

Chapter 16

Stress Distribution in an Infinite Plate with Circular Hole by Modified Body Force Method



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Abstract Stress distribution in an infinite plate with circular hole subjected to uniform tension is determined by employing a modified body force method. In this method, the problem of a plate with a hole under uniform tension is considered as a plate with an imaginary hole. The boundary of the imaginary hole is divided into a number of divisions. At the mid-point of each division, concentrated forces known as body forces are applied. The magnitudes of these body forces are computed from complex potential functions, and stress at an arbitrary point is obtained by the summation of stresses due to these body forces applied at the mid-point of each division and stresses due to applied load. Results obtained from the modified body force method show trends in line with theoretical results. However, more accurate results can be obtained by using better estimate of body forces which satisfy boundary conditions at the circular hole. Setting Poisson's ratio $\nu = 0$ has little effect on the computed stress distribution.

Keywords Body force method · Boundary force method · Stress concentration

16.1 Introduction

The solution to the problem of stress distribution in an infinite plate with circular hole subjected to uniaxial loading was first obtained by Kirsch [1]. The details of the analytical solution are found in Timoshenko [2] and Wang [3]. Complex variables approach was first introduced into plane elastic problems in 1909 by Kolosov [4, 5], which was further utilised in solving various problems in elastostatics by Muskhelishvili [6] and others. With improvements in computer performance, numerical techniques like finite element method (FEM) and boundary element method (BEM)

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became popular methods of solving various engineering problems. However, FEM and BEM have the limitation of increased computing cost and accuracy when discontinuities like holes and cracks are present in the material. Body force method (BFM) originally proposed by Nisitani [7, 8] is a boundary type technique for stress analysis. The software program named BFM2D, developed by Nisitani [8] and based on BFM is reported to provide accurate results even with coarse mesh. More recently, Manjunath and Ramakrishna have applied BFM and solved various problems including Flamant and Melan problem [9–11].

16.2 Body Force Method

Body force method is a boundary type technique based on the principle of superposition. For a plate with circular hole subjected to uniform tension as shown in Fig. 16.1a, the circular hole is considered as imaginary hole, which is divided into a number of divisions (imaginary hole is divided into four divisions in Fig. 16.1b), at the mid-point of each division, body forces ρ_{xi} , ρ_{yi} are applied in x - and y -directions, the magnitudes of which are unknown to begin with. Applying equilibrium condition to each division in x - and y -directions due to the body forces and external applied load, we obtain a set of linear equations which can be expressed in the form, $\mathbf{Ax} = \mathbf{b}$ where \mathbf{A} is influence coefficient matrix obtained from Kelvin's problem (point load in an infinite plane) where unit loads are applied in x - and y -directions, \mathbf{x} is the unknown body force vector and \mathbf{b} is a vector consisting of forces on each division in x - and y -directions due to external applied load. The stress at an arbitrary point $\mathbf{P}(x, y)$ (Fig. 16.1a) is obtained by the summation of stresses at point $\mathbf{P}(x, y)$ due to body forces ρ_{xi} , ρ_{yi} and stresses due to the applied uniform load as shown in Fig. 16.1b.

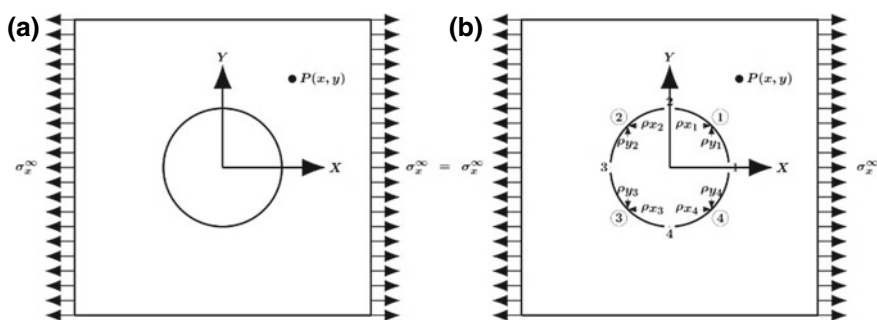


Fig. 16.1 Plate with a circular hole subjected to uniform tension and its equivalence in BFM

