Adaptive Synchronization of Nonlinear Complex Dynamical Networks with Time-Delays and Sampled-Data

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Abstract. This paper investigates the adaptive synchronization of nonlinear complex dynamical networks with time-delays and sampled-data by proposing a new adaptive strategy to coupling strengths and feedback gains. According to Lyapunov theorem, it is testified that the agents of sub-groups can converge those synchronous states respectively under some special conditions. In addition, some simulations are proposed to illustrate the theoretical results.

Keywords: Complex dynamical networks · Adaptive synchronization · Time-delays · Sampled-data

1 Introduction

Synchronization is a ubiquitous phenomenon in nature, such as the consistency of fireflies twinkling, Synchronized chirping of crickets and Synchronization of beating rhythm of cardiac myocytes. In recent years, the synchronization problems of nonlinear complex dynamical networks have attracted great attention and emerged a good deal of excellent works [1–4].

Synchronous methods of complex networks have emerged as the times require, and one of the most significant methods is to design advisable adaptive strategies for the relevant parameters, such as the coupling strengths and the feedback gains [3–8]. In [5], adaptive synchronization of complex dynamical networks was studied. Liu et al. [7] studied the adaptive synchronization of complex dynamical networks governed by local Lipschitz nonlinearlity on switching topology. For nonlinear complex dynamics networks, information interaction between agents can be considered as sampled information [9–12]. In [9], the author gave the

This work was supported in part by the National Natural Science Foundation of China Grant (No. 61773023), 2018 Beijing Education Commission Basic Science Research Expenses Project, "The-Great-Wall-Scholar" Candidate Training Plan of NCUT (XN070006), Construction Plan for Innovative Research Team of NCUT (XN018010), "Yu Xiu" Talents of NCUT.

L. Pan et al. (Eds.): BIC-TA 2019, CCIS 1160, pp. 43–57, 2020.
https://doi.org/10.1007/978-981-15-3415-7_4

necessary and sufficient conditions for solving consensus problems of double-integrator dynamics via sampled control and group synchronization of nonlinear complex dynamics networks with sampled data was investigated in [11]. However, time-delay is widespread in communication information among agents in real life [1–3,5,10]. Sampled-data based consensus of continuous-time multi-agent systems with time-varying topology was studied in [10]. The authors divided the whole group into some sub-groups to research those synchronization, that is group synchronization [11–14]. In [13], the authors considered the group synchronization of complex network with nonlinear dynamics via pinning control.

Inspired by these literatures, we will consider the adaptive synchronization of nonlinear complex dynamical network with time-delays and sampled-data in this paper. The contribution of this paper are twofold. We first design effective adaptive strategies for the coupling strengths and the feedback gains, and present a stability analysis of adaptive synchronization of networks with time-delays and sampled-data. The second contribution is that the influence of adaptive strategies, time-delay and coupling on synchronization of nonlinear complex dynamical networks are considered.

An outline of this paper is organized as follows. Section 2 declares the model of nonlinear complex dynamics network and gives some preliminaries. In Sect. 3, we study the adaptive synchronization of nonlinear complex dynamics network with time-delays and sampled-data. The simulation results are presented in Sect. 4. Finally, Sect. 5 concludes this paper.

2 Model and Preliminaries

Consider a complex dynamical network of $N + M$ nodes with time-delays and sampled-data described by:

$$\dot{x}_i(t) = \begin{cases} f(x_i(t_k), x_i(t_k - \tau(t_k))) + \sum\limits_{j\in\mathscr{N}_{1i}} c_{ij}(t_k)a_{ij}(x_j(t_k - \tau(t_k)) - x_i(t_k - \tau(t_k))) \\ \quad + \sum\limits_{j\in\mathscr{N}_{2i}} d_{ij}(t_k)b_{ij}x_j(t_k - \tau(t_k)) + \mu_i, \quad \forall i \in \ell_1, \quad \forall t \in [t_k, t_{k+1}] \\ f(x_i(t_k), x_i(t_k - \tau(t_k))) + \sum\limits_{j\in\mathscr{N}_{2i}} c_{ij}(t_k)a_{ij}(x_j(t_k - \tau(t_k)) - x_i(t_k - \tau(t_k))) \\ \quad + \sum\limits_{j\in\mathscr{N}_{1i}} d_{ij}(t_k)b_{ij}x_j(t_k - \tau(t_k)) + \mu_i, \quad \forall i \in \ell_2, \quad \forall t \in [t_k, t_{k+1}] \end{cases}, \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ denotes position vectors of the node i at time t, $f(\cdot) \in R^n$ describes the intrinsic dynamics of network and it is continuously differentiable, $c_{ij}(t_k), d_{ij}(t_k)$ represent the coupling strengths, and $\tau(t_k)$ denotes time-varying delays in transmission process. In this network, $\ell_1 = 1, 2, \cdots, N$, $\ell_2 = N+1, N+2, \cdots, N+M$, and $X_1 = \{x_i | i \in \ell_1\}$, $X_2 = \{x_i | i \in \ell_2\}$. $\mathscr{N}_i$ is the neighbor of node i, $\mathscr{N}_i \in \mathscr{N}_{1i} \cup \mathscr{N}_{2i}$, where $\mathscr{N}_{1i} \bigcap \mathscr{N}_{2i} = \emptyset$, $\mathscr{N}_{1i} = \{x_j \in X_1 | a_{ij} > 0, i, j \in \ell_1\}$, $\mathscr{N}_{2i} = \{x_j \in X_2 | a_{ij} > 0, i, j \in \ell_2\}$. If node i can get information from node j in the same group, then $a_{ij} > 0$; otherwise $a_{ij} = 0$; If node i can get information from node j between different groups, then

$b_{ij} \neq 0$; otherwise $b_{ij} = 0$. Thus the coupled matrix $A \in R^{(N+M)\times(N+M)}$ can be written as

$$A = \begin{bmatrix} A_{11}^{N\times N} & B_{12}^{N\times M} \\ B_{21}^{M\times N} & A_{22}^{M\times M} \end{bmatrix},$$

where let $S_i \triangleq a_{i1} + a_{i2} + \ldots + a_{iN}$, then $A_{11} = \begin{bmatrix} a_{11} - S_1 & \ldots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \ldots & a_{NN} - S_N \end{bmatrix}_{N\times N}$,

and

$$A_{22} = \begin{bmatrix} a_{(N+1)(N+1)} - S_{N+1} & \ldots & a_{(N+1)(N+M)} \\ \vdots & \ddots & \vdots \\ a_{(N+M)(N+1)} & \ldots & a_{(N+M)(N+M)} - S_{N+M} \end{bmatrix}_{M\times M}$$

represent the coupling configuration of the subgroups, respectively.

In system (1), the controller is

$$\mu_i = \begin{cases} -c_i(t_k)h_i(x_i(t_k - \tau(t_k)) - \bar{x}_1(t_k - \tau(t_k))), & i \in \ell_1 \\ -c_i(t_k)h_i(x_i(t_k - \tau(t_k)) - \bar{x}_2(t_k - \tau(t_k))), & i \in \ell_2 \end{cases}, \tag{2}$$

where $\bar{x}_1(t),\ \bar{x}_2(t) \in R^n$ are the synchronous states, h_i is an on-off control. If the system is sampled date, then $h_i = 1$; otherwise $h_i = 0$.

