A New Algorithm for Reduction of High Order Commensurate Non-integer Interval Systems

Kalyana Kiran Kumar, Kurman Sangeeta and Chongala Prasad

Abstract This note presents a novel methodology for reduction of high order linear time-invariant commensurate non-integer interval systems. It is shown first that the fractional-order interval system is reconstructed to integer interval system and further a hybrid technique is applied as a model reduction scheme. In this scheme, the reduced denominator is acquired by applying a modified least square method and the numerator is achieved by time moment matching. This formulated reduced interval integer model is reconverted to a reduced fractional interval model. As a final point, the results of a numerical illustration are verified to show the relevance and superiority of the proposed technique.

Keywords Non-integer interval systems · Kharitonov's theorem · Model reduction · Commensurate fractional

1 Introduction

A structure whose parameters are two-dimensional having lower and upper bounds is defined as an uncertain system. For example, interval parameters $\left[a_i^-, a_i^+\right]$ and $\left[b_i^-, b_i^+\right]$ mean, the parameters a_i and b_i can take independently any values in respective intervals $\left[a_i^-, a_i^+\right]$ and $\left[b_i^-, b_i^+\right]$. The width of the interval coefficient is equal to the difference between the upper and the lower boundary parameters. Some coefficients of an interval system may have zero interval width, i.e., equal values for both lower and upper bounds. These systems are successfully handled with interval arithmetic. The interval system having all the coefficients with zero width is called a fixed system. Moore's first book printed on interval analysis was the product of his Ph.D. thesis. He primarily focused on solutions for ordinary differential equations problems.

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Recently, fractional-order control systems are fetching progressively prominent in the control community. Fractional systems are governed by fractional-order derivatives and integrals which are basically infinite order linear operators. These fractional-order systems extend the notion of our integer order concepts in control and improvise on the existing results. So in essence, both fractional calculus and control systems have been applied to improve the performance of existing control theories [1]. The processes that we want to estimate the real-world objects are in general of fractional order [2–4]. These fractional-order systems on applying Laplace transform with zero initial conditions obtain a transfer function of unlimited order and therefore require infinite memory [5]. In order to reduce the infinite memory occupied by the fractional systems, it is possible to apply the concept of approximations [6]. Some of the frequency-based domain reduction methods have been suggested [7–9]. These approaches are used to generate a reduced fractional model whose characteristics are akin to the original fractional system. Since the obtained model is compact compared to the complex original fractional system. These approaches are called as model reduction techniques.

MOR is a well-recognized and long-standing concept applied for systems of integer order but an insufficient work being implemented for systems fractional order. [10]. this instigated us to select this particular zone and put an effort to introduce a unique reduction methodology for the commensurate non-integer interval systems [7, 11, 12]. The present study has been put on display with three examples that the determined reduced fractional interval model by the recommended method approximates well for stable class of high order nonminimum, non-integer fractional-order integer systems. To the author's knowledge, the proposed reduction technique applied to commensurate fractional-order interval system has not been proclaimed so far. The concept and the process endorsed in the proposed scheme would be a foundation stone for the evolution of new MOR schemes for fractional interval systems in the future. Two different theories, time moment matching and least square optimization, are blended together in this method [13, 14] to get the reduced model. The paper is organized as follows: To make a proper background, the proposed method is explained in Sect. 2. The capability of the proposed approach is shown in Sect. 3 through three test systems and the paper are concluded in Sect. 4.

2 Proposed Method

Step 1: Let the high order fractional-order interval system (FOIS) represented as a transfer function as cited in [15]

$$G(s)\frac{\sum_{i=0}^{m}\left[n_i^-,n_i^+\right]s^{\beta_i}}{\sum_{i=0}^{n}[d_i^-,d_i^+]s^{\alpha_i}} \tag{1}$$

where $\beta_m > \beta_{m-1} > \cdots > \beta_1 \geq 0$ and $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 \geq 0$.

