Estimation from Censored Sample: Size-Biased Lomax Distribution

A. Naga Durgamamba and Kanti Sahu

Abstract In this work, the scale parameter is derived with famous shape parameter from the censored sample victimization of the maximum likelihood technique for the size-biased Lomax distribution (SBLD). The predicting equations are changed to urge less-complicated and economical predictors. Two ways of modification are steered. The results are given.

Keywords Likelihood function · Size-biased Lomax distribution · Order statistics · Censored sample

1 Introduction

The probability density and distribution functions of the SBLD is given by

$$f(t) = \frac{\lambda(\lambda-1)}{\theta}\frac{t}{\theta}\left(1+\frac{t}{\theta}\right)^{-(\lambda+1)} \quad \text{for} \quad t \geq 0, \lambda > 1, \theta > 0 \tag{1}$$

$$F(t) = 1 - \left(1+\frac{\lambda t}{\theta}\right)\left(1+\frac{t}{\theta}\right)^{-\lambda}; \quad t \geq 0, \lambda > 1, \theta > 0 \tag{2}$$

where λ and θ are shape and scale parameters of the distribution.

The maximum likelihood technique of estimation of the parameters λ and θ from censored sample is established once, whereas one of the parameters is thought and the opposite is unknown. Some modifications to maximum likelihood method from censored sample are suggested as we solve estimating equations by numerical iterative techniques. Such studies primarily based on modified maximum likelihood estimation which includes from [1–11].

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In this paper, we used two methods to estimate the scale parameter in size-biased Lomax distribution from censored sample. We present these in the following Sections. Since the method of estimation involves loads of numerical computations, all such results are given within the variety of numerical tables toward the tip of the paper with applicable identification and labels.

1.1 Modified ML Estimation of a Scale Parameter from Censored Samples

Censoring a given sample in life testing experiments now and then becomes essential to avoid delay and value of experimentation. One of the major, not unusual schemes of censoring may be a failure-censored sample, whereby prearranged n things are a place to checking out existence, and additionally, the test is terminated as rapidly as a predefined observation (say) 'r' is mentioned below ($r < n$). In such matters, we have a tendency to which are left with 'r' actual observations say $x_1 < x_2 < x_3 < \cdots < x_r$ and also the lifetimes of the remaining $(n - r)$ things are over x_r. This kind of sample is termed Type II right-censored sample.

Allow $x_1 < x_2 < x_3 < \cdots < x_r$ to be a Type II right-censored sample from a SBLD all through a deliberately stochastic pattern of length 'n'. The likelihood operation of the above-censored sample is

$$\prod_{i=1}^{r} f(t_i; \lambda, \theta).[1 - F(t_r; \lambda, \theta)]^{n-r} \tag{3}$$

here f (.) and F(.) severally represent the pdf and cdf of SBLD.

The log-likelihood operation to estimate θ from the given censored sample is given by

$$L \propto \left[\prod_{i=1}^{r} \frac{\lambda(\lambda - 1)}{\theta^2}.t_i.\left(1 + \frac{t_i}{\theta}\right)^{-(\lambda+1)}\right]\left[\left(1 + \frac{t_r}{\theta}\right)^{-\lambda}\left(1 + \frac{\lambda t_r}{\theta}\right)\right]^{n-r}$$

$$\begin{aligned}\text{Log } L &= \text{constant} - 2r\ \log\theta \\ &\quad - (\lambda + 1)\sum_{i=1}^{r}\log\left(1 + \frac{t_i}{\theta}\right) - \lambda(n - r)\log\left(1 + \frac{t_r}{\theta}\right) \\ &\quad + (n - r)\log\left(1 + \frac{\lambda \cdot t_r}{\theta}\right)\end{aligned}$$

where the constant is independent of the parameters to be estimated. Differentiating with respect to 'θ', we acquire the predicting equation for the parameter 'θ'

$$\frac{\partial \log L}{\partial \theta} = 0 \Rightarrow \frac{2r}{\theta} - \frac{\lambda+1}{\theta}\sum_{i=1}^{r}\frac{\frac{t_i}{\theta}}{1+\frac{t_i}{\theta}} - \frac{\lambda(n-r)}{\theta}\frac{\frac{t_r}{\theta}}{1+\frac{t_r}{\theta}} + \frac{n-r}{\theta}\frac{\lambda\frac{t_r}{\theta}}{1+\lambda\frac{t_r}{\theta}} = 0$$

where $Z_i = \frac{t_i}{\theta}$ and $Z_r = \frac{t_r}{\theta}$

$$2r - (\lambda+1)\sum_{i=1}^{r}\frac{Z_i}{1+Z_i} - \lambda(n-r)\frac{Z_r}{1+Z_r} + (n-r)\frac{\lambda Z_r}{1+\lambda Z_r} = 0 \tag{4}$$