The adaptive strategies on coupling strengths and feedback gains designed as:

$$\begin{aligned}
\dot{c}_{ij}(t_k) &= \begin{cases} a_{ij}k_{ij}[(x_i(t_k - \tau(t_k)) - x_j(t_k - \tau(t_k)))^T(x_i(t_k - \tau(t_k)) - x_j(t_k - \tau(t_k))) \\ \quad + (\dot{x}_i(t) - \dot{x}_j(t))^T(\dot{x}_i(t) - \dot{x}_j(t))] \quad i, j \in \ell_1 \\ a_{ij}k_{ij}[(x_i(t_k - \tau(t_k)) - x_j(t_k - \tau(t_k)))^T(x_i(t_k - \tau(t_k)) - x_j(t_k - \tau(t_k))) \\ \quad + (\dot{x}_i(t) - \dot{x}_j(t))^T(\dot{x}_i(t) - \dot{x}_j(t))] \quad i, j \in \ell_2 \end{cases}, \\
\dot{d}_{ij}(t_k) &= \begin{cases} b_{ij}k_{ij}(x_j(t_k - \tau(t_k)) - \bar{x}_2(t_k - \tau(t_k)))^T(x_j(t_k - \tau(t_k)) - \bar{x}_2(t_k - \tau(t_k))) \\ i \in \ell_1, j \in \ell_2 \\ b_{ij}k_{ij}(x_j(t_k - \tau(t_k)) - \bar{x}_1(t_k - \tau(t_k)))^T(x_j(t_k - \tau(t_k)) - \bar{x}_1(t_k - \tau(t_k))) \\ i \in \ell_2, j \in \ell_1 \end{cases}, \\
\dot{c}_i(t_k) &= \begin{cases} h_ik_i[(x_i(t_k - \tau(t_k)) - \bar{x}_1(t_k - \tau(t_k)))^T(x_i(t_k - \tau(t_k)) - \bar{x}_1(t_k - \tau(t_k))) \\ \quad + (\dot{x}_i(t_k) - \dot{\bar{x}}_1(t_k))^T(\dot{x}_i(t_k) - \dot{\bar{x}}_1(t_k))] \quad i \in \ell_1 \\ h_ik_i[(x_i(t_k - \tau(t_k)) - \bar{x}_2(t_k - \tau(t_k)))^T(x_i(t_k - \tau(t_k)) - \bar{x}_2(t_k - \tau(t_k))) \\ \quad + (\dot{x}_i(t_k) - \dot{\bar{x}}_2(t_k))^T(\dot{x}_i(t_k) - \dot{\bar{x}}_2(t_k))] \quad i \in \ell_2 \end{cases},
\end{aligned} \tag{3}$$

where $c_{ij} \geq 0,\ c_i \geq 0$, the constants $k_{ij} > 0$ and $k_i > 0$ are the weights of the $c_{ij}(t)$ and $c_i(t)$, respectively.

In the following, we will analyze the sampling period, which is an important factor in the sampling information. Given a positive real number α and a sample periodic T, we suppose that (see [12] in more detail)

$$t_{i+1} - t_i = \alpha T_i, \qquad \forall i = 0, 1, 2, \cdots,$$

where $t_0 < t_1 < \cdots$ are the discrete times; the node j can obtain information from its neighbors and positive integer T_i is a sampled time about the ith time

($\forall i = 0, 1, 2, \cdots$) satisfying $T_i \leq T$. Under this condition, a linear consensus protocol based on a linear estimation-based sampling period is designed as follows:

$$\begin{cases} \dot{x}_i(t_k+\alpha) = \dot{x}_i(t_k) - \frac{1}{T}\dot{x}_i(t_k) = \big(1-\frac{1}{T}\big)\dot{x}_i(t_k) \\ \dot{x}_i(t_k+2\alpha) = \dot{x}_i(t_k+\alpha) + (\dot{x}_i(t_k+\alpha) - \dot{x}_i(t_k)) = \big(1-\frac{2}{T}\big)\dot{x}_i(t_k) \\ \vdots \\ \dot{x}_i(t_{k+1}-\alpha) = \dot{x}_i(t_k+T_k\alpha-\alpha) = \big(1-\frac{T_k-1}{T}\big)\dot{x}_i(t_k) \end{cases}. \quad (4)$$

Substituting system (4) into system (1), and let $h = 0, 1, \cdots, T_k - 1$, we can have

$$\dot{x}_i(t) = \begin{cases} (1-\frac{h}{T}) \times \Big[f(x_i(t_k), x_i(t_k-\tau(t_k))) + \sum_{j\in\mathscr{N}_{1i}} c_{ij}(t_k)a_{ij}(x_j(t_k-\tau(t_k)) - x_i(t_k-\tau(t_k))) \\ + \sum_{j\in\mathscr{N}_{2i}} d_{ij}(t_k)b_{ij}x_j(t_k-\tau(t_k)) + \mu_i\Big] \quad \forall i \in \ell_1, \quad \forall t \in [t_k, t_{k+1}]; \\ \\ (1-\frac{h}{T}) \times \Big[f(x_i(t_k), x_i(t_k-\tau(t_k))) + \sum_{j\in\mathscr{N}_{2i}} c_{ij}(t_k)a_{ij}(x_j(t_k-\tau(t_k)) - x_i(t_k-\tau(t_k))) \\ + \sum_{j\in\mathscr{N}_{1i}} d_{ij}(t_k)b_{ij}x_j(t_k-\tau(t_k)) + \mu_i\Big] \quad \forall i \in \ell_2, \quad \forall t \in [t_k, t_{k+1}], \end{cases} \quad (5)$$

In order to solve the synchronization problem, we give the following assumptions and lemmas.

Assumption 1 [11]**.** *There exist nonnegative constants ρ_1 and ρ_2 such that*

$$\|f(a,b) - f(c,d)\| \leq \rho_1\|a-c\| + \rho_2\|b-d\|, \quad \forall a, b, c, d \in R^n.$$

Assumption 2. *The coupling strengths and feedback gains are bounded, which means that*

$$\|c_{ij}(t_k)\| \leq c_{ij}, \qquad \|d_{ij}(t_k)\| \leq d_{ij}, \qquad \|c_i(t_k)\| \leq c_i.$$

Definition 1. *Network is said to group synchronization if*

$$\lim_{t\to\infty} \|(x_i(t) - x_j(t))\| = 0, \quad \forall i, j \in \ell_1, \qquad \lim_{t\to\infty} \|(x_i(t) - x_j(t))\| = 0, \quad \forall i, j \in \ell_2.$$

Lemma 1 [3]**.** *Suppose that $x, y \in R^n$ are vectors, and in matrix M, the following inequality holds:*

$$2x^T y \leq x^T M x + y^T M^{-1} y.$$

Lemma 2 [5]**.** *For any real differentiable vector function $x(t) \in R^n$ and any $n \times n$ constant matrix $W = W^T > 0$, we have the following inequality:*

$$\left[\int_{t-\tau(t)}^{t} x(s)ds\right]^T W \left[\int_{t-\tau(t)}^{t} x(s)ds\right] \leq \tau \int_{t-\tau(t)}^{t} x^T(s)Wx(s)ds, t \geq 0,$$

where $0 \leq \tau(t) \leq \tau$.

