $\{e_{n+1}, \ldots, e_{2m+1}\}$, an orthonormal basis of the normal bundle $T^{\perp}M$. We put

$$h^r_{ij} = g(h(e_i, e_j), e_r) \text{ and } \|h\|^2 = g(h(e_i, e_j), h(e_i, e_j)), \tag{2.8}$$

for $r \in \{n+1, \ldots, 2m+1\}$, $i, j = 1, 2, \ldots, n$.

For a differentiable function f on M, the gradient $\boldsymbol{\nabla} f$ is defined by

$$g(\boldsymbol{\nabla} f, X) = Xf, \tag{2.9}$$

for any $X \in \Gamma(TM)$. As a consequence, we get

$$\|\nabla f\|^2 = \sum_{i=1}^{n} (e_i(f))^2. \tag{2.10}$$

For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we can write

$$\text{(a) } \phi X = PX + QX, \quad \text{(b) } \phi V = bV + cV \tag{2.11}$$

where $PX,\ bV$ are the tangential components and $QX,\ cV$ are the normal components of ϕX, respectively.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be slant if for each nonzero vector $X \in T_pM$, the angle θ between ϕX and T_pM is constant, i.e. it does not depend on the choice of $p \in M$.

A submanifold M of an almost contact metric manifold $\bar{M}$ is said to be pointwise slant [14] if for any nonzero vector $X \in T_pM$ at $p \in M$, such that X is not proportional to ξ_p, the angle $\theta(X)$ between ϕX and $T_p^*M = T_pM - \{0\}$ is independent of the choice of nonzero $X \in T_p^*M$.

For pointwise-slant submanifold, θ is a function on M, which is known as a slant function of M. Invariant and anti-invariant submanifolds are particular cases of pointwise-slant submanifolds with slant function $\theta = 0$ and $\frac{\pi}{2}$, respectively. Also a pointwise-slant submanifold M will be slant if and only if θ is constant on M. Thus a pointwise-slant submanifold is proper if neither $\theta = 0,\ \frac{\pi}{2}$ nor constant. It may be noted that [26] M is pointwise slant if and only if, $\exists$ a constant $\lambda \in [0, 1]$ such that

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.12}$$

Furthermore, $\lambda = \cos^2\theta$ for slant function θ. If M is a pointwise-slant submanifold of an almost contact metric manifold $\bar{M}$, the following relations hold: If M is a pointwise-slant submanifold of $\bar{M}$, then, from [35], we know that

$$bQX = \sin^2\theta\{-X + \eta(X)\xi\}, \quad cQX = -QPX. \tag{2.13}$$

Let $M_1,\ M_2,\ M_3$ be Riemannian manifolds and let $M = M_1 \times_{f_1} M_2 \times_{f_2} M_3$ be the product manifold of $M_1,\ M_2,\ M_3$ such that $f_1,\ f_2 : M_1 \to \mathbb{R}^+$ are real-valued smooth functions. For each i, denote by $\pi_i : M \to M_i$ the canonical projection of M onto M_i, $i = 1, 2, 3$. Then the metric on M, called a bi-warped metric, is given by

$$g(X, Y) = g(\pi_{1_*}X, \pi_{2_*}Y) + (f_1 \circ \pi_1)^2 g(\pi_{2_*}X, \pi_{2_*}Y) + (f_2 \circ \pi_1)^2 g(\pi_{3_*}X, \pi_{3_*}Y)$$

for any $X,\ Y \in \Gamma(TM)$ and $*$ denotes the symbol for tangent maps. The manifold M endowed with this product metric is called a bi-warped product manifold. Here $f_1,\ f_2$ are non-constant functions, called warping functions on M. Clearly, if both $f_1,\ f_2$ are constant on M, then M is simply a Riemannian product manifold and if any one of the functions is constant, then M is a single warped product manifold. If neither f_1 nor f_2 is constant, then M is a proper bi-warped product manifold.

Let $M = M_1 \times_{f_1} M_2 \times_{f_2} M_3$ be a warped product submanifold of $\bar{M}$. Then, one obtains [36]

$$\nabla_X Z = \sum_{i=1}^{2} (X(\ln f_i)) Z^i \tag{2.14}$$

for any $X \in \mathcal{D}^1$, the tangent space of M_1 and $Z \in \Gamma(TM)$, where $M = M_1 \times_{f_1} M_2 \times \times_{f_2} M_3$ and Z^i is M_i components of Z for each $i = 2, 3$ and ∇ is the Levi-Civita connection on M.

3 Submanifolds of $\bar{M}$

In this section, we consider submanifold M of $\bar{M}$ such that

$$TM = \mathcal{D}_1^\theta \oplus \mathcal{D}_2^\theta \oplus \mathcal{D}_3^\theta \oplus \langle \xi \rangle,$$

$$T^\perp M = Q\mathcal{D}_1^\theta \oplus Q\mathcal{D}_2^\theta \oplus Q\mathcal{D}_3^\theta \oplus \nu,$$

where ν is a ϕ-invariant normal subbundle of $T^\perp M$.

If M is such submanifold of $\bar{M}$ then for any $X \in \Gamma(TM)$, we have

$$X = T_1 X + T_2 X + T_3 X, \tag{3.1}$$

where T_1, T_2 and T_3 are the projections from TM onto $\mathcal{D}_1^\theta$, $\mathcal{D}_2^\theta$ and $\mathcal{D}_3^\theta$, respectively.

If we put $P_1 = T_1 \circ P$, $P_2 = T_2 \circ P$ and $P_3 = T_3 \circ P$ then from (3.1), we get

$$\phi X = P_1 X + P_2 X + P_3 X + QX, \tag{3.2}$$

for $X \in \Gamma(TM)$.

From (2.12) and (3.2), we get

$$P_i^2 = \cos^2 \theta_i (-I + \eta \otimes \xi), \quad \text{for } \ i = 1, 2, 3. \tag{3.3}$$

Here we obtain some useful results for further study.

Lemma 3.1 *Let M be a submanifold of $\bar{M}$ where $TM = \mathcal{D}_1^\theta \oplus \mathcal{D}_2^\theta \oplus \mathcal{D}_3^\theta$ such that $\xi \in \Gamma(\mathcal{D}_1^\theta \oplus \mathcal{D}_2^\theta)$, then the following relations hold:*

$$\begin{aligned}(\cos^2 \theta_3 - \cos^2 \theta_1) g(\nabla_{X_1} Y_1, X_3) = {} & g(A_{QP_1Y_1} X_3 - A_{QY_1} P_3 X_3, X_1) \\ & + g(A_{QP_3X_3} Y_1 - A_{QX_3} P_1 Y_1, X_1) \\ & + \sin^2 \theta_1 \eta(Y_1) g(P_3 X_3, X_1),\end{aligned} \tag{3.4}$$

$$(\cos^2\theta_3 - \cos^2\theta_2)g(\nabla_{X_2}Y_2, X_3) = g(A_{QP_2Y_2}X_3 - A_{QY_2}P_3X_3, X_2) \\ +g(A_{QP_3X_3}Y_2 - A_{QX_3}P_2Y_2, X_2) \\ + \sin^2\theta_2\eta(Y_2)g(P_3X_3, X_2), \tag{3.5}$$

$$(\cos^2\theta_3 - \cos^2\theta_2)g(\nabla_{X_1}X_2, X_3) = g(A_{QP_2X_2}X_3 - A_{QX_2}P_3X_3, X_1) \\ +g(A_{QP_3X_3}X_2 - A_{QX_3}P_2X_2, X_1) \\ + \sin^2\theta_2\eta(X_2)g(P_3X_3, X_1), \tag{3.6}$$

$$(\cos^2\theta_3 - \cos^2\theta_1)g(\nabla_{X_2}X_1, X_3) = g(A_{QP_1X_1}X_3 - A_{QX_1}P_3X_3, X_2) \\ +g(A_{QP_3X_3}X_1 - A_{QX_3}P_1X_1, X_2) \\ + \sin^2\theta_1\eta(Y_1)g(P_3X_3, X_2) \tag{3.7}$$

for any $X_1, Y_1 \in \Gamma(\mathcal{D}_1^\theta)$, $X_2, Y_2 \in \Gamma(\mathcal{D}_2^\theta)$ *and* $X_3 \in \Gamma(\mathcal{D}_3^\theta)$.

Proof For any $X_1, Y_1 \in \Gamma(\mathcal{D}_1^\theta)$ and $X_3 \in \Gamma(\mathcal{D}_3^\theta)$, we have

$$g(\nabla_{X_1}Y_1, X_3) = g(\bar{\nabla}_{X_1}\phi Y_1, \phi X_3) - g((\bar{\nabla}_{X_1}\phi)Y_1, \phi X_3).$$

Then by virtue of (2.5), (2.6), (2.11)–(2.13), (3.2) and (3.3), we get from the above equation

$$(\cos^2\theta_3 - \cos^2\theta_1)g(\nabla_{X_1}Y_1, X_3) = g(h(X_1, X_3), QP_1X_1) - g(h(X_1, P_3X_3), QY_1) \\ +g(h(X_1, Y_1), QP_3X_3) - g(h(X_1, P_1Y_1), QX_3) \\ + \sin^2\theta_1\eta(Y_1)g(X_1, P_3Y_3)$$

from which using the relation between the second fundamental form and shape operator, we can get (3.4.)

Similarly, we obtain (3.5)–(3.7). □

Lemma 3.2 *Let* M *be a submanifold of* $\bar{M}$ *where* $TM = \mathcal{D}_1^\theta \oplus \mathcal{D}_2^\theta \oplus \mathcal{D}_3^\theta$ *such that* $\xi \in \Gamma(\mathcal{D}_1^\theta \oplus \mathcal{D}_2^\theta)$, *then the following relations hold:*

$$(\sin^2\theta_3 - \sin^2\theta_1)g(\nabla_{X_3}Y_3, X_1) = g(A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1, X_3) \\ +g(A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3, X_3) + \sin^2\theta_1\eta(X_1)g(X_3, P_3Y_3), \tag{3.8}$$

$$(\sin^2\theta_3 - \sin^2\theta_2)g(\nabla_{X_3}Y_3, X_2) = g(A_{QP_3Y_3}X_2 - A_{QY_3}P_2X_2, X_3) \\ +g(A_{QP_2X_2}Y_3 - A_{QX_2}P_3Y_3, X_3) + \sin^2\theta_2\eta(X_2)g(X_3, P_3Y_3) \tag{3.9}$$

for any $X_1 \in \Gamma(\mathcal{D}_1^\theta)$, $X_2 \in \Gamma(\mathcal{D}_2^\theta)$ *and* $X_3, Y_3 \in \Gamma(\mathcal{D}_3^\theta)$.

Proof For any $X_1 \in \Gamma(\mathcal{D}_1^\theta)$ and $X_3, Y_3 \in \Gamma(\mathcal{D}_3^\theta)$, we have

$$g(\nabla_{X_3}Y_3, X_1) = g(\phi\bar{\nabla}_{X_3}Y_3, \phi X_1) + \eta(\bar{\nabla}_{X_3}Y_3)\eta(X_1).$$

Now, using (2.5), (2.12), (2.13), (3.2) and (3.3) we compute

$$\begin{aligned} g(\nabla_{X_3}Y_3, X_1) = \cos^2\theta_3 g(\bar{\nabla}_{X_3}Y_3, X_1) - g(\bar{\nabla}_{X_3}QP_3Y_3, X_1) \\ + g(\bar{\nabla}_{X_3}QY_3, P_1X_1) - \sin^2\theta_1 g(\bar{\nabla}_{X_3}X_1, Y_3) + g(\bar{\nabla}_{X_3}QX_1, P_3Y_3) \\ + g(\bar{\nabla}_{X_3}QP_1X_1, Y_3) + \sin^2\theta_1\eta(X_1)g(X_3, P_3Y_3). \end{aligned}$$

In view of (2.6) and the relation $g(h(X, Y), V) = g(A_V X, Y)$, the above equation yields (3.8). Following the same computational procedure, for any $X_2 \in \Gamma(\mathcal{D}_2^\theta)$ and $X_3, Y_3 \in \Gamma(\mathcal{D}_3^\theta)$, we can establish relation (3.9). And hence, the lemma is proved. □

4 Warped Product Submanifolds of Sasakian Manifolds

In this section, we study warped product submanifolds of the form $M = M_5 \times_f M_{\theta_3}$ of $\bar{M}$ where $M_5 = M_{\theta_1} \times M_{\theta_2}$ such that ξ is orthogonal to M_{θ_3}. Here M_{θ_1}, M_{θ_2} represent proper slant submanifolds of $\bar{M}$ with slant angles θ_1, θ_2, respectively, and M_{θ_3} represents pointwise-slant submanifolds of $\bar{M}$ with slant function θ_3.

Now we construct two examples of a nontrivial warped product pointwise-slant submanifold of the form $M = M_5 \times_f M_{\theta_3}$ of $\bar{M}$.

We obtain the following useful results.

Lemma 4.1 *Let $M = M_5 \times_f M_{\theta_3}$ be a warped product submanifold of $\bar{M}$, where $M_5 = M_{\theta_1} \times M_{\theta_2}$ and $M_{\theta_1}, M_{\theta_2}$ are proper slant submanifolds and M_{θ_3} is pointwise-slant submanifold of $\bar{M}$, then*

$$\xi lnf = 0, \tag{4.1}$$

$$g(h(X_1, Y_1), QX_3) = g(h(X_1, X_3), QY_1), \tag{4.2}$$

$$g(h(X_2, Y_2), QX_3) = g(h(X_2, X_3), QY_2), \tag{4.3}$$

$$g(h(X_1, X_3), QX_2) = g(h(X_1, X_2), QX_3) = g(h(X_2, X_3), QX_1) \tag{4.4}$$

for $X_1, Y_1 \in M_{\theta_1}, X_2, Y_2 \in M_{\theta_2}$ and $X_3, Y_3 \in M_{\theta_3}$.

Proof The proof of (4.1) is already done in [15].
Now, for $X_1, Y_1 \in M_{\theta_1}$ and $X_3 \in M_{\theta_3}$, we have

$$g(h(X_1, X_3), QY_1) = g(\bar{\nabla}_{X1}X_3, \phi Y_1) - g(\bar{\nabla}_{X_1}X_3, P_1Y_1).$$

Then using (2.5), (2.6), (2.14), (3.2) and orthogonality of the distributions, we get (4.2). Proceeding the same way, for any $X_2, Y_2 \in M_{\theta_2}$ and $X_3 \in M_{\theta_3}$, we get (4.3).

Again, for any $X_1 \in M_{\theta_1}$, $X_2 \in M_{\theta_2}$ and $X_3 \in M_{\theta_3}$, we have

$$g(h(X_1, X_3), QX_2) = g(\bar{\nabla}_{X_3} X_1, \phi X_2) - g(\bar{\nabla}_{X_3} X_1, P_2 X_2).$$

Using (1.2), (2.5), (2.6), (3.2) and orthogonality of the distributions we get

$$g(h(X_1, X_3), QX_2) = g(h(X_2, X_3), QX_1). \tag{4.5}$$

And also we know

$$g(h(X_1, X_2), QX_3) = g(\bar{\nabla}_{X_1} X_2, \phi X_3) - g(\bar{\nabla}_{X_1} X_2, P_3 X_3)$$

from which using (1.2), (2.5), (2.6), (3.2) and orthogonality of the distributions, we compute

$$g(h(X_1, X_2), QX_3) = g(h(X_1, X_2), QX_3). \tag{4.6}$$

Combining (4.5) and (4.6) we get (4.4). This completes the proof. □

Lemma 4.2 *Let $M = M_5 \times_f M_{\theta_3}$ be a warped product submanifold of $\bar{M}$, where $M_5 = M_{\theta_1} \times M_{\theta_2}$ and M_{θ_1}, M_{θ_2} are proper slant submanifolds and M_{θ_3} is pointwise-slant submanifold of $\bar{M}$, then*

$$\begin{aligned} &g(h(X_3, Y_3), QX_1) - g(h(X_3, X_1), QY_3) \\ &= (X_1 lnf) g(P_3 Y_3, X_3) + [(P_1 X_1 lnf) + \eta(X_1)] g(X_3, Y_3), \end{aligned} \tag{4.7}$$

$$\begin{aligned} &g(h(X_3, Y_3), QX_2) - g(h(X_3, X_2), QY_3) \\ &= (X_1 lnf) g(P_3 Y_3, X_3) + [(P_2 X_2 lnf) + \eta(X_2)] g(X_3, Y_3), \end{aligned} \tag{4.8}$$

$$\begin{aligned} &g(h(X_3, Y_3), QP_1 X_1) - g(h(P_3 Y_3, X_3), QX_1) \\ &+ g(h(X_1, X_3), QP_3 Y_3) - g(h(P_1 X_1, X_3), QY_3) \\ &= (\cos^2\theta_3 - \cos^2\theta_1)(X_1 lnf) g(X_3, Y_3) - \sin^2\theta_1 \eta(X_1) g(X_3, P_3 Y_3), \end{aligned} \tag{4.9}$$

$$\begin{aligned} &g(h(X_3, Y_3), QP_2 X_2) - g(h(P_3 Y_3, X_3), QX_2) \\ &+ g(h(X_2, X_3), QP_3 Y_3) - g(h(P_2 X_2, X_3), QY_3) \\ &= (\cos^2\theta_3 - \cos^2\theta_2)(X_2 lnf) g(X_3, Y_3) - \sin^2\theta_2 \eta(X_2) g(X_3, P_3 Y_3), \end{aligned} \tag{4.10}$$

for $X_1 \in M_{\theta_1}$, $X_2 \in M_{\theta_2}$ and $X_3, Y_3 \in M_{\theta_3}$.

Proof In view of (2.6), for any $X_1 \in M_{\theta_1}$ and $X_3, Y_3 \in M_{\theta_3}$, we have

$$g(h(X_3, X_1), QY_3) = g(\bar{\nabla}_{X_3} X_1, \phi Y_3) - g(\bar{\nabla}_{X_3} X_1, P_3 Y_3).$$

By virtue of (1.2), (2.5), (2.6), (3.2) and orthogonality of the distributions, we compute

$$g(h(X_3, X_1), QY_3) = g(h(X_3, Y_3), QX_1)$$
$$\eta(X_1)g(X_3, Y_3) + (P_1X_1 lnf)g(X_3, Y_3) + (X_1 lnf)g(P_3Y_3, X_3).$$

From the above, Eq. (4.7) follows. Following the same procedure, for any $X_2 \in M_{\theta_2}$ and $X_3, Y_3 \in M_{\theta_3}$, (4.8) is obtained.

In (4.7), replacing Y_3 by P_3Y_3 and using (3.3), we get

$$g(h(X_3, P_3Y_3), QX_1) - g(h(X_1, X_3), QP_3Y_3) = -(X_1 lnf)\cos^2\theta_3 g(X_3, Y_3) + \{(P_1X_1 lnf) + \eta(X_1)\}g(X_3, P_3Y_3). \tag{4.11}$$

Again, we have

$$g(h(P_1X_1, X_3), QY_3) = g(\bar{\nabla}_{X_3}P_1X_1, \phi Y_3) - g(\bar{\nabla}_{X_3}P_1X_1, P_3Y_3).$$

By virtue of (1.2), (2.11), (3.2) and (3.3), we get

$$g(h(P_1X_1, X_3), QY_3) - g(h(X_3, X_3), QP_1X_1) = \cos^2\theta_1(X_1 lnf)g(X_3, Y_3) - \cos^2\theta_1\eta(X_1)g(X_3, P_3Y_3) - (P_1X_1 lnf)g(X_3, P_3Y_3). \tag{4.12}$$

Adding (4.11) and (4.12) and by simple computation, (4.9) is obtained and following the same technique for any $X_2 \in M_{\theta_2}$ and $X_3, Y_3 \in M_{\theta_3}$, we get (4.10). □

5 Characterization of Warped Product Pointwise-Slant Submanifolds

In this section, we have characterized warped product pointwise-slant submanifold M of Sasakian manifold $\bar{M}$.

Theorem 5.1 *Let M be a submanifold of a Sasakian manifold $\bar{M}$ such that $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3}$ with ξ orthogonal to $\mathcal{D}^{\theta_3}$, then M is locally a warped product submanifold of the form $M = M_5 \times_f M_{\theta_3}$ where $M_5 = M_{\theta_1} \times M_{\theta_2}$ if and only if*

$$A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1 = (\cos^2\theta_3 - \cos^2\theta_1)X_1\mu Y_3 - \sin^2\theta_1\eta(X_1)P_3Y_3, \tag{5.1}$$

$$A_{QP_2X_2}Y_3 - A_{QX_2}P_3Y_3 + A_{QP_3Y_3}X_2 - A_{QY_3}P_2X_2 = (\cos^2\theta_3 - \cos^2\theta_2)X_2\mu Y_3 - \sin^2\theta_2\eta(X_2)P_3Y_3, \tag{5.2}$$

$$\xi\mu = 0 \tag{5.3}$$

for every $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, $X_2 \in \Gamma(\mathcal{D}^{\theta_2})$, $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$ *and for some smooth function* μ *on* M *satisfying where* $(Y_3\mu) = 0$ *for any* $Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$.

Proof Let $M = M_4 \times_f M_{\theta_3}$ be a proper warped product submanifold of a Kenmotsu manifold $\bar{M}$ such that $M_4 = M_{\theta_1} \times M_{\theta_2}$. Denote the tangent space of M_{θ_1}, M_{θ_2} and M_{θ_3} by $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\mathcal{D}^{\theta_3}$, respectively. Then from (4.2), we get

$$g(A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1, X_1) = 0. \tag{5.4}$$

Similarly, from (4.4) we get

$$g(A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1, X_2) = 0. \tag{5.5}$$

So, from (5.4) and (5.5) we conclude that

$$A_{QP_1X_1}Y_3 - A_{QX_1}P_3Y_3 + A_{QP_3Y_3}X_1 - A_{QY_3}P_1X_1 \in \mathcal{D}^{\theta_3}. \tag{5.6}$$

Hence, from (4.9) and (5.6), relation (5.1) follows.

In similar way, in view of (4.3), (4.4) and (4.10) we get (5.2). The relation (5.3) is directly obtained from (4.1).

Conversely, let M be a submanifold of $\bar{M}$ such that $TM = \mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \mathcal{D}^{\theta_3}$ with ξ orthogonal to $\mathcal{D}^{\theta_3}$ and the conditions (5.1)–(5.3) satisfied. Then from (3.4) and (3.7), in view of (5.1), respectively we get

$$g(\nabla_{X_1}Y_1, X_3) = 0, \quad \text{and} \quad g(\nabla_{X_2}X_1, X_3) = 0, \tag{5.7}$$

and also from (3.5), (3.6) in view of (5.2), respectively we get

$$g(\nabla_{X_2}Y_2, X_3) = 0, \quad \text{and} \quad g(\nabla_{X_1}X_2, X_3) = 0. \tag{5.8}$$

Thus from (5.7), (5.8) and the fact that $\nabla_{X_3}\xi = 0$ we conclude that $g(\nabla_E F, X_3) = 0$ for every $E, F \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$. Hence the leaves of $\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle$ are totally geodesic in M.

Now, by virtue of (3.8), (5.1) yields

$$g([X_3, Y_3], X_1) = 0 \tag{5.9}$$

and by virtue of (3.9), (5.2) yields

$$g([X_3, Y_3], X_2) = 0. \tag{5.10}$$

Hence, from (5.9), (5.10) and the fact that $h(A, \xi) = 0, \ \forall A \in TM$, we conclude that

$$g([X_3, Y_3], E) = 0 \qquad \forall X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3}) \tag{5.11}$$

and $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$, consequently $\mathcal{D}^{\theta_3}$ is integrable.

Let h^{θ_3} be the second fundamental form of M_{θ_3} in $\bar{M}$. Then for any $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and $X_1 \in \Gamma(\mathcal{D}^{\theta_1})$, we get from (3.8) that

$$g(h^{\theta_3}(X_3, Y_3), X_1) = -(X_1\mu)g(X_3, Y_3). \tag{5.12}$$

Similarly, for $X_2 \in \Gamma(\mathcal{D}^{\theta_2})$, from (3.9) we get

$$g(h^{\theta_3}(X_3, Y_3), X_2) = -(X_2\mu)g(X_3, Y_3). \tag{5.13}$$

Again, for any $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$, in view of (5.3) we have

$$g(h^{\theta_3}(X_3, Y_3), \xi) = -(\xi\mu)g(X_3, Y_3). \tag{5.14}$$

Hence, from (5.12)–(5.14) we conclude that

$$g(h^{\theta}(X_3, Y_3), E) = -g(\nabla\mu, E)g(X_3, Y_3)$$

for every $X_3, Y_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus, \langle\xi\rangle)$. Consequently, M_{θ_3} is totally umbilical in $\bar{M}$ with mean curvature vector $H^{\theta_3} = -\nabla\mu$.

Finally, we will show that H^{θ_3} is parallel with respect to the normal connection D^N of M_{θ_3} in M. We take $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_3} \oplus \langle\xi\rangle)$ and $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$, then we have

$$g(D^N_{X_3}\nabla\mu, E) = g(\nabla_{X_3}\nabla^{\theta_1}\mu, X_1) + g(\nabla_{X_3}\nabla^{\theta_2}\mu, X_2) + g(\nabla_{X_3}\nabla^{\xi}\mu, \xi),$$

where ∇^{θ_1}, ∇^{θ_2} and ∇^{ξ} are the gradient components of μ on M along $\mathcal{D}^{\theta_1}$, $\mathcal{D}^{\theta_2}$ and $\langle\xi\rangle$, respectively. Then by the property of Riemannian metric, the above equation reduces to

$$\begin{aligned}
g(D^N_U\nabla\mu, E) &= X_3 g(\nabla^{\theta_1}\mu, X_1) - g(\nabla^{\theta_1}\mu, \nabla_{X_3}X_1) + X_3 g(\nabla^{\theta_2}\mu, X_2)\\
&\quad - g(\nabla^{\theta_2}\mu, \nabla_{X_3}X_2) + X_3 g(\nabla^{\xi}\mu, \xi) - g(\nabla^{\xi}\mu, \nabla_{X_3}\xi)\\
&= X_3(X_1\mu) - g(\nabla^{\theta_1}\mu, [X_3, X_1]) - g(\nabla^{\theta_1}\mu, \nabla_{X_1}X_3)\\
&\quad + X_3(X_2\mu) - g(\nabla^{\theta_2}\mu, [X_3, X_2]) - g(\nabla^{\theta_2}\mu, \nabla_{X_2}X_3)\\
&\quad + X_3(\xi\mu) - g(\nabla^{\xi}\mu, [X_3, \xi]) - g(\nabla^{\xi}\mu, \nabla_{\xi}X_3)\\
&= X_1(X_3\mu) + g(\nabla_{X_1}\nabla^{\theta_1}\mu, X_3) + X_2(X_3\mu) + X_2(X_3\mu)\\
&\quad + g(\nabla_{X_2}\nabla^{\theta_2}\mu, X_3) + \xi(X_3\mu) - g(\nabla_{\xi}\nabla^{\xi}\mu, X_3)\\
&= 0,
\end{aligned}$$

since $(X_3\mu) = 0$ for every $X_3 \in \Gamma(\mathcal{D}^{\theta_3})$ and $\nabla_{X_1}\nabla^{\theta_1}\mu + \nabla_{X_2}\nabla^{\theta_2}\mu + \nabla_{\xi}\nabla^{\xi}\mu = \nabla_E\nabla\mu$ is orthogonal to $\mathcal{D}^{\theta_3}$ for any $E \in \Gamma(\mathcal{D}^{\theta_1} \oplus \mathcal{D}^{\theta_2} \oplus \langle\xi\rangle)$, and $\nabla\mu$ is the gradient along M_5, and M_5 is totally geodesic in $\bar{M}$. Hence, the mean curvature vector H^{θ_3} of M_{θ_3} is parallel. Thus, M_{θ_3} is an extrinsic sphere in M. Hence by Hiepko's

Theorem (see [16]), M is locally a warped product submanifold. Thus the proof is complete. □

As an application of Theorem 5.1, let us consider $m_2 = 0$, $\theta_1 = 0$ and $\theta_3 = \theta =$ constant, then we see that Theorem 5.1 generalizes Theorem 5 of [26].

Corollary 5.1 (Theorem 5 of [26]) *Let M be a pointwise semi-slant submanifold of Sasakian manifold $\bar{M}$. Then M is locally a nontrivial warped product submanifold of the form $M_T \times_f M_\theta$, where M_T is an invariant submanifold and M_θ is proper pointwise-slant submanifold of $\bar{M}$ if and only if*

$$A_{QY_3}\phi X_1 - A_{QP_3Y_3}X_1 = \sin^2\theta X_1\mu Y_3$$

where $X_1 \in \Gamma(\mathcal{D}^T)$ and $Y_3 \in \Gamma(\mathcal{D}^\theta)$.

Again considering $\theta_1 = \frac{\pi}{2}$ and $m_2 = 0$, $\theta_3 = \theta =$ constant, then we can get Theorem 4.7 of [38] from our characterization Theorem 5.1.

Corollary 5.2 (Theorem 4.7 of [38]) *Let M be a pointwise pseudo-slant submanifold of a Sasakian manifold $\bar{M}$. Then M is locally warped product submanifold of the form $M_\perp \times_f M_\theta$, where $M_\perp$ is an anti-invariant submanifold and M_θ is proper pointwise-slant submanifold of $\bar{M}$ if and only if*

$$A_{QX_1}P_3Y_3 - A_{QP_3Y_3}X_1 = \eta(X_1)P_3Y_3 - \cos^2\theta X_1\mu Y_3$$

where $X_1 \in \Gamma(\mathcal{D}^T)$ and $Y_3 \in \Gamma(\mathcal{D}^\theta)$.

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Hypersurfaces of a Metallic Riemannian Manifold

Mobin Ahmad, Jae-Bok Jun, and Mohammad Aamir Qayyoom

Abstract In the present paper, we study hypersurfaces of a metallic Riemannian manifold. We find some properties of induced structure on hypersurfaces by metallic Riemannian structure. The totally geodesic and totally umbilical hypersurfaces in metallic Riemannian manifolds are analyzed and an example of hypersurfaces in a metallic Riemannian manifold is constructed.

Keywords Metallic structure · Riemannian manifold · Invariant hypersurface · Non-invariant hypersurface · Totally geodesic · Umbilical hypersurface · Normal induced structure · Killing vector fields

1 Introduction

The theory of submanifolds of Riemannian manifolds is a very interesting topic in differential geometry. It is an active and vast research field playing an important role in the development of modern differential geometry. Investigating the submanifold theory on manifolds endowed with various geometric structures provides a fruitful study field. Many geometers studied hypersurfaces in different spaces including [2, 4–6]. Recently, Riemannian manifold with metallic structure provides many geometric results to characterized submanifold of such ambient manifolds.

M. Ahmad (✉) · M. A. Qayyoom
Faculty of Science, Department of Mathematics & Statistics, Integral University, Lucknow 226026, India
e-mail: mobinahmad68@gmail.com

M. A. Qayyoom
e-mail: aamir.qayyoom@gmail.com

J.-B. Jun
Department of Mathematics, College of Natural Science, Kookmin University, Seoul 136702, Korea
e-mail: jbjun@kookmin.edu.kr

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The notion of metallic structure in Riemannian manifold was introduced by Hretcanu and Crasmareanu in [18]. A metallic structure is a polynomial structure as defined by Goldberg and Yano in [14] with structural polynomial $J^2 = pJ + qI$, where p and q are positive integers. Metallic structure on Riemannian manifolds provides important geometrical results on submanifolds. Invariant, anti-invariant, slant and semi-slant submanifolds of metallic Riemannian manifolds are studied in [7, 15, 16]. Light like hypersurfaces of metallic semi-Riemannian manifold were studied by Acet in [1].

One of the most important subclasses of metallic Riemannian manifolds is provided by golden Riemannian manifold. Ahmad and Qayyoom [3] studied submanifolds immersed in golden Riemannian manifold. Many authors studied golden Riemannian manifolds and their submanifolds in recent years (see [9–13, 17]). Motivated by the studies on submanifolds of metallic Riemannian manifolds, in this paper, we study hypersurfaces of metallic Riemannian manifolds. The outline of this paper is as follows: In Sect. 2, we recall the notion of metallic structure on a Riemannian manifold. In Sect. 3, we focus on the geometry of hypersurfaces endowed with structures induced by metallic Riemannian structures. In the last section, we focus on properties of induced structures on hypersurfaces in metallic Riemannian manifolds with a special view toward totally geodesics, minimal and totally umbilical hypersurfaces. An example of metallic Riemannian structure is constructed on the Euclidean space and its hypersphere is analyzed with the tools of the previous section.

2 Metallic Riemannian Manifolds

In this section, a class of polynomial structures, namely metallic structures, is introduced in Riemannian manifolds.

Definition 2.1 A polynomial structure on a manifold M is called a metallic structure if it is determined by an (1,1) tensor field J which satisfies the equation

$$J^2 = pJ + qI,$$

where p, q are positive integers and I is the identity operator on the Lie algebra $\chi(M)$ of the vector fields on M. Since the Riemannian geometry is the most used framework of the differential geometry, let us add a metric to our study. We say that a Riemannian metric g is J-compatible if

$$g(JX, Y) = g(X, JY)$$

for every $X, Y \epsilon \chi(M)$, which means that J is a self-adjoint operator with respect to g. This condition is equivalent to our framework with

$$g(JX, JY) = p.g(X, JY) + q.g(X, Y).$$

Definition 2.2 A Riemannian manifold (M,g) endowed with a metallic structure J so that the Riemannian metric g is $J-compatible$ is named a metallic Riemannian manifold and (g, J) is called a metallic Riemannian structure on M.

3 Hypersurfaces Immersed in a Metallic Riemannian Manifold

Let M be an n-dimensional hypersurface isometrically immersed in an $(n+1)$-dimensional metallic Riemannian manifold $(\overline{M}, g, J)$ with $n \epsilon N$. We denote by $T_x M$ the tangent space of M at a point $x \epsilon M$ and $\underline{T_x^{\perp}} M$ the normal space of x in M. Let i be the differential of immersion $i : M \to \overline{M}$. The induced Riemannian metric g on M is given by $g(X, Y) = g(iX, iY)$ for every $X, Y \epsilon \chi(M)$. We consider a local orthonormal basis N of the normal space $T_x^{\perp} M$. For every $X \epsilon T_x M$, the vector fields $J(iX)$ and $J(N)$ can be decomposed into tangential and normal components as follows:

$$J(iX) = i(PX) + u(X)N, \tag{3.1}$$

$$J(N) = i(\xi) + aN. \tag{3.2}$$

We denote the covariant differential in $\overline{M}$ by $\overline{\nabla}$ and covariant differential in M determined by the induced metric g on M by ∇. We denote by A the Weingarten operator on $T(M)$ with respect to the local unit normal vector field N of M in $\overline{M}$. The Gauss and Weingarten formulae are given by [8]

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\overline{\nabla}_X N = -AX,$$

where $h(X, Y) = g(AX, Y)$ is second fundamental form in $T^{\perp} M$.

Proposition 3.1 ([18]) *The structure* $\Sigma = (P, g, u, \xi, a)$ *verifies the equalities:*

$$P^2(X) = pP(X) + qX - u(X)\xi, \tag{3.3}$$

$$u(PX) = (p-a)u(X), \tag{3.4}$$

$$u(\xi) = q + pa - a^2, \tag{3.5}$$

$$P(\xi) = (p-a)\xi, \tag{3.6}$$

$$u(X) = g(X, \xi), \tag{3.7}$$

$$g(PX, Y) = g(X, PY), \tag{3.8}$$

$$g(PX, PY) = g(PX, Y) + g(X, Y) - u(X)u(Y) \tag{3.9}$$

for every $X, Y \epsilon \chi(M)$.

Proposition 3.2 ([18]) *If* M *is a hypersurface in a metallic Riemannian manifold* $(\overline{M}, g, J)$ *and* J *is parallel with respect to the Levi Civita connection* $\overline{\nabla}$ *on* $\overline{M}$ $(\overline{\nabla} J = 0)$, *then the elements of the structure* $\sum = (P, g, u, \xi, a)$ *have the following properties:*

$$(\nabla_X P)(Y) = g(AX, Y)\xi + u(Y)AX, \tag{3.10}$$

$$(\nabla_X u)(Y) = -g(AX, PY) + ag(AX, Y), \tag{3.11}$$

$$\nabla_X \xi = -P(AX) + aAX, \tag{3.12}$$

$$X(a) = -2u(AX) = -2g(AX, \xi) = -2g(X, A\xi) \tag{3.13}$$

for every $X, Y \epsilon T_x M$.

Theorem 3.1 ([18]) *If* $\sum = (P, g, \xi, u, a)$ *is the induced structure on an umbilical hypersurface* M *in a metallic Riemannian manifold* $(\overline{M}, g, J)$ *with* $\nabla_X J = 0$, *we have for any* $X, Y \epsilon \chi(M)$

$$(\nabla_X P)(Y) = \lambda[g(X, Y)\xi + g(Y, \xi)X], \tag{3.14}$$

$$(\nabla_X u)(Y) = \lambda[ag(X, Y) - g(X, PY)], \tag{3.15}$$

$$(\nabla_X u)\xi = \lambda(aX - P(X)), \tag{3.16}$$

$$\nabla_\xi \xi = \lambda(2a - 1)\xi, \tag{3.17}$$

$$X(a) = -2\lambda g(X, \xi) \tag{3.18}$$

for any $X, Y \epsilon \chi(M)$.

Corollary 3.1 *Let* $\overline{M}$ *be a totally umbilical hypersurface in a metallic Riemannian manifold* (M, g, J) *with induced structure* (P, g, ξ, u, a) *and* $\nabla J = 0$. *Then it follows that*

$$(\nabla_X P)(\xi) = \lambda(q + pa - a^2)X, \tag{3.19}$$

$$(\nabla_\xi P)(X) = 2\lambda g(X, \xi)\xi), \tag{3.20}$$

$$(\nabla_X u)\xi = 2a\lambda g(X, \xi) - p\lambda g(X, \xi) \tag{3.21}$$

for any $X \epsilon \chi(M)$.

Proof We know that in totally umbilical manifold, $A = \lambda I$, then Eq. (3.10) reduces to

$$(\nabla_X P)(Y) = g(\lambda X, Y)\xi + \lambda u(Y)X. \tag{3.22}$$

For $Y = \xi$, the above equation gives

$$(\nabla_X P)(\xi) = g(\lambda X, \xi)\xi + \lambda u(\xi)X. \tag{3.23}$$

Using equality (3.5), we obtain

$$(\nabla_X P)(\xi) = g(AX, \xi)\xi + \lambda(pa + q - a^2). \tag{3.24}$$

By using Weingarten formula for hypersurface, we have

$$(\nabla_X P)\xi = g(-\overline{\nabla}_X N, \xi)\xi + \lambda(pa + q - a^2)$$

$$(\nabla_X P)\xi = \lambda(pa + q - a^2)$$

which gives (3.19).

Using equality (3.7) and if $X = \xi$, we obtain

$$(\nabla_\xi P)(Y) = \lambda g(Y, \xi)\xi + \lambda g(\xi, Y)\xi \tag{3.25}$$

For $Y = X$ in above equation, we get

$$(\nabla_\xi P)(Y) = \lambda g(X, \xi)\xi + \lambda g(\xi, X)\xi. \tag{3.26}$$

Since g is symmetric,

$$(\nabla_\xi P)(X) = \lambda g(\xi, X)\xi + \lambda g(\xi, X)\xi \tag{3.27}$$

$$(\nabla_\xi P)(X) = 2\lambda g(\xi, X)\xi,$$

which gives (3.20).

Using equality (3.11) and $A = \lambda I$, we get

$$(\nabla_X u)(Y) = -g(\lambda X, PY) + ag(\lambda X, Y)$$

$$(\nabla_X u)(Y) = \lambda[ag(X, \xi) - g(X, PY)]. \tag{3.28}$$

For $Y = \xi$, we get

$$(\nabla_X u)(\xi) = \lambda[ag(X, \xi) - g(X, P\xi)].$$

Using equality (3.6) in above equation, we have

$$(\nabla_X u)(\xi) = \lambda[ag(X, \xi) - g(X, (p - a)\xi)]$$

$$(\nabla_X u)(\xi) = \lambda[ag(X, \xi) - g(X, p\xi) + g(X, a\xi)]$$

$$(\nabla_X u)(\xi) = 2\lambda ag(X, \xi) - pg(X, \xi),$$

which gives (3.21).

Proposition 3.3 *If M is a hypersurface in a metallic Riemannian manifold* $(\overline{M}, g, J)$ *with structure* (P, g, ξ, u, a) *induced on* M *by* J*, then the following equations are equivalent:*

$$\nabla_X u = 0 \Leftrightarrow \nabla_X \xi = 0 \tag{3.29}$$

for each $X \epsilon \chi(M)$.

Proof If $\nabla_X u = 0$, using equality (3.11), we get

$$g(AX, PY) = ag((AX), Y). \tag{3.30}$$

Using (3.8) in above equation, we get

$$g(P(AX) - aAX, Y) = 0$$

for any $X, Y \epsilon \chi(M)$. Using (3.12), we have

$$g(\nabla_X \xi, Y) = 0$$

as $y \neq 0$, we get

$$\nabla_X \xi = 0. \tag{3.31}$$

Conversely, we suppose that $\nabla \xi = 0$ and we have

$$g(\nabla_X \xi, Y) = 0.$$

By using equality (3.12), we get

$$g(P(AX) - aAX, Y) = 0$$

$$g(P(AX), Y) - g(aAX, Y) = 0$$

$$g(AX, PY) - ag(AX, Y) = 0. \tag{3.32}$$

In view of (3.11), above equation becomes

$$\nabla_X u = 0.$$

Proposition 3.4 *Let* M *be a hypersurface of a metallic Riemannian manifold* $(\overline{M}, g, J)$ *with structure* (P, g, ξ, u, a) *induced on* M *by the structure* J *with* $\xi \neq 0$.

A necessary and sufficient condition for M to the totally geodesics in M is that $\nabla_X P = 0$ *for any* $X \epsilon \chi(M)$.

Proof If M is totally geodesic, then $A = 0$.

Using equality (3.11), we obtain

$$(\nabla_X P)(Y) = g(AX, Y)\xi.$$

That is,

$$\nabla_X P = 0. \tag{3.33}$$

Conversely, we suppose that $\nabla_X P = 0$ and from (3.10), we have

$$g(AX, Y)\xi + g(Y, \xi) = 0. \tag{3.34}$$

We may have one of the following conditions:

(i) If AX and ξ are linearly dependent vectors fields, then there exists a real number α such that $AX = \alpha\xi$ and from this, we obtain

$$g(\alpha\xi, Y)\xi + g(Y, \xi)\alpha\xi = 0. \tag{3.35}$$

That is,

$$g(Y, \xi) = 0$$

for any $Y \epsilon \chi(M)$. Then for $Y = \xi$, we obtain $g(\xi, \xi) = 0$, which is equivalent with $\xi = 0$. But this is impossible.

(ii) If AX and ξ are linearly independent vector fields, then

$$g(AX, Y) = 0 \tag{3.36}$$

for any $X, Y \epsilon \chi(M)$. Thus $A = 0$ and from this, we have that M is a totally geodesic hypersurface in $\overline{M}$.

Proposition 3.5 *If M is a totally umbilical hypersurface in a metallic Riemannian manifold* $(\overline{M}, g, J)$ *with the induced structure* (P, g, ξ, u, a). *Then the* $1 - form\ u$ *is closed.*

Proof As M is totally umbilical hypersurface, that is $A = \lambda I$. Then from (3.15) and $du(X, Y) = (\nabla_X u)(Y) - (\nabla_Y u)(X)$, we get

$$du(X, Y) = [-\lambda g(X, PY) + a\lambda g(X, Y)] - [-\lambda g(Y, PX) + a\lambda g(Y, X)] \tag{3.37}$$

$$du(X, Y) = -\lambda g(X, PY) + \lambda g(Y, PX)$$

$$du(X, Y) = \lambda[-g(PX, Y) + g(PX, Y)]$$

$$du(X, Y) = 0.$$

Thus, $1 - form\ u$ is closed.

Proposition 3.6 *Let M be a totally umbilical hypersurface of a metallic Riemannian manifold* $(\overline{M}, g, J)$ *with* $\nabla J = 0$ *and induced structure* (P, g, ξ, u, a) *on* M *and* ξ *is a Killing vector field. If* $a \neq \sigma$, *then* $rank A = 1$ *and* ξ *is an eigen vector of the Weingarten operator A with the eigen value* $\frac{\xi(a)}{2(a^2 - pa - q)}$.

Proof In view of proposition 3.5, we have $PA = AP$ and therefore

$$P^2(AX) = a^2(AX) \tag{3.38}$$

for all $X \epsilon \chi(M)$.

Using equality (3.3) in (3.38), we have

$$pP(AX) + qAX - u(AX)\xi = a^2(AX)$$

$$p(PA)X + qAX - u(AX)\xi = a^2(AX)$$

$$paAX + qAX - u(AX)\xi = a^2(AX)$$

$$-u(AX)\xi = (a^2 - pa - q)AX. \tag{3.39}$$

Using equality (3.13) in (3.39), we get

$$\frac{X(a)}{2}\xi = (a^2 - pa - q)AX$$

$$AX = \frac{X(a)}{2(a^2 - pa - q)}\xi$$

$\forall X \epsilon \chi(M)$. If we put $X = \xi$ in above equation, we obtain

$$A(\xi) = \frac{\xi(a)}{2(a^2 - pa - q)}\xi \tag{3.40}$$

$\forall X \epsilon \chi(M)$.

Thus, ξ is an eigen value of Weingarten operator A, and its eigen value is $\frac{\xi(a)}{2(a^2 - pa - q)}$.

Proposition 3.7 *Let* M *be a hypersurface in a metallic Riemannian manifold* $(\overline{M}, g, J)$ *with* $\nabla J = 0$ *and* (P, g, ξ, u, a) *induced structure on* M. *Then* ξ *is a Killing vector field with respect to g on M if and only if we have*

$$2aA = PA + AP, \tag{3.41}$$

where A is the Weingarten operator on M.

Proof We have that ξ is a Killing vector field on M if and only if

$$(L_\xi g)(Y, Z) = 0 \tag{3.42}$$

for all $Y, Z \epsilon \chi(M)$.

$$L_\xi g(Y, Z) - g(L_\xi Y, Z) - g(Y, L_\xi Z) = 0 \tag{3.43}$$

$$\xi g(Y, Z) - g([\xi, Y], Z) - g(Y, [\xi, Z]) = 0$$

$$\xi g(Y, Z) - g(\nabla_\xi Y - \nabla_Y \xi, Z) - g(Y, \nabla_\xi Z - \nabla_Z \xi) = 0$$

$$\xi g(Y, Z) - g(\nabla_\xi Y, Z) - g(Y, \nabla_\xi Z) + g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) = 0 \tag{3.44}$$

So,

$$g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) = 0$$

$\forall Y, Z \epsilon \chi(M)$.

Using the equality (3.12), we get

$$g(-PAY + aAY, Z) + g(-PAZ + aAZ, Y) = 0$$

$$g(-PAY + aAY, Z) + g((-PA + aA)Y, Z) = 0$$

$$g(2aAY - PAY - APY, Z) = 0$$

$$g(2aAY - PAY - APY, Z) = 0$$

$\forall Y, Z \epsilon \chi(M)$. Thus, we get

$$2aA = PA + AP,$$

which is (3.41).

Theorem 3.2 *If M is an invariant hypersurface of a metallic Riemannian manifold $(\overline{M}, g, J)$. Then it is necessary and sufficient that the normal of M is an eigen vector of the matrix J.*

Proof Suppose $a = \sigma$, using equality (3.5), we have

$$u(\xi) = q + p\sigma - \sigma^2 = 0$$

or

$$u(\xi) = g(X, \xi).$$

which is equivalent to $X \perp \xi$.

Since X, ξ are both tangential, then there is only one possibility that $\xi = 0$ because $X \neq 0$. From (3.2), we have

$$J(N) = \xi + aN$$

$$J(N) = aN \tag{3.45}$$

for $\xi = 0$. Hence N is the eigen vector for J.

Example 3.1 We construct an example of hypersurfaces of metallic Riemannian manifold which is based on an example in [18].

We suppose that the ambient space is E^{2a+b} $(a, b \in N^*)$ and for any point of E^{2a+b}, we have its coordinates

$$(x^1, ..., x^a, y^1, ..., y^a, z^1, ..., z^b) = (x^i, y^i, z^j),$$

where $i \in 1, ..., a$ and $j \in 1, ...b$. The tangent space $T_x(E^{2a+b})$ is isomorphic with E^{2a+b}. Let $J : E^{2a+b} \to E^{2a+b}$ be a metallic structure on E^{2a+b} such that

$$J(x^1, ..., x^a, y^1, ..., y^a, z^1, ..., z^b) =$$
$$(\sigma x^1, ..., \sigma x^a, \sigma y^1, ..., \sigma y^a, (p - \sigma)z^1, ..., (p - \sigma)z^b). \tag{3.46}$$

Then

$$J^2(x^1, ..., x^a, y^1, ..., y^a, z^1, ..., z^b) =$$
$$(\sigma^2 x^1, ..., \sigma^2 x^a, \sigma^2 y^1, ..., \sigma^2 y^a, (p - \sigma)^2 z^1, ..., (p - \sigma)^2 z^b).$$

Since σ and $(p - \sigma)$ are roots of $x^2 = px + q$, then $\sigma^2 = p\sigma + q$ and $(p - \sigma)^2 = (p - \sigma) + 1$. Then, (3.46) gives

$$J^2(x^1, ..., x^a, y^1, ..., y^a, z^1, ..., z^b) =$$
$$((p\sigma + q)x^1, ..., (p\sigma + q)x^a, (q + p\sigma)y^1, ..., (q + p\sigma)y^a,$$
$$(p(p - \sigma) + q)z^1, ..., (p(p - q\sigma) + q)z^b)$$
$$J^2(x^1, ..., x^a, y^1, ..., y^a, z^1, ..., z^b) =$$
$$(p\sigma x^1, ..., p\sigma x^a, p\sigma y^1, ...p\sigma y^a p(p - \sigma)z^1, ..., p(p - \sigma)z^b)$$
$$+(qx^1, ..., qx^a, qy^1, ..., qy^a, qz^1, ..., qz^b)$$
$$J^2(x^i, y^i, z^j) = pJ(x^i, y^i, z^j) + q(x^i, y^i, z^j)$$

or

$$J^2 = pJ + qI.$$

It follows that $(E^{2a+b}, <, >, J)$ is a metallic Riemannian manifold.

In E^{2a+b}, we can get the hypersphere

$$S^{2a+b-1}(R) = \{(x^1, ..., x^a, y^1, ..., y^a, ..., z^1, ..., z^b), \sum_{i=1}^{a} {x^i}^2 + \sum_{i=1}^{a} {y^i}^2 + \sum_{j=1}^{b} {z^j}^2 = R^2\},$$

where R is its radius and $(x^1, ..., x^a, y^1, ..., y^a, z^1, ..., z^b)$ are the coordinates of any point of $S^{2a+b-1}(R)$. We use the following notations

$\sum_{i=1}^{a}(x^i)^2 = r_1^2$, $\sum_{i=1}^{a}(y^i)^2 = r_2^2$, $\sum_{j=1}^{b}(z^j)^2 = r_3^2$ and $r_1^2 + r_2^2 = r^2$. Thus, we have $r^2 + r_3^2 = R^2$.

We remark that $N_1 = \frac{1}{R}(x^i, y^i, z^j)$, $i \in (1, ..., a)$, $j \in (1, ..., b)$ is a unit normal vector field on sphere $S^{2a+b-1}(R)$ and

$$J(N_1) = \frac{1}{R}(\sigma x^i, \sigma y^i, (p - \sigma)z^j).$$

For a tangent vector field X on $S^{2a+b-1}(R)$, we use the following notation

$$X = (X^1, ..., X^a, Y^1, ..., Y^a, Z^1, ..., Z^b) = (X^i, Y^i, Z^j).$$

Hence, we have

$$\sum_{i=1}^{a} x^i Y^i + \sum_{i=1}^{a} y^i Y^i + \sum_{j=1}^{b} z^j Z^j = 0.$$

If we decompose $J(N)$ and $J(X^i, Y^i, Z^j)$, respectively, into tangential and normal components on $T_(x, y, z)S^{2a+b-1}(R)$, we obtain

$$J(N) = \xi + AN, J(X^i, Y^i, Z^j) = P(X^i, Y^i, Z^j) + u(X^i, Y^i, Z^j),$$

where (X^i, Y^i, Z^j) is a tangent vector field on $S^{2a+b-1}(R)$, u is $1 - form$ on $S^{2a+b-1}(R)$ and A is smooth real function on $S^{2a+b-1}(R)$.

Using $A = \langle J(N), N\rangle, \xi = J(N) - AN, u(X^i, Y^i, Z^j) = \langle (X^i, Y^i, Z^j), \xi\rangle$ and $P(X^i, Y^i, Z^j) = J(X^i, Y^i, Z^J) - u(X^i, Y^i, Z^j)N$, the elements of the induced structure $\sum = (P, (,), \xi, u, A)$ on $S^{2a+b-1}(R)$ by the metallic Riemannian structure $(J, (,))$ on E^{2a+b} are given as follows:

(i) $A = \frac{\sigma r^2 + (p-\sigma) r_3^2}{R^2}$,
(ii) $\xi = \frac{2\sigma - p}{R^3}[r_3^2 x^i, r_3^2 y^i, -r^2 z^j]$,
(iii) $u(X) = \frac{1}{R}[\sigma \sum_{i=1}^{a}(x^i X^i + y^i Y^j) + (p - \sigma) \sum_{j=1}^{b} z^j Z^j]$,
(iv) $P(X) = (\sigma X^i - \frac{1}{R}u(X)x^i, \sigma Y^i - \frac{1}{R}u(X)y^i, (p - \sigma)Z^j - \frac{1}{R}u(X)z^j)$.

Now, we have

$$A = \langle J(N), N\rangle$$

$$\langle J(N), N\rangle = \langle \xi + AN, N\rangle.$$

$$\frac{1}{R^2}\langle J(x^i, y^i, z^j), (x^i, y^i, z^j)\rangle = A\langle N, N\rangle$$

$$\frac{1}{R^2}\langle(\sigma x^1, ..., \sigma x^a, \sigma y^1, ..., \sigma y^a, (p-\sigma)z^1, ..., (p-\sigma)z^b), (x^1, ..., x^a, y^1, ..., y^a, z^1, ..., z^b)\rangle$$

$$= \frac{A}{R^2}\langle(x^i, y^i, z^j), (x^i, y^i, z^j)\rangle$$

$$\frac{1}{R^2}\left(\sigma\sum_{i=1}^{a}(x^i)^2 + \sum_{i=1}^{a}(y^i)^2 + (p-\sigma)\sum_{j=1}^{b}(z^j)^2 = \frac{A}{R^2}(\sum_{i=1}^{a}(x^i)^2 + \sum_{i=1}^{a}(y^i)^2 + \sum_{j=1}^{a}(z^j)^2\right)$$

$$\sigma r_1^2 + \sigma r_2^2 + (p-\sigma)r_3^2 = A(r_1^2 + r_2^2 + r_3^2)$$

$$\sigma r^2 + (p-\sigma)r_3^2 = AR^2$$

or

$$A = \frac{\sigma r^2 + (p-\sigma)r_3^2}{R^2}.$$

Since

$$\xi = J(N) - AN$$

$$\xi = J\left(\frac{1}{R}(x^i, y^i, z^j) - A(\frac{1}{R}(x^i, y^i, z^j)\right)$$

$$\xi = \frac{1}{R}\left[(\sigma x^1, ..., \sigma x^a, \sigma y^1, ..., \sigma y^a, (p-\sigma)z^1, ..., (p-\sigma)z^b) - A\left(\frac{1}{R}(x^i, y^i, z^j)\right)\right]$$

$$\xi = \frac{1}{R}\left((\sigma x^i, \sigma y^i, (p-\sigma)z^j) - \frac{A}{R}(x^i, y^i, z^j)\right)$$

$$\xi = \frac{1}{R}\Bigg[(\sigma - \frac{\sigma r^2 + (p-\sigma)r_3^2}{R^2})x^i, (\sigma - \frac{\sigma r^2 + (p-\sigma)r_3^2}{R^2})y^i,$$

$$\left((p-\sigma) - \frac{\sigma r^2 + (p-\sigma)r_3^2}{R^2}\right)z^i\Bigg]$$

$$A = \frac{\sigma r^2 + (p-\sigma)r_3^2}{R^2}.$$

Also,

$$u(X) = u(X^i, Y^i, Z^j) = \langle (X^i, Y^i, Z^j), \xi \rangle$$

$$u(X) = \langle (X^i, Y^i, Z^j), J(N) - AN \rangle$$

$$u(X) = \langle (X^1, ..., X^a, Y^1, ..., Y^a, Z^1, ..., Z^b), \frac{1}{R}$$

$$(\sigma x^1, ..., \sigma x^a, \sigma y^1, ..., \sigma y^a, (p-\sigma)z^1, ..., (p-\sigma)z^b) \rangle$$

$$u(X) = \frac{1}{R}\left(\sum_{i=1}^{a} \sigma x^i X^i + \sum_{i=1}^{a} \sigma y^i Y^i + \sum_{j=1}^{b} (p-\sigma) z^j Z^j\right)$$

$$u(X) = \frac{1}{R}\left[\sigma \sum_{i=1}^{a} (x^i X^i + y^i Y^i) + (p-\sigma) \sum_{j=1}^{b} z^j Z^j)\right].$$

Since

$$P(X) = P(X^i, Y^i, Z^j)$$

$$P(X) = J(X^i, Y^i, Z^j) - u(X^i, Y^i, Z^j)N$$

$$P(X) = (\sigma X^i, \sigma Y^i, (p-\sigma)Z^j) - \frac{1}{R} u(X)(x^i, y^i, z^j)$$

$$P(X) = \left(\sigma X^i - \frac{1}{R} u(X) x^i, \sigma Y^i - \frac{1}{R} u(X) y^i, (p-\sigma) Z^j - \frac{1}{R} u(X) z^j\right),$$

where $X = (X^i, Y^i, Z^j)$ is a tangent vector at sphere in any point (x^i, y^i, z^j). Therefore, from the above relations, we have $(P, <>, \xi, u, a)$ induced structure by J from E^{2a+b} on the sphere $S^{2a+b-1}(R)$ of codimension 1 in Euclidean space $E^{2a+b}(R)$.