If all the powers of the derivatives of the above equation are integer multiples of Υ, that is, α_i, β_i = $i\,\Upsilon$. Such that $\Upsilon \in R+$ and i = 0, 1, 2, 3… then it is considered as commensurate fractional-order interval system. Therefore, Eq. (1) can be formulated as

$$G(s)=\frac{\sum_{i=0}^{m}\left[n_i^-,n_i^+\right]s^{\Upsilon i}}{\sum_{i=0}^{n}[d_i^-,d_i^+]s^{\Upsilon i}}=\frac{\sum_{i=0}^{m}\left[n_i^-,n_i^+\right]\left(s^{\Upsilon}\right)^i}{\sum_{i=0}^{n}[d_i^-,d_i^+](s^{\Upsilon})^i} \tag{2}$$

By substituting $s^{\Upsilon}=\lambda$, in the above equation becomes integer order interval transfer function as,

$$G(\lambda)=\frac{\sum_{i=0}^{m}\left[n_i^-,n_i^+\right]\lambda^i}{\sum_{i=0}^{n}[d_i^-,d_i^+]\lambda^i} \tag{3}$$

where n = order of the system > m, i = 0, 1, 2, 3, … n and $\left[n_i^-,n_i^+\right]$, $\left[d_i^-,d_i^+\right]$ are the specified lower and upper boundaries of the ith perturbation.

Step 2: Formulation of four Kharitonov transfer functions from the above equation as

$$K_1(\lambda)=\frac{n_0^-+n_1^-\lambda+n_2^+\lambda^2+n_3^+\lambda^3+\cdots}{d_0^-+d_1^-\lambda+d_2^+\lambda^2+d_3^+\lambda^3+\cdots} \tag{4}$$

$$K_2(\lambda)=\frac{n_0^++n_1^+\lambda+n_2^-\lambda^2+n_3^-\lambda^3+\cdots}{d_0^++d_1^+\lambda+d_2^-\lambda^2+d_3^-\lambda^3+\cdots} \tag{5}$$

$$K_3(\lambda)=\frac{n_0^++n_1^-\lambda+n_2^-\lambda^2+n_3^+\lambda^3+\cdots}{d_0^++d_1^-\lambda+d_2^-\lambda^2+d_3^+\lambda^3+\cdots} \tag{6}$$

$$K_4(\lambda)=\frac{n_0^-+n_1^+\lambda+n_2^+\lambda^2+n_3^-\lambda^3+\cdots}{d_0^-+d_1^+\lambda+d_2^+\lambda^2+d_3^-\lambda^3+\cdots} \tag{7}$$

The above four nth order Kharitonov transfer functions can be generalized as

$$K_w(\lambda)=\frac{\sum_{i=0}^{n-1}x_{wi}\lambda^i}{\sum_{i=0}^{n}y_{wi}\lambda^i}=\frac{x_{w0}+x_{w1}\lambda+x_{w2}\lambda^2+\cdots+x_{w(n-1)}\lambda^{n-1}}{y_{w0}+y_{w1}\lambda+y_{w2}\lambda^2+\cdots+y_{wn}\lambda^n} \tag{8}$$

such that equate the coefficients of Eqs. (4)–(7) individually with Eq. (8) at w = 1, 2, 3, 4, respectively.

Step 3: Determination of the denominator coefficients of the reduced order interval model.

If $K_w(\lambda)$ is extended about $\lambda = 0$, then the time moment proportions, v_{wi}, are assumed to be

$$K_w(\lambda) = \sum_{i=0}^{\infty} v_{wi}\lambda^i = v_{w0}\lambda^0 + v_{w1}\lambda^1 + v_{w2}\lambda^2 + \cdots + v_{w\infty}\lambda^{\infty} \tag{9}$$

Let the corresponding rth order generalized reduced models of $K_w(\lambda)$ represented as

$$K_{wr}(\lambda) = \frac{p_{wr}(\lambda)}{q_{wr}(\lambda)} = \frac{\sum_{i=0}^{r-1} p_{wi}\lambda^i}{\sum_{i=0}^{r} q_{wi}\lambda^i} = \frac{p_{w0} + p_{w1}\lambda + \cdots + p_{w(r-1)}\lambda^{r-1}}{q_{w0} + q_{w1}\lambda + \cdots + q_{w(r-1)}\lambda^{r-1} + \lambda^r} \tag{10}$$

where w = 1, 2, 3, 4.