Lemma 3 [9]**.** *If matrix $A = (a_{ij}) \in R^{N \times N}$ is the symmetric irreducible matrix, where $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$, then all eigenvalues of matrix $A - E$ are negative numbers, where matrix $E = diag(e, 0, \cdots, 0)$ with $e > 0$.*

3 Main Results

In this section, we consider adaptive synchronization of nonlinear complex dynamics network with time-delays and sampled-data. We have the following theorems.

Theorem 1. *Under Assumptions 1–2 and Lemmas 1–3, if coupled matrix A_{11}, A_{22} are the symmetric irreducible matrices, suppose that topology graph of system (1) is connected and the time-delays are bound, then system (1) can be steered to the synchronous state $\bar{x}_1(t)$ and $\bar{x}_2(t)$ by the adaptive strategies (3).*

Proof. Define

$$\begin{cases} \tilde{x}_i(t_k) \triangleq x_i(t_k) - \bar{x}_1(t_k), \quad \tilde{x}_i(t_k - \tau(t_k)) \triangleq x_i(t_k - \tau(t_k)) - \bar{x}_1(t_k - \tau(t_k)), \\ \dot{\tilde{x}}_i(t_k) \triangleq \dot{x}_i(t_k) - \dot{\bar{x}}_1(t_k), \quad \dot{\tilde{x}}_i(t_k - \tau(t_k)) \triangleq \dot{x}_i(t_k - \tau(t_k)) - \dot{\bar{x}}_1(t_k - \tau(t_k)), \\ i = 1, 2, \ldots, N, \\ \\ \tilde{x}_i(t_k) \triangleq x_i(t_k) - \bar{x}_2(t_k), \quad \tilde{x}_i(t_k - \tau(t_k)) \triangleq x_i(t_k - \tau(t_k)) - \bar{x}_2(t_k - \tau(t_k)), \\ \dot{\tilde{x}}_i(t_k) \triangleq \dot{x}_i(t_k) - \dot{\bar{x}}_2(t_k), \quad \dot{\tilde{x}}_i(t_k - \tau(t_k)) \triangleq \dot{x}_i(t_k - \tau(t_k)) - \dot{\bar{x}}_2(t_k - \tau(t_k)), \\ i = N+1, N+2, \ldots, N+M, \end{cases}$$

then

$$\dot{\tilde{x}}_i(t) = \begin{cases} (1 - \frac{h}{T}) \times \Big[f(x_i(t_k), x_i(t_k - \tau(t_k))) - f(\bar{x}_1(t_k), \bar{x}_1(t_k - \tau(t_k))) \\ + \sum_{j \in \mathscr{N}_{1i}} c_{ij}(t_k) a_{ij} (\tilde{x}_j(t_k - \tau(t_k)) - \tilde{x}_i(t_k - \tau(t_k))) + \sum_{j \in \mathscr{N}_{2i}} d_{ij}(t_k) b_{ij} \tilde{x}_j(t_k - \tau(t_k)) + \mu_i \Big] \\ \forall i \in \ell_1, \quad \forall t \in [t_k, t_{k+1}]; \\ \\ (1 - \frac{h}{T}) \times \Big[f(x_i(t_k), x_i(t_k - \tau(t_k))) - f(\bar{x}_2(t_k), \bar{x}_2(t_k - \tau(t_k))) \\ + \sum_{j \in \mathscr{N}_{2i}} c_{ij}(t_k) a_{ij} (\tilde{x}_j(t_k - \tau(t_k)) - \tilde{x}_i(t_k - \tau(t_k))) + \sum_{j \in \mathscr{N}_{1i}} d_{ij}(t_k) b_{ij} \tilde{x}_j(t_k - \tau(t_k)) + \mu_i \Big] \\ \forall i \in \ell_2, \quad \forall t \in [t_k, t_{k+1}], \end{cases} \tag{6}$$

Construct a Lyapunov function as follow:

$$V(t_k) = V_1(t_k) + V_2(t_k) + V_3(t_k),$$

where

$$
\begin{aligned}
V_1(t_k) &= \frac{1}{2}\sum_{i\in\mathcal{N}_{1i}} \tilde{x}_i^T(t_k)\tilde{x}_i(t_k) + \sum_{i\in\mathcal{N}_{1i}}\sum_{j\in\mathcal{N}_{1i}} \frac{(c_{ij}(t_k)-2c_{ij}-p)^2}{4k_{ij}} \\
&\quad + \sum_{i\in\mathcal{N}_{1i}}\sum_{j\in\mathcal{N}_{2i}} \frac{(d_{ij}(t_k)-2d_{ij}-1)^2}{4k_{ij}} + \sum_{i\in\mathcal{N}_{1i}} \frac{(c_i(t_k)-\frac{3}{2}c_i-p)^2}{2k_i}, \\
V_2(t_k) &= \frac{1}{2}\sum_{i\in\mathcal{N}_{2i}} \tilde{x}_i^T(t_k)\tilde{x}_i(t_k) + \sum_{i\in\mathcal{N}_{2i}}\sum_{j\in\mathcal{N}_{2i}} \frac{(c_{ij}(t_k)-2c_{ij}-p)^2}{4k_{ij}} \\
&\quad + \sum_{i\in\mathcal{N}_{2i}}\sum_{j\in\mathcal{N}_{1i}} \frac{(d_{ij}(t_k)-2d_{ij}-1)^2}{4k_{ij}} + \sum_{i\in\mathcal{N}_{2i}} \frac{(c_i(t_k)-\frac{3}{2}c_i-p)^2}{2k_i}, \\
V_3(t_k) &= \tau \sum_{i\in\mathcal{N}_{1i}} \big(2\rho_1+\rho_2+\sum_{i\in\mathcal{N}_{1i}} c_{ij}a_{ij} \\
&\qquad + \sum_{i\in\mathcal{N}_{2i}} d_{ij}b_{ij} + h_i c_i\big)\int_{t_k-\tau(t_k)}^{t_k}(s-t_k+\tau)\dot{\tilde{x}}_i^T(s)\dot{\tilde{x}}_i(s)ds \\
&\quad + \tau \sum_{i\in\mathcal{N}_{2i}} \big(2\rho_3+\rho_4+\sum_{i\in\mathcal{N}_{2i}} c_{ij}a_{ij} \\
&\qquad + \sum_{i\in\mathcal{N}_{1i}} d_{ij}b_{ij} + h_i c_i\big)\int_{t_k-\tau(t_k)}^{t_k}(s-t_k+\tau)\dot{\tilde{x}}_i^T(s)\dot{\tilde{x}}_i(s)ds,
\end{aligned}
$$