In conclusion, $S^{a+b-1}(r)$ is a totally umbilical hypersurface in E^{2a+b}.

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Willmore Surfaces in Three-Dimensional Simply Isotropic Spaces $\mathbb{I}_3^1$

Mohamd Saleem Lone

Abstract A Willmore surface is a generalization of a minimal surface satisfying $\Delta\mathbf{H} + 2\mathbf{H}(\mathbf{H}^2 - \mathbf{K}) = 0$, where $\mathbf{H}$ and $\mathbf{K}$ are mean curvature and Gaussian curvature, respectively. In this paper, we study the translation and factorable surfaces in three-dimensional simply isotropic space. We obtain explicit forms of Willmore translation and Willmore factorable surfaces in three-dimensional simply isotropic space $\mathbb{I}_3^1$.

Keywords Factorable surface · Laplace Beltrami operator · Simply isotropic space · Translation surface · Willmore surface

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1 Introduction

Let $g : \mathbf{M} \to \mathbb{R}^3$ be a smooth immersed surface, then Willmore functional is described as [23]

$$\mathrm{W}(g) = \int_{\mathbf{M}} \mathbf{H}^2 d\mu_h,$$

where h is the Riemannian metric, $d\mu_h = \sqrt{\det(h)}dx$ is the induced surface element and $\mathbf{H} = \frac{(\kappa_1+\kappa_2)}{2}$ is the mean curvature with κ_1 and κ_2 as the principal curvatures. One of the important features of Willmore functional is that it is invariant under the full Mobius group of $\mathbb{R}^3$.

Willmore surfaces are the critical points of the Willmore functional satisfying the Willmore equation given by

$$\Delta\mathbf{H} + 2\mathbf{H}(\mathbf{H}^2 - \mathbf{K}) = 0. \tag{1.1}$$

M. S. Lone (✉)
International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru 560089, India
e-mail: saleemraja2008@gmail.com

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Let $\tilde{w} = (\tilde{w}_{ij})(G)$ be the matrix comprising the components of the induced metric on $\mathbf{M}$, and $\tilde{w}^{-1} = (\tilde{w}^{ij})$ be the inverse matrix of $(\tilde{w}_{ij})$. Then the Laplace Beltrami operator Δ on $\mathbf{M}$ is given by [16]

$$\Delta = -\frac{1}{\sqrt{|G|}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{|G|} \tilde{w}^{ij} \frac{\partial}{\partial x^j} \right). \tag{1.2}$$

Bryant [5] proved a duality theorem for Willmore surfaces. After that, Li [12] discussed the Willmore surfaces in S^n, establishing an integral inequality for compact Willmore surfaces in S^n. Luo and Sun [15] proved that every entire two-dimensional Willmore graph in $\mathbb{R}^3$ with square integrable mean curvature is a plane. Acqua et al. [1] studied the Willmore unstable revolution surfaces with natural boundary conditions. Chen and Lamm [7] proved the plane nature of every two-dimensional graphical solution of Willmore equation with square integrable second fundamental form.

2 Preliminaries

Simply isotropic space $\mathbb{I}_3^1$ is one among the nine Cayley–Klein geometries. It is obtained by subtracting a certain triplet (ϖ, J_1, J_2) from a projective space $\mathcal{P}(R^3)$. Here ϖ is a plane in $\mathcal{P}(R^3)$ called *absolute plane* and (J_1, J_2) is a pair of complex conjugate straight lines in ϖ called *absolute lines*. The triplet (ϖ, J_1, J_2) is called *absolute figure* of $\mathbb{I}_3^1$ [22]. Let $\left(\tilde{t}, \tilde{p}, \tilde{q}, \tilde{r}\right) \neq (0:0:0:0)$ be the projective coordinates. Then the plane ϖ is parametrized by $\tilde{t} = 0$, and J_1 and J_2 are parametrized by $\tilde{t} = \tilde{p} \pm i\tilde{q} = 0$. The intersection point $\mathbb{P}(0:0:0:1)$ of J_1 and J_2 is called the *absolute point*. The motion group of $\mathbb{I}_3^1$ is a group of six parameters defined in affine coordinates $p = \frac{\tilde{p}}{\tilde{t}}, q = \frac{\tilde{q}}{\tilde{t}}, r = \frac{\tilde{r}}{\tilde{t}}$ by

$$(p, q, r) \longmapsto (\tilde{p}, \tilde{q}, \tilde{r}) : \begin{cases} \tilde{p} = a + p\cos\theta - q\sin\theta, \\ \tilde{q} = b + p\sin\theta + q\cos\theta, \\ \tilde{r} = c + c_1 p + c_2 q + r, \end{cases} \tag{2.1}$$

where $a, b, c, c_1, c_2, \theta \in \mathbb{R}$. Affine transformations of this type are called *isotropic congruence transformations* or *i-motions*.

Consider two points $\mathbf{P} = (p_1, p_2, p_3)$ and $\mathbf{Q} = (q_1, q_2, q_3)$ in $\mathbb{I}_3^1$, then the isotropic distance between $\mathbf{P}$ and $\mathbf{Q}$ is defined as

$$||\mathbf{P} - \mathbf{Q}||_i = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}.$$

The isotropic scalar product of the vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined as

$$\mathbf{u}\cdot\mathbf{v}=\begin{cases}u_1v_1+u_2v_2 & \text{if at least one of } u_i \text{ or } v_i \text{ is nonzero, } i=1,2,\\ u_3v_3 & \text{if } u_i=0=v_i \text{ for } i=1,2.\end{cases}$$

In isotropic geometry, we have different types of lines and planes with respect to the nature of absolute figure. If the points at infinity of a line do not coincide with the point $\mathbb{P}$, then the line is called non-isotropic and is called isotropic if the points at infinity of a line coincide with the point $\mathbb{P}$. Similarly a plane is called non-isotropic if its line at infinity does not contain $\mathbb{P}$ and isotropic if its line at infinity contains $\mathbb{P}$. For example, $ap+bq+cr=0 \quad (a,b,c\in\mathbb{R}), \quad c\neq 0$ is a non-isotropic plane whereas $ap+bq=0$ is an isotropic plane. Thus in this affine model isotropic lines and isotropic planes appear vertical, i.e., parallel to $r-$axis. In isotropic space $\mathbb{I}_3^1$, the metric ds is defined as

$$ds^2=dp^2+dq^2.$$

A surface $\mathbf{M}$ immersed in $\mathbb{I}_3^1$ with no isotropic tangent plane $T_p(\mathbf{M})$ at each point p is called an admissible surface, thus basically has an Euclidean metric. The first fundamental form coefficients E, F, G are obtained by the induced metric of $\mathbb{I}_3^1$ on $\mathbf{M}$. The unit isotropic vector $U=(0,0,1)$ parallel to $r-$direction is assumed as normal vector field of $\mathbf{M}$, which is, in fact, orthogonal to all vectors in $T_p(\mathbf{M})$ for all $p\in\mathbf{M}$. Therefore, the components of the second fundamental form are obtained with respect to U, as

$$L=\frac{\det(f_{pp},f_p,f_q)}{\sqrt{EG-F^2}},\quad M=\frac{\det(f_{pq},f_p,f_q)}{\sqrt{EG-F^2}},\quad N=\frac{\det(f_{qq},f_p,f_q)}{\sqrt{EG-F^2}},$$

where $f(p,q)$ is a local parametrization of $\mathbf{M}$.

The isotropic Gaussian curvature $\mathbf{K}$ and isotropic mean curvature $\mathbf{H}$ are defined as

$$\mathbf{K}=\frac{LN-M^2}{EG-F^2},\qquad \mathbf{H}=\frac{EN-2FM+GL}{2(EG-F^2)}, \tag{2.2}$$

respectively. If the isotropic Gaussian curvature $\mathbf{K}$ of $\mathbf{M}$ vanishes, then $\mathbf{M}$ is called isotropic flat, and if the isotropic mean curvature $\mathbf{H}$ of $\mathbf{M}$ vanishes, then $\mathbf{M}$ is called isotropic minimal. Similarly a surface $\mathbf{M}$ is said to be $CIMC$ ($CIGC$) if $\mathbf{H}$(resp.$\mathbf{K}$) is constant on whole $\mathbf{M}$. For more general references on isotropic spaces, we refer the reader to [6, 8, 17–21].

A surface obtained by translating a curve over the other gives rise to a surface known as translation surface. Similarly taking the product of two curves gives rise to a surface known as factorable surface. For a long time, various authors have studied translation and factorable surfaces in different ambient spaces [3, 9, 10, 13, 14, 24, 25]. Aydin [2] completely studied the translation surfaces generated by a space curve and a planar curve in the isotropic space $\mathbb{I}_3^1$. Our framework is to impose the Willmore functional property on translation and factorable surfaces in $\mathbb{I}_3^1$. Therefore, in the present paper, we find the explicit forms of Willmore translation and factorable surface.

3 Translation and Factorable Surfaces in $\mathbb{I}_3^1$

When we translate two planar curves $f(u_1)$ and $g(u_2)$, we obtain a translation surface of the form:

$$\mathbf{M}(u_1, u_2) = f(u_1) + g(u_2).$$

Depending upon the absolute figure, there are three different possibilities according to the position of curves:

(1) Both the curves lie in isotropic planes, which can be obtained by setting $p = 0$ and $q = 0$, respectively;
(2) Mixed type, i.e., one curve lies in isotropic plane and the other in non-isotropic plane by setting ($p = 0$ or $q = 0$) and $r = 0$, respectively;
(3) Both the curves lie in perpendicular non-isotropic planes, which can be obtained by setting $q - r = \pi$ and $q + r = \pi$, respectively.

Thus there are the following parametrization of translation surfaces [11, 22]:

Type I: The parametrization of a surface **M** obtained by translating $f(u_1) = (u_1, 0, \alpha(u_1))$ and $g(u_2) = (0, u_2, \beta(u_2))$ is given by

$$\mathbf{M}(u_1, u_2) = (u_1, v, \alpha(u_1) + \beta(u_2)). \tag{3.1}$$

Type II: The parametrization of a surface **M** obtained by translating $f(u_1) = (u_1, \alpha(u_1), 0)$ and $g(u_2) = (0, \beta(u_2), u_2)$ is given by

$$\mathbf{M}(u_1, u_2) = (u_1, \alpha(u_1) + \beta(u_2), u_2). \tag{3.2}$$

For an admissible surface, we assume $\beta'(u_2) \neq 0$, i.e., $\beta(u_2) \neq$ constant.

Type III: The parametrization of a surface **M** obtained by translating $f(u_1) = (\alpha(u_1), u_1 + \frac{\pi}{2}, u_1 - \frac{\pi}{2})$ and $g(u_2) = (\beta(u_2), \frac{\pi}{2} - u_2, \frac{\pi}{2} + u_2)$ is given by

$$\mathbf{M}(u_1, u_2) = \frac{1}{2}(\alpha(u_1) + \beta(u_2), u_1 - u_2 + \pi, u_1 + u_2). \tag{3.3}$$

For an admissible surface, we assume $\alpha' + \beta' \neq 0$, i.e., $\alpha'(u_1) \neq -\beta'(u_2) \neq a = constant$.

A surface obtained as a graph of product of two curves is a factorable surface. A three-dimensional simply isotropic space $\mathbb{I}_3^1$ provides two different types of factorable surfaces. It is indeed a product of the $pq-$plane and the isotropic $r-$direction with degenerate metric. Due to the isotropic axis in $\mathbb{I}_3^1$, there are two types of factorable surfaces in $\mathbb{I}_3^1$ [4].

Type I: The parametrization of factorable surface of type-I is given by

$$\mathbf{M}(u_1, u_2) = (u_1, u_2, \alpha(u_1)\beta(u_2)). \tag{3.4}$$

Type II: The parametrization of factorable surface of type-II is given by

$$\mathbf{M}(u_1, u_3) = (u_1, \alpha(u_1)\beta(u_3), u_3), \tag{3.5}$$

or,

$$\mathbf{M}(u_2, u_3) = (\alpha(u_2)\beta(u_3), u_2, u_3). \tag{3.6}$$

4 Willmore Translation Surfaces of Type-I in Simply Isotropic Space

The coefficients of the first and second fundamental form for the translation surface of type-I in (3.1) are given by

$$E = G = 1, \quad F = M = 0, \tag{4.1}$$

$$L = \alpha'', \quad N = \beta''. \tag{4.2}$$

Thus the Gaussian curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are obtained as

$$\mathbf{K} = \alpha''(u_1)\beta''(u_2), \quad \mathbf{H} = \frac{\alpha''(u_1) + \beta''(u_2)}{2}, \tag{4.3}$$

assuming that the Gaussian curvature of the surface does not vanish, i.e.,

$$\alpha''(u_1)\beta''(u_2) \neq 0, \forall u_1, u_2 \in I.$$

From (1.2), it is easy to find out

$$\Delta\mathbf{H} = \left(0, 0, \frac{1}{2}(-\alpha^{(4)} - \beta^{(4)})\right).$$

Therefore, from (1.1), (4.3) and the above equation, we can find the following system of equations.

$$\frac{1}{4}(\alpha'' - \beta'')^2(\alpha'' + \beta'') = 0, \tag{4.4}$$

$$\frac{1}{4}\left[\alpha''^3 - \alpha''^2\beta'' - \alpha''\beta''^2 + \beta''^3 - 2(\alpha^{(4)} + \beta^{(4)})\right] = 0. \tag{4.5}$$

From (4.4), following two cases arise:

Case 1: $(\alpha''(u_1) - \beta''(u_2))^2 = 0$.

Since α and β are functions of two independent variables, the above equation can be written as

$$\alpha'' = \beta'' = c, \quad c \in \mathbb{R}.$$

Thus, we get

$$\alpha(u_1) = \frac{c}{2}u_1^2 + c_1u_1 + c_2, \qquad \beta(u_2) = \frac{c}{2}u_2^2 + c_3u_2 + c_4, \quad c_i \in \mathbb{R}.$$

The above solution also satisfies (4.5), hence **M** is a Willmore surface with following parametrization

$$\mathbf{M}(u_1, u_2) = \left(u_1, u_2, \left(\frac{c}{2}u_1^2 + c_1u_1 + c_2\right) + \left(\frac{c}{2}u_2^2 + c_3u_2 + c_4\right)\right).$$

Case 2: $\alpha'' + \beta'' = 0$.

The above condition is equivalent to minimality of **M**. Since α and β are functions of two independent variables, the above equation can be written as

$$\alpha'' = -\beta'' = c, \quad c \in \mathbb{R}.$$

Thus, we get

$$\alpha(u_1) = \frac{c}{2}u_1^2 + c_1u_1 + c_2, \qquad \beta(u_2) = -\frac{c}{2}u_2^2 + c_3u_2 + c_4, \quad c_i \in \mathbb{R}.$$

Since every minimal surface is a Willmore surface, **M** is parametrized by

$$\mathbf{M}(u_1, u_2) = \left(u_1, u_2, \left(\frac{c}{2}u_1^2 + c_1u_1 + c_2\right) + \left(-\frac{c}{2}u_2^2 + c_3u_2 + c_4\right)\right).$$

Theorem 4.1 *Let* ***M*** *be a non-minimal Willmore translation surface of type-I in simply isotropic space* $\mathbb{I}_3^1$*, then by the translation of* $\mathbb{I}_3^1$*,* ***M*** *is congruent to*

$$\boldsymbol{M}(u_1, u_2) = \left(u_1, u_2, \left(\frac{c}{2}u_1^2 + c_1u_1\right) + \left(\frac{c}{2}u_2^2 + c_3u_2\right)\right).$$

Theorem 4.2 *Let* ***M*** *be a minimal surface of type-I in simply isotropic space* $\mathbb{I}_3^1$*, then by the translation of* $\mathbb{I}_3^1$*,* ***M*** *is a Willmore translation surface if it is congruent to*

$$\boldsymbol{M}(u_1, u_2) = \left(u_1, u_2, \left(\frac{c}{2}u_1^2 + c_1u_1\right) + \left(-\frac{c}{2}u_2^2 + c_3u_2\right)\right).$$

5 Willmore Translation Surfaces of Type-II in Simply Isotropic Space

The coefficients of the first and second fundamental form for the translation surface of type-II in (3.2) are given by

$$E = 1 + \alpha'(u_1)^2, \quad F = \alpha'(u_1)\beta'(u_2), \quad G = \beta'(u_2)^2, \tag{5.1}$$

$$L = -\frac{\alpha''(u_1)}{\beta'(u_2)}, \quad N = -\frac{\beta''(u_2)}{\beta'(u_2)}, \quad M = 0. \tag{5.2}$$

Then the Gaussian curvature $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are obtained as

$$\mathbf{K} = \frac{\alpha''(u_1)\beta''(u_2)}{\beta'(u_2)^4}, \quad \mathbf{H} = -\frac{\beta'(u_2)^2\alpha''(u_1) + (1 + \alpha'(u_1)^2)\beta''(u_2)}{2\beta'(u_2)^3}. \tag{5.3}$$

Suppose that the surface has a nonzero Gaussian curvature, i.e.,

$$\alpha''(u_1)\beta''(u_2) \neq 0, \forall u, v \in I.$$

From (1.2), it is easy to find out

$$\Delta\mathbf{H} = \left\{ \begin{array}{c} 0, \\ 0, \\ \frac{1}{2\beta'^7}\Big[15(1+\alpha'^2)^2\beta''^3 + \beta'^4\beta''(3\alpha''^2 + 4\alpha'\alpha^{(3)}) \\ -2(1+3\alpha'^2)\beta'^3\alpha''\beta^{(3)} - 10(1+\alpha'^2)^2\beta'\beta''\beta^{(3)} \\ +\beta^{(6)}\alpha^{(4)} + \beta'^2(6(1+3\alpha'^2)\alpha''\beta''^2 + (1+\alpha'^2)^2\beta^{(4)}) \Big] \end{array} \right\}.$$

Therefore, from (1.1), (5.3) and the above equation, we can find the following system of equations.

$$-\frac{1}{4\beta'^9}(\beta'^2\alpha'' + \beta'' + \alpha'^2\beta'')(\beta'^4\alpha''^2 - 2\beta'^2\alpha''\beta'' + 2\alpha'^2\beta'^2\alpha''\beta'' + \beta''^2 + 2\alpha'^2\beta''^2 + \alpha'^4\beta''^2) = 0. \tag{5.4}$$

$$\begin{aligned} &\frac{1}{4\beta'^9}(-\beta'^6\alpha''^2 + \beta'^4\alpha''^2\beta'' - 3\alpha'^2\beta'^4\alpha''^2\beta'' + 6\beta'^6\alpha''^2\beta'' + \beta'^2\alpha''\beta''^2 \\ &-2\alpha'^2\beta'^2\alpha''\beta''^2 - 3\alpha'^4\beta'^2\alpha''\beta''^2 + 12\beta'^4\alpha''\beta''^2 + 36\alpha'^2\beta'^4\alpha''\beta''^2 - \beta''^3 \\ &-3\alpha'^2\beta''^3 - 3\alpha'^4\beta''^3 - \alpha'^6\beta''^3 + 30\beta'^2\beta''^3 + 60\alpha''^2\beta''^2\beta''^3 + 30\alpha'^4\beta'^2\beta''^3 \\ &+8\alpha'\beta'^6\beta''\alpha^{(3)} - 4\beta'^5\alpha''\beta^{(3)} - 12\alpha'^2\beta'^5\alpha''\beta^{(3)} - 20\beta'^3\beta''\beta^{(3)} - 40\alpha'^2\beta'^3\beta''\beta^{(3)} \\ &-20\alpha'^4\beta'^3\beta''\beta^{(3)} + 2\beta'^8\alpha^{(4)} + 2\beta'^4\beta^{(4)} + 4\alpha'^2\beta'^4\beta^{(4)} + 2\alpha'^4\beta'^4\beta^{(4)}) = 0. \end{aligned} \tag{5.5}$$

From (5.4), we have the following two cases:

Case 1: $\beta'^2\alpha'' + \beta'' + \alpha'^2\beta'' = 0$.

The above condition is equal to the minimality of $\mathbf{M}$, therefore, from the above equation, we have

$$\frac{\alpha''}{(1+\alpha'^2)} = -\frac{\beta''}{\beta'^2} = c.$$

Thus, we get

$$\alpha(u_1) = c_2 - \frac{\log(\cos(cu_1 + c_1))}{c}, \quad \beta(u_2) = c_2 + \frac{\log(cu_2 - c_1)}{c}.$$

Since every minimal surface is a Willmore surface, **M** has a parametrization given by

$$\left(u_1, u_2, \left(c_2 - \frac{\log(\cos(cu_1 + c_1))}{c}\right) + \left(c_2 + \frac{\log(cu_2 - c_1)}{c}\right)\right), \quad c \in \mathbb{R}_0, c_1, c_2 \in \mathbb{R}.$$

Case 2:

$$\beta'^4\alpha''^2 - 2\beta'^2\alpha''\beta'' + 2\alpha'^2\beta'^2\alpha''\beta'' + \beta''^2 + 2\alpha'^2\beta''^2 + \alpha'^4\beta''^2 = 0. \tag{5.6}$$

Equation (5.6) can be rearranged as

$$2\beta'^2\alpha''\beta'' - \beta'^4\alpha''^2 - 2\alpha'^2\beta'^2\alpha''\beta'' = \beta''^2 + 2\alpha'^2\beta''^2 + \alpha'^4\beta''^2.$$

Since $\mathbf{K} \neq 0$, dividing the above equation by β''^2, we get

$$2\beta'^2\frac{\alpha''}{\beta''} - \beta'^4\frac{\alpha''^2}{\beta''^2} - 2\alpha'^2\beta'^2\frac{\alpha''}{\beta''} = 1 + 2\alpha'^2 + \alpha'^4. \tag{5.7}$$

Differentiating (5.7) w.r.t. u_2, we get

$$\alpha''\left[2\left(\frac{\beta'^2}{\beta''}\right)' - \alpha''\left(\frac{\beta'^4}{\beta''^2}\right)' - 2\alpha'^2\left(\frac{\beta'^2}{\beta''}\right)'\right] = 0,$$

or,

$$2\left(\frac{\beta'^2}{\beta''}\right)' - \alpha''\left(\frac{\beta'^4}{\beta''^2}\right)' - 2\alpha'^2\left(\frac{\beta'^2}{\beta''}\right)' = 0. \tag{5.8}$$

Now, differentiating (5.8), w.r.t. u_1, we get

$$-\alpha'''\left(\frac{\beta'^4}{\beta''^2}\right)' - 2\alpha'\alpha''\left(\frac{\beta'^2}{\beta''}\right)' = 0.$$

The above equation reduces to

$$\frac{\beta''}{\beta'^2} + \frac{\alpha'''}{\alpha'\alpha''} = 0. \tag{5.9}$$

Since α and β are functions of two independent variables, therefore, from (5.9), we must have

$$\frac{\beta''}{\beta'^2} = -c, \quad \frac{\alpha'''}{\alpha'\alpha''} = c.$$

Thus, we get

$$\alpha(u_1) = c_3 - \frac{2}{c}\log\left(\cos\left(\frac{\sqrt{cc_1}(u_1 + c_2)}{\sqrt{2}}\right)\right), \quad \beta(u_2) = c_2 + \frac{\log(cu_2 - c_1)}{c},$$
$$c \in \mathbb{R}_0, c_i \in \mathbb{R}, i = 1, 2, 3.$$

The above solution does not satisfy (5.5). Hence, we conclude the following.

Theorem 5.1 *There are no non-minimal Willmore translation surfaces of type-II in simply isotropic 3-space* $\mathbb{I}_3^1$.

Theorem 5.2 *Let* $\boldsymbol{M}$ *be a minimal translation surface of type-II in* $\mathbb{I}_3^1$*, then by the translation and dilation of* $\mathbb{I}_3^1$*,* $\boldsymbol{M}$ *is a Willmore surface if* $\boldsymbol{M}$ *is congruent to*

$$\boldsymbol{M}(u_1, u_2) = (u_1, u_2, (c_2 - \log\cos(u_1)) + (c_2 + \log\cos(u_2))).$$

6 Willmore Translation Surfaces of Type-III in Simply Isotropic Space

The coefficients of the first and second fundamental form for the translation surface of type-III in (3.3) are given by

$$E = \frac{1 + \alpha'(u_1)^2}{4}, \quad F = \frac{\alpha'(u_1)\beta'(u_2) - 1}{4}, \quad G = \frac{1 + \beta'(u_2)^2}{4}, \tag{6.1}$$

$$L = \frac{\alpha'(u_1) + \beta'(u_2)}{\alpha''(u_1)}, \quad M = 0, \quad N = \frac{\alpha'(u_1) + \beta'(u_2)}{\beta''(u_2)}. \tag{6.2}$$

Then the Gaussian curvature **K** and the mean curvature **H** are obtained as

$$\mathbf{K} = \frac{16\alpha''(u_1)\beta''(u_2)}{(\alpha'(u_1) + \beta'(u_2))^4}, \quad \mathbf{H} = 2\frac{(1 + \beta'(u_2)^2)\alpha''(u_1) + (1 + \alpha'(u_1)^2)\beta''(u_2)}{(\alpha'(u_1) + \beta'(u_2))^3}. \tag{6.3}$$

Suppose that the Gaussian curvature of the surface is non-vanishing, i.e.,

$$\alpha''(u_1)\beta''(u_2) \neq 0, \forall u_1, u_2 \in I.$$

From (1.2), it is easy to find out

$$\Delta\mathbf{H} = \left\{ \begin{array}{c} 0, \\ 0, \\ \frac{-1}{(\alpha'+\beta')^7} 8[15(1+\beta'^2)^2\alpha''^2 + 3(15+12\beta'^2+\beta'^4 - 8\alpha'\beta'(2+\beta'^2) \\ +\alpha'^2(2+6\beta'^2))\alpha''^2\beta'' + 15(1+\alpha'^2)^2\beta''^3 - 2(\alpha'+\beta')\beta''((5+3\beta'^2 \\ -2\alpha'\beta'(3+\beta'^2) + \alpha'^2(1+3\beta'^2))\alpha^{(3)} + 5(1+\alpha'^2)^2\beta^{(3)}) \\ +\alpha''(3(15+\alpha'^4 - 16\alpha'\beta' - 8\alpha'^3\beta' + 2\beta'^2 + 6\alpha'^2(2+\beta'^2))\beta''^2 \\ +2(\alpha'+\beta')(-5(1+\beta'^2)^2\alpha^{(3)} + (-5+6\alpha'\beta' + 2\alpha'^3\beta' - \beta'^2 \\ -3\alpha'^2(1+\beta'^2))\beta^{(3)})) + (\alpha'+\beta')^2((1+\beta'^2)^2\alpha^{(4)} + (1+\alpha'^2)^2\beta^{(4)})] \end{array} \right\}.$$

Therefore from (1.1), (6.3) and the above equation, we can find the following system of equations.

$$\frac{1}{(\alpha'+\beta')^9} 16\{(1+\beta'^2)\alpha'' + (1+\alpha'^2)\beta''\}\{-4(\alpha'+\beta')^2\alpha''\beta'' + ((1+\beta'^2)\alpha'' + (1+\alpha'^2)\beta'')\} = 0. \tag{6.4}$$

$$\begin{aligned} &\frac{1}{(\alpha'+\beta')^9} 16\{(1+\beta'^2)\alpha'' + (1+\alpha'^2)\beta''\}\{-4(\alpha'+\beta')^2\alpha''\beta'' + ((1+\beta'^2)\alpha'' \\ &+(1+\alpha'^2)\beta'')\} - \frac{1}{(\alpha'+\beta')^7} 8\{15(1+\beta'^2)^2\alpha''^3 + 3(15+12\beta'^2+\beta'^4 \\ &-8\alpha'\beta'(2+\beta'^2) + \alpha'^2(2+6\beta'^2))\alpha''^2\beta'' + 15(1+\alpha'^2)^2\beta''^3 \\ &-2(\alpha'+\beta')\beta''\{(5+3\beta'^2 - 2\alpha'\beta'(3+\beta'^2) + \alpha'^2(1+3\beta'^2))\alpha^{(3)} \\ &+5(1+\alpha'^2)^2\beta^{(3)}\} + \alpha''\{3(15+\alpha'^4 - 16\alpha'\beta' - 8\alpha'^3\beta' + 2\beta'^2 \\ &+6\alpha'^2(2+\beta'^2))\beta''^2 + 2(\alpha'+\beta')(-5(1+\beta'^2)^2\alpha^{(3)} + (-5+6\alpha'\beta' + 2\alpha'^3\beta' \\ &-\beta'^2 - 3\alpha'^2(1+\beta'^2))\beta^{(3)})\} + (\alpha'+\beta')^2((1+\beta'^2)^2\alpha^{(4)} \\ &+(1+\alpha'^2)^2\beta^{(4)})\} = 0. \end{aligned} \tag{6.5}$$

From (6.4), we have the following two cases:

Case 1:

$$(1+\beta'^2)\alpha'' + (1+\alpha'^2)\beta'' = 0. \tag{6.6}$$

Since α and β are functions of two independent variables, Eq. (6.6) can be written as

$$\frac{\beta''}{(1+\beta'^2)} = -\frac{\alpha''}{(1+\alpha'^2)} = c.$$

Thus, we get

$$\alpha(u_1) = c_2 + \frac{\log(\cos(cu_1 + c_1))}{c}, \quad \beta(u_2) = c_2 - \frac{\log(\cos(cu_2 + c_1))}{c}, \quad c \in \mathbb{R}_0, c_1, c_2 \in \mathbb{R}.$$

From (6.3), we see that the condition in case 1 is equal to the minimality of **M**. Since we know that every minimal surface is a Willmore surface, therefore, **M** is parametrized by

$$\left(u_1, u_2, \left(c_2 + \frac{\log(\cos(cu_1 + c_1))}{c}\right) + \left(c_2 - \frac{\log(\cos(cu_2 + c_1))}{c}\right)\right).$$

Case 2:

$$-4(\alpha' + \beta')^2\alpha''\beta'' + (1 + {\beta'}^2)\alpha'' + (1 + {\alpha'}^2)\beta'' = 0.$$

Using (6.3), the above reduces to

$$-(\alpha' + \beta')^2[8\alpha''\beta'' - \mathbf{H}] = 0. \tag{6.7}$$

Since **M** is an admissible surface, i.e., $\alpha' + \beta' \neq 0$, Eq. (6.7) reduces to

$$8\alpha''\beta'' - \mathbf{H} = 0.$$

The above equation can be rewritten as

$$\alpha'' - \frac{\mathbf{H}}{8\beta''} = 0.$$

Since α and β are functions of two independent variables u_1 and u_2, respectively, we get

$$\alpha(u_1) = cu_1^2 + c_1u_1 + c_2, \quad \beta(u_2) = \frac{\mathbf{H}}{8c}u_2^2 + c_3u_2 + c_4.$$

The above found solution does not satisfy (6.5), therefore we conclude the following.

Theorem 6.1 *There are no non-minimal Willmore translation surfaces in simply isotropic 3-spaces* $\mathbb{I}_3^1$.

Theorem 6.2 *Let* $\boldsymbol{M}$ *be a minimal translation surface of type 3 in simply isotropic 3-space, then by the translation and dilation of* $\mathbb{I}_3^1$, $\boldsymbol{M}$ *is a Willmore surface if it is congruent to*

$$\boldsymbol{M}(u_1, u_2) = (u_1, u_2, (\log(\cos(cu_1 + c_1)) - \log(\cos(cu_2 + c_1)))\,.$$

7 Willmore Factorable Surfaces of Type-I in Simply Isotropic Space

The first and second fundamental form coefficients for the factorable surface of type-I in (3.4) are given by

$$E = G = 1 \qquad F = 0, \tag{7.1}$$

$$E = \alpha''\beta, \qquad M = \alpha'\beta', \qquad N = \alpha\beta''. \tag{7.2}$$

Then the Gaussian and the mean curvature are given by

$$\mathbf{K} = -\alpha'^2\beta'^2 + \alpha\beta\alpha''\beta'', \qquad \mathbf{H} = \frac{1}{2}(\beta\alpha'' + \alpha\beta''). \tag{7.3}$$

From (1.2), it is easy to find out

$$\Delta\mathbf{H} = \left(0, 0, -\frac{1}{2}(2\alpha''\beta' + \beta\alpha^{(4)} + \alpha\beta^{(4)})\right). \tag{7.4}$$

Therefore, from (1.1), (7.3) and the above equation, we can find the following system of equations.

$$\frac{1}{4}(\beta\alpha'' + \alpha\beta'')(4\alpha'^2\beta'^2 + (\beta\alpha'' + \alpha\beta'')^2) = 0. \tag{7.5}$$

$$\frac{1}{4}\left[(\beta\alpha'' + \alpha\beta'')(4\alpha'^2\beta'^2 + (\beta\alpha'' + \alpha\beta'')^2) - 2(2(\alpha''\beta''\beta\alpha^{(4)} + \alpha\beta^{(4)}))\right] = 0. \tag{7.6}$$

Thus for **M** being of Willmore type, we have to find the simultaneous solutions of (7.5) and (7.6). Therefore, we have the following cases:

Case 1: M is minimal, i.e., $\beta\alpha'' + \alpha\beta'' = 0$. Then, we have to find the simultaneous solutions of the following equations

$$\beta\alpha'' + \alpha\beta'' = 0, \tag{7.7}$$

$$\frac{1}{2}\left[2(\alpha''\beta''\beta\alpha^{(4)} + \alpha\beta^{(4)})\right] = 0. \tag{7.8}$$

Depending upon the choices of α, β, we have the following:

Case 1.1: Suppose α or β be a nonzero constant, we have the following sub-cases:

Case 1.1.1: Let $\alpha = c$, from (7.7), we obtain $\beta = c_1u_2 + c_2$, which satisfies (7.7) and (7.8).

Case 1.1.2: Let $\alpha = c$, from (7.8), we obtain $\beta = c_1u_2^3 + c_2u_2^2 + c_3u_2 + c_4$, which does not satisfy (7.7), where $c, c_1 \in \mathbb{R}_0$, $c_2, c_3, c_4 \in \mathbb{R}$.

Case 1.2: Suppose α, β are linear functions, then (7.7) and (7.8) both are satisfied.

Case 1.3: Suppose α, β are nonlinear, we have the following sub-cases:

Case 1.3.1: From (7.7), we obtain

$$\frac{\alpha''}{\alpha} = -\frac{\beta''}{\beta} = c.$$

Thus we get,

$$\alpha = c_1 e^{\sqrt{c}u_1} + c_2 e^{-\sqrt{c}u_1}, \quad \beta = c_1 \cos(\sqrt{c}u_2) + c_2 \sin(\sqrt{c}u_2), \tag{7.9}$$

where $c_i \in \mathbb{R}$.

The found forms of α and β in (7.9) do not satisfy (7.6).

Case 1.3.2: From (7.8), we obtain

$$2\frac{\alpha''\beta''}{\alpha\beta} + \frac{\alpha^{(4)}}{\alpha} + \frac{\beta^{(4)}}{\beta} = 0. \tag{7.10}$$

Differentiating (7.10) with respect to u_1, we get

$$2\left(\frac{\alpha''}{\alpha}\right)' \frac{\beta''}{\beta} = -\left(\frac{\alpha^{(4)}}{\alpha}\right)'. \tag{7.11}$$

Since the R.H.S. of (7.11) is either a constant or a function of u_1, for $-\left(\frac{\alpha^{(4)}}{\alpha}\right)' = c$, (7.11) reduces to the following equation:

$$2\left(\frac{\alpha''}{\alpha}\right)' \frac{\beta''}{\beta} = c. \tag{7.12}$$

If $\frac{\beta''}{\beta} = 0$, then β is linear, which is a contradiction. Therefore from (7.12), we have

$$2\left(\frac{\alpha''}{\alpha}\right)' = -c\frac{\beta}{\beta''} = k, \quad k \in \mathbb{R}_0.$$

Thus, we get

$$\alpha(u_1) = c_1 Ai\left(\sqrt[3]{\frac{k}{2}}u_1\right) + c_2 Bi\left(\sqrt[3]{\frac{k}{2}}u_1\right),$$

$$\beta(u_2) = c_1 \cos\left(\sqrt{\frac{c}{k}}\right) u_2 + c_2 \sin\left(\sqrt{\frac{c}{k}}\right) u_2,$$

where Ai and Bi are airy functions.

The above found forms of α, β do not satisfy (7.7).

For $-\left(\frac{\alpha^{(4)}}{\alpha}\right)'$ being a function of u_1, (7.11) reduces to the following equation

$$\frac{2\left(\frac{\alpha''}{\alpha}\right)'}{\left(\frac{\alpha^{(4)}}{\alpha}\right)'} = -\frac{\beta}{\beta''}. \tag{7.13}$$

From (7.13), R.H.S. is either a constant or a function of u_2, whereas L.H.S. is a function of u_1, which is a contradiction in any case.

Case 2: $\mathbf{M}$ is not minimal, i.e., $4\alpha'^2\beta'^2 + (\beta\alpha'' - \alpha\beta'')^2 = 0$. Then, we have to find the simultaneous solutions of the following equations

$$4\alpha'^2\beta'^2 + (\beta\alpha'' - \alpha\beta'')^2 = 0, \tag{7.14}$$

$$\frac{1}{2}\left[2(\alpha''\beta''\beta\alpha^{(4)} + \alpha\beta^{(4)})\right] = 0. \tag{7.15}$$

Depending upon the choices of α, β, we have following cases:

Case 2.1: Suppose α or β be a nonzero constant, we have the following:

Case 2.1.1: Let $\alpha = c$, then from (7.14), we obtain $\beta = c_1u_2 + c_2$, which satisfies (7.14) and (7.15).

Case 2.1.2: Let $\alpha = c$, then from (7.15), we obtain $\beta = c_1u_2^3 + c_2u_2^2 + c_3u_2 + c_4$, which does not satisfy (7.14).

Case 2.2: Suppose α, β are linear functions, i.e., $\alpha = c_1u_1 + c_2$, $\beta = c_1u_2 + c_2$, then (7.14) is not satisfied, where $c, c_1 \in \mathbb{R}_0, c_2, c_3, c_4 \in \mathbb{R}$.

Case 2.3: Suppose α, β are nonlinear functions, we have the following sub-cases:

Case 2.3.1: From (7.14), we get

$$\left(\frac{2\alpha'\beta'}{\alpha\beta}\right)^2 + \left(\frac{\alpha''}{\alpha} - \frac{\beta''}{\beta}\right)^2 = 0. \tag{7.16}$$

We see that (7.16) is a sum of two positive quantities equal to zero, therefore, each term must be zero itself. From the first part of (7.16), we have either $f = c$ or $g = c$, which is a contradiction to the nonlinearity of α and β.

Case 2.3.2: This is similar to Case 1.3.2.

Theorem 7.1 *Let* $\mathbf{M}$ *be a minimal factorable surface of type 1 in simply isotropic space* $\mathbb{I}_3^1$*, then by the translation and dilation of* $\mathbb{I}_3^1$*,* $\mathbf{M}$ *is of Willmore type if it is congruent to*

$$(u_1, u_2, (c)(c_1u_2 + c_2)), \ \textit{or}, \ \ (u_1, u_2, u_1u_2).$$

Theorem 7.2 *Let* $\mathbf{M}$ *be a non-minimal factorable surface of type 1 in simply isotropic space* $\mathbb{I}_3^1$*, then* $\mathbf{M}$ *is of Willmore type if it is congruent to*

$$(u_1, u_2, (c)(c_1u_2 + c_2)).$$

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Algebra

Product of Generalized Derivations with Commuting Values on a Lie Ideal

Luisa Carini, Vincenzo De Filippis, and Giovanni Scudo

Abstract Let R be a non-commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, L a non-central Lie ideal of R, F and G two nonzero generalized derivations of R. If $[F(u)G(u), u] = 0$ for all $u \in L$, then one of the following holds:

1. There exist $u, v \in U$ such that $uv \in C$ and $F(x) = xu$, $G(x) = vx$, for all $x \in R$;
2. $R \subseteq M_2(C)$.

Keywords Prime rings · Differential identities · Generalized derivations

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1 Introduction

Let R be a prime ring of characteristic different from 2, $Z(R)$ the center of R, U the Utumi quotient ring of R and $C = Z(U)$, the center of U (C is usually called the extended centroid of R). An additive map $G : R \to R$ is called generalized derivation of R if there exists a derivation d of R such that $G(xy) = G(x)y + xd(y)$, for all $x, y \in R$. The simplest example of generalized derivation is a map of the form $g(x) = ax + xb$, for some $a, b \in R$: such generalized derivations are called inner. Generalized inner derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see

L. Carini
MIFT, University of Messina, 98166 Messina, Italy
e-mail: lcarini@unime.it

V. De Filippis (✉) · G. Scudo
Department of Engineering, University of Messina, 98166 Messina, Italy
e-mail: defilippis@unime.it

G. Scudo
e-mail: gscudo@unime.it

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for example [15, 18, 20]). Here we will consider some related problems concerning identities with generalized derivations in prime rings.

In [5] it is proved that if R is a prime ring with infinite extended centroid and $d_1, \dots, d_n$ are derivations of R such that $d_1(x)d_2(x)\cdots d_n(x) = 0$, for all $x \in R$, then at least one d_i is trivial.

Later in [22] Vukman extended the result to α-derivations, in the case $n = 2$. More precisely an additive mapping $d : R \to R$ is called α-derivation if $d(xy) = d(x)\alpha(y) + xd(y)$, for all $x, y \in R$ and for a fixed automorphism α of R. In light of this definition, Theorem 3 in [22] proves that if d and g are α-derivations of R such that $d(x)g(x) = 0$ for all $x \in R$, then either $d = 0$ or $g = 0$.

Recently in [23] this result has been generalized to the case of (α, β)-derivations. We recall that an additive $d : R \to R$ is called (α, β)-derivation if $d(xy)=d(x)\alpha(y) + \beta(x)d(y)$, for all $x, y \in R$ and for fixed automorphisms α, β of R. In [23] it is proved that if d and g are (α, β)-derivations such that either d or g commutes with α and β, and $d(x)g(x) = 0$ for all $x \in R$, then either $d = 0$ or $g = 0$.

More recently in [13], M. Fošner and Vukman have considered an analogous problem, where derivations and (α, β)-derivations are replaced by generalized derivations. In [13, Theorem 3] they prove that if F_1 and F_2 are generalized derivations of a prime ring R of characteristic different from 2, such that $F_1(x)F_2(x) = 0$ for all $x \in R$, then there exist p, q elements of the Martindale quotient ring Q of R, such that $F_1(x) = xp$ and $F_2(x) = qx$ for all $x \in R$ and $pq = 0$, except when at least one F_i is zero. In [6] this last result has been extended to the case when generalized derivations act on multilinear polynomials.

Here our aim is to generalize the result in [13] to the case when the product of two generalized derivations is commuting on a Lie ideal L of R. More precisely we will prove the following:

Theorem 1 *Let R be a non-commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, L a non-central Lie ideal of R, F and G two nonzero generalized derivations of R. If $[F(u)G(u), u] = 0$ for all $u \in L$, then one of the following holds:*

1. *There exist $u, v \in U$ such that $uv \in C$ and $F(x) = xu$, $G(x) = vx$, for all $x \in R$;*
2. *$R \subseteq M_2(C)$.*

2 The Case of Inner Generalized Derivations

In this section we firstly assume there are elements $a, b, c, q, u \in R$ such that R satisfies the generalized polynomial identity

$$\Big[a[x_1, x_2]c[x_1, x_2] + a[x_1, x_2]^2 q + [x_1, x_2]u[x_1, x_2] + [x_1, x_2]b[x_1, x_2]q, [x_1, x_2]\Big]. \quad (1)$$

Here we will prove the following result:

Proposition 1 *Let R be a prime ring of characteristic different from 2. If (1) is a generalized polynomial identity for R, then one of the following holds:*

1. $a = -b \in C$ *and* $ac + u \in C$;
2. $c = -q \in C$ *and* $bq + u \in C$;
3. $a, q \in C$ *and* $bq + ac + u \in C$;
4. $R \subseteq M_2(C)$.

Then we suppose there are elements $a, b, c, q \in R$ such that R satisfies the generalized polynomial identity

$$\Big[a[x_1, x_2]c[x_1, x_2] + a[x_1, x_2]^2 q + [x_1, x_2]bc[x_1, x_2] + [x_1, x_2]b[x_1, x_2]q, [x_1, x_2]\Big]. \tag{2}$$

As a consequence of Proposition 1 we will also prove the following:

Proposition 2 *Let R be a prime ring of characteristic different from 2. If (2) is a generalized polynomial identity for R, then one of the following holds:*

1. $a = -b \in C$;
2. $c = -q \in C$;
3. $a, q \in C$ *and* $(a + b)(c + q) \in C$;
4. $R \subseteq M_2(C)$.

We begin with the following:

Lemma 2 *Let R be a prime ring and $u \in R$ such that*

$$\Big[[x_1, x_2]\big(u[x_1, x_2] + [x_1, x_2]v\big), [x_1, x_2]\Big]$$

is a generalized polynomial identity for R. Then either $u, v \in C$ or $R \subseteq M_2(C)$.

Proof It is a consequence of Lemma 4.3 in [7]. □

Lemma 3 *Let R be a prime ring and $u, v \in R$ such that*

$$\Big[\big(u[x_1, x_2] + [x_1, x_2]v\big)[x_1, x_2], [x_1, x_2]\Big]$$

is a generalized polynomial identity for R. Then either $u, v \in C$ or $R \subseteq M_2(C)$.

Proof It follows from Lemma 2.2 in [1]. □

Lemma 4 *Let R be a prime ring and $u \in R$ such that $\Big[u[r_1, r_2], [r_1, r_2]\Big] = 0$, for any $r_1, r_2 \in R$. Then $u \in Z(R)$.*

Proof It is an immediate consequence of main result in [19]. □

Lemma 5 *Let R be a prime ring of characteristic different from 2, a, b, c, q, u elements of R. Assume that $a, q \in Z(R)$. If R satisfies (1), then either $bq + ac + u \in Z(R)$ or $R \subseteq M_2(C)$.*

Proof Under these hypothesis and by (1) we have that R satisfies

$$\Big[[x_1, x_2](ac + aq + u + bq)[x_1, x_2], [x_1, x_2]\Big]$$

and by Lemma 3 we get the required conclusions. □

Lemma 6 *Let R be a prime ring of characteristic different from 2, a, b, c, q, u elements of R. Assume that $a, b \in Z(R)$. If R satisfies (1), then one of the following holds:*

1. *$a + b = 0$ and $ac + u \in Z(R)$;*
2. *$q \in Z(R)$ and $ac + u \in Z(R)$;*
3. *$R \subseteq M_2(C)$.*

Proof In this case relation (1) reduces to

$$\Big[[x_1, x_2]\big((ac + u)[x_1, x_2] + [x_1, x_2](a + b)q\big), [x_1, x_2]\Big].$$

By Lemma 2 we have $R \subseteq M_2(C)$ unless when $ac + u \in Z(R)$ and $(a + b)q \in Z(R)$. In particular, if $a + b \neq 0$, then $q \in Z(R)$. □

Lemma 7 *Let R be a prime ring of characteristic different from 2, a, b, c, q, u elements of R. Assume that $c, q \in Z(R)$. If R satisfies (1), then one of the following holds:*

1. *$c + q = 0$ and $bq + u \in Z(R)$;*
2. *$a \in Z(R)$ and $bq + u \in Z(R)$;*
3. *$R \subseteq M_2(C)$.*

Proof Here relation (1) reduces to

$$\Big[\big(a(c + q)[x_1, x_2] + [x_1, x_2](bq + u)\big)[x_1, x_2], [x_1, x_2]\Big].$$

By Lemma 3 we have $R \subseteq M_2(C)$ unless when $bq + u \in Z(R)$ and $a(c + q) \in Z(R)$. In particular, if $c + q \neq 0$, then $a \in Z(R)$. □

In the next Lemmas we study the case of ring of matrices. We start with the following:

Lemma 8 *Let K be a field of characteristic different from 2, let $R = M_m(K)$ be the algebra of $m \times m$ matrices over K, $Z(R)$ the center of R, a, b, c, q, u elements of R. Assume that $a \in Z(R)$. If $m \geq 3$ and R satisfies (1), then one of the following holds:*

1. $b \in Z(R)$*,* $a + b = 0$ *and* $ac + u \in Z(R)$*;*
2. $q \in Z(R)$ *and* $ac + bq + u \in Z(R)$*.*

Proof Since $a \in Z(R)$, relation (1) reduces to

$$\Big[[x_1, x_2](ac + u)[x_1, x_2] + a[x_1, x_2]^2 q + [x_1, x_2]b[x_1, x_2]q, [x_1, x_2]\Big]. \quad (3)$$

Let e_{ij} the usual matrix unit, with 1 in the (i, j)-entry and zero elsewhere. Say $b = \sum_{kl} b_{kl}e_{kl}$ and $q = \sum_{kl} q_{kl}e_{kl}$, for $b_{kl}, q_{kl} \in K$.

Since $e_{ij} \in [R, R]$ for all $i \neq j$, then by (3) we get

$$[e_{ij}(ac + u)e_{ij} + e_{ij}be_{ij}q, e_{ij}] = 0.$$

In particular, $e_{ij}be_{ij}qe_{ij} = 0$, which implies $b_{ji}q_{ji} = 0$, for any $i \neq j$. As an application of [10, Proposition 1], it follows either $b \in Z(R)$ or $q \in Z(R)$.

Consider the case $b \in Z(R)$ and assume $a + b \neq 0$. Then, by (3), R satisfies

$$\Big[[x_1, x_2]\big((ac + u)[x_1, x_2] + [x_1, x_2](a + b)q\big), [x_1, x_2]\Big].$$

By Lemma 2, it follows that both $ac + u \in Z(R)$ and $(a + b)q \in Z(R)$. In particular, since $0 \neq a + b \in Z(R)$, we get $q \in Z(R)$. On the other hand, if $a + b = 0$, then R satisfies

$$\Big[[x_1, x_2](ac + u)[x_1, x_2]\big), [x_1, x_2]\Big]$$

and, as above, it follows that $ac + u \in Z(R)$.

Let now $q \in Z(R)$. By (3) and since $aq \in Z(R)$, we have that R satisfies

$$\Big[[x_1, x_2](bq + ac + u)[x_1, x_2], [x_1, x_2]\Big].$$

By the same above argument, we get $bq + ac + u \in Z(R)$. □

Lemma 9 *Let* K *be a field of characteristic different from* 2*, let* $R = M_m(K)$ *be the algebra of* $m \times m$ *matrices over* K*,* $Z(R)$ *the center of* R*,* a, b, c, q, u *elements of* R*. If* $m \geq 3$ *and* R *satisfies (1), then one of the following holds:*

1. $a = -b \in Z(R)$ *and* $ac + u \in Z(R)$*;*
2. $c = -q \in Z(R)$ *and* $bq + u \in Z(R)$*;*
3. $a, q \in Z(R)$ *and* $bq + ac + u \in Z(R)$*.*

Proof Say $a = \sum_{kl} a_{kl}e_{kl}$, $c = \sum_{kl} c_{kl}e_{kl}$ and $q = \sum_{kl} q_{kl}e_{kl}$, for $0 \neq a_{kl}, c_{kl}, q_{kl} \in K$. Let i, j, k three different indices and choose $[x_1, x_2] = e_{ii} - e_{jj}$ in relation (1). Right multiplying by e_{ii} and left multiplying by e_{kk}, it follows that

$$a_{ki}(c_{ii}+q_{ii})+a_{kj}(q_{ji}-c_{ji})=0. \tag{4}$$

One can see that, for any inner automorphism φ of $M_m(K)$,

$$\Big[\varphi(a)[x_1,x_2]\varphi(c)[x_1,x_2]+\varphi(a)[x_1,x_2]^2\varphi(q)+[x_1,x_2]\varphi(u)[x_1,x_2]+ \\ +[x_1,x_2]\varphi(b)[x_1,x_2]\varphi(q),[x_1,x_2]\Big] \tag{5}$$

is a generalized polynomial identity for R. In particular, let $\varphi(x)=(1+e_{ik})x(1-e_{ik})$, for any $x\in R$. If we denote $\varphi(a)=\sum_{kl}a'_{kl}e_{kl}$, $\varphi(c)=\sum_{kl}c'_{kl}e_{kl}$ and $\varphi(q)=\sum_{kl}q'_{kl}e_{kl}$, for $a'_{kl},c'_{kl},q'_{kl}\in K$, and using relation (4), it follows that

$$a'_{ki}(c'_{ii}+q'_{ii})+a'_{kj}(q'_{ji}-c'_{ji})=0$$

that is

$$a_{ki}(c_{ki}+q_{ki})=0. \tag{6}$$

As above, by [10, Proposition 1], it follows either $a\in Z(R)$ or $c+q\in Z(R)$. In the first case we conclude by Lemma 8. Thus we may assume $a\notin Z(R)$ and $c+q\in Z(R)$.

Now we consider the automorphism $\chi(x)=(1+e_{jk})x(1-e_{jk})$, for any $x\in R$ and denote $\chi(a)=\sum_{kl}a''_{kl}e_{kl}$, $\chi(c)=\sum_{kl}c''_{kl}e_{kl}$ and $\chi(q)=\sum_{kl}q''_{kl}e_{kl}$, for $a''_{kl},c''_{kl},q''_{kl}\in K$. Using again relation (4), it follows that

$$a''_{ki}(c''_{ii}+q''_{ii})+a''_{kj}(q''_{ji}-c''_{ji})=0$$

that is

$$a_{kj}(q_{ki}-c_{ki})=0. \tag{7}$$

Once again, by applying [10, Proposition 1], and since we assume $a\notin Z(R)$, it follows $q-c\in Z(R)$. Therefore both $q-c\in Z(R)$ and $q+c\in Z(R)$, that is, $c,q\in Z(R)$. In this case the conclusion follows from Lemma 7. □

Lemma 10 *Let R be a prime ring of characteristic different from 2. If (1) is a trivial generalized polynomial identity for R, then one of the following holds:*

1. *$a=-b\in C$ and $ac+u\in C$;*
2. *$c=-q\in C$ and $bq+u\in C$.*
3. *$a,q\in C$ and $bq+ac+u\in C$.*

Proof Since R and U satisfy the same generalized polynomial identities [3, Theorem 2], we have that (1) is satisfied by U. Since (1) is a trivial generalized polynomial identity for U, either $\{a,1\}$ is a linearly C-dependent set or U satisfies

$$[x_1,x_2]\Big(c[x_1,x_2]+[x_1,x_2]q\Big)[x_1,x_2]. \tag{8}$$

In this last case, since (8) is trivial, it follows $q = -c \in C$. Therefore, by relation (1), U satisfies

$$\Big[[x_1, x_2](u + bq)[x_1, x_2], [x_1, x_2]\Big].$$

Since it is a trivial GPI for U, the element $bq + u$ must fall in the extended centroid C.

On the other hand, if $\{a, 1\}$ is a linearly C-dependent set, then $a \in C$ and (1) reduces to

$$\Big[[x_1, x_2]ac[x_1, x_2] + [x_1, x_2]^2aq + [x_1, x_2]u[x_1, x_2] + [x_1, x_2]b[x_1, x_2]q, [x_1, x_2]\Big]. \tag{9}$$

Again, since (9) is a trivial generalized polynomial identity for U, either $\{q, 1\}$ are linearly C-dependent or U satisfies

$$[x_1, x_2]^2\Big([x_1, x_2](a + b)[x_1, x_2]q\Big). \tag{10}$$

In the last case, since (10) is trivial, it follows $a + b \in C$ (that is $b \in C$) and $(a + b)q = 0$. Hence, by (9), we have that

$$\Big[[x_1, x_2](ac + u), [x_1, x_2]\Big][x_1, x_2] \tag{11}$$

is a trivial generalized polyonomial identity for U. This implies $ac + u \in C$.

Thus we may assume that $\{q, 1\}$ are linearly C-dependent, that is, $q \in C$. Under this assumption, and by (9), one has that U satisfies

$$[x_1, x_2](ac + u + bq)[x_1, x_2]^2 - [x_1, x_2]^2(ac + u + bq)[x_1, x_2]. \tag{12}$$

Finally, again since (12) is trivial, we have that $ac + u + bq \in C$. □

Proof of Proposition 1 Firstly we notice that the Proposition is trivially true if one of the following holds:

- $a, b \in C$ (see Lemma 6);
- $c, q \in C$ (see Lemma 7);
- $a, q \in C$ (see Lemma 5).

Hence, for the rest of the proof we suppose that the following hold simultaneously:

- either $a \notin C$ or $b \notin C$;
- either $c \notin C$ or $q \notin C$;
- either $a \notin C$ or $q \notin C$.

Our aim is to prove that a number of contradiction follow.

Notice that, by Lemma 10 we may assume that (1) is a non-trivial generalized polynomial identity for R. By a theorem due to Beidar [3, Theorem 2] this generalized polynomial identity is also satisfied by U. In case C is infinite, we obtain that $U \bigotimes_C \overline{C}$ satisfies (3), where $\overline{C}$ is the algebraic closure of C. Since both U and $U \bigotimes_C \overline{C}$ are centrally closed [11, Theorems 2.5 and 3.5], we may replace R by either U or $U \bigotimes_C \overline{C}$ according to whether C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem [21], R is a primitive ring having a nonzero socle with C as the associated division ring. In light of Jacobson's theorem [14, page 75] R is isomorphic to a dense ring of linear transformations on some vector space V over C. If $dim_C V = k$ is finite, then $R \cong M_k(C)$, the ring of $k \times k$ matrice over C, the conclusion follows from Lemma 9.

Thus we assume $dim_C V = \infty$.

Notice that eUe satisfies (1), for all $e^2 = e \in Soc(U) = H$. By the above remark, we have that

- one of a, b doesn't centralize the nonzero ideal H of R;
- one of c, q doesn't centralize H;
- one of a, q doesn't centralize H;

Thus there exist $h_1, h_2, h_3 \in H$ such that

- either $[a, h_1] \neq 0$ or $[b, h_1] \neq 0$;
- either $[c, h_2] \neq 0$ or $[q, h_2] \neq 0$;
- either $[a, h_3] \neq 0$ or $[q, h_3] \neq 0$.