It may be seen that Eq. (4) cannot be solved analytically for θ. The MLE of θ has got to be obtained as an associate in nursing repetitious answer of (4). We have a tendency to approximate the expression

$$h(Z_i) = \frac{Z_i}{1+Z_i},\ h(Z_r) = \frac{Z_r}{1+Z_r},\ h^*(Z_r) = \frac{\lambda z_r}{1+\lambda z_r}$$

of the likelihood Eq. (4) for estimating θ by a linear expression as

$$h(z_i) = \gamma_i + \delta_i z_i,\ h(Z_r) = \gamma_r + \delta_r z_r,\ h^*(Z_r) = \gamma_r^* + \delta_r^* z_r \tag{5}$$

where γ_i and δ_i, γ_r and δ_r and γ_r^* and δ_r^* are to be fitly found, to induce a changed MLE of θ. As per the constant quantity specifications, we tend to take $\lambda = 3$.

$$\frac{\partial \log l}{\partial \theta} = 0 \Rightarrow 2r - 4\sum_{i=1}^{r-1}(\gamma_i + \delta_i Z_i) - (4+3(n-r)) (\gamma_r + \delta_r Z_r) + (n-r)\left(\gamma_r^* + \delta_r^* Z_r\right) = 0$$

$$\hat{\theta} = \frac{4\sum_{i=1}^{r-1}\delta_i t_i + [4+3(n-r)]\delta_r t_r - (n-r)\delta_r^* t_r}{2r - 4\sum_{i=1}^{r-1}\gamma_i - [4+3(n-r)]\gamma_r + (n-r)\gamma_r^*} \tag{6}$$

The ensuing MMLE's of θ from the censored sample can be obtained by using two ways namely Tiku (1967) and [1]. We propose two ways for finding γ_i and δ_i, γ_r and δ_r and γ_r^* and δ_r^* of Eq. (5).

Method I

Let $P_r = \frac{r}{n+1},\ r = 1, 2, 3, \ldots n$ and $q_r = 1 - P_r$

Let Z_r^*, Z_r^{**} be the solutions of the following equations

$$F\left(Z_r^*\right) = p_r^* \text{ and } F\left(Z_r^{**}\right) = p_r^{**}$$

where $p_r^* = p_r - \sqrt{\frac{p_r q_r}{n}},\ p_r^{**} = p_r$

The solutions of Z_r^*, Z_r^{**} in our size-biased Lomax model are

$$F\left(Z_r^*\right) = P_r^* \Rightarrow Z_r^* = F^{-1}\left(P_r^*\right) \quad F\left(Z_r^{**}\right) = P_r^{**} \Rightarrow Z_r^{**} = F^{-1}\left(P_r^{**}\right)$$

$$h\left(Z_r^*\right) = \frac{Z_r^*}{1 + Z_r^*} \text{ and } h\left(Z_r^{**}\right) = \frac{Z_r^{**}}{1 + Z_r^{**}}$$

$$h^*\left(Z_r^*\right) = \frac{3Z_r^*}{1 + 3Z_r^*} \text{ and } h^*\left(Z_r^{**}\right) = \frac{3Z_r^{**}}{1 + 3Z_r^{**}}$$

The intercepts γ_r, γ_r^* and slopes δ_r, δ_r^* of Eq. (5) are given below

$$\delta_r = \frac{h\left(Z_r^{**}\right) - h\left(Z_r^*\right)}{Z_r^{**} - Z_r^*} \text{ and } \gamma_r = h\left(Z_r^*\right) - \delta_r Z_r^* \tag{7}$$

$$\delta_r^* = \frac{h^*\left(Z_r^{**}\right) - h^*\left(Z_r^*\right)}{Z_r^{**} - Z_r^*} \text{ and } \gamma_r^* = h^*\left(Z_r^*\right) - \delta_r^* Z_r^* \tag{8}$$

Method II

Consider Taylor's expansion of $h(\xi_r) = \frac{\xi_r}{1+\xi_r}$ within the neighborhood of the rth quantile of SBLD. We have a tendency to get another linear approximation for $h(\xi_r)$ which is given by

$$h(\xi_r) = \gamma_r + \delta_r \xi_r \tag{9}$$

where $\delta_r = h^1(\xi_r)$ with ξ_r as the rth quantile for the population is given by

$$F(\xi_r) = P_r \Rightarrow \xi_r = F^{-1}(P_r), \quad P_r = \frac{r}{n+1}\ \gamma_r = h(\xi_r) - \delta_r \xi_r$$

$$h^*(\xi_r) = \frac{3\xi_r}{1 + 3\xi_r} \quad h^*(\xi_r) = \gamma_r^* + \delta_r^* \xi_r$$

where

$$\delta_r^* = h^{*^1}(\xi_r)\ \gamma_r^* = h^*(\xi_r) - \delta_r^* \xi_r \tag{10}$$

The relevant values of slopes and intercepts are calculated for $n = 5$, 10, 15 and 20 for 20% censored sample of r.