16.3 Modified Body Force Method

Body forces ρ_{xi}, ρ_{yi} can also be computed from complex potential functions which describe the given loading and boundary conditions. The complex potential functions for an infinite plate subjected to uniform tensile stress in x -direction are well known [10]. In this study, body forces are obtained from complex potential function given by Honein [10]. These body forces are applied at the mid-point of each division. The stress at an arbitrary point $\mathbf{P}(x, y)$ is obtained by the summation of stresses due to the computed body forces and stresses due to the applied load.

Stresses are related to complex potential functions $\varphi(z)$ and $\psi(z)$ [2, 3, 5, 6] by the following relations:

$$\sigma_x + \sigma_y = 2(\varphi'(z) + \overline{\varphi'(z)}) = 4\Re\{\varphi'(z)\} \quad (16.1)$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2(\bar{z}\varphi''(z) + \psi'(z)) \quad (16.2)$$

Forces acting on an arc from A to B are related to complex potential functions by the following relations [12]:

$$P_x + P_y = -i[\varphi(z) + z \cdot \overline{\varphi'(z)} + \overline{\psi(z)}]_A^B \quad (16.3)$$

Complex potential functions for an infinite plate subjected to uniform tensile stress in x -direction are as follows [10]:

$$\varphi(z) = \frac{z}{4} \cdot \sigma_x^\infty \quad (16.4)$$

$$\psi(z) = -\frac{z}{2} \cdot \sigma_x^\infty \quad (16.5)$$

Complex potential functions $\varphi(z)$ and $\psi(z)$ from Eqs. (16.4) and (16.5) when substituted in Eqs. (16.1) and (16.2), the following stresses are obtained:

$$\sigma_x = \sigma_x^\infty \quad (16.6)$$

$$\sigma_y = 0 \quad (16.7)$$

$$\tau_{xy} = 0 \quad (16.8)$$

Equations (16.6), (16.7) and (16.8) accurately describe the state of stress at arbitrary point $\mathbf{P}(x, y)$ in an infinite plate without any discontinuity.

Complex potential functions from Eqs. (16.4) and (16.5) when substituted in Eq. (16.3), the following forces are obtained:

Table 16.1 Body force values for four divisions of the imaginary circle

Body forces	P_{x1}	P_{y1}	P_{x2}	P_{y2}	P_{x3}	P_{y3}	P_{x4}	P_{y4}
Numerical values	500	0	-500	0	-500	0	500	0

$$P_x = \sigma_x^\infty \cdot [y_B - y_A] \quad (16.9)$$

$$P_y = 0 \quad (16.10)$$

Body forces are computed for an infinite plate with unit thickness having a circular hole of 5 mm radius subjected to uniform tensile stress of 100 MPa. Table 16.1 shows numeric values of the body forces obtained from Eqs. (16.9) and (16.10), for four divisions of the imaginary circle.

16.4 Theoretical Results

In an infinite plane with thickness h , the stress in polar coordinates at an arbitrary point $z = x + i \cdot y$ due to concentrated load $P_x + i \cdot P_y$ acting at location $z = x_0 + i \cdot y_0$ is reproduced from Wang [13].

$$\sigma_r = -\frac{(3 + \nu)}{4\pi h} \left(\frac{P_x \cos \theta + P_y \sin \theta}{r} \right) \quad (16.11)$$

$$\sigma_\theta = \frac{(1 - \nu)}{4\pi h} \left(\frac{P_x \cos \theta + P_y \sin \theta}{r} \right) \quad (16.12)$$

$$\tau_{r\theta} = \frac{(1 - \nu)}{4\pi h} \left(\frac{P_x \sin \theta - P_y \cos \theta}{r} \right) \quad (16.13)$$

Stress at arbitrary point $\mathbf{P}(x, y)$ is obtained by summing stresses (Eqs. 16.11–16.13) due to body forces obtained for Eqs. (16.9) and (16.10) and stresses due to applied load.