Moreover, because of the infinite dimensionality, H does not satisfy the polynomial $s_4(x_1, \ldots, x_4)$, that is, there exist $t_1, t_2, t_3, t_4 \in H$ such that $s_4(t_1, \ldots, t_4) \neq 0$. By Litoff's theorem in [12] there exists $e^2 = e \in H$ such that

$$ah_1, h_1a, bh_1, h_1b, ch_2, h_2c, qh_2, h_2q, ah_3, h_3a, qh_3, h_3q \in eUe$$
$$\text{and also} \quad t_1, t_2, t_3, t_4 \in eUe$$

moreover eUe is a central simple algebra finite dimensional over its center. Since $s_4(t_1, \ldots, t_4) \neq 0$, then $eUe \cong M_m(C)$, for $m \geq 3$. By (1) we know that

$$\Big[(eae)[x_1, x_2](ece)[x_1, x_2] + (eae)[x_1, x_2]^2(eqe) + [x_1, x_2](eue)[x_1, x_2] + [x_1, x_2](ebe)[x_1, x_2](eqe), [x_1, x_2]\Big] \tag{13}$$

is a generalized polynomial identity for eUe, then by the finite dimensional case, we have that one of the following holds:

1. $eae, ebe \in Ce$;
2. $ece, eqe \in Ce$;
3. $eae, eqe \in Ce$.

Thus one of the following is a contradiction:

$$ah_1 = eah_1 = eaeh_1 = h_1eae = h_1ae = h_1a$$

$$bh_1 = ebh_1 = ebeh_1 = h_1ebe = h_1be = h_1b$$

$$ch_2 = ech_2 = eceh_2 = h_2ece = h_2ce = h_2c$$

$$qh_2 = eqh_2 = eqeh_2 = h_2eqe = h_2qe = h_2q$$

$$ah_3 = eah_3 = eaeh_3 = h_3eae = h_3ae = h_3a$$

$$qh_3 = eqh_3 = eqeh_3 = h_3eqe = h_3qe = h_3q.$$

We are ready to prove the main result of this Section:

Proof of Proposition 2 Here we simply apply the conclusion of Proposition 1 to the relation (2). Thus one of the following holds:

1. $a = -b \in C$ and $ac + bc \in C$;
2. $c = -q \in C$ and $bq + bc \in C$;
3. $a, q \in C$ and $bq + ac + bc \in C$;
4. $R \subseteq M_2(C)$,

that is, one of the following holds:

1. $a = -b \in C$;
2. $c = -q \in C$;
3. $a, q \in C$ and $(a + b)(c + q) \in C$;
4. $R \subseteq M_2(C)$.

as required.

3 The Main Theorem

In order to prove our main Theorem, we need to recall some well known results in literature concerning the relationship between the differential identities satisfied by a prime ring R and the ones satisfied by its Utumi quotient ring U. More precisely, it is well known that both every derivation and every generalized derivation of R can be uniquely extended to the Utumi quotient ring U. Moreover any generalized derivation g of R assumes the form $g(x) = qx + d(x)$ for some $q \in U$ and d a derivation on U ([4, Proposition 2.5.1], [18, Theorem 3]).

In this sense, if $\Phi(x_i^{\Delta_j})$ is a differential identity for R having coefficients in U, where every Δ_j is a derivation word of the form $\Delta_j = d_{j1}d_{j2}\dots d_{jm}$ and each d_{ji} is a derivation of R, then $\Phi(x_i^{\Delta_j})$ is a differential identity for U ([17]).

We also recall that R and U satisfy the same generalized polynomial identity with coefficients in U [8].

For a complete description of the theory of generalized polynomial identities and differential identities we refer the reader to [4, Chap. 7].

Proof of Theorem 1 In light of previous remarks, we assume

$$F(x) = ax + d(x) \quad \text{and} \quad G(x) = cx + \delta(x)$$

where $a, c \in U$ and d, δ are derivations of U. Moreover, we may assume that

$$[F(X) \cdot G(X), X] = 0 \quad \text{for all} \quad X = [x_1, x_2] \in [U, U]. \tag{14}$$

In order to prove our result, we have to examine the following three cases:

Case 1: We firstly assume that d and δ are inner derivations of U. Thus there exist $b, q \in U$ such that $d(x) = [b, x]$, $\delta(x) = [q, x]$, $F(x) = ax + [b, x] = (a + b)x - xb$ and $G(x) = cx + [q, x] = (c + q)x - xq$. Let $a' = a + b$, $b' = -b$, $c' = c + q$ and $q' = -q$, so that $F(x) = a'x + xb'$ and $G(x) = c'x + xq'$, for all $x \in U$. Thus we have that, for all $X = [x_1, x_2] \in [U, U]$,

$$\left[\left(a'X + Xb'\right)\left(c'X + Xq'\right), X\right] = 0.$$

Application of Proposition 2 implies that one of the following holds:

1. $a' = -b' \in C$, that is, $F = 0$, a contradiction;
2. $c' = -q' \in C$, that is, $G = 0$, a contradiction;
3. $a', q' \in C$ and $(a' + b')(c' + q') \in C$. This implies $F(x) = xu$ and $G(x) = vx$, for all $x \in R$, for some $u, v \in U$, where $uv \in C$.
4. $R \subseteq M_2(C)$

Case 2: Let now d and δ be linearly C-independent modulo U-inner derivations. Since U satisfies

$$\Big[\Big(a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)]\Big) \cdot \Big(c[x_1, x_2] + [\delta(x_1), x_2] + [x_1, \delta(x_2)]\Big), [x_1, x_2]\Big]$$

and applying Kharchenko's theory in [16], we have that U satisfies

$$\Big[\Big(a[x_1, x_2] + [y_1, x_2] + [x_1, y_2]\Big) \cdot \Big(c[x_1, x_2] + [z_1, x_2] + [x_1, z_2]\Big), [x_1, x_2]\Big].$$

Fix $y_1 = 0$, then U satisfies the blended component

$$\left[[y_1, x_2] \cdot \left(c[x_1, x_2] + [z_1, x_2] + [x_1, z_2]\right), [x_1, x_2]\right].$$

For $z_1 = 0$, consider again the blended component, so that

$$\left[[y_1, x_2] \cdot [z_1, x_2], [x_1, x_2]\right] = 0$$

for all $x_1, x_2, y_1, z_1 \in U$. Since U satisfies a polynomial identity, there exist a field K and an integer m such that the ring of all matrices $M_m(K)$ satisfies the same polynomial identities. Since U is not commutative, we may assume $m > 1$. For

$$x_1 = e_{12},\ z_1 = e_{12},\ y_1 = e_{22},\ x_2 = e_{21},$$

we obtain $2e_{21} = 0$, which is a contradiction, since $char(R) \neq 2$.

Case 3: Finally we suppose that

- d and δ are not both inner derivations (otherwise we are done by Case 1);
- d and δ are C-dependent modulo U-inner derivations (otherwise we are done by Case 2).

Therefore there exist $\alpha, \beta \in C$ and $p \in U$ such that $\alpha d(x) + \beta\delta(x) = [p, x]$. In this case we will prove that a number of contradictions occurs, unless when $R \subseteq M_2(C)$.

We firstly consider the case $\alpha = 0$. Thus $\delta(x) = [q, x]$, for all $x \in U$, where $q = \beta^{-1} p$ and d is not an inner derivation (if not we are done by Case 1). Hence by (14) we have that U satisfies

$$\left[\left(a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)]\right) \cdot \left((c + q)[x_1, x_2] - [x_1, x_2]q\right), [x_1, x_2]\right]$$

Since d is not inner, by [16], U satisfies

$$\left[\left(a[x_1, x_2] + [y_1, x_2] + [x_1, y_2]\right) \cdot \left((c + q)[x_1, x_2] - [x_1, x_2]q\right), [x_1, x_2]\right]$$

and in particular, U satisfies

$$\left[[y_1, x_2] \cdot \left((c + q)[x_1, x_2] - [x_1, x_2]q\right), [x_1, x_2]\right]$$

For $y_1 = x_1$, U satisfies

$$[x_1, x_2] \cdot \left[(c + q)[x_1, x_2] - [x_1, x_2]q, [x_1, x_2]\right] \tag{15}$$

By applying Lemma 5 in [2], either $R \subseteq M_2(C)$ or one of the following holds:

1. $c = 0$, $q \in C$, which implies $G = 0$;
2. $c, q \in C$ and $[x_1, x_2]^2$ is central valued on U. By [9], it follows that either $c = 0$ (so that $G = 0$) or $[x_1, x_2]^2$ is an identity for U.

In any case we get a contradiction.

The case $\beta = 0$ is fully in line with that of $\alpha = 0$, so that we omit it for brevity.

As a consequence we may consider both $\alpha \neq 0$ and $\beta \neq 0$. The derivation δ assumes the form $\delta(x) = [q, x] + \gamma d(x)$ for all $x \in U$, where $q = \beta^{-1}p$ and $\gamma = -\beta^{-1}\alpha \neq 0$. We also remark that d cannot be inner. Therefore by (14), U satisfies

$$\Big[\Big(a[x_1, x_2] + [y_1, x_2] + [x_1, y_2]\Big) \cdot \Big((c + q)[x_1, x_2] - [x_1, x_2]q + \gamma[y_1, x_2] + \gamma[x_1, y_2]\Big), [x_1, x_2]\Big]$$

For $y_1 = 0$, U satisfies the blended component

$$\Big[\gamma\Big(a[x_1, x_2] + [x_1, y_2]\Big) \cdot [y_1, x_2] + [y_1, x_2] \cdot \Big((c + q)[x_1, x_2] - [x_1, x_2]q + \gamma[y_1, x_2] + \gamma[x_1, y_2]\Big), [x_1, x_2]\Big]$$

Analogously, for $y_2 = 0$, U satisfies the blended component

$$\gamma\Big[[x_1, y_2][y_1, x_2] + [y_1, x_2][x_1, y_2], [x_1, x_2]\Big]. \tag{16}$$

Since U satisfies a polynomial identity, there exists an integer m such that $M_m(K)$, the ring of all matrices over a suitable field K, satisfies the same polynomial identities. Notice that, if $m \leq 2$, then U satisfies the standard identity $s_4(x_1, \ldots, x_4)$. In other words, we get $U \subseteq M_2(C)$, as required. Hence we may suppose $m \geq 3$ and choose

$$x_1 = e_{12}, \ y_2 = e_{23}, \ y_1 = e_{22}, \ x_2 = e_{21}.$$

Thus, by (16) we obtain the contradiction

$$0 = \gamma[e_{23}, e_{11} - e_{22}] = \gamma e_{23} \neq 0.$$

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Some Extension Theorems in the Ring of Quotients of $*$-Prime Rings

Clause Haetinger, Mohammad Aslam Siddeeque, and Mohammad Ashraf

Abstract Let R be a semiprime ring with an involution '$*$'. Let Q_{mr} and Q_s denote its right Utumi quotient ring and right symmetric Martindale quotient ring, respectively. In the present paper, the following extension problems have been obtained: (i) an involution of a semiprime ring can be uniquely extended to its right symmetric Martindale quotient ring; (ii) if R is a $*$-prime ring, then so is its right symmetric Martindale quotient ring; (iii) every $*$-derivation of a commutative semiprime ring can be uniquely extended to its right symmetric Martindale quotient ring. Finally, we have also discussed C-dependence of any two nonzero elements of right symmetric Martindale quotient ring of $*$-prime ring R, where C is the extended centroid of R.

Keywords Right Utumi quotient ring · Right symmetric Martindale quotient ring · $*$-prime ring · C-dependence and independence

Mathematics Subject Classification (2010): 16N60 · 16W10 · 16W25

1 Introduction

Throughout the paper, unless otherwise stated, R will represent a semiprime ring with center Z. R is called a prime ring if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. It is called semiprime if $xRx = \{0\}$ implies $x = 0$. An additive mapping $x \mapsto x^*$ of R into itself is called an involution on R if it satisfies the conditions: (i) $(x^*)^* = x$,

C. Haetinger
Center of Exact and Technological Sciences, Vale do Taquari University- Univates, Lajeado, RS 95914-014, Brazil
e-mail: chaet@univates.br

M. A. Siddeeque (✉) · M. Ashraf
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
e-mail: aslamsiddeeque@gmail.com

M. Ashraf
e-mail: mashrf80@hotmail.com

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(ii) $(xy)^* = y^*x^*$ for all $x, y \in R$. A ring R equipped with an involution '$*$' is called a ring with involution or a $*$-ring. A ring R with involution '$*$' is said to be $*$-prime if $aRb = aRb^* = \{0\}$, where $a, b \in R$ (equivalently $aRb = a^*Rb = \{0\}$) implies that either $a = 0$ or $b = 0$. It is to be noted that every prime ring having an involution '$*$' is $*$-prime but the converse is not true in general. Of course, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. Let R be a $*$-ring, then an additive mapping $d : R \longrightarrow R$ is said to be a $*$-derivation on R if $d(xy) = d(x)y^* + xd(y)$ holds for all $x, y \in R$. Let R be a $*$-prime ring, $a \in R$ and $aRa = \{0\}$. This implies that $aRaRa^* = \{0\}$ also. Now $*$-primeness of R insures that $a = 0$ or $aRa^* = \{0\}$. $aRa^* = \{0\}$ together with $aRa = \{0\}$ gives us $a = 0$. Thus we conclude that every $*$-prime ring is a semiprime ring.

A right ideal I of R is called a dense right ideal if given any $0 \neq r_1 \in R, r_2 \in R$ there exists $r \in R$ such that $r_1r \neq 0$ and $r_2r \in I$. Similarly, a dense left ideal can also be defined in an analogous fashion. An ideal I of R is called a dense ideal if it is both a dense left ideal and a dense right ideal. The collection of all dense right ideals of R will be denoted by $\mathcal{D} = \mathcal{D}(R)$. We set $\mathcal{I} = \mathcal{I}(R)$=$\{I \mid I$ is an ideal of R and $l(I) = \{0\}\}$, where $l(I)$ denotes the left annihilator of the ideal I in the ring R. It is obvious to observe that $\mathcal{I}$ consists of precisely the dense ideals of R. Let Q_{mr} and Q_s denote right Utumi quotient ring and right symmetric Martindale quotient ring of R, respectively. Let us review some important facts about the rings Q_{mr} and Q_s (see [1] for details). The ring Q_{mr} can be characterized by the following four properties:

(i) R is a subring of Q_{mr};
(ii) For all $q \in Q_{mr}$, there exists $J \in \mathcal{D}$ such that $qJ \subseteq R$;
(iii) For all $q \in Q_{mr}$ and $J \in \mathcal{D}$, $qJ = \{0\}$ if and only if $q = 0$;
(iv) If $J \in \mathcal{D}$ and $f : J_R \longrightarrow R_R$ is a homomorphism of right R-modules, then there exists $q \in Q_{mr}$ such that $f(x) = qx$ for all $x \in J$.

The ring Q_s can be characterized by the following four properties:

(i) R is a subring of Q_s;
(ii) For all $q \in Q_s$, there exists $J \in \mathcal{I}$ such that $qJ \cup Jq \subseteq R$;
(iii) For all $q \in Q_s$ and $J \in \mathcal{I}$, $qJ = \{0\}$ (or $Jq = \{0\}$) if and only if $q = 0$;
(iv) If $J \in \mathcal{I}$, $f : J_R \longrightarrow R_R$ and $g :_R J \longrightarrow_R R$ are homomorphism of right R-modules and homomorphism of left R-modules, respectively, such that $xf(y) = g(x)y$ for all $x, y \in J$, then there exists $q \in Q_s$ such that $f(x) = qx$ and $g(x) = xq$ for all $x \in J$.

Further, it is well known that Q_s=$\{q \in Q_{mr} \mid qJ \cup Jq \subseteq R$ for some $J \in \mathcal{I}\}$. Let $q_1, q_2, \ldots, q_n \in Q_{mr}$, then the set $T = \{q_1, q_2, \ldots, q_n\}$ is called C-dependent if there exist $c_1, c_2, \ldots, c_n \in C$ not all zero such that $c_1q_1 + c_2q_2 + \cdots + c_nq_n = 0$. On the other hand if T is not C-dependent, then it is called C-independent. It is also well known that C is a field if R is prime and on the other hand if C of a semiprime R is a field, then R must be prime. Further, it is to be noted that if R is a prime ring then two nonzero elements $q_1, q_2 \in Q_{mr}$ will be C-dependent if and only if $q_1 = \lambda q_2$

for some $\lambda \in C$. In the present paper, we have proved a sufficient condition under which two nonzero elements of Q_s become C-dependent if Q_s is the right symmetric Martindale quotient ring of a $*$-prime ring R.

There has been a great deal of work on extension problems of a semiprime ring R to its different types of quotient rings, i.e., Q_{mr} and Q_s, etc. For example, we know the following: (i) an automorphism (resp. antiautomorphism) of a semiprime ring R can be uniquely extended to Q_{mr} and Q_s (resp. Q_s). (ii) Let d be a derivation of a semiprime ring R, then d can be uniquely extended to Q_{mr} and Q_s. (iii) Let R be a prime ring with involution '$*$', then '$*$' can be uniquely extended to an involution of its right symmetric Martindale quotient ring. (iv) Let R be a prime (resp. semiprime) ring, then so are its quotient rings Q_{mr} and Q_s (See [1, 3] for further details).

Motivated by the above nice extensions, we have obtained some possible analogous for $*$-prime rings as follows: (i) an involution of a semiprime ring can be uniquely extended to its right symmetric Martindale quotient ring; (ii) if R is a $*$-prime ring, then so is its right symmetric Martindale quotient ring; (iii) every $*$-derivation of a $*$-prime ring can be uniquely extended to its right symmetric Martindale quotient ring.

2 Preliminary Results

We begin with the following lemmas which are essential for developing the proof of our main results. The proof of the Lemma 2.1 can be found in ([1], Theorem 2.3.3).

Lemma 2.1 *Let R be a semiprime ring, $Q = Q_{mr}(R)$, $C = Z(Q)$ and $q_1, q_2, \ldots, q_n \in Q$. Suppose that $q_1 \notin \sum_{i=2}^{n} Cq_i$. Then there exists an element $p = \sum_{i=1}^{m} l_{a_i} r_{b_i} \in R_{(l)} R_{(r)}$ such that $q_1 p = \sum_{i=1}^{m} a_i q_1 b_i \neq 0$ and $q_j p = 0$ for $j \geq 2$. Here $R_{(l)}$ (resp. $R_{(r)}$) denotes the subring of $End_C(Q)$ generated by all left (resp., by all right) multiplications by elements of R, where $End_C(Q)$ denotes the ring of all homomorphisms of Q as left-C modules.*

In the year 1989, M. Bresǎr and J.Vukman ([2], Proposition 1) proved that if a prime $*$-ring R admits a nonzero $*$-derivation, then R is commutative. We have shown that this result holds even for $*$-prime rings. In fact, we have obtained the following.

Lemma 2.2 *Let R be a $*$-prime ring. If it admits a nonzero $*$-derivation d, then R is commutative.*

Proof By hypothesis we have, for all $x, y, z \in R$

$$\begin{aligned} d((xy)z) &= d(xy)z^* + xyd(z) \\ &= \{d(x)y^* + xd(y)\}z^* + xyd(z) \\ &= d(x)y^*z^* + xd(y)z^* + xyd(z). \end{aligned}$$

Also

$$\begin{aligned} d(x(yz)) &= d(x)(yz)^* + xd(yz) \\ &= d(x)z^*y^* + x\{d(y)z^* + yd(z)\} \\ &= d(x)z^*y^* + xd(y)z^* + xyd(z). \end{aligned}$$

We get

$d(x)y^*z^* = d(x)z^*y^*$ for all $x, y, z \in R$.

Putting y^* and z^* in the places of y and z, respectively, we find that

$$d(x)yz = d(x)zy. \tag{2.1}$$

Now replacing y by yr where $r \in R$, in the relation (2.1) and using it again we arrive at $d(x)yrz = d(x)yzr$, i.e.,

$$d(x)R[r, z] = \{0\} \tag{2.2}$$

for all $x, z, r \in R$. Replacing r and z by r^* and z^*, respectively, in the relation (2.2), we also obtain that

$$d(x)R[r, z]^* = \{0\} \tag{2.3}$$

for all $x, z, r \in R$. Since $d \neq 0$ and R is a $*$-prime ring, using the relations (2.2) and (2.3), we conclude that $rz = zr$ for all $z, r \in R$. Therefore, R is commutative.

The following example demonstrates that the $*$-primeness in the hypothesis of Lemma 2.2 cannot be omitted.

Example 2.1 Let $\mathbb{H}$ and $\mathbb{C}$ be the ring of real quaternions and complex numbers, respectively. Assume $R = \mathbb{H} \times \mathbb{C}$ is the ring of Cartesian product of $\mathbb{H}$ and $\mathbb{C}$ with regard to componentwise addition and multiplication. Let $*_1$, $*_2$ and $*$ denote the involutions of rings $\mathbb{H}$, $\mathbb{C}$ and R, respectively, defined by $q^{*_1} = \alpha - \beta\imath - \gamma\jmath - \delta k$, where $q = \alpha + \beta\imath + \gamma\jmath + \delta k \in \mathbb{H}$; $z^{*_2} = x - \imath y$, where $z = x + \imath y \in \mathbb{C}$ and $(q, z)^* = (q^{*_1}, z^{*_2})$ for all $(q, z) \in R$. Define $d : R \longrightarrow R$ such that $d(q, z) = (0, \eta(z - z^{*_2}))$ where η is any fixed complex number. It can be easily verified that R is a semiprime ring but not a $*$-prime ring and d is a nonzero $*$-derivation of R. However, R is not commutative.

If R is a prime ring with involution $*$, then we know that R is a $*$-prime ring. Using this fact, we get the following:

Corollary 2.1 ([2], Proposition 1) *If a prime $*$-ring R admits a nonzero $*$-derivation, then R is commutative.*

3 Main Results

In the year 1989, C. L. Chung ([3], Theorem (Kharchenko)) proved that if a prime ring R is endowed with involution '$*$', then '$*$' can be uniquely extended to an involution of its left symmetric Martindale quotient ring. We have shown that this extension is possible even in semiprime ring with involution. In fact we obtained the following:

Theorem 3.1 *Let R be a semiprime ring with involution '$*$'. Then '$*$' can be uniquely extended to an involution of its right symmetric Martindale quotient ring.*

Proof Since R is a semiprime ring, Q_{mr} and Q_s will exist. We will also denote the extension of '$*$', the involution of R to $Q_s = Q$ by the same '$*$'. Let $q \in Q$. This implies that $q \in Q_{mr}$ and there exists $I \in \mathcal{I}$ such that $qI \cup Iq \subseteq R$. It is easy to see that I^* is also a dense ideal, and therefore $I^* \in \mathcal{I}$. Now we define a relation $f : I^* \longrightarrow R$ such that $f(i^*) = (iq)^*$. It is easy to check that f is a well defined map and in addition it is a homomorphism of right R-modules. Therefore $[f; I^*] \in Q_{mr}$. Let us say $q^* = [f; I^*]$. Consider $q^*i^* = [f; I^*][l_{i^*}; R] = [fl_{i^*}; l_{i^*}^{-1}(I^*)] = [l_{(iq)^*}; R] = (iq)^*$ for all $i \in I$. Also consider $i^*q^* = [l_{i^*}; R][f; I^*] = [l_{i^*}f; f^{-1}(R)] = [l_{i^*}f; I^*] = [l_{(qi)^*}; R] = (qi)^*$ for all $i \in I$. Now we obtain the following two relations:

$$q^*i^* = (iq)^* \tag{3.1}$$

for all $i \in I$ and

$$i^*q^* = (qi)^* \tag{3.2}$$

for all $i \in I$. From the above two relations, it is clear that $q^*I^* \cup I^*q^* \subseteq R$. Therefore $q^* \in Q$.

Next we define a mapping $q \mapsto q^*$ of Q into itself, where $q^* = [f; I^*]$. We will prove that this is our required unique extension of involution '$*$' of R. Let $q_1, q_2 \in Q$. This implies that $q_1 + q_2 \in Q$. There exists a dense ideal J of R, i.e., $J \in \mathcal{I}$ such that $q_1J \cup Jq_1$, $q_2J \cup Jq_2$, $(q_1 + q_2)J \cup J(q_1 + q_2)$ are all contained in R. It is obvious that relations (3.1) and (3.2) will be true if we replace q by q_1, q_2 or $(q_1 + q_2)$ and I by J. Therefore for all $j \in J$, we have $(q_1 + q_2)^* j^* = (j(q_1 + q_2))^* = (jq_1 + jq_2)^* = (jq_1)^* + (jq_2)^* = (q_1^* + q_2^*)j^*$. Finally, we arrive at $\{(q_1 + q_2)^* - (q_1^* + q_2^*)\}J^* = \{0\}$. Since $J^* \in \mathcal{I}$, by characterization of Q_s we conclude that $(q_1 + q_2)^* = q_1^* + q_2^*$ showing that '$*$' is an additive map. Let $q_1, q_2 \in Q$. This implies that $q_1q_2 \in Q$. There exists a dense ideal K of R, i.e., $K \in \mathcal{I}$ such that $q_1K \cup Kq_1, q_2K \cup Kq_2, (q_1q_2)K \cup K(q_1q_2)$ are all contained in R and let $L = K^2$. Then $q_1L, Lq_1, q_2L, Lq_2 \subseteq K$. It is obvious that $L \in \mathcal{I}$. Like above for all $l \in L$, we have $(q_1q_2)^*l^* = (lq_1q_2)^* = q_2^*(lq_1)^* = q_2^*q_1^*l^*$. This implies that $(q_1q_2)^* = q_2^*q_1^*$. Operating '$*$' on both sides of relation (3.1), we obtain that $i(q^*)^* = iq$ for all $i \in I$ since '$*$' is the involution of R. Now we arrive at $I\{(q^*)^* - q\} = \{0\}$. Since $I \in \mathcal{I}$ and $\{(q^*)^* - q\} \in Q$, we conclude that $(q^*)^* = q$. Including all the above arguments, we obtain that '$*$' is an involution of Q.

Finally, we have to prove that this extension is unique. Let us suppose that 'ϕ' and '$*$' be two extensions of the involution of R. From the above arguments, it is clear that for any $q \in Q$, there exists $I \in \mathcal{I}$ such that $qI \cup Iq \subseteq R$. It is obvious that for all $i \in I$, $qi \in R$. Using the fact that $r^\phi = r^*$ for all $r \in R$, we obtain that $(qi)^\phi = (qi)^*$ for all $i \in I$. This implies that $i^\phi q^\phi = i^* q^*$ for all $i \in I$. Now we conclude that $I^*(q^\phi - q^*) = \{0\}$. But $I^* \in \mathcal{I}$, therefore using the characterization of Q, we arrive at $q^* = q^\phi$ for all $q \in Q$, hence this is a unique extension.

It is well known that if R is a prime (resp. semiprime) ring, then its quotient rings Q_{mr} and Q_s are also prime (resp. semiprime). Let R be a $*$-prime ring, then it is natural to investigate the $*$-primeness nature of its quotient rings. Of course, we proved the following:

Theorem 3.2 *Let R be a $*$-prime ring. Then its right symmetric Martindale quotient ring is also a $*$-prime ring.*

Proof Since R is a $*$-prime ring, it must be a semiprime ring also. Therefore, its right symmetric Martindale quotient ring Q_s will exist. By the above theorem, it is clear that involution '$*$' of R can be uniquely lifted to an involution of Q_s. Therefore, we can assume that '$*$' is defined on whole of Q_s. Finally, we conclude that Q_s is a $*$-ring. Now we have to prove that $Q = Q_s$ is also a $*$-prime ring. Suppose that $q_1, q_2 \in Q$ such that $q_1 Q q_2 = \{0\}$ and $q_1 Q q_2^* = \{0\}$, then we have to prove that either $q_1 = 0$ or $q_2 = 0$. Suppose on contrary that $q_1 \neq 0$ and $q_2 \neq 0$. There exist dense ideals $J_1, J_2 \in \mathcal{I}$ such that $q_1 J_1 \cup J_1 q_1 \subseteq R$ and $q_2 J_2 \cup J_2 q_2 \subseteq R$. By characterization of Q_s, we have $x \in J_1$ and $y \in J_2$ such that $0 \neq q_1 x \in R$ and $0 \neq q_2 y \in R$. But now by using hypothesis, we have $(q_1 x)R(q_2 y) = \{0\}$ and $(q_1 x)R(q_2 y)^* = \{0\}$. Contradicting the fact that R is a $*$-prime ring. Finally, we conclude that Q is a $*$-prime ring.

Theorem 3.3 *Let R be a commutative semiprime ring with involution '$*$' admitting a $*$-derivation d. Then d can be uniquely extended to a $*$-derivation of its right symmetric Martindale quotient ring.*

Proof Since R is a commutative semiprime ring, its right symmetric Martindale quotient ring $Q = Q_s$ will exist and will also be commutative. For this case, we will also have $Q_{mr} = Q_s$. By Theorem 3.1, involution '$*$' of R can be uniquely extended to an involution of Q. Therefore, we can assume that '$*$' is defined on whole Q. We shall let d also denote its extension to Q. $d(q)$, where $q \in Q$ will be denoted by q^d.

Given any $q \in Q$. This implies that $q \in Q_{mr}$ and there exists $J \in \mathcal{I}$ such that $qJ \cup Jq \subseteq R$. It is also obvious that J is a dense right ideal of R. Now we set $J_d = \sum_{x \in J} x\{(x^d : J)_R\}^*$. Since $x^d \in R$, $(x^d : J)_R$ is a dense right ideal. Here '$*$' is an automorphism of R; therefore, $\{(x^d : J)_R\}^*$ is also a dense right ideal of R. Next we claim that J_d is a dense right ideal of R. It is obvious to observe that J_d is a right ideal of R. Let $0 \neq r_1, r_2 \in R$. Since J is a dense right ideal of R, $0 \neq r_1 s$ and $r_2 s \in J$ for some $s \in R$. As we already know that $\{((r_2 s)^d : J)_R\}^*$ is a dense right ideal of R. Therefore, $0 \neq r_1 st$ for some $t \in \{((r_2 s)^d : J)_R\}^*$, it is due to the fact that the left annihilator of any dense right ideal in a semiprime ring vanishes. Clearly $r_2st \in J_d$ and so our claim stands proved. Also we observe that $J_d \subseteq J$ and $(J_d)^d \subseteq J$. Since J_d is a dense right ideal of R, $(J_d)^*$ is also a dense right ideal of R. We define $f : (J_d)^* \longrightarrow R$ by the rule $f(x^*) = (qx)^d - qx^d$ for all $x^* \in (J_d)^*$. It is easy to see that f is a well defined map and additive also. For all $x^* \in (J_d)^*$ and $r \in R$, we have $f(x^*r) = f(xl)^* = (qxl)^d - q(xl)^d = (qx)^d l^* + qxl^d - qx^d l^* - qxl^d = (qx)^d l^* - qx^d l^* = \{(qx)^d - qx^d\}l^* = f(x^*)r$; where $r = l^*$ for some $l \in R$. Arguments given above show that f is a homomorphism of right R-modules. Therefore $[f; (J_d)^*] \in Q_{mr}$. Now we put $q^d = [f; (J_d)^*]$. Due to commutativity of Q, it is trivial to see that $(J_d)^* \in \mathcal{I}$ and $q^d(J_d)^* \cup (J_d)^* q^d \subseteq R$. Finally, we arrive at $q^d \in Q = Q_s$.

Let us define a map $q \mapsto q^d$ of Q into itself, where $q^d = [f; (J_d)^*]$. We will prove that this is our required unique extension of ∗-derivation d of R. First we compute the following: $q^d x^* = [f; (J_d)^*][l_{x^*}; R] = [f l_{x^*}; {l_{x^*}}^{-1}(J_d)^*] = [l_{(qx)^d - qx^d}; R] = (qx)^d - qx^d$ for all $x \in J_d$. Now we get the following relation

$$q^d x^* = (qx)^d - qx^d \tag{3.3}$$

for all $x \in J_d$. Let $q_1, q_2 \in Q$. This implies that $q_1 + q_2 \in Q$. There exists a $K \in \mathcal{I}$ such that $q_1 K \cup K q_1$, $q_2 K \cup K q_2$, $(q_1 + q_2)K \cup K(q_1 + q_2)$ are all contained in R. It is obvious that relation (3.3) will be true if we replace q by q_1, q_2 or $q_1 + q_2$ and J_d by K_d where $K_d = \sum_{x \in K} x\{(x^d : K)_R\}^*$. Therefore for all $k \in K_d$, we have $(q_1 + q_2)^d k^* = ((q_1 + q_2)k)^d - (q_1 + q_2)k^d = (q_1 k)^d + (q_2 k)^d - q_1 k^d - q_2 k^d = \{q_1^d + q_2^d\}k^*$. Finally, we arrive at $\{(q_1 + q_2)^d - (q_1^d + q_2^d)\}(K_d)^* = \{0\}$. Since $(K_d)^* \in \mathcal{I}$, using characterization of Q_s, we conclude that $(q_1 + q_2)^d = q_1^d + q_2^d$ showing that d is an additive map. Let $q_1, q_2 \in Q$. This implies that $q_1 q_2 \in Q$. By above arguments, it is clear that there exists $T_d \in \mathcal{I}$ such that $q_1 T_d \cup T_d q_1$, $q_2 T_d \cup T_d q_2$, $q_1 q_2 T_d \cup T_d q_1 q_2$ are all contained in R. It is obvious that relation (3.3) will be true if we replace q by q_1, q_2 or $q_1 q_2$ and J_d by T_d. Let $I = (T_d)^2$. Then $q_1 I, I q_1, q_2 I, I q_2 \subseteq T_d$. For all $i \in I$, we have $(q_1 q_2)^d i^* = (q_1 q_2 i)^d - q_1 q_2 i^d = q_1^d (q_2 i)^* + q_1 (q_2 i)^d - q_1 q_2 i^d = q_1^d q_2^* i^* + q_1 q_2^d i^* + q_1 q_2 i^d - q_1 q_2 i^d = q_1^d q_2^* i^* + q_1 q_2^d i^*$. Finally, we arrive at $\{(q_1 q_2)^d - q_1^d q_2^* - q_1 q_2^d\} I^* = \{0\}$. But $I^* \in \mathcal{I}$, by characterization of Q, we conclude that $(q_1 q_2)^d = q_1^d q_2^* + q_1 q_2^d$. Therefore, d is a ∗-derivation of Q.

Finally, we have to prove that this extension is unique. Let us suppose that δ and d be two extensions of the ∗-derivation of R. From the above arguments, it is clear that for any $q \in Q$, there exists $J \in \mathcal{I}$ such that $qJ \cup Jq \subseteq R$. It is obvious that for all $j \in J, qj \in R$. Using the fact that $r^\delta = r^d$ for all $r \in R$, we obtain that $(qj)^\delta = (qj)^d$ for all $j \in J$. This implies that $q^\delta j^* + qj^\delta = q^d j^* + qj^d$ for all $j \in J$. Now we infer that $(q^\delta - q^d)J^* = \{0\}$. But $J^* \in \mathcal{I}$, therefore by characterization of Q, we find that $q^\delta = q^d$ for all $q \in Q$, thus this extension is unique.

In the light of Lemma 2.1 and ([2], Proposition 1), we obtained the following:

Corollary 3.1 *Let R be a $*$-prime ring (resp. prime ring with involution '$*$') admitting a $*$-derivation d. Then d can be uniquely extended to a $*$-derivation of its right symmetric Martindale quotient ring.*

It has been proved in ([1], Theorem 2.3.4) that if R is a prime ring, $Q = Q_{mr}$ and $a, b \in Q$. Suppose that $axb = bxa$ for all $x \in R$. Then a and b are C-dependent. We have extended this result in the setting of $*$-prime rings as follows:

Theorem 3.4 *Let R be a $*$-prime ring, $Q = Q_s$ and $0 \neq a, 0 \neq b \in Q$. Suppose that $axb^* = bxa$ and $a^*xb^* = b^*xa$ for all $x \in R$. Then $a \in Cb^*$ and hence a and b^* are C-dependent.*

Proof Since R is a $*$-prime ring, it will be a semiprime also and $Q = Q_s$ exists. By Theorem 3.1, '$*$' can be assumed to be defined on whole of Q. We have to prove that $a \in Cb^*$. Suppose on contrary, i.e., $a \notin Cb^*$, then by Lemma 2.1 there exists an element $p = \sum_{i=1}^{n} l_{x_i} r_{y_i} \in R_{(l)} R_{(r)}$ such that $d = ap \neq 0$ and $b^* p = 0$. Using the condition $axb^* = bxa$ for all $x \in R$, we have $0 = ar \sum_{i=1}^{n} x_i b^* y_i = \sum_{i=1}^{n} (arx_i b^*) y_i = \sum_{i=1}^{n} (brx_i a) y_i = br \sum_{i=1}^{n} (x_i a y_i) = brd$ for all $r \in R$ and hence we obtain that

$$bRd = \{0\}. \tag{3.4}$$

If we use the condition $a^*xb^* = b^*xa$ for all $x \in R$, on the other hand we also obtain that $0 = a^*r \sum_{i=1}^{n} x_i b^* y_i = \sum_{i=1}^{n} (a^*rx_i b^*) y_i = \sum_{i=1}^{n} (b^*rx_i a) y_i = b^*r \sum_{i=1}^{n} (x_i a y_i) = b^*rd$ for all $r \in R$ and therefore

$$b^*Rd = \{0\}. \tag{3.5}$$

It is given that $0 \neq b$ and $0 \neq d \in Q_s$. By characterization of Q_s, we conclude that there exist $J, U \in \mathcal{I}$ such that $0 \neq bj \in R$ and $0 \neq du \in R$ for some $j \in J$ and $u \in U$. Using the relations (3.4) and (3.5), we also conclude that $(bj)R(du) = \{0\}$ and $(bj)^*R(du) = \{0\}$, leading to a contradiction due to the fact that R is a $*$-prime ring.

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Rings in Which Every 2-Absorbing Ideal Is Prime

Mohamed Issoual and Najib Mahdou

Abstract In this article, we study those rings in which every 2-absorbing ideal is prime. We investigate the stability of this property under homorphic image and localization, and its transfer to various contexts of constructions such as direct products, trivial ring extensions and amalgamated algebra along an ideal.

Keywords 2-absorbing ideal · Prime ideal · Trivial ring extension

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1 Introduction

Throughout this work, all rings are commutative with identity element, and all modules are unitary.

Over the past several years, there has been considerable attention in the literature to n-absorbing ideals of commutative rings and their generalizations, for example see [2–7, 9, 13–15, 21–24, 28, 29, 31–34]. We recall from [2] that a proper ideal I of R is called a *2-absorbing ideal* of R if $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Clearly, every prime ideal is 2-absorbing ideal. That a 2-absorbing ideal need not be a prime ideal as shown by an example in [3].

M. Issoual · N. Mahdou (✉)
Laboratory of Modelling and Mathematical Structures, Department of Mathematics, Faculty of Science and Technology of Fez, University S.M. Ben Abdellah, Box 2202, Fez, Morocco
e-mail: mahdou@hotmail.com

M. Issoual
e-mail: issoual2@yahoo.fr

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In [10], Bennis and Fahid introduced and studied a ring A satisfying the following condition: every 2-absorbing ideal of R is prime. We call such a ring as $2_A B$ ring. They showed that a ring R is a $2_A B$ ring if and only if (1) every prime ideals of R are comparable; in particular, R is quasi-local with maximal ideal M and (2) if P is minimal prime over a 2-absorbing ideal I, then $IM = P$. They also characterize under valuation domain that a ring R is a $2_A B$ ring if and only if $P = P^2$ for every prime ideal P of R.

Recall that a nonzero prime ideal P of a ring R is called *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$. An integral domain R is said to be *divided domain* if every prime ideal of R is a divided prime ideal. An integral domain R is said to be a *valuation domain* if $x \mid y$ (in R) or $y \mid x$ (in R) for every nonzero $x, y \in R$. It is known that a valuation domain is a divided domain. If I is an ideal of R, I is said semi-primary if $\sqrt{I}$ is a prime ideal of R. The set of minimal 2-absorbing ideals over an ideal I will be denoted by $2 - \mathrm{Min}_R(I)$.

In [10], the author ask the following question [10, Question 2]: Is it true that $P^2 = P$ for any prime ideal P of a $2 - AB$ ring. In section three, we study this conjecture which we call a $(\star)$ conjecture.

Let A be a ring and E an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R = A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. Recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J = I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A [1, Theorem 3.2]. Suitable background on commutative trivial ring extensions is [1, 7, 8, 26–28, 30].

Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$

$$A \bowtie^f B = \{(a, f(a) + j) | a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f. Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation (see [18, Examples 2.5 and 2.6]) and other classical constructions, such as the Nagatas idealization (cf. [35, p. 2]), and the CPI extensions (in the sense of Boisen and Sheldon [11]) are strictly related to it (see [18, Example 2.7 and Remark 2.8]). A particular case of this construction is the amalgamated duplication of a ring along an ideal I (introduced and studied by D'Anna and Fontana in [16, 17]). Let A be a ring, and let I be an ideal of A. $A \bowtie I := \{(a, a + i) : a \in A, i \in i\}$ is called the amalgamated duplication of A along the ideal I. See for instance [12, 16–20].

In this article, we study those rings in which every 2-absorbing ideal is prime. We investigate the stability of this property under trivial ring extensions and amalgamated algebra along an ideal in Sects. 2 and 3, respectively. In Sect. 3, we discuss the validity of the equation $P^2 = P$ for any prime ideal P in a $2 - AB$ ring.

2 Trivial Ring Extension Defined by 2-AB Ring Property

Recall that a ring R is said to be a $2AB$-ring if every 2-absorbing ideal of R is prime. Our first result studies the possible transfer of the $2AB$-ring property between a ring A and a trivial ring extension $A \propto E$.

Theorem 1 *Let A be a ring, E be an A-module and $R = A \propto E$ be a trivial ring extension of A by E. Then*

(1) If R is a $2 - AB$ ring, then so is A.
(2) Assume that A is a 2-AB ring and let J be a 2-absorbing ideal of R. Then J is either a prime ideal or a semi-primary ideal of R.
(3) Assume that A is an integral domain with quotient field K and E is a K-vector space. Then $R := A \propto E$ is a $2 - AB$ ring if and only if so is A and $E = 0$.
(4) Assume that E is a finitely generated A-module. Then R is $2 - AB$ ring if and only if so is A and $E = 0$.

The proof of this theorem involves the following lemma with independent interest.

Lemma 2 *(1) Any homomorphic image of a $2 - AB$ ring is a $2 - AB$ ring.*
(2) Any localisation of a $2 - AB$ ring is a $2 - AB$ ring.

Proof Let A be a commutative ring and I be an ideal of A.

(1) Let K be a 2-absorbing ideal of A/I, then there exists a 2-absorbing ideal J of R such that $J \supset I$ and $K = J/I$. Hence, J is a prime ideal of R (since R is 2-AB ring) and so K is a prime ideal of A/I.
(2) Let S be a multiplicative subset of R and let $S^{-1}I$ be a 2-absorbing ideal of $S^{-1}R$, such that $I \cap S = \emptyset$. Then I is a prime ideal of A and so $S^{-1}I$ is a prime ideal of $S^{-1}A$. □

Proof of Theorem 1

(1) Assume that $A \propto E$ is a 2-AB ring. Hence, A is a 2-AB ring by the Lemma 2(1) (since $A \propto E/0 \propto E \simeq A$).
(2) Assume that A is a 2-AB ring and let J be a 2-absorbing ideal of R. Two cases are then possible:
Case 1: $0 \propto E \subseteq J$.
In this case, $J = I \propto E$ for some ideal I of A from the [1, Theorem 3.1]. By [28, Theorem 2.1], I is a 2-absorbing ideal of A and hence I is a prime ideal of A and consequently J is a prime ideal of R.

Case 2: $0 \propto E \nsubseteq J$.
By [3, Theorem 2.1], $\sqrt{J}$ is a 2-absorbing ideal of R which containing $0 \propto E$; by the [1, Theorem], there exists some ideal K of A such that $\sqrt{J} = K \propto E$. The ideal K is a 2-absorbing ideal of A; therefore, K is a prime and hence $\sqrt{J}$ is prime ideal of R.

(3) Recall that if A is an integral domain with quotient field K and E be a divisible A-module, the ideals of $A \propto E$ are of the form $I \propto E$ where I is an ideal of A, or $0 \propto F$ where F is a A-submodule of E.
If A is a $2AB$ ring and $E = O$, then $R \cong A$ is a $2AB$ ring. Conversely, assume that R is a $2 - AB$ ring. It is clear that A is a 2-AB ring by (1). We know from [28, Theorem 2.2] that if F is a A-submodule of E, then $0 \propto F$ is 2-absorbing ideal of R if and only if F is a K-vector subspace of E. Thus for $F = 0$, we get $0 \propto 0$ is a 2-absorbing ideal, thus a prime ideal of R. We conclude that R is an integral domain, thus $E = 0$.
(4) Assume that E is finitely generated. If A is a $2AB$ ring and $E = O$, then $R \cong A$ is a $2AB$ ring. Conversely, assume that R is a $2 - AB$ ring. Then A is a $2 - AB$ ring by (1) and A is a quasi-local ring with a maximal ideal M such that $M^2 = M$ by [10, Corollary 2.4]. On the other hand, R is a quasi-local ring with maximal ideal $M \propto E$ such that $(M \propto E)^2 = M \propto E$, so $ME = E$, and by Nakayama's lemma, we conclude that $E = 0$. This completes the Proof of Theorem. □

Corollary 3 *Let A be a Noetherian module, E be a finitely generated A-module and $R = A \propto E$. Then, R is a 2-AB ring if and only if so is A and $E = 0$.*

Proof If A is a $2AB$ ring and $E = O$, then $R \cong A$ is a $2AB$ ring. Conversely, assume that R is a 2-AB ring. Then, A is a field from [10, Corollary 2.5] and so E is A-vector space. The result follows from Theorem 1(3). □

In general, the converse of Theorem 1(1) is false even if $E = A$, as the following examples shows:

Example 4 Let A be an integral domain, $R = A \propto A$ be the trivial ring extension of A by A, and set $I = 0 \propto P$, where P is a prime ideal of A. Then, I is a non-prime 2-absorbing ideal of $A \propto A$ by [2, Theorem 4.10].

Example 5 Let P be a prime ideal of a ring A. Then, $P \propto P$ is a non-prime 2-absorbing ideal of $A \propto A$ by [28, Corollary 2.8].

3 Amalagamated Algebras Along an Ideal Defined by 2-AB Ring

In this section, we study the transfer of a $2 - AB$ ring property in amalgamated algebras along an ideal.

Theorem 6 *Let $f : A \to B$ be a rings homomorphism, J be an ideal of B, and $A \bowtie^f J$ be the amalgamated algebra. Then the following statements hold:*

(1) If $A \bowtie^f J$ is a $2 - AB$ ring, then A and $f(A) + J$ are a $2 - AB$ rings.
(2) Assume that J is a finitely generated A-module. Then $A \bowtie^f J$ is a $2 - AB$ ring if and only if so is A and $J = 0$.

(3) Assume that $f^{-1}(J) = \{0\}$. Then $A \bowtie^f J$ is a $2 - AB$ ring if and only if so is $f(A) + J$.

(4) Assume that J is a nonzero idempotent ideal of B such that $J \subset Nil(B)$. Then $A \bowtie^f J$ is $2 - AB$ ring if and only if so is A.

The proof of this theorem involves the following lemmas.

Lemma 7 *Let $f : A \to B$ be a rings homomorphism and let J be an ideal of B. If $A \bowtie^f J$ is a $2 - AB$ ring, then A is a local ring with maximal ideal M, $J \subset Jac(B)$ and $(f(M) + J)J = 0$.*

Proof Since $A \bowtie^f J$ is a $2 - AB$ ring, then $A \bowtie^f J$ is a local ring by [10, Theorem 2.3] and so A is a local ring with maximal ideal M and $J \subset \mathrm{Jac}(B)$. On the other hand, the maximal ideal of $A \bowtie^f J$ is $M \bowtie^f J$. Hence, by [10, Corollary 2.4], $(M \bowtie^f J)^2 = M^2 \bowtie^f (f(M)J + J^2) = M \bowtie^f J$ and so $f(M)J + J^2 = J$. □

Recall that the set of minimal 2-absorbing ideals over an ideal I will be denoted by $2 - \mathrm{Min}_R(I)$.

Lemma 8 ([10, Theorem 2.9]) *A ring R is a 2-AB ring if and only if the two following conditions holds:*

(1) A prime ideals of R are comparable. In particular, R is quasi-local.

(2) For every prime ideal P of R, $2 - Min_R(P^2) = P$.

Proof of Theorem 6

(1) Clear by Lemma 2(1) since $\dfrac{A \bowtie^f J}{\{0\} \times J} \simeq A$ and $\dfrac{A \bowtie^f J}{f^{-1}(J) \times D} \cong f(A) + J$.

(2) Assume that J is a finitely generated A-module. If A is a $2 - AB$ ring and $J = 0$, then $\dfrac{A \bowtie^f J}{\{0\} \times J} \simeq A$. Conversely, assume that $A \bowtie^f J$ is a $2 - AB$ ring. Hence, A is a $2 - AB$ ring by (1). On the other hand, A is a local ring with maximal ideal M such that $f(M)J + J^2 = J$ by Lemma 7. It is well known that $f(M) + J$ is a maximal ideal of the quasi-local ring $f(A) + J$. On the other hand, since J is a finitely generated A-module, it is a finitely generated $(f(A) + J)$-module. By Nakayam's lemma, we conclude that $J = 0$.

(3) Clear since $A \bowtie^f J \simeq f(A) + J$ (since $f^{-1}(J) = 0$).

(4) If $A \bowtie^f J$ is a $2 - AB$-ring, then so is A by (1). Conversely, assume that A is a $2 - AB$ ring and J a nonzero idempotent ideal of B such that $J \subseteq Nil(B)$. Hence, A is a local ring with maximal ideal M and so $A \bowtie^f J$ is a local ring with maximal ideal $M \bowtie^f J$ (since $J \subset \mathrm{Jac}(B)$). On the other hand, since $J \subset \mathrm{Nil}(B)$, the prime ideals of $A \bowtie^f J$ are of the form $P \bowtie^f J$, where P is a prime ideal of A. Since the prime ideals are comparable, it is clear that the prime ideals of $A \bowtie^f J$ are comparable.
Now we will shows that $2 - \min_{A \bowtie^f J}\left((P \bowtie^f J)^2\right) = P \bowtie^f J$. Indeed let K be a 2-absorbing ideal of $A \bowtie^f J$ and suppose that there exists a 2-absorbing

ideal L of $A \bowtie^f J$ such that $(P \bowtie^f J)^2 \subset L \subset K$. But $(P \bowtie^f J)^2 = P^2 \bowtie^f (f(P)J + J^2) = P^2 \bowtie^f J \subset L \subset K$. Hence, $\{0\} \times J \subset K$ and $\{0\} \times J \subset L$, thus there exist prime ideals K_1 and L_1 of A such that $L = L_1 \bowtie^f J$ and $K = K_1 \bowtie^f J$. Then $P^2 \subset L_1 \subset K_1$ and so $K_1 = L_1 = P$ (since A is a $2-AB$ ring), thus $K = L = P \bowtie^f J$. Then $A \bowtie^f J$ is a $2-AB$ ring by Lemma 8 and this completes the Proof of Theorem 6. □

Corollary 9 *Let $f : A \to B$ be a rings homomorphism, B be a local ring with maximal ideal M, and let J be a nonzero idempotent ideal of B. Then $A \bowtie^f J$ is a 2-AB ring if and only if A is a 2-AB ring.*

Proof The proof is clear by Theorem 6 since $J \subset M = \mathrm{Jac}(B) = \mathrm{Nil}(B)$. □

Example 10 Let A be an integral domain with quotient field K, and E be a K-vector space. Set $B := A \propto E$, and $J := 0 \propto E$. We consider $f : A \to B$ the canonical homomorphism defined by $f(a) = (a, 0)$ for every $a \in A$.

Then $A \bowtie^f J$ is a 2-AB ring if and only if so is A and $J = 0$.

Proof Indeed, assume that $A \bowtie^f J$ be a 2-AB ring, then A so is by Theorem 6. On the other hand, we have $f(M)J = (M \propto 0)(0 \propto E) = 0$ and $J^2 = 0$, thus $J = 0$. The converse is clear. □

Let $f : A \to B$ be a rings homomorphism, J be an ideal of B. If A is 2-AB ring and $J \subset \mathrm{Jac}(B)$, the ring $A \bowtie^f J$ need not be a $2 - AB$ ring as the following example shows.

Example 11 Let A be a local ring with maximal ideal M and E be a nonzero A-module such that $ME = 0$. Set $B := A \propto E$ and $f : A \to B$ be the canonical homomorphism defined by $f(a) = (a, 0)$ for every $a \in A$ and set $J = 0 \propto E$. Then $A \bowtie^f J$ is not a $2 - AB$ ring.

Proof It is clear that J is an ideal of B and $J \subset \mathrm{Jac}(B)$ since $J^2 = 0$. On the other hand, $f(M)J = 0$ but $J \neq 0$. Therefore, $A \bowtie^f J$ is not a $2 - AB$ ring. □

Now we will determine the 2-absorbing ideals in a particular case $A \bowtie^f XK[[X]]$, where A is an integral domain with quotient field K.

Theorem 12 (The 2-absorbing ideal in $A \bowtie^f J$ in case $J := XK[[X]]$) *Let A be an integral domain with quotient field K, $f : A \hookrightarrow K[[X]]$ be the natural embedding and $J := XK[[X]]$. Assume that A is a 2-AB ring, and let L be a 2-absorbing ideal of $A \bowtie^f J$. Then one of the following statements holds:*

(1) L is a prime ideal of $A \bowtie^f J$.
(2) $L = 0 \bowtie^f X^2K[[X]]$ which is a non-prime 2-absorbing ideal of $A \bowtie^f J$.

Proof First we show that $0 \bowtie^f J^2$ is a 2-absorbing ideal of R. Let $x_i = (a_i, a_i + XS_i(X)) \in R$, where $S_i \in K[[X]]$, $1 \leq i \leq 3$. Suppose $x_1x_2x_3 \in 0 \bowtie^f J^2$. Since A is an integral domain, we may assume that $a_3 = 0$. Two cases are then possible:

Case 1: Assume that $a_1a_2 = 0$. Then $x_1x_3 = (0, X^2S_1(X)S_3(X)) \in 0 \bowtie^f J^2$ or $x_2x_3 \in 0 \bowtie^f J^2$.

Case 2: $a_1a_2 \neq 0$. Then $x_1x_2x_3 = (0, XS_3(X)(a_1, a_1+XS_1(X))(a_2, a_2+XS_2(X)) \in 0 \bowtie^f J^2$, and so $XS_3(X) \in J^2$ (since J^2 is an ideal of $K[[X]]$, and $a_1a_1 \neq 0$). Hence, $x_1x_3 = (0, XS_3(X)(a_1, a_1 + XS_1(X)) \in 0 \bowtie^f J^2$.

Therefore, $0 \bowtie^f J^2$ is a 2-absorbing ideal of R. On the other hand, $0 \bowtie^f J$ and $I \bowtie^f J$ are 2-absorbing ideals of $A \bowtie^f J$ by Theorem 1.

Conversely, let L be a 2-absorbing ideal of $A \bowtie^f J$. Then $u(L)$ is an 2-absorbing ideal of $A + XK[[X]]$. Using theorem [2, Theorem 4.17], $u(L) = I + XK[[X]]$ for some 2-absorbing ideal I of A, or $u(L) = XK[[X]]$, or $u(L) = X^2K[[X]]$, where $u : A \bowtie^f J \to A + XK[[X]]$ is a canonical isomorphism. Hence, $L = u^{-1}(I + J) = I \bowtie^f J$ or $L = u^{-1}(J) = 0 \bowtie^f J$ or $L = u^{-1}(J^2) = 0 \bowtie^f J^2$. Assume that $L = I \bowtie^f J$, where I is a 2-absorbing ideal of A. Then I is a prime ideal of A (since A is a $2 - AB$ ring) and so L is a prime ideal of $A \bowtie^f J$. Now, if $L = 0 \bowtie^f X^2K[[X]]$, by [2, Theorem 4.17] and the fact that $L \simeq X^2K[[X]]$, we conclude that L is a non-prime 2-absorbing ideal of $A \bowtie^f J$. □

4 About (⋆) Conjecture

In [10], Bennis and Fahid conjecture that in every 2-AB ring A, we have $P^2 = P$ for every prime ideal P of A which we call a (⋆) conjecture. It is proved in [10, Proposition 2.6] that a valuation domain satisfies (⋆) conjecture. In the next proposition, we give a positive answer to (⋆) conjecture in particular case.

Proposition 13 *Let R be a ring. Then*

(1) Assume that R is a divided domain. Then R is a $2 - AB$ ring if and only if $P^2 = P$ for every prime ideal of R.
(2) Assume that every semi-primary ideal of R is a primary ideal of R and R is a 2-AB ring. Then for every prime ideal P of R, we have $P^2 = P$.

Proof

(1) Assume that R is a 2-AB divided domain and let P be a nonzero prime ideal. Hence, P^2 is a 2-absorbing ideal of R (by [3, Corollary 3.8]) and so it is a prime ideal of R. Then $P^2 = P$.
Conversely, let I be a nonzero 2-absorbing proper ideal of R. Since R is divided domain, we have $\sqrt{I} = P$ is a nonzero divided prime ideal, and by [3, Theorem 3.6], I is P-primary and $P^2 \subset I \subset P$. Thus $I = P$ is a prime ideal of R, as desired.

(2) Assume that R is a 2-AB ring and let P be a prime ideal of R. By the assumption P^2 is P-primary ideal of R, P^2 is a 2-absorbing ideal of R by [2, Theorem 3.1]. We conclude that P^2 is a prime ideal of R, and $P^2 = P$, as desired. □

In [25], Gilmer studies the ring in which every semi-primary ideal is a primary ideal.