Differentiating $V_1(t_k)$, we can know

$$
\begin{aligned}
\dot{V}_1(t_k) &= \big(1-\frac{h}{T}\big)\sum_{i\in\mathcal{N}_{1i}} \tilde{x}_i^T(t_k)\big[f(x_i(t_k),x_i(t_k-\tau(t_k))) - f(\bar{x}_1(t_k),\bar{x}_1(t_k-\tau(t_k)))\big] \\
&\quad + \big(1-\frac{h}{T}\big)\sum_{i\in\mathcal{N}_{1i}} \tilde{x}_i^T(t_k)\Big[\sum_{j\in\mathcal{N}_{1i}} c_{ij}(t_k)a_{ij}(\tilde{x}_j(t_k-\tau(t_k)) - \tilde{x}_i(t_k-\tau(t_k))) \\
&\quad + \sum_{j\in\mathcal{N}_{2i}} d_{ij}(t_k)b_{ij}\tilde{x}_j(t_k-\tau(t_k)) - c_i(t_k)h_i(\tilde{x}_i(t_k-\tau(t_k))\Big] \\
&\quad + \frac{1}{2}\sum_{i\in\mathcal{N}_{1i}}\sum_{j\in\mathcal{N}_{1i}} (c_{ij}(t_k)-2c_{ij}-p)a_{ij}\big[(\dot{\tilde{x}}_i(t_k)-\dot{\tilde{x}}_j(t_k))^T(\dot{\tilde{x}}_i(t_k)-\dot{\tilde{x}}_j(t_k)) \\
&\quad + (\tilde{x}_i(t_k-\tau(t_k))-\tilde{x}_j(t_k-\tau(t_k)))^T(\tilde{x}_i(t_k-\tau(t_k))-\tilde{x}_j(t_k-\tau(t_k)))\big] \\
&\quad + \frac{1}{2}\sum_{i\in\mathcal{N}_{1i}}\sum_{j\in\mathcal{N}_{2i}} (d_{ij}(t_k)-2d_{ij}-1)b_{ij}\tilde{x}_j^T(t_k-\tau(t_k))\tilde{x}_j(t_k-\tau(t_k)) \\
&\quad + \sum_{i\in\mathcal{N}_{1i}} (c_i(t_k)-\frac{3}{2}c_i-p)h_i\big[\tilde{x}_i^T(t_k-\tau(t_k))\tilde{x}_i(t_k-\tau(t_k)) + \dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k)\big].
\end{aligned} \tag{7}
$$

Under Assumption 1 and using Lemma 1, we can have

$$\begin{aligned}
\dot{V}_1(t_k) \leq\ & (1-\frac{h}{T})(\rho_1+\frac{1}{2}\rho_2)\sum_{i\in\mathscr{N}_{1i}}\tilde{x}_i^T(t_k)\tilde{x}_i(t_k)+(1-\frac{h}{T})\frac{1}{2}\rho_2\sum_{i\in\mathscr{N}_{1i}}\tilde{x}_i^T(t_k-\tau(t_k)))\tilde{x}_i(t_k-\tau(t_k))) \\
& +(1-\frac{h}{T})\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}}c_{ij}(t_k)a_{ij}(\tilde{x}_j(t_k-\tau(t_k)) \\
& -\tilde{x}_i(t_k-\tau(t_k)))^T(\tilde{x}_j(t_k-\tau(t_k))-\tilde{x}_i(t_k-\tau(t_k))) \\
& +(1-\frac{h}{T})\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}}c_{ij}(t_k)a_{ij}\tilde{x}_i^T(t_k)\tilde{x}_i(t_k)+(1-\frac{h}{T})\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{2i}}d_{ij}(t_k)b_{ij}\tilde{x}_i^T(t_k)\tilde{x}_i(t_k) \\
& +(1-\frac{h}{T})\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{2i}}d_{ij}(t_k)b_{ij}\tilde{x}_j^T(t_k-\tau(t_k))\tilde{x}_j(t_k-\tau(t_k)) \\
& +(1-\frac{h}{T})\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}c_i(t_k)h_i\tilde{x}_i^T(t_k)\tilde{x}_i(t_k)+(1-\frac{h}{T})\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}c_i(t_k)h_i\tilde{x}_i^T(t_k-\tau(t_k))\tilde{x}_i(t_k-\tau(t_k)) \\
& +\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}}(c_{ij}(t_k)-2c_{ij}-p)a_{ij}\big[(\dot{\tilde{x}}_i(t_k)-\dot{\tilde{x}}_j(t_k))^T(\dot{\tilde{x}}_i(t_k)-\dot{\tilde{x}}_j(t_k)) \\
& +(\tilde{x}_i(t_k-\tau(t_k))-\tilde{x}_j(t_k-\tau(t_k)))^T(\tilde{x}_i(t_k-\tau(t_k))-\tilde{x}_j(t_k-\tau(t_k)))\big] \\
& +\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{2i}}(d_{ij}(t_k)-2d_{ij}-1)b_{ij}\tilde{x}_j^T(t_k-\tau(t_k))\tilde{x}_j(t_k-\tau(t_k)) \\
& +\sum_{i\in\mathscr{N}_{1,i}}(c_i(t_k)-\frac{3}{2}c_i-p)h_i\big[\tilde{x}_i^T(t_k-\tau(t_k))\tilde{x}_i(t_k-\tau(t_k))+\dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k)\big].
\end{aligned}$$

As a result of $h=0,1,\cdots,T_k-1<T$, and Assumption 2, we can get

$$\begin{aligned}
\dot{V}_1(t_k) \leq\ & \big(\rho_1+\frac{1}{2}\rho_2\big)\sum_{i\in\mathscr{N}_{1i}}\tilde{x}_i^T(t_k)\tilde{x}_i^T(t_k)+\frac{1}{2}\rho_2\sum_{i\in\mathscr{N}_{1i}}\tilde{x}_i^T(t_k-\tau(t_k))\tilde{x}_i^T(t_k-\tau(t_k)) \\
& +\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}}c_{ij}a_{ij}\tilde{x}_i^T(t_k)\tilde{x}_i(t_k)+\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{2i}}d_{ij}b_{ij}\tilde{x}_i^T(t_k)\tilde{x}_i(t_k) \\
& +\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}c_ih_i\tilde{x}_i^T(t_k)\tilde{x}_i(t_k) \\
& -\frac{p}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}}a_{ij}(\tilde{x}_i(t_k-\tau(t_k))-\tilde{x}_j(t_k-\tau(t_k)))^T(\tilde{x}_i(t_k-\tau(t_k))-\tilde{x}_j(t_k-\tau(t_k))) \\
& -\frac{p}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}}a_{ij}(\dot{\tilde{x}}_i(t_k)-\dot{\tilde{x}}_j(t_k))^T(\dot{\tilde{x}}_i(t_k)-\dot{\tilde{x}}_j(t_k)) \\
& -\frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{2i}}b_{ij}\tilde{x}_j^T(t_k-\tau(t_k))\tilde{x}_j(t_k-\tau(t_k)) \\
& -p\sum_{i\in\mathscr{N}_{1i}}h_i\tilde{x}_i^T(t_k-\tau(t_k))\tilde{x}_i(t_k-\tau(t_k))-p\sum_{i\in\mathscr{N}_{1i}}h_i\dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k).
\end{aligned}$$

Using the Leibniz-Newton formula:

$$x(t_k)-x(t_k-\tau(t_k))=\int_{t_k-\tau(t_k)}^{t_k}\dot{x}(s)ds,$$

we can get that

$$x^T(t_k)=x^T(t_k-\tau(t_k))+\bigg(\int_{t_k-\tau(t_k)}^{t_k}\dot{x}(s)ds\bigg)^T.$$

