Coupled Chaotic Systems and Extreme Ecologic-Economic Outcomes

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Abstract In sympathy with work of Akio Matsumoto, this essay reviews models that consider how the coupling of systems within ecologic-economic contexts can generate not only chaotic dynamics, but lead to outcomes that exhibit kurototic outcomes rather than reflecting Gaussian distributions. This aligns with arguments made by Martin Weitzmann regarding the global climate system. The models considered included one where climate and economic systems are separately non-chaotic but chaotic when combined and another where the economic system is chaotic and when combined with climate generates kurtotic outcomes through flare attractors. Likewise, similarly coupled models involving fisheries and forestry dynamics are considered where coupling leads to chaotic dynamics. Multi-level systems with such dynamics are then considered with the governance issues involved with such systems are examined.

1 Introduction

Akio Matsumoto has long studied coupled dynamical systems exhibiting various forms of complex dynamics, often involving lags (Matsumoto 1997, 1999; Matsumoto and Szidarovszky 2015). In addition, he has had an interest in implications of such models connecting economics with environmental problems (Matsumoto et al. 2018; Ishikawa et al. 2019). A theme of his work on these topics has indeed been that both coupling and lags tend to increase the complexities arising from such systems. This might appear to run counter to another theme of his work, that sometimes chaotic dynamics "can be beneficial" (Matsumoto 2001, 2003). However, those models involved one-dimensional systems of price dynamics without coupling or lags or other complications that could undermine their relatively sunny outcomes. Nevertheless, this insight of Matsumoto's that chaotic dynamics are not necessarily "bad" has not been fully appreciated.

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In appreciation of these themes of Matsumoto's we shall consider how coupled ecologic and economic chaotic systems can generate extreme events, kurtotic "fat tails." While there are various such possible applications, including to fisheries and forests, arguably the most important involves global warming, a more accurate term that this observer prefers to the more anodyne and widely used "climate change." While most of the models underlying official IPCC reports have assumed Gaussian distributions of outcomes, Martin Weitzman (2009, 2011, 2012, 2014) has argued that underlying nonlinear dynamics of the global climate system in interaction with the global economic system is subject to power law or other distributions that exhibit kurtosis and thus a higher probability of extreme outcomes than appearing in the more conventional models. Indeed, Lorenz (1963) first identified a strange attractor associated with sensitive dependence on initial conditions in a chaotic model of climate dynamics. It is thus completely appropriate to consider how such models can bring about these outcomes that Weitzman considered to be so important.

This raises the question of how policy should be carried out in the face of such phenomena, especially as this happens in the context of complications such as the hierarchical complexity of ecologic-economic systems and the bounded rationality of policy makers (Rosser and Rosser 2006, 2015). Such analysis is deeply in synch with the spirit and tradition of the work of Akio Matsumoto.

2 A Coupled Climate-Economy Model

As already noted, Lorenz (1963) modeled climate dynamics as being chaotic, although that term was not yet in use at that time. However, the chaotic nature of climate dynamics is widely accepted, with the "butterfly effect" of sensitive dependence on initial conditions being widely viewed as a reason why weather forecasting has only a fairly short range of reliability, even though longer term averages and trends may be forecasted.

While many theoretical models of chaotic economic dynamics have been proposed (Rosser 2011, Appendix A), solid empirical verification of such dynamics in economic systems has been lacking, although a variety of complex nonlinear dynamics have been accepted as happening in economic systems. However, as studied in Rosser (2002) two systems that by themselves may not exhibit chaotic dynamics can do so when coupled together. This draws on work of Chen (1997), which draws on simple underlying sub-systems.

This simple system has two sectors in its economic part, agricultural and manufacturing. These sectors are each related to global average temperature, T. For agriculture, temperature is a negative input. For manufacturing, it is a positive input to global average temperature. Each sub-system is very simple, but the coupled system can show chaotic dynamics.