Remark 14 Let R be a ring in which every semi-primary is primary. By [25, Corollary 2.2], for every nonmaximal prime ideal P of R, we have $P^2 = P$. Now Let M be a maximal ideal of R, it is well known that M^2 is a 2-absorbing ideal of R by [3, Theorem 3.1]. If R is a 2-AB ring, then R is quasi-local ring with a maximal ideal M by [10, Theorem 2.9]. We conclude that M^2 is a prime ideal of R, and $M^2 = M$. Thus for every prime ideal I of R, we have $I^2 = I$.

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A Note on Skew-Commutators with Derivations on Ideals

Nadeem Ur Rehman, Mohd Arif Raza, and Muzibur Rahman Mozumder

Abstract Let R be a prime ring of characteristic different from two, Q be the Martindale quotient ring of R and C be the extended centroid of R. Suppose that d is a derivation of R, I a nonzero ideal of R and $m, k \in \mathbb{Z}^+$ where $m > 1$. If

$$[(x \circ_k y)^d, x \circ_k y]^m = [(x \circ_k y)^d, x \circ_k y]$$

for all $x, y \in I$, then either $d = 0$ or R satisfies S_4, the standard identity in four variables. We also extend the result to semiprime rings.

Keywords Prime ring · Derivations · Martindale ring of quotients · Differential identities

2010 Mathematics Subject Classification 16N60 · 16R50 · 16W25

1 Introduction

Let R be a ring and $1 < m \in \mathbb{Z}^+$. An element $x \in R$ is said to be m-potent, if $x^m = x$. Also, it is known as an idempotent element, for $m = 2$. A lot of work has been done for m-potent matrices and m-potent preserving maps in the literature. In this direction, the work of an eminent algebraist Jacobson [18] should be mentioned who established

N. Ur Rehman (✉) · M. R. Mozumder
Department of Mathematics, Aligarh Muslim University, Aligarh 202002, Uttar Pradesh, India
e-mail: nu.rehman.mm@amu.ac.in

M. R. Mozumder
e-mail: muzibamu81@gmail.com

M. A. Raza
Department of Mathematics, Faculty of Science and Arts-Rabigh, King Abdulaziz University, Jeddah, Saudi Arabia
e-mail: arifraza03@gmail.com

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that: Any ring in which every element x is m-potent, i.e., $x^m = x,\ 1 < m \in \mathbb{Z}^+$ must be commutative. In [16], Herstein discussed the m-potent commutators of rings. More precisely, he established the commutativity of a ring and proved that: A ring R must be commutative if it has m-potent commutators, i.e., R satisfies $[x, y]^m = [x, y]$ for every $x, y \in R$, where $1 < m \in \mathbb{Z}^+$. In 1994, Giambruno et al. [15] established that a ring must be commutative if it satisfies $[x, y]_k^n = [x, y]_k$. Motivated by above mentioned results, several identities have been investigated in various directions. For results concerning derivations and their generalizations on similar differential identities, we refer the reader to [1, 3, 4, 8, 12, 13, 23, 24, 26–28].

For each $x, y \in R$ and each $k \geq 0$, define $x \circ_k y$ inductively by $x \circ_0 y = x$, $x \circ_1 y = xy + yx$ and $x \circ_k y = (x \circ_{k-1} y) \circ y$ for $k \geq 1$. The ring R is said to satisfy an Engel condition if there exists a positive integer k such that $[x, y]_k = 0$. Note that an Engel condition is a polynomial $[x, y]_k = \sum_{n=0}^{k}(-1)^n \binom{k}{n} y^n x y^{k-n}$ (for skew-commutator $x \circ_k y = \sum_{n=0}^{k} \binom{k}{n} y^n x y^{k-n}$) in non-commuting indeterminates x, y. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. In particular, d is an inner derivation induced by an element $a \in R$, if $d(x) = [a, x]$ for all $x \in R$. More information about derivations/generalized derivations having skew-commutators conditions on rings/algebras and their applications can be found in [2, 5, 9, 17, 24, 25, 29, 30].

Our purpose here is to continue on this line of investigation and consider the case of m-potent commutators involving derivations on kth skew-commutator $x \circ_k y$, for a fixed positive integer $k \geq 1$, which is not multilinear.

2 Main Results

In what follows, unless stated otherwise, R will be a prime ring. We denote the Martindale quotient ring of R by Q. Let C be the center of Q, which is called the extended centroid of R. The definitions, the axiomatic formulations and the properties of these quotient rings can be found [7]. Note that Q is also prime ring with identity and C is a field.

Before proving our main theorem, we will fix some notations and collect some known result which will be of use in the sequel.

Fact 1 *Let R be a prime ring and I a two sided ideal of R, then I, R, U satisfy the same generalized polynomial identities with coefficient in U (see [10]).*

Fact 2 *Every derivation d of R can be uniquely extended to a derivation on U (see Proposition 2.4.1 of [7]).*

Fact 3 ([6, Lemma 7.1]) *Let ${}_{\mathcal{D}}\mathcal{M}$ be a left vector space over a division ring $\mathcal{D}$ with $dim_{\mathcal{D}}\mathcal{M} \geq 2$ and $\mathcal{T} \in End(\mathcal{M})$. If x and $\mathcal{T}x$ are $\mathcal{D}$-dependent for every $x \in \mathcal{M}$, then there exists $\lambda \in \mathcal{D}$ such that $\mathcal{T}x = \lambda x$ for all $x \in \mathcal{M}$.*

We are now ready to prove the main result:

Theorem 1 *Let $m, k \in \mathbb{Z}^+$, where $m > 1$. Next, let R be a noncommutative prime ring with characteristic not two, I be a nonzero ideal of R and d be a derivation of R. If $[(x \circ_k y)^d, x \circ_k y]^m = [(x \circ_k y)^d, x \circ_k y]$ holds for every $x, y \in I$, then either $d = 0$ or R satisfies s_4, the standard identity in four variables.*

Proof By our hypothesis, $[(x \circ_k y)^d, x \circ_k y]^m = [(x \circ_k y)^d, x \circ_k y]$ for all $x, y \in I$, that is, I satisfies the differential identity

$$\begin{aligned}\Bigg[\sum_{k=1}^{n}\binom{n}{k}\Big(\sum_{i+j=k-1} y^i y^d y^j\Big)xy^{n-k} + \sum_{k=0}^{n}\binom{n}{k}y^k x^d y^{n-k}\\ + \sum_{k=0}^{n-1}\binom{n}{k}y^k x\Big(\sum_{r+s=m-k-1} y^r y^d y^s\Big), x \circ_k y\Bigg]^m\\ = \Bigg[\sum_{k=1}^{n}\binom{n}{k}\Big(\sum_{i+j=k-1} y^i y^d y^j\Big)xy^{n-k} + \sum_{k=0}^{n}\binom{n}{k}y^k x^d y^{n-k}\\ + \sum_{k=0}^{n-1}\binom{n}{k}y^k x\Big(\sum_{r+s=m-k-1} y^r y^d y^s\Big), x \circ_k y\Bigg]\end{aligned}$$

for all $x, y \in I$. In the light of Kharchenko's theory [20], we split the proof into two cases:

Firstly we assume that d is an inner derivation induced by an element $q \in Q$ such that $x^d = [q, x]$ for all $x \in R$. Thus, we have

$$\begin{aligned}\Bigg[\sum_{k=1}^{n}\binom{n}{k}\Big(\sum_{i+j=k-1} y^i [q, y] y^j\Big)xy^{n-k} + \sum_{k=0}^{n}\binom{n}{k}y^k [q, x] y^{n-k}\\ + \sum_{k=0}^{n-1}\binom{n}{k}y^k x\Big(\sum_{r+s=m-k-1} y^r [q, y] y^s\Big), x \circ_k y\Bigg]^m\\ = \Bigg[\sum_{k=1}^{n}\binom{n}{k}\Big(\sum_{i+j=k-1} y^i [q, y] y^j\Big)xy^{n-k} + \sum_{k=0}^{n}\binom{n}{k}y^k [q, x] y^{n-k}\\ + \sum_{k=0}^{n-1}\binom{n}{k}y^k x\Big(\sum_{r+s=m-k-1} y^r [q, y] y^s\Big), x \circ_k y\Bigg]\end{aligned}$$

for all $x, y \in I$. One can see that

$$\Big[q(\sum_{n=0}^{k}\binom{k}{n}y^nxy^{k-n}) - \Big(\sum_{n=0}^{k}\binom{k}{n}y^nxy^{k-n}\Big)q, x\circ_k y]^m$$
$$= \Big[q(\sum_{n=0}^{k}\binom{k}{n}y^nxy^{k-n}) - \Big(\sum_{n=0}^{k}\binom{k}{n}y^nxy^{k-n}\Big)q, x\circ_k y]$$

for all $x, y \in I$. And hence we can write $[[q, x\circ_k y], x\circ_k y]^m = [[q, x\circ_k y], x\circ_k y]$ for all $x, y \in I$. Since $q \notin C$ and hence I satisfies nontrivial generalized polynomial identity (GPI). By Chuang [10, Theorem 2], I and Q satisfy the same generalized polynomial identities, thus we have

$$[[q, x\circ_k y], x\circ_k y]^m = [[q, x\circ_k y], x\circ_k y]$$

for all $x, y \in Q$. In case the center C of Q is infinite, we have

$$[[q, x\circ_k y], x\circ_k y]^m = [[q, x\circ_k y], x\circ_k y]$$

for all $x, y \in Q\otimes_C \overline{C}$, where $\overline{C}$ is algebraic closure of C. Since both Q and $Q\otimes_C \overline{C}$ are prime and centrally closed [14, Theorems 2.5 and 3.5], we may replace R by Q or $Q\otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC = R$) which is either finite or algebraically closed and $[[q, x\circ_k y], x\circ_k y]^m = [[q, x\circ_k y], x\circ_k y]$ for all $x, y \in R$. By Martindale [22, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H with $\mathcal{D}$ as the associated division ring.

Hence by Jacobson's theorem [19, p. 75], R is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $\mathcal{D}$ and H consists of the finite rank linear transformations in R. If $\mathcal{V}$ is a finite dimensional over $\mathcal{D}$, then the density of R on $\mathcal{V}$ implies that $R \cong M_t(\mathcal{D})$, where $t = dim_{\mathcal{D}}\mathcal{V}$. Assume first that $dim_{\mathcal{D}}\mathcal{V} \geq 3$.

We want to show that, for any $v \in \mathcal{V}$, v and qv are linearly $\mathcal{D}$-dependent. Since $dim_{\mathcal{D}}\mathcal{V} \geq 3$, then there exists $w \in \mathcal{V}$ such that $\{v, qv, w\}$ is also linearly $\mathcal{D}$-independent. By the density of R, there exist $x, y \in R$ such that

$$xv = v,\ xqv = w,\ xw = w$$
$$yv = 0,\ yqv = 0,\ \ yw = w.$$

These imply that $0 = ([[q, x\circ_k y], x\circ_k y]^m - [[q, x\circ_k y], x\circ_k y])v = (2^{mk} - 2^k)w \neq 0$, a contradiction as $char(R) \neq 2$. So, we conclude that $\{v, qv\}$ is linearly $\mathcal{D}$-dependent, for all $v \in \mathcal{V}$. Thus, by Fact 3, there exists $\lambda \in C$ such that $qv = v\lambda$ for any $v \in \mathcal{V}$. Let now for $r \in R$, $v \in \mathcal{V}$. We can write $qv = v\alpha$, $r(qv) = r(v\alpha)$, and also $q(rv) = (rv)\alpha$. Thus $0 = [q, r]v$, for any $v \in \mathcal{V}$, that is $[q, r]\mathcal{V} = 0$. Since $\mathcal{V}$ is a left faithful irreducible R-module, hence $[q, r] = 0$, for all $r \in R$, i.e., $q \in Z(R)$ and $d = 0$, which contradicts our hypothesis.

Therefore $dim_{\mathcal{D}}\mathcal{V}$ must be ≤ 2. In this case, R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [21, Lemma 2], it follows that there exists a suitable field $\mathbb{F}$ such that $R \subseteq M_t(\mathbb{F})$, the ring of all $t \times t$ matrices over $\mathbb{F}$, and moreover, $M_t(\mathbb{F})$ satisfies the same generalized polynomial identity of R.

If we assume $t \geq 3$, then by the same argument as presented above, we get a contradiction. As R is noncommutative, so we may assume that $t = 2$, i.e., $R \subseteq M_2(\mathbb{F})$, where $M_2(\mathbb{F})$ satisfies $[[q, x \circ_k y], x \circ_k y]^m = [[q, x \circ_k y], x \circ_k y]$. Denote by e_{ij} the usual unit matrix with 1 in (i, j)-entry and zero elsewhere. Since by choosing $x = e_{ij}, y = e_{jj}$. In this case, we have $[[q, e_{ij}], e_{ij}]^m = [[q, e_{ij}], e_{ij}]$. One can see that $-2e_{ij}qe_{ij} = 0$. Now set $q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. By calculation, we find that $-2\begin{pmatrix} 0 & q_{ji} \\ 0 & 0 \end{pmatrix} = 0$, which implies that $q_{ji} = 0$. In the same manner, we can see that $q_{ij} = 0$. Thus we conclude that q is a diagonal matrix in $M_2(\mathbb{F})$. Let $\chi \in Aut(M_2(\mathbb{F}))$. Since $[[\chi(q), \chi(x) \circ_k \chi(y)], \chi(x) \circ_k \chi(y)]^m = [[\chi(q), \chi(x) \circ_k \chi(y)], \chi(x) \circ_k \chi(y)]$. So, $\chi(q)$ must be diagonal matrix in $M_2(\mathbb{F})$. In particular, let $\chi(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$. Then $\chi(q) = q + (q_{ii} - q_{jj})e_{ij}$, that is, $q_{ii} = q_{jj}$ for $i \neq j$. This implies that q is central in $M_2(\mathbb{F})$, which leads to $d = 0$.

Secondly, we assume that d is an outer derivation on Q. Now by Kharchenko's theorem [20], I satisfies the generalized polynomial identity

$$\begin{aligned}
\Bigg[\sum_{k=1}^{n}\binom{n}{k}\Bigg(\sum_{i+j=k-1} y^i z y^j\Bigg) x y^{n-k} + \sum_{k=0}^{n}\binom{n}{k} y^k w y^{n-k} \\
+ \sum_{k=0}^{n-1}\binom{n}{k} y^k x\Bigg(\sum_{r+s=m-k-1} y^r z y^s\Bigg), x \circ_k y\Bigg]^m \\
= \Bigg[\sum_{k=1}^{n}\binom{n}{k}\Bigg(\sum_{i+j=k-1} y^i z y^j\Bigg) x y^{n-k} + \sum_{k=0}^{n}\binom{n}{k} y^k w y^{n-k} \\
+ \sum_{k=0}^{n-1}\binom{n}{k} y^k x\Bigg(\sum_{r+s=m-k-1} y^r z y^s\Bigg), x \circ_k y\Bigg]
\end{aligned}$$

for all $x, y \in I$ and in particular I satisfies the polynomial identity

$$\Bigg[\sum_{k=0}^{n}\binom{n}{k} y^k w y^{n-k}, x \circ_k y\Bigg]^m = \Bigg[\sum_{k=0}^{n}\binom{n}{k} y^k w y^{n-k}, x \circ_k y\Bigg]$$

for all $x, y, w \in I$ and hence it is satisfied by R. The last identity is a polynomial identity so that there exists a field $\mathbb{F}$ such that R and $\mathbb{F}_t$ satisfy the same identities. Thus pick $x = e_{12}, y = e_{22}, w = e_{23}$, we see that $x \circ_k y = e_{12}$ and $\sum_{k=0}^{n}\binom{n}{k} y^k w y^{n-k} = e_{23}$.

Therefore

$$e_{13} = \left[\sum_{k=0}^{n} \binom{n}{k} y^k w y^{n-k}, x \circ_k y \right]^m - \left[\sum_{k=0}^{n} \binom{n}{k} y^k w y^{n-k}, x \circ_k y \right] \neq 0$$

a contradiction. Therefore $t \leq 2$ and R satisfies s_4. This completes the proof of the theorem. □

We immediately get the following corollary from the above theorem:

Corollary 1 *Let* $1 < m \in \mathbb{Z}^+$*. Next, let* R *be a prime ring with characteristic not two,* I *be a nonzero ideal of* R *and* d *be a derivation of* R*. If* $[(x \circ y)^d, x \circ y]^m = [(x \circ y)^d, x \circ y]$ *holds for every* $x, y \in I$*, then either* $d = 0$ *or* R *satisfies* s_4*.*

From now on, R is a semiprime ring and U is the left Utumi quotient ring of R. In order to prove the main results of this section, we will make use of the following facts:

Fact 4 ([7, Proposition 2.5.1]) *Any derivation of a semiprime ring* R *can be uniquely extended to a derivation of its left Utumi quotient ring* U*, and so any derivation of* R *can be defined on the whole* U*.*

Fact 5 ([11, p. 38]) *If* R *is semiprime, then so is its left Utumi quotient ring. The extended centroid* C *of a semiprime ring coincides with the center of its left Utumi quotient ring.*

Fact 6 ([11, p. 42]) *Let* B *be the set of all the idempotents in* C*, the extended centroid of* R*. Suppose that* R *is an orthogonally complete* B*-algebra. For any maximal ideal* P *of* B*,* PR *forms a minimal prime ideal of* R*, which is invariant under any derivation of* R*.*

Now we are ready to prove the following:

Theorem 2 *Let* $m, k \in \mathbb{Z}^+$*, where* $m > 1$*. Next, let* R *be a noncommutative semiprime ring with characteristic not two,* U *the left Utumi quotient ring of* R *and* d *be a derivation of* R*. If* $[(x \circ_k y)^d, x \circ_k y]^m = [(x \circ_k y)^d, x \circ_k y]$ *holds for every* $x, y \in R$*, then there exists a central idempotent element* $e \in U$*, such that* $U = eU \bigoplus (1 - e)U$*,* $d(eU) = 0$ *and* $(1 - e)U$ *satisfies* $s_4(x_1, x_2, x_3, x_4)$*.*

Proof By Fact 5, $Z(U) = C$, the extended centroid of R, and by Fact 4, the derivation d can be uniquely extended on U. Since R and U satisfy the same differential identities, then $[(x \circ_k y)^d, x \circ_k y]^m = [(x \circ_k y)^d, x \circ_k y]$ for all $x, y \in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. Since U is an orthogonally complete B-algebra [11, p. 42], thus by Fact 6, MU is a prime ideal of U, which is d-invariant. Let $\overline{d}$ be the derivation induced by d on $\overline{U} = U/MU$. Since $Z(\overline{U}) = (C + MU)/MU = C/MU$, then $[(x \circ_k y)^d, x \circ_k y]^m = [(x \circ_k y)^d, x \circ_k y]$, for all $x, y \in \overline{U}$. Moreover $\overline{U}$ is prime, hence we may conclude, by Theorem 1, either $\overline{d} = 0$ in $\overline{U}$ or $\overline{U}$ satisfies s_4. This implies that, for any maximal ideal M of B, either $d(U) \subseteq MU$ or $s_4(x_1, x_2, x_3, x_4) \subseteq MU$, for all $x_1, x_2, x_3, x_4 \in U$. In any case $d(U)s_4(x_1, x_2, x_3, x_4) \subseteq \bigcap_M MU = 0$. From [7, Chap. 3], there exists a central idempotent element e of U, the left Utumi quotient ring of R, such that on the direct sum decomposition $U = eU \oplus (1-e)U$, $d(eU) = 0$ and the ring $(1-e)U$ satisfies s_4. This completes the proof of the theorem. □

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A Brief Survey on Semiprime and Weakly Compressible Modules

N. Dehghani and M. R. Vedadi

Abstract This is a brief survey on a module generalization of the concepts of prime (semiprime) for a ring. An R-module M is called weakly compressible if $\mathrm{Hom}_R(M, N)N$ is nonzero for every $0 \neq N \leq M_R$. They are semiprime (i.e., $M \in \mathrm{Cog}(N)$ for every $N \leq_{ess} M_R$). We show that there exist semiprime modules which are not weakly compressible. Further, we investigate when a semiprime module is weakly compressible over any ring R. For certain rings R, including prime hereditary Noetherian rings, weakly compressible (semiprime) modules are characterized. In continue, we study rings R whose every semiprime module is weakly compressible (say in this case, R is a SW ring). Duo Noetherian SW rings are shown to be local. If R is Morita equivalent to a Dedekind domain, then R is SW if and only if it is either simple Artinian or $J(R) \neq 0$.

Keywords Semiprime · Weakly compressible · Semi-Artinian · Krull dimension SW ring

1 Introduction

Throughout this paper, rings will have a nonzero identity, modules will be right and unitary. In the literature, there are several module generalizations of a prime (semiprime) ring R, see [14, Sects. 13 and 14] for an excellent reference. These generalizations introduce various concepts of semiprime (prime) modules and many important theories on semiprime (prime) rings are generalized to modules by them; see Behboodi [2], Smith et al. [8], Haghany and Vedadi [9], Lomp [11] and Zelmanowitz

N. Dehghani (✉)
Department of Mathematics, Persian Gulf University, 75169-13817 Boushehr, Iran
e-mail: n.dehghani@pgu.ac.ir

M. R. Vedadi
Department of Mathematical Sciences, Isfahan University of Technology, 84156-83111 Isfahan, Iran
e-mail: mrvedadi@cc.iut.ac.ir

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[15]. A generalization of prime rings was introduced and studied by Bican et al. in [3]. They called a module M_R $\star$-prime provided $\mathrm{Hom}_R(M, N)K =: N \star K = 0$ implies that $N = 0$ or $K = 0$ for all submodules N and K of M_R (equivalently for any $0 \neq L \subseteq M_R$, there exists a set Λ such that M embeds in L^Λ). Modules M_R in which $N \star N \neq 0$ for all $0 \neq N \subseteq M_R$ was considered by Avraamova in [1] as weakly compressible modules. Lomp in [11] called an R-module M semiprime if for every essential submodule N of M_R, there exists a set Λ such that M embeds in N^Λ. It is easy to see that both semiprime and weakly compressible concepts are coincidence for any ring R. Weakly compressible modules have been applied in different situations. For example, in the study of weakly semisimple modules by Zelmanowitz [15] and modules which have semiprime right Goldie endomorphism rings by Haghany and Vedadi [10, Theorem 2.6]. They have also appeared in the Cohen-Fishmans question about the semiprimeness of the smash product $A\#H$ when H is a semisimple Hopf algebra and A is a semiprime H-module algebra. Lomp in [11, Corollary 7.6] showed that for certain semisimple Hopf algebra H, $A\#H$ is a semiprime ring if and only if the $A\#H$-module A is weakly compressible.

In this survey, we state salient results by main contributors on this topic and some recent results by the authors. In order to study semiprime modules, Lomp in 2005 has shown that every semiprime module is weakly compressible. However, the converse has been asked as the following question.

Question 1 ([11, Open problem 2, p. 92]) Is a semiprime module weakly compressible?

In this survey, we give a complete answer to this question by showing that there are semiprime modules which are not weakly compressible. Furthermore, we investigate under what conditions a semiprime module is weakly compressible. For example, we show that over a commutative ring R, every semiprime R-module with finite uniform dimension is weakly compressible (Theorem 6). In continue, we investigate rings R whose every semiprime module is weakly compressible, say SW ring. To investigate SW rings, first we consider the case that all R-modules are weakly compressible. For many rings R, including commutative rings, it is proved that R is a SW ring if and only if the class of weakly compressible modules is enveloping for mod-R (Theorem 15). Every right semi-Artinian ring is SW. Duo Noetherian SW rings are shown to be local (Theorem 16). If R is Morita equivalent to a Dedekind domain, then R is SW if and only if R is simple Artinian or $\mathrm{J}(R) \neq 0$ (Theorem 18). Semiprime and weakly compressible modules have been characterized over certain rings R, including prime hereditary Noetherian rings (Theorems 10 and 11).

Let M be an R-module. The notations $N \subseteq M$, $N \leq M$, $N \leq_{ess} K$, or $N \leq_\oplus M$ mean that N is a subset, a submodule, an essential submodule, or a direct summand of M_R, respectively. The notation M^A means $\prod_{i \in I} M_i$ where A is a set and each $M_i \simeq M$. If $X \subseteq M$, then the right annihilator X in R will be denoted by $\mathrm{Ann}_R(X)$. The singular (second singular) submodule of M_R is denoted by $\mathrm{Z}(M)$ ($\mathrm{Z}_2(M)$). Also, if X and Y are R-modules, then $\cap\{\ker f \mid f : X_R \to Y_R\}$ and $\cap\{\ker f \mid f : X_R \to T_R \text{ where } T_R \text{ is simple}\}$ are denoted by $\mathrm{Rej}(X, Y)$ and $\mathrm{Rad}(X)$, respectively.

If there exists an R-monomorphism from X to Y, we write $X \hookrightarrow Y$. The module X is cogenerated by Y (write $X \in \mathrm{Cog}(Y)$) if $\mathrm{Rej}(X, Y) = 0$. Any unexplained terminology and all the basic results on rings and modules that are used in the sequel can be found in [12].

2 Main Results

We carry out a study of semiprime and weakly compressible modules. Below we explain the relation between them that was proved by Lomp.

Theorem 1 ([11, Theorem 5.5]) *Consider the following statements on an R-module M.*
(a) *M_R is weakly compressible;*
(b) *M_R is semiprime;*
(c) *M is a subdirect product of prime modules;*
(d) $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(M)$ *for every essential submodule N of M;*
(e) $\mathrm{Ann}_R(M)$ *is semiprime.*

Proof Then the implications (a)⇒ (b)⇒ (c) ⇒ (d) ⇒ (e) hold. If M is a torsionless as $R/\mathrm{Ann}(M)$-Module, then all statements (a)–(e) are equivalent. □

By the above result, every semiprime module is weakly compressible. However, the converse is asked as Question 1. In the following two examples, we give a positive answer to it by showing that there are semiprime modules which are not weakly compressible.

Example 2 Let P be the set of all prime integer numbers and $p \in P$. Consider the $\mathbb{Z}$-module $N = \{m/p^n \mid m, n \in \mathbb{Z}, n \geq 1\}$. Then for each $q \in P \setminus \{p\}$, qN is a maximal submodule of $N_{\mathbb{Z}}$. To see this, note that $qN \neq N$ and suppose that K is any submodule of $N_{\mathbb{Z}}$ such that $qN \subsetneq K$ and $m/p^t \in K \setminus qN$. Hence $(m, q) = 1$. Also, if $a/p^r \in K$ for some $r \geq 1$ and $(a, q) = 1$, then $1/p^r \in K$. It follows that $1/p^n \in K$ for all $n \geq 1$ (take $n \geq t$ or $n \leq t$). Therefore $K = N$ and so qN is a maximal submodule. Clearly $\bigcap_{q \neq p} qN = 0$ and so N is isomorphic to a submodule of $\prod_{q \in P} N/qN$. Therefore $\prod_{q \in P} N/qN$ is a semiprime module [5, Proposition 2.1(e)]. We show that $N_{\mathbb{Z}}$ is not weakly compressible. Because if $N_{\mathbb{Z}}$ is weakly compressible, then $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \neq 0$ and since N is uniform, we must have $N \hookrightarrow \mathbb{Z}$, contradiction. Thus by [5, Proposition 2.1 (f)], $\prod_{q \in P} N/qN$ is not weakly compressible.

A ring R is called *right semi-Artinian* if every factor ring of R has a nonzero socle.

Example 3 Let R be a commutative regular ring which is not semi-Artinian (for example $R = \prod \mathbb{Z}_2$). Since R is a regular ring, $\mathrm{Rad}(M) = 0$ for all R-modules. Hence every R-module embeds in a product of simple modules. Note that the class of semiprime R-modules is closed under products [5, Proposition 2.1(e)]. On the other hand, since R is not semi-Artinian, there exists an R-module M which is not weakly compressible by Theorem 13 later in this section. Now if M embeds in a semiprime R-module L, then L is not weakly compressible by [5, Proposition 2.1(f)].

Now it is a natural question "under what conditions, a semiprime module is weakly compressible". In the following, the authors showed that under some conditions, it happens. An R-module M is called *torsionless* if $M \hookrightarrow R^A$ where A is a set.

Theorem 4 ([6, Proposition 2.2]) *Every torsionless module over a semiprime ring is weakly compressible.*

We say that an R-module M is dual semiprime and denoted as d-semiprime in short whenever the condition $M \in \mathrm{Cog}(N)$ implies $N \leq_{ess} M$ for any nonzero submodule N of M_R. A semisimple module M_R is d-semiprime if and only if its homogenous components have length 1. Clearly, every uniform module is a d-semiprime module which is not necessarily a semiprime module. For example, the module $\mathbb{Q}_{\mathbb{Z}}$ is not semiprime because $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. Further examples of d-semiprime modules include Duo modules (i.e., a module M_R is duo if every submodule of M_R is fully invariant).

Theorem 5 ([4, Theorem 2.2]) (a) *Every semiprime d-semiprime R-module is a weakly compressible module.*
(b) *Duo semiprime R-modules are weakly compressible.*
(c) *Every semiprime polyform R-module is weakly compressible.*

We say that a ring R is right (essentially) duo if every nonzero (essential) right ideal of R is an ideal. It can be shown that $n \times n$ (upper triangular) matrix rings over a field are right essentially duo and that are not right duo when $n > 1$. In the following, we investigate when a semiprime module with finite uniform dimension is weakly compressible.

Theorem 6 ([4, Theorems 2.8 and 2.9]) *Let R be a ring. Then every semiprime R-module with finite uniform dimension is weakly compressible if one of the following cases occurs.*
(a) *R is a right essentially duo ring;*
(b) *R is a commutative ring;*
(c) *R is a right FBN ring.*

In the following, authors study the class of semiprime and weakly compressible modules separately and characterize them on certain rings. Some equivalent conditions for weakly compressible modules have been obtained by Lomp.

Theorem 7 ([11, Theorem 5.1]) *The following statements are equivalent for an R-module M.*
(a) *M_R is weakly compressible;*
(b) $\mathrm{Hom}_R(M, N)^2 \neq 0$ *for all nonzero $N \leq M_R$;*
(c) $N \cap \mathrm{Rej}(M, N) = 0$ *for any nonzero $N \leq M_R$.*

In the following, authors give more equivalent conditions for a nonzero module M to be weakly compressible. Let M be an R-module and N be a submodule of M_R. We say that M is N-weakly compressible if for each nonzero submodule K of N, there exists an R-homomorphism $f : M \to K$ such that $f(K) \neq 0$.

Theorem 8 ([6, Theorem 2.5]) *The following conditions are equivalent for a nonzero R-module M.*
(a) *M_R is weakly compressible;*
(b) *For every nonzero $N \leq M$, there exists $f \in \mathrm{Hom}_R(M, N)$ such that $f^2 \neq 0$;*
(c) *$N \not\hookrightarrow \mathrm{Rej}(M, N)$, for every nonzero $N \leq M_R$;*
(d) *$M_1 \not\hookrightarrow \mathrm{Rej}(M, M_2)$ for all nonzero isomorphic R-modules M_1 and M_2;*
(e) *There exists an essential submodule N of M_R such that M is N-weakly compressible;*
(f) *There exists submodule N of M_R such that M is N-weakly compressible and M/N is weakly compressible;*
(g) *There exists a semiprime ideal I of R such that $MI = 0$ and M is Rej$(M, R/I)$-weakly compressible;*
(h) *M is $Z(M)$-weakly compressible and $M/\mathrm{Z}_2(M) \in \mathrm{Cog}(R/I)$ for some semiprime ideal $I \subseteq \mathrm{ann}_R(M)$.*

The condition (h) in Theorem 8 shows that the study of weakly compressible modules can be reduced to the study of such modules when they are either singular or nonsingular. A characterization of weakly compressible abelian groups has been obtained by Samsonova in [13].

Theorem 9 ([13, Main Theorem]) *A $\mathbb{Z}$-module M is weakly compressible if and only if $\mathrm{Z}(M)$ is semisimple and $M/\mathrm{Z}(M)$ is torsionless.*

In the next two results, semiprime and weakly compressible modules have been characterized over certain rings including prime hereditary Noetherian rings. If R is a hereditary Noetherian ring, then by [12, Proposition 5.4.5], every nonzero singular R-module has a nonzero socle. We call such rings R right singular semi-Artinian.

Theorem 10 ([6, Theorem 4.1]) *Suppose that R is a right singular semi-Artinian ring, M_R is nonzero and $MI = 0$ for some ideal I of R. If M_R is semiprime, then $M \in \mathrm{Cog}(\mathrm{Soc}(M) \oplus R/I)$. The converse holds if I is a prime ideal of R.*

Theorem 11 ([6, Theorem 4.5]) *Suppose that M is a nonzero R-module and $MI = 0$ for some semiprime ideal I of R. If $M \in \mathrm{Cog}(\mathrm{Soc}(M) \oplus R/I)$ and $M/\mathrm{Soc}(M) \in \mathrm{Cog}(R/I)$, then M_R is weakly compressible. The converse holds if R is a right singular semi-Artinian ring such that every cyclic R-module has a finitely generated socle or acc on direct summands.*

The result of Samsonova has been extended to PID by authors.

Theorem 12 ([6, Corollary 4.6]) *Let R be a prime right singular semi-Artinian ring such that cyclic R-modules have finite uniform dimensions. Then the following statements hold for M_R.*
(a) $M \in \text{Cog}(\text{Soc}(M) \oplus R)$ *and* $M/\text{Soc}(M) \in \text{Cog}(R)$ *if and only if* M_R *is weakly compressible;*
(b) $M \in \text{Cog}(\text{Soc}(M) \oplus R)$ *if and only if* M_R *is semiprime;*
(c) *If* M_R *is semiprime, then either* M_R *is semisimple or* $\text{Z}(M) = \text{Soc}(M)$*;*
(d) *Furthermore, if* R *is a PID, then* $M/\text{Soc}(M) \in \text{Cog}(R)$ *if and only if* M_R *is weakly compressible.*

In continue, we investigated rings whose every semiprime module is weakly compressible. Rings with the latter property are called SW. In order to study SW rings, they first investigate the stronger cases "when all R-modules are weakly compressible?" A ring R is called right duo (right quasi-duo) if every right ideal (right maximal ideal) of R is a two-sided ideal.

Theorem 13 ([5, Proposition 3.7]) *Let R be a ring. Consider the following conditions.*
(a) *R is a right semi-Artinian right V-ring;*
(b) *All R-modules are weakly compressible;*
(c) *All R-modules are semiprime;*
Then (a)$\Rightarrow$ (b) $\Leftrightarrow$ (c). All conditions are equivalent if the ring R is Morita equivalent to a right quasi-duo ring.

Theorem 14 ([5, Proposition 3.5 and Corollary 3.9]) (a) *If R is a right semi-Artinian ring, then R is SW.*
(b) *Let R be a strongly regular ring. Then R is SW if and only if R is a semi-Artinian ring.*

A class $\mathcal{C}$ of R-modules is said to be enveloping if for every R-module M there exist $E \in \mathcal{C}$ and $\theta : M \to E$ such that $(E,)$ is an envelop for M (i.e., (i) For every $\theta : M \to E' \in \mathcal{C}$ there exists $\varepsilon : E \to E$ with $\varepsilon\theta = \theta$. (ii) If in (i), $\varepsilon\theta = \theta$ then ε is an automorphism); see [7].

Theorem 15 ([5, Theorem 3.3]) *Let R be Morita equivalent to a right essentially duo ring,* $\text{Spec}(R)$ *the set of all prime ideals of R and* $\Lambda = \{P \in \text{Spec}(R) \mid P \leq_{ess} R_R\}$*. The following conditions are equivalent.*
(a) *R is SW;*
(b) *The R-module $\prod_{P\in\Lambda} R/P$ is weakly compressible;*
(c) *The class of weakly compressible R-modules is enveloping.*

A ring R is called semilocal if $\text{Max}(R)$ is a finite set where $\text{Max}(R) = \{P$ is an ideal of $R \mid R/P$ is a simple ring$\}$. In the following, we study Noetherian SW rings.

Theorem 16 ([5, Theorem 3.11]) *If R is a right duo right Noetherian SW ring, then R is a semilocal ring.*

The classical Krull dimension of a ring R, if exists, is denoted by $\dim(R)$. For a ring R, we have $\dim(R) = 0$ which means every prime ideal of R is a maximal ideal, see for example [12, 6.4.3]. Finally, we show that for Noetherian duo rings R with $\dim(R) \leqslant 1$, including integral Dedekind domains, the converse of Theorem 16 holds.

Theorem 17 ([5, Theorem 3.12]) *Let R be a right duo right Noetherian ring with* $\dim(R) \leqslant 1$. *Then the following statements are equivalent.*
(a) *R is a SW ring;*
(b) *R is a semilocal ring;*
(c) $\mathrm{Spec}(R)$ *is a finite set;*
(d) $\forall P \in \mathrm{Min}(R) \setminus \mathrm{Max}(R), \mathrm{J}(R/P) \neq 0$.

We end the paper by giving a characterization of SW rings that are the Dedekind domain.

Theorem 18 ([5, Corollary 3.13]) *Let R be a Dedekind domain. Then the class of weakly compressible R-modules is enveloping if and only if R is either simple Artinian or* $\mathrm{J}(R) \neq 0$.

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On Commutativity of Prime Rings with Involution Involving Pair of Derivations

Husain Alhazmi, Shakir Ali, Adnan Abbasi, and Muzibur Rahman Mozumder

Abstract The purpose of this paper is to investigate the commutativity of prime rings with involution '$*$' of the second kind in which a pair of derivations satisfy certain $*$-differential identities.

Keywords Prime ring · Involution · Commutativity · Derivation

Mathematics Subject Classification (2010): 16W10 · 16N60 · 16W25

1 Introduction

Throughout this article, R will represent an associative ring with center $Z(R)$. We denote $[x, y] = xy - yx$, the commutator of x and y and $x \circ y = xy + yx$, the anti-commutator of x and y. A ring R is said to 2-torsion free if $2x = 0$ (where $x \in R$) implies $x = 0$. A ring R is said to be prime if $aRb = (0)$ (where $a, b \in R$) implies either $a = 0$ or $b = 0$, and is called semiprime ring if $aRa = (0)$ (where $a \in R$) implies $a = 0$. A mapping $* : R \to R$ is called an involution if it satisfies the conditions $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-

H. Alhazmi
Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
e-mail: alhazmih@yahoo.com

S. Ali (✉) · A. Abbasi · M. R. Mozumder
Department of Mathematics, Aligarh Muslim University, Aligarh, India
e-mail: shakir.ali.mm@amu.ac.in; drshakir1971@gmail.com

A. Abbasi
e-mail: adnan.abbasi001@gmail.com

M. R. Mozumder
e-mail: muzibamu81@gmail.com

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hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. A ring equipped with an involution is known as ring with involution or $*$-ring. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case, $S(R) \cap Z(R) \neq (0)$. If R is 2-torsion free, then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. Note that in this case x is normal, i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring. An example is the ring of quaternions. A description of such rings can be found in [14], where further references can be found. A derivation on R is an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. Over the last 30 years, several authors have investigated the relationship between commutativity of prime and semiprime rings and certain special types of maps on R. The first result in this direction is due to Divinsky [13], who proved that a simple Artinian ring is commutative if it has a commuting non-trivial automorphism. Two years later, Posner [19] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the last few decades, several authors have refined and extended these results in various directions (see [5, 7–9, 11] where further references can be found).

In [1], the second author together with Dar established a $*$-version of classical theorem due to Posner's [19]. In fact, they proved that, if R is a prime ring with involution '$*$' such that char$(R) \neq 2$ and d is a nonzero derivation of R satisfying $[d(x), x^*] = 0$ for all $x \in R$ and $d(Z(R)) \neq (0)$, then R is normal. The mentioned theorem attracts attention of many researchers in the classical ring theory viz.; [2, 12, 15–18] where further references can be found. Very recently, Nejjar et al. [16] improved the result proved in [1] and established the commutativity of prime rings with involution in more general setting. The main purpose of this paper is to establish a corresponding result obtained in [1, 16], respectively, for a pair of derivations.

2 Main Results

Our first theorem is motivated by the $*$-version of Posner's theorem proved in [1]. In fact, we prove the following result for a pair of derivations.

Theorem 2.1 *Let R be a prime ring with involution '$*$' of the second kind such that char$(R) \neq 2$. Let d_1 and d_2 be nonzero derivations of R. Then following statements are equivalent:*

(i) $d_1(x)x^* - x^*d_2(x) = 0$ *for all* $x \in R$,
(ii) *R is a commutative integral domain and* $d_1 = d_2$.

Proof We need only to prove nontrivial implication.
$(i) \implies (ii)$ By the assumption, we have

$$d_1(x)x^* - x^*d_2(x) = 0 \text{ for all } x \in R. \tag{2.1}$$

On linearization, we get

$$d_1(x)y^* + d_1(y)x^* - x^*d_2(y) - y^*d_2(x) = 0 \text{ for all } x, y \in R. \tag{2.2}$$

Replacing y by hy, where $y \in R$ and $h \in H(R) \cap Z(R)$, we get

$$d_1(x)y^*h + d_1(y)x^*h + yd_1(h)x^* - x^*d_2(y)h - x^*yd_2(h) - y^*hd_2(x) = 0$$

for all $x, y \in R$. Using (2.2), we obtain

$$d_1(h)yx^* - x^*yd_2(h) = 0 \text{ for all } x, y \in R. \tag{2.3}$$

Taking $x = y = k$ where $k \in S(R) \cap Z(R)$ in (2.3), we get

$$(d_1(h) - d_2(h))k^2 = 0 \text{ for all } h \in H(R) \cap Z(R) \text{ and } k \in S(R) \cap Z(R). \tag{2.4}$$

Using the primeness of R, we have $d_1(h) - d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$ or $k^2 = 0$ for all $k \in S(R) \cap Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, we are forced to conclude that $d_1(h) = d_2(h)$ for all $h \in H(R) \cap Z(R)$. Thus in view of (2.3), we get $(yx^* - x^*y)d_1(h) = 0$ for all $x, y \in R$. Again by the primeness of R either $yx^* - x^*y = 0$ for all $x, y \in R$ or $d_1(h) = 0$ for all $h \in H(R) \cap Z(R)$. Now $yx^* - x^*y = 0$ for all $x, y \in R$ implies that R is commutative. Therefore, we are left with $d_1(h) = 0$ for all $h \in H(R) \cap Z(R)$. Replace y by ky in (2.2) where $k \in S(R) \cap Z(R)$ and add with k times (2.2), we get $2(d_1(y)x^* - x^*d_2(y))k = 0$ for all $x, y \in R$. Invoking the primeness of R and using the fact that char$(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we arrive at $d_1(y)x^* - x^*d_2(y) = 0$ for all $x, y \in R$. Hence in view of [10, Theorem 4.1], R is commutative. Then (2.1) becomes $(d_1(x) - d_2(x))x^* = 0$ for all $x \in R$, that is

$$D(x)x^* = 0 \text{ for all } x \in R, \tag{2.5}$$

where D is a derivation defined by $D = d_1 - d_2$. Moreover, it is easy to see that (2.5) implies $D = 0$ and thus $d_1 = d_2$. This completes the proof of the theorem. □

Remark 2.1 In the above theorem, if we replace d_2 by $-d_2$, we get $d_1(x)x^* + x^*d_2(x) = 0$ for all $x \in R$. Therefore, if R satisfies the $*$-differential identity $d_1(x)x^* + x^*d_2(x) = 0$ for all $x \in R$, then R is a commutative integral domain and in this particular situation, we conclude that $d_1 = -d_2$.

Theorem 2.2 *Let R be a prime ring with involution '$*$' of the second kind such that char$(R) \neq 2$. Let d_1 and d_2 be derivations of R. Then following statements are equivalent:*

(i) $d_1(x) \circ d_2(x^*) = [x, x^*]$ *for all* $x \in R$,
(ii) *R is a commutative integral domain and either $d_1 = 0$ or $d_2 = 0$.*

Proof $(i) \implies (ii)$. If either d_1 or d_2 is zero, then we have either $[x, x^*] = 0$ for all $x \in R$. Then in view of [16, Lemma 2.1], R is commutative. Next we assume that both d_1 and d_2 are nonzero. Then by the given assumption, we have

$$d_1(x) \circ d_2(x^*) = [x, x^*] \text{ for all } x \in R. \tag{2.6}$$

This can be written as

$$d_1(x)d_2(x^*) + d_2(x^*)d_1(x) = xx^* - x^*x \text{ for all } x \in R. \tag{2.7}$$

Replacing x by $x + y$ in (2.7), we get

$$\begin{aligned} &d_1(x)d_2(x^*) + d_1(x)d_2(y^*) + d_1(y)d_2(x^*) + d_1(y)d_2(y^*) \\ &+d_2(x^*)d_1(x) + d_2(x^*)d_1(y) + d_2(y^*)d_1(x) + d_2(y^*)d_1(y) \\ &= xx^* + xy^* + yx^* + yy^* - x^*x - x^*y - y^*x - y^*y \text{ for all } x, y \in R. \end{aligned} \tag{2.8}$$

Using (2.7) in (2.8), we have

$$d_1(x)d_2(y^*) + d_1(y)d_2(x^*) + d_2(x^*)d_1(y) + d_2(y^*)d_1(x) = xy^* + yx^* - x^*y - y^*x \tag{2.9}$$

for all $x, y \in R$. Substituting hy by y in (2.9) where $h \in H(R) \cap Z(R)$, we obtain

$$\begin{aligned} &d_1(x)d_2(y^*)h + d_1(x)y^*d_2(h) + d_1(h)yd_2(x^*) + hd_1(y)d_2(x^*) \\ &+d_2(x^*)d_1(h)y + d_2(x^*)hd_1(y) + d_2(y^*)hd_1(x) + y^*d_2(h)d_1(x) \\ &= (xy^* + yx^* - x^*y - y^*x)h \text{ for all } x, y \in R. \end{aligned}$$

Multiplying (2.9) by h and combining it with (2.9), we get

$$(d_1(x) \circ y^*)d_2(h) + d_1(h)(y \circ d_2(x^*)) = 0 \tag{2.10}$$

for all $x, y \in R$. Replacing y by ky in (2.10) where $k \in S(R) \cap Z(R)$, we arrive at

$$-(d_1(x) \circ y^*)kd_2(h) + d_1(h)k(y \circ d_2(x^*)) = 0$$

for all $x, y \in R$. Multiplying (2.10) by k where $k \in S(R) \cap Z(R)$ and adding with last relation, we get

$$2d_1(h)k(y \circ d_2(x^*)) = 0 \text{ for all } x, y \in R.$$

Since $\text{char}(R) \neq 2$ and R is prime, we get either $d_1(h) = 0$ or $y \circ d_2(x^*) = 0$. Now suppose $y \circ d_2(x^*) = 0$. This further implies that $y \circ d_2(x) = 0$ for all $x, y \in R$.

Thus R is commutative in view of [20, Theorem 1]. On the other hand, suppose $d_1(h) = 0$ for all $h \in H(R) \cap Z(R)$. This in turn implies that $d_1(z) = 0$ for all $z \in Z(R)$. Thus in view of (2.10), we get $(d_1(x) \circ y^*)d_2(h) = 0$ for all $x, y \in R$. Again using primeness, we get either $(d_1(x) \circ y^*)$ for all $x, y \in R$ or $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. Suppose $(d_1(x) \circ y^*) = 0$ for all $x, y \in R$, then R is commutative [20, Theorem 1]. Now consider the case $d_2(h) = 0$, which further implies that $d_2(z) = 0$ for all $z \in Z(R)$. Replacing y by ky in (2.9) where $y \in R$ and $k \in S(R) \cap Z(R)$, we get

$$(-d_1(x)d_2(y^*) + d_1(y)d_2(x^*) + d_2(x^*)d_1(y) - d_2(y^*)d_1(x))k$$

$$= (-xy^* + yx^* - x^*y + y^*x)k \text{ for all } x, y \in R.$$

Multiplying (2.9) by k where $k \in S(R) \cap Z(R)$ and adding with the previous equation, we finally arrive at

$$2(d_1(y)d_2(x^*) + d_2(x^*)d_1(y) - [y, x^*]) = 0 \text{ for all } x, y \in R.$$

Substituting x^* for x, and using the fact that char$(R) \neq 2$, we get

$$(d_1(y)d_2(x) + d_2(x)d_1(y) - [y, x]) = 0 \text{ for all } x, y \in R.$$

This implies that

$$(d_1(y) \circ d_2(x) - [y, x]) = 0 \text{ for all } x, y \in R. \tag{2.11}$$

Thus in view of [4, Theorem 2.4 for $n = m = 1$], we get that R is commutative. Henceforward, the relation (2.11) reduces to $2d_1(y)d_2(x) = 0$ for all $x, y \in R$. Since char$(R) \neq 2$, the last relation gives that $d_1(y)d_2(x) = 0$ for all $x, y \in R$. This implies that either $d_1 = 0$ or $d_2 = 0$, which leads a contradiction, i.e., we are forced to conclude that at least one of them d_1 and d_2 must be zero. This completes the proof. □

Theorem 2.3 *Let R be prime ring with involution '$*$' of the second kind such that char$(R) \neq 2$. Let d_1 and d_2 be two nonzero derivations of R such that $[d_1(x), x^*d_2(x)] = 0$ for all $x \in R$, then R is a commutative integral domain.*

Proof Assume that

$$[d_1(x), x^*d_2(x)] = 0 \text{ for all } x \in R. \tag{2.12}$$

Linearization of (2.12) gives us

$$[d_1(x), x^*d_2(x)] + [d_1(x), x^*d_2(y)] + [d_1(x), y^*d_2(x)] + [d_1(x), y^*d_2(y)]$$

$$+[d_1(y), x^*d_2(x)] + [d_1(y), x^*d_2(y)] + [d_1(y), y^*d_2(x)] + [d_1(y), y^*d_2(y)] = 0$$

for all $x, y \in R$. Application of (2.12) yields that

$$[d_1(x), x^*d_2(y)] + [d_1(x), y^*d_2(x)] + [d_1(x), y^*d_2(y)] \tag{2.13}$$

$$+[d_1(y), x^*d_2(x)] + [d_1(y), x^*d_2(y)] + [d_1(y), y^*d_2(x)] = 0 \text{ for all } x, y \in R.$$

Replacing y by h where $h \in H(R) \cap Z(R)$, we get

$$[d_1(x), x^*]d_2(h) + h[d_1(x), d_2(x)] = 0 \text{ for all } x \in R. \tag{2.14}$$

Linearization on x gives

$$([d_1(x), y^*] + [d_1(y), x^*])d_2(h) + h([d_1(x), d_2(y)] + [d_1(y), d_2(x)]) = 0 \tag{2.15}$$

for all $x, y \in R$. Now taking hy for y where $y \in R$ and $h \in H(R) \cap Z(R)$, we obtain

$$[d_1(x), y^*]hd_2(h) + [d_1(y), x^*]hd_2(h) + [y, x^*]d_1(h)d_2(h) + h^2[d_1(x), d_2(y)] \tag{2.16}$$

$$+h[d_1(x), y]d_2(h) + h^2[d_1(y), d_2(x)] + hd_1(h)[y, d_2(x)] = 0 \text{ for all } x, y \in R.$$

Multiplying (2.15) by h and using in (2.16), we get

$$[y, x^*]d_1(h)d_2(h) + h[d_1(x), y]d_2(h) + hd_1(h)[y, d_2(x)] = 0 \tag{2.17}$$

for $x, y \in R$ and $h \in H(R) \cap Z(R)$. Replacing x by kx where $x \in R$ and $k \in S(R) \cap Z(R)$, we arrive at

$$-[y, x^*]kd_1(h)d_2(h) + hk[d_1(x), y]d_2(h) + h[x, y]d_1(k)d_2(h) \tag{2.18}$$

$$+hd_1(h)[y, d_2(x)]k + hd_1(h)[y, x]d_2(k) = 0 \text{ for all } x, y \in R.$$

Multiplying (2.17) by k where $k \in S(R) \cap Z(R)$ and adding with (2.18), we get

$$2hk([d_1(x), y]d_2(h) + [y, d_2(x)]d_1(h)) + h[x, y]d_1(k)d_2(h) + hd_1(h)[y, x]d_2(k) = 0$$

for all $x, y \in R$. Taking $y = x$, the last expression gives

$$2hk([d_1(x), x]d_2(h) + [x, d_2(x)]d_1(h)) = 0 \text{ for all } x \in R$$

Since $\text{char}(R) \neq 2$, $S(R) \cap Z(R) \neq (0)$, and R is prime, we conclude that

$$[d_1(x), x]d_2(h) + [x, d_2(x)]d_1(h) = 0 \text{ for all } x \in R. \tag{2.19}$$

Replacing y by x in (2.17), we get

$$[x, x^*]d_1(h)d_2(h) + h[d_1(x), x]d_2(h) + hd_1(h)[x, d_2(x)] = 0 \qquad (2.20)$$

for $x \in R$. Using (2.19) in (2.20), we get $[x, x^*]d_1(h)d_2(h) = 0$ for all $x \in R$. By the primeness of R, we get either $[x, x^*] = 0$ for all $x \in R$ or $d_1(h)d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. If we consider $[x, x^*] = 0$ for all $x \in R$. Hence, this implies that R is normal. Then in view of [1, Lemma 2.1], R is commutative. Now consider the second part $d_1(h)d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. Using primeness condition, we get either $d_1(h) = 0$ or $d_2(h) = 0$. If consider $d_1(h) = 0$ for all $h \in H(R) \cap Z(R)$, then by (2.17), we get $h[d_1(x), y]d_2(h) = 0$ for all $x, y \in R$. Further, the primeness condition yields $[d_1(x), y]d_2(h) = 0$ for all $x, y \in R$. Again by the primeness condition, we have either $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$ or $[d_1(x), y] = 0$ for all $x, y \in R$. If we consider $[d_1(x), y] = 0$ for all $x, y \in R$. This gives R is commutative by Posner's result [19]. Therefore, we are left with the case $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. Similarly in view of (2.17), we get either $d_1(h) = 0$ for all $h \in H(R) \cap Z(R)$. Replacing y by h where $h \in H(R) \cap Z(R)$ in (2.13) and using $d_1(h) = 0$ and $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$, we get $h[d_1(x), d_2(x)] = 0$ for all $x \in R$. Now using the primeness and $S(R) \cap Z(R) \neq (0)$ conditions, we get

$$[d_1(x), d_2(x)] = 0 \text{ for all } x \in R. \qquad (2.21)$$

Replacing x by $x + y^*$, we get

$$[d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)] = 0 \text{ for all } x, y \in R. \qquad (2.22)$$

Taking yk for y in (2.22), where $k \in S(R) \cap Z(R)$, we get $2[d_1(y), d_2(x^*)]k = 0$ for all $x, y \in R$, since char$(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get $[d_1(y), d_2(x)] = 0$ for all $x, y \in R$. Thus in view of [3, Theorem 3.1], we obtain R is commutative. □

Theorem 2.4 *Let R be a prime ring with involution '$*$' of the second kind such that char$(R) \neq 2$. There are no derivations d_1 and d_2 of R such that $[d_1(x), d_2(x^*)] = x \circ x^*$ for all $x \in R$.*

Proof If either d_1 or d_2 is zero, then we have $x \circ x^* = 0$ for all $x \in R$, which assures that $x = 0$ for all $x \in R$ so that R is a trivial ring $R = \{0\}$, which is impossible. Henceforth, we assume that both d_1 and d_2 are nonzero, and we have

$$[d_1(x), d_2(x^*)] = x \circ x^* \text{ for all } x \in R. \qquad (2.23)$$

Replacing x by $x + y$ in (2.23), we get

$$[d_1(x), d_2(x^*)] + [d_1(y), d_2(y^*)] + [d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)]$$

$$= x \circ x^* + y \circ y^* + xy^* + yx^* + x^*y + y^*x$$

for all $x, y \in R$. Application of (2.23) gives

$$[d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)] = xy^* + yx^* + x^*y + y^*x \tag{2.24}$$

for all $x, y \in R$. Taking hy for y in (2.24) where $x, y \in R$ and $h \in H(R) \cap Z(R)$, we have

$$\begin{aligned} h([d_1(x), d_2(y^*)] + [d_1(y), d_2(x^*)]) + [d_1(x), y^*]d_2(h) + d_1(h)[y, d_2(x^*)] \\ = h(xy^* + yx^* + x^*y + y^*x) \text{ for all } x, y \in R. \end{aligned} \tag{2.25}$$

Using (2.24), (2.25) reduces to

$$[d_1(x), y^*]d_2(h) + d_1(h)[y, d_2(x^*)] = 0 \text{ for all } x, y \in R. \tag{2.26}$$

Replacing y by ky, where $k \in S(R) \cap Z(R)$, we get

$$-[d_1(x), y^*]kd_2(h) + d_1(h)k[y, d_2(x^*)] = 0 \text{ for all } x, y \in R. \tag{2.27}$$

Multiplying (2.26) by k and adding with (2.27), we obtain

$$2d_1(h)k[y, d_2(x^*)] = 0 \text{ for all } x, y \in R.$$

Since $\text{char}(R) \neq 2$, the last expression gives

$$d_1(h)k[y, d_2(x^*)] = 0 \text{ for all } x, y \in R.$$

Invoking the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we get

$$\text{either } d_1(h) = 0 \text{ or } [y, d_2(x^*)] = 0 \text{ for all } x, y \in R. \tag{2.28}$$

If $[y, d_2(x^*)] = 0$ implies that R is commutative in view of Posner's result [19]. Therefore, we are left with $d_1(h) = 0$ for all $h \in H(R) \cap Z(R)$. Using this in (2.27), we obtain

$$-[d_1(x), y^*]kd_2(h) = 0 \text{ for all } x, y \in R.$$

Using the primeness, we get

$$d_2(h) = 0 \text{ for all } h \in H(R) \cap Z(R). \tag{2.29}$$

or

$$[d_1(x), y^*] = 0 \text{ for all } x, y \in R.$$

Again if $[d_1(x), y^*] = 0$, using Posner's result [19], we conclude that R is commutative. Therefore, we are left with the case $d_2(h) = 0$ for all $h \in H(R) \cap Z(R)$. This implies that $d_2(k) = 0$ for all $k \in S(R) \cap Z(R)$ and hence $d_2(Z(R)) = (0)$. Similarly in view of (2.28), we get $d_1(Z(R)) = (0)$. Now replace y by ky in (2.24) where $k \in S(R) \cap Z(R)$, we get

$$\begin{aligned} &- [d_1(x), d_2(y^*)]k - [d_1(x), y^*]d_2(k) + d_1(k)[y, d_2(x^*)] + k[d_1(y), d_2(x^*)] \\ &= -xy^*k + kyx^* + x^*ky - ky^*x \end{aligned} \tag{2.30}$$

for all $x, y \in R$ and $k \in S(R) \cap Z(R)$. Now multiplying (2.24) by $k \in S(R) \cap Z(R)$ and adding with (2.30), we get

$$2[d_1(y), d_2(x^*)]k = 2k(yx^* + x^*y) \text{ for all } x, y \in R.$$

Since $\text{char}(R) \neq 2$, the last relation implies that

$$k([d_1(y), d_2(x^*)] - (y \circ x^*)) = 0 \text{ for all } x, y \in R.$$

The primeness of R yields that

$$[d_1(y), d_2(x^*)] - (y \circ x^*) = 0 \text{ for all } x, y \in R.$$

Now replace x by x^* to get

$$[d_1(y), d_2(x)] - (y \circ x) = 0 \text{ for all } x, y \in R.$$

Taking y by h, where $h \in H(R) \cap Z(R)$, we get $2xh = 0$ for all $x \in R$. Since $\text{char}(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we arrive at $x = 0$ for all $x \in R$, which gives a contradiction. Hence R is commutative. Then, Eq. (2.23) gives $2xx^* = 0$ for all $x \in R$. Since $\text{char}(R) \neq 2$, this implies that $xx^* = 0$ for all $x \in R$, which obviously leads to $x = 0$ for all $x \in R$, which is impossible. Hence, there exist no such derivations for which $[d_1(x), d_2(x^*)] = x \circ x^*$ for all $x \in R$. This completes the proof of the theorem. □

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Analysis

Approximation of a Common Fixed Point of Two Nonlinear Mappings with Nonsummable Errors in a Banach Space

Takanori Ibaraki, Shunsuke Kajiba, and Yasunori Kimura

Abstract In this paper, we study an iterative scheme for two nonlinear mappings with errors in a uniformly convex and uniformly smooth Banach space. We consider that the sequence including errors for obtaining the value of metric projection still has an efficient property for approximating a common fixed point of two mappings. Moreover, we prove a convergence theorem for a common fixed point of two mappings in a Banach space.