Using Lemma 1, we can have

$$\tilde{x}^T(t_k)\tilde{x}(t_k) = \Big[\tilde{x}^T(t_k - \tau(t_k)) + \Big(\int_{t_k-\tau(t_k)}^{t_k} \dot{\tilde{x}}(s)ds\Big)^T\Big]\Big[\tilde{x}(t_k - \tau(t_k)) + \Big(\int_{t_k-\tau(t_k)}^{t_k} \dot{\tilde{x}}(s)ds\Big)\Big]$$
$$\leq 2\tilde{x}^T(t_k - \tau(t_k))\tilde{x}(t_k - \tau(t_k)) + 2\Big(\int_{t_k-\tau(t_k)}^{t_k} \dot{\tilde{x}}(s)ds\Big)^T\Big(\int_{t_k-\tau(t_k)}^{t_k} \dot{\tilde{x}}(s)ds\Big)$$
$$\leq 2\tilde{x}^T(t_k - \tau(t_k))\tilde{x}(t_k - \tau(t_k)) + 2\tau\int_{t_k-\tau(t_k)}^{t_k} \dot{\tilde{x}}_i^T(s)\dot{\tilde{x}}_i(s)ds. \tag{8}$$

Thus,

$$\dot{V}_1(t_k) \leq \sum_{i\in\mathscr{N}_{1i}} \big[(2\rho_1 + \frac{3}{2}\rho_2) + \sum_{j\in\mathscr{N}_{1i}} c_{ij}a_{ij} + \sum_{j\in\mathscr{N}_{2i}} d_{ij}b_{ij} + c_i h_i\big]\tilde{x}^T(t_k - \tau(t_k))\tilde{x}(t_k - \tau(t_k))$$
$$+ \sum_{i\in\mathscr{N}_{1i}} \big[(2\rho_1 + \rho_2) + \sum_{j\in\mathscr{N}_{1i}} c_{ij}a_{ij} + \sum_{j\in\mathscr{N}_{2i}} d_{ij}b_{ij} + c_i h_i\big]\tau\int_{t_k-\tau(t_k)}^{t_k} \dot{\tilde{x}}_i^T(s)\dot{\tilde{x}}_i(s)ds$$
$$+ p(\|\tilde{x}_1(t_k - \tau(t_k))\|, \|\tilde{x}_2(t_k - \tau(t_k))\|, \cdots, \|\tilde{x}_N(t_k - \tau(t_k))\|)$$
$$(A_{11} - H_1)\begin{pmatrix} \|\tilde{x}_1(t_k - \tau(t_k))\| \\ \|\tilde{x}_2(t_k - \tau(t_k))\| \\ \cdots \\ \|\tilde{x}_N(t_k - \tau(t_k))\| \end{pmatrix}$$
$$+ P(\|\dot{\tilde{x}}_1(t_k)\|, \|\dot{\tilde{x}}_2(t_k)\|, \cdots, \|\dot{\tilde{x}}_N(t_k)\|)(A_{11} - H_1)\begin{pmatrix} \|\dot{\tilde{x}}_1(t_k)\| \\ \|\dot{\tilde{x}}_2(t_k)\| \\ \cdots \\ \|\dot{\tilde{x}}_N(t_k)\| \end{pmatrix}$$
$$- \frac{1}{2}\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{2i}} b_{ij}(\tilde{x}_j(t_k - \tau(t_k)))^T(\tilde{x}_j(t_k - \tau(t_k))).$$

Similarly, differentiating $V_2(t_k)$, we can have

$$\dot{V}_2(t_k) \leq \sum_{i\in\mathscr{N}_{2i}} \big(2\rho_3 + \frac{3}{2}\rho_4 + \sum_{j\in\mathscr{N}_{2i}} c_{ij}a_{ij} + \sum_{j\in\mathscr{N}_{1i}} d_{ij}b_{ij} + c_i h_i\big)(\tilde{x}(t_k - \tau(t_k)))^T(\tilde{x}(t_k - \tau(t_k)))$$
$$+ \sum_{i\in\mathscr{N}_{2i}} \big(2\rho_3 + \rho_4 + \sum_{j\in\mathscr{N}_{2i}} c_{ij}a_{ij} + \sum_{j\in\mathscr{N}_{1i}} d_{ij}b_{ij} + c_i h_i\big)\tau\int_{t_k-\tau(t_k)}^{t_k} \dot{\tilde{x}}_i^T(s)\dot{\tilde{x}}_i(s)ds$$
$$+ p(\|\tilde{x}_{N+1}(t_k - \tau(t_k))\|, \|\tilde{x}_{N+2}(t_k - \tau(t_k))\|, \cdots, \|\tilde{x}_{N+M}(t_k - \tau(t_k))\|)$$
$$(A_{22} - H_2)\begin{pmatrix} \|\tilde{x}_{N+1}(t_k - \tau(t_k))\| \\ \|\tilde{x}_{N+2}(t_k - \tau(t_k))\| \\ \cdots \\ \|\tilde{x}_{N+M}(t_k - \tau(t_k))\| \end{pmatrix}$$
$$+ P(\|\dot{\tilde{x}}_{N+1}(t_k)\|, \|\dot{\tilde{x}}_{N+2}(t_k)\|, \cdots, \|\dot{\tilde{x}}_{N+M}(t_k)\|)(A_{22} - H_2)\begin{pmatrix} \|\dot{\tilde{x}}_{N+1}(t_k)\| \\ \|\dot{\tilde{x}}_{N+2}(t_k)\| \\ \cdots \\ \|\dot{\tilde{x}}_{N+M}(t_k)\| \end{pmatrix}$$
$$- \frac{1}{2}\sum_{i\in\mathscr{N}_{2i}}\sum_{j\in\mathscr{N}_{1i}} b_{ij}(\tilde{x}_j(t_k - \tau(t_k)))^T(\tilde{x}_j(t_k - \tau(t_k))).$$

Differentiating $V_3(t_k)$, we get

$$\begin{aligned}\dot{V}_3(t_k) = \tau^2 \sum_{i\in\mathscr{N}_{1i}} \big(2\rho_1 + \rho_2 + \sum_{i\in\mathscr{N}_{1i}} c_{ij}a_{ij} + \sum_{i\in\mathscr{N}_{2i}} d_{ij}b_{ij} + h_ic_i\big)\dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k) \\ - \sum_{i\in\mathscr{N}_{1i}} \big(2\rho_1 + \rho_2 + \sum_{i\in\mathscr{N}_{1i}} c_{ij}a_{ij} + \sum_{i\in\mathscr{N}_{2i}} d_{ij}b_{ij} + h_ic_i\big) \int_{t_k-\tau(t_k)}^{t_k} \dot{\tilde{x}}_i^T(s)\dot{\tilde{x}}_i(s)ds \\ + \tau^2 \sum_{i\in\mathscr{N}_{2i}} \big(2\rho_3 + \rho_4 + \sum_{i\in\mathscr{N}_{2i}} c_{ij}a_{ij} + \sum_{i\in\mathscr{N}_{1i}} d_{ij}b_{ij} + h_ic_i\big)\dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k) \\ - \sum_{i\in\mathscr{N}_{2i}} \big(2\rho_3 + \rho_4 + \sum_{i\in\mathscr{N}_{2i}} c_{ij}a_{ij} + \sum_{i\in\mathscr{N}_{1i}} d_{ij}b_{ij} + h_ic_i\big) \int_{t_k-\tau(t_k)}^{t_k} \dot{\tilde{x}}_i^T(s)\dot{\tilde{x}}_i(s)ds,\end{aligned}$$