The expression for stresses in polar coordinates for an infinite plate with circular hole (Fig. 16.1a) is available in Timoshenko [12]. These equations are reproduced here from [12].

$$\sigma_r = \frac{\sigma_x^\infty}{2} \left(1 - \frac{a^2}{r^2} \right) + \frac{\sigma_x^\infty}{2} \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2} \right) \cos 2\theta \quad (16.14)$$

$$\sigma_\theta = \frac{\sigma_x^\infty}{2} \left(1 + \frac{a^2}{r^2} \right) - \frac{\sigma_x^\infty}{2} \left(1 + \frac{3a^4}{r^4} \right) \cos 2\theta \quad (16.15)$$

$$\tau_{r\theta} = -\frac{\sigma_x^\infty}{2} \left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2} \right) \sin 2\theta \tag{16.16}$$

16.5 Numerical Results

A plate with unit thickness having a circular hole of 5 mm radius is considered with an applied uniform tensile stress of 100 MPa. The radial, hoop and shear stresses along radial direction at angles 0°, 45° and 90° are computed using body force Eqs. (16.9) and (16.10) (legend BFM) and are compared with analytical results (Eqs. 16.14–16.16).

Figures 16.2, 16.3 and 16.4 show stresses along 0° radial line. Radial and hoop stresses show trends in line with the theoretical results.

Figures 16.5, 16.6 and 16.7 show stresses along 45° radial line. Radial, hoop and shear stresses show trends in line with the theoretical results. However, it is observed that the magnitudes of these stresses in the proximity of the circular hole deviate from theoretical values.

Figures 16.8, 16.9 and 16.10 show stresses along 90° radial line. Radial and hoop stresses show trends in line with the theoretical results.

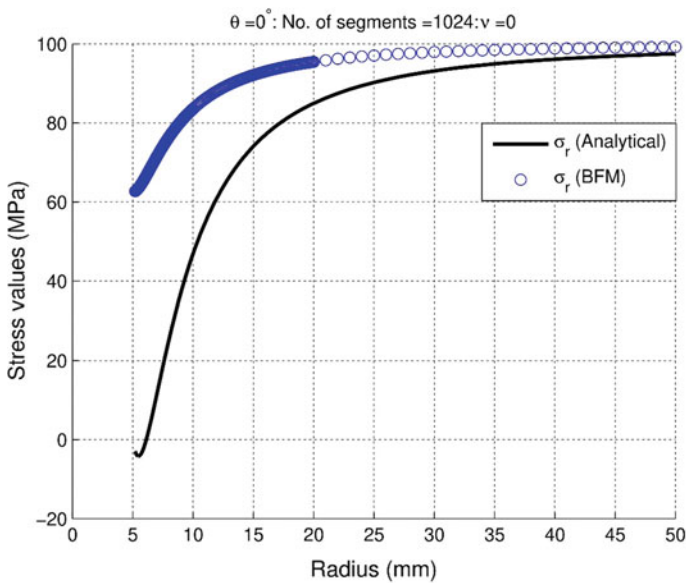


Fig. 16.2 Radial stress (σ_r) along radial line making 0° with x-axis, and circle is divided into 1024 divisions