Keywords Common fixed point · Relatively nonexpansive · Quasinonexpansive · Shrinking projection method · Iterative scheme · Metric projection · Error term

Mathematics Subject Classification (2010) 47H05 · 47H09 · 47J25

1 Introduction

Let E be a real Banach space. We say that a mapping T from a nonempty closed convex subset C of E to E is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. The fixed point problem for a nonexpansive mapping is defined so as to find a fixed point of T, that is, a point $z \in C$ satisfying that $z = Tz$. We may apply

T. Ibaraki (✉)
Department of Mathematics Education, Yokohama National University, Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan
e-mail: ibaraki@ynu.ac.jp

S. Kajiba
Chiba Reimei High school, 625 Ho, Yachimata-shi,, Chiba 289-1115, Japan
e-mail: kajiba@reimei.ac.jp

Y. Kimura
Department of Information Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan
e-mail: yasunori@is.sci.toho-u.ac.jp

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it to various types of nonlinear problems, such as convex minimization problems, variational inequality problems, equilibrium problems, and others.

For this problem, we consider not only the existence but also the approximation methods of its solution. In the metric fixed point theory, iterative methods for nonlinear mapping are one of the most important topics and a number of researchers have been dealing with this issue in recent research. In particular, the shrinking projection method proposed by Takahashi et al. [16] is a remarkable result.

Theorem 1.1 (Takahashi-Takeuchi–Kubota [16]) *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) = \{p \in C : p = Tp\}$ is nonempty. Let $\{\alpha_n\}$ be a nonnegative real sequence in $[0, a[$, where $0 < a < 1$. For a point x_0 in H chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $C_1 = C$, $x_1 = P_{C_1}x_0$, and*

$$
\begin{aligned}
y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n,\\
C_{n+1} &= \{z \in C : \|z - y_n\| \le \|z - x_n\|\} \cap C_n,\\
x_{n+1} &= P_{C_{n+1}}x_0
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H.

We should remark that the original result of this method is a convergence theorem to a common fixed point of a family of nonexpansive mappings.

In the original shrinking projection method, we use metric projections for each step to generate an iterative sequence. However, to calculate the exact value of metric projection becomes much more difficult as the iteration proceeds since the shape of the corresponding convex sets gets complicated gradually.

Recently, Kimura [6] considers an error to obtain the value of metric projections and proves that the sequence still has an efficient property for approximating a fixed point of the mapping. Later, this method has been generalized to several different cases in the setting of a Hilbert space and a Banach space; see [3, 4, 7, 8] and others.

In this paper, we study an iterative scheme for two nonlinear mappings of nonexpansive type with errors in a uniformly convex and uniformly smooth Banach space. We discuss an approximation method of a common fixed point by using shrinking projection method with errors and prove a strong convergence theorem to a common fixed point of two mappings.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and the dual space E^* and let $S_E := \{x \in E : \|x\| = 1\}$. We denote strong convergence and weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. E is said to be strictly convex if $\|x + y\| < 2$ for all $x, y \in S_E$ with $x \neq y$ and E is said to be uniformly convex if $\lim_n \|x_n - y_n\| = 0$ where $\{x_n\}, \{y_n\} \subset S_E$ such that $\lim_n \|x_n + y_n\| = 2$. E is said to have the Kadec–Klee property if a sequence $\{x_n\}$ of E converges strongly to x_0 whenever it satisfies $x_n \rightharpoonup x_0$ and $\|x_n\| \to \|x_0\|$.

A Banach space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S_E$. E is said to be uniformly smooth if the limit (2.1) converges uniformly for all $x, y \in S_E$; see [14, 15] for more details.

The normalized duality mapping $J : E \to E^*$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. We know the following properties; see [2, 14, 15] for more details.

(1) Jx is nonempty for each $x \in E$;
(2) $\langle x - y, x^* - y^* \rangle \geq 0$ for each $(x, x^*), (y, y^*) \in J$;
(3) E is strictly convex if and only if J is one-to-one;
(4) E is reflexive if and only if J is surjective;
(5) E is smooth if and only if J is single-valued;
(6) if E is reflexive, smooth, and strictly convex, then the duality mapping $J^* : E^* \to E$ is the inverse of J, that is, $J^* = J^{-1}$;
(7) E is uniformly smooth if and only if E^* is uniformly convex;
(8) if E is uniformly convex, then E is reflexive, strictly convex, and having the Kadec–Klee property.

Let E be a reflexive and strictly convex Banach space and let C be a nonempty closed convex subset of E. It is known that for each $x \in E$, there exists a unique point $z \in C$ such that

$$\|x - z\| = \min_{y \in C} \|x - y\|.$$

Such a point z is denoted by $P_C x$, and P_C is called the metric projection of E onto C.

Let E be a Banach space and let $C_1, C_2, C_3, \ldots$ be a sequence of nonempty closed convex subset of E. We denote by $\text{s-Li}_n C_n$ the set of limit points of $\{C_n\}$, that is, $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $x_n \to x$ and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, we denote by $\text{w-Ls}\, C_n$ the set of cluster points of $\{C_n\}$; $y \in \text{w-Ls}\, C_n$ if and only if there exists $\{y_{n_i}\}$ such that $y_{n_i} \rightharpoonup y$ and that $y_{n_i} \in C_{n_i}$ for all $i \in \mathbb{N}$. Using these definitions, we define the Mosco convergence [12] of $\{C_n\}$. If C_0 satisfies

$$\operatorname*{s-Li}_n C_n = C_0 = \operatorname*{w-Ls}_n C_n,$$

we say that $\{C_n\}$ is a Mosco convergent sequence to C_0 and write

$$C_0 = \operatorname*{M-lim}_{n\to\infty} C_n.$$

Notice that the inclusion s-Li$_n$ $C_n \subset$ w-Ls C_n is always true. Therefore, to show the existence of M-lim$_n$ C_n, it is sufficient to prove w-Ls$_n$ $C_n \subset$ s-Li$_n$ C_n; see [12] for more details. In 1984, Tsukada [17] proved the following theorem for the metric projections in a Banach space.

Theorem 2.1 (Tsukada [17]) *Let E be a reflexive and strictly convex Banach space having the Kadec–Klee property and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E. If $C_0 =$ M-lim$_n$ C_n exists and is nonempty, then $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$.*

One of the simplest of the sequence $\{C_n\}$ satisfying the condition in the theorem above is a decreasing sequence with respect to inclusion; $C_{n+1} \subset C_n$ for every $n \in \mathbb{N}$. In this case, M-lim$_n$ $C_n = \bigcap_{n=1}^{\infty} C_n$.

Let E be a smooth Banach space, define a function $V : E \times E \to \mathbb{R}$ by

$$V(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,$$

where $E \times E$ is the product set of E and itself. We know the following properties; see [1, 5, 11] for more details.

(1) $(\|x\| - \|y\|)^2 \le V(x, y) \le (\|x\| + \|y\|)^2$ for each $x, y \in E$;
(2) $V(x, y) + V(y, x) = 2\langle x - y, Jx - Jy\rangle$ for each $x, y \in E$;
(3) $V(x, y) = V(x, z) + V(z, y) + 2\langle x - z, Jz - Jy\rangle$ for each $x, y, z \in E$;
(4) if E is additionally assumed to be strictly convex, then $V(x, y) = 0$ if and only if $x = y$ for each $x, y \in E$.

Let $B_r = \{x \in E : \|x\| \le r\}$, where r is a positive real number. The following results show that the existence of mappings $\underline{g}_r$, $\overline{g}_r$, $\underline{g}_r^*$, and $\overline{g}_r^*$, related to the convex structures of a Banach space E and its dual space. These mappings play important roles in our result.

Theorem 2.2 (Xu [18]) *Let E be a Banach space and $r > 0$. Then,*

(i) *if E is uniformly convex, then there exists continuous, strictly increasing, and convex function $\underline{g}_r : [0, 2r] \to [0, \infty[$ with $\underline{g}_r(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \le \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\underline{g}_r(\|x - y\|)$$

for all $x, y \in B_r$ and $0 \le \alpha \le 1$.

(ii) *if E is uniformly smooth, then there exists continuous, strictly increasing, and convex function $\overline{g}_r : [0, 2r] \to [0, \infty[$ with $\overline{g}_r(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \ge \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\overline{g}_r(\|x - y\|)$$

for all $x, y \in B_r$ and $0 \le \alpha \le 1$.

As a direct consequence of Theorem 2.2, we obtain the following result.

Theorem 2.3 *Let E be a reflexive, smooth, and strictly convex Banach space and $r > 0$. Then,*

(i) *if E is uniformly smooth, then there exists continuous, strictly increasing, and convex function $\underline{g}_r^* : [0, 2r] \to [0, \infty[$ with $\underline{g}_r^*(0) = 0$ such that*

$$\|\alpha Jx + (1-\alpha)Jy\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\underline{g}_r^*(\|Jx - Jy\|)$$

for all $x, y \in B_r$ and $0 \leq \alpha \leq 1$.

(i) *if E is uniformly convex, then there exists continuous, strictly increasing, and convex function $\overline{g}_r^* : [0, 2r] \to [0, \infty[$ with $\overline{g}_r^*(0) = 0$ such that*

$$\|\alpha Jx + (1-\alpha)Jy\|^2 \geq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\overline{g}_r^*(\|Jx - Jy\|)$$

for all $x, y \in B_r$ and $0 \leq \alpha \leq 1$.

From Theorems 2.2 and 2.3, we can show the following results.

Theorem 2.4 (Kimura [8]) *Let E be a uniformly smooth and uniformly convex Banach space and $r > 0$. Then, the functions $\underline{g}_r$ and $\overline{g}_r$ in Theorem 2.2 satisfy*

$$\underline{g}_r(\|x - y\|) \leq V(x, y) \leq \overline{g}_r(\|x - y\|)$$

for all $x, y \in B_r$.

Theorem 2.5 (Ibaraki–Kimura [4]) *Let E be a uniformly smooth and uniformly convex Banach space and $r > 0$. Then, the functions $\underline{g}_r^*$ and $\overline{g}_r^*$ in Theorem 2.3 satisfy*

$$\underline{g}_r^*(\|Jx - Jy\|) \leq V(x, y) \leq \overline{g}_r^*(\|Jx - Jy\|)$$

for all $x, y \in B_r$.

Let C be a nonempty closed convex subset of a smooth Banach space E. A mapping $T : C \to E$ is said to be relatively nonexpansive [10, 11] if $\hat{F}(T) = F(T) \neq \emptyset$ and

$$V(u, Tx) \leq V(u, x)$$

for every $u \in F(T)$ and $x \in C$, where $F(T)$ is the set of all fixed points of T and $\hat{F}(T)$ is the set of all asymptotic fixed points of T by [13] . Namely, $u \in \hat{F}(T)$ if and only if there exists a sequence $\{u_n\} \subset C$ such that $\{u_n\}$ converges weakly to u and that $\lim_n \|u_n - Tu_n\| = 0$. It was proved in [11] that, if E is smooth and strictly convex Banach space, then $F(T)$ is closed and convex. We note that there are a lot of important examples of relatively nonexpansive mappings; see [11]. In our main result, we use relatively nonexpansive mappings without the condition

$\hat{F}(T) = F(T)$. Instead of it, we assume the closedness of $I - T$ at zero, that is, if a sequence $\{x_n\}$ converges strongly to x_0 and $\{x_n - Tx_n\}$ converges strongly to 0, then $x_0 - Tx_0 = 0$.

3 The Main Result

In this section, we propose an approximation theorem for two nonlinear mappings in a uniformly convex and uniformly smooth Banach space.

Theorem 3.1 *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty bounded closed convex subset of E and let $r > 0$ such that $C \subset B_r$. Let T_1, T_2 be mappings from C to E such that $F(T_1) \cap F(T_2) \neq \emptyset$ and $V(p, T_i x) \leq V(p, x)$ for every $x \in C$ and $p \in F(T_i)$ for $i \in \{1, 2\}$. Let $\{\alpha_{n,i} : n \in \mathbb{N}, i \in \{1, 2\}\}$ be a family of positive real numbers such that $\alpha_{n,1} + \alpha_{n,2} = 1$, and $\liminf_n \alpha_{n,i} > 0$ for $i \in \{1, 2\}$. Let $\{\delta_n\}$ be a nonnegative real sequence and $\delta_0 = \limsup_n \delta_n$. For given $u \in E$, generate an iterative sequence $\{x_n\}$ as follows: $x_1 \in C$, $C_1 = C$,*

$$\begin{aligned} y_n &= J^{-1}(\alpha_{n,1} JT_1 x_n + \alpha_{n,2} JT_2 x_n), \\ C_{n+1} &= \{z \in C : V(z, y_n) \leq V(z, x_n)\} \cap C_n, \\ x_{n+1} &\in \{z \in C : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}\} \cap C_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - T_i x_n\| \leq 2\underline{g}_r^{-1}\left(\frac{1}{2}\delta_0 + \frac{1}{2\underline{\alpha}_i}(\zeta_0 + \eta_0)\right)$$

for each $i \in \{1, 2\}$, where

$$\zeta_0 = \overline{g}_r(\underline{g}_r^{-1}(\delta_0)),$$

$$\eta_0 = \overline{\alpha}_1\overline{\alpha}_2\overline{g}_r\left(\underline{g}_r^{*-1}\left(\frac{1}{\underline{\alpha}_1\underline{\alpha}_2}\overline{g}_r\left(\underline{g}_r^{-1}(\delta_0)\right) + \frac{8r}{\underline{\alpha}_1\underline{\alpha}_2}\underline{g}_r^{*-1}\left(\overline{g}_r(\underline{g}_r^{-1}(\delta_0))\right)\right)\right),$$

$$\underline{\alpha}_i = \liminf_{n\to\infty} \alpha_{n,i}, \quad \overline{\alpha}_i = \limsup_{n\to\infty} \alpha_{n,i}$$

for each $i \in \{1, 2\}$. Moreover, if $\delta_0 = 0$ and $I - T_i$ is closed at zero for each $i \in \{1, 2\}$, then $\{x_n\}$ converges strongly to $P_{F(T_1)\cap F(T_2)}u$.

Proof Let $F := F(T_1) \cap F(T_2)$. We first show that C_n is a closed convex subset such that $F \subset C_n$ for each $n \in \mathbb{N}$ by induction. It is trivial that $F \subset C_1 = C$ and a given point x_1 is defined. It is also obvious that C_n is closed and convex for all $n \in \mathbb{N}$. Suppose that each of $C_1, C_2, \ldots, C_k$ contains F. Then, since C_k is nonempty, we can choose a point $x_k \in C_k$ satisfying the condition in the theorem. Then y_k and C_{k+1} are also defined. Let $p \in F$. Since it follows from the convexity of $\|\cdot\|^2$ that

$$\begin{aligned} V(p, y_k) &= V(p, J^{-1}(\alpha_{k,1} JT_1 x_k + \alpha_{k,2} JT_2 x_k)) \\ &\leq \|p\|^2 - 2\alpha_{k,1}\langle p, JT_1 x_k\rangle - 2\alpha_{k,2}\langle p, JT_2 x_k\rangle \\ &\qquad + \alpha_{k,1}\|JT_1 x_k\|^2 + \alpha_{k,2}\|JT_2 x_k\|^2 \\ &= \alpha_{k,1} V(p, T_1 x_k) + \alpha_{k,2} V(p, T_2 x_k) \\ &\leq \alpha_{k,1} V(p, x_k) + \alpha_{k,2} V(p, x_k) = V(p, x_k), \end{aligned}$$

we have $p \in C_{k+1}$. Thus we have $F \subset C_{k+1}$. Hence $\{C_n\}$ is a sequence of nonempty closed convex subset of E such that $F \subset \bigcap_{n=1}^{\infty} C_n$. Let $C_0 = \bigcap_{n=1}^{\infty} C_n$ and let $p_n = P_{C_n} u$ for all $n \in \mathbb{N}$. Then, since $\{C_n\}$ is decreasing with respect to inclusion, by Theorem 2.1, it follows that $\{p_n\}$ converges strongly to $p_0 = P_{C_0} u$. From $x_n \in C_n$ and $d(u, C_n) = \|u - p_n\|$, it follows that

$$\|u - x_n\|^2 \leq \|u - p_n\|^2 + \delta_n$$

for all $n \in \mathbb{N} \setminus \{1\}$. From Theorem 2.2(i), we have, for $\alpha \in]0, 1[$

$$\begin{aligned} \|p_n - u\|^2 &\leq \|\alpha p_n + (1-\alpha) x_n - u\|^2 \\ &\leq \alpha\|p_n - u\|^2 + (1-\alpha)\|x_n - u\|^2 - \alpha(1-\alpha)\underline{g}_r(\|p_n - x_n\|) \end{aligned}$$

and thus

$$\alpha \underline{g}_r(\|p_n - x_n\|) \leq \|x_n - u\|^2 - \|p_n - u\|^2 \leq \delta_n.$$

Tending $\alpha \to 1$, we have

$$\underline{g}_r(\|p_n - x_n\|) \leq \delta_n$$

and thus

$$\|p_n - x_n\| \leq \underline{g}_r^{-1}(\delta_n) \tag{3.1}$$

for all $n \in \mathbb{N} \setminus \{1\}$. Using the definition of V and Theorem 2.3 (ii), it follows that

$$\begin{aligned} V(p_n, y_n) &= V(p_n, J^{-1}(\alpha_{n,1} JT_1 x_n + \alpha_{n,2} JT_2 x_n)) \\ &= \|p_n\|^2 - 2\langle p_n, \alpha_{n,1} JT_1 x_n + \alpha_{n,2} JT_2 x_n\rangle + \|\alpha_{n,1} JT_1 x_n + \alpha_{n,2} JT_2 x_n\|^2 \\ &\geq \alpha_{n,1}\|p_n\|^2 - 2\alpha_{n,1}\langle p_n, JT_1 x_n\rangle + \alpha_{n,1}\|T_1 x_n\|^2 \\ &\qquad + \alpha_{n,2}\|p_n\|^2 - 2\alpha_{n,2}\langle p_n, JT_2 x_n\rangle + \alpha_{n,2}\|T_2 x_n\|^2 \\ &\qquad\quad - \alpha_{n,1}\alpha_{n,2}\overline{g}_r^*(\|JT_1 x_n - JT_2 x_n\|) \\ &= \alpha_{n,1} V(p_n, T_1 x_n) + \alpha_{n,2} V(p_n, T_2 x_n) - \alpha_{n,1}\alpha_{n,2}\overline{g}_r^*(\|JT_1 x_n - JT_2 x_n\|) \end{aligned}$$

for all $n \in \mathbb{N} \setminus \{1\}$. Therefore, we get

$$\alpha_{n,1} V(p_n, T_1 x_n) + \alpha_{n,2} V(p_n, T_2 x_n) \leq V(p_n, y_n) + \alpha_{n,1}\alpha_{n,2}\overline{g}_r^*(\|JT_1 x_n - JT_2 x_n\|)$$

and thus

$$\alpha_{n,i}\underline{g}_r(\|p_n - T_i x_n\|) \leq V(p_n, y_n) + \alpha_{n,1}\alpha_{n,2}\overline{g}_r^*(\|JT_1x_n - JT_2x_n\|) \tag{3.2}$$

for each $i \in \{1, 2\}$. Let $p \in F$. It follows from Theorem 2.3 (i) that

$$\begin{aligned}
V(p, y_n) &= V(p, J^{-1}(\alpha_{n,1}JT_1x_n + \alpha_{n,2}JT_2x_n)) \\
&= \|p\|^2 - 2\langle p_n, \alpha_{n,1}JT_1x_n + \alpha_{n,2}JT_2x_n\rangle + \|\alpha_{n,1}JT_1x_n + \alpha_{n,2}JT_2x_n\|^2 \\
&\leq \alpha_{n,1}\|p\|^2 - 2\alpha_{n,1}\langle p_n, JT_1x_n\rangle + \alpha_{n,1}\|T_1x_n\|^2 \\
&\qquad + \alpha_{n,2}\|p\|^2 - 2\alpha_{n,2}\langle p_n, JT_2x_n\rangle + \alpha_{n,2}\|T_2x_n\|^2 \\
&\qquad\quad - \alpha_{n,1}\alpha_{n,2}\underline{g}_r^*(\|JT_1x_n - JT_2x_n\|) \\
&= \alpha_{n,1}V(p, T_1x_n) + \alpha_{n,2}V(p, T_2x_n) - \alpha_{n,1}\alpha_{n,2}\underline{g}_r^*(\|JT_1x_n - JT_2x_n\|) \\
&\leq \alpha_{n,1}V(p, x_n) + \alpha_{n,2}V(p, x_n) - \alpha_{n,1}\alpha_{n,2}\underline{g}_r^*(\|JT_1x_n - JT_2x_n\|) \\
&= V(p, x_n) - \alpha_{n,1}\alpha_{n,2}\underline{g}_r^*(\|JT_1x_n - JT_2x_n\|)
\end{aligned}$$

for all $n \in \mathbb{N} \setminus \{1\}$. Therefore, we obtain

$$\alpha_{n,1}\alpha_{n,2}\underline{g}_r^*(\|JT_1x_n - JT_2x_n\|) \leq V(p, x_n) - V(p, y_n). \tag{3.3}$$

From the property of V, it follows that

$$\begin{aligned}
V(p, x_n) - V(p, y_n) &= V(p, p_n) + V(p_n, x_n) + \langle p - p_n, Jp_n - Jx_n\rangle \\
&\qquad - V(p, p_n) - V(p_n, y_n) - \langle p - p_n, Jp_n - Jy_n\rangle \\
&= V(p_n, x_n) - V(p_n, y_n) + 2\langle p - p_n, Jy_n - Jx_n\rangle \\
&\leq V(p_n, x_n) + 2\|p - p_n\|\|Jx_n - Jy_n\| \\
&\leq \overline{g}_r(\|p_n - x_n\|) + 2(\|p\| + \|p_n\|)\|Jx_n - Jy_n\| \\
&\leq \overline{g}_r(\underline{g}_r^{-1}(\delta_n)) + 4r\|Jx_n - Jy_n\|.
\end{aligned} \tag{3.4}$$

Put $\epsilon_n = \|p_n - p_{n+1}\|$ for all $n \in \mathbb{N} \setminus \{1\}$. From Theorems 2.4 and 2.5, it follows that

$$\begin{aligned}
\|Jx_n - Jy_n\| &\leq \|Jx_n - Jp_{n+1}\| + \|Jp_{n+1} - Jy_n\| \\
&\leq \underline{g}_r^{*-1}\Big(V(p_{n+1}, x_n)\Big) + \underline{g}_r^{*-1}\Big(V(p_{n+1}, y_n)\Big) \\
&\leq \underline{g}_r^{*-1}\Big(V(p_{n+1}, x_n)\Big) + \underline{g}_r^{*-1}\Big(V(p_{n+1}, x_n)\Big) \\
&= 2\underline{g}_r^{*-1}\Big(V(p_{n+1}, p_n) + V(p_n, x_n) + 2\langle p_{n+1} - p_n, Jp_n - Jx_n\rangle\Big) \\
&\leq 2\underline{g}_r^{*-1}\Big(\overline{g}_r(\epsilon_n) + \overline{g}_r(\|p_n - x_n\|) + 2\epsilon_n\|Jp_n - Jx_n\|\Big)
\end{aligned}$$

$$\leq 2\underline{g}_r^{*-1}\Big(\overline{g}_r(\epsilon_n) + \overline{g}_r(\underline{g}_r^{-1}(\delta_n)) + 2\epsilon_n \underline{g}_r^{*-1}(V(p_n, x_n))\Big)$$
$$\leq 2\underline{g}_r^{*-1}\left(\overline{g}_r(\epsilon_n) + \overline{g}_r(\underline{g}_r^{-1}(\delta_n)) + 2\epsilon_n \underline{g}_r^{*-1}(\overline{g}_r(\underline{g}_r^{-1}(\delta_n)))\right) \tag{3.5}$$

for all $n \in \mathbb{N} \setminus \{1\}$. From (3.3), (3.4), and (3.5), we have

$$\alpha_{n,1}\alpha_{n,2}\underline{g}_r^{*}(\|JT_1x_n - JT_2x_n\|)$$
$$\leq \overline{g}_r\left(\underline{g}_r^{-1}(\delta_n)\right) + 8r\underline{g}_r^{*-1}\left(\overline{g}_r(\epsilon_n) + \overline{g}_r(\underline{g}_r^{-1}(\delta_n)) + 2\epsilon_n \underline{g}_r^{*-1}\big(\overline{g}_r(\underline{g}_r^{-1}(\delta_n))\big)\right)$$

and thus

$$\|JT_1x_n - JT_2x_n\| \leq \underline{g}_r^{*-1}\Bigg(\frac{1}{\alpha_{n,1}\alpha_{n,2}}\overline{g}_r\left(\underline{g}_r^{-1}(\delta_n)\right)$$
$$+ \frac{8r}{\alpha_{n,1}\alpha_{n,2}}\underline{g}_r^{*-1}\Big(\overline{g}_r(\epsilon_n) + \overline{g}_r(\underline{g}_r^{-1}(\delta_n)) + 2\epsilon_n \underline{g}_r^{*-1}\big(\overline{g}_r(\underline{g}_r^{-1}(\delta_n))\big)\Big)\Bigg).$$

Therefore, we have

$$\limsup_{n\to\infty} \|JT_1x_n - JT_2x_n\| \tag{3.6}$$
$$\leq \underline{g}^{*-1}\left(\frac{1}{\underline{\alpha}_1\underline{\alpha}_2}\overline{g}_r(\underline{g}_r^{-1}(\delta_0)) + \frac{8r}{\underline{\alpha}_1\underline{\alpha}_2}\underline{g}_r^{*-1}\big(\overline{g}_r(\underline{g}_r^{-1}(\delta_0))\big)\right).$$

From the property of V, it follows that

$$\begin{aligned}
V(p_n, y_n) &= V(p_n, p_{n+1}) + V(p_{n+1}, y_n) + 2\langle p_n - p_{n+1}, Jp_{n+1} - Jy_n\rangle \\
&\leq V(p_n, p_{n+1}) + V(p_{n+1}, x_n) + 2\|p_n - p_{n+1}\|\|Jp_{n+1} - Jy_n\| \\
&\leq V(p_n, p_{n+1}) + V(p_{n+1}, p_n) + V(p_n, x_n) + 2\langle p_{n+1} - p_n, p_n - x_n\rangle \\
&\quad + 2\|p_n - p_{n+1}\|(\|Jp_{n+1}\| + \|Jy_n\|) \\
&\leq V(p_n, p_{n+1}) + V(p_{n+1}, p_n) + V(p_n, x_n) + 2\|p_{n+1} - p_n\|\|p_n - x_n\| \\
&\quad + 4r\|p_n - p_{n+1}\|.
\end{aligned}$$

From Theorem 2.4 and (3.1), we have

$$\limsup_{n\to\infty} V(p_n, y_n) \leq \limsup_{n\to\infty} V(p_n, x_n) \leq \overline{g}_r\left(\underline{g}_r^{-1}(\delta_0)\right). \tag{3.7}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\underline{g}_r\left(\frac{\|x_n - T_i x_n\|}{2}\right)\\
&\leq \underline{g}_r\left(\frac{\|x_n - p_n\| + \|p_n - T_i x_n\|}{2}\right)\\
&\leq \frac{1}{2}\underline{g}_r(\|x_n - p_n\|) + \frac{1}{2}\underline{g}_r(\|p_n - T_i x_n\|)\\
&\leq \frac{1}{2}\delta_n + \frac{1}{2\alpha_{n,i}}\Big(V(p_n, y_n) + \alpha_{n,1}\alpha_{n,2}\overline{g}_r^*(\|JT_1 x_n - JT_2 x_n\|)\Big)
\end{aligned}$$

for each $i \in \{1, 2\}$ and for all $n \in \mathbb{N} \setminus \{1\}$. From (3.6) and (3.7), it follows that

$$\limsup_{n\to\infty} \|x_n - T_i x_n\| \leq 2\underline{g}_r^{-1}\left(\frac{1}{2}\delta_0 + \frac{1}{2\underline{\alpha}_i}(\zeta_0 + \eta_0)\right)$$

for each $i \in \{1, 2\}$, where $\zeta_0 = \overline{g}_r(\underline{g}_r^{-1}(\delta_0))$ and

$$\eta_0 = \overline{\alpha}_1\overline{\alpha}_2\overline{g}_r^*\left(\underline{g}_r^{*-1}\Big(\frac{1}{\underline{\alpha}_1\underline{\alpha}_2}\overline{g}_r(\underline{g}_r^{-1}(\delta_0)) + \frac{8r}{\underline{\alpha}_1\underline{\alpha}_2}\underline{g}_r^{*-1}\big(\overline{g}_r(\underline{g}_r^{-1}(\delta_0))\big)\Big)\right).$$

For latter part of the theorem, suppose that $\delta_0 = 0$. Then

$$\limsup_{n\to\infty} \underline{g}_r(\|p_n - x_n\|) \leq \limsup_{n\to\infty} \delta_n = 0.$$

It implies that $\lim_n \|p_n - x_n\| = 0$ and thus $\{x_n\}$ converges strongly to $p_0 = P_{C_0}u$. We also have $\zeta_0 = \overline{g}_r(\underline{g}_r^{-1}(0)) = 0$ and

$$\eta_0 = \overline{\alpha}_1\overline{\alpha}_2\overline{g}_r^*\left(\underline{g}_r^{*-1}\Big(\frac{1}{\underline{\alpha}_1\underline{\alpha}_2}\overline{g}_r(\underline{g}_r^{-1}(0)) + \frac{8r}{\underline{\alpha}_1\underline{\alpha}_2}\underline{g}_r^{*-1}\big(\overline{g}_r(\underline{g}_r^{-1}(0))\big)\Big)\right) = 0,$$

and thus

$$\lim_{n\to\infty} \|(I - T_i)x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 2\underline{g}_r^{-1}\left(\frac{1}{2}0 + \frac{1}{2\underline{\alpha}_i}(0 + 0)\right) = 0$$

for $i \in \{1, 2\}$. By the closedness of $I - T_i$ at zero, it follows that $(I - T_i)p_0 = 0$ for each $i \in \{1, 2\}$, that is, $p_0 \in F$. From $F \subset C_0$, we get that $p_0 = P_{C_0}u = P_F u$, which is the desired result.

In this case where E is a Hilbert space, the functions $\underline{g}_r$, $\overline{g}_r$, $\underline{g}_r^*$, and $\overline{g}_r^*$ become $\underline{g}_r = \overline{g}_r = \underline{g}_r^* = \overline{g}_r^* = |\cdot|^2$ for all $r > 0$. We also know that a relatively nonexpansive mapping T without the condition $\hat{F}(T) = F(T)$ is quasinonexpansive, that is, $\|u - Tx\| \leq \|u - x\|$ for every $x \in C$ and $u \in F(T) \neq \emptyset$. Therefore, we obtain the following:

Corollary 3.2 *Let H be a Hilbert space, let C be a nonempty bounded closed convex subset of H and let $r > 0$ such that $C \subset B_r$. Let T_1, T_2 be quasinonexpansive mappings from C to H such that $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{\alpha_{n,i} : n \in \mathbb{N}, i \in \{1, 2\}\}$ be a family of positive real numbers such that $\alpha_{n,1} + \alpha_{n,2} = 1$, and $\liminf_n \alpha_{n,i} > 0$ for $i \in \{1, 2\}$. Let $\{\delta_n\}$ be a nonnegative real sequence and $\delta_0 = \limsup_n \delta_n$. For given point $u \in H$, generate an iterative sequence $\{x_n\}$ as follows: $x_1 = x \in C$, $C_1 = C$,*

$$\begin{aligned}
&y_n = \alpha_{n,1} T_1 x_n + \alpha_{n,2} T_2 x_n, \\
&C_{n+1} = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\} \cap C_n, \\
&x_{n+1} \in \{z \in C : \|u - z\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}\} \cap C_{n+1}
\end{aligned}$$

for all $n \in \mathbb{N}$. Then,

$$\limsup_{n\to\infty} \|x_n - T_i x_n\| \leq 2\sqrt{\left\{\frac{1}{2} + \frac{1}{2\underline{\alpha}_i}\left(1 + \frac{\overline{\alpha}_1\overline{\alpha}_2}{\underline{\alpha}_1\underline{\alpha}_2}\right)\right\}\delta_0 + \frac{4r\overline{\alpha}_1\overline{\alpha}_2}{\underline{\alpha}_i\underline{\alpha}_1\underline{\alpha}_2}\sqrt{\delta_0}} \tag{3.8}$$

for $i \in \{1, 2\}$, where

$$\underline{\alpha}_i = \liminf_{n\to\infty} \alpha_{n,i}, \quad \overline{\alpha}_i = \limsup_{n\to\infty} \alpha_{n,i}$$

for $i \in \{1, 2\}$. Moreover, if $\delta_0 = 0$ and $I - T_i$ is closed at zero for each $i \in \{1, 2\}$, then $\{x_n\}$ converges strongly to $P_{F(T_1)\cap F(T_2)}u$.

Remark 3.3 In corollary 3.2, if $\liminf_n \alpha_{n,i} = \limsup_n \alpha_{n,i}$ for each $i \in \{1, 2\}$, then $\lim_n \alpha_{n,i} (= \alpha_i)$ exists and

$$\limsup_{n\to\infty} \|x_n - T_i x_n\| \leq 2\sqrt{\left(\frac{1}{2} + \frac{1}{\alpha_i}\right)\delta_0 + \frac{4r}{\alpha_i}\sqrt{\delta_0}}.$$

for $i \in \{1, 2\}$. In this case, upper bound of $\limsup_n \|x_n - Tx_n\|$ is simpler than the (3.8).

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Convex Minimization Problems on Geodesic Spaces and the Shrinking Projection Method with Errors

Yasunori Kimura

Abstract To find a minimizer of a given convex function defined on a complete geodesic space, we apply the shrinking projection method with the resolvent operator for the function and obtain an approximate sequence to its minimizing point. We also consider the error terms in the scheme when we calculate metric projections.

Keywords Convex minimization problem · Resolvent · Shrinking projection method · Fixed point · Approximation

Mathematics Subject Classification (2010) 47H09, 47J05, 52A41

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space X. We consider the problem of finding a minimizer of a given convex function $f : C \to]-\infty, +\infty]$ under several conditions. This problem is called a convex minimization problem and it is one of the most fundamental and important problems in convex analysis. It is known that we may apply the theory of nonexpansive mappings for this problem by using the notion of resolvents for the function f, which is defined as follows: Suppose that f is proper, lower semicontinuous, and convex. Then, for $x \in X$, there exists a unique point $y_x \in C$ such that

$$f(y_x) + \|y_x - x\|^2 = \inf_{y \in C}(f(y) + \|y - x\|^2).$$

The resolvent operator $J_f : X \to C$ is defined by $J_f x = y_x$ for any $x \in X$. Namely, $J_f x$ is a unique minimizer of $g(y) = f(y) + \|y - x\|^2$ on C. Since J_f is firmly

Y. Kimura (✉)
Department of Information Science, Toho University,
Miyama, Funabashi, Chiba 274-8510, Japan
e-mail: yasunori@is.sci.toho-u.ac.jp

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nonexpansive and its set of fixed points coincides with the set of minimizers of f, we can make good use of approximation method of fixed points for nonexpansive mappings.

One of the most remarkable results for this problem is the proximal point algorithm, which was originally introduced by Martinet [1] and studied by Rockafellar [2]. The approximation sequence generated by this algorithm is guaranteed to be weakly convergent to a minimizer of the function.

On the other hand, Takahashi et al. [3] proposed the shrinking projection method to generate a sequence strongly convergent to a fixed point of a given nonexpansive mapping. The following is a simple version of their result.

Theorem 1.1 (Takahashi et al. [3]) *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping such that the set* $\operatorname{Fix} T$ *of fixed points of T is nonempty. For a given point $u \in H$, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$C_{n+1} = \{z \in C : \|z - Tx_n\| \leq \|z - x_n\|\} \cap C_n,$$
$$x_{n+1} = P_{C_{n+1}} u$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{\operatorname{Fix} T} u \in C$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H.

This method was firstly applied to an equilibrium problem for a bifunction defined on a Banach space by Takahashi and Takahashi [4]. Note that the equilibrium problems are a generalization of convex minimization problems; see [5] for the detail.

Since this new method was established, a large number of generalized results have been proposed. In 2012, Kimura and Satô [6] proved the following convergence theorem for a mapping defined on a subset of the unit sphere of a Hilbert space. Note that the class of complete CAT(1) spaces, which we will consider in this work, contains this set.

Theorem 1.2 (Kimura and Satô [6]) *Let S_H be the unit sphere of a real Hilbert space H with the metric d defined by a length of minimal great arc, and C a closed convex subset of S_H such that $d(u, v) < \pi/2$ for every $u, v \in C$. Let $T : C \to C$ be a nonexpansive mapping such that* $\operatorname{Fix} T$ *is nonempty. For a given point $u \in C$, generate a sequence $\{x_n\}$ as follows: $x_1 \in C$, $C_1 = C$, and*

$$C_{n+1} = \{z \in C : d(Tx_n, z) \leq d(x_n, z)\} \cap C_n,$$
$$x_{n+1} = P_{C_{n+1}} u,$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_{\operatorname{Fix} T} u \in C$.

Recently, Kimura and Kohsaka [7] extended the concept of resolvents of convex functions defined on a subset of a Hilbert space to that defined on a complete CAT(1) space. In this work, we apply the shrinking projection method for this operator and obtain an approximate sequence to a minimizer of the convex function. We also consider the error terms in the scheme when we calculate metric projections. We employ the technique developed in [8].

2 Preliminaries

Let (X, d) be a metric space. For $x, y \in X$, we define a mapping $c : [0, l] \to X$ called a geodesic with the endpoints x and y as follows: c satisfies that $c(0) = x$, $c(1) = y$, and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, l]$. We say that X is a geodesic metric space if for every $x, y \in X$, there exists a geodesic with the endpoints $x, y \in X$. In what follows, we assume that a geodesic is unique for every choice of endpoints $x, y \in X$.

Denote the image of a geodesic with the endpoints $x, y \in X$ by $[x, y]$. A point $p = c(s) \in [x, y]$ can be regarded as a dividing point between x and y with the ratio $s : (l - s)$, and therefore we use the notation $p = (1 - t)x \oplus ty$, where $t = s/l$.

Let X be a geodesic metric space such that $d(u, v) < \pi/2$ for every $u, v \in X$. Then, for any geodesic triangle $\triangle(x, y, z) = [y, z] \cup [z, x] \cup [x, y]$ with vertices $x, y, z \in X$, there exist points $\overline{x}, \overline{y}, \overline{z} \in \mathbb{S}^2$ such that $d(y, z) = d_{\mathbb{S}^2}(\overline{y}, \overline{z})$, $d(z, x) = d_{\mathbb{S}^2}(\overline{z}, \overline{x})$, and $d(x, y) = d_{\mathbb{S}^2}(\overline{x}, \overline{y})$, where $d_{\mathbb{S}^2}$ is the spherical metric defined on the two-dimensional unit sphere $\mathbb{S}^2$. We call the triangle $\triangle(\overline{x}, \overline{y}, \overline{z}) \subset \mathbb{S}^2$ with the vertices $\overline{x}, \overline{y}, \overline{z}$ the comparison triangle of $\triangle(x, y, z)$. Note that the comparison triangle is unique up to rotation and reflection. For a point p on a geodesic triangle $\triangle(x, y, z)$, we denote its comparison point by $\overline{p}$; if $p = (1 - t)x \oplus ty \in [x, y] \subset \triangle(x, y, z) \subset X$ and $t \in [0, 1]$, then $\overline{p} = (1 - t)\overline{x} \oplus t\overline{y} \in [\overline{x}, \overline{y}] \subset \triangle(\overline{x}, \overline{y}, \overline{z}) \subset \mathbb{S}^2$.

We say that X is a CAT(1) space if for each geodesic triangle on X is as thin as its comparison triangle on the two-dimensional unit sphere $\mathbb{S}^2$. Namely, every $p, q \in X$ on the edges of the triangle $x, y, z \in X$ and their comparison points $\overline{p}, \overline{q} \in \mathbb{S}^2$ on the edges of the comparison triangle $\overline{x}, \overline{y}, \overline{z} \in \mathbb{S}^2$ satisfy the CAT(1) inequality

$$d(p, q) \le d_{\mathbb{S}^2}(\overline{p}, \overline{q}).$$

We know that if X is a CAT(1) space such that $d(u, v) < \pi/2$ for all $u, v \in X$, then for $x, y, z \in X$ and $t \in [0, 1]$, the following important inequality holds; see [9].

$$\begin{aligned}\sin d(x, y) \cos d(tx \oplus (1 - t)y, z)& \\ \ge \sin(td(x, y)) \cos d(x, z) &+ \sin((1 - t)d(x, y)) \cos d(y, z).\end{aligned}$$

Let C be a nonempty closed convex subset of a complete CAT(1) space and suppose that $d(u, v) < \pi/2$ for every $u, v \in X$. Then, for each $x \in X$, there exists a unique $y_x \in C$ such that $d(x, y_x) = d(x, C) = \inf_{y \in C} d(x, y)$. We define a mapping $P_C : X \to C$ by $P_C x = y_x$ for $x \in X$ and we call it the metric projection of X onto C.

The following lemma is essentially obtained from the result in [9].

Lemma 2.1 (Kimura and Satô [9]) *Let X be a complete* CAT(1) *space such that $d(u, v) < \pi/2$ for every $u, v \in X$. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of X such that $C_{n+1} \subset C_n$ for every $n \in \mathbb{N}$ and $C_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$. Let $\{P_{C_n}\}$ be a sequence of metric projections corresponding to $\{C_n\}$. Then, for $u \in X$, a sequence $\{P_{C_n} u\}$ converges to $P_{C_0} u \in X$.*

Let X be a complete CAT(1) space and $T : X \to X$ a mapping. The set $\operatorname{Fix} T$ of all fixed points of T is defined by

$$\operatorname{Fix} T = \{z \in X : z = Tz\}.$$

Then T is said to be quasinonexpansive if $\operatorname{Fix} T \neq \emptyset$ and

$$d(Tx, z) \leq d(x, z)$$

for any $x \in X$ and $z \in \operatorname{Fix} T$.

Let X be a complete CAT(1) space and f a proper lower semicontinuous convex function of X into $]-\infty, +\infty]$. Then there exists a unique point $y_x \in X$ such that

$$f(y_x) + \tan d(y_x, x) \sin d(y_x, x) = \inf_{y \in X} (f(y) + \tan d(y, x) \sin d(y, x)).$$

Using this fact, we define a resolvent $J_f : X \to X$ by

$$J_f x = y_x = \operatorname{argmin}_{y \in X} (f(y) + \tan d(y, x) \sin d(y, x))$$

for all $x \in X$. Resolvent operators satisfy the following fundamental properties:

Lemma 2.2 (Kimura and Kohsaka [7]) *Let X be a complete* CAT(1) *space such that $d(u, v) < \pi/2$ for every $u, v \in X$, $f : X \to]-\infty, +\infty]$ a proper lower semicontinuous convex function, and $J_f : X \to X$ the resolvent of f. Then the following holds:*

(i) $\operatorname{Fix} J_f = \operatorname{argmin}_{y \in X} f(y)$;
(ii) *J_f is firmly spherically nonspreading in the sense that*

$$(\cos d(J_f x, x) + \cos d(J_f y, y)) \cos^2 d(J_f x, J_f y) \geq 2 \cos d(J_f x, y) \cos d(x, J_f y)$$

for all $x, y \in X$.

It is also easy to see that J_f is quasinonexpansive. Indeed, for $x \in X$ and $z \in \operatorname{Fix} J_f$, using the firm spherical nonspreadingness of J_f, we obtain

$$(\cos d(J_f x, x) + 1) \cos^2 d(J_f x, z) \geq 2 \cos d(J_f x, z) \cos d(x, z).$$

Since $\cos d(J_f x, x) \leq 1$, we have

$$2\cos^2 d(J_f x, z) \geq 2\cos d(J_f x, z)\cos d(x, z)$$

and thus $\cos d(J_f x, z) \geq \cos d(x, z)$, or equivalently, $d(J_f x, z) \leq d(x, z)$.

3 Approximation of a Minimizer of a Convex Function

In this section, we propose an approximation method for finding a minimizer of a convex function defined on a complete CAT(1) space. This method includes computation error term so that it is easy to apply for practical calculation.

Theorem 3.1 *Let X be a complete* CAT(1) *space satisfying the following conditions:*

(i) *$d(u, v) < \pi/2$ for every $u, v \in X$;*
(ii) *a subset $\{z \in X : d(v, z) \leq d(u, z)\}$ is convex for every $u, v \in X$.*

Let $f : X \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function such that the set of its minimizers $S = \operatorname{argmin}_{x \in X} f(x)$ is nonempty. Let $\{\rho_n\}$ be a sequence in $]0, +\infty[$ such that $\rho_0 = \inf_{n\in\mathbb{N}} \rho_n > 0$ and let $\{\epsilon_n\}$ be a sequence in $[0, +\infty[$ with $\epsilon_0 = \limsup_{n\to\infty} \epsilon_n$. For a given point $u \in X$, generate a sequence $\{x_n\}$ as follows: $x_1 = u$, $C_1 = X$, and

$$C_{n+1} = \{z \in X : d(J_{\rho_n f} x_n, z) \leq d(x_n, z)\} \cap C_n,$$
$$x_{n+1} \in C_{n+1} \text{ such that } \cos d(u, x_{n+1}) \geq \cos d(u, C_{n+1}) \cos \epsilon_{n+1},$$

for each $n \in \mathbb{N}$, where $J_{\rho_n f}$ is the resolvent for $\rho_n f$. Then

$$\limsup_{n\to\infty} d(x_n, J_{\rho_n f} x_n) \leq 2\epsilon_0$$

and

$$\begin{aligned} f(p) &\leq \liminf_{n\to\infty} f(J_{\rho_n f} x_n) \\ &\leq \limsup_{n\to\infty} f(J_{\rho_n f} x_n) \leq f(p) + \frac{\pi}{\rho_0}\left(\frac{1}{\cos^2(2\epsilon_0)} + 1\right)\sin\epsilon_0, \end{aligned}$$

where $p = P_S u$ and P_S is the metric projection of X onto S. Moreover, if $\epsilon_0 = 0$, then $\{x_n\}$ converges to $P_S u$.

Proof We first prove that the sequence $\{x_n\}$ is well defined. To show this, we will see that every C_n is a closed convex subset containing S by induction. For $n = 1$, it is obvious since $C_1 = X$ by definition. Suppose that C_k is a nonempty closed convex subset of X and $S \subset C_k$ for fixed $k \in \mathbb{N}$. Let $z \in S$. Since $J_{\rho_n f}$ is quasi nonexpansive with $\operatorname{Fix} J_{\rho_n f} = S$, we have $d(J_{\rho_n f} x_k, z) \leq d(x_k, z)$ and $z \in S \subset C_k$. This implies that $z \in C_{k+1}$, and hence $S \subset C_{k+1}$. It is easy to see that C_{k+1} is closed since the metric d is continuous. The convexity of C_{k+1} follows from the assumption of the space. We also know that there exists at least one point $y \in C_k$ such that $\cos d(u, y) \geq \cos d(u, C_k) \cos \epsilon_k$. Indeed, taking $y = P_{C_k} u \in X$, we have

$$\cos d(u, y) = \cos d(u, P_{C_k} u) = \cos d(u, C_k) \geq \cos d(u, C_k) \cos \epsilon_k.$$

Thus we can choose $x_{k+1} \in C_{k+1}$. Therefore, C_n is a closed convex subset containing S for all $n \in \mathbb{N}$ and $\{x_n\}$ is well defined.

For $n \in \mathbb{N}$, we can define the metric projection P_{C_n} of X onto C_n. Let $p_n = P_{C_n} u$ for all $n \in \mathbb{N}$ and $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then, by Lemma 2.1, $\{p_n\}$ converges to $p_0 = P_{C_0} u$. Since $x_n \in C_n$, we have

$$\cos d(u, x_n) \geq \cos d(u, C_n) \cos \epsilon_n$$

for every $n \in \mathbb{N}$. Using the convexity of C_n, we have $\alpha p_n \oplus (1 - \alpha) x_n \in C_n$ for $\alpha \in]0, 1[$. It follows that

$$\begin{aligned}
&\sin d(p_n, x_n) \cos d(p_n, u) \\
&\geq \sin d(p_n, x_n) \cos d(\alpha p_n \oplus (1 - \alpha) x_n, u) \\
&\geq \sin(\alpha d(p_n, x_n)) \cos d(p_n, u) + \sin((1 - \alpha) d(p_n, x_n)) \cos d(x_n, u)
\end{aligned}$$

and thus

$$\sin d(p_n, x_n) - \sin(\alpha d(p_n, x_n)) \geq \sin((1 - \alpha) d(p_n, x_n)) \frac{\cos d(x_n, u)}{\cos d(p_n, u)}.$$

On the other hand, we have

$$\sin d(p_n, x_n) - \sin(\alpha d(p_n, x_n)) = 2 \cos\left(\frac{1 + \alpha}{2} d(p_n, x_n)\right) \sin\left(\frac{1 - \alpha}{2} d(p_n, x_n)\right)$$

and

$$\sin((1 - \alpha) d(p_n, x_n)) = 2 \cos\left(\frac{1 - \alpha}{2} d(p_n, x_n)\right) \sin\left(\frac{1 - \alpha}{2} d(p_n, x_n)\right).$$

Suppose that $p_n \neq x_n$. Then using the three equations above, we obtain

$$\cos\left(\frac{1 + \alpha}{2} d(p_n, x_n)\right) \geq \cos\left(\frac{1 - \alpha}{2} d(p_n, x_n)\right) \frac{\cos d(x_n, u)}{\cos d(p_n, u)}.$$

As $\alpha \to 1$, we have

$$\cos d(p_n, x_n) \geq \frac{\cos d(x_n, u)}{\cos d(p_n, u)} = \frac{\cos d(x_n, u)}{\cos d(u, C_n)} \geq \cos \epsilon_n,$$

and it follows that $d(p_n, x_n) \leq \epsilon_n$ for every $n \in \mathbb{N}$. Note that this inequality trivially holds if $p_n = x_n$. Since $p_{n+1} \in C_{n+1}$, we get

$$\begin{aligned} d(J_{\rho_n f} x_n, x_n) &\leq d(J_{\rho_n f} x_n, p_{n+1}) + d(p_{n+1}, x_n) \\ &\leq 2d(x_n, p_{n+1}) \\ &\leq 2(d(x_n, p_n) + d(p_n, p_{n+1})) \\ &\leq 2(\epsilon_n + d(p_n, p_{n+1})) \end{aligned}$$

for every $n \in \mathbb{N}$. Tending $n \to \infty$, we obtain

$$\limsup_{n\to\infty} d(J_{\rho_n f} x_n, x_n) \leq 2\epsilon_0.$$

For $\alpha \in]0, 1[$, let $z_\alpha = \alpha J_{\rho_n f} x_n \oplus (1-\alpha)p$. By the definition of the resolvent $J_{\rho_n f}$, we have

$$\begin{aligned} &\rho_n f(J_{\rho_n f} x_n) + \tan d(J_{\rho_n f} x_n, x_n) \sin d(J_{\rho_n f} x_n, x_n) \\ &\leq \rho_n f(z_\alpha) + \tan d(z_\alpha, x_n) \sin d(z_\alpha, x_n) \\ &\leq \alpha \rho_n f(J_{\rho_n f} x_n) + (1-\alpha)\rho_n f(p) + \tan d(z_\alpha, x_n) \sin d(z_\alpha, x_n). \end{aligned}$$

Since $\tan t \sin t = 1/\cos t - \cos t$, it follows that

$$\begin{aligned} &(1-\alpha)\rho_n (f(J_{\rho_n f} x_n) - f(p)) \\ &\leq \tan d(z_\alpha, x_n) \sin d(z_\alpha, x_n) - \tan d(J_{\rho_n f} x_n, x_n) \sin d(J_{\rho_n f} x_n, x_n) \\ &= \left(\frac{1}{\cos d(z_\alpha, x_n)} - \frac{1}{\cos d(J_{\rho_n f} x_n, x_n)} \right) - (\cos d(z_\alpha, x_n) - \cos d(J_{\rho_n f} x_n, x_n)) \\ &= \left(\frac{1}{\cos d(z_\alpha, x_n) \cos d(J_{\rho_n f} x_n, x_n)} + 1 \right) (\cos d(J_{\rho_n f} x_n, x_n) - \cos d(z_\alpha, x_n)). \end{aligned}$$

Let $D_n = d(J_{\rho_n f} x_n, p)$ for $n \in \mathbb{N}$. If $D_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then since $J_{\rho_{n_0} f} x_{n_0} = p \in S = \operatorname{Fix} J_{\rho_{n_0} f}$, we have $f(J_{\rho_{n_0} f} x_{n_0}) = f(p)$. In this case, the conclusions of the theorem obviously hold. Thus we may suppose that $D_n > 0$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} &(\cos d(J_{\rho_n f} x_n, x_n) - \cos d(z_\alpha, x_n)) \sin D_n \\ &= \cos d(J_{\rho_n f} x_n, x_n) \sin D_n - \cos d(\alpha J_{\rho_n f} x_n \oplus (1-\alpha)p, x_n) \sin D_n \\ &\leq \cos d(J_{\rho_n f} x_n, x_n) \sin D_n \\ &\quad - \cos d(J_{\rho_n f} x_n, x_n) \sin(\alpha D_n) - \cos d(p, x_n) \sin((1-\alpha)D_n) \\ &= \cos d(J_{\rho_n f} x_n, x_n)(\sin D_n - \sin(\alpha D_n)) - \cos d(p, x_n) \sin((1-\alpha)D_n) \\ &\leq (\sin D_n - \sin(\alpha D_n)) - \cos d(p, x_n) \sin((1-\alpha)D_n) \\ &= 2\cos\frac{(1+\alpha)D_n}{2} \sin\frac{(1-\alpha)D_n}{2} - \cos d(p, x_n) \sin((1-\alpha)D_n), \end{aligned}$$

and therefore

$$\begin{aligned}
&\rho_n(f(J_{\rho_n f}x_n) - f(p))\frac{\sin D_n}{D_n} \\
&\leq E_{\alpha,n}\left(\frac{\sin((1-\alpha)D_n/2)}{(1-\alpha)D_n/2}\cos\frac{(1+\alpha)D_n}{2} - \frac{\sin((1-\alpha)D_n)}{(1-\alpha)D_n}\cos d(p, x_n)\right),
\end{aligned}$$

where

$$E_{\alpha,n} = \frac{1}{\cos d(z_\alpha, x_n)\cos d(J_{\rho_n f}x_n, x_n)} + 1.$$

Since $E_{\alpha,n} \to 1/\cos^2 d(J_{\rho_n f}x_n, x_n) + 1$ as $\alpha \uparrow 1$, we have

$$\begin{aligned}
&\rho_n(f(J_{\rho_n f}x_n) - f(p))\frac{\sin D_n}{D_n} \\
&\leq \left(\frac{1}{\cos^2 d(J_{\rho_n f}x_n, x_n)} + 1\right)(\cos D_n - \cos d(p, x_n)) \\
&= \left(\frac{1}{\cos^2 d(J_{\rho_n f}x_n, x_n)} + 1\right)(\cos d(J_{\rho_n f}x_n, p) - \cos d(x_n, p)).
\end{aligned}$$

We further calculate that

$$\begin{aligned}
&\cos d(J_{\rho_n f}x_n, p) - \cos d(x_n, p) \\
&= 2\sin\frac{d(x_n, p) + d(J_{\rho_n f}x_n, p)}{2}\sin\frac{d(x_n, p) - d(J_{\rho_n f}x_n, p)}{2} \\
&\leq 2\sin\frac{d(x_n, p) - d(J_{\rho_n f}x_n, p)}{2} \\
&\leq 2\sin\frac{d(J_{\rho_n f}x_n, x_n)}{2}
\end{aligned}$$

and thus

$$\rho_n(f(J_{\rho_n f}x_n) - f(p))\frac{\sin D_n}{D_n} \leq 2\left(\frac{1}{\cos^2 d(J_{\rho_n f}x_n, x_n)} + 1\right)\sin\frac{d(J_{\rho_n f}x_n, x_n)}{2}.$$

Since $\rho_n \geq \rho_0$ and $\sin D_n/D_n \geq 2/\pi$ for all $n \in \mathbb{N}$, taking the upper limit as $n \to \infty$, we have

$$\begin{aligned}
&\frac{2\rho_0}{\pi}\left(\limsup_{n\to\infty} f(J_{\rho_n f}x_n) - f(p)\right) \\
&\leq 2\left(\frac{1}{\cos^2(2\epsilon_0)} + 1\right)\limsup_{n\to\infty}\sin\frac{d(J_{\rho_n f}x_n, x_n)}{2} \\
&\leq 2\left(\frac{1}{\cos^2(2\epsilon_0)} + 1\right)\sin\epsilon_0.
\end{aligned}$$

Hence, we have

$$\limsup_{n\to\infty} f(J_{\rho_n f} x_n) - f(p) \le \frac{\pi}{\rho_0}\left(\frac{1}{\cos^2(2\epsilon_0)} + 1\right)\sin\epsilon_0,$$

which implies that

$$\begin{aligned} f(p) &\le \liminf_{n\to\infty} f(J_{\rho_n f} x_n) \\ &\le \limsup_{n\to\infty} f(J_{\rho_n f} x_n) \le f(p) + \frac{\pi}{\rho_0}\left(\frac{1}{\cos^2(2\epsilon_0)} + 1\right)\sin\epsilon_0, \end{aligned}$$

the desired result.