On the economic side demand is given by a CES utility function of agriculture, A, and manufacturing, M.

$$U(A,\ M) = (A^{\rho} + M^{\rho})^{1/\rho}. \tag{1}$$

We are assuming equilibrium on the economic side so that consumption of each good equals its output. This gives the elasticity of substitution as is standard for CES functions to be

$$\sigma = 1/(1-\rho) < 1. \tag{2}$$

Both production functions are linear in labor, L, with total labor normalized to unity, so that

$$L(A) + L(M) = 1. \tag{3}$$

Besides a positive constant and the labor input, agricultural production also includes a negative quadratic term for global average temperature, so that

$$A = (-\alpha T^2 + \beta T + 1)L(A). \tag{4}$$

Manufacturing output is given by

$$M = bL(M). \tag{5}$$

This generates a market clearing manufacturing price of

$$P = (-\alpha T^2 + \beta T + 1)/b. \tag{6}$$

The climate model draws on one due to Henderson-Sellers and McGuffie (1987). This now involves dynamics with time subscripts as temperature in a succeeding time period that is determined by the temperature in the current one along with a long-run normal temperature, $T_{n,}$ as well as a positive linear function of manufacturing output. With c in the unit interval and $g > 0$, this is given by

$$T_{t+1} = (1-c)(T_t - T_n) + T_n + gM_t. \tag{7}$$

Combining with the economic sub-system generates an equilibrium motion for global temperature that is given by

$$T_{t+1} = (1-c)T_t + g(bp_t^{1-\sigma})/(1 + p_t^{1-\sigma}). \tag{8}$$

Chen simulated this model setting $\sigma = 0.5$, $\alpha = 8$, $\beta = 7$, $b = 1$, and $g = 0.6$. The climatic tuning parameter, c, for this set of other parameter values, generates a unique and stable steady state for values in (0.233, 1). As c declines below 0.233, period-doubling bifurcations appear, and aperiodic chaotic dynamics appear after it goes below $c = 0.209$. The system also exhibits sensitive dependence on initial conditions ("butterfly effect") below this level as well.

3 Flare Attractors and Extreme Ecologic-Economic Outcomes

A related model that can bring about an outcome of a combined ecologic-economic system with a chaotic driver, if differing in important details from the model in the preceding section, involves *flare attractors*. These are key to the not-fully developed *econochemistry* concept. Initially conceived by Otto Rössler and Georg Hartmannn (1995) to study solar flares and various autocatalytic chemical reactions, they came to be applied to economics as well, initially for entrepreneurial activities (Hartmann and Rössler 1998) and then for asset price volatility (Rosser et al. 2003).

This approach differs from that in the previous section by having the underlying fundamental process being chaotic rather than becoming chaotic as a result of the coupling aspect. In the case of this model the "flaring" kurtotic outcomes, sudden bursts coming almost from nowhere, are the result of the coupled second layer deriving from the underlying driving chaotic process. This also involves an introduction of heterogeneous agents into the system. Ironically as one moves from the original model of solar flares to the model of climatic outbursts of extreme temperatures, we see a return to an original physical chemistry application after passing through an economics application that explored financial market dynamics.

The underlying mathematics of this model were developed by Rössler et al. (1995). The attractors involved are extensions of the continuous chaotic attractor model of Rössler (1976) as special cases that are continuous-but-nowhere-differentiable and also exhibit "riddled basins." The full explication of such attractors is due to Milnor (1985).

Here we shall extend this model to an application not previously made, to the problem of global warming, or more generally, extreme outcomes of climate change. The previous model due to Chen (1997), had the ecologic-economic interactions more direct, which arguably reflects a longer run perspective. Here we shall focus more on a shorter-term perspective of economic-to-climate interactions. The coupling aspect involves the second-tier aspect of heterogeneous agents responding to the underlying economic model already assumes an environmental limit on economic growth. This limit is not connected to the higher level global warming issue, but a narrower limit more locally determined. The model is one of the earliest chaotic economic models due to Day (1982). His model involves a logistic equation, which relates to the original model of chaotic application. This was due to May (1976). Such a model depends on a hard upper limit of growth along with a lower bound.

The underlying economic model, due to Day (1982) is a modified Solow growth model. It has the labor exponent as α, and β the capital exponent, y being per capita output, λ being the population growth rate, with m being a "capital-congestion" saturation coefficient, which ultimately drives the logistic formulation that has an upper limit, and which resembles the model of May (1976). The modified production function is given by

$$f(k) = \beta k^{\beta}(m-k)^{y}. \tag{9}$$

Assuming a constant savings rate, the capital-labor ratio implies the following difference growth equation,

$$k_{t+1} = \alpha\beta k_t^{\beta}(m - k_t)^{y}/(1 + \lambda). \tag{10}$$

This formulation coincides with that of May (1976), who made clear the parameter values of this model for which chaotic dynamics will occur. Rosser et al. (2003, p. 80) assumed that

$$A\beta/(1 + \lambda) = 3.99 = k_{t+1}/(1 - k_t). \tag{11}$$

This formulation provides a chaotic dynamical process as k changes. This process assumes that the capital share remains constant.