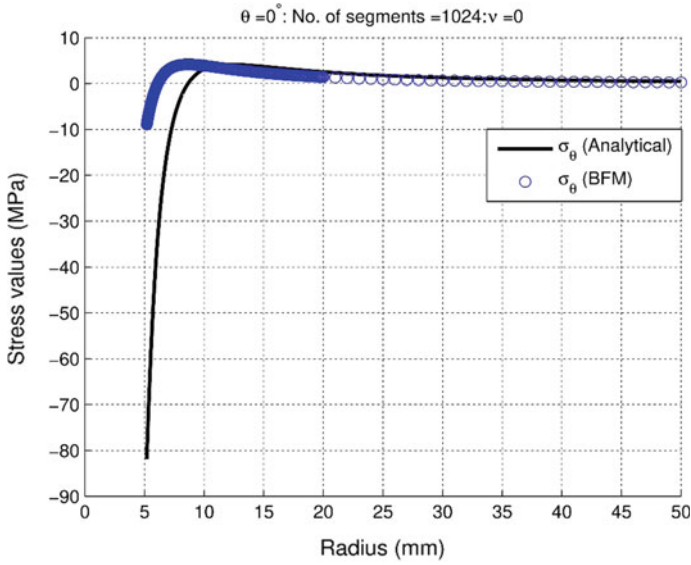


Fig. 16.3 Hoop stress (σ_θ) along radial line making 0° with x -axis, and circle is divided into 1024 divisions

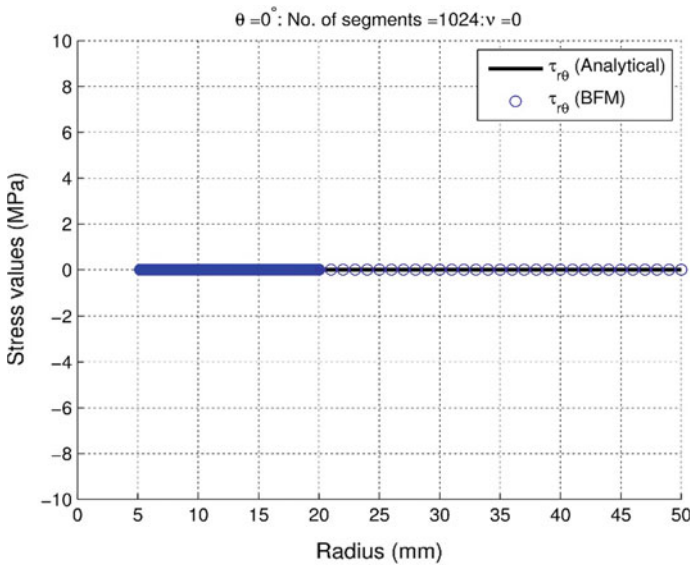


Fig. 16.4 Shear stress ($\tau_{r\theta}$) along radial line making 0° with x -axis, and circle is divided into 1024 divisions

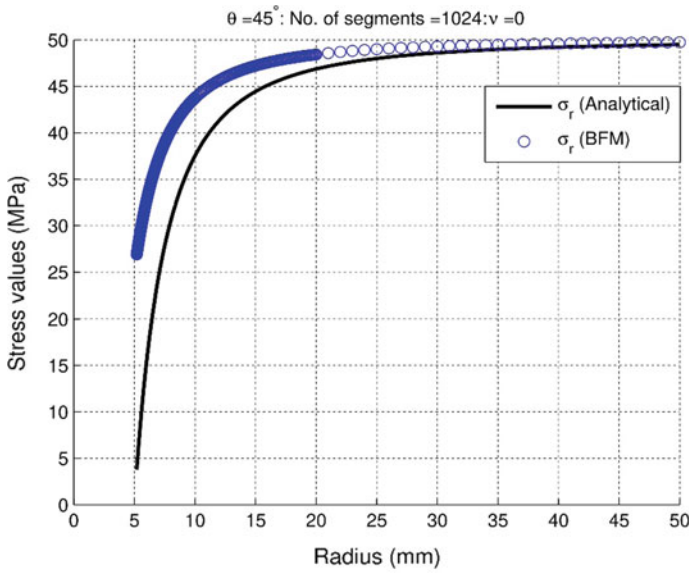


Fig. 16.5 Radial stress (σ_r) along radial line making 45° with x -axis, and circle is divided into 1024 divisions