In the two ways referred higher than the fundamental principle, the expressions $\frac{Z_i}{1+Z_i}$, $\frac{Z_r}{1+Z_r}$ and $\frac{\lambda Z_r}{1+\lambda Z_r}$ are approximated by several linear functions in some neighborhood of the population. It may be seen that the development of the neighborhood over a definite performance is linearized based on the sample size additionally. Larger the size nearer is going to be the approximation, that is, the accuracy of the approximation becomes finer and finer for big values of n. Hence, the approximate log-likelihood equation and also the precise log-likelihood equation vary by tiny quantities for big n. Therefore, the solutions of the precise equation and approximate log-likelihood equations tend to every different as $n \to \infty$. Hence, the precise and changed MLEs are asymptotically identical (Tiku et al. 1986). However, identical cannot be aforementioned in little samples. At the identical time, the little sample variance of changed MLE is not mathematically tractable. We, therefore, compared these estimates in little samples through Monte Carlo simulation.

For mounted worth of $\lambda = 3$, the values of δ_r, γ_r and δ_r^*, γ_r^* for techniques I and II are bestowed within Tables 1 and 2. The bias, variance, and MSE of the estimates by the two ways of modification are obtained through simulation for $n = 5$ (5) 20 with all attainable issues of censored samples are given in Table 3. The subsequent conclusions are drawn from the simulated sampling characteristics supported 1000 Monte Carlo simulation runs.

2 Conclusion

The variances of modified MLEs from censored samples are determined to be over the analogous variances of modified MLEs from the complete sample as set up by means of the relationships among Tables 3 and 4. Because of censored samples, there is a loss of information leading to fallen efficiency and therefore raising variance.

Table 1 Slope ($\delta_{r,}$, δ_r^*) and intercept (γ_r, γ_r^*) in modification to MLE by technique I

r	n	$\lambda = 3$				r	n	$\lambda = 3$			
		$\delta_{r,}$	γ_r	δ_r^*	γ_r^*			$\delta_{r,}$	γ_r	δ_r^*	γ_r^*
1	5	0.454	0.1	0.506	0.335	11	15	0.128	0.392	0.075	0.692
2	5	0.368	0.116	0.373	0.352	12	15	0.095	0.455	0.052	0.739
3	5	0.226	0.226	0.174	0.522	13	15	0.064	0.528	0.032	0.787
4	5	0.116	0.368	0.07	0.665	14	15	0.035	0.62	0.016	0.839
1	10	0.593	0.051	0.839	0.215	1	20	0.703	0.025	1.207	0.13
2	10	0.54	0.055	0.72	0.218	2	20	0.669	0.026	1.104	0.129
3	10	0.431	0.101	0.471	0.33	3	20	0.588	0.047	0.838	0.201
4	10	0.344	0.151	0.321	0.423	4	20	0.523	0.069	0.663	0.262
5	10	0.271	0.208	0.22	0.505	5	20	0.467	0.092	0.535	0.317
6	10	0.208	0.271	0.148	0.58	6	20	0.418	0.116	0.438	0.366
7	10	0.151	0.344	0.095	0.651	7	20	0.374	0.142	0.361	0.412
8	10	0.101	0.431	0.056	0.72	8	20	0.333	0.169	0.298	0.455
9	10	0.055	0.54	0.027	0.793	9	20	0.296	0.198	0.247	0.495
1	15	0.66	0.034	1.053	0.161	10	20	0.261	0.228	0.203	0.534
2	15	0.62	0.035	0.943	0.162	11	20	0.228	0.261	0.167	0.572
3	15	0.528	0.064	0.68	0.249	12	20	0.198	0.296	0.136	0.608
4	15	0.455	0.095	0.513	0.321	13	20	0.169	0.333	0.109	0.643
5	15	0.392	0.128	0.394	0.386	14	20	0.142	0.374	0.086	0.678
6	15	0.337	0.163	0.306	0.444	15	20	0.116	0.418	0.066	0.713
7	15	0.287	0.201	0.237	0.499	16	20	0.092	0.467	0.049	0.748