The minimum time moments required for a reduced model to retain is $2r$ time moments, 'r' indicates the order of the reduced model and the coefficient q_{wi}, p_{wi} in Eq. (10) are derived from following a set of equations as:

$$\left.\begin{array}{l}
p_{w0} = q_{w0}v_{w0} \\
p_{w1} = q_{w1}v_{w0} + q_{w0}v_{w1} \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
p_{w(r-1)} = q_{w(r-1)}v_{w0} + \ldots + q_{w0}v_{w(r-1)} \\
-v_{w0} = q_{w(r-1)}v_{w1} + \ldots + q_{w1}v_{w(r-1)} + q_{w0}v_{w(r)} \\
-v_{w1} = q_{w(r-1)}v_{w2} + \ldots + q_{w1}v_{wr} + q_{w0}v_{w(r+1)} \\
\quad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \quad \vdots \qquad \vdots \quad \vdots \\
-v_{w(t-r-1)} = q_{w(r-1)}v_{w(t-r)} + \ldots + q_{w0}v_{w(t-1)}
\end{array}\right\} \tag{11}$$

Equating Eqs. (9) and (10) the linear set of equations to preserve 't' time moments of the original system, the solution of represented as

$$\underbrace{\begin{bmatrix} v_{w(r)} & v_{w(r-1)} & \cdots & v_{w1} \\ v_{w(r+1)} & v_{w(r)} & \cdots & v_{w2} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ v_{w(t-1)} & v_{w(t-2)} & \cdots & v_{w(t-r)} \end{bmatrix}}_{H_w} * \underbrace{\begin{bmatrix} q_{w0} \\ q_{w1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ q_{w(r-1)} \end{bmatrix}}_{Q_w} = \underbrace{\begin{bmatrix} -v_{w0} \\ -v_{w1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ -v_{w(t-r-1)} \end{bmatrix}}_{V_w} \tag{12}$$

(Or) $H_w * Q_w = V_w$ is the matrix-vector form, where $w = 1, 2, 3, 4$. The 'Q_w' vector can be solved in the least squares sense using the generalized inverse method. This gives the denominator vector estimate 'Q_w' as:

$$Q_w = \left(H_w^T H_w\right)^{-1} H_w^T V_w \tag{13}$$

If the coefficients 'q_{wi}' of 'Q_w' vector obtained by the solution of Eq. (12) does not constitute a stable denominator, then by adding an additional equation to this set to get stable denominator, then the solution of the linear set is:

$$\begin{bmatrix} v_{w(t)} & v_{w(t-1)} & \cdots & v_{w(t-r+1)} \end{bmatrix} \text{ and } \begin{bmatrix} -v_{w(t-r)} \end{bmatrix} \tag{14}$$

This process is continued until the Q_w vector gets stable and accurate denominator coefficients.

Finally, the reduced denominator obtained as

$$q_{wr}(\lambda) = q_{w0} + q_{w1}\lambda + \cdots + q_{w(r-1)}\lambda^{r-1} + \lambda^r \tag{15}$$

Step 4: Determination of numerator coefficients of reduced model substitute the reduced denominator coefficients in the Eq. (10) and replace it about $\lambda = 0$. Then, the time moments proportions, f_{wi} of the reduced model, is given as

$$K_{wr}(\lambda) = \sum_{i=0}^{\infty} f_{wi}\lambda^i \tag{16}$$

By equating the time moment proportionals of original system Eq. (9) with reduced model Eq. (16), the reduced numerator coefficients are found thus the reduced numerator is obtained as

$$p_{wr}(\lambda) = p_{w0} + p_{w1}\lambda + \cdots + p_{w(r-1)}\lambda^{r-1}$$

Thus, the corresponding four rth order reduced models obtained from the above as:

$$K_{1r}(\lambda) = \frac{p_{10} + p_{11}\lambda + \cdots + p_{1(r-1)}\lambda^{r-1}}{q_{10} + q_{11}\lambda + \cdots + q_{1(r-1)}\lambda^{r-1} + \lambda^r}$$

$$K_{2r}(\lambda) = \frac{p_{20} + p_{21}\lambda + \cdots + p_{2(r-1)}\lambda^{r-1}}{q_{20} + q_{21}\lambda + \cdots + q_{2(r-1)}\lambda^{r-1} + \lambda^r}$$

$$K_{3r}(\lambda) = \frac{p_{30} + p_{31}\lambda + \cdots + p_{3(r-1)}\lambda^{r-1}}{q_{30} + q_{31}\lambda + \cdots + q_{3(r-1)}\lambda^{r-1} + \lambda^r}$$

$$K_{4r}(\lambda) = \frac{p_{40} + p_{41}\lambda + \cdots + p_{4(r-1)}\lambda^{r-1}}{q_{40} + q_{41}\lambda + \cdots + q_{4(r-1)}\lambda^{r-1} + \lambda^r}$$

Step 5: From the above four transfer functions the minimum and maximum values of the coefficients are considered to formulate the reduced interval model represented as

$$\begin{aligned} G_r(\lambda) &= \frac{[p_{w0\min}, p_{w0\max}] + [p_{w1\min}, p_{w1\max}]\lambda + \cdots + [p_{wr-1\min}, p_{wr-1\max}]\lambda^{r-1}}{[q_{w0\min}, q_{w0\max}] + [q_{w1\min}, q_{w1\max}]\lambda + \cdots + [q_{wr\min}, q_{wr\max}]\lambda^r} \\ &= \frac{\sum_{i=0}^{r-1}[p_i^-, p_i^+]\lambda^i}{\sum_{i=0}^{r}[q_i^-, q_i^+]\lambda^i} \end{aligned}$$

Step 6: Re-convert the above integer order interval transfer function into its fractional commensurate interval form of the transfer function by substitute the $\lambda = s^{\Upsilon}$

$$G_r(s) = \frac{[p_0^-, p_0^+] + [p_1^-, p_1^+]s^{\Upsilon} + \cdots + [p_{r-1}^-, p_{r-1}^+]s^{\Upsilon r-1}}{[q_0^-, q_0^+] + [q_1^-, q_1^+]s^{\Upsilon} + \cdots + [q_{r-1}^-, q_{r-1}^+]s^{\Upsilon r-1} + [q_r^-, q_r^+]s^{\Upsilon r}} \quad (17)$$

The integral performance indices of the original system related to its reduced model are expressed to calculate and measure the goodness of the reduced order models, by means of the relative integral square error criterion, which are given as

$\mathrm{ISE}_{\mathrm{impulse},}$ $\mathrm{RISE}_{\mathrm{impulse},}$ $\mathrm{ISE}_{\mathrm{step}}$, $\mathrm{RISE}_{\mathrm{step}}$, $\mathrm{IAE}_{\mathrm{step}}$, $\mathrm{ITAE}_{\mathrm{step}}$ and are defined as follows

$$\mathrm{ISE}_{\mathrm{impulse}} = \int_{0}^{t\mathrm{sim}} [g(t) - g_r(t)]^2 \mathrm{d}t \tag{18}$$

$$\mathrm{RISE}_{\mathrm{impulse}} = \int_{0}^{t\mathrm{sim}} [g(t) - g_r(t)]^2 \mathrm{d}t \Bigg/ \int_{0}^{t\mathrm{sim}} g^2(t)\mathrm{d}t \tag{19}$$

$$\mathrm{ISE}_{\mathrm{step}} = \int_{0}^{t\mathrm{sim}} [y(t) - y_r(t)]^2 \mathrm{d}t \tag{20}$$

$$\mathrm{RISE}_{\mathrm{step}} = \int_{0}^{t\mathrm{sim}} [y(t) - y_r(t)]^2 \mathrm{d}t \Bigg/ \int_{0}^{t\mathrm{sim}} [y(t) - y(\infty)]^2 \mathrm{d}t \tag{21}$$