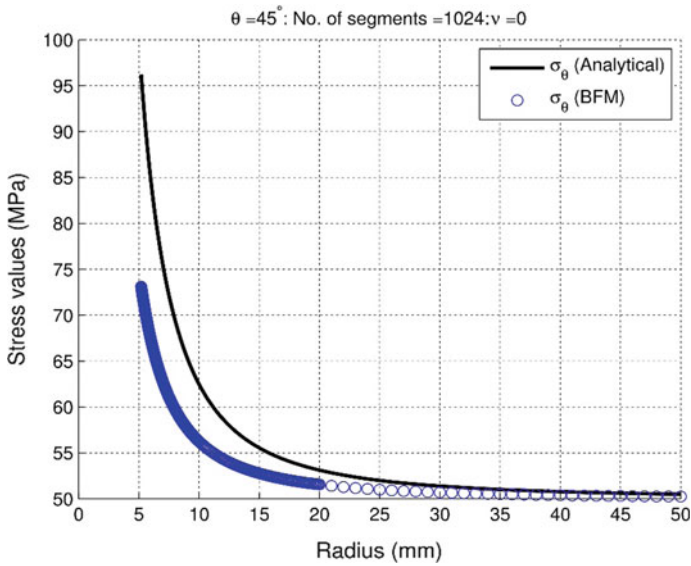


Fig. 16.6 Hoop stress (σ_θ) along radial line making 45° with x -axis, and circle is divided into 1024 divisions

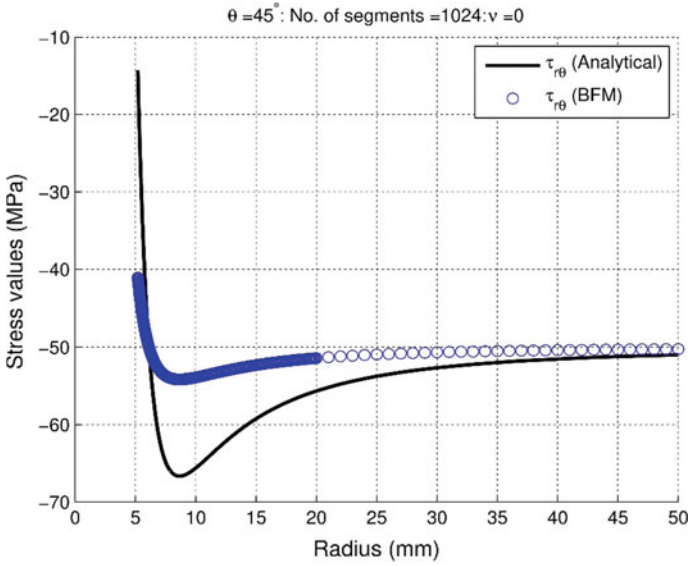


Fig. 16.7 Shear stress ($\tau_{r\theta}$) along radial line making 45° with x -axis, and circle is divided into 1024 divisions

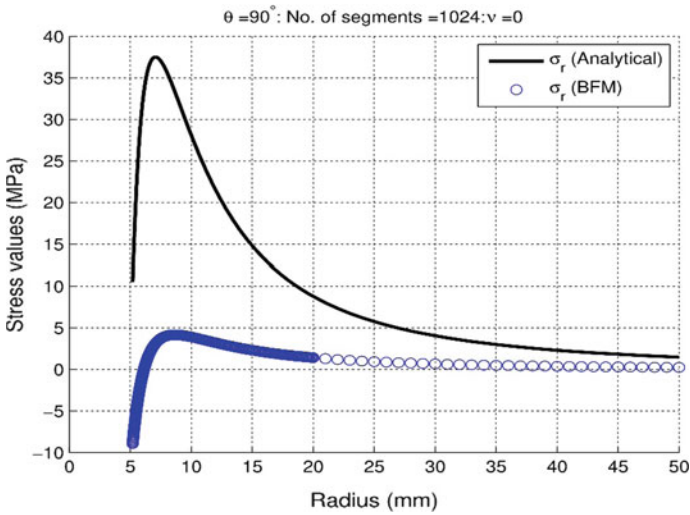


Fig. 16.8 Radial stress (σ_r) along radial line making 90° with x -axis, and circle is divided into 1024 divisions