Earlier literature has posited at this point that the specification of heterogeneous agents involved human agents responding differently to the underlying system. For this case we follow Hartmann and Rössler (1998) for giving a general form of the agent reaction function. The difference between this formulation and earlier work by these authors in physical chemistry is that while here the agents are heterogeneous individuals or organizations, in this case implicitly the agents are nations or regions of the world subject to climatic variation.

What goes on here is that we have a set of locations that have a varying relation with the exogenous chaotic driving force. In particular there will be a switching value of a, a function of k, beyond which there will be a substantial increase in temperature. Whereas in the asset model of Rosser et al. (2003) these agent reaction functions represent behavior of human agents, including human organizations, in this case these represent locations on the planet with their respective situations that imply heterogeneous behavior. The appearance of an outburst reflects a sufficient number of these agents/locations crossing their critical value of $1 > a > 0$.

The general form of the reaction function for an agent/location of I type out of n, assuming agents/locations, and $c > 0$, and will be given by

$$B_{t+1}^{I} = b_t^{I} + b_t^{I}\left(a^{I} - k_t^{I}\right) - cb_t^{(I)2} + cs_t. \tag{12}$$

The first term in (12) is an autoregressive component. The second is the switching term. The third provides a stabilizing component. The fourth is the destabilizing element coming from the buildup of previous trends, representing the ongoing overall state of the system determined by overall demand s, and is given by

$$S_{t+1} = b_t^1 + b_t^2 + \cdots + b_t^n. \tag{13}$$

In Rosser et al. (2003), assuming certain values of the parameters allows for a simulation that provides a sequence of outcomes that exhibit scattered kurtotic outbursts consistent with the Weitzman scenario for global warming.

4 Coupled Chaotic Dynamics in Renewable Resource Markets

While we have seen that coupled chaotic dynamics can happen at the global level scale of the climate-economy system, such coupled chaotic dynamics can also happen in lower level ecologic-economic systems. Two examples are in fisheries and also in forestry, although the former have been modeled more clearly, with Conklin and Kohlberg (1994) initially showing the possibility for chaotic dynamics within a fishery in a non-optimizing setting. Central to such dynamics in these systems is when supply curves bend backwards, a result first suggested for fisheries without a formal model by Copes (1970). More complex dynamics for fisheries than those presented below are presented in Foroni et al. (2003).

Hommes and Rosser (2001) have demonstrated the possibility of this for fisheries in what they label a "Gordon-Schaefer-Clark" model of an optimally managed fishery. This assumes on the ecological side a Schaefer (1957) yield function, $f(x)$, with x the fish biomass, which in equilibrium will also be the harvest function, $h(x)$, with r the natural growth rate of the fish and k the carrying capacity of the fishery is given by

$$h(x) = f(x) = rx(1 - x/k). \tag{14}$$

Following Clark (1985) the economic side is given by an effort function linear in time fishing, E, with costs $C(E)$, without fixed costs, constant marginal costs, c, and a catchability coefficient, q, with p the price of fish, and R the rent, output Y is given by

$$Y = qEx = h(x). \tag{15}$$

This implies that rent which the present value of which is to be maximized is

$$R(Y) = pqEx - cE. \tag{16}$$

In the optimization non-equilibrium must be allowed where harvest may not equal the yield function. Solving the intertemporal optimal control problem with non-negativity constraints on x and h and a constant discount rate, δ, (Hommes and Rosser 2001) leads to

$$f(x) = \delta = [cf(x)]/(p - c). \tag{19}$$

From this optimal discounted supply curve is given by

$$x(p, \delta) = k/4[1 + (c/pqk) - (\delta/r) + (1 + c/pqk - (\delta/r)^2 + 8c\delta/pqkr)^{1/2}]. \tag{20}$$

The crucial variable determining system dynamics is the discount rate, δ. At zero with no discounting of future rents, the supply curve slopes upwards, but as it increases beyond about 0.02, the supply curve bends backwards, allowing for catastrophic collapses of the fishery. As it goes to infinity implying not counting the future at all, the curve bends backwards the most and becomes identical to that for an open access fishery subject to "tragedy" (Gordon 1954), the problem dealt with by Ostrom (1990) and others. The open access supply curve is given by

$$x(p, \infty) = (rc/pq)(1-c/pqk). \tag{21}$$

Assuming lags in behavior by fishers turns this into a form of a cobweb model. Somewhat similarly to the finding of Matsumoto (1997), Hommes and Rosser (2001) show for an appropriate demand curve and for intermediate values of the discount rate, chaotic dynamics can emerge in this coupled fishery system.