Count up $\dot{V}_1(t_k),\ \dot{V}_2(t_k),\ \dot{V}_3(t_k)$, we know that

$$\begin{aligned}\dot{V}(t_k) \le \sum_{i\in\mathscr{N}_{1i}} \big(2\rho_1 + \frac{3}{2}\rho_2 + \sum_{j\in\mathscr{N}_{1i}} c_{ij}a_{ij} + \sum_{j\in\mathscr{N}_{2i}} d_{ij}b_{ij} + c_ih_i \\ - \frac{1}{2} + p\lambda_1\big)(\tilde{x}(t_k - \tau(t_k)))^T(\tilde{x}(t_k - \tau(t_k))) \\ + \tau^2 \sum_{i\in\mathscr{N}_{1i}} \big(2\rho_1 + \rho_2 + \sum_{i\in\mathscr{N}_{1i}} c_{ij}a_{ij} + \sum_{i\in\mathscr{N}_{2i}} d_{ij}b_{ij} + h_ic_i + p\lambda_1\big)\dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k) \\ + \sum_{i\in\mathscr{N}_{2i}} \big(2\rho_3 + \frac{3}{2}\rho_4 + \sum_{j\in\mathscr{N}_{2i}} c_{ij}a_{ij} + \sum_{j\in\mathscr{N}_{1i}} d_{ij}b_{ij} + c_ih_i \\ - \frac{1}{2} + p\lambda_2\big)(\tilde{x}(t_k - \tau(t_k)))^T(\tilde{x}(t_k - \tau(t_k))) \\ + \tau^2 \sum_{i\in\mathscr{N}_{2i}} \big(2\rho_3 + \rho_4 + \sum_{i\in\mathscr{N}_{2i}} c_{ij}a_{ij} + \sum_{i\in\mathscr{N}_{1i}} d_{ij}b_{ij} + h_ic_i + p\lambda_2\big)\dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k),\end{aligned}$$

where $\lambda_1,\ \lambda_2$ are the minimum eigenvalue of $A_{11} - H_1,\ A_{22} - H_2$, respectively, with

$$H_1 = diag\{h_i\}\ \ \forall i \in \ell_1, \qquad H_2 = diag\{h_i\}\ \ \forall i \in \ell_2.$$

From the conditions of Theorem 1, we known that the matrix H_1, H_2 are diagonal matrices with at least one element equaling to 1. Since A_{11}, A_{22} are symmetric, all eigenvalues of $A_{11} - H_1,\ A_{22} - H_2$ are negative from Lemma 3. So $\lambda_1 < 0,\ \lambda_2 < 0$ and $p > 0$ is sufficiently large, therefore, $\dot{V}(t_k) < 0$, all the nodes of system (1) with time-delays and sampled-data can converge to their own synchronous states asymptotically. The proof is completed.

Theorem 1 shows that when coupled matrix A_{11}, A_{22} are the symmetric irreducible matrices, system (1) can synchronize. If coupled matrix A_{11}, A_{22} are the asymmetric irreducible matrices, the synchronization of the system (1) is given by the following result.

Theorem 2. *Under Assumptions 1–2 and Lemmas 1–3, if coupled matrix A_{11}, A_{22} are the asymmetric irreducible matrices, suppose that system (1) is connected and the time-delays are bound, then system (1) is steered to the synchronous state $\bar{x}_1(t)$ and $\bar{x}_2(t)$ by the adaptive strategies (3).*

Proof.

$$
\begin{aligned}
&-\frac{1}{2}p\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}} a_{ij}[(\tilde{x}_i(t_k-\tau(t_k))-\tilde{x}_j(t_k-\tau(t_k)))^T(\tilde{x}_i(t_k-\tau(t_k))-\tilde{x}_j(t_k-\tau(t_k)))]\\
&=-\frac{1}{2}p\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}} a_{ij}(\tilde{x}_i(t_k-\tau(t_k))^T(\tilde{x}_i(t_k-\tau(t_k))\\
&+\frac{1}{2}p\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}} a_{ij}(\tilde{x}_i(t_k-\tau(t_k))^T(\tilde{x}_j(t_k-\tau(t_k))\\
&-\frac{1}{2}p\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}} a_{ij}(\tilde{x}_j(t_k-\tau(t_k))^T(\tilde{x}_j(t_k-\tau(t_k))\\
&+\frac{1}{2}p\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}} a_{ij}(\tilde{x}_j(t_k-\tau(t_k))^T(\tilde{x}_i(t_k-\tau(t_k))\\
&=p(\|\tilde{x}_1(t_k-\tau(t_k))\|,\|\tilde{x}_2(t_k-\tau(t_k))\|,\cdots,\|\tilde{x}_N(t_k-\tau(t_k))\|)(\frac{A_{11}+A_{11}^T}{2})\begin{pmatrix}\|\tilde{x}_1(t_k-\tau(t_k))\|\\ \|\tilde{x}_2(t_k-\tau(t_k))\|\\ \cdots\\ \|\tilde{x}_N(t_k-\tau(t_k))\|\end{pmatrix}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&-\frac{1}{2}p\sum_{i\in\mathscr{N}_{2i}}\sum_{j\in\mathscr{N}_{2i}} a_{ij}[(\tilde{x}_i(t_k-\tau(t_k))-\tilde{x}_j(t_k-\tau(t_k)))^T(\tilde{x}_i(t_k-\tau(t_k))-\tilde{x}_j(t_k-\tau(t_k)))]\\
&=p(\|\tilde{x}_{N+1}(t_k-\tau(t_k))\|,\|\tilde{x}_{N+2}(t_k-\tau(t_k))\|,\cdots,\|\tilde{x}_{N+M}(t_k-\tau(t_k))\|)\\
&\qquad(\frac{A_{22}+A_{22}^T}{2})\begin{pmatrix}\|\tilde{x}_{N+1}(t_k-\tau(t_k))\|\\ \|\tilde{x}_{N+2}(t_k-\tau(t_k))\|\\ \cdots\\ \|\tilde{x}_{N+M}(t_k-\tau(t_k))\|\end{pmatrix},\\
&-\frac{1}{2}p\sum_{i\in\mathscr{N}_{1i}}\sum_{j\in\mathscr{N}_{1i}} a_{ij}[(\dot{\tilde{x}}_i(t_k)-\dot{\tilde{x}}_j(t_k))^T(\dot{\tilde{x}}_i(t_k)-\dot{\tilde{x}}_j(t_k))]\\
&=P(\|\dot{\tilde{x}}_1(t_k)\|,\|\dot{\tilde{x}}_2(t_k)\|,\cdots,\|\dot{\tilde{x}}_N(t_k)\|)(\frac{A_{11}+A_{11}^T}{2})\begin{pmatrix}\|\dot{\tilde{x}}_1(t_k)\|\\ \|\dot{\tilde{x}}_2(t_k)\|\\ \cdots\\ \|\dot{\tilde{x}}_N(t_k)\|\end{pmatrix},\\
&-\frac{1}{2}p\sum_{i\in\mathscr{N}_{2i}}\sum_{j\in\mathscr{N}_{2i}} a_{ij}[(\dot{\tilde{x}}_i(t_k)-\dot{\tilde{x}}_j(t_k))^T(\dot{\tilde{x}}_i(t_k)-\dot{\tilde{x}}_j(t_k))]\\
&=P(\|\dot{\tilde{x}}_{N+1}(t_k)\|,\|\dot{\tilde{x}}_{N+2}(t_k)\|,\cdots,\|\dot{\tilde{x}}_{N+M}(t_k)\|)(\frac{A_{22}+A_{22}^T}{2})\begin{pmatrix}\|\dot{\tilde{x}}_{N+1}(t_k)\|\\ \|\dot{\tilde{x}}_{N+2}(t_k)\|\\ \cdots\\ \|\dot{\tilde{x}}_{N+M}(t_k)\|\end{pmatrix}.
\end{aligned}
$$