For the latter part of the theorem, suppose $\epsilon_0 = 0$. Then, since $d(x_n, p_n) \le \epsilon_n$ and $\lim_{n\to\infty} \epsilon_n = \epsilon_0 = 0$, we get $\{x_n\}$ and $\{p_n\}$ have the identical limit $p_0 = P_{C_0} u$. Moreover, since $\limsup_{n\to\infty} d(J_{\rho_n f} x_n, x_n) \le 2\epsilon_0 = 0$, $\{J_{\rho_n f} x_n\}$ also converges to p_0. Using the lower semicontinuity of f, we have that

$$\begin{aligned} f(p) &\le f(p_0) \\ &\le \liminf_{n\to\infty} f(J_{\rho_n f} x_n) \\ &\le f(p) + \frac{\pi}{\rho_0}\left(\frac{1}{\cos^2 0} + 1\right)\sin 0 \\ &= f(p). \end{aligned}$$

Therefore p_0 is a minimizer of f, that is, $p_0 \in S$. Since $S \subset C_0$ and $p_0 = P_{C_0} u$, we have $p_0 = P_S u$ and we complete the proof. □

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An Extension of Integrals

Toshiharu Kawasaki

Abstract Extension of Denjoy–Perron–Henstock–Kurzweil integral has been done by replacing the derivative with the approximate derivative or the distributional derivative. However even in these integrals, for instance, it is impossible to calculate the integral of $f(x) = \frac{1}{x}$. Therefore we need to extend integrations in another direction. The continuity of integral hinders the extension of integrals. In this paper, we propose an extension of integrals that relaxes the continuity of integral and describe properties of this integral. Lastly, we consider applications.

Keywords Henstock–Kurzweil integral · Extended integral

2010 Mathematics Subject Classification 47H10

1 Introduction

Each integral has double-facedness of a limit of certain sum and an inverse operation of derivative. A representative example of the former is the Lebesgue integral. The Lebesgue integral is an extension concept of length, area, and volume and brings practically useful results such as convergence theorem. However, the Lebesgue integral is inadequate with respect to the inverse operation with respect to the derivative, that is, f' is not necessarily integrable even if there exists the derivative f' of a continuous function f. On the other hand, a representative example of the later is the Newton integral. A function f is Newton integrable if there exists a continuous function F such that $F' = f$. Therefore, obviously, it is true that the primitive of f' is f for the Newton integral. Unfortunately, the set of the Lebesgue integrable functions

T. Kawasaki (✉)
College of Engineering, Nihon University, Fukushima 963–8642, Japan
e-mail: toshiharu.kawasaki@nifty.ne.jp

Faculty of Engineering, Tamagawa University, Tokyo 194–8610, Japan

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and the set of the Newton integrable functions cannot be included with each other. Therefore an integral including both integrals is needed. The Denjoy integral, the Perron integral, and the Henstock–Kurzweil integral include both of the Lebesgue integral and the Newton integral and are equivalent [7, 12].

There are many studies on these integrals. Recently the minimal integral, called the C-integral [1–3], including the Lebesgue integral and the Newton integral was found. Moreover there are many studies on the C-integral [11] and on related integrals [4, 9, 10].

Extensions of Denjoy–Perron–Henstock–Kurzweil integral have been done by replacing the derivative with the approximate derivative [7, 12] or the distributional derivative [19].

In general, an operator T is an integral on $D \subset \mathbb{R}$ if it satisfies the following conditions:

(I1) Let

$$\mathcal{I}_0 = \{[a, b] \mid a, b \in \mathbb{R}\} \\ \cup\{(-\infty, b] \mid b \in \mathbb{R}\} \cup \{[a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, \infty)\},$$

let $D \in \mathcal{I}_0$ and let

$$\mathcal{I} = \{I \mid I \in \mathcal{I}_0, I \subset D\}.$$

We consider a linear space $\mathcal{F}$ consisting of functions from D into $\mathbb{R}$. Then T is an operator from $\mathcal{F} \times \mathcal{I}$ into $\mathbb{R}$.

(I2) T is linear with respect to the first argument, that is,

$$T(\alpha f + \beta g, D) = \alpha T(f, D) + \beta T(g, D)$$

for any $\alpha, \beta \in \mathbb{R}$ and for any $f, g \in \mathcal{F}$.

(I3) T has the additivity of intervals, that is, if $I_1 \cap I_2$ consists the only one point, then

$$T(f, I_1) + T(f, I_2) = T(f, I_1 \cup I_2)$$

for any $f \in \mathcal{F}$ and for any $I_1, I_2 \in \mathcal{I}$.

(I4) Let $f \in \mathcal{F}$ and let $[a, x] \in \mathcal{I}$. We consider that $T(f, [a, x])$ is a function of the variable x. Then it is continuous.

Then $f \in \mathcal{F}$ is said to be T-integrable on D.

However even in these above, for instance, it is impossible to calculate the integral of $f(x) = \frac{1}{x}$. Therefore we need to extend integrations in another direction. The continuity of integral (I4) hinders the extension of integrals. In this paper, we propose an extension of integrals that relaxes the continuity of integral and describe properties of this integral. Lastly, we consider applications.

2 Preliminaries

In this paper, we consider an extension of integrals. However in later sections, we consider the Henstock–Kurzweil integral as a concrete integral. In this section, we prepare some results for Henstock–Kurzweil integral.

Let $[a, b] \subset \mathbb{R}$. A function δ from $[a, b]$ into $(0, \infty)$ is called a gauge. A collection $\{(I_k, \xi_k) \mid k = 1, \ldots, K\}$ of pair of closed intervals and points is called a partial δ-fine Perron partition if

$$\xi_k \in I_k \subset (\xi_k - \delta(\xi_k), \xi_k + \delta(\xi_k))$$

for any k, $I_{k_1}^\circ \cap I_{k_2}^\circ = \emptyset$ for any k_1, k_2 with $k_1 \neq k_2$ and

$$\sum_{k=1}^{K} |I_k| \leq b - a,$$

where $|I|$ is the Lebesgue measure of I. If a partial δ-fine Perron partition $\{(I_k, \xi_k) \mid k = 1, \ldots, K\}$ satisfies

$$\sum_{k=1}^{K} |I_k| = b - a,$$

then it is called a δ-fine Perron partition. A function f from $[a, b]$ into $\mathbb{R}$ is said to be Henstock–Kurzweil integrable on $[a, b]$ if there exists $A \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a gauge δ such that

$$\left| \sum_{k=1}^{K} f(\xi_k)|I_k| - A \right| < \varepsilon$$

for any δ-fine Perron partition $\{(I_k, \xi_k) \mid k = 1, \ldots, K\}$ of $[a, b]$. Then we denote the constant A by

$$(HK) \int_{[a,b]} f(x)dx = A.$$

The following are, for instance, in [7]:

Theorem 2.1 *If a function f from $[a, b]$ into $\mathbb{R}$ is Henstock–Kurzweil integrable on $[a, b]$, then for any $\varepsilon > 0$ there exists a gauge δ such that*

$$\sum_{k=1}^{K} \left| f(\xi_k)|I_k| - (HK) \int_{I_k} f(x)dx \right| < \varepsilon$$

for any partial δ-fine Perron partition $\{(I_k, \xi_k) \mid k = 1, \dots, K\}$ of $[a, b]$.

Theorem 2.2 *Let $\{f_n\}$ be a sequence of Henstock–Kurzweil integrable functions on $[a, b]$. Suppose that $\{f_n\}$ is pointwisely convergent to f and that there exist Henstock–Kurzweil integrable functions g and h such that $g \leq f_n \leq h$ for any n. Then f is Henstock–Kurzweil integrable and*

$$(HK)\int_{[a,b]} f(x)dx = \lim_{n\to\infty}(HK)\int_{[a,b]} f_n(x)dx.$$

A function f from $E \subset \mathbb{R}$ into $\mathbb{R}$ is said to be absolutely continuous in the restricted sense if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{K} \sup\{|f(v) - f(u)| \mid u, v \in [c_k, d_k]\} < \varepsilon$$

whenever $\{[c_k, d_k] \mid k = 1, \dots, K\}$ satisfies $c_k, d_k \in E$ for any k, $(c_{k_1}, d_{k_1}) \cap (c_{k_2}, d_{k_2}) = \emptyset$ for any k_1, k_2 with $k_1 \neq k_2$ and

$$\sum_{k=1}^{K} (d_k - c_k) < \delta.$$

A function f from E into $\mathbb{R}$ is said to be generalized absolutely continuous in the restricted sense if there exists a sequence $\{E_m \mid m \in \mathbb{N}\}$ of subsets of E such that $\bigcup_{m=1}^{\infty} E_m = E$ and $f|_{E_m}$ is absolutely continuous in the restricted sense. If E is bounded and f is generalized absolutely continuous in the restricted sense, then f is differentiable almost everywhere, for instance, see [7, 12].

Theorem 2.3 *Let f be a Henstock–Kurzweil integrable function on $[a, b]$ and*

$$F(x) = (HK)\int_{[a,x]} f(t)dt$$

for any $x \in [a, b]$. Then F is generalized absolutely continuous in the restricted sense and hence differentiable almost everywhere. Moreover $F' = f$ holds almost everywhere and hence F' is Henstock–Kurzweil integrable.

3 An Extension of Integrals

In this section, firstly, we introduce an extension of integrals. Let T be an integral from $\mathcal{F} \times \mathcal{I}$ into $\mathbb{R}$ on $D \in \mathcal{I}_0$. We consider a linear space $\mathcal{F}^*$ including $\mathcal{F}$. Moreover let

$$\mathcal{N}_f = \{z \in D \mid f \text{ is not } T\text{-integrable on any } I \in \mathcal{I} \text{ satisfying } z \in I\}.$$

Suppose that $\mathcal{N}_f$ is a closed null set for any $f \in \mathcal{F}^*$. Let $\{(a_p, b_p) \mid p \in \mathbb{N}\}$ be the set of all components of $D^\circ \setminus \mathcal{N}_f$ and let

$$\mathcal{I}_f = \{[a,b] \mid \text{there exists } p \in \mathbb{N} \text{ such that } [a,b] \subset (a_p, b_p)\}.$$

Since f is T-integrable on I for any $f \in \mathcal{F}^*$ and for any $I \in \mathcal{I}_f$, an operator T^* from $\bigcup_{f \in \mathcal{F}^*} (\{f\} \times \mathcal{I}_f)$ into $\mathbb{R}$ is defined by $T^*(f, I) = T(f, I)$.

The extension of integral has the following properties:

Theorem 3.1 *Let T^* be the extension of the integral T on D. If f is T-integrable on D, then f is also T^*-integrable.*

Proof Let f be T-integrable and put $\mathcal{N}_f = \emptyset$. Then f is T^*-integrable. □

Theorem 3.2 *Let T^* be the extension of the integral T on D, let $\alpha, \beta \in \mathbb{R}$, let $f, g \in \mathcal{F}^*$ and let $I \in \mathcal{I}_f \cap \mathcal{I}_g$. Then*

$$T^*(\alpha f + \beta g, I) = \alpha T^*(f, I) + \beta T^*(g, I).$$

Proof Since f and g are also T-integrable on I, from (I2) we obtain

$$\begin{aligned} T^*(\alpha f + \beta g, I) &= T(\alpha f + \beta g, I) \\ &= \alpha T(f, I) + \beta T(g, I) \\ &= \alpha T^*(f, I) + \beta T^*(g, I). \end{aligned}$$

□

Theorem 3.3 *Let T^* be the extension of the integral T on D, let $f \in \mathcal{F}^*$ and let $I_1, I_2 \in \mathcal{I}_f \cap \mathcal{I}_g$. Suppose that $I_1 \cap I_2$ consists of the only one point. Then*

$$T^*(f, I_1) + T^*(f, I_2) = T^*(f, I_1 \cup I_2).$$

Proof Let $\{(a_p, b_p) \mid p \in \mathbb{N}\}$ be the set of all components of $D^\circ \setminus \mathcal{N}_f$. Since $I_1 \cap I_2 \neq \emptyset$, I_1 and I_2 are subintervals containing the same component (a_p, b_p), there exists $T^*(f, I_1 \cup I_2)$. Since f is also T-integrable on I_1 and I_2, from (I3) we obtain

$$\begin{aligned} T^*(f, I_1) + T^*(f, I_2) &= T(f, I_1) + T(f, I_2) \\ &= T(f, I_1 \cup I_2) \\ &= T^*(f, I_1 \cup I_2). \end{aligned}$$

□

Usually any closed interval $[a,b]$ satisfies $a \leq b$. However, when b must be considered a variable, we should consider the case of $a > b$. As is often used, if $a > b$, we define $T^*(f, [a,b]) = -T^*(f, [b,a])$. Moreover, even in the case of $a > b$, if $[b,a] \in \mathcal{I}_f$, we determine $[a,b] \in \mathcal{I}_f$.

Theorem 3.4 *Let T^* be the extension of the integral T on D, let $f \in \mathcal{F}^*$, let $\{(a_p, b_p) \mid p \in \mathbb{N}\}$ be the set of all components of $D^\circ \setminus \mathcal{N}_f$ and let $c_p \in (a_p, b_p)$ for any p. If $[c_p, x] \in \mathcal{I}_f$, then $T^*(f, [c_p, x])$ is the function of the variable x and continuous on $D^\circ \setminus \mathcal{N}_f$.*

Proof Let $\{(a_p, b_p) \mid p \in \mathbb{N}\}$ be the set of all components of $D^\circ \setminus \mathcal{N}_f$. Note that $[a, x]$ is a subinterval of the fixed component (a_p, b_p). Since f is also T-integrable on $[a, x]$ for any x satisfying $[a, x] \subset (a_p, b_p)$, $T^*(f, [a, x])$ is the function of the variable x and by (I4) continuous. □

4 An Extension of Henstock–Kurzweil Integral

In this section, we consider the Henstock–Kurzweil integral as integral in the previous section. We call the extension of Henstock–Kurzweil integral the extended Henstock–Kurzweil integral and denote by

$$(HK)^* \int_I f(x)dx$$

the extended Henstock–Kurzweil integral of f on $I \in \mathcal{I}_f$. The Saks-Henstock lemma is very important. Firstly, we consider the Saks-Henstock lemma for the extended Henstock–Kurzweil integral.

Theorem 4.1 *Suppose that a function f from D into $\mathbb{R}$ is extended Henstock–Kurzweil integrable on D. Then for any $I \in \mathcal{I}_f$ and for any $\varepsilon > 0$ there exists a gauge δ such that*

$$\sum_{k=1}^{K} \left| f(\xi_k)|I_k| - (HK)^* \int_{I_k} f(x)dx \right| < \varepsilon$$

for any partial δ-fine Perron partition $\{(I_k, \xi_k) \mid k = 1, \ldots, K\}$ of I.

Proof Since f is Henstock–Kurzweil integrable on I, by Theorem 2.1 we obtain the desired result. □

Next we consider a convergence theorem.

Theorem 4.2 *Let $\{f_n\}$ be a sequence of extended Henstock–Kurzweil integrable functions on $[a, b]$. Suppose that $\{f_n\}$ is pointwisely convergent to f and that there exist extended Henstock–Kurzweil integrable functions g and h such that $g \le f_n \le h$ for any n. Suppose that the closure $\mathcal{N}$ of $\bigcup_{n=1}^{\infty} \mathcal{N}_{f_n}$ is a null set. Let $\mathcal{N}_f = \mathcal{N} \cup \mathcal{N}_g \cup \mathcal{N}_h$. Then f is extended Henstock–Kurzweil integrable and*

$$(HK)^* \int_I f(x)dx = \lim_{n\to\infty} (HK)^* \int_I f_n(x)dx$$

for any $I \in \mathcal{I}_f$.

Proof Since f_n, g and h are Henstock–Kurzweil integrable on I, by Theorem 2.2 we obtain the desired result. □

Lastly, we consider the relation with the derivative.

Theorem 4.3 *Let f be an extended Henstock–Kurzweil integrable function on $[a, b]$ and*

$$F(x) = (HK)^* \int_{[c,x]} f(t)dt$$

for any $[c, x] \in \mathcal{I}_f$. Then F is generalized absolutely continuous in the restricted sense and hence differentiable almost everywhere. Moreover $F' = f$ holds almost everywhere and hence F' is extended Henstock–Kurzweil integrable.

Proof Let $\{(a_p, b_p) \mid p \in \mathbb{N}\}$ be the set of all components of $(a, b) \setminus \mathcal{N}_f$. Then F is the function from $\bigcup_{p=1}^{\infty}(a_p, b_p) = (a, b) \setminus \mathcal{N}_f$ into $\mathbb{R}$. Note that, if $x \in (a_p, b_p)$, then the constant $c \in (a_p, b_p)$ also. Let

$$E_{p,m} = \left[a_p + \frac{b_p - a_p}{2^m}, b_p - \frac{b_p - a_p}{2^m}\right]$$

for any p and for any $m \in \mathbb{N}$. Since $F|_{E_{p,m}}$ is generalized absolutely continuous in the restricted sense, there exists $\{E_{p,m,\ell} \mid \ell \in \mathbb{N}\}$ such that

$$\bigcup_{\ell=1}^{\infty} E_{p,m,\ell} = E_{p,m}$$

and $F|_{E_{p,m,\ell}}$ is absolutely continuous in the restricted sense. Since

$$\begin{aligned}\bigcup_{p=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{\ell=1}^{\infty} E_{p,m,\ell} &= \bigcup_{p=1}^{\infty}\bigcup_{m=1}^{\infty} E_{p,m} \\ &= \bigcup_{p=1}^{\infty}(a_p, b_p) \\ &= (a, b) \setminus \mathcal{N}_f,\end{aligned}$$

F is generalized absolutely continuous in the restricted sense.

Moreover, since f is Henstock–Kurzweil integrable on $[c, x]$, by Theorem 2.3 $F' = f$ holds almost everywhere. □

5 Applications

In this section, firstly, we consider the integral of $f(x) = \frac{1}{x}$. If $a < 0 < b$, then f is not Henstock–Kurzweil integrable on $[a, b]$.

Let $\mathcal{N}_f = \{0\}$. Then

$$\mathcal{I}_f = \{[a, b] \mid [a, b] \subset (-\infty, 0) \cup (0, \infty)\}$$

and

$$(HK)^* \int_{[a,x]} \frac{dt}{t} \left(= (HK) \int_{[a,x]} \frac{dt}{t}\right) = \log|x| - \log|a|$$

for any $[a, x] \in \mathcal{I}_f$.

Next we consider the following initial value problem of differential equation:

$$\begin{cases} u'(t) = 1 + u(t)^2, \\ u(0) = 0. \end{cases}$$

This problem can be solved as follows. Since $u'(t) = 1 + u(t)^2$,

$$\int_{[c,t]} \frac{u'(s)}{1 + u(s)^2} ds = \int_{[c,t]} ds.$$

By integration by substitution we obtain

$$\int_{[c,t]} \frac{u'(s)}{1 + u(s)^2} ds = \int_{[u(c),u(t)]} \frac{dv}{1 + v^2} = \arctan u(t) - \arctan u(c)$$

and hence

$$\arctan u(t) - \arctan u(c) = t - c.$$

Therefore we obtain

$$u(t) = \tan(t - c + \arctan u(c)), \ t \in \left(\frac{\pi(2n-1)}{2}, \frac{\pi(2n+1)}{2}\right)$$

for any $n \in \mathbb{Z}$. From $u(0) = 0$ we obtain

$$0 = u(0) = \tan(-c + \arctan u(c))$$

and hence

$$-c + \arctan u(c) = \pi m$$

for any $m \in \mathbb{Z}$. Therefore

$$u(t) = \tan(t + \pi m) = \tan t,\ t \in \left(\frac{\pi(2n-1)}{2}, \frac{\pi(2n+1)}{2}\right)$$

for any $n \in \mathbb{Z}$. Since the domain of u must include 0 and the solution blows up at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, $n = 0$ must be used, that is,

$$u(t) = \tan t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

The solution cannot be extended anymore.

On the other hand, we replace the integral to the extended integral. Let $\mathcal{N}_f = \left\{ \frac{\pi(2n+1)}{2} \,\middle|\, n \in \mathbb{Z} \right\}$. Then

$$\mathcal{I}_f = \left\{ [a, b] \,\middle|\, \text{there exists } n \in \mathbb{Z} \text{ such that } [a, b] \subset \left(\frac{\pi(2n-1)}{2}, \frac{\pi(2n+1)}{2}\right) \right\}.$$

We obtain

$$(HK)^* \int_{[c,t]} \frac{u'(s)}{1 + u(s)^2} ds = (HK)^* \int_{[c,t]} ds$$

for any $[c, t] \in \mathcal{I}_f$. In the same way as above, we obtain

$$u(t) = \tan(t - c + \arctan u(c)),\ c, t \in \left(\frac{\pi(2n-1)}{2}, \frac{\pi(2n+1)}{2}\right)$$

for any $n \in \mathbb{Z}$. In the case of $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, from $u(0) = 0$ we obtain

$$0 = u(0) = \tan(-c + \arctan u(c))$$

and hence

$$-c + \arctan u(c) = 0.$$

Therefore

$$u(t) = \tan t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

In the case of $c \in \left(\frac{\pi(2n-1)}{2}, \frac{\pi(2n+1)}{2}\right)$, since u is continuous on $\left(\frac{\pi(2n-1)}{2}, \frac{\pi(2n+1)}{2}\right)$, we obtain

$$-c + \arctan u(c) = \pi n.$$

Therefore

$$u(t) = \tan(t + \pi n) = \tan t, \ t \in \left(\frac{\pi(2n-1)}{2}, \frac{\pi(2n+1)}{2} \right)$$

for any $n \in \mathbb{Z}$. Using the extended integral, we do not need to consider the blow-up of the solution. Therefore n can be arbitrary.

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A Higher Order Finite Difference Method for Numerical Solution of the Kuramoto–Sivashinsky Equation

Deepti Kaur and R. K. Mohanty

Abstract The Kuramoto–Sivashinsky equation is a fundamental fourth-order partial differential equation modeling various nonlinear physical phenomena in unstable systems. It occupies a considerable position in explaining the motion of a fluid going down a vertical wall, a spatially uniform oscillating chemical reaction in a homogeneous medium and unstable drift waves in plasmas. The analytical treatment of this nonlinear differential equation is too involved a process and requires application of advanced mathematical tools, so it is required to develop efficient numerical techniques whose solutions are of great significance to scientists and engineers. One way of solving this equation is the application of compact finite difference method which is steadily acquiring popularity owing to its high accuracy and easy implementation. In this paper, a novel two-level implicit compact finite difference method to the solution of the one-dimensional Kuramoto–Sivashinsky equation subject to appropriate initial and boundary conditions is presented using coupled approach. The method is fourth-order accurate in space and second-order accurate in time. It is based on only three-spatial grid points of a compact stencil without the need to discretize the boundary conditions. Computational results are presented to illustrate the applicability and efficiency of the proposed method.

Keywords Kuramoto–Sivashinsky equation · Nonlinear · Two-level · Compact finite difference method · Tri-diagonal

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D. Kaur (✉)
Department of Mathematics, Maitreyi College, University of Delhi,
New Delhi 110021, India
e-mail: deeptimaths15@gmail.com

R. K. Mohanty
Department of Applied Mathematics, Faculty of Mathematics
and Computer Science, South Asian University, Akbar Bhawan,
Chanakyapuri, New Delhi 110021, India
e-mail: rmohanty@sau.ac.in

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1 Introduction

The Kuramoto–Sivashinsky equation (KSE) reveals complex chaotic behavior often encountered in the study of continuous media having the following form:

$$\phi_t + \phi\,\phi_x + \alpha\,\phi_{xx} + \gamma\,\phi_{xxxx} = 0, \quad x \in [a,b],\; t \in (0,T], \tag{1}$$

subject to the following initial and boundary conditions:

$$\phi(x,0) = \phi_0(x), \quad a \le x \le b, \tag{2.1}$$

$$\phi(a,t) = g_0(t), \quad \phi(b,t) = g_1(t), \quad t > 0, \tag{2.2}$$

$$\phi_{xx}(a,t) = h_0(t), \quad \phi_{xx}(b,t) = h_1(t), \quad t > 0,, \tag{2.3}$$

where α and γ are constants associated with the growth of linear stability and surface tension, respectively. The KSE arises in a wide range of physical processes such as in representing long waves on the interface between two viscous fluids [1], in flame propagation [2], in reaction–diffusion combustion dynamics and in the study of unstable drift waves in plasmas [3]. In this framework, it was introduced independently by Kuramoto and Tsuzuki [3] in order to study dissipative structure of reaction–diffusion and by Sivashinsky [2] as a model for analyzing flame front propagation during mild combustion processes and instabilities induced by thermal conduction of the considered gas. The KSE comprises of high-order dissipation term ϕ_{xxxx}, nonlinear advection $\phi\,\phi_x$ and linear growth term ϕ_{xx}. Thus, in general the KSE comprises a nonlinear initial valued problem concerning fourth-order spatial derivative. When $\gamma = 0$, the surface tension term is eliminated and the KSE reduces to Burger's equation. It has emerged as a primary evolution equation for explaining highly nonlinear physical phenomena in unstable systems.

Due to its vast applications, considerable attention has been given to determine its analytical and numerical solutions. In recent years, the solution of KSE has been found by applying various methods including finite difference methods [4, 5], tanh-function method [6], local discontinuous Galerkin method [7], Chebyshev spectral collocation method [8], radial basis function based on mesh free method [9], He's variational iteration method [10], quintic B-spline collocation scheme [11] and lattice Boltzmann model [12]. In [13], B-spline-based finite element approach to the solution of the one-dimensional KSE is presented. Recently, in [14] the exponential cubic B-spline collocation method is used for the numerical treatment of KSE. Compact finite difference method which is restricted to a patch of cells immediately surrounding any given mesh point is one of the attractive means for obtaining approximate solutions to KSE and is steadily gaining recognition due to the relative ease of implementation and flexibility. Apart from this, the advantage of developing a compact scheme is its suitability to be used directly adjacent to the boundary without introducing any extra nodes outside the boundary of the domain.

In this article, we present a new two-level implicit method of accuracy two in time and four in space for the solution of KSE (1). In this method, we use only three spatial grid points. We do not require any fictitious point for computation. The numerical solution has been computed without transforming the equation and without using linearization. The paper is arranged as follows: In Sect. 2, we describe the compact finite difference method. In Sect. 3, we discuss the stability of the linear part of KSE. In Sect. 4, numerical results are presented for different test problems with tabular and graphical illustrations. Conclusions are given in Sect. 5.

2 Description of the Method

In this section, we will carry on the space and temporal discretization of the time-dependent one-dimensional KSE (1). We define a new variable $\psi = \phi_{xx}$, then the problem (1) is decomposed into a system of two second-order PDEs as

$$\phi_{xx} = \psi, \quad x \in [a, b], \tag{3.1}$$

$$\gamma\psi_{xx} + \phi_t = -\phi\,\phi_x - \alpha\,\psi, \quad x \in [a, b],\ t \in (0, T], \tag{3.2}$$

subject to initial and boundary conditions

$$\phi(x, 0) = \phi_0(x), \quad \psi(x, 0) = \phi_0^{''}(x), \quad a \le x \le b \tag{4.1}$$

$$\phi(a, t) = g_0(t), \quad \phi(b, t) = g_1(t), \quad t > 0 \tag{4.2}$$

$$\psi(a, t) = h_0(t), \quad \psi(b, t) = h_1(t), \quad t > 0 \tag{4.3}$$

It is noted that via the initial condition (2.1), the values of all the successive tangential partial derivatives $\phi_x, \phi_{xx}, \ldots$ can be found at $t = 0$. Since $\psi(x, 0) = \phi_{xx}(x, 0)$, hence ψ is also determined at $t = 0$.

We assume that $\phi(x, t)$ is sufficiently smooth and its required high order derivatives exist in the solution region $\Omega \equiv \{(x, t) \mid a < x < b,\ t > 0\}$. Let $h > 0$ and $k > 0$ be the mesh spacing in the space and time directions, respectively. Spatial knots are equally distributed over the solution domain $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b,$$

with mesh spacing $h = (b - a)/(N + 1)$ and $t_j = jk, 0 < j < J$, where N and J are positive integers. Let $\lambda = (k/h^2) > 0$ be the mesh ratio parameter. We replace the region Ω by a set of grid points (x_l, t_j). Let the exact solution values of $\phi(x, t)$ and $\psi(x, t)$ at the grid point (x_l, t_j) be denoted by Φ_l^j and Ψ_l^j, respectively, and ϕ_l^j and ψ_l^j denote their approximate solution values, respectively.

We denote $f(x,t) = -\phi\,\phi_x - \alpha\,\psi$. Then at the grid point (x_l, t_j), the differential equations (3.1)–(3.2) can be written as

$$\phi^j_{xx_l} = \psi^j_l, \tag{5.1}$$

$$\gamma\psi^j_{xx_l} + \phi^j_{t_l} = -\phi^j_l\,\phi^j_{x_l} - \alpha\,\psi^j_l \equiv f^j_l.. \tag{5.2}$$

At the grid point (x_l, t_j), for $S = \phi,\ \psi$, we denote

$$S_{ab} = \left(\frac{\partial^{a+b} S}{\partial x^a \partial t^b}\right)_{(x_l,t_j)}.$$

Differentiating (3.1)–(3.2) with respect to 't' at the grid point (x_l, t_j), we obtain

$$\phi_{21} = \psi_{01}, \tag{6.1}$$

$$\gamma\psi_{21} + \phi_{02} = -\phi_{01}\phi_{10} - \phi_{00}\phi_{11} - \alpha\psi_{01}. \tag{6.2}$$

Further the functions ψ^j_l and f^j_l defined by (5.1)–(5.2) satisfies

$$\delta^2_x\phi^j_l = \frac{h^2}{12}[\psi^j_{l+1} + 10\psi^j_l + \psi^j_{l-1}] + O(h^6), \tag{7.1}$$

$$\gamma\delta^2_x\psi^j_l + \frac{h^2}{12}[\phi^j_{t_{l+1}} + 10\phi^j_{t_l} + \phi^j_{t_{l-1}}] = \frac{h^2}{12}[f^j_{l+1} + 10f^j_l + f^j_{l-1}] + O(h^6), \quad , \tag{7.2}$$

where δ_x denotes the central difference operator in the spatial direction.

For $p = 0, \pm 1$, we consider the following approximations:

$$\bar{t}_j = t_j + \frac{k}{2}, \tag{8.1}$$

$$\overline{\phi}^j_{l+p} = (\phi^{j+1}_{l+p} + \phi^j_{l+p})/2, \tag{8.2}$$

$$\overline{\psi}^j_{l+p} = (\psi^{j+1}_{l+p} + \psi^j_{l+p})/2, \tag{8.3}$$

$$\overline{\phi}^j_{t_{l+p}} = (\phi^{j+1}_{l+p} - \phi^j_{l+p})/k, \tag{8.4}$$

$$\overline{\phi}^j_{x_l} = (\overline{\phi}^j_{l+1} - \overline{\phi}^j_{l-1})/2h, \tag{8.5}$$

$$\overline{\phi}^j_{x_{l\pm1}} = (\pm 3\overline{\phi}^j_{l\pm1} \mp 4\overline{\phi}^j_l \pm \overline{\phi}^j_{l\mp1})/2h, \tag{8.6}$$

$$\overline{f}^j_{l\pm1} = -\overline{\phi}^j_{l\pm1}\,\overline{\phi}^j_{x_{l\pm1}} - \alpha\,\overline{\psi}^j_{l\pm1}. \tag{8.7}$$

Simplifying above approximations, we obtain

$$\overline{\phi}_l^j = \phi_l^j + \frac{k}{2}\phi_{01} + O(k^2), \tag{9.1}$$

$$\overline{\psi}_l^j = \psi_l^j + \frac{k}{2}\psi_{01} + O(k^2), \tag{9.2}$$

$$\overline{\phi}_{l\pm1}^j = \phi_{l\pm1}^j + \frac{k}{2}\phi_{01} + O(k^2 \pm kh), \tag{9.3}$$

$$\overline{\psi}_{l\pm1}^j = \psi_{l\pm1}^j + \frac{k}{2}\psi_{01} + O(k^2 \pm kh), \tag{9.4}$$

$$\overline{\phi}_{t_l}^j = \phi_{t_l}^j + \frac{k}{2}\phi_{02} + O(k^2), \tag{9.5}$$

$$\overline{\phi}_{t_{l\pm1}}^j = \phi_{t_{l\pm1}}^j + \frac{k}{2}\phi_{02} + O(k^2 \pm kh), \tag{9.6}$$

$$\overline{\phi}_{x_l}^j = \phi_{x_l}^j + \frac{k}{2}\phi_{11} + \frac{h^2}{6}\phi_{30} + O(k^2 + kh^2 + h^4), \tag{9.7}$$

$$\overline{\phi}_{x_{l\pm1}}^j = \phi_{x_{l\pm1}}^j + \frac{k}{2}\phi_{11} - \frac{h^2}{3}\phi_{30} \pm O(kh + h^3), \tag{9.8}$$

$$\overline{f}_{l\pm1}^j = f_{l\pm1}^j + \frac{k}{2}\big(-\phi_{01}\phi_{10} - \phi_{00}\phi_{11} - \alpha\psi_{01}\big) + \frac{h^2}{3}\phi_{00}\phi_{30} \pm O(kh + h^3). \tag{9.9}$$

In order to derive $O(k^2 + kh^2 + h^4)$ difference method, we require $O(k + h^2)$ approximation for the first-order partial derivative ϕ_x. Let

$$\overline{\overline{\phi}}_{x_l}^j = \overline{\phi}_{x_l}^j + a\,h\big[\,\overline{\psi}_{l+1}^j - \overline{\psi}_{l-1}^j\,\big], \tag{10}$$

where ‘a’ is a free parameter to be determined.

By the help of the approximations (9.4) and (9.7), from (10), we obtain

$$\overline{\overline{\phi}}_{x_l}^j = \phi_{x_l}^j + \frac{k}{2}\phi_{11} + \frac{h^2}{6}(1 + 12a)\phi_{30} + O(k^2 + kh^2 + h^4). \tag{11}$$

Next, we define

$$\overline{\overline{f}}_l^j = -\overline{\phi}_l^j\,\overline{\overline{\phi}}_{x_l}^j - \alpha\,\overline{\psi}_l^j. \tag{12}$$

Finally, by the help of approximations (9.1), (9.2) and (11), from (12), we obtain

$$\overline{\overline{f}}_l^j = f_l^j + \frac{k}{2}\big(-\phi_{01}\phi_{10} - \phi_{00}\phi_{11} - \alpha\psi_{01}\big) - \frac{h^2}{6}(1 + 12a)\phi_{00}\phi_{30} + O(k^2 + kh^2 + h^4). \tag{13}$$

Then the numerical method may be written as

$$\delta_x^2 \overline{\phi}_l^j = \frac{h^2}{12}[\overline{\psi}_{l+1}^j + 10\,\overline{\psi}_l^j + \overline{\psi}_{l-1}^j], \tag{14.1}$$

$$\gamma\delta_x^2 \overline{\psi}_l^j + \frac{h^2}{12}[\overline{\phi}_{t_{l+1}}^j + 10\,\overline{\phi}_{t_l}^j + \overline{\phi}_{t_{l-1}}^j] = \frac{h^2}{12}[\overline{f}_{l+1}^j + 10\,\overline{\overline{f}}_l^j + \overline{f}_{l-1}^j],\ l = 1, 2, \ldots, N;\ j = 0, 1, 2, \ldots\quad . \tag{14.2}$$

The local truncation error (LTE_1) associated with (14.1) may be obtained as (LTE_1)= $O(k^2h^2 + kh^4 + h^6)$ using the relations (6.1) and (7.1). We will see that the local truncation error (LTE_2) associated with (14.2) is obtained as (LTE_2) $= O(k^2h^2 + kh^4 + h^6)$, implying that the accuracy of the method is of $O(k^2 + kh^2 + h^4)$. When $k \propto h^2$, that is, for a fixed value of λ, the accuracy of the method becomes $O(h^4)$ in space.

By the help of the approximations (9.2), (9.4)–(9.6), (9.9) and (13), from (7.2) and (14.2) we obtain:

$$\begin{aligned}&\gamma\delta_x^2 \psi_l^j + \frac{h^2}{12}[\phi_{t_{l+1}}^j + 10\phi_{t_l}^j + \phi_{t_{l-1}}^j] + \frac{kh^2}{2}(\gamma\psi_{21} + \phi_{02}) + O(k^2h^2 + kh^4 + h^6)\\ &= \frac{h^2}{12}[f_{l+1}^j + 10f_l^j + f_{l-1}^j] + \frac{kh^2}{2}(-\phi_{01}\phi_{10} - \phi_{00}\phi_{11} - \alpha\psi_{01}) - \frac{h^4}{12}(1 + 20a)\phi_{00}\phi_{30} + (\mathrm{LTE}_2)\end{aligned} \tag{15}$$

Substituting (6.2) into (15), we obtain the local truncation error

$$(\mathrm{LTE}_2) = \frac{h^4}{12}(1 + 20a)\phi_{00}\phi_{30} + O(k^2h^2 + kh^4 + h^6). \tag{16}$$

For the proposed method (14.2) to be of $O(k^2 + kh^2 + h^4)$, the coefficient of h^4 in (16) must be zero. Thus we obtain the value of the parameter $a = -1/20$ and the local truncation error reduces to (LTE_2) $= O(k^2h^2 + kh^4 + h^6)$.

Incorporating the given initial and boundary conditions (2.1)–(2.3), the three-point compact difference scheme results in a tri-diagonal nonlinear system. To solve such a system, we could apply Newton's nonlinear iteration procedure [15, 16].

3 Stability Analysis

Consider the following the higher order linear part of the KSE:

$$\phi_t + \alpha\,\phi_{xx} + \gamma\,\phi_{xxxx} = 0,\quad x \in [a, b],\ t \in (0, T] \tag{17}$$

subject to the initial and boundary conditions (2.1)–(2.3). The matrix form of the difference formula (14.1)–(14.2) when applied to the model Eq. (17) is

$$\mathbf{A}\,\mathbf{y}^{j+1} = (-\mathbf{A} + \mathbf{B})\,\mathbf{y}^j + \mathbf{c}, \tag{18}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix}$$

The submatrices are given by

$$\begin{aligned} &\mathbf{A}_{11} = 12[1, -2, 1], \;\; \mathbf{A}_{12} = -h^2[1, 10, 1], \;\; \mathbf{A}_{21} = [1, 10, 1], \\ &\mathbf{A}_{22} = \lambda\left[6\gamma + \frac{\alpha h^2}{2}, -12\gamma + 5\alpha h^2, 6\gamma + \frac{\alpha h^2}{2}\right], \\ &\mathbf{B}_{11} = \mathbf{B}_{12} = \mathbf{B}_{22} = [0, 0, 0], \;\; \mathbf{B}_{21} = 2[1, 10, 1], \end{aligned}$$

where

$$[a, b, c] = \begin{bmatrix} b & c & 0 & \cdots & 0 \\ a & b & c & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & a & b & c \\ 0 & \cdots & 0 & a & b \end{bmatrix}$$

is the Nth-order tri-diagonal matrix having eigenvalues $b + 2\sqrt{a\,c}\cos(2\theta), \; 2\,\theta = (s\pi)/(N+1), \; s = 1, \ldots, N$. Œ, are solution vectors and vectors $\mathbf{c}_1, \; \mathbf{c}_2$ consist of homogenous functions, initial and boundary values of the block system (18).

For discussing stability, we must consider the homogenous part of (18) which may be written as

$$\mathbf{y}^{j+1} = (-\mathbf{I} + \mathbf{A}^{-1}\,\mathbf{B})\,\mathbf{y}^j \tag{19}$$

Let $\mathbf{E}^j = \mathbf{y}^j - \mathbf{Y}^j$ (In the absence of round-off errors) be the error vector at jth iterate and

$$\mathbf{Y}^j = \begin{bmatrix} \Phi \\ \Psi \end{bmatrix}^j,$$

where Φ, Ψ are exact solution vectors.

The error equation may be written as

$$\boldsymbol{E}^{j+1} = \mathbf{H}\boldsymbol{E}^j,$$

where $\mathbf{H} = -\mathbf{I} + \mathbf{A}^{-1}\,\mathbf{B}$ is the amplification matrix. The eigenvalues of $\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{21}$ and $\mathbf{A}_{22}$ are given by $-48\sin^2\phi$, $-h^2(12 - 4\sin^2\phi)$, $12 - 4\sin^2\phi$ and $\lambda(6\alpha h^2 - 2(12\gamma + \alpha h^2)\sin^2\phi)$, respectively.

If ξ denotes the eigenvalue of the matrix $\mathbf{A}$, then it satisfies the characteristic equation:

$$\det\begin{bmatrix} -48\sin^2\phi-\xi & -h^2(12-4\sin^2\phi) \\ 12-4\sin^2\phi & \lambda(6\alpha h^2-2(12\gamma+\alpha h^2)\sin^2\phi)-\xi \end{bmatrix}=0 \tag{20}$$

which on simplification gives the equation

$$\xi^2+[48\sin^2\phi-\lambda(6\alpha h^2-2(12\gamma+\alpha h^2)\sin^2\phi)]\xi-48\lambda\sin^2\phi(6\alpha h^2-2(12\gamma+\alpha h^2)\sin^2\phi) \\ +h^2(12-4\sin^2\phi)^2=0 \tag{21}$$

Further, the eigenvalues of $\mathbf{B}_{11}$, $\mathbf{B}_{12}$, $\mathbf{B}_{21}$ and $\mathbf{B}_{22}$ are given by 0, 0, $2(12-4\sin^2\phi)$ and 0, respectively. Consequently, 0 is the only eigenvalue of matrix $\mathbf{B}$. Let τ be the eigenvalue of $\mathbf{A}^{-1}\mathbf{B}$, where ξ and 0 are the eigenvalues of $\mathbf{A}$ and $\mathbf{B}$, respectively. Then, $\tau-1$ is the eigenvalue of the amplification matrix $\mathbf{H}$. Hence the scheme (14.1)–(14.2) is stable as long as $0<\tau<2$.

4 Numerical Validation

To see the efficiency and versatility of the proposed method, two numerical examples are studied in this section. The accuracy of the method is evaluated in terms of the global relative error (GRE) defined as

$$\text{GRE}=\frac{\sum_{l=1}^{N}|\phi(x_l,t)-\Phi(x_l,t)|}{\sum_{l=1}^{N}|\Phi(x_l,t)|}$$

Example 1 We obtain the numerical solution of the KSE (1) for $\alpha=1$ and $\gamma=1$ with the exact solution given by

$$\phi(x,t)=b+\frac{15}{19}\sqrt{\frac{11}{19}}(-9\tanh(K(x-bt-x_0))+11\tanh^3(K(x-bt-x_0))).$$

The initial and boundary conditions are taken from the exact solution. The above solution models the shock wave propagation with speed b and initial position x_0. This example is studied in [11, 12, 14]. For reason of comparison with the corresponding works in the literature, we take the same physical parameters as in [11, 12, 14]: $b=5$, $K=\frac{1}{2}\sqrt{\frac{11}{19}}$, $x_0=-12$ and the solution domain is taken as $[-30,30]$ with number of partitions as 150 and $k=0.01$. Results are reported in Table 1. The two-dimensional visual comparison of exact and numerical solutions at different time intervals is presented in Fig. 1. Also, in Table 2, the GRE is compared for various

Table 1 Comparison of GRE for Example 1 at different times, $N + 1 = 150$

t	Proposed scheme (14.1)–(14.2)	Method discussed in [14]	Method discussed in [11]	Method discussed in [12]
1	6.0297(-05)	3.3291(-04)	3.8173(-04)	6.7923(-04)
2	9.9303(-05)	5.5636(-04)	5.5114(-04)	1.1503(-03)
3	1.3064(-04)	8.7489(-04)	7.0398(-04)	1.5941(-03)
4	1.6060(-04)	1.2516(-03)	8.6366(-04)	2.0075(-03)

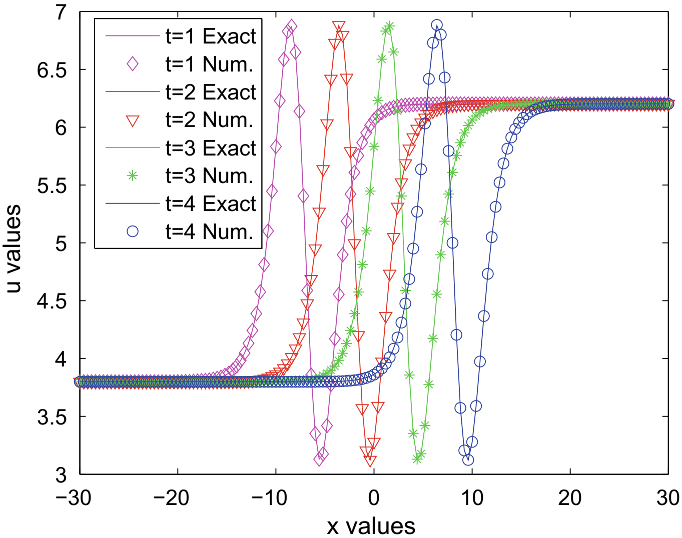


Fig. 1 Example 1: Numerical versus exact solution at various time intervals

spatial partitions at different time intervals with those of [11] to present the effect of change in the number of grid points.

Example 2 We obtain the numerical solution of the KSE (1) for $\alpha = -1$ and $\gamma = 1$ with the exact solution given by

$$u(x,t) = b + \frac{15}{19\sqrt{19}}(-3\tanh(K(x - bt - x_0)) + \tanh^3(K(x - bt - x_0))).$$

For comparison, we have run the proposed algorithm (14.1)–(14.2) with the parameters: $b = 5,\ K = \frac{1}{2\sqrt{19}}$, $x_0 = -25$ and the solution domain is taken as $[-50, 50]$ with number of partitions as 200 and $k = 0.01$. The GREs are reported in Table 2 and comparison is made with the B-spline collocation method of [11], the lattice Boltzmann method of [12] and exponential B-spline collocation algorithm of [14]. Also in Table 3, we have compared the global relative error for different number of

Table 2 Comparison of GRE for Example 1 with change in number of partitions at different times

t	$N + 1 = 200$		$N + 1 = 300$		$N + 1 = 400$	
	Proposed scheme (14.1)–(14.2)	Method discussed in [11]	Proposed scheme (14.1)–(14.2)	Method discussed in [11]	Proposed scheme (14.1)–(14.2)	Method discussed in [11]
1	3.3445(-05)	2.1335(-04)	2.4382(-05)	1.2335(-04)	2.2991(-05)	6.6956(-05)
2	5.8065(-05)	3.0874(-04)	4.4914(-05)	1.6780(-04)	4.2944(-05)	9.6417(-05)
3	7.9875(-05)	3.9500(-04)	6.4010(-05)	2.0791(-04)	6.1833(-05)	1.0947(-04)
4	9.7756(-05)	4.8479(-04)	8.1239(-05)	2.5018(-04)	7.9068(-05)	1.2600(-04)

Table 3 Comparison of GRE for Example 2 at different times, $N + 1 = 200$

t	Proposed scheme (14.1)–(14.2)	Method discussed in [14]	Method discussed in [11]	Method discussed in [12]
6	8.9929(-08)	9.3379(-06)	6.5093(-06)	7.8808(-06)
8	2.3011(-07)	1.5717(-05)	7.1315(-06)	9.5324(-06)
10	3.3576(-07)	2.3730(-05)	7.3103(-06)	1.0891(-05)
12	5.2537(-07)	3.3337(-05)	8.7766(-06)	1.1793(-05)

partitions for various time intervals with those of [11, 12, 14] to illustrate the effect of change in the number of mesh points. The two-dimensional graph of numerical solution versus exact solution is plotted in Fig. 2 for $-50 < x < 50$ at various time intervals.

Example 3 (*Non-homogenous Kuramoto–Sivashinsky equation*)

$$\phi_t + \phi\,\phi_x + \phi_{xx} + \phi_{xxxx} = g(x,t) \quad 0 < x < 1, \quad t > 0 \tag{22}$$

The exact solution of the above problem is $\phi(x,t) = \sinh(t)\sin(\pi x)$. Here

$$g(x,t) = \left(\pi^4 - \pi^2 + \pi\sinh(t)\cos(\pi x)\right)\sinh(t)\sin(\pi x) + \cosh(t)\sin(\pi x).$$

The numerical solution of differential equation (22) is computed using proposed difference method with number of partitions 8, 16 and 32 at time $t = 1$ and 2. The maximum absolute errors defined using the formula

$$\text{MAE} = \max_{1\le l\le N} |\phi(x_l,t) - \Phi(x_l,t)|$$

are computed. For each spatial mesh length h, the corresponding time step size is chosen as $k \propto h^2$. With this choice of time step size, the theoretical order of convergence becomes $O(h^4)$, i.e., the method is fourth-order accurate in space, which is verified using the formula

$$(\log(e_{h_1}) - \log(e_{h_2}))/(\log(h_1) - \log(h_2))$$

where e_{h_1} and e_{h_2} are errors corresponding to two uniform mesh lengths h_1 and h_2, respectively. The maximum absolute error and the order of convergence are tabulated in Table 4. The 3D graphs of numerical and analytical solutions are plotted in Fig. 3 for $h = \frac{1}{16}$ at $t = 1.0$ for $0 < x < 1$.

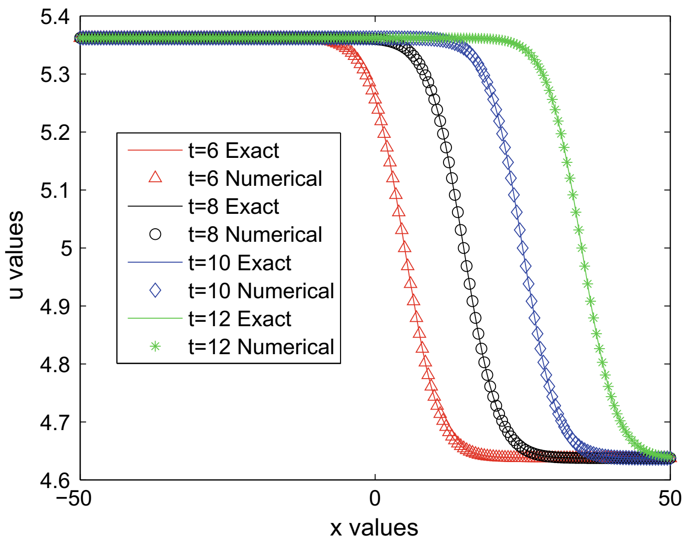
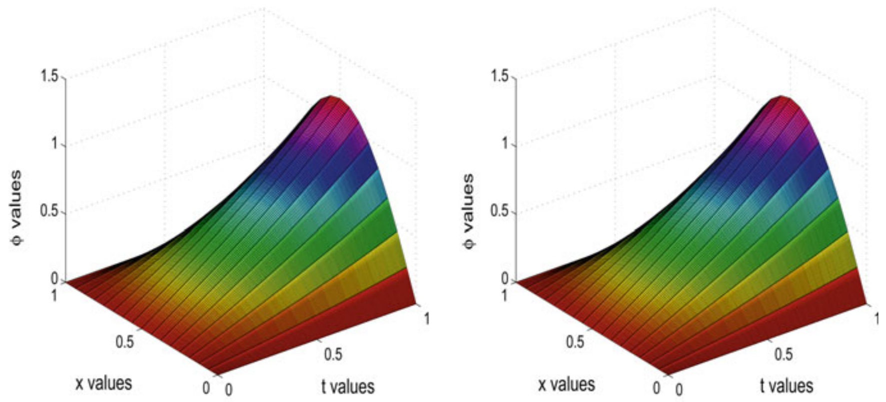


Fig. 2 Example 2: Numerical versus exact solution at various time intervals

Table 4 The maximum absolute errors and order of convergence of proposed method for Example 3 at $t = 1$ and 2 for a fixed $\lambda = (k/h^2) = 1.6$

		$t = 1$		$t = 2$	
h		MAE	Order	MAE	Order
1/8	ϕ	1.5299(-04)	–	4.7467(-04)	–
	ϕ_{xx}	3.6268(-04)	–	1.3747(-03)	–
1/16	ϕ	9.4912(-06)	4.0107	2.9448(-05)	4.0107
	ϕ_{xx}	2.2481(-05)	4.0119	8.4283(-05)	4.0277
1/32	ϕ	5.9413(-07)	3.9977	1.8392(-06)	4.0010
	ϕ_{xx}	1.2354(-06)	4.1857	4.2829(-06)	4.2986



(a) Numerical Solution (b) Exact Solution

Fig. 3 The graph of numerical and exact solution at $t = 1$ and $h = \frac{1}{16}$ for $0 < x < 1$: Example 3

5 Concluding Remarks

In essence, the current work presents highly accurate numerical approximation of order two in time and four in space for the one-dimensional time-dependent KSE. The method is derived using three spatial uniform grid points. Extensive numerical results of diverse scenarios for the KSE illustrate the superiority of our approach. It is observed that the simulating results are in good agreement with both the exact and existing numerical solutions. The results indicate that the errors in our method are much less than the errors in Refs. [11, 12, 14]. We are currently working to extend this technique to solve the time-dependent KSE in two and three space dimensions.

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An Extension of the Robe's Problem

Bhavneet Kaur

Abstract The aim of the paper is to extend the Robe's restricted three-body problem to $2 + 2$ bodies. We study the possible generalizations of the above problem by taking the shape of the first primary as a spherical shell and the second an oblate body. Next, the hydrostatic equilibrium figure of the first primary is taken as a Roche Ellipsoid. We have taken into consideration all the three components of the pressure field in deriving the expression for the buoyancy force, viz., (i) due to the own gravitational field of the fluid (ii) that originating in the attraction of m_2 (iii) that arising from the centrifugal force. We study the equations of motion in each case and interpret them.

Keywords Robe's restricted problem · Buoyancy · Equilibrium solutions · Roche Ellipsoid

1 Introduction

Robe [12] has formulated a new kind of restricted three-body problem in which the bigger primary of mass m_1 is a rigid spherical shell filled with a homogeneous incompressible fluid of density ρ_1. The smaller primary is a mass point m_2 outside the shell. The third body of mass m_3 which is supposed to be moving inside the shell is assumed to be a small solid sphere of density ρ_3. The mass and the radius of the third body are infinitesimal. He assumed the motion of m_2 around the mass m_1 to be Keplerian. He has discussed the linear stability of the equilibrium solutions.

Hallan and Rana [7] extended the [12] and proved that there exist other equilibrium points too. They [6] studied the effect of oblateness on the location and stability of equilibrium points in the Robe's circular problem.

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B. Kaur (✉)
Lady Shri Ram College for Women, Delhi University, Delhi, India
e-mail: bhavneet.lsr@gmail.com

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Hallan and Mangang [4] considered the first primary as an oblate spheroid in the Robe's restricted three-body problem. They [5] studied the nonlinear stability of equilibrium point in the perturbed Robe's restricted circular three-body problem.

Whipple [16] studied equilibrium solutions of the restricted problem of 2 + 2 bodies. He further studied the linear stability of all the equilibrium solutions. Motivated by the work of Robe and Whipple, [8] extended the Robe's restricted three-body problem to 2 + 2 bodies as a foremost initiative. Later, [1] deliberated the existence and the linear stability of the equilibrium solutions in the Robe's restricted problem of 2 + 2 bodies when the bigger primary is a spherical shell and the smaller an oblate body.

Plastino and Plastino [11] considered the Robe's problem by taking the shape of the fluid body as Roche's ellipsoid [2]. They discussed the linear stability of the equilibrium solution, which is the centre of the ellipsoid. Giordano et al. [3] discussed the effect of drag force on the stability of the equilibrium point, both in the [12] problem and the problem studied by [11]. Intrigued by the work of Plastino, [9] contemplated the location and linear stability of the equilibrium points in Robe's restricted problem of 2 + 2 bodies when the bigger primary is a Roche ellipsoid. Kaur and Aggarwal [10] unfolded [9] problem considering the smaller primary as an oblate body. Singh and Omale [13] studied the motion of an infinitesimal mass in the Robe's circular restricted three-body problem in two cases. In the first case, the first primary is an oblate spheroid, while in the second case the first primary is a Roche ellipsoid and the full buoyancy of the fluid is taken into account.

In our present paper, we shall analyze the restricted problem of 2 + 2 bodies in the Robe's setup and study all the generalizations to the problem by comparing their equations of motion. Such a model may be useful to study the motion of submarines due to the attraction of Earth and Moon.