While no one has shown specifically chaotic dynamics in a forest-harvesting model, under certain situations an optimally managed forest can also exhibit backward-bending supply curves for sufficiently high discount rates. This was first proposed by Hyde (1980) with empirical support for backward-bending forestry supply curves found in the Amazon rain forest for certain circumstances (Amacher et al. 2009). Drawing on Colin Clark's fishery model, Binkley (1986) developed a model that formally showed how such a backward-bending supply curve could arise in a forestry model, with this further studied by Rosser (2013). These models are all for a single output, timber from cut trees, with Binkley finding tentative empirical support for the long run supply of loblolly pines in the southeastern US. The basic canonical optimal forestry management model accounting for multiple uses and infinite time horizon is given by Hartman (1976).

Letting most variables be identical to the above fishery model, the main new variable that appears in the system is T, the optimal rotation age for the forest, the time that trees should be cut and then replanting of them occurs. This T depends on the discount rate and also p, the price of timber, and unlike the fishery, the yield function is a function of time since the last replanting, $f(t)$, with the growth at optimal rotation age given by $f(T(p))$. From all this an optimal inverse supply function for p as a function of T and δ is given by

$$p = c/[f(T) - f'(t)(1 - e^{-\delta t})/\delta)]. \tag{22}$$

This is consistent with the possibility of a backward-bending supply curve for certain parameter values. Binkley (1986, p. 173) provides an intuitive explanation of what is happening in such situations.

> High stumpage prices imply not only that the output from the forest has high value, but also that the capital in the form of growing stock has a high opportunity cost. At high prices, it is optimal to conserve on the use of capital and therefore to reduce the stock inventory by reducing the rotation age.

While it has not been shown explicitly that this model can generate chaotic dynamics, I am reasonably certain that with appropriate lags for forester behavior, such will occur for certain situations. I close this discussion by observing that chaotic dynamics have been found for a variety of both harvested biological populations (Sakai 2001) as well as non-harvested ones (Zimmer 1999; Turchin 2003; Solé and Bascompte 2006).

5 Policy in Complex Multi-level Hierarchies with Bounded Rationality

The difficulty of managing such dynamically complex coupled systems is complicated when they exist within hierarchical ecologic-economic contexts (Radner 1992; Rosser 1995, 2001). This complexity enforces the necessary reliance on bounded rationality as posed by Simon (1957, 1962; Rosser and Rosser 2015). It also involves positing the appropriate level of the system as the locus of such policymaking in order to overcome the difficulties of common property resources that arise in such situations (Netting 1976; Ostrom 1990; Bromley 1991; Rosser and Rosser 2006; Rosser 2016).

While Simon (1962) formalized the discussion of hierarchy in complex systems, his arguments for dynamical systems were prefigured in general systems theory (von Bertalanffy 1962) and its predecessor, tektology (Bogdanov 1925–1929). These have been more fully generalized for ecological systems by Holling (1992). A deep issue is the relation between higher and lower levels of such systems. While it is generally argued that higher levels dominate or at least constrain lower levels (Radner 1992), it may be possible for changes in lower levels to lead to changes in higher levels, or even the complex emergence of higher levels through hypercyclic morphogenesis (Rosser 1991).

We can consider such systems that allow for ultimately flexible relations with both fast and slow dynamics in the formalization of synergetics as developed by Haken (1977). Let there be a well-defined hierarchy with n levels. Higher levels constrain more rapidly oscillating lower levels under normal conditions. Thus fast dynamics operate at lower levels and slow dynamics operate at higher levels.