Define the same Lyapunov function as Theorem 1, we can obtain

$$
\begin{aligned}
\dot{V}(t_k) \le & \sum_{i\in\mathscr{N}_{1i}} (2\rho_1 + \frac{3}{2}\rho_2 + \sum_{j\in\mathscr{N}_{1i}} c_{ij}a_{ij} + \sum_{j\in\mathscr{N}_{2i}} d_{ij}b_{ij} + c_i h_i - \frac{1}{2})(\tilde{x}(t_k-\tau(t_k)))^T(\tilde{x}(t_k-\tau(t_k))) \\
& + p(\|\tilde{x}_1(t_k-\tau(t_k))\|, \|\tilde{x}_2(t_k-\tau(t_k))\|, \cdots, \|\tilde{x}_N(t_k-\tau(t_k))\|) \\
& \qquad (\frac{A_{11}+A_{11}^T}{2} - H_1)\begin{pmatrix} \|\tilde{x}_1(t_k-\tau(t_k))\| \\ \|\tilde{x}_2(t_k-\tau(t_k))\| \\ \cdots \\ \|\tilde{x}_N(t_k-\tau(t_k))\| \end{pmatrix} \\
& + \tau^2 \sum_{i\in\mathscr{N}_{1i}} (2\rho_1 + \rho_2 + \sum_{i\in\mathscr{N}_{1i}} c_{ij}a_{ij} + \sum_{i\in\mathscr{N}_{2i}} d_{ij}b_{ij} + h_i c_i)\dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k) \\
& + P(\|\dot{\tilde{x}}_1(t_k)\|, \|\dot{\tilde{x}}_2(t_k)\|, \cdots, \|\dot{\tilde{x}}_N(t_k)\|)(\frac{A_{11}+A_{11}^T}{2} - H_1)\begin{pmatrix} \|\dot{\tilde{x}}_1(t_k)\| \\ \|\dot{\tilde{x}}_2(t_k)\| \\ \cdots \\ \|\dot{\tilde{x}}_N(t_k)\| \end{pmatrix} \\
& + \sum_{i\in\mathscr{N}_{2i}} (2\rho_3 + \frac{3}{2}\rho_4 + \sum_{j\in\mathscr{N}_{2i}} c_{ij}a_{ij} + \sum_{j\in\mathscr{N}_{1i}} d_{ij}b_{ij} + c_i h_i - \frac{1}{2})(\tilde{x}(t_k-\tau(t_k)))^T(\tilde{x}(t_k-\tau(t_k))) \\
& + p(\|\tilde{x}_{N+1}(t_k-\tau(t_k))\|, \|\tilde{x}_{N+2}(t_k-\tau(t_k))\|, \cdots, \|\tilde{x}_{N+M}(t_k-\tau(t_k))\|) \\
& \qquad (\frac{A_{22}+A_{22}^T}{2} - H_2)\begin{pmatrix} \|\tilde{x}_{N+1}(t_k-\tau(t_k))\| \\ \|\tilde{x}_{N+2}(t_k-\tau(t_k))\| \\ \cdots \\ \|\tilde{x}_{N+M}(t_k-\tau(t_k))\| \end{pmatrix} \\
& + \tau^2 \sum_{i\in\mathscr{N}_{2i}} (2\rho_3 + \rho_4 + (\sum_{i\in\mathscr{N}_{2i}} c_{ij}a_{ij}) + (\sum_{i\in\mathscr{N}_{1i}} d_{ij}b_{ij}) + h_i c_i)\dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k) \\
& + P(\|\dot{\tilde{x}}_{N+1}(t_k)\|, \|\dot{\tilde{x}}_{N+2}(t_k)\|, \cdots, \|\dot{\tilde{x}}_{N+M}(t_k)\|)(\frac{A_{22}+A_{22}^T}{2} - H_2)\begin{pmatrix} \|\dot{\tilde{x}}_{N+1}(t_k)\| \\ \|\dot{\tilde{x}}_{N+2}(t_k)\| \\ \cdots \\ \|\dot{\tilde{x}}_{N+M}(t_k)\| \end{pmatrix} \\
\le & \sum_{i\in\mathscr{N}_{1i}} (2\rho_1 + \frac{3}{2}\rho_2 + \sum_{j\in\mathscr{N}_{1i}} c_{ij}a_{ij} + \sum_{j\in\mathscr{N}_{2i}} d_{ij}b_{ij} + c_i h_i - \frac{1}{2} + p\lambda_1)\tilde{x}^T(t_k-\tau(t_k))\tilde{x}(t_k-\tau(t_k)) \\
& + \tau^2 \sum_{i\in\mathscr{N}_{1i}} (2\rho_1 + \rho_2 + \sum_{i\in\mathscr{N}_{1i}} c_{ij}a_{ij} + \sum_{i\in\mathscr{N}_{2i}} d_{ij}b_{ij} + h_i c_i + p\lambda_1)\dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k) \\
& + \sum_{i\in\mathscr{N}_{2i}} (2\rho_3 + \frac{3}{2}\rho_4 + \sum_{j\in\mathscr{N}_{2i}} c_{ij}a_{ij} + \sum_{j\in\mathscr{N}_{1i}} d_{ij}b_{ij} + c_i h_i - \frac{1}{2} + p\lambda_2)\tilde{x}^T(t_k-\tau(t_k))\tilde{x}(t_k-\tau(t_k)) \\
& + \tau^2 \sum_{i\in\mathscr{N}_{2i}} (2\rho_3 + \rho_4 + \sum_{i\in\mathscr{N}_{2i}} c_{ij}a_{ij} + \sum_{i\in\mathscr{N}_{1i}} d_{ij}b_{ij} + h_i c_i + p\lambda_2)\dot{\tilde{x}}_i^T(t_k)\dot{\tilde{x}}_i(t_k),
\end{aligned}
$$

where $\lambda_1,\ \lambda_2$ are the minimum eigenvalue of $\frac{A_{11}+A_{11}^T}{2} - H_1,\ \frac{A_{22}+A_{22}^T}{2} - H_2$, respectively, with $H_1 = diag\{h_i\}\ \ \forall i \in \ell_1, \quad H_2 = diag\{h_i\}\ \ \forall i \in \ell_2$.

Even though matrix A_{11}, A_{22} are asymmetric, matrix $\frac{A_{11}+A_{11}^T}{2}, \frac{A_{22}+A_{22}^T}{2}$ are symmetric, thus, all eigenvalues of $\frac{A_{11}+A_{11}^T}{2} - H_1,\ \ \frac{A_{22}+A_{22}^T}{2} - H_2$ are negative from Lemma 3. So $\lambda_1 < 0,\ \lambda_2 < 0$ and $p > 0$ is sufficiently large, therefore, $\dot{V}(t_k) < 0$. Similar to Theorem 1, all the nodes of system (1) with time-delays and sampled-data can converge to their own synchronous states asymptotically. The proof is completed.

