Mathematical Analysis of Stiffness of Orthotropic Beam with Hollow Circular and Rectangular Cross-sections

Michael C. Agarana, Esther T. Akinlabi, and Anuoluwapo M. Olanrewaju

Abstract Stiffness and strength of mechanical materials have to be thoroughly investigated to avoid functional and physical failures. This study investigates, analytically, the stiffness of wood beam, which is an example of orthotropic material, with different cross-sectional areas. The moment of inertia for each type of wood, considered, was analysed. The axial stiffness and bending stiffness were also computed for different cross-sectional areas. Comparisons were made and results are consistent with the ones in the literature. Specifically, it was observed that the cross-sectional areas of wood have a lot of influence on its mechanical properties. The axial stiffness, bending stiffness, shear stresses and moment of inertia of the orthotropic solid materials, considered, depend on their cross-sectional areas, to a large extent.

Keywords Stiffness · Orthotropic beam · Mathematical analysis · Hollow cross-sections

1 Introduction

In material science and solid mechanics, orthotropic materials have material properties that differ along three mutually-orthogonal twofold axes of rotational symmetry. They are a subset of anisotropic materials because their properties change when

M. C. Agarana (✉) · A. M. Olanrewaju
Department of Mathematics, Covenant University, Ota, Ogun, Nigeria
e-mail: michael.agarana@covenantuniversity.edu.ng

A. M. Olanrewaju
e-mail: anu.olanrewaju@covenantuiversity.edu.ng

M. C. Agarana · E. T. Akinlabi
Department of Mechanical Engineering Science, University of Johannesburg, Auckland Park Kingsway Campus, P.O. Box 524, Johannesburg 2006, South Africa
e-mail: etakinlabi@gmail.com

E. T. Akinlabi
Department of Mechanical Engineering, Covenant University, Ota, Ogun, Nigeria

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measured from different directions. So every orthotropic material is anisotropic. A familiar example of an orthotropic material is wood [1, 2].

Stiffness is the ability of the material to resist deformation or deflection (functional failure). Understanding the roles that stiffness play is essential to the decision process when choosing foundational support to minimise risk [2, 3]. Stiffness is different from strength. Strength is a measure of the anxiety that can be connected to a material before it for all time disfigures (yield quality) or breaks (rigidity). In the event that the forced stress is not as much as the yield strength, the material comes back to its unique shape when the stress is expelled. On the off chance that the connected anxiety surpasses the yield quality, plastic or perpetual twisting happens, and the material can never again come back to its unique shape once the load is evacuated [3–5].

A polygon is a two-dimensional shape with no less than three sides that are straight and create interior angles where they meet. Polygons can be regular or irregular. A regular polygon is a polygon with all sides of equivalent length and every interior angle of equivalent measure. An irregular polygon, on the other hand, is a polygon that is not regular. That is, an irregular polygon is a polygon that does not have all sides of equivalent lengths and all interior angles equal. Polygons can have any number of sides insofar as there are at least three of them. A lot of n-sided polygons have special names [5–8]. A polygon with three sides is a triangle. A polygon with four sides is known as a quadrilateral and one with five sides is a pentagon. A polygon with six sides is a hexagon, one with seven sides is known as a heptagon, and a polygon with eight sides is an octagon, and so on [8, 9].

The difference between bending moment and moment of inertial is as follows: the reaction induced in a structural element when an external force or moment is applied to the element causing the element to bend is referred to as the bending moment, while moment of inertia is the capacity of a cross-section to resist bending [9–12].

Rigidity of an object, otherwise known as bending stiffness, is the measurement of the extent to which the member resists bending deformation in response to an applied load. This paper analyses, mathematically, the stiffness and strength of wood, an orthotropic material, with a regular diagonal cross-sectional area of different number of vertices.

2 Formulation of Problem

$$E \equiv \sigma(\varepsilon)\varepsilon = F/A\Delta L/L_0 = FL_0 A\Delta L E \equiv \frac{\sigma(\varepsilon)}{\varepsilon} = \frac{F/A}{\Delta L/L_0} = \frac{FL_0}{A\Delta L}$$

Strain can be expressed as

$$\xi = \Delta L/L \tag{1}$$

where

ξ strain (m/m) (in/in)
ΔL elongation or compression (offset) of the object (m) (in)
L length of the object (m) (in).