(continued)

Table 1 (continued)

r	n	$\lambda = 3$				r	n	$\lambda = 3$			
		$\delta_{r,}$	γ_r	δ_r^*	γ_r^*			$\delta_{r,}$	γ_r	δ_r^*	γ_r^*
8	15	0.242	0.242	0.183	0.55	17	20	0.069	0.523	0.035	0.785
9	15	0.201	0.287	0.14	0.599	18	20	0.047	0.588	0.022	0.823
10	15	0.163	0.337	0.104	0.646	19	20	0.026	0.669	0.011	0.865

Table 2 Slope (δ_r, δ_r^*) and intercept (γ_r, γ_r^*) in modification to MLE by technique II

r	n	$\lambda = 3$				r	n	$\lambda = 3$			
		$\delta_{r,}$	γ_r	δ_r^*	γ_r^*			$\delta_{r,}$	γ_r	δ_r^*	γ_r^*
1	5	0.549	0.067	0.714	0.262	11	15	0.207	0.297	0.142	0.612
2	5	0.611	0.048	0.887	0.208	12	15	0.166	0.352	0.104	0.663
3	5	0.428	0.12	0.448	0.376	13	15	0.126	0.416	0.072	0.714
4	5	0.28	0.221	0.223	0.529	14	15	0.087	0.496	0.045	0.77
1	10	0.663	0.035	1.056	0.165	1	20	0.754	0.017	1.415	0.098
2	10	0.725	0.022	1.291	0.118	2	20	0.805	0.011	1.66	0.066
3	10	0.597	0.052	0.845	0.22	3	20	0.714	0.024	1.248	0.126
4	10	0.495	0.088	0.585	0.312	4	20	0.642	0.039	0.987	0.182
5	10	0.408	0.13	0.413	0.396	5	20	0.581	0.057	0.8	0.234
6	10	0.331	0.18	0.29	0.475	6	20	0.526	0.075	0.658	0.283
7	10	0.261	0.24	0.2	0.551	7	20	0.477	0.096	0.546	0.329
8	10	0.196	0.311	0.131	0.626	8	20	0.432	0.118	0.455	0.373
9	10	0.134	0.402	0.078	0.703	9	20	0.389	0.141	0.38	0.415
1	15	0.719	0.023	1.267	0.123	10	20	0.35	0.167	0.318	0.455
2	15	0.775	0.014	1.513	0.084	11	20	0.312	0.195	0.264	0.494
3	15	0.67	0.033	1.083	0.159	12	20	0.277	0.225	0.219	0.533
4	15	0.587	0.055	0.819	0.228	13	20	0.243	0.257	0.18	0.57
5	15	0.517	0.079	0.635	0.292	14	20	0.211	0.293	0.146	0.608
6	15	0.454	0.107	0.498	0.351	15	20	0.18	0.332	0.116	0.645
7	15	0.397	0.137	0.393	0.407	16	20	0.15	0.376	0.091	0.683

(continued)

Table 2 (continued)

r	n	$\lambda = 3$				r	n	$\lambda = 3$			
		$\delta_{r,}$	γ_r	δ_r^*	γ_r^*			$\delta_{r,}$	γ_r	δ_r^*	γ_r^*
8	15	0.345	0.171	0.31	0.461	17	20	0.121	0.426	0.068	0.721
9	15	0.296	0.208	0.243	0.512	18	20	0.092	0.484	0.049	0.762
10	15	0.25	0.25	0.188	0.563	19	20	0.065	0.556	0.031	0.807

Table 3 Sample characteristics of MMLE of θ from expurgated samples' techniques I and II

λ	n	Bias		Variance	MSE	
		MMLE-I	MMLE-II	MMLE-II	MMLE-I	MMLE-II
3	5	0.101	0.094	0.6858	0.705	0.6946
3	10	0.041	0.071	0.418	0.3917	0.4231
3	15	0.006	0.042	0.2975	0.2722	0.2992
3	20	0.021	0.053	0.1576	0.1487	0.1604

Table 4 Sample characteristics of MMLE of θ from expurgated samples' technique I and technique II

λ	n	Bias		Variance	MSE	
		MMLE-I	MMLE-II	MMLE-II	MMLE-I	MMLE-II
3	5	0.257	0.594	1.1352	0.8279	1.4876
3	10	0.089	0.297	0.3963	0.3007	0.4843
3	15	0.045	0.199	0.18	0.1379	0.2195
3	20	0.033	0.154	0.1058	0.0837	0.1295

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