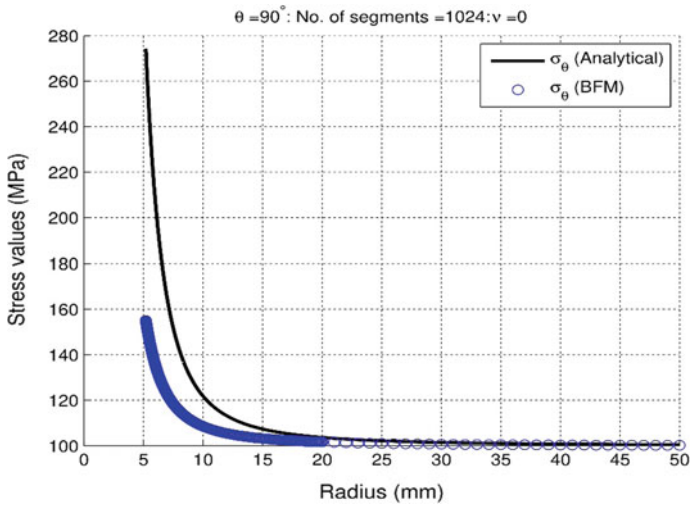


Fig. 16.9 Hoop stress (σ_θ) along radial line making 90° with x -axis, and circle is divided into 1024 divisions

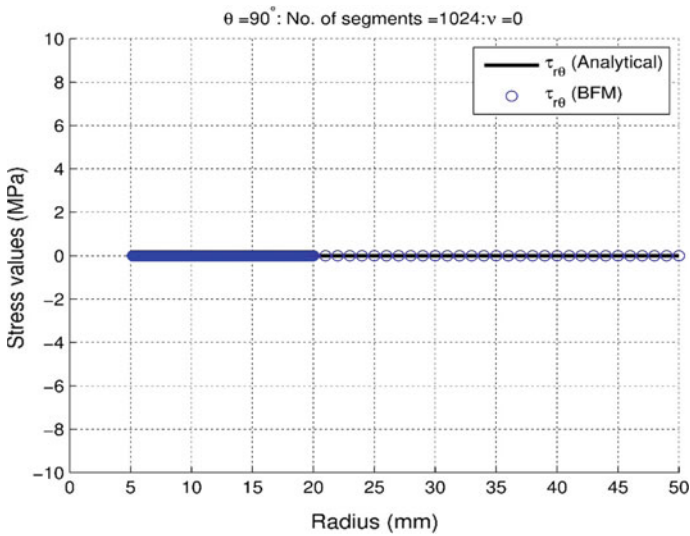


Fig. 16.10 Shear stress ($\tau_{r\theta}$) along radial line making 90° with x -axis, and circle is divided into 1024 divisions

Figures 16.11, 16.12 and 16.13 show radial, hoop and shear stresses along 0° radial line. Here, circle is divided into 1024 divisions, with Poisson's ratio (a) $\nu = 0$, (b) $\nu = 0.32$. It is observed that taking Poisson's ratio as zero has little effect on