Stress can be expressed as

$$\sigma(\xi) = F/A \tag{2}$$

where

$\sigma(\xi)$ stress (N/m^2, lb/in^2, psi)
F force (N, lb)
A area of object (m^2, in^2).

Young's modulus (E) can be expressed as

E stress/strain.

$$E = \frac{\sigma(\xi)}{\xi} = \frac{F}{A}.\frac{L_0}{\Delta L} = \frac{FL_0}{A\Delta L} \tag{3}$$

where

$\sigma(\xi)$ is the tensile stress;
ξ is the engineering extentional strain;
E is the Young's modulus (modulus of elasticity);
F is the force exerted on an object under tension;
A is the actual cross-sectional area, which equals the area of the cross-section perpendicular to the applied force;
ΔL is the amount by which the length of the object changes (ΔL is positive if the material is stretched and negative when the material is compressed);
L_0 is the original length of the object.

The Young's modulus of a material can be used to calculate the force it exerts under specific strain

$$F = EA\Delta L L_0 \quad F = \frac{EA\Delta L}{L_0}$$

F is the force exerted by the material when contracted or stretched by ΔL. The force F is given as

$$F = \frac{EA\Delta L}{L_0} \tag{4}$$

The axial stiffness is given as

$$k = \frac{AE}{L} \tag{5}$$

The bending stiffness is given as

$$K = \frac{F}{w} \tag{6}$$

where

F is the applied force
w is the deflection.

Substituting Eq. (6) into (8) gives

$$K = \frac{\frac{EA\Delta L}{L_0}}{w} = \frac{EA\Delta L}{wL_0} \tag{7}$$

The moment of inertia of the wood beam, along the x-axis, with different cross-sections are given as follows:

$$I_{xc} = \frac{\pi(d)^4}{64} \tag{8}$$

$$I_{xr} = \frac{1}{2}bh^3 \tag{9}$$

$$I_{xch} = \frac{\pi}{64}\left[(d_2)^4 - (d_1)^4\right] \tag{10}$$

$$I_{xrh} = \frac{1}{12}BH^3 - \frac{1}{12}bh^3 \tag{11}$$

where

d is thediameter of the circular cross-section
b is the horizontal distance of the cross-section
h is the height of the cross-section
d_2 is the diameter of the outer circle of the cross-section
d_1 is the diameter of the inner circle of the cross-section
B and H are the horizontal distance and height of the outer rectangle of the cross-section
B and h are the horizontal distance and height of the inner rectangle of the cross-section
I_{xc} is the moment of inertia of wood beam with a circular cross-section.
I_{xr} is the moment of inertia of wood beam with a rectangular cross-section.
I_{xch} is the moment of inertia of wood beam with a circular hollow cross-section.
I_{xrh} moment of inertia of wood beam with a rectangular hollow cross-section

3 Numerical Example

Considering uniform blocks of wood beam with the following cross-sectional areas: circular, rectangular, hollow circular and hollow rectangular, stiffness of such structures of wood is calculated and compared. Different values length (L), cross-sectional area (A) and value of Young's modulus of wood, were used to carry out the numerical analysis of the stiffness of block of wood beam with different cross-sectional areas and lengths. For brevity, only the cases of Oak wood beam with four different cross-sections were considered in this paper.

The moment of inertia of the wood hollow circular beam is obtained using Eq. (10), converting inches to meters, as follows:

$$I_{xch} = \frac{\pi}{64}\left[(d_2)^4 - (d_1)^4\right] = \approx 0.0000008549\,\text{kg m}^2 \tag{12}$$

The axial stiffness of the wood hollow circular beam is obtained by adopting Eq. (5) and taken the Young's modulus of Oak wood beam as 12,300 MPa:

$$\begin{aligned} k &= \frac{A_2E - A_1E}{L} \\ &= \pi r_2^2\frac{E}{L} - \pi r_1^2\frac{E}{L} \\ &= \frac{22}{7}\frac{E}{L}\left[r_2^2 - r_1^2\right] \\ &= \frac{22}{7}\left(\frac{12{,}300}{40 \times 0.03}\right)\left[\left(\frac{5.0}{2} \times 0.03\right)^2 - \left(\frac{3.6}{2} \times 0.03\right)^2\right] \\ &\approx 87.27\,\text{N/m} \end{aligned} \tag{13}$$