4 Simulations

Consider a complex dynamical network with $N+M$ nodes, where $N = 3,\ M = 3$. Let the initial value of the 6 nodes is $X(0) = [29\ 12\ 20\ 17\ 25\ -7\ 22\ 9]$, the initial values of the coupling strengths and the feedback gains are $c_{ij}(0) = d_{ij}(0) = c_i(0) = 0.01$.

Take $A_{11},\ A_{22}$ be symmetric as

$$A_{11} = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} * 0.1, \qquad A_{22} = \begin{bmatrix} -3 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -3 \end{bmatrix} * 0.05;$$

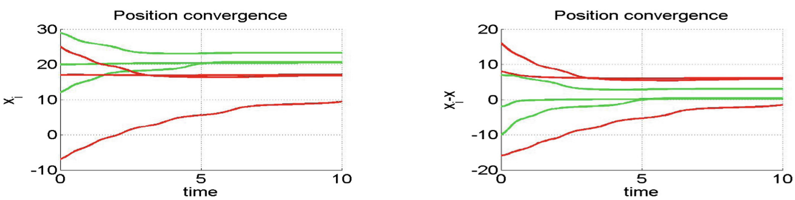
and $A_{11},\ A_{22}$ be asymmetric as,

$$A_{11} = \begin{bmatrix} -3 & 0 & 3 \\ 2 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} * 0.1, \qquad A_{22} = \begin{bmatrix} -2 & 2 & 0 \\ 3 & -6 & 3 \\ 1 & 2 & -3 \end{bmatrix} * 0.05;$$

respectively.

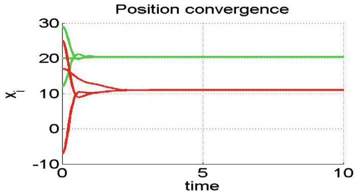
Take $B_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} * 0.1$ and $B_{21} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} * 0.1.$

Figures 1 and 2 present that the effects of adaptive strategies for the synchronization of nonlinear complex dynamical networks. Figure 1 shows the simulation results on the synchronization of system (1) without adaptive strategies, where coupling matrices of sub-groups are symmetric. Figure 2 shows the simulation results on the adaptive synchronization of system (1) with $\tau = 0.2$, where coupling matrices of sub-groups are symmetric. Figures 2 and 3 show the simulation results on the group synchronization of system (1) with $\tau = 0.2$ and $\tau = 0.6$, where coupling matrices of sub-groups are symmetric as Figs. 2 and 3, respectively. From Figs. 2 and 3, we can see that all nodes of system achieve synchronization and the coupling strengths and the feedback gains also tend to be consistent. Figures 3 and 4 show the simulation results on the group synchronization of system (1) with $\tau = 0.6$, where coupling matrices of sub-groups are symmetric or symmetric presented as Figs. 3 and 4, respectively. Similarly, all nodes of system achieve synchronization and the coupling strengths and the feedback gains also tend to be consistent.

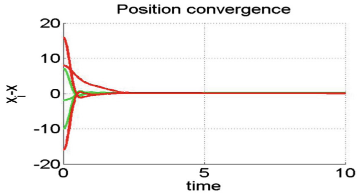


(a) positions (b) error

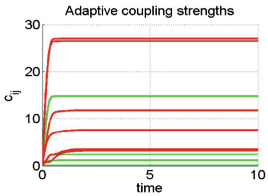
Fig. 1. Without adaptive strategies and time delay $\tau = 0.2$ when intra-group coupling matrix is symmetric.



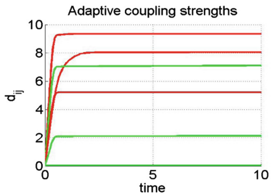
(a) positions



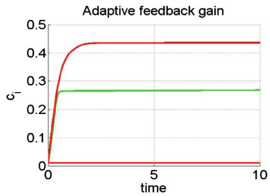
(b) error



(c) c-ij

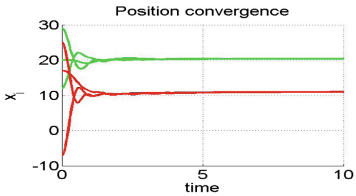


(d) d-ij

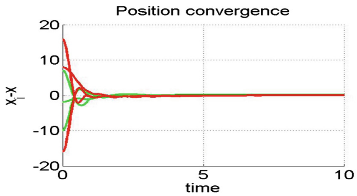


(e) c-i

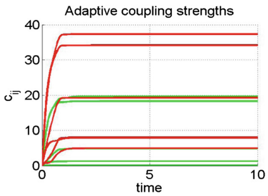
Fig. 2. Intra-group coupling matrix is symmetric and time delay $\tau = 0.2$.



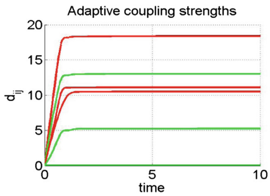
(a) positions



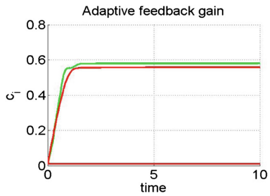
(b) error



(c) c-ij

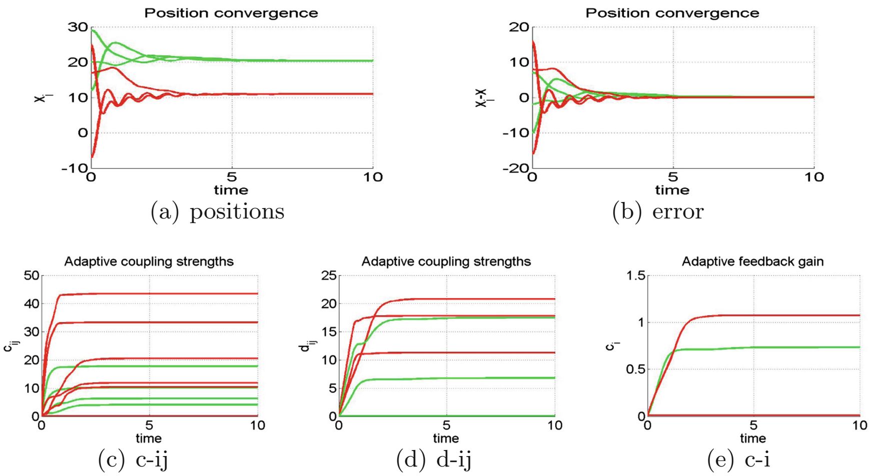


(d) d-ij



(e) c-i

Fig. 3. Intra-group coupling matrix is symmetric and time delay $\tau = 0.6$.



(a) positions (b) error
(c) c-ij (d) d-ij (e) c-i

Fig. 4. Intra-group coupling matrix is asymmetric and time delay $\tau = 0.6$.

5 Conclusion

In this paper, we have studied the adaptive synchronization of nonlinear complex dynamical networks with time-delays and sampled-data. Whether the coupled matrix A_{11}, A_{22} are symmetric or not, we have obtained the sufficient conditions satisfying the local Lipschitz condition.

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