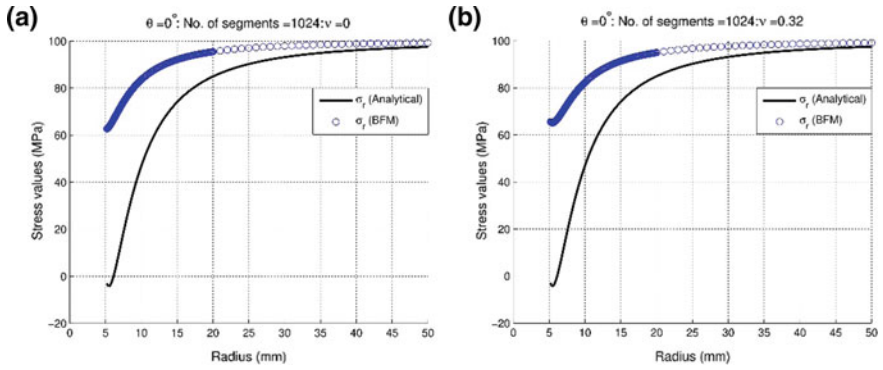


Fig. 16.11 Radial stress along radial line making 0° with x -axis, circle is divided into 1024 divisions, **a** $\nu = 0$, **b** $\nu = 0.32$

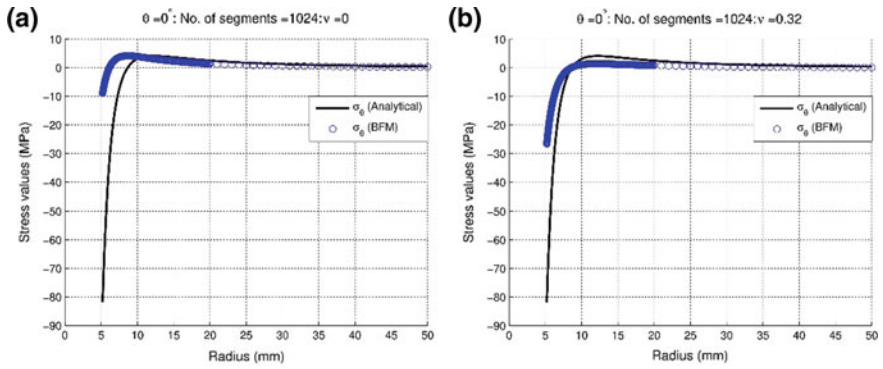


Fig. 16.12 Hoop stress along radial line making 0° with x -axis, circle is divided into 1024 divisions, **a** $\nu = 0$, **b** $\nu = 0.32$

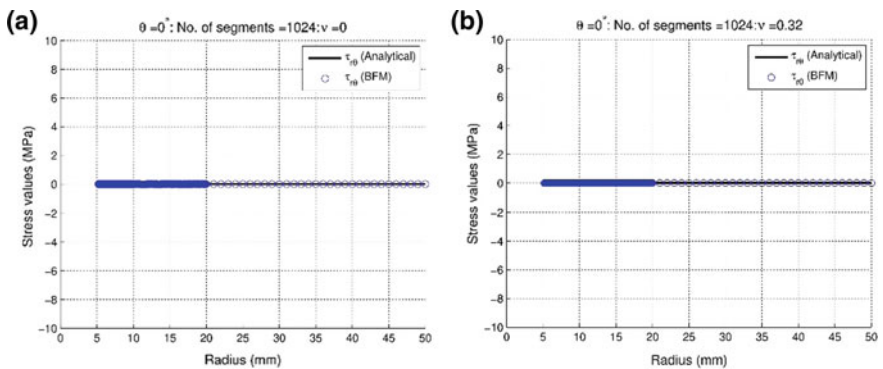


Fig. 16.13 Shear stress along radial line making 0° with x -axis, circle is divided into 1024 divisions, **a** $\nu = 0$, **b** $\nu = 0.32$

the radial and shear stresses. With nonzero Poisson's ratio, hoop stress shows little improvement in its trend towards theoretical results.