2 Statement of the Problem and Equations of Motion: Robe's Restricted Problem of 2 + 2 Bodies

In the problem of 2 + 2 bodies in the Robe's setup, we consider the bigger primary of mass m_1^* as a rigid spherical shell filled with homogeneous incompressible fluid of density ρ_1. The smaller primary is a mass point m_2 outside the shell. The third and the fourth bodies (of mass m_3 and m_4, respectively) are small solid spheres of density ρ_3 and ρ_4, respectively, inside the shell, with the assumption that the mass and the radius of the third and the fourth body are infinitesimal. We assume that (i) m_3 and m_4 never reach the surface of the shell (ii) their position vector at any time t are not the same. Let m_2 describe a circle around m_1^* with constant angular velocity ω (say). The masses m_3 and m_4 mutually attract each other do not influence the motion of m_1^* and m_2 but are influenced by them. We also assume masses m_3 and m_4 are moving in the plane of motion of mass m_2.

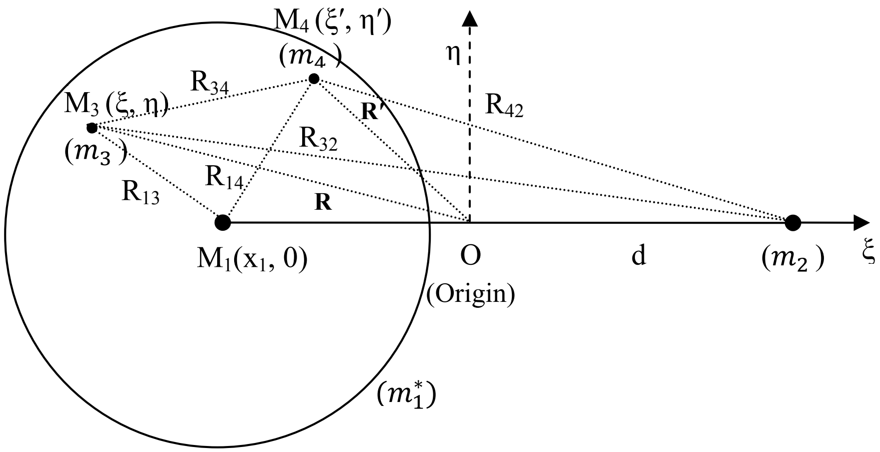

Fig. 1 Geometry of the Robe's restricted problem of 2 + 2 bodies

As in the case of classical restricted problem [15], let the orbital plane of m_2 around m_1^* (i.e., shell with its fluid of density ρ_1) be taken as the $\xi\eta$ plane and let the origin of the coordinate system be at the centre of mass O of the two finite bodies. The coordinate system $O\xi\eta$ are as shown in Fig. 1.

Various forces acting on m_3 are as follows:

1. The gravitational force $\mathbf{F_{32}}$ acting on m_3 due to m_2 is

$$\mathbf{F_{32}} = \frac{G m_3 m_2 \mathbf{R_{32}}}{R_{32}^3}. \tag{1}$$

2. The gravitational force $\mathbf{F_{34}}$ acting on m_3 due to m_4 is

$$\mathbf{F_{34}} = \frac{G m_3 m_4 \mathbf{R_{34}}}{R_{34}^3}. \tag{2}$$

3. The gravitational force $\mathbf{F_A}$ exerted by the fluid of density ρ_1 on m_3 is

$$\mathbf{F_A} = -\left(\frac{4}{3}\right)\pi G \rho_1 m_3 \mathbf{R_{13}}, \tag{3}$$

provided $|R_{13}| < a'$ where a' is the radius of m_1^* and $\mathbf{R_{ij}} = \mathbf{M_i M_j}$, M_1 is the centre of the shell m_1^* and M_3 the centre of m_3. 4. The buoyancy force $\mathbf{F_B}$ acting on m_3 is

$$\mathbf{F_B} = \left(\frac{4}{3}\right)\pi \frac{G \rho_1^2 m_3 \mathbf{R_{13}}}{\rho_3}. \tag{4}$$

provided $|R_{13}| < a'$.

The last expression derived is taking into account that m_3 is a sphere of very small radius b, so that the pressure of the fluid ρ_1 inside the shell keeps its spherical symmetry around M_1. The buoyancy force is then $\left(\frac{4}{3}\pi b^3\right)\rho_1\mathbf{g}$, where $\mathbf{g}$ is the gravity of the fluid ρ_1 at M_3, i.e.,

$$\mathbf{g} = \left(\frac{4}{3}\right)\pi G\rho_1\mathbf{R_{13}}$$

with

$$m_3 = \left(\frac{4}{3}\pi b^3\right)\rho_3.$$

The equation of motion of m_3 in the inertial system is

$$\ddot{\mathbf{R}} = \frac{Gm_2\mathbf{R_{32}}}{R_{32}^3} + \frac{Gm_4\mathbf{R_{34}}}{R_{34}^3} - \frac{4}{3}\pi G\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right)\mathbf{R_{13}},$$

where $\mathbf{R} = \mathbf{OM_3}$ and $\mathbf{R_{ij}} = \mathbf{M_iM_j}$.

Now, we determine the equation of motion of m_3 in the rotating (synodic) system. Let us suppose that the coordinate system $O\xi\eta$ rotates with angular velocity $\boldsymbol{\omega}$. This is the same as the angular velocity of m_2 which is describing a circle around m_1^*.

In the synodic system, the equation of motion of m_3 is

$$\begin{aligned}&\frac{\partial^2\mathbf{r}}{\partial t^2} + 2\boldsymbol{\omega}\times\frac{\partial\mathbf{r}}{\partial t} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = \\ &\qquad\frac{Gm_2\mathbf{R_{32}}}{R_{32}^3} + \frac{Gm_4\mathbf{R_{34}}}{R_{34}^3} - \frac{4}{3}\pi G\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right)\mathbf{R_{13}}\end{aligned} \tag{5}$$

where $\mathbf{r} = \mathbf{OM_3}$ and $\boldsymbol{\omega} = \omega\hat{\mathbf{k}} =$ (constant).

Let the coordinates of m_3 and m_4 be (ξ, η) and (ξ', η'), respectively.

The equations of motion of m_3 in cartesian coordinates are

$$\begin{aligned}\ddot{\xi} - 2\omega\dot{\eta} = &-\frac{Gm_2(\xi - x_2)}{\left[(\xi - x_2)^2 + \eta^2\right]^{\frac{3}{2}}} - \frac{Gm_4\left(\xi - \xi'\right)}{\left[(\xi - \xi')^2 + (\eta - \eta')^2\right]^{\frac{3}{2}}} \\ &-\frac{4}{3}\pi G\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right)(\xi - x_1) + \omega^2\xi,\end{aligned} \tag{6}$$

$$\begin{aligned}\ddot{\eta} + 2\omega\dot{\xi} = &-\frac{Gm_2\eta}{\left[(\xi - x_2)^2 + \eta^2\right]^{\frac{3}{2}}} - \frac{Gm_4\left(\eta - \eta'\right)}{\left[(\xi - \xi')^2 + (\eta - \eta')^2\right]^{\frac{3}{2}}} \\ &-\frac{4}{3}\pi G\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right)\eta + \omega^2\eta.\end{aligned} \tag{7}$$

Now, as m_2 is moving around m_1^* in a circle of radius d with angular velocity $\boldsymbol{\omega}$, we have

$$\omega = \sqrt{\frac{G\left(m_1^* + m_2\right)}{d^3}}. \tag{8}$$

We, now, fix the units such that the sum of the masses of the primaries is unity, i.e., $m_1^* + m_2 = 1$, the distance between the primaries is unity, i.e., $d = 1$. The unit of time t is chosen in such a way that $G = 1$ [14].

We further take

$$\mu_1 = \frac{m_1^*}{m_1^* + m_2}, \qquad \mu_2 = \frac{m_2}{m_1^* + m_2}.$$

Let $\mu_2 = \mu$, (say), then $\mu_1 = 1 - \mu$.

Since the centre of mass of the primaries divides the line joining them in the ratio of the masses, so $x_1 = -\mu$ and $x_2 = 1 - \mu$.

Thus, the coordinates of m_1^* and m_2 are $(-\mu, 0)$, $(1 - \mu, 0)$.

In the new units, the angular velocity ω given by the Eq. (8) becomes unity, i.e., $\omega = 1$.

The equations of motion of m_3 in the dimensionless cartesian coordinates are

$$\begin{aligned}\ddot{\xi} - 2\dot{\eta} = &- \frac{\mu\left(\xi - (1-\mu)\right)}{\left[(\xi - (1-\mu))^2 + \eta^2\right]^{\frac{3}{2}}} - \frac{\mu_4\left(\xi - \xi'\right)}{\left[(\xi - \xi')^2 + (\eta - \eta')^2\right]^{\frac{3}{2}}} \\ &- \frac{4}{3}\pi\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right)(\xi + \mu) + \xi,\end{aligned} \tag{9}$$

$$\begin{aligned}\ddot{\eta} + 2\dot{\xi} = &- \frac{\mu\eta}{\left[(\xi - (1-\mu))^2 + \eta^2\right]^{\frac{3}{2}}} - \frac{\mu_4\left(\eta - \eta'\right)}{\left[(\xi - \xi')^2 + (\eta - \eta')^2\right]^{\frac{3}{2}}} \\ &- \frac{4}{3}\pi\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right)\eta + \eta,\end{aligned} \tag{10}$$

where

$$\mu_4 = \frac{m_4}{m_1^* + m_2} \ll 1. \tag{11}$$

Thus, the equations of motion of m_3 in the dimensionless cartesian coordinates can be rewritten as

$$\ddot{\xi} - 2\dot{\eta} = V_\xi, \tag{12}$$

$$\ddot{\eta} + 2\dot{\xi} = V_\eta, \tag{13}$$

where

$$V = \frac{1}{2}\left(\xi^2 + \eta^2\right) + \frac{\mu}{R_{32}} + \frac{\mu_4}{R_{34}} - \frac{K}{2}\left[(\xi + \mu)^2 + \eta^2\right] \tag{14}$$

and

$$K = \frac{4}{3}\pi\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right). \tag{15}$$

Here V_ξ, V_η denote the partial derivatives of V with respect to ξ and η, respectively.

Now with the similar conditions on m_4 as mentioned for m_3 in Eqs. (1), (2), (3) and (4), the equations of motion of m_4 in the synodic system in the dimensionless cartesian coordinates are

$$\ddot{\xi}' - 2\dot{\eta}' = V'_{\xi'}, \tag{16}$$

$$\ddot{\eta}' + 2\dot{\xi}' = V'_{\eta'}, \tag{17}$$

where

$$V' = \frac{1}{2}\left(\xi'^2 + \eta'^2\right) + \frac{\mu}{R_{42}} + \frac{\mu_3}{R_{43}} - \frac{K'}{2}\left[\left(\xi' + \mu\right)^2 + \eta'^2\right] \tag{18}$$

and

$$\mu_3 = \frac{m_3}{m_1^* + m_2} \ll 1, \quad K' = \frac{4}{3}\pi\rho_1\left(1 - \frac{\rho_1}{\rho_4}\right). \tag{19}$$

3 Statement of the Problem and Equations of Motion: Robe's Restricted Problem of 2 + 2 Bodies with One of the Primaries an Oblate Body

The bodies in the restricted three-body problem are strictly spherical in shape, but in nature, the celestial bodies are not perfect spheres. They are either oblate or triaxial. The Earth, Jupiter, Saturn, Regulus, Neutron stars and black dwarfs are oblate. The Moon, Pluto and its moon Charon are triaxial. It is therefore essential that we concentrate on primaries which are axis-symmetric bodies and preferably on oblate bodies. Many authors have worked taking primaries as oblate bodies.

In this problem, one of the primaries of mass m_1^* is a rigid spherical shell filled with homogeneous incompressible fluid of density ρ_1. The second primary of mass $m_2 (m_1^* > m_2)$ is an oblate body outside the shell. The third and the fourth body (of mass m_3 and m_4, respectively) are small solid spheres of density ρ_3 and ρ_4, respectively inside the shell, with the assumption that the mass and radius of the third and the fourth body are infinitesimal. We also assume that m_2 is moving around m_1^* with angular velocity $\boldsymbol{\omega}$ (say) in a circular orbit. The masses m_3 and m_4 mutually attract each other are influenced by the motions of m_1^* and m_2 but do not influence them. Lastly, we assume that masses m_3 and m_4 are moving in the plane of motion of mass m_2.

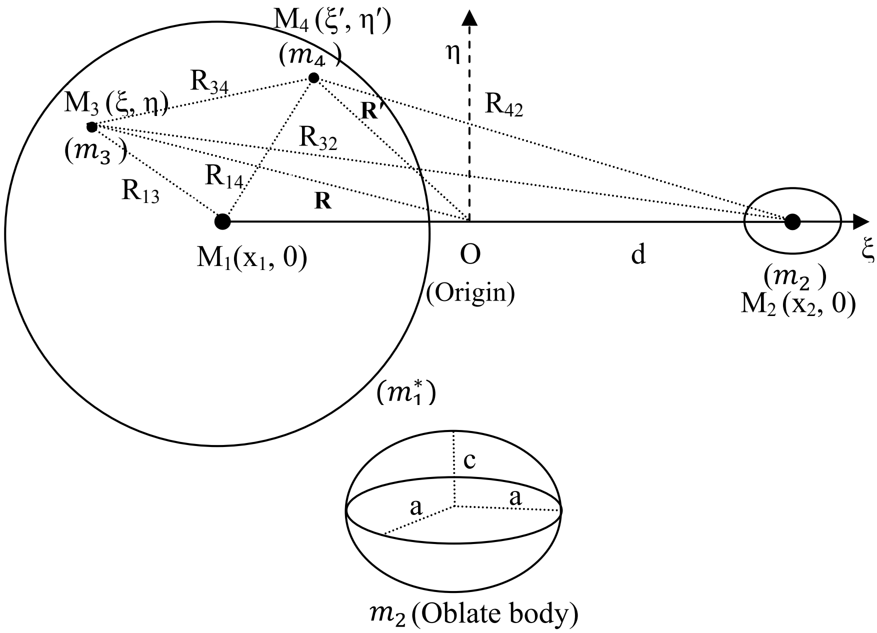

Fig. 2 Geometry of the Robe's restricted problem of 2 + 2 bodies with the smaller primary m_2 an oblate body

Let the orbital plane of m_2 around m_1^* be taken as the $\xi\eta$ plane and the origin of the coordinate system be at the centre of mass O of the two finite bodies. The coordinate system $O\xi\eta$ is as shown in Fig. 2. Let the synodic system of coordinates initially coincident with the inertial system rotate with angular velocity $\boldsymbol{\omega}$. This is the same as the angular velocity of m_2 which is describing a circle around m_1^*. Let initially the principal axes of m_2 be parallel to the synodic axes and their axes of symmetry be perpendicular to the plane of motion. Since m_2 is revolving without rotation about m_1^* with the same angular velocity as that of the synodic axes, the principal axes of m_2 will remain parallel to them throughout the motion.

Let the coordinates of m_3 and m_4 be (ξ, η) and (ξ', η'), respectively.

The equations of motion of m_3 in the dimensionless cartesian coordinates are

$$\ddot{\xi} - 2\omega\dot{\eta} = V_\xi, \tag{20}$$

$$\ddot{\eta} + 2\omega\dot{\xi} = V_\eta, , \tag{21}$$

where

$$V = \frac{\omega^2}{2}\left(\xi^2 + \eta^2\right) + \frac{\mu}{R_{32}} + \frac{\mu_4}{R_{34}} - \frac{K}{2}\left((\xi + \mu)^2 + \eta^2\right) + A\frac{\mu}{2R_{32}^3} \tag{22}$$

and

$$\mu_4 = \frac{m_4}{m_1^* + m_2} \ll 1, \; K = \frac{4}{3}\pi\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right) \tag{23}$$

and

$$\omega^2 = 1 + \frac{3}{2}A, \tag{24}$$

where

$$A = \frac{a^2 - c^2}{5d^2}. \tag{25}$$

4 Statement of the Problem and Equations of Motion: Robe's Restricted Problem of 2 + 2 Bodies When the Bigger Primary Is a Roche Ellipsoid

In deriving the expression for the buoyancy force, Robe assumed that the pressure field of the fluid ρ_1 has spherical symmetry about the centre of the shell, in accordance with its assumed shape and he took into account just one of the three components of the pressure field, that is, due to the own gravitational field of the fluid ρ_1 itself. The remaining two components are (i) that originating in the attraction of m_2, (ii) that arising from the centrifugal force. Plastino and Plastino [11] revisited the Robe's problem by associating the above two contributions (i) and (ii) of the pressure field, when the second primary moves in a circular orbit around the first primary. They assumed the hydrostatic equilibrium figure of the first primary as Roche Ellipsoid [2].

In this setup, one of the primaries of mass m_1 is described by a Roche Ellipsoid filled with a homogeneous incompressible fluid of density ρ_1. The second primary of mass $m_2 (m_1 > m_2)$ is a mass point outside the Ellipsoid. The third and the fourth body (of mass m_3 and m_4 respectively) are small solid spheres of density ρ_3 and ρ_4, respectively, inside the Ellipsoid, with the assumption that the mass and radius of the third and the fourth body are infinitesimal. Let d be the distance between the centres of mass of m_1 and m_2. We assume that m_2 describes a circular orbit of radius d around m_1 with constant angular velocity $\boldsymbol{\omega}$. The masses m_3 and m_4 mutually attract each other are influenced by the motions of m_1 and m_2 but do not influence them. We adopt a uniformly rotating coordinate system $Ox_1x_2x_3$, with origin of the coordinate system at the centre of the bigger primary, Ox_1 pointing towards m_2 and Ox_1x_2 being the orbital plane of m_2 around m_1. The coordinate system $Ox_1x_2x_3$ are as shown in Fig. 3.

The hydrodynamical equations of motion of the fluid elements of m_1 are

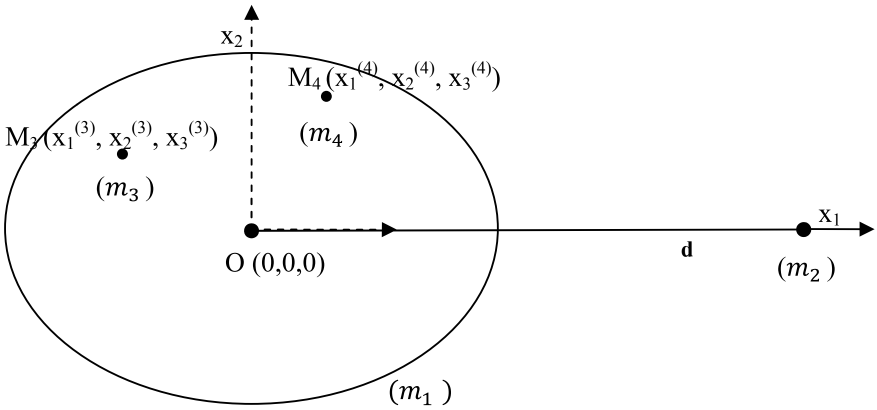

Fig. 3 Geometry of the Robe's restricted problem of 2 + 2 bodies when the bigger primary m_1 is considered as Roche Ellipsoid

$$\rho_1 \frac{du_i}{dt} = -\frac{\partial P}{\partial x_i} + \rho_1 \frac{\partial}{\partial x_i} \left[B + B' + \frac{1}{2}\omega^2 \left(x_1 - \frac{m_2 d}{m_1 + m_2} \right)^2 + x_2^2 \right]$$
$$+ 2\rho_1 \Omega_m \epsilon_{i,lm} u_l \quad (i = 1, 2, 3) \text{ (Chandrasekhar [1987])} \tag{26}$$

where u_i are the components of the fluid velocity field in the rotating frame, P and B are, respectively, the pressure field and the gravitational potential due to the fluid mass, $\epsilon_{i,lm}$ is the distance between two neighbouring fluid elements with velocities u_l and u_m, respectively. They are moving around each other with angular velocity of Ω_m.

The gravitational potential (or tide generating potential) B' at a point L due to m_2 (Fig. 4) is given by

$$\begin{aligned} B' &= \frac{Gm_2}{\rho} \\ &= \frac{Gm_2}{\sqrt{\rho'^2 + d^2 - 2\rho' d Cos\psi}} \\ &= \frac{Gm_2}{d} \left[1 + \frac{\rho' Cos\psi}{d} + \frac{\rho'^2 \left\{ 3Cos^2\psi - 1 \right\}}{2d^2} + \cdots \right] \\ &= \frac{Gm_2}{d} \left(1 + \frac{x_1}{d} + \frac{x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2}{d^2} + \cdots \right). \end{aligned} \tag{27}$$

Roche's approximation is based on keeping in the Taylor's expansion for B' only terms up to the second order in $x_i's$, $i = 1, 2, 3$.
Under this assumption, the equations of motion become

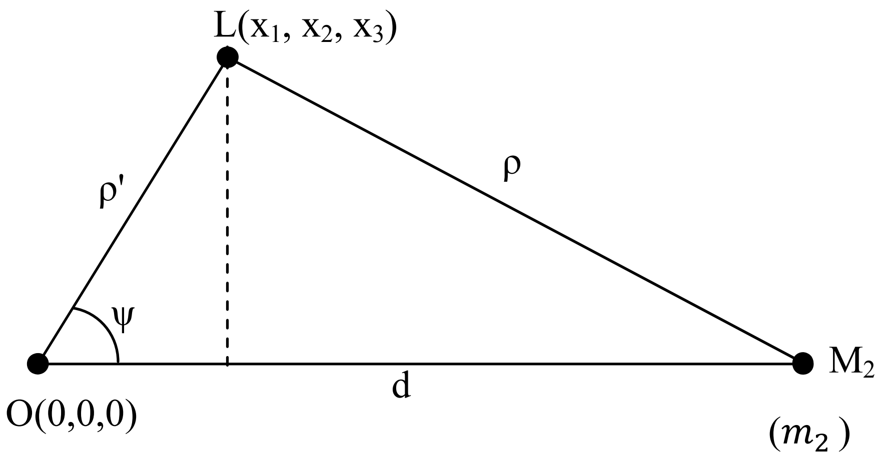

Fig. 4 Tide generating potential at a point L due to m_2

$$\rho_1 \frac{du_i}{dt} = -\frac{\partial P}{\partial x_i} + \rho_1 \frac{\partial}{\partial x_i}\left[B + \frac{1}{2}\omega^2\left(x_1^2 + x_2^2\right) + \mu\left(x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2\right)\right] + 2\rho_1\Omega_m\epsilon_{i,lm}u_l \quad (i = 1, 2, 3), \tag{28}$$

where

$$\mu = \frac{Gm_2}{d^3}, \tag{29}$$

and the angular velocity ω is given by

$$\omega^2 = \frac{G(m_1 + m_2)}{d^3}. \tag{30}$$

For hydrostatic equilibrium, the Eq. (28) becomes

$$\nabla\left[B + \frac{1}{2}\omega^2\left(x_1^2 + x_2^2\right) + \mu\left(x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_3^2\right) - \frac{P}{\rho_1}\right] = 0. \tag{31}$$

Roche Ellipsoid constitute the solutions to the Eq. (31). They are ellipsoidal figures with semi axes a_1, a_2, a_3 parallel, respectively, to the coordinate system Ox_1, Ox_2, Ox_3.

The potential B at any internal point x_i of the homogeneous ellipsoid is given by

$$B = \pi G\rho_1\left(I' - A_1x_1^2 - A_2x_2^2 - A_3x_3^2\right), \tag{32}$$

where

$$I' = a_1^2A_1 + a_2^2A_2 + a_3^2A_3 \tag{33}$$

and

$$A_i = a_1 a_2 a_3 \int_0^\infty \frac{du}{\delta\left(a_i^2 + u\right)} \; (i = 1, 2, 3)$$

with

$$\delta^2 = \left(a_1^2 + u\right)\left(a_2^2 + u\right)\left(a_3^2 + u\right). \tag{34}$$

Let the coordinates of m_3 and m_4 be $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$ $(i = 3, 4)$, respectively.

We describe the motion of a small mass m_3 within the Roche Ellipsoid.

Various forces (per unit mass) acting on m_3 are as follows:

1. The gravitational force due to m_4

$$\mathbf{F_{34}} = \frac{Gm_4\mathbf{R_{34}}}{R_{34}^3} \tag{35}$$

where $\mathbf{R_{ij}} = \mathbf{M_i M_j}$, M_3 is the centre of the shell m_3 and M_4 the centre of m_4.

2. The attraction $\mathbf{C}$ of the fluid ρ_1.

$$\mathbf{C} = \nabla B_3, \tag{36}$$

where B_3 is the gravitational potential due to the fluid mass.

This equation holds provided

$$\frac{\left(x_1^{(3)}\right)^2}{a_1^2} + \frac{\left(x_2^{(3)}\right)^2}{a_2^2} + \frac{\left(x_3^{(3)}\right)^2}{a_3^2} < 1 \tag{37}$$

3. The gravitational field $\mathbf{D}$ due to the point mass m_2.

$$\mathbf{D} = \nabla\left[\mu d x_1^{(3)} + \mu\left\{\left(x_1^{(3)}\right)^2 - \frac{1}{2}\left(x_2^{(3)}\right)^2 - \frac{1}{2}\left(x_3^{(3)}\right)^2\right\}\right]. \tag{38}$$

4. The buoyancy force$\mathbf{V}$ per unit mass arising in the fluid.

$$\mathbf{V} = -\frac{\rho_1}{\rho_3}\nabla\big[B_3 + \frac{1}{2}\omega^2\left\{\left(x_1^{(3)}\right)^2 + \left(x_2^{(3)}\right)^2\right\} + \mu\left\{\left(x_1^{(3)}\right)^2 - \frac{1}{2}\left(x_2^{(3)}\right)^2 - \frac{1}{2}\left(x_3^{(3)}\right)^2\right\}\big]. \tag{39}$$

This equation holds provided

$$\frac{\left(x_1^{(3)}\right)^2}{a_1^2} + \frac{\left(x_2^{(3)}\right)^2}{a_2^2} + \frac{\left(x_3^{(3)}\right)^2}{a_3^2} < 1. \tag{40}$$

The equation of motion of m_3 in the inertial system is

$$m_3\ddot{\mathbf{R}} = \mathbf{F_{34}} + \mathbf{C} + \mathbf{D} + \mathbf{V}$$

or

$$\begin{aligned}\ddot{\mathbf{R}} = &\frac{Gm_4\mathbf{R_{34}}}{R_{34}^3} + \nabla\left[B_3 + \mu d x_1^{(3)} + \mu\left\{\left(x_1^{(3)}\right)^2 - \frac{1}{2}\left(x_2^{(3)}\right)^2 - \frac{1}{2}\left(x_3^{(3)}\right)^2\right\}\right] \\ &- \frac{\rho_1}{\rho_3}\nabla\left[B_3 + \frac{1}{2}\omega^2\left\{\left(x_1^{(3)}\right)^2 + \left(x_2^{(3)}\right)^2\right\}\right. \\ &\left.+\mu\left\{\left(x_1^{(3)}\right)^2 - \frac{1}{2}\left(x_2^{(3)}\right)^2 - \frac{1}{2}\left(x_3^{(3)}\right)^2\right\}\right], \end{aligned} \tag{41}$$

where $\mathbf{R} = \mathbf{OM_3}$ and $\mathbf{R_{ij}} = \mathbf{M_iM_j}$.

Now, we determine the equation of motion of m_3 in the synodic system. Let us suppose that the coordinate system rotates with angular velocity $\boldsymbol{\omega}$. This is the same as the angular velocity of m_2 which is describing a circle around m_1.

In the rotating (synodic) system, the equation of motion of m_3 is

$$\begin{aligned}&\frac{\partial^2\mathbf{r}}{\partial t^2} + 2\boldsymbol{\omega} \times \frac{\partial \mathbf{r}}{\partial t} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \\ &\frac{Gm_4\mathbf{R_{34}}}{R_{34}^3} + \nabla\left[B_3 + \mu d x_1^{(3)} + \mu\left\{\left(x_1^{(3)}\right)^2 - \frac{1}{2}\left(x_2^{(3)}\right)^2 - \frac{1}{2}\left(x_3^{(3)}\right)^2\right\}\right] \\ &- \frac{\rho_1}{\rho_3}\nabla\left[B_3 + \frac{1}{2}\omega^2\left\{\left(x_1^{(3)}\right)^2 + \left(x_2^{(3)}\right)^2\right\}\right. \\ &\left.+\mu\left\{\left(x_1^{(3)}\right)^2 - \frac{1}{2}\left(x_2^{(3)}\right)^2 - \frac{1}{2}\left(x_3^{(3)}\right)^2\right\}\right], \end{aligned} \tag{42}$$

where $\mathbf{r} = \mathbf{OM_3}$ and $\boldsymbol{\omega} = \omega\hat{\mathbf{k}} =$ (constant).

We, now, fix the units such that the sum of the masses of the primaries is unity, i.e., $m_1 + m_2 = 1$, the distance between the primaries is unity, i.e., $d = 1$. The unit of time t is chosen in such a way that $G = 1$ [14].

The quantity μ of the Eq. (29) becomes numerically equal to the ratio

$$\frac{m_2}{m_1 + m_2}.$$

In the new units, the angular velocity ω given by the Eq. (30) becomes unity, i.e., $\omega = 1$.

The equations of motion of m_3 and similarly of m_4 in the dimensionless cartesian coordinates are

$$\ddot{x}_1^{(i)} - 2\dot{x}_2^{(i)} = V^{(i)}_{x_1^{(i)}}, \quad (43)$$

$$\ddot{x}_2^{(i)} + 2\dot{x}_1^{(i)} = V^{(i)}_{x_2^{(i)}}, \quad (44)$$

$$\ddot{x}_3^{(i)} = V^{(i)}_{x_3^{(i)}}, , \quad (45)$$

where

$$V^{(i)} = \frac{\mu_j}{R_{ij}} + D_i \left[B_i + \frac{1}{2}\omega^2 \left\{ \left(x_1^{(i)}\right)^2 + \left(x_2^{(i)}\right)^2 \right\} + \mu \left\{ \left(x_1^{(i)}\right)^2 - \frac{1}{2}\left(x_2^{(i)}\right)^2 - \frac{1}{2}\left(x_3^{(i)}\right)^2 \right\} \right] + \frac{\mu^2}{2} \quad (46)$$

and

$$\mu_j = \frac{m_j}{m_1 + m_2}, \quad D_i = \left(1 - \frac{\rho_1}{\rho_i}\right) \quad i, j = 3, 4; i \neq j \quad (47)$$

$$B_i = \pi G \rho_1 \left(I' - A_1 \left(x_1^{(i)}\right)^2 - A_2 \left(x_2^{(i)}\right)^2 - A_3 \left(x_3^{(i)}\right)^2 \right).$$

5 Conclusion

In this paper, we have extended the Robe's problem to $2+2$ bodies taking two infinitesimal masses within the spherical shell. The problem proposed by Robe had an application to the oscillations of the Earth's core due to the Moon's Attraction. Motivated by his work, we have studied the equations of motion of two infinitesimal masses m_3 and m_4 supposed moving inside m_1, taking m_1 as spherical shell and m_2 a point mass. We can see the presence of the infinitesimal mass m_4 in the equations of motion of m_3 and vice versa. This problem can be seen to have an application to the motion of submarines in the Earth–Moon System. Moving from a simple model towards realism, next, we extend the problem by taking the shape of m_2 as an oblate body as most celestial bodies are oblate or axis-symmetric. We can notice the presence of oblateness of m_2 in the equations of motion. The results coincide with the [8] when the oblateness is absent and with those of [12] in case both oblateness and an infinitesimal mass are absent. In the last case, we have taken into consideration all the three components of the pressure field in deriving the expression for the buoyancy force, viz., due to the own gravitational field of the fluid, that originating in the attraction of m_2 and that arising from the centrifugal force, taking the shape of the first primary as a Roche Ellipsoid. The results are in tune with [1] if we ignore the components of buoyancy and take the oblateness coefficient $A = 0$.

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Better Rate of Convergence by Modified Integral Type Operators

Nadeem Rao, Abdul Wafi, and Salma Khatoon

Abstract In this article, we introduce Chlodowsky Integral type operators with the help of generalized exponential function with two unbounded and non-negative real number sequences a_n and b_n. We study their basic estimates and investigate local and global approximation results with the aid of second-order modulus of continuity, Peetre's K-functional, Lipschitz-type class and rth-order Lipschitz-type maximal function. In the last, statistical approximation results are studied.

Keywords Szász operators · Linear positive operators · Modulus of continuity · Rate of convergence · Dunkl analogue

Mathematics Subject Classification (2010) 41A10 · 41A25 · 41A28 · 41A35 · 41A36

1 Introduction

The Szász-type operators via generalized exponential functions was presented by Sucu [25] as

N. Rao (✉) · A. Wafi
Department of Mathematics, Jamia Millia Islamia, New Delhi 110025, India
e-mail: nadeemrao1990@gmail.com

A. Wafi
e-mail: awafi@jmi.ac.in

S. Khatoon
Directorate of Education, Sarvodaya Kanya Vidyalya Noor Nagar, Okhla, New Delhi 110025, India
e-mail: dr.sjsiddiqui@gmail.com

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$$S_n(g;u) := \frac{1}{e_\nu(nu)} \sum_{k=0}^{\infty} \frac{(nu)^k}{\gamma_\nu(k)} g\left(\frac{k+2\nu\theta_k}{n}\right), \tag{1}$$

where the generating function [23] is given as

$$e_\mu(x) = \sum_{\nu=0}^{\infty} \frac{x^\nu}{\gamma_\mu(\nu)}, \tag{2}$$

with coefficients $\gamma_\mu(kgiven)$ are introduced as

For $\mu > -1/2$ and $k \in \mathbb{N}_0 = \{0\} \bigcup \mathbb{N}$, we have

$$\gamma_\mu(2k) = \frac{2^{2k}k!\Gamma(k+\mu+1/2)}{\Gamma(\mu+1/2)}, \gamma_\mu(2k+1) = \frac{2^{2k+1}k!\Gamma(k+\mu+3/2)}{\Gamma(\mu+1/2)}.$$

Recursive relation is given as

$$\gamma_\mu(k+1) = (k+1+2\mu\theta_{k+1})\gamma_\mu(k),\ \ k \in \mathbb{N}_0, \tag{3}$$

with θ_k is given to be 0 if $k \in 2\mathbb{N}$ and 1 if $k \in 2\mathbb{N}+1$. The operators presented in (1) are restricted to approximate the continuous functions only. Wafi and Rao [8] constructed a sequence of positive linear operators to discuss the approximation results for the Lebesgue measurable functions as

$$D_n(f;x) = \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} \frac{n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)} \int_0^\infty t^{k+2\mu\theta_k+\lambda} e^{-nt} f(t)dt. \tag{4}$$

Several mathematicians researched in this direction to approximate the continuous functions only and Lebesgue measurable functions, i.e. Wafi and Rao [7] and Mursaleen et al. [9–11], Karaisa et al. [14] and Icoz et al. [12, 13] Motivated by the above, we present a Chlodowsky Integral type operators via Dunkl analogue as

$$A_n(f;x) = \frac{1}{e_\mu(a_n x)} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{\gamma_\mu(k)} \frac{b_n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)} \int_0^\infty t^{k+2\mu\theta_k+\lambda} e^{-b_n t} f(t)dt, \tag{5}$$

where a_n and b_n are unbounded and increasing sequences of real numbers such that

$$\lim_{n\to\infty} b_n = \infty \text{ and } \lim_{n\to\infty} \frac{b_n}{n} = 0. \tag{6}$$

In the subsequent sections, we prove some basic lemmas and proposition which shows the uniform convergence of the operators (5). Further, we study the pointwise approximation results and global approximation results. In the last part of this manuscript, statistical approximation results are investigated.

2 Approximation Properties of $A_n(f;x)$

Lemma 2.1 *Let $\mu \geq -\frac{1}{2}$ and $x \geq 0$. Then with the aid of generalized exponential function given in (2), one has*

$$\frac{1}{e_\mu(a_n x)}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{\gamma_\mu(k)} = 1,$$

$$\frac{1}{e_\mu(a_n x)}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{\gamma_\mu(k)}(k+2\mu\theta_k) = a_n x,$$

$$\frac{1}{e_\mu(a_n x)}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{\gamma_\mu(k)}(k+2\mu\theta_k)^2 = a_n^2x^2 + \left(1+2\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)a_n x,$$

$$\frac{1}{e_\mu(a_n x)}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{\gamma_\mu(k)}(k+2\mu\theta_k)^3 = a_n^3x^3 + \left(3-2\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)a_n^2x^2 + \left(1+4\mu^2+2\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)a_n x,$$

$$\frac{1}{e_\mu(a_n x)}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{\gamma_\mu(k)}(k+2\mu\theta_k)^4 = a_n^4x^4 + \left(6+4\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)a_n^3x^3 + \left(7+4\mu^2-8\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)a_n^2x^2 + \left(1+12\mu^2+6\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}+8\mu^3\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)a_n x.$$

Proof With the help of Eq. (2) and $\theta_{k+1} = (-1)^k + \theta_k$, one can easily prove Lemma 2.1. □

In order to discuss the basic properties of the operators introduced by the Eq. (5), we consider $e_\nu(t) = t^\nu, \nu \in \{0,1,2,3,4\}$ and $\psi_x^\nu(t) = (t-x)^\nu, \nu \in \{1,2,3,4\}$, respectively.

Lemma 2.2 *Let $A_n(f;x)$ be the operators defined in (5). Then one has*

$$A_n(e_0;x) = 1,$$

$$A_n(e_1;x) = \frac{a_n}{b_n}x + \frac{\lambda+1}{b_n},$$

$$A_n(e_2;x) = \frac{a_n^2}{b_n^2}x^2 + \left(4+2\lambda+2\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)\frac{a_n x}{b_n^2} + \frac{(\lambda+1)(\lambda+2)}{b_n^2},$$

$$A_n(e_3;x) = \frac{a_n^3}{b_n^3}x^3 + \left(9+3\lambda-2\mu\frac{e_\mu(-a_n x)}{e_\mu(a_n x)}\right)\frac{a_n^2}{b_n^3}x^2 + \Bigg(18+k\lambda(\lambda+5)+4\mu^2$$

$$+ 2\mu(8+3\lambda)\frac{e_\mu(-a_nx)}{e_\mu(a_nx)}\bigg)\frac{a_n}{b_n^3}x + \frac{\lambda^3+6\lambda^2+11\lambda+6}{n^3},$$

$$A_n(e_4;x) = \frac{a_n^4}{b_n^4}x^4 + \left(16+4\lambda+4\mu\frac{e_\mu(-a_nx)}{e_\mu(a_nx)}\right)\frac{a_n^3}{b_n^4} + o\left(\frac{1}{b_n^2}\right).$$

Proof Using Lemma 2.1, we have for $\nu = 0$

$$\begin{aligned}
A_n(e_0;x) &= \frac{1}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\frac{b_n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)}\int_0^\infty t^{k+2\mu\theta_k+\lambda}e^{-b_nt}dt\\
&= \frac{1}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\frac{\Gamma(k+2\mu\theta_k+\lambda+1)}{\Gamma(k+2\mu\theta_k+\lambda+1)}\\
&= 1.
\end{aligned}$$

For $\nu = 1$

$$\begin{aligned}
A_n(e_1;x) &= \frac{1}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\frac{b_n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)}\int_0^\infty t^{k+2\mu\theta_k+\lambda+1}e^{-b_nt}dt,\\
&= \frac{1}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\frac{\Gamma(k+2\mu\theta_k+\lambda+2)}{b_n\Gamma(k+2\mu\theta_k+\lambda+1)}\\
&= \frac{1}{b_ne_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}(k+2\mu\theta_k+\lambda+1)\frac{\Gamma(k+2\mu\theta_k+\lambda+1)}{\Gamma(k+2\mu\theta_k+\lambda+1)}\\
&= \frac{a_n}{b_n}x + \frac{\lambda+1}{b_n}.
\end{aligned}$$

For $\nu = 2$

$$\begin{aligned}
A_n(e_2;x) &= \frac{1}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\frac{b_n^{k+2\mu\theta_k+\lambda+1}}{\Gamma(k+2\mu\theta_k+\lambda+1)}\int_0^\infty t^{k+2\mu\theta_k+\lambda+2}e^{-b_nt}dt\\
&= \frac{1}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\frac{\Gamma(k+2\mu\theta_k+\lambda+3)}{b_n^2\Gamma(k+2\mu\theta_k+\lambda+1)}\\
&= \frac{1}{e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}\frac{(k+2\mu\theta_k+\lambda+2)(k+2\mu\theta_k+\lambda+1)}{b_n^2}\\
&= \frac{1}{b_n^2e_\mu(a_nx)}\sum_{k=0}^{\infty}\frac{(a_nx)^k}{\gamma_\mu(k)}((k+2\mu\theta_k)^2+(2\lambda+3)(k+2\mu\theta_k)\\
&+ (\lambda+1)(\lambda+2))\\
&= \frac{a_n^2}{b_n^2}x^2 + \left(4+2\lambda+2\mu\frac{e_\mu(-a_nx)}{e_\mu(a_nx)}\right)\frac{a_n}{b_n^2}x + \frac{(\lambda+1)(\lambda+2)}{b_n^2}.
\end{aligned}$$

Similarly, the rest part of the Lemma 2.2 can be easily proved. □

Lemma 2.3 *Let the $A_n(f;x)$ be the operators given in (5). Then we have*

$$A_n(\psi_x^1;x) = \frac{\lambda+1}{n},$$
$$A_n(\psi_x^2;x) = \left(2+2i\frac{e_i(-nx)}{e_i(nx)}\right)\frac{x}{n} + \frac{(\lambda+1)(\lambda+2)}{n^2},$$
$$A_n(\psi_x^4;x) = o\left(\frac{1}{n}\right).$$

Proof In view of Lemma 2.2 and linearity property, one has

$$A_n(\psi_x^1;x) = A_n(t;x) - xA_n(1;x),$$
$$A_n(\psi_x^2;x) = A_n(t^2;x) - 2xA_n(t;x) + x^2A_n(1;x),$$
$$A_n(\psi_x^4;x) = A_n(t^4;x) - 4xA_n(t^3;x) + 6x^2A_n(t^2;x) - 4x^3A_n(t;x) + x^4A_n(1;x).$$

In the light of Lemma 2.2, we prove the Lemma 2.3. □

Proposition 2.4 *For the operators A_n given in (2) and for every $f \in C[0,\infty)$, A_n converges to f uniformly on $[0,a]$, $a>0$.*

Proof From Korovkin Theorem 4.1.4 in [1], it is sufficient to show that

$$A_n(e_\nu;x) \to e_\nu(x), \text{ for } \nu = 0,1,2.$$

Lemma 2.2 implies that $A_n(e_0;x) \to e_0(x)$ as $n \to \infty$. For $\nu = 1$

$$\lim_{n\to\infty} A_n(e_1;x) = \lim_{n\to\infty}\left(\left(\frac{a_n}{b_n}-2\right)x + \frac{1}{b_n}\right) = e_1(x).$$

In the similar manner, one can show that for $\nu = 2$, $A_n(e_2;x) \to e_2$ which completes the proof of Proposition 2.4. □

3 Pointwise Approximation Results of A_n

Here, we recall some notations from DeVore and Lorentz [2] as $C_B[0,\infty)$ be the space of bounded and real valued continuous functions endowed with the norm $\|f\| = \sup_{0\le x<\infty} |f(x)|$. Let the function $f \in C_B[0,\infty)$ and $\delta > 0$. Then, the Peetre's K-functional is given by

$$K_2(f,\delta) = inf\{\|f-g\| + \delta\|g''\| : g \in C_B^2[0,\infty)\},$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. From DeVore and Lorentz [[2], p.177, Theorem 2.4], there exits an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}). \tag{7}$$

Consider the auxiliary operator $\widehat{A}_n^*$ as

$$\widehat{A}_n(f;x) = A_n(f;x) + f(x) - f\left(\frac{a_n}{b_n}x + \frac{1}{b_n}\right). \tag{8}$$

Lemma 3.1 *For $g \in C_B^2[0,\infty)$ and $x \geq 0$, one has*

$$|\widehat{A}_n(g;x) - g(x)| \leq \xi_n(x)\|g''\|,$$

where

$$\xi_n(x) = A_n(\psi_x^2;x) + \left(A_n(\psi_x^1;x)\right)^2.$$

Proof With the aid of auxiliary operators given in (8), we get

$$\widehat{A}_n(1;x) = 1,\ \widehat{A}_n(\psi_x;x) = 0 \text{ and } |\widehat{A}_n(f;x)| \leq 3\|f\|. \tag{9}$$

From Taylor's series expansion, for every $g \in C_B^2[0,\infty)$, we obtain

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-v)g''(v)dv. \tag{10}$$

Applying auxiliary operators $\widehat{A}_n$ in Eq. 10, we have

$$\widehat{A}_n(g;x) - g(x) = g'(x)\widehat{A}_n(t-x;x) + \widehat{A}_n^*\Big(\int_x^t (t-v)g''(v)dv;x\Big).$$

From (8) and (9), we have

$$\begin{aligned}\widehat{A}_n(g;x) - g(x) &= \widehat{A}_n\Big(\int_x^t (t-v)g''(v)dv;x\Big)\\ &= A_n\Big(\int_x^t (t-v)g''(v)dv;x\Big) - \int_x^{\frac{a_n}{b_n}x+\frac{1}{b_n}} \left(\frac{a_n}{b_n}x + \frac{1}{b_n} - v\right)g''(v)dv.\end{aligned}$$

$$|\widehat{A}_n(g;x)-g(x)| \leq \left|A_n\Big(\int\limits_x^t (t-v)g''(v)dv;x\Big)\right| + \left|\int\limits_x^{\frac{a_n}{b_n}x+\frac{1}{b_n}} \Big(\frac{a_n}{b_n}x+\frac{1}{b_n}-v\Big)g''(v)dv\right|. \quad (11)$$

Since

$$\left|\int\limits_x^t (t-v)g''(v)dv\right| \leq (t-x)^2 \parallel g'' \parallel . \quad (12)$$

Then

$$\left|\int\limits_x^{\frac{a_n}{b_n}x+\frac{1}{b_n}} \Big(\frac{a_n}{b_n}x+\frac{1}{b_n}-v\Big)g''(v)dv\right| \leq \Big(\frac{a_n}{b_n}x+\frac{1}{b_n}-x\Big)^2 \parallel g'' \parallel . \quad (13)$$

Using (12) and (13) in (11), we deduce

$$\begin{aligned}|\widehat{A}_n^*(g;x)-g(x)| &\leq \left\{A_n((t-x)^2;x)+\Big(\frac{a_n}{b_n}x+\frac{1}{b_n}-x\Big)^2\right\}\|g''\| \\ &= \xi_n(x)\|g''\|.\end{aligned}$$

Hence, the proof of Lemma 3.1 is completed. □

Theorem 3.2 *For $f \in C_B^2[0,\infty)$, we have*

$$| A_n(f;x)-f(x) |\leq C\omega_2\big(f;\sqrt{\xi_n(x)}\big)+\omega(f;A_n(\psi_x;x)),$$

where $\xi_n(x)$ is calculated in Lemma 3.1 and $C>0$ is a constant.

Proof For $f \in C_B[0,\infty)$ and $g \in C_B^2[0,\infty)$ and the operators $\widehat{A}_n$, we get

$$\begin{aligned}|A_n(f;x)-f(x)| &\leq |\widehat{A}_n(f-g;x)|+|(f-g)(x)|+|\widehat{A}_n(g;x)-g(x)| \\ &+\left|f\Big(\frac{a_n}{b_n}x+\frac{1}{b_n}\Big)-f(x)\right|.\end{aligned}$$

From Lemma 3.1 and identities given by (9), we deduce

$$\begin{aligned}|A_n(f;x)-f(x)| &\leq 4\|f-g\|+|\widehat{A}_n(g;x)-g(x)|+\left|f\Big(\frac{a_n}{b_n}x+\frac{1}{b_n}\Big)-f(x)\right| \\ &\leq 4\|f-g\|+\xi_n(x)\|g''\|+\omega\Big(f;A_n(\psi_x;x)\Big).\end{aligned}$$

Using Peetre's K-functional, one obtains

$$|A_n(f;x) - f(x)| \leq C\omega_2\big(f; \sqrt{\xi_n(x)}\big) + \omega(f; A_n(\psi_x; x)),$$

which completes the required result. □

Consider the Lipschitz-type space [22] as

$$Lip_M^{\beta_1,\beta_2}(\gamma) := \left\{ f \in C_B[0,\infty) : |f(t) - f(x)| \leq M \frac{|t-x|^\gamma}{(t+\beta_1 x + \beta_2 x^2)^{\frac{\gamma}{2}}} : x, t \in (0,\infty) \right\},$$

where $\beta_1, \beta_2 > 0$ are two fixed real numbers, M is a positive constant and $0 < \gamma \leq 1$.

Theorem 3.3 *For the operators defined by (6) and for every* $f \in Lip_M^{\beta_1,\beta_2}(\gamma)$, $0 < x < \infty$)*, we have*

$$|A_n(f;x) - f(x)| \leq M\left(\frac{\eta_n(x)}{\beta_1 x + \beta_2 x^2}\right)^{\frac{\gamma}{2}}, \tag{14}$$

where $\gamma \in (0,1]$ *and* $\eta_n(x) = A_n(\psi_x^2; x)$.

Proof For $\gamma = 1$ and $x \in (0,\infty)$, we have

$$\begin{aligned}|A_n(f;x) - f(x)| &\leq A_n(|f(t) - f(x)|; x)\\ &\leq MA_n\left(\frac{|t-x|}{(t+\beta_1 x+\beta_2 x^2)^{\frac{1}{2}}}; x\right).\end{aligned}$$

It is obvious that $\frac{1}{t+\beta_1 x+\beta_2 x^2} < \frac{1}{\beta_1 x+\beta_2 x^2}$ for all $0 \leq x < \infty$, we obtain

$$\begin{aligned}|A_n(f;x) - f(x)| &\leq \frac{M}{(\beta_1 x+\beta_2 x^2)^{\frac{1}{2}}}(A_n((t-x)^2; x))^{\frac{1}{2}}\\ &\leq M\left(\frac{\eta_n(x)}{\beta_1 x+\beta_2 x^2}\right)^{\frac{1}{2}}.\end{aligned}$$

This shows that the Theorem 3.3 satisfies for $\gamma = 1$. Next, for $\gamma \in (0,\infty)$ and with the aid of Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we deduce

$$\begin{aligned}|A_n(f;x) - f(x)| &\leq \big(A_n(|f(t) - f(x)|^{\frac{2}{\gamma}}; x)\big)^{\frac{\gamma}{2}}\\ &\leq M\left(A_n\left(\frac{|t-x|^2}{(t+\beta_1 x+\beta_2 x^2)}; x\right)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

Since $\frac{1}{t+\beta_1 x+\beta_2 x^2} < \frac{1}{\beta_1 x+\beta_2 x^2}$ for all $x \in (0,\infty)$, we have

$$|A_n(f;x) - f(x)| \leq M\left(\frac{A_n(|t-x|^2;x)}{\beta_1 x + \beta_2 x^2}\right)^{\frac{\gamma}{2}}$$
$$\leq M\left(\frac{\eta_n(x)}{\beta_1 x + \beta_2 x^2}\right)^{\frac{\gamma}{2}}.$$

Hence, the proof of the Theorem 3.3 is completed. □

Here, we recall rth-order Lipschitz-type maximal function introduced by Lenze [16] as

$$\widetilde{\omega}_r(f;x) = \sup_{t \neq x, t \in (0,\infty)} \frac{|f(t) - f(x)|}{|t-x|^r}, \; x \in [0,\infty) \text{ and } r \in (0,1]. \tag{15}$$

Then, we get the next result

Theorem 3.4 *For $f \in C_B[0,\infty)$ and $0 < r \leq 1$, $0 \leq x < \infty$, we have*

$$|A_n(f;x) - f(x)| \leq \widetilde{\omega}_r(f;x)\Big(\eta_n(x)\Big)^{\frac{r}{2}}.$$

Proof It is clear that

$$|A_n(f;x) - f(x)| \leq A_n(|f(t) - f(x)|;x).$$

From Eq. (15), we have

$$|A_n(f;x) - f(x)| \leq \widetilde{\omega}_r(f;x) A_n(|t-x|^r;x).$$

From Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, we have

$$|A_n(f;x) - f(x)| \leq \widetilde{\omega}_r(f;x)\big(A_n(|t-x|^2;x)\big)^{\frac{r}{2}},$$

which proves the desired result. □

4 Global Approximation Results

Here, we recall some notations from [5] to prove next result. Let $B_{1+x^2}[0,\infty) = \{f(x) : |f(x)| \leq M_f(1+x^2), 1+x^2$ is weight function, M_f is a constant depending on f and $x \in [0,\infty)\}$, $C_{1+x^2}[0,\infty)$ is the space of continuous function in $B_{1+x^2}[0,\infty)$ with the norm $\|f(x)\|_{1+x^2} = \sup\limits_{x \in [0,\infty)} \frac{|f(x)|}{1+x^2}$ and $C^k_{1+x^2}[0,\infty) = \{f \in C_{1+x^2} : \lim\limits_{|x| \to \infty} \frac{f(x)}{1+x^2} = k$, where k is a constant depending on $f\}$.

Theorem 4.1 *Let A_n be the operators defined by (6) from $C^k_{1+x^2}[0,\infty)$ to $B_{1+x^2}[0,\infty)$ satisfying the conditions*

$$\lim_{n\to\infty} \|A_n(e_i;x) - x^i\|_{1+x^2} = 0,\ i = 0,1,2.$$

Then for each $C^k_{1+x^2}[0,\infty)$

$$\lim_{n\to\infty} \|A_n(f;x) - f\|_{1+x^2} = 0.$$

Proof In order to prove this result, it is sufficient to show that

$$\lim_{n\to\infty} \|A_n(e_i;x) - x^i\|_{1+x^2} = 0,\ i = 0,1,2.$$

From Lemma 2.2, we have

$$\|A_n(e_0;x) - x^0\|_{1+x^2} = \sup_{x\in[0,\infty)} \frac{|A_n(1;x)-1|}{1+x^2} = 0 \text{ for } i = 0.$$

For $i = 1$

$$\begin{aligned}\|A_n(e_1;x) - x^1\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{|\frac{a_n}{b_n}x + \frac{1}{b_n} - x|}{1+x^2}\\ &= \left(\frac{a_n}{b_n} - 1\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2} + \frac{1}{b_n}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}.\end{aligned}$$

This implies that $\|A_n(e_1;x) - x^1\|_{1+x^2} \to 0$ an $n \to \infty$.
For $i = 2$

$$\begin{aligned}\|A_n(e_2;x) - x^2\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{\left|\left(\frac{a_n^2}{b_n^2} - 1\right)x^2 + \frac{a_n}{b_n^2}\left(2 + 2\mu\frac{e_\mu(-a_nx)}{e_\mu(a_nx)}\right)x + \frac{1}{3b_n^2}\right|}{1+x^2}\\ &= \left(\frac{a_n^2}{b_n^2} - 1\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2} + \frac{a_n}{b_n^2}\left(2 + 2\mu\frac{e_\mu(-a_nx)}{e_\mu(a_nx)}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\ &+ \frac{1}{3b_n^2}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}.\end{aligned}$$

Which shows that $\|A_n(e_2;x) - x^2\|_{1+x^2} \to 0$ an $n \to \infty$. □

In the next result, we discuss a result to approximate each function belongs to $C^k_{1+x^2}[0,\infty)$.

Theorem 4.2 *Let $f \in C^k_{1+x^2}[0,\infty)$ and $\gamma > 0$. Then, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|A_n(f;x)|-f(x)}{(1+x^2)^{1+\gamma}}=0.$$

Proof For $x_0>0$, a fixed number, we have

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{|A_n(f;x)|-f(x)}{(1+x^2)^{1+\gamma}}&\le\sup_{x\le x_0}\frac{|A_n(f;x)|-f(x)}{(1+x^2)^{1+\gamma}}+\sup_{x\ge x_0}\frac{|A_n(f;x)|-f(x)}{(1+x^2)^{1+\gamma}}\\&\le\||A_n(f;x)|-f(x)\|_{C[0,x_0]}\\&+\|f\|_{1+x^2}\sup_{x\ge x_0}\frac{|A_n(1+t^2;x)|}{(1+x^2)^{1+\gamma}}+\sup_{x\ge x_0}\frac{|f(x)|}{(1+x^2)^{1+\gamma}}\\&=J_1+J_2+J_3,\ say.\end{aligned}\tag{16}$$

Since $|f(x)|\le\|f\|_{1+x^2}(1+x^2)$, we have

$$\begin{aligned}J_3&=\sup_{x\ge x_0}\frac{|f(x)|}{(1+x^2)^{1+\gamma}}\\&\le\sup_{x\ge x_0}\frac{\|f\|_{1+x^2}(1+x^2)}{(1+x^2)^{1+\gamma}}\le\frac{\|f\|_{1+x^2}}{(1+x^2)^{\gamma}}.\end{aligned}$$

Let $\epsilon>0$ be arbitrary real number. Then, from Proposition 2.4, there exists $n_1\in\mathbb{N}$ such that

$$\begin{aligned}J_2&<\frac{1}{(1+x^2)^{\gamma}}\|f\|_{1+x^2}\left(1+x^2+\frac{\epsilon}{3\|f\|_{1+x^2}}\right)\text{ for all }n_1\ge n,\\&<\frac{\|f\|_{1+x^2}}{(1+x^2)^{\gamma}}+\frac{\epsilon}{3}\text{ for all }n_1\ge n.\end{aligned}$$

This implies that

$$J_2+J_3<2\frac{\|f\|_{1+x^2}}{(1+x^2)^{\gamma}}+\frac{\epsilon}{3}.$$

Next, let for a large value of x_0, we have $\frac{\|f\|_{1+x^2}}{(1+x^2)^{\gamma}}<\frac{\epsilon}{6}$.

$$J_2+J_3<\frac{2\epsilon}{3}\text{ for all }n_1\ge n.\tag{17}$$

Using Theorem 4.1, there exists $n_2>n$ such that

$$J_1=\|A_n(f)-f\|_{C[0,x_0]}<\frac{\epsilon}{3}\text{ for all }n_2\ge n.\tag{18}$$

For $n_3=max(n_1,n_2)$, using (16), (17) and (18), we obtain

$$\sup_{x\in[0,\infty)} \frac{|A_n(f;x)| - f(x)}{(1+x^2)^{1+\gamma}} < \epsilon.$$

Hence, we arrive at the desired results. □

5 A-Statistical Approximation Results

Here, we recall some notations [5, 6] as Let $A = (a_{nk})$ be a non-negative infinite sumability matrix. For a given sequence $x := (x_k)$, the A-transform of x denoted by $Ax : ((Ax)_n)$ is and defined as

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k,$$

provided the series converges for each n. A is said to be regular if $\lim(Ax)_n = L$ whenever $\lim x = L$. Then $x = (x_n)$ is said to be a A-statistically convergent to L, i.e. $st_A - \lim x = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|x_k - L|\geq \epsilon} a_{nk} = 0$.