At a given level let q be the vector of fast variable dynamics and F the vector of slow variable dynamics, with A, B, and C being matrices and $\varepsilon(t)$ be an i.i.d. stochastic fluctuations term. Then the fast dynamics are given by

$$dq/dt = Aq + B(F)q + C(F) + \varepsilon(t). \tag{23}$$

Haken argues that such a system can be simplified by rearranging this equation in order to exhibit adiabatic approximation in which the fast dynamics are shown to depend solely upon the slow dynamics based on order parameters in F. This is given by

$$dq/dt = -(A + B(F))^{-1}C(F). \tag{24}$$

The order parameters are the variables in F and can be ranked in inverse order of the absolute values of the variables in $A + B(F)$. Curiously these order parameters are unstable in the sense that they possess positive real parts of their eigenvalues. Other variables are the slaved variables and have negative real parts of their eigenvalues.

Structural changes in the sense of Holling can come from either the bottom or the top in such a system. Bottom-up changes can come about through a slaved variable destabilizing by having the real part of its eigenvalue going positive in a process known as "the revolt of the slaved variables" (Diener and Poston 1984). Haken saw this as a key to the emergence of chaotic dynamics in a structured system. An example might be the outbreak of the Great Plague in Europe in the mid-14th century as an accumulation of malnutrition weakening population immune systems reached a critical mass such that the plague could sweep through the population (Braudel 1967).

The top-down mechanism can happen through the emergence of a new constraining higher level of the system, such as the emergence of a city in an urban hierarchy much larger than previous ones that dominates them through the appearance of new economic activities (Rosser 1994). The mechanisms for such anagenetic moments of hypercyclic morphogenesis can arise from frequency entrainment as modeled by Nicolis (1986). Another way may be through the appearance of cooperative forms leading to multi-level evolution in an evolutionary process (Crow 1955).

The policy problem must confront this hierarchical complexity. This is an issue that Ostrom (1990) and others have tried to confront. A clear outcome is that governance should operate at the most crucial level that determines the crucial dynamics of the system. In light of the analysis above of synergetic systems, it may not always be obvious what that level is (Wilson et al. 1999). A system apparently dominated by the highest level may actually be dependent on dynamics at the bottom and vice versa. More generally, focusing policy on an ecologic-economic hierarchy level that is not crucial to the system dynamics can lead to worse outcomes than doing nothing.

Indeed, for the most difficult problems the complex links mean that actions may need to be taken at several levels. This would seem to be especially the case for the global climate system, where in fact given the coupled nonlinear dynamics involved it would seem that multiple levels are involved. Global agreements are necessary for setting overall goals. But individual nations must set goals and establish specific policies. But many of these policies end up being carried out at lower levels. Likewise it is not just the political and economic elements that have this multi-level aspect, but also the ecological and climatological. The ecologic-economic system functions at levels ranging from almost minutely local to the totally global.

A further complication due to the complexities associated especially with chaotic dynamics is that when a system is decomposed from the global to the regional level, it may be subject to severe effects due to sensitive dependence on initial conditions. Thus Massetti and Di Lorenzo (2019) have considered in detail the regional level forecasts from simulations of global level climate models used by the United Nations IPCC for projecting possible future climate outcomes. In particular they ran simulations slightly varying initial starting values for certain variables and indeed found substantial sensitive dependence for regional level projections. Thus for the west-central portion of the United States some projections would have substantial warming while others actually found cooling happening even as the global average was for warming, again for starting values only slightly apart, thus replicating the old result found by Lorenz (1963) for climate models. Needless to say, this seriously complicates knowing what to do at more local levels for such situations.

These multi-layered complexities involve deep uncertainties about all the matters noted above and more. These include ongoing debates about underlying science issues, as well as the full nature of the interactions between the economic and climatological aspects. That the elements of this involve chaotic dynamics subject to sensitive dependence on initial conditions makes the whole matter that much more difficult to understand. All this leads to the inability of any observer or agent to reliably understand in full detail how it works. This means that inevitably bounded rationality is the best that can be hoped for to be used in analyzing such a system.

6 Conclusions

The coupled global ecologic-economic system deeply involves chaotic dynamics. This means the system is subject to sensitive dependence on initial conditions. Also it may be subject to flare phenomena. These involve kurtotic outbursts that increase the dangers involved in understanding the system and increase the risks involved in the analysis. These issues extend to other kinds of coupled dynamical ecologic-economic systems such as those involving fisheries and forests. As a multi-layered complex system, where management must apply at the appropriate level, decision makers are limited to bounded rationality in dealing with it. The ideas involved in these matters are deeply linked to ideas that Akio Matsumoto has studied in his lifetime of research.

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