Chapter 8
Granular Computing Based on m-Polar Fuzzy Hypergraphs

An m-polar fuzzy model, as an extension of fuzzy and bipolar fuzzy models, plays a vital role in modeling of real-world problems that involve multi-attribute, multipolar information, and uncertainty. The m-polar fuzzy models give increasing precision and flexibility to the system as compared to the fuzzy and bipolar fuzzy models. An m-polar fuzzy set assigns the membership degree to an object belonging to $[0, 1]^m$ describing the m distinct attributes of that element. Granular computing deals with representing and processing information in the form of information granules. These information granules are collections of elements combined together due to their similarity and functional/physical adjacency. In this chapter, we illustrate the formation of granular structures using m-polar fuzzy hypergraphs and level hypergraphs. Further, we define m-polar fuzzy hierarchical quotient space structures. The mappings between the m-polar fuzzy hypergraphs depict the relationships among granules that occurred in different levels. The consequences reveal that the representation of partition of universal set is more efficient through m-polar fuzzy hypergraphs as compared to crisp hypergraphs. We also present some examples and a real-world problem to signify the validity of our proposed model. This chapter is due to [11, 12, 18].

8.1 Introduction

Granular computing is defined as an identification of techniques, methodologies, tools, and theories that yields the advantages of clusters, groups or classes, i.e., the granules. The terminology was first introduced by Lin [15]. The fundamental concepts of granular computing are utilized in various disciplines, including machine learning, rough set theory, cluster analysis, and artificial intelligence. Different models have been proposed to study the various issues occurring in granular computing, including classification of the universe, illustration of granules, and the identification of relations among granules. For example, the procedure of problem-solving through

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granular computing can be considered as subdivisions of the problem at multilevels and these levels are linked together to construct a hierarchical space structure. Thus, this is a way of dealing with the formation of granules and the switching between different granularities. Here, the word "hierarchy" implies the methodology of hierarchical analysis in solving a problem and human activities [32]. To understand this methodology, let us consider an example of national administration in which the complete nation is subdivided into various provinces. Further, we divide every province into various divisions and so on. The human activities and problem-solving involve the simplification of original complicated problem by ignoring some details rather than thinking about all points of the problem. This rationalize model is then further refined till the issue is completely solved. Thus, we resolve and interpret the complex problems from weaker grain to stronger one or from highest rank to lowest or from universal to particular, etc. This technique is called the hierarchical problem-solving. This is further acknowledged that hierarchical strategy is the only technique which is used by humans to deal with complicated problems and it enhances the competence and efficiency. This strategy is also known as the multi-granular computing.

Hypergraphs, as an extension of classical graphs, experience various properties which appear very effective and useful as the basis of different techniques in many fields, including problem-solving, declustering, and databases [10]. The real-world problems which are represented and solved using hypergraphs have been achieved very good impacts. The formation of hypergraphs is same as that of granule structures and the relations between the vertices and hyperedges of hypergraphs can depict the relationships of granules and objects. A hyperedge can contain n vertices representing n-ary relations and hence can provide more effective analysis and description of granules. Many researchers have used hypergraph methods to study the clustering of complex documentation by means of granular computing and investigated the database techniques [16, 22]. Chen et al. [11] proposed a model of granular computing based on crisp hypergraph. They related a crisp hypergraph to a set of granules and represented the hierarchical structures using series of hypergraphs. They proved a hypergraph model as a visual description of granular computing.

Zadeh's [25] fuzzy set has been acquired greater attention by researchers in a wide range of scientific areas, including management sciences, robotics, decision theory, and many other disciplines. Zhang [29] generalized the idea of fuzzy sets to the concept of bipolar fuzzy sets whose membership degrees range over the interval $[-1, 1]$. An m-polar fuzzy set, as an extension of fuzzy set and bipolar fuzzy set, was proposed by Chen et al. [12] and it proved that 2-polar fuzzy sets and bipolar fuzzy sets are equivalent concepts in mathematics. An m-polar fuzzy set corresponds to the existence of "multipolar information" because there are many real-world problems which take data or information from n agents ($n \geq 2$). For example, in the case of telecommunication safety, the exact membership degree lies in the interval $[0, 1]^n$ ($n \approx 7 \times 10^9$) as the distinct members are monitored at different times. Similarly, there are many problems which are based on n logic implication operators ($n \geq 2$), including rough measures, ordering results of magazines and fuzziness measures, etc. To handle uncertainty in the representation of different objects or in the relationships between them, fuzzy graphs were defined by Rosenfeld [20]. m-polar fuzzy graphs and their interesting properties were discussed by Akram et al. [1] to deal with the network models possessing multi-attribute and multipolar data. As an extension of fuzzy graphs, Kaufmann [13] defined fuzzy hypergraphs. Although, many researchers have been explored the construction of granular structures using hypergraphs in various fields. However, there are many graph theoretic problems which may contain uncertainty and vagueness. To overcome the problems of uncertainty in models of granular computing, Wang and Gong [21] studied the construction of granular structures by means of fuzzy hypergraphs. They concluded that the representation of granules and partition is much efficient through the fuzzy hypergraphs. Novel applications and transversals of m-polar fuzzy hypergraphs were defined by Akram and Sarwar [5, 6]. Further, Akram and Shahzadi [7] studied various operations on m-polar fuzzy hypergraphs. Akram and Luqman [3, 4] introduced intuitionistic single-valued and bipolar neutrosophic hypergraphs. The basic purpose of this work is to develop an interpretation of granular structures using m-polar fuzzy hypergraphs. In the proposed model, the vertex of m-polar fuzzy hypergraph denotes an object and an m-polar fuzzy hyperedge represents a granule. The "refinement" and "coarsening" operators are defined to switch the different granularities from coarser to finer and vice versa, respectively.