Theorem 5.1 *For $A = (a_{nk})$, a non-negative regular sumability matrix and for all $f \in C^k_{1+x^{2+\lambda}}[0,\infty)$ with $\lambda > 0$, we have*

$$st_A - \lim_n \|A_n(f;x) - f\|_{1+x^{2+\lambda}} = 0.$$

Proof In the light of [3], p. 191, Th. 3, it is enough to show that for $\lambda = 0$

$$st_A - \lim_n \|A_n(e_i;x) - e_i\|_{1+x^2} = 0, \text{ for } i \in \{0,1,2\}. \tag{19}$$

In view of Lemma 2.3, we get

$$\begin{aligned}\|A_n(e_1;x) - x\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{a_n}{b_n}x + \frac{1}{b_n} - x\right| \\ &\leq \left(\frac{a_n}{b_n} + \frac{1}{b_n} - 1\right).\end{aligned}$$

Now, for a given $\epsilon > 0$, we define the following sets

$$E_1 := \left\{ n : \|A_n(e_1; x) - x\| \geq \epsilon \right\}$$

$$E_2 := \left\{ n : \left(\frac{a_n}{b_n} x + \frac{1}{b_n} - 1 \right) \geq \epsilon \right\}.$$

This implies that $E_1 \subseteq E_2$, which shows that $\sum_{k \in E_1} a_{nk} \leq \sum_{k \in E_2} a_{nk}$. Therefore, we get

$$st_A - \lim_n \|A_n(e_1; x) - x\|_{1+x^2} = 0. \tag{20}$$

For $i = 2$ and using Lemma 2.3, we have

$$\|A_n(e_2; x) - x^2\|_{1+x^2} = \sup_{x \in [0,\infty)} \frac{1}{1+x^2} \left| \left(\frac{a_n^2}{b_n^2} - 1 \right) x^2 + \frac{a_n}{b_n^2} \left(2 + 2\mu \frac{e_\mu(-a_n x)}{e_\mu(a_n x)} \right) x + \frac{1}{3b_n^2} \right|$$
$$\leq \left(\frac{a_n^2}{b_n^2} - 1 \right) + \frac{a_n}{b_n^2} \left(2 + 2\mu \frac{e_\mu(-a_n x)}{e_\mu(a_n x)} \right) + \frac{1}{3b_n^2}.$$

For a given $\varepsilon > 0$, we have the following sets

$$H_1 := \left\{ n : \left\| A_n(e_2; x) - x^2 \right\| \geq \epsilon \right\}$$

$$H_2 := \left\{ n : \left(\frac{a_n^2}{b_n^2} - 1 \right) \geq \frac{\epsilon}{3} \right\}$$

$$H_3 := \left\{ n : \frac{a_n}{b_n^2} \left(2 + 2\mu \frac{e_\mu(-a_n x)}{e_\mu(a_n x)} \right) \geq \frac{\epsilon}{3} \right\}$$

$$H_4 := \left\{ n : \frac{1}{3b_n^2} \geq \frac{\epsilon}{3} \right\}.$$

This implies that $H_1 \subseteq H_2 \bigcup H_3 \bigcup H_4$. By which, we obtained

$$\sum_{k \in H_1} a_{nk} \leq \sum_{k \in H_2} a_{nk} + \sum_{k \in H_3} a_{nk} + \sum_{k \in H_4} a_{nk}.$$

As $n \to \infty$, we get

$$st_A - \lim_n \|A_n(e_2; x) - x^2\|_{1+x^2} = 0. \tag{21}$$

Therefore, the proof of Theorem 5.1 is completed. □

Theorem 5.2 *Let $f \in C_B^2[0, \infty)$. Then*

$$st_A - \lim_n \|A_n(f) - f\|_{C_B[0,\infty)} = 0.$$

Proof With the aid of Taylor's series expansion, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(\xi)(t-x)^2,$$

where $t \leq \xi \leq x$. Applying A_n, we have

$$A_n(f;x) - f(x) = f'(x)A_n(\psi_x;x) + \frac{1}{2}f''(\xi)A_n(\psi_x^2;x).$$

This implies that

$$\begin{aligned} \|A_n(f) - f\|_{C_B[0,\infty)} &\leq \|f'\|_{C_B[0,\infty)}\|A_n(e_1-,.)\|_{C_B[0,\infty)} \\ &+ \|f''\|_{C_B[0,\infty)}\|A_n(e_1-,.)^2\|_{C_B[0,\infty)} \\ &= I_1 + I_2, say. \end{aligned} \tag{22}$$

From (19), one has

$$\lim_n \sum_{k\in\mathbb{N}:I_1\geq\frac{\epsilon}{2}} a_{nk} = 0,$$
$$\lim_n \sum_{k\in\mathbb{N}:I_2\geq\frac{\epsilon}{2}} a_{nk} = 0.$$

From (22), we have

$$\lim_n \sum_{k\in\mathbb{N}:\|A_n(f)-f\|_{C_B[0,\infty)}\geq\epsilon} a_{nk} \leq \lim_n \sum_{k\in\mathbb{N}:I_1\geq\frac{\epsilon}{2}} a_{nk} + \lim_n \sum_{k\in\mathbb{N}:I_2\geq\frac{\epsilon}{2}} a_{nk}.$$

Thus $st_A - \lim_n \|A_n(f) - f\|_{C_B[0,\infty)} \to 0.$ as $n \to \infty$. Hence, we arrive at the required result. □

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Generalized Composition Operators and Evaluation Kernel on Weighted Hardy Spaces

Rohit Gandhi, Sunil Kumar Sharma, and B. S. Komal

Abstract In this paper, we study generalized composition operators by using evaluation kernel on weighted Hardy spaces. The properties of generalized composition operators, generalized multiplication operators and generalized weighted composition operators like adjoint and boundedness are obtained with the help of evaluation kernel function.

Keywords Evaluation Kernel · Generalized composition operator · Generalized multiplication operator · Generalized weighted composition operator · Weighted Hardy space

1 Introduction

Let $\{\beta_n\}$ be a sequence of positive real numbers with $\beta(0) = 1$. For $1 \leq p < \infty$, let $H^p(\beta)$ be the space of formal series $\{f : f(z) = \sum_{n=0}^{\infty} f_n z^n, \sum_{n=0}^{\infty} |f_n|^p \beta_n^p < \infty\}$, where $\{f_n\}_{n=0}^{\infty}$ is a sequence of complex numbers such that $\sum_{n=0}^{\infty} |f_n|^p \beta_n^p < \infty$. Then $H^p(\beta)$ is a Banach space under the norm $||f||_\beta^p = \sum_{n=0}^{\infty} |f_n|^p \beta_n^p < \infty$. For p = 2, the space $H^2(\beta)$ is a Hilbert space under the inner product defined as

R. Gandhi
Lovely Professional University, Phagwara, Punjab, India
e-mail: rohitglpu@gmail.com

S. K. Sharma (✉)
NSCBM Government College, Hamirpur, Himachal Pradesh, India
e-mail: sunilshrm167@gmail.com

B. S. Komal
M.I.E.T, Kot Bhalwal, Jammu, India
e-mail: bskomal2@yahoo.co.in

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$\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \bar{b}_n \beta_n^2$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ are elements of $H^2(\beta)$. The space $H^2(\beta)$ is known as weighted Hardy space.

If $\beta_n = 1$ for all $N \cup \{0\}$, then $H^2(\beta)$ is the classical Hardy space. If $\beta_n = \frac{1}{\sqrt{n+1}}$ for all $N \cup \{0\}$, then $H^2(\beta)$ is the Bergman space. Further, if $\beta_n = \sqrt{n+1}$ for all $N \cup \{0\}$, then $H^2(\beta)$ is the Dirichlet space. For $\beta_n = n!$ for all $N \cup \{0\}$, then $H^2(\beta)$ consist of entire functions and it is known as Fischer space.

Let Ω be a region in open unit disk in the complex plane C. Let $\theta : \Omega \to C$ and $\phi : \Omega \to \Omega$ be the analytic maps. Then a generalized composition operator $C_\phi^d : H^2(\beta) \to H^2(\beta)$ is defined by $C_\phi^d f = f' o \phi$, where f' is the derivative of f. A generalized multiplication operator $M_\theta^d : H^2(\beta) \to H^2(\beta)$ is defined by $M_\theta^d f = \theta . f'$. Further, a generalized weighted composition operator $W_{\theta,\phi}^d : H^2(\beta) \to H^2(\beta)$ is defined by $W_{\theta,\phi}^d f = \theta . f' o \phi$. If $\phi(z) = z$ for every $z \in \Omega$, then $W_{\theta,\phi}^d = M_\theta^d$ which is a generalized multiplication operator.

Definition Let w be a point in the open unit disk. Define $K_w(z) = \sum_{n=0}^{\infty} \frac{z^n \bar{w}^n}{\beta_n^2}$ for every $z \in C$. Then $K_w(z)$ is in $H^2(\beta)$ whenever $|w| < 1$. The function $K_w(z)$ is known as point evaluation kernel at w. Now $K_w(z) = \sum_{n=0}^{\infty} (\frac{\bar{w}^n}{\beta_n^2}) z^n$ and so

$$||K_w||^2 = \sum_{n=0}^{\infty} \left(\frac{|\bar{w}|^n}{\beta_n^2} \right)^2 \beta_n^2 = \sum_{n=0}^{\infty} \frac{|w|^{2n}}{\beta_n^2}.$$

Clearly $||K_w||$ is an increasing function of $|w|$ and $\langle f, K_w\rangle = f(w)$ for all $f \in H^2(\beta)$.

The systematic study of composition operators on spaces of analytic functions began with the paper of Nordgen [4]. The work on composition operators is then followed by several mathematicians in several directions. To mention a few of them are Zorboska [14], Schwartz [6], Ridge [5], Shapiro [7], Singh [13], Singh and Komal [12]. They have done commendable work on composition operators. Cowen and MacCluer [1], Zorboska [14] initiated the study of composition operators on weighted Hardy spaces. Weighted composition operators are studied by Gunatillake [3], Sharma and Komal [9–11]. Sharma [8] has studied generalized weighted composition operators on Bergman space. The main purpose of the present paper is to study generalized composition operators, generalized multiplication operators and generalized weighted composition operators on weighted Hardy spaces.

2 Adjoint of a Generalized Composition Operator Using Evaluation Kernel on a Weighted Hardy Space

Boundedness of the generalized composition operators on weighted Hardy space is characterized in Sharma and Komal [9]. In this section, we shall compute the adjoint of generalized composition operator using evaluation kernel. For the sake of convenience, we give here Theorem[2.16] of Cowen and MacCluer [1] in the form of Lemma 1.

Lemma 1 *Let* $f \in H^2(\beta)$ *and* $K_w(z)$ *be a point evaluation function. Then* $\langle f, K_w^{[1]}\rangle = f^{'}(w)$ *where* $K_w^{[1]}(z) = \sum_{n=1}^{\infty} \frac{n(\bar{w})^{n-1}}{\beta_n^2} z^n$.

Proof Given that $f \in H^2(\beta)$,
So put, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $K_w(z) = \sum_{n=0}^{\infty} \frac{z^n(\bar{w})^n}{\beta_n^2}$
Now $K_w^{[1]}(z) = \sum_{n=1}^{\infty} \frac{z^n n(\bar{w})^{n-1}}{\beta_n^2}$
Clearly, it can be seen that

$$\langle f, K_w^{[1]}\rangle = f^{'}(w)$$

Hence, the result.

Theorem 1 *Let* $C_\phi^d \in B(H^2(\beta))$. *Then* $C_\phi^{d^*} K_w = K_{\phi(w)}^{[1]}$, *where* $C_\phi^{d^*}$ *is the adjoint of* C_ϕ^d.

Proof For every $f \in H^2(\beta)$,
we have

$$\langle f, C_\phi^{d^*} K_w\rangle = \langle C_\phi^d f, K_w\rangle = \langle f^{'} o\phi, K_w\rangle = f^{'}(\phi(w)) = \langle f, K_{\phi(w)}^{[1]}\rangle$$

Hence,

$$C_\phi^{d^*} K_w = K_{\phi(w)}^{[1]}$$

Example 1 Let $\Omega = \{z \in \mathbb{C} : |z| < e^{-1}\}$ and $\phi(z) = z^2$ for all $z \in \Omega$. Then $\phi : \Omega \to \Omega$ is an analytic map. For every $n \in \mathbb{N} \cup \{0\}$ define $\beta_n = e^{-n}$. Then $H^2(\beta) \neq \emptyset$ as $e_1 \in H^2(\beta)$. We first show that $C_\phi^d : H^2(\beta) \to H^2(\beta)$ is a bounded operator. Take $f(z) = \sum_{n=0}^{\infty} f_n z^n \quad in \quad H^2(\beta)$. Then

$$||f||^2 = \sum_{n=0}^{\infty} |f_n|^2 \beta_n^2 < \infty$$

Now

$$C_\phi^d f = \sum_{n=1}^{\infty} n f_n (\phi(z))^{n-1} = \sum_{n=1}^{\infty} n f_n z^{2n-2}$$

Therefore,

$$||C_\phi^d f||^2 = \sum_{n=0}^{\infty} (n+1)^2 |f_{n+1}|^2 \beta_{2n}^2 = \sum_{n=0}^{\infty} \left((n+1) \frac{\beta_{2n}}{\beta_{n+1}} \right)^2 |f_{n+1}|^2 \beta_{n+1}^2 \quad (1)$$

But

$$(n+1) \frac{\beta_{2n}}{\beta_{n+1}} = \frac{(n+1)e^{n+1}}{e^{2n}} = \frac{(n+1)}{e^{n-1}} \leq e \quad \text{for every} \quad n = 0, 1, 2, 3, \ldots$$

Therefore, from (1),

$$||C_\phi^d f||^2 \leq e^2 \sum_{n=0}^{\infty} |f_{n+1}|^2 \beta_{n+1}^2 \leq e^2 ||f||^2$$

or

$$||C_\phi^d f|| \leq e||f|| \quad \text{for every} \quad f \in H^2(\beta)$$

Hence, C_ϕ^d is a bounded operator.

Consider

$$\langle f, C_\phi^{d^*} K_w \rangle = \langle C_\phi^d f, K_w \rangle = f^{'}(\phi(w)) = f^{'}(w^2) = \sum_{n=0}^{\infty} (n+1) f_{n+1} w^{2n}.$$

Also it can be seen easily that

$$\langle f, K_{\phi(w)}^{[1]} \rangle = \langle f, K_{w^2}^{[1]} \rangle = \sum_{n=0}^{\infty} (n+1) f_{n+1} w^{2n}.$$

Hence,

$$C_\phi^{d^*} K_w = K_{\phi(w)}^{[1]}.$$

3 Adjoint and Norm Estimate of Generalized Multiplication Operator by Using Evaluation Kernel

In this section, we have to find the adjoint and norm estimate of generalized multiplication operator by using evaluation kernel.

Theorem 2 *Let $M_\theta^d \in B(H^2(\beta))$. Then $M_\theta^{d^*} K_w = \overline{\theta(w)} K_w^{[1]}$, where $M_\theta^{d^*}$ is the adjoint of M_θ^d.*

Proof Let $f \in H^2(\beta)$,
we have

$$\langle f, M_\theta^{d^*} K_w \rangle = \langle M_\theta^d f, K_w \rangle = \langle \theta f^{'}, K_w \rangle = \theta(w) f^{'}(w) = \theta(w) \langle f, K_w^{[1]} \rangle$$

This proves that

$$M_\theta^{d^*} K_w = \overline{\theta(w)} K_w^{[1]}$$

Example 2 Let M_θ^d be a bounded operator. For $\theta(z) = z$,
consider

$$\langle f, M_\theta^{d^*} K_w \rangle = \langle M_\theta^d f, K_w \rangle = \langle \theta f^{'}, K_w \rangle = w f^{'}(w) = \sum_{n=0}^{\infty} n f_n w^n$$

Now it can be easily seen that $\langle f, \overline{\theta(w)} K_w^{[1]} \rangle = \sum_{n=0}^{\infty} n f_n w^n$.

Theorem 3 *Let Ω be the open unit disk in complex plane $\mathbb{C}$. If $M_\theta^d \in B(H^2(\beta))$, then $|\theta(w)| \leq ||M_\theta^d|| \frac{||K_w||}{||K_w^{[1]}||}$ for each $w \in \Omega$.*

Proof Let $f_w = \frac{K_w}{||K_w||}$, Then $||f_w|| = 1$
Since M_θ^d is bounded, so $||M_\theta^{d^*} f_w|| \leq ||M_\theta^d||$

$$||M_\theta^{d^*} \frac{K_w}{||K_w||}|| \leq ||M_\theta^d||$$

$$||M_\theta^{d^*} K_w|| \leq ||M_\theta^d|| ||K_w||$$

By using theorem (2), we have

$$||\overline{\theta(w)} K_w^{[1]}|| \leq ||M_\theta^d|| ||K_w||$$

Hence, the result

$$|\theta(w)| \leq ||M_\theta^d|| \frac{||K_w||}{||K_w^{[1]}||}$$

Theorem 4 *Suppose that* $M^d_\theta \in B(H^2(\beta))$, *where* $\sum_{n=0}^{\infty} \frac{1}{\beta_n^2} < \infty$.

Then

$$\frac{|\theta(w)|}{\beta_1 \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}} < ||M^d_\theta||$$

.

Proof By theorem (3),

$$|\theta(w)| \leq ||M^d_\theta|| \frac{||K_w||}{||K^{[1]}_w||} \tag{2}$$

Now $\quad ||K_w||^2 = \sum_{n=0}^{\infty} \frac{|w|^{2n}}{\beta_n^2}$

For any $|w| < 1$, it is easy to see that $||K_w|| < \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}$

As

$$||K^{[1]}_w||^2 = \sum_{n=1}^{\infty} \frac{n^2 |w|^{2(n-1)}}{\beta_n^2} > \frac{1}{\beta_1^2}$$

Therefore, $\quad ||K^{[1]}_w|| > \frac{1}{\beta_1}$ implies $\quad \frac{1}{||K^{[1]}_w||} < \beta_1$

Therefore, from inequality (2), we have $|\theta(w)| < ||M^d_\theta|| \beta_1 \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}$

This proves that

$$\frac{|\theta(w)|}{\beta_1 \sqrt{\sum_{n=0}^{\infty} \frac{1}{\beta_n^2}}} < ||M^d_\theta||$$

4 Adjoint of Generalized Weighted Composition Operator by Using Evaluation Kernel

In this section, we shall obtain the adjoint of generalized weighted composition operator by using evaluation kernel.

Theorem 5 *Suppose* $W^d_{\theta,\phi} \in B(H^2(\beta))$*. Then* $W^{d^*}_{\theta,\phi} K_w = \overline{\theta(w)} K^{[1]}_{\phi(w)}$*, where* $W^{d^*}_{\theta,\phi}$ *is the adjoint of* $W^d_{\theta,\phi}$*.*

Proof A straight forward calculation shows that $W^{d^*}_{\theta,\phi} K_w = \overline{\theta(w)} K^{[1]}_{\phi(w)}$, see Gandhi et al. [2].

Example 3 Let $\Omega = \{z \in \mathbb{C} : |z| < e^{-1}\}$. For every $n \in \mathbb{N} \cup \{0\}$, let $\beta_n = e^{-n}$. Let $\phi : \Omega \to \Omega$ be defined by $\phi(z) = z^2$. Let $\theta : \Omega \to \mathbb{C}$ be defined by $\theta(z) = z^2$. We first prove that $W^d_{\theta,\phi}$ is a bounded operator.

Take $f(z) = \sum_{n=0}^{\infty} f_n z^n$ in $H^2(\beta)$. Then $||f||^2 = \sum_{n=0}^{\infty} |f_n|^2 \beta_n^2 < \infty$

Now

$$(W^d_{\theta,\phi} f)(z) = \theta(z) f^{'}(\phi(z)) = z^2 \sum_{n=1}^{\infty} n f_n z^{2n-2} = \sum_{n=1}^{\infty} n f_n z^{2n}$$

Therefore,

$$||W^d_{\theta,\phi} f||^2 = \sum_{n=0}^{\infty} n^2 |f_n|^2 \beta_{2n}^2 = \sum_{n=0}^{\infty} \left(n \frac{\beta_{2n}}{\beta_n} \right)^2 |f_n|^2 \beta_n^2 \tag{3}$$

But

$$\frac{n\beta_{2n}}{\beta_n} = \frac{ne^n}{e^{2n}} = \frac{n}{e^n} < 1$$

Hence, from Eq. (3), $||W^d_{\theta,\phi} f||^2 < ||f||^2$

This proves that $W^d_{\theta,\phi} f$ is a bounded operator.

For the adjoint of $W^d_{\theta,\phi}$, consider

$$\langle f, W^{d^*}_{\theta,\phi} K_w \rangle = \langle W^d_{\theta,\phi} f, K_w \rangle = \theta(w).f^{'}(\phi(w)) = w^2.f^{'}(w^2) = \sum_{n=1}^{\infty} n f_n w^{2n}$$

Also, it can be seen easily that

$$\langle f, \overline{\theta(w)} K^{[1]}_{\phi(w)} \rangle = \sum_{n=1}^{\infty} n f_n w^{2n}$$

Hence,

$$W^{d^*}_{\theta,\phi} K_w = \overline{\theta(w)} K^{[1]}_{\phi(w)}$$

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Common Solution to Generalized General Variational-Like Inequality and Hierarchical Fixed Point Problems

Rehan Ali and Mohammad Shahzad

Abstract This paper deals with a strong convergence theorem for a hybrid iterative algorithm to approximate the common solution of generalized general variational-like inequality problem for generalized relaxed α-monotone mapping and hierarchical fixed point problem for nonexpansive mapping in real Hilbert spaces. Some consequences of the strong convergence theorem are also derived. Finally, we give a numerical example to justify the main result. The method and results presented in this paper generalize and unify previously known corresponding results of this area.

Keywords Generalized general variational-like inequality problem · Hierarchical fixed point problem · Hybrid iterative algorithm · Strong convergence

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$ and let $C \subset H$ be a nonempty, closed and convex set.

It is well known that the theory of variational inequalities plays an important role in optimization, economics and engineering sciences. Because of its vast range applicability, various extensions and generalizations of variational inequality problems have been made and analysed in various directions for the past several years. One of the important generalizations is variational-like inequality problem introduced by Parida et al. [17] which has applications in optimization.

R. Ali (✉)
Department of Mathematics, Jamia Millia Islamia, New Delhi 110025, India
e-mail: rehan08amu@gmail.com

M. Shahzad
SHSS School, Aligarh Muslim University, Aligarh 202002, India
e-mail: mdshahzad10@gmail.com

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In 2006, Preda et al. [18] introduced and studied the general variational-like inequality problem (in, short GVLIP) of finding $x^* \in C$ such that

$$F(x, x^*; x^*) \geq 0, \ \forall x \in C, \tag{1}$$

which has applications in mathematical and equilibrium programming, see for example [22].

Very recently, Kazmi and Ali [10] introduced the generalized general variational-like inequality problem (in, short GGVLIP) which is to find $x^* \in C$ such that

$$F(x, x^*; x^*) + \phi(x, x^*) - \phi(x^*, x^*) \geq 0, \ \forall x \in C. \tag{2}$$

The solution set of GGVLIP (2) is denoted by $\Omega = \text{Sol}(\text{GGVLIP}(1.2))$.They proved an existence theorem for GGVLIP (2) and proved strong convergence theorem for an iterative method for approximating a common solution to a system of GGVLIPs and a common fixed point problem in Banach space.

If we set $F(x, x^*; x^*) = \langle fx^* + gx^*, \eta(x, x^*)\rangle$ where $f, g : C \to H$ and $\eta : C \times C \to H$, then GGVLIP (2) is reduced to the mixed variational-like inequality problem introduced and studied by Noor [16].

Further, if we set $F(x, x^*; x^*) = \langle fx^*, \eta(x, x^*)\rangle$ where $f : C \to H$ and $\eta : C \times C \to H$ and $\phi = 0$, then GGVLIP (2) is reduced to the variational-like inequality problem of finding $x^* \in C$ such that

$$\langle fx^*, \eta(x, x^*)\rangle \geq 0, \ \forall x \in C,$$

introduced and studied by Parida et al. [17], which has applications in mathematical programming problems.

Moreover if $\eta(x, x^*) = x - x^*$ for all $x, x^* \in C$, then variational-like inequality problem is reduced to the classical variational inequality problem of finding $x^* \in C$ such that

$$\langle fx^*, x - x^*\rangle \geq 0, \ \forall x \in C,$$

introduced and studied by Hartman and Stampacchia [7].

On the other hand, hierarchical fixed point problem (in short, HFPP) is an important problem which covers a number of problems like monotone variational inequality on fixed point sets, minimization problems over equilibrium constraints, hierarchical minimization problems, fixed point problem, etc., see [14]. Recall that a mapping $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. It is known that if $\text{Fix}(T) := \{x \in C : Tx = x\} \neq \emptyset$, then $\text{Fix}(T)$ is closed and convex. Assume that $\text{Fix}(T) \neq \emptyset$. The HFPP for nonexpansive mappings is defined as follows: Find $x^* \in \text{Fix}(T)$ such that

$$\langle x^* - Sx^*, x^* - x\rangle \leq 0, \ \forall x \in \text{Fix}(T), \tag{3}$$

where $S : C \to C$ is a nonexpansive mapping. HFPP (3) was initially considered and studied by Moudafi and Mainge [14]. This amounts to saying that $x^* \in \mathrm{Fix}(T)$ satisfies a variational inequality depending on a given criterion S, namely: Find $x^* \in C$ such that

$$0 \in (I - S)x^* + N_{\mathrm{Fix}(T)}(x^*), \tag{4}$$

where I is the identity mapping on C and $N_{\mathrm{Fix}(T)}$ is the normal cone to $\mathrm{Fix}(T)$ defined by

$$N_{\mathrm{Fix}(T)} = \begin{cases} \{z \in H : \langle y - x, z\rangle \leq 0,\ \forall\, y \in \mathrm{Fix}(T)\}, & \text{if } x \in \mathrm{Fix}(T), \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is easy to see that HFPP (3) is equivalent to the following fixed point problem: Find $x^* \in C$ such that

$$x^* = P_{\mathrm{Fix}(T)} \circ Sx^*, \tag{5}$$

where $P_{\mathrm{Fix}(T)}$ is the metric projection of H onto $\mathrm{Fix}(T)$. The solution set of HFPP (3) is denoted by $\Phi := \{x^* \in C : x^* = (P_{\mathrm{Fix}(T)} \circ S)x^*\}$.
If we set $S = I$, the solution set of HFPP (3) is just $\mathrm{Fix}(T)$.

By setting $S = I - \gamma f$, where f is η-Lipschitz continuous and k-strongly monotone with $\gamma \in \left(0, \dfrac{2k}{\eta^2}\right]$, then HFPP (3) reduces to the following variational inequality problem over $\mathrm{Fix}(T)$, so-called hierarchical variational inequality problem (in short, HVIP): Find $x^* \in \mathrm{Fix}(T)$ such that

$$\langle f(x^*), x - x^*\rangle \geq 0,\ \forall x \in \mathrm{Fix}(T), \tag{6}$$

which has been studied by Yamada and Ogura [21].
By setting $T = J_\lambda^M := (I + \lambda M)^{-1}$ for $\lambda > 0$, and $S = I - \gamma\nabla\psi$, where ψ is convex and Gateaux differentiable function such that $\nabla\psi$ is η-Lipschitz continuous with $\gamma \in \left(0, \dfrac{2}{\eta}\right]$, and using the fact that $\mathrm{Fix}(J_\lambda^M) = M^{-1}(0)$, then HFPP (3) reduces to the following mathematical programming problem with generalized equation constraint considered by Luo et al. [12]:

$$\min_{0 \in M(x^*)} \psi(x^*). \tag{7}$$

By taking $M = \partial\varphi$, where $\partial\varphi$ is the subdifferential of a lower semicontinuous and convex function, then problem (7) reduces to the following hierarchical minimization problem, considered by Cabot [4]:

$$\min_{x^* \in \arg\min \varphi} \psi(x^*). \tag{8}$$

Note that based on relation (5), HFPP (3) has the iterative algorithm $x_{n+1} = P_{\mathrm{Fix}(T)}(Sx_n)$. It will converge if a fixed point of the operator $P_{\mathrm{Fix}(T)} \circ S$ exists, and if S is averaged, not just nonexpansive. But, in this case, the computing $P_{\mathrm{Fix}(T)} \circ S$ is not easy, so it would be nice if one has an iterative algorithm that uses T itself, rather than $P_{\mathrm{Fix}(T)} \circ S$. For this purpose, Moudafi [13] introduced the following Krasnoselski–Mann algorithm for solving HFPP (3):

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(\sigma_n Sx_n + (1 - \sigma_n)Tx_n), \ \forall n \geq 0, \tag{9}$$

where $\{\alpha_n\}$ and $\{\sigma_n\}$ are control sequences in $(0, 1)$. It is worth to mention that some algorithms in signal processing and image reconstruction may be written as the Krasnoselski–Mann iterative algorithm which provides a unified frame for analysing various concrete algorithms; see for instance [3, 19]. For further study of some generalizations of iterative algorithm (9), see [5, 8, 9, 11, 14, 15, 20].

It is also worth to mention that the most of the results for HFPP (3) available in the literature are related to the weak convergence of iterative algorithms for solving HFPP (3). Therefore, we focus our attention to propose an iterative algorithm to find a common solution of GGVLIP (2) and HFPP (3) and prove a strong convergence theorem.

In this paper, motivated by iterative algorithms (9), we propose the following hybrid iterative algorithm to approximate a common solution of GGVLIP (2) and HFPP (3).

Iterative algorithm: Choose initial values $x_0, z_0 \in C_0 = C$ arbitrarily. Let the sequences $\{x_n\}$ and $\{z_n\}$ be generated by the scheme:

$$\left.\begin{array}{l} F(w, w_n; w_n) + \phi(w, w_n) - \phi(w_n, w_n) + \dfrac{1}{r_n}\langle w - w_n, w_n - x_n\rangle \geq 0, \ \ \forall w \in C; \\ z_n = (1 - \alpha_n)w_n + \alpha_n(\sigma_n Sw_n + (1 - \sigma_n)Tw_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap Q_n} x_0, \end{array}\right\} \tag{10}$$

where $\{r_n\} \subset (0, \infty)$; $\{\alpha_n\} \subseteq [0, 1]$, $\{\sigma_n\} \subseteq [0, 1]$ are control sequences. We can easily observe that a number of iterative algorithms can be obtained from iterative algorithm (10).

In the next section, we present some concepts and results which are needed in the proof of the main result while in Section 3, we prove a strong convergence theorem for iterative algorithm (10). Further, we derive some consequences from the main result. Finally, we give a numerical example to justify the main result. The method and results presented in this paper extend, improve and unify the corresponding known results in the literature.

2 Preliminaries

Throughout the paper, we denote the strong and weak convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We recall the following concepts and results which are needed in sequel.

It is well known that a real Hilbert space H satisfies

(i) the identity

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{11}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$;

(ii) Opial's condition if for any sequence $\{x_n\}$ in H such that $x_n \rightharpoonup x$, for some $x \in H$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $x \neq y$;

(iii) the Kadec–Klee property [6], i.e., if $\{x_n\}$ is a sequence in H which satisfies $x_n \rightharpoonup x \in H$ and $\|x_n\| \to \|x\|$ as $n \to \infty$, then $\|x_n - x\| \to 0$.

For every point $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

The mapping P_C is called the *metric projection* of H onto C. It is well known that P_C is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{12}$$

Moreover, $P_C x$ is characterized by the fact $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall y \in C, \tag{13}$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H,\ y \in C. \tag{14}$$

Definition 2.1 ([2]) An operator $M : H \to 2^H$ is said to be

(i) monotone if

$$\langle u - v, x - y\rangle \geq 0, \ \textit{whenever}\ u \in M(x),\ v \in M(y);$$

(ii) maximal monotone if M is monotone and the graph, $\text{graph}(M) := \{(x, y) \in H \times H : y \in M(x)\}$, is not properly contained in the graph of any other monotone operator.

Lemma 2.1 *Let T is a nonexpansive mapping on H then*

(i) [3] *$I - T$ is a maximal monotone mapping on H;*
(ii) [6] *T is demiclosed on H in the sense that if $\{x_n\}$ converges weakly to $x \in H$ and $\{x_n - Tx_n\}$ converges strongly to 0 then $x \in \text{Fix}(T)$.*

Lemma 2.2 (Corollary 4.15 [1]) *Let $C \subset H$ be a nonempty, closed and convex set and let $T : C \to H$ be a nonexpansive mapping. Then* $\text{Fix}(T)$ *is closed and convex.*

Assumption 2.1 Let F and ϕ satisfy the following conditions:

(i) $F(x, y; z) = 0$ if $x = y$ for any $x, y, z \in C$;
(ii) F is generalized relaxed α-monotone, i.e., for any $x, y \in C$ and $t \in (0, 1]$, we have
$$F(y, x; y) - F(y, x; x) \geq \alpha(x, y),$$
where $\alpha : H \times H \to \mathbb{R}$ such that
$$\lim_{t \to 0} \frac{\alpha(x, ty + (1 - t)x)}{t} = 0;$$
(iii) $F(y, x; \cdot)$ is hemicontinuous for any fixed $x, y \in C$;
(iv) $F(\cdot, x; z)$ is convex and lower semicontinuous for any fixed $x, z \in C$;
(v) $F(x, y; z) + F(y, x; z) = 0$ for any $x, y, z \in C$;
(vi) $\phi(\cdot, \cdot)$ is weakly continuous and $\phi(\cdot, y)$ is convex for any fixed $y \in C$;
(vii) ϕ is skew-symmetric, i.e., $\phi(x, x) - \phi(x, y) + \phi(y, y) - \phi(y, x) \geq 0$, $\forall x, y \in C$.

For a given $r \geq 0$, define a mapping $T_r^F : H \to C$ as follows:
$$T_r^F(x) = \left\{ z \in C : F(y, z; z) + \frac{1}{r}\langle y - z, z - x\rangle + \phi(z, y) - \phi(z, z) \geq 0, \forall y \in C \right\}, \forall x \in H. \tag{15}$$
The following lemma is a special case of Lemma 3.1–3.3 due to [10] in real Hilbert space.

Lemma 2.3 ([10]) *Assume that $F : C \times C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ satisfy Assumption 2.1. Suppose the mapping $T_r^F : H \to C$ be defined as in (15). Then the following hold:*

(i) $T_r^F(x) \neq \emptyset$ for each $x \in H$;
(ii) T_r^F is single valued;
(iii) T_r^F is firmly nonexpansive, i.e.,

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle, \quad \forall x, y \in H;$$

(iv) $\mathrm{Fix}(T_r^F) = \mathrm{Sol}(\mathrm{GGVLIP}(1.2))$;

(v) $\mathrm{Sol}(\mathrm{GGVLIP}(1.2))$ *is closed and convex.*

3 Strong Convergence Theorem

We prove a strong convergence theorem for Iterative algorithm (10) to approximate a common solution of GGVLIP (2) and HFPP (3).

Theorem 3.1 *Let H be a real Hilbert space and let $C \subseteq H$ be a nonempty, closed and convex set. Let $T, S : C \to C$ be nonexpansive mappings; let $F : C \times C \times C \to \mathbb{R}$ is a trifunction and $\phi : C \times C \to \mathbb{R}$ is a bifunctions satisfying Assumption 2.1 with $F(x, \cdot; x)$ is weakly continuous. Assume that $\Gamma = \Omega \bigcap \Phi \bigcap \mathrm{Fix(S)} \neq \emptyset$. Let the sequences $\{x_n\}$, $\{z_n\}$ generated by Iterative algorithm (10) and the control sequences $\{\alpha_n\}, \{\sigma_n\}$ be such that $\{\alpha_n\} \in [c, 1),\ c \in (0, 1),\ \{\sigma_n\} \in [a, b],\ \ a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$, $\liminf_{n\to\infty} r_n = r > 0$. Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $x^* \in \Gamma$, where $x^* = P_\Gamma x_0$.*

Proof Since $\Gamma \neq \emptyset$, $T_{r_n}^F$ is nonexpansive and hence, it follows from Lemma 2.3 that $\mathrm{Fix}(T_{r_n}^F) = \Omega$ is a closed and convex set. Clearly Φ is closed and convex, since $\Phi = \mathrm{Fix}(P_{\mathrm{Fix}(T)} \circ S) \neq \emptyset$. Thus, Γ is nonempty, closed and convex and hence $P_\Gamma x_0$ is then well defined. Next, we show that $C_n \cap Q_n$ is closed and convex. For any $z \in C_n$, we have

$$\begin{aligned} & \|z_n - z\|^2 \leq \|x_n - z\|^2 \\ \Leftrightarrow \quad & \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle \leq 0. \end{aligned} \tag{16}$$

Now, we can easily observe that C_n is closed and convex for all $n \geq 0$. Further, evidently Q_n is closed and convex for all $n \geq 0$. Consequently, $C_n \cap Q_n$ is closed and convex for all $n \geq 0$. Next, we claim that $\Gamma \subset C_n \bigcap Q_n,\ \forall\, n \geq 0$. Indeed, for any $p \in \Gamma$, i.e., $p \in \Omega$, we have $p = T_{r_n}^F p$ and we estimate

$$\begin{aligned} \|w_n - p\|^2 &= \|T_{r_n}^F x_n - p\|^2 \\ &\leq \|x_n - p\|^2 \end{aligned} \tag{17}$$

Since, for any $p \in \Gamma$, and using (17) in the following inequality

$$
\begin{aligned}
\|z_n - p\|^2 &= \|(1-\alpha_n)w_n + \alpha_n(\sigma_n S w_n + (1-\sigma_n)T w_n) - p\|^2 \\
&= \|(1-\alpha_n)(w_n - p) + \alpha_n\left(\sigma_n(S w_n - p) + (1-\sigma_n)(T w_n - p)\right)\|^2 \\
&\le (1-\alpha_n)\|w_n - p\|^2 + \alpha_n(\sigma_n\|S w_n - p\|^2 + (1-\sigma_n)\|T w_n - p\|^2 \\
&\quad -\sigma_n(1-\sigma_n)\|S w_n - T w_n\|^2) \\
&\le (1-\alpha_n)\|w_n - p\|^2 + \alpha_n(\sigma_n\|w_n - p\|^2 + (1-\sigma_n)\|w_n - p\|^2 \\
&\quad -\sigma_n(1-\sigma_n)\|S w_n - T w_n\|^2) \\
&\le \|w_n - p\|^2 - \alpha_n\sigma_n(1-\sigma_n)\|S w_n - T w_n\|^2. && (18)\\
&\le \|x_n - p\|^2 - \alpha_n\sigma_n(1-\sigma_n)\|S w_n - T w_n\|^2. && (19)\\
&\le \|x_n - p\|^2. && (20)
\end{aligned}
$$

This implies that $p \in C_n$ and hence $\Gamma \subset C_n,\ \forall n \ge 0$. Further, by induction method, we show that $\Gamma \subset Q_n,\ \forall n \ge 0$. For $n = 0$, evidently $\Gamma \subset Q_0 = H$, and hence $\Gamma \subset C_0 \cap Q_0$. Therefore $x_1 = P_{C_0 \cap Q_0} x_0$ is well defined. Now, assume that $\Gamma \subset C_{n-1} \cap Q_{n-1}$, for some $n > 1$. Let $x_n = P_{C_{n-1}\cap Q_{n-1}} x_0$ then for any $p \in \Gamma$, it follows from (13) that $\langle x_0 - x_n, x_n - p\rangle = \langle x_0 - P_{C_{n-1}\cap Q_{n-1}} x_0,\ P_{C_{n-1}\cap Q_{n-1}} x_0 - p\rangle \ge 0$, and hence $p \in Q_n$. Therefore $\Gamma \subset C_n \cap Q_n$. Consequently $C_n \cap Q_n$ is nonempty, closed and convex and hence $x_{n+1} = P_{C_n \cap Q_n} x_0$ is well defined for all $n \ge 0$. Thus the sequence $\{x_n\}$ is well defined.

Since $\Gamma \neq \emptyset$ then it is a closed and convex subset of C. From the definition of Q_n and $\Gamma \subset Q_n$, we have

$$
\|x_n - x_0\| \le \|p - x_0\|,\ \forall p \in \Gamma,\ \forall n \ge 0. \tag{21}
$$

This implies that $\{x_n\}$ is bounded and hence in particular,

$$
\|x_n - x_0\| \le \|q - x_0\|,\ \forall n \ge 0, \tag{22}
$$

where $q = P_\Gamma x_0$. Since $x_n = P_{Q_n} x_0$ and $x_{n+1} \in Q_n$, using (14), we have

$$
\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \forall n \ge 0. \tag{23}
$$

Hence, it follows from (22) and (23) that

$$
\begin{aligned}
\sum_{n=1}^{N} \|x_{n+1} - x_n\|^2 &\le \sum_{n=1}^{N} \left(\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2\right) \\
&= \|x_{N+1} - x_0\|^2 - \|x_1 - x_0\|^2 \\
&\le \|q - x_0\|^2 - \|x_1 - x_0\|^2, && (24)
\end{aligned}
$$

which implies that $\sum\limits_{n=1}^{\infty} \|x_{n+1} - x_n\|^2$ is convergent and hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{25}$$

Since $x_{n+1} = P_{C_n \bigcap Q_n} x_0 \in C_n$, it follows that

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2. \tag{26}$$

It follows from (25) and (26) that

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0. \tag{27}$$

Furthermore, we have

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\|. \tag{28}$$

Hence, it follows from (25), (27) and (28) that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{29}$$

Since $T_{r_n}^F$ is firmly nonexpansive, we have

$$\begin{aligned} \|w_n - p\|^2 &= \|T_{r_n}^F x_n - T_{r_n}^F p\|^2 \\ &\leq \langle x_n - p, w_n - p\rangle \\ &\leq \frac{1}{2}\big[\|x_n - p\|^2 + \|w_n - p\|^2 - \|x_n - w_n\|^2\big], \end{aligned}$$

which in turn yields

$$\|w_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - w_n\|^2, \tag{30}$$

and this together with (18) gives that

$$\begin{aligned} \|x_n - w_n\|^2 &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\leq (\|z_n - p\| + \|x_n - p\|)\|x_n - z_n\| \\ &\leq L_1\|x_n - z_n\| \end{aligned} \tag{31}$$

where $L_1 := \sup_n\{\|x_n - p\| + \|z_n - p\|\}$. Hence, it follows from (29) and (31) that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{32}$$

Since

$$\|z_n - w_n\| \leq \|z_n - x_n\| + \|x_n - w_n\|, \tag{33}$$

it follows from (29), (32) and (33) that

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{34}$$

From (19), we have

$$\begin{aligned}\alpha_n\sigma_n(1-\sigma_n)\|Sw_n - Tw_n\|^2 &= \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\leq \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) \\ &= L_1\|x_n - z_n\|. \end{aligned}\tag{35}$$

By using (29) and $\{\alpha_n\} \in [c, 1),\ c \in (0, 1),\ \{\sigma_n\} \in [a, b] \subset (0, 1)$ in (35), we get

$$\lim_{n\to\infty} \|Sw_n - Tw_n\| = 0. \tag{36}$$

Further, from (10), we have

$$\alpha_n\|Tw_n - w_n\| \leq \|z_n - w_n\| + \alpha_n\sigma_n\|Tw_n - Sw_n\| \tag{37}$$

$$\|Tw_n - w_n\| \leq \frac{1}{\alpha_n}\|z_n - w_n\| + \alpha_n\sigma_n\|Tw_n - Sw_n\| \tag{38}$$

It follows from $\{\alpha_n\} \in [c, 1),\ c \in (0, 1),\ \{\sigma_n\} \in [a, b] \subset (0, 1)$, (34), (36) and (38) that

$$\lim_{n\to\infty} \|Tw_n - w_n\| = 0. \tag{39}$$

From (36) and (39), we have

$$\lim_{n\to\infty} \|Sw_n - w_n\| = 0. \tag{40}$$

Since

$$\begin{aligned}\|Tx_n - x_n\| &\leq \|Tx_n - Tw_n\| + \|Tw_n - w_n\| + \|w_n - x_n\| \\ &\leq 2\|w_n - x_n\| + \|Tw_n - w_n\| \end{aligned}\tag{41}$$

Hence, it follows from (32), (39) and (41) that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{42}$$

Therefore, it follows from Lemma 2.1(ii) and (42) that every weak limit point of $\{x_n\}$ is a fixed point of the mapping T, i.e., $\omega_w(x_n) \subset \mathrm{Fix}(T)$. Since every Hilbert space satisfies Opial's condition, Opial's condition guarantees that $\omega_w(x_n)$ is singleton. Thus $\{x_n\}$ converges weakly to $x^* \in \mathrm{Fix}(T)$. Further, it follows from (32) that the sequences $\{x_n\}$ and $\{w_n\}$ have the same asymptotic behaviour and hence $\{w_n\}$ converges weakly to $x^* \in \text{Fix}(S)$.

Next, we show that $x^* \in \Phi$. Since

$$z_n - w_n = \alpha_n(\sigma_n(Sw_n - w_n) + (1 - \sigma_n)(Tw_n - w_n)) \tag{43}$$

and hence

$$\frac{1}{\alpha_n \sigma_n}(w_n - z_n) = (I - S)w_n + \left(\frac{1 - \sigma_n}{\sigma_n}\right)(I - T)w_n, \tag{44}$$

and hence for all $z \in \text{Fix}(T)$ and using monotonicity of $I - S$, we have

$$\begin{aligned}\langle \frac{w_n - z_n}{\alpha_n \sigma_n}, w_n - z\rangle &= \langle (I - S)w_n - (I - S)z, w_n - z\rangle + \langle (I - S)z, w_n - z\rangle \\ &\quad + \frac{1 - \sigma_n}{\sigma_n}\langle w_n - Tw_n, w_n - z\rangle \\ &\geq \langle (I - S)z, w_n - z\rangle + \frac{1 - \sigma_n}{\sigma_n}\langle w_n - Tw_n, w_n - z\rangle. \end{aligned} \tag{45}$$

Using (34), (39), conditions on α_n and σ_n in (45), we have

$$\limsup_{n\to\infty}\langle z - Sz, w_n - z\rangle \leq 0, \quad \forall z \in \text{Fix}(T). \tag{46}$$

Since $w_n \rightharpoonup x^*$ then, it follows from (46)

$$\langle (I - S)z, x^* - z\rangle \leq 0, \quad \forall z \in \text{Fix}(T). \tag{47}$$

Since $\text{Fix}(T)$ is convex, $\lambda z + (1 - \lambda)x^* \in \text{Fix}(T)$ for $\lambda \in (0, 1)$ and hence

$$\begin{aligned}&\langle (I - S)(\lambda z + (1 - \lambda)x^*), x^* - (\lambda z + (1 - \lambda)x^*)\rangle \\ &= \lambda\langle (I - S)(\lambda z + (1 - \lambda)x^*), x^* - z\rangle \\ &\leq 0 \ \forall z \in \text{Fix}(T),\end{aligned} \tag{48}$$

which implies

$$\langle (I - S)(\lambda z + (1 - \lambda)x^*), x^* - z\rangle \leq 0 \ \forall z \in \text{Fix}(T).$$

On taking limits $\lambda \to 0_+$, we have

$$\langle (I - S)x^*, x^* - z\rangle \leq 0 \ \forall z \in \text{Fix}(T). \tag{49}$$

That is $x^* \in \Phi$. Next, we show that $x^* \in \mathrm{Sol(GGVLIP(1.2))} = \Omega$. Since $w_n = T_{r_n}^F x_n$, we have

$$F(w, w_n; w_n) + \phi(w, w_n) - \phi(w_n, w_n) + \frac{1}{r_n}\langle w - w_n, w_n - x_n\rangle \geq 0, \quad \forall w \in C.$$

It follows from generalized relaxed α-monotonicity of F, and above inequality implies that

$$\phi(w, w_n) - \phi(w_n, w_n) + \langle w - w_n, \frac{w_n - x_n}{r_n}\rangle \geq -F(w, w_n; w) + \alpha(w_n, w), \quad \forall w \in C. \tag{50}$$

Since $\liminf_{n\to\infty} r_n > 0$, then there exists a real number $r > 0$ such that $r_n \geq r, \ \forall\, n$ and hence, we have

$$\frac{\|w_n - x_n\|}{r_n} \leq \frac{\|w_n - x_n\|}{r}.$$

It follows from (32) that

$$\lim_{n\to\infty} \frac{\|w_n - x_n\|}{r_n} \leq \frac{1}{r}\lim_{n\to\infty} \|w_n - x_n\| = 0.$$

Since α is lower semicontinuous in the first argument, ϕ is weakly continuous and $F(u, \cdot; u)$ is weakly continuous then on taking $n \to \infty$ in (50), we get

$$\alpha(x^*, w) - F(w, x^*; w) - \phi(w, x^*) + \phi(x^*, x^*) \leq 0, \quad \forall w \in C. \tag{51}$$

For t with $0 < t \leq 1$ and $w \in C$, set $w_t = tw + (1-t)x^*$. Since C is convex set, $w_t \in C$, then from (51), we have

$$\alpha(x^*, w_t) - F(w_t, x^*; w_t) - \phi(w_t, x^*) + \phi(x^*, x^*) \leq 0, \tag{52}$$

which implies that

$$\begin{aligned}\alpha(x^*, w_t) &\leq F(w_t, x^*; w_t) - \phi(x^*, x^*) + \phi(w_t, x^*)\\ &\leq tF(u, x^*; w_t) + (1-t)F(x^*, x^*; w_t) - \phi(x^*, x^*) + t\phi(u, x^*) + (1-t)\phi(x^*, x^*)\\ &\leq t[F(u, x^*; w_t) + \phi(u, x^*) - \phi(x^*, x^*)].\end{aligned} \tag{53}$$

Since $F(u, x^*; \cdot)$ is hemicontinuous and letting $t \to 0$, we have

$$\lim_{t\to 0}\{F(u, x^*; w_t) + \phi(u, x^*) - \phi(x^*, x^*)\} \geq \lim_{t\to 0} \frac{\alpha(x^*, w_t)}{t}, \tag{54}$$

which implies

$$F(u, x^*; x^*) + \phi(u, x^*) - \phi(x^*, x^*) \geq 0. \tag{55}$$

This implies that $x^* \in \mathrm{Sol(GGVLIP(1.2))} = \Omega$ and thus $x^* \in \Gamma$.

Finally, we show that $x_n \to x^*$, where $x^* = P_\Gamma x_0$. Since $x_n = P_{Q_n} x_0$ and $x^* \in \Gamma \subset Q_n$, we have

$$\|x_n - x_0\| \le \|x^* - x_0\|.$$

It follows from $q = P_\Gamma x_0$, (22) and the lower semicontinuity of the norm that

$$\|q - x_0\| \le \|x^* - x_0\| \le \liminf_{n\to\infty} \|x_n - x_0\| \le \limsup_{n\to\infty} \|x_n - x_0\| \le \|q - x_0\|.$$

This implies that $\lim_{n\to\infty} \|x_n - x_0\| = \|q - x_0\| = \|x^* - x_0\|$. Since $x_n - x_0 \rightharpoonup x^* - x_0$ and $\|x_n - x_0\| \to \|x^* - x_0\|$ then from the Kadec–Klee property of H, we have $\lim_{n\to\infty} x_n = x^* = q$. Thus, we conclude that $\{x_n\}$ converges strongly to x^* where $x^* = P_\Gamma x_0$. □

Now, we give the following consequence of Theorem 3.1.

If we set $S = I$ and $\sigma_n = 0, \forall n$ in Iterative algorithm (10), we have the following strong convergence theorem for finding a common element of solution set of GGVLIP (2) and $\mathrm{Fix}(T)$.

Corollary 3.1 *Let H be a real Hilbert space and let $C \subseteq H$ be a nonempty, closed and convex set. Let $T : C \to C$ be nonexpansive mappings; let $F : C \times C \times C \to \mathbb{R}$ is a trifunction and $\phi : C \times C \to \mathbb{R}$ is a bifunctions satisfying Assumption 2.1 with $F(x, \cdot; x)$ is weakly continuous. Assume that $\Gamma' = \Omega \bigcap \mathrm{Fix(T)} \neq \emptyset$. Let the sequences $\{x_n\}$, $\{z_n\}$ generated by Iterative algorithm (10) and the control sequences $\{\alpha_n\}$, $\{\sigma_n\}$ be such that $\{\alpha_n\} \in [c, 1),\ c \in (0, 1)$ and $\lim_{n\to\infty} r_n = r > 0$.*

$$\left.\begin{array}{l}
F(w, w_n; w_n) + \phi(w, w_n) - \phi(w_n, w_n) + \dfrac{1}{r_n}\langle w - w_n, w_n - x_n\rangle \ge 0, \quad \forall w \in C;\\
z_n = (1 - \alpha_n) w_n + \alpha_n T w_n,\\
C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2\},\\
Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
x_{n+1} = P_{C_n \bigcap Q_n} x_0,
\end{array}\right\}$$

Then the sequences $\{x_n\}$ and $\{z_n\}$ converge strongly to $x^* \in \Gamma'$, where $x^* = P_{\Gamma'} x_0$.

4 Numerical Example

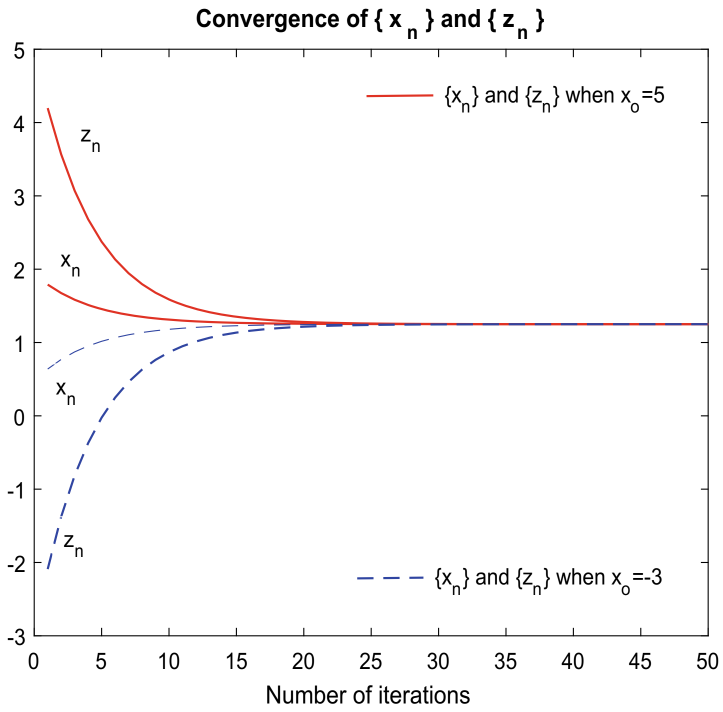
We give a numerical example which justify Theorem 3.1.

Example 4.1 Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy,\ \forall\, x, y \in \mathbb{R}$, and induced usual norm $|\cdot|$. Let $C = (-\infty, +\infty)$; let $F : C \times C \times C \to \mathbb{R}$ and $\phi : C \times C \to \mathbb{R}$ be defined by $F(y, x; x) = (x - \frac{5}{2})(y - x)$, with $\alpha(x, y) = (y - x)^2, \forall x, y \in C$ and $\phi(x, y) = xy,\ \forall x, y \in C$; let the mapping $T : C \to C$ be defined by $Tx = x$, and let the mapping $S : C \to C$ be defined by $Sx = \frac{x+5}{5}, \forall x \in C$. Setting $\{\alpha_n\} = 0.9$, $\{\sigma_n\} = 0.9$ and $\{r_n\} = 0.25$, $\forall n \geq 1$. Then $\{x_n\}$ and $\{z_n\}$ generated by the hybrid iterative scheme (10) converge to a point $x^* = \{\frac{5}{4}\} \in \Gamma$.

Proof It is easy to prove that the trifunction F and bifunction ϕ satisfy Assumption 2.1. It is also easy to observe that T, S are nonexpansive mappings with $\text{Fix}(T) = \left\{\frac{5}{4}\right\}$, $\text{Fix}(S) = \left\{\frac{5}{4}\right\}$, and hence $\Phi = \text{Sol(HFPP(1.3))} = \left\{\frac{5}{4}\right\}$. Furthermore, it is easy to prove that $\Omega = \text{GGVLIP(1.2))} = \left\{\frac{5}{4}\right\}$. Therefore, $\Gamma = \Omega \bigcap \Phi \bigcap \text{Fix(S)} = \left\{\frac{5}{4}\right\} \neq \emptyset$. After simplification, hybrid iterative scheme (10) reduces to the following scheme: Given initial value x_0,

$$\begin{aligned}
&w_n = \frac{8x_n + 5}{12}; \quad z_n = (1-\alpha_n)w_n + \tfrac{\alpha_n}{5}(5\sigma_n + 5w_n - 4\sigma_n w_n);\\
&C_n = [e_n, \infty), \quad \text{where } \; e_n := \frac{z_n + x_n}{2};\\
&Q_n = [x_n, \infty);\\
&x_{n+1} = P_{C_n \bigcap Q_n} x_0, \; n \geq 0.
\end{aligned}$$

Finally, using the software MATLAB 7.8.0, we have the following figure and table which show that $\{x_n\}$ and $\{z_n\}$ converge to $x^* = \frac{5}{4}$ as $n \to +\infty$.



This completes the proof. □

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