16.6 Conclusion

In this study, stress distribution is obtained by summation of stresses due to body forces derived from complex potential functions for a plate with uniform load in x -direction and stresses due to applied load (modified body force method). Results obtained from this method show trends in line with theoretical solution. However, numerical values obtained deviate from the theoretical solution. One of the possible reasons for this deviation in numerical values is that the boundary of the imaginary circle being stress-free may not be completely satisfied. Better results can be obtained when accurate estimates of the body forces on the boundary of the imaginary circle are made. Alternately, body forces obtained from a set of linear equations $A\mathbf{x} = \mathbf{b}$ (body force method) can be used to obtain the stress distribution.

Modified body force method may be useful in the instances where theoretical results are unavailable and this method can give values of stresses approaching actual stresses.

As suggested by Nisitani [14], it is acceptable to take Poisson's ratio $\nu = 0$ since the marginal improvement in computed stresses is observed when nonzero Poisson's ratio is used.

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References

1. Kirsch, E.G.: Die Theorie der Elastizität und die Bedürfnisse der Festigkeitslehre. *Zeitschrift des Vereines deutscher Ingenieure* **42**, 797–807 (1898)
2. Timoshenko, S.P., Goodier, J.N.: *Theory of Elasticity*, 3rd edn. McGraw-Hill, New York (1970)
3. Wang, C.-T.: *Applied Elasticity*, pp. 171–208. McGraw-Hill, New York (1953)
4. Kolosov, G.V.: On an application of complex function theory to a plane problem of the mathematical theory of elasticity. Doctoral thesis, Yuriev (1909)
5. England, A.H.: *Complex Variable Methods in Elasticity*, Dover edn. Dover Publications, Mineola, N.Y. (2003)
6. Muskhelishvili, N.I.: *Some Basic Problems of the Mathematical Theory of Elasticity*, 2nd edn. Noordhoff International Publishing, Leyden (1977). <https://doi.org/10.1007/978-94-017-3034-1>
7. Nisitani, H., Saimoto, A.: Short history of body force method and its application to various problems of stress analysis. *Mater. Sci. Forum* **440–441**, 161–168 (2003). <https://doi.org/10.4028/www.scientific.net/MSF.440-441.161>
8. Nisitani, H., Saimoto, A.: Effectiveness of two-dimensional versatile program based on body force method and its application to crack problems. *Key Eng. Mater.* **251–252**, 97–102 (2003). <https://doi.org/10.4028/www.scientific.net/KEM.251-252.97>

9. Manjunath, B.S., Ramakrishna, D.S.: Body force method for flamant problem using complex potentials. In: ASME Engineering Systems Design and Analysis, Volume 4: Fatigue and Fracture, Heat Transfer, Internal Combustion Engines, Manufacturing, and Technology and Society, pp. 99–103 (2006). <https://doi.org/10.1115/esda2006-95303>
10. Manjunath, B.S., Ramakrishna, D.S.: Body force method for melan problem with hole using complex potentials. In: ASME International Mechanical Engineering Congress and Exposition, Volume 10: Mechanics of Solids and Structures, Parts A and B: pp. 861–865 (2007). <https://doi.org/10.1115/imece2007-42885>
11. Manjunath, B.S.: Body force method in the field of stress analysis. Ph.D. thesis, Visvesvaraya Technological University, Belgaum (2009)
12. Nisitani, H., Chen, D.: Body force method and its applications to numerical and theoretical problems in fracture and damage. *Comput. Mech.* **19**(6), 470–480 (1997). <https://doi.org/10.1007/s004660050195>
13. Honein, T, Herrmann, G.: The Involution correspondence in plane elastostatics for regions bounded by a circle. *J. Appl. Mech.* **55**(3), 566–573 (1988). <https://doi.org/10.1115/1.3125831>
14. Nisitani, H.: Stress analysis of notch problem by body force method. In: Sih G.C. (ed.) *Mechanics of Fracture 5*, Sijthoff & Noordhoff, Chapter 1, pp. 1–68 (1978)
15. Chen, D., Nisitani, H.: *Int. J. Fract.* **86**(1–2), 161–189 (1997). <https://doi.org/10.1023/A:1007337210078>