The bending stiffness of the wood hollow circular beam is obtained by adopting Eq. (6) and taken the Young's modulus of Oak wood beam as 12,300 MPa:

$$K = \frac{EA_2\Delta L}{wL_0} - \frac{EA_1\Delta L}{wL_0} = \frac{\Delta LE}{wL_0}(A_2 - A_1) \tag{14}$$

The deflection w is given as

$$w = \frac{L^3F}{3EI_x} \tag{15}$$

Substituting Eq. (4) into (15) gives

$$w = \frac{L^3\frac{EA\Delta L}{L_0}}{3EI_x} \tag{16}$$

Substituting Eq. (16) into Eq. (14) and simplifying gives

$$K = \frac{3EI_x}{L^3} \tag{17}$$

Substituting the value of the parameters gives

$$\begin{aligned} K &= \frac{3(12{,}300)(0.0000008549)}{(40 \times 0.03)^3} \\ &\approx 0.02\,\text{Nm}^2 \end{aligned} \tag{18}$$

The moment of inertia of the wood hollow rectangular beam cross-section is obtained using Eq. (11):

$$\begin{aligned} I_{xrh} &= \frac{1}{12}BH^3 - \frac{1}{12}bh^3 \\ &= \frac{1}{12}\left[6(0.0254).\{12(0.0254)\}^3\right] - \frac{1}{12}\left[4(0.0254).\{10(0.0254)\}^3\right] \\ &\approx 0.0002222\,\text{kg m}^2 \end{aligned} \tag{19}$$

The axial stiffness of the wood hollow rectangular beam, whose cross-section can be obtained by adopting Eq. (5) and taken the Young's modulus of Oak wood beam as 12,300 MPa:

The area of the outer rectangle A_2 is

$$\begin{aligned} &12(0.0254) \times 6(0.0254) \\ &= 1.83\,\text{m}^2 \end{aligned} \tag{20}$$

The area of the inner rectangle A_1 is

$$\begin{aligned} &10(0.0254) \times 4(0.0254) \\ &= 1.02\,\text{m}^2 \end{aligned} \tag{21}$$

Therefore,

$$\begin{aligned} k &= \frac{A_2E - A_1E}{L} \\ &= \frac{E(1.83 - 1.02)}{0.03(40)} \\ &= \frac{12{,}300(0.81)}{1.2} \\ &\approx 8.3\,\text{N/m} \end{aligned} \tag{22}$$

The bending stiffness of the wood hollow rectangular beam with length 40 inches, whose cross-section can be obtained by adopting Eq. (17) and taken the Young's modulus of Oak wood beam as 12,300 MPa:

$$\begin{aligned} K &= \frac{3EI_x}{L^3} \\ &= \frac{3(12{,}300)(0.0002222)}{(40 \times 0.03)^3} \\ &\approx 4.74\,\mathrm{Nm}^2 \end{aligned} \tag{23}$$

Now, in order to compare with the hollow solid shapes, the moment of inertia, axial stiffness and bending stiffness of both wood beam with a circular cross-section and wood beam with rectangular cross-section are considered in this section. The moment of inertia of the wood circular beam, assuming the hollow is not there, is obtained using Eq. (8), and converting inches to meters, as follows:

$$\begin{aligned} I_{xc} &= \frac{\pi(d)^4}{64} \\ &= \frac{22}{7}\frac{(5.0 \times 0.03)^4}{64} \\ &\approx 0.00002486\,\mathrm{kg\,m}^2 \end{aligned} \tag{24}$$

The moment of inertia of the wood rectangular beam, assuming the hollow is not there, is obtained using Eq. (9), and converting inches to meters, as follows:

$$\begin{aligned} I_{xr} &= \frac{1}{2}bh^3 \\ &= \frac{1}{2}(6 \times 0.03)(12 \times 0.03)^3 \\ &\approx 0.004199\,\mathrm{kg\,m}^2 \end{aligned} \tag{25}$$

The axial stiffness of the wood circular and wood rectangular beam, assuming there is no hollow, are obtained by adopting Eq. (5) and taken the Young's modulus of Oak wood beam as 12,300 MPa:

For circular cross-section

$$\begin{aligned} k &= \frac{AE}{L} \\ &= \frac{\pi r^2(12{,}300)}{(40 \times 0.03)} \\ &= \frac{22}{7}\left(\frac{5}{2}\right)^2\left(\frac{12{,}300}{1.2}\right) \\ &\approx 201339.29\,\mathrm{N/m} \end{aligned} \tag{26}$$

For rectangular cross-section

$$
\begin{aligned}
k &= \frac{AE}{L} \\
&= \frac{(12 \times 6 \times 0.03)(12{,}300)}{(40 \times 0.03)} \\
&\approx 22.14\,\mathrm{N/m}
\end{aligned} \tag{27}
$$

The bending stiffness of the wood circular and rectangular beam with length 40 inches, whose cross-section, assuming the hollows are not there, can be obtained by adopting Eq. (20) and taken the Young's modulus of Oak wood beam as 12,300 MPa:

For Circular cross-section

$$
\begin{aligned}
K &= \frac{3EI_x}{L^3} \\
&= \frac{3(12{,}300)(0.00002486)}{(40 \times 0.03)^3} \\
&\approx 0.53\,\mathrm{Nm}^2
\end{aligned} \tag{28}
$$

For Rectangular cross-section

$$
\begin{aligned}
K &= \frac{3EI_x}{L^3} \\
&= \frac{3(12{,}300)(0.004199)}{(40 \times 0.03)^3} \\
&\approx 89.67\,\mathrm{Nm}^2
\end{aligned} \tag{29}
$$

4 Results and Discussion

From the analysis result values, depicted in Table 1, it can be seen that the value of the

Table 1 The stiffness and moment of inertia for Oak wood beam with different cross-sections

S/N	Cross-section	I_{xc} (kg m^2)	I_{xr} (kg m^2)	I_{xch} (kg m^2)	I_{xrh} (kg m^2)	k (N/m)	K (Nm^2)
1	Circular hollow	–	–	0.0000008549	–	87.27	0.02
2	Rectangular hollow	–	–	–	0.0002222	8.3	4.74
3	Circular	0.00002486	–	–	–	201,339.29	0.53
4	Rectangular	–	0.004199	–	–	22.14	89.67

moment of inertia for both the rectangular cross-section (I_{xr}) and rectangular hollow cross-section (I_{xrh}) are greater than that of the circular cross-section (I_{xc}) and circular hollow cross-section (I_{xch}). This implies that the wood beam with a rectangular cross-section can resist rotational force than the wood beam with a circular cross-section. This also suggests that the stress in the wood circular beam is less than that in the wood rectangular beam, going by the formula for stress of a beam: The axial stiffness (k) value for both circular and circular hollow cross-sections is higher than that of rectangular and rectangular hollow cross-sections. This implies that more force is required to produce unit axial deformation in the wood circular beam than in the wood rectangular beam. However, the value of the bending stiffness (K) for both rectangular cross-section and rectangular hollow cross-section is higher than that of circular and circular hollow cross-sections. This implies that the former is more rigid than the later. It shows that the resistance of a member against bending deformation is higher in the wood rectangular beam than in wood circular beam. It is also observed that the moment of inertia of all the cross-sections considered in this paper is less than one ($I_{xch} < I_{xrh} < I_{xc} < I_{xr} < 1$). The wood beams without hollow cross-section seem to resist rotation than the ones with hollow cross-section.

5 Conclusion

This study set out to analyse mathematically the stiffness of orthotropic beam of different cross-sections. Wood, an example of orthotropic material, was considered. Also, Oak, a type of wood, with Young's modulus of 12,300 MPa, was considered. The moment of inertia, axial stiffness and bending stiffness for an Oak wood beam with different cross-sections were analysed. The special cross-sections considered are circular cross-section, rectangular cross-section, hollow circular cross-section and hollow rectangular cross-section. This study revealed that the type of cross-section of an Oak wood beam affects its stiffness. It was noticed among the types of cross-sections considered, that an Oak wood beam with rectangular cross-section has the highest bending stiffness, and the one with hollow circular cross-section has the least bending moment. Also, the highest moment of inertia goes to the Oak wood beam with rectangular cross-section, and the lowest goes to the beam with the hollow circular cross-section. Therefore this would help a structural engineer to make an informed decision on the type of wood beam to use in construction, as regards the cross-section, in order to minimise the risk of bending due to eternal forces.

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