$$\mathrm{IAE}_{\mathrm{step}} = \int_{0}^{t\mathrm{sim}} |y(t) - y_r(t)| \mathrm{d}t \tag{22}$$

$$\mathrm{ITAE}_{\mathrm{step}} = \int_{0}^{t\mathrm{sim}} t|y(t) - y_r(t)| \mathrm{d}t \tag{23}$$

where $g(t)$ and $y(t)$ are the impulse and step responses of the original system, respectively, and $g_r(t)$, $y_r(t)$ are that of their approximants. $t$sim indicates the time required for the responses to reach the final steady state value.

3 Examples and Graphs

In this section, three test systems are deliberated and then these systems are reduced by the proposed method to show the superiority and effectiveness of this proposed new algorithm and the results are successfully verified using MATLAB. A step-by-step procedure is given for test systems.

Test system-1

Let us consider a commensurate fractional-order interval system as:

$$G_1(s) = \frac{[1,1]s^{2.4} + [7,8]s^{1.6} + [24,25]s^{0.8} + [24,25]}{[1,1]s^{3.2} + [10,11]s^{2.4} + [35,36]s^{1.6} + [50,51]s^{0.8} + [24,25]} \quad (24)$$

By substituting $\lambda = s^{0.8,}$ a fourth-order integer interval transfer function framed out in terms of λ as:

$$G_1(\lambda) = \frac{[1,1]\lambda^3 + [7,8]\lambda^2 + [24,25]\lambda^1 + [24,25]}{[1,1]\lambda^4 + [10,11]\lambda^3 + [35,36]\lambda^2 + [50,51]\lambda^1 + [24,25]} \quad (25)$$

Using Eqs. (4)–(17) the proposed reduced fractional interval model obtained as

$$G_r(s) = \frac{[0.34444, 0.87745]s^{0.8} + [1.294117, 5.277799]}{[1,1]s^{1.6} + [2.27941, 5.8333525]s^{0.8} + [1.294117, 5.277799]} \quad (26)$$

The trajectories of the original fractional interval system and the proposed reduced fractional interval model are compared in Fig. 1 and 2 for four-time moments shown in Table 1. The integral performance indices showed in Table 2 which are given as $ISE_{impulse}$, $RISE_{impulse}$, ISE_{step}, $RISE_{step}$, IAE_{step}, $ITAE_{step}$ of the original system related to its reduced model are expressed to strengthen the superiority of the proposed method resulting in fewer error values. It can be observed from Figs. 1 and 2 that the proposed method generates stable reduced fractional

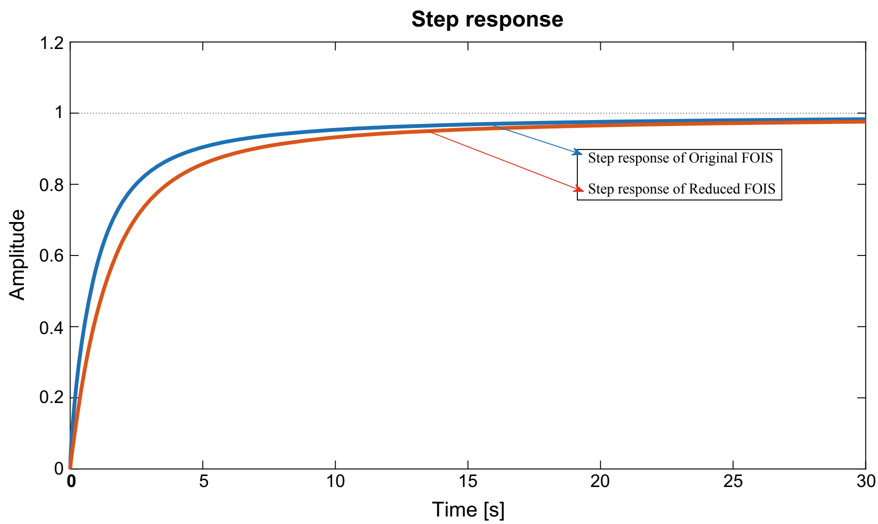

Fig. 1 The lower boundary step response of the reduced fractional order model and original system