For further terminologies and studies on m-polar fuzzy hypergraphs, readers are referred to [2, 8, 9, 14, 17, 19, 23, 26–28].

8.2 Fundamental Features of m-Polar Fuzzy Hypergraphs

Definition 8.1 An *m-polar fuzzy set* M on a universal set X is defined as a mapping $M{:}X \to [0,1]^m$. The membership degree of each element $z \in X$ is represented by $M(z)=(\mathscr{P}_1 \circ M(z), \mathscr{P}_2 \circ M(z), \mathscr{P}_3 \circ M(z), \ldots, \mathscr{P}_m \circ M(z))$, where $\mathscr{P}_j \circ M(z) : [0,1]^m \to [0,1]$ is defined as j-th projection mapping.

Note that, the m-th power of [0, 1] (i.e., $[0,1]^m$) is regarded as a partially ordered set with the point-wise order $\leq$, where m is considered as an ordinal number ($m = n|n < m$ when $m > 0$), $\leq$ is defined as $z_1 \leq z_2$ if and only if $\mathscr{P}_j(z_1) \leq \mathscr{P}_j(z_2)$, for every $1 \leq j \leq m$. $\mathbf{0} = (0, 0, \ldots, 0)$ and $\mathbf{1} = (1, 1, \ldots, 1)$ are the smallest and largest values in $[0,1]^m$, respectively.

Definition 8.2 Let M be an m-polar fuzzy set on X. An *m-polar fuzzy relation* $N = (\mathscr{P}_1 \circ N, \mathscr{P}_2 \circ N, \mathscr{P}_3 \circ N, \ldots, \mathscr{P}_m \circ N)$ on M is a mapping $N : M \to M$ such that $N(z_1z_2) \leq \inf\{M(z_1), M(z_2)\}$, for all $z_1, z_2 \in X$, i.e., for each $1 \leq j \leq m$, $\mathscr{P}_j \circ N(z_1z_2) \leq \inf\{\mathscr{P}_j \circ M(z_1), \mathscr{P}_j \circ M(z_2)\}$, where $\mathscr{P}_j \circ M(z)$ and $\mathscr{P}_j \circ N(z_1z_2)$ denote the j-th membership degree of an element $z \in X$ and the pair z_1z_2, respectively.

Definition 8.3 An *m-polar fuzzy graph* on X is defined as an ordered pair of functions $G = (C, D)$, where $C : X \to [0,1]^m$ is an m-polar vertex set and $D : X \times X \to [0,1]^m$ is an m-polar edge set of G such that $D(wz) \leq \inf\{C(w), C(z)\}$, i.e., $\mathscr{P}_j \circ D(wz) \leq \inf\{\mathscr{P}_j \circ C(w), \mathscr{P}_j \circ C(z)\}$, for all $w, z \in X$ and $1 \leq j \leq m$.

Definition 8.4 An *m-polar fuzzy hypergraph* on a non-empty set X is a pair $H = (A, B)$, where $A = \{M_1, M_2, \ldots, M_r\}$ is a finite family of m-polar fuzzy sets on X and B is an m-polar fuzzy relation on m-polar fuzzy sets M_k such that

- $B(E_k) = B(\{z_1, z_2, \ldots, z_l\}) \leq \inf\{M_k(z_1), M_k(z_2), \ldots, M_k(z_l)\}$,
- $\bigcup_{k=1}^{r} supp(M_k) = X$, for all $M_k \in A$ and for all $z_1, z_2, \ldots, z_l \in X$.

Definition 8.5 Let $H = (A, B)$ be an m-polar fuzzy hypergraph and $\tau \in [0, 1]^m$. Then the τ-cut level set of an m-polar fuzzy set M is defined as $M_\tau = \{z | \mathscr{P}_j \circ M(z) \geq t_j, 1 \leq j \leq m\}, \tau = (t_1, t_2, \ldots, t_m)$.

$H_\tau = (A_\tau, B_\tau)$ is called a *τ-cut level hypergraph* of H, where $A_\tau = \bigcup_{i=1}^{r} M_{i\tau}$.

8.2.1 Uncertainty Measures of m-Polar Fuzzy Hierarchical Quotient Space Structure

The question of distinct membership degrees of same object from different scholars is arisen because of various ways of thinking about the interpretation of different functions dealing with the same problem. To resolve this issue, fuzzy set was structurally defined by Zhang and Zhang [31] which was based on quotient space theory and fuzzy equivalence relation [30]. This definition provides some new initiatives regarding to membership degree, called a hierarchical quotient space structure of a fuzzy equivalence relation. By following the same concept, we develop a hierarchical quotient space structure of an m-polar fuzzy equivalence relation.

Definition 8.6 An m-polar fuzzy equivalence relation on a non-empty finite set X is called an *m-polar fuzzy similarity relation* if it satisfies,

1. $N(z, z) = (\mathscr{P}_1 \circ N(z, z), \mathscr{P}_2 \circ N(z, z), \ldots, \mathscr{P}_m \circ N(z, z)) = (1, 1, \ldots, 1)$, for all $z \in X$,
2. $N(u, w) = (\mathscr{P}_1 \circ N(u, w), \mathscr{P}_2 \circ N(u, w), \ldots, \mathscr{P}_m \circ N(u, w)) = (\mathscr{P}_1 \circ N(w, u), \mathscr{P}_2 \circ N(w, u), \ldots, \mathscr{P}_m \circ N(w, u)) = N(w, u)$, for all $u, w \in X$.

Definition 8.7 An m-polar fuzzy equivalence relation on a non-empty finite set X is