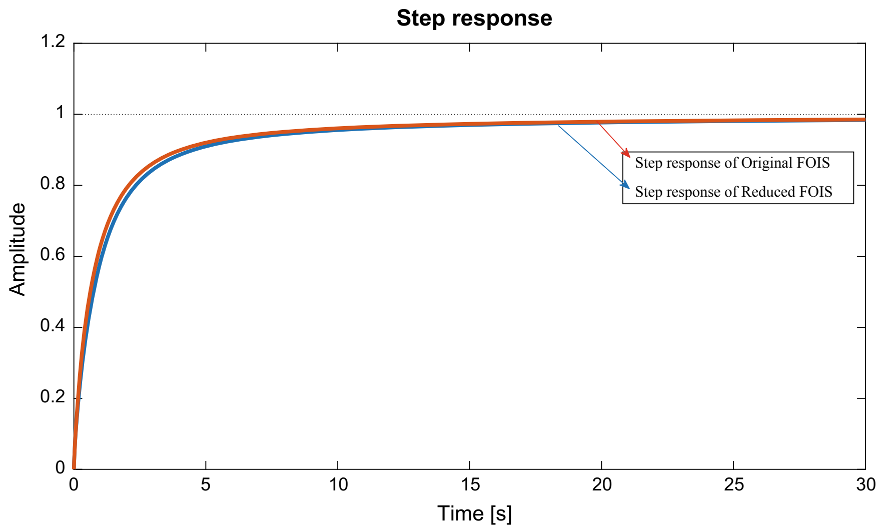

Fig. 2 The upper boundary step response of the reduced fractional order model and original system

Table 1 Time moments of four Kharitonov equations

$K_1(\lambda)$	$K_2(\lambda)$	$K_3(\lambda)$	$K_4(\lambda)$
$v_1 = 1$ $v_2 = -1.0833$ $v_3 = 1.09028$ $v_4 = -1.06308$	$v_1 = 1$ $v_2 = -1.04$ $v_3 = 1.0016$ $v_4 = -0.947264$	$v_1 = 1$ $v_2 = -1.04$ $v_3 = 0.96$ $v_4 = -0.86$	$v_1 = 1$ $v_2 = -1.0833$ $v_3 = 1.135417$ $v_4 = -1.162761$

Table 2 Performance indices of commensurate fractional-order interval systems

Examples		Performance indices					
		ISE for impulse response	RISE for impulse response	ISE for step response	RISE for step response	IAE for step response	ITAE for step response
Example 1	LB	0.052081	0.117601	0.44806	0.078574	0.411871	0.770593
	UB	0.009488	0.020671	0.004004	0.007276	0.100991	0.153221

interval order models with a good approximation for a stable original high order interval system. The transient and the steady state step response of the reduced model obtained by matching four-time moments closely follow the original high order commensurate fractional interval system. The proposed method provides the flexibility to the user in selecting the number and type of extra constraints to be taken into account for better approximations.

4 Conclusion

It is significant to study the behavior of non-integer interval system as they have a huge demand in applying for various control applications. Investigation still needs to be done vast as a part of research in this field. This approach tried to give solutions to some queries related to non-integer interval systems. This note recommended by the author states about the reduction of commensurate non-integer interval system. A blended model order reduction mechanism is implemented to extract a diminishing fractional non-integer model which constitutes a least squares and time moment matching algorithms. As the method associated with matrix-vector algebra, its estimation is clear, efficient and also develops a precise stable reduce interval model. Simulation results validate the dominance of this scheme where the original system may be retrieved with the approximate model thereby shorten the design procedure. Extending the results of this paper to be applicable for a designing of controllers to fractional-order interval systems can be noticed as an interesting research topic as future scope of work. The beauty of this technique is that it provides a great advantage to the engineering practitioners working with large and complex real-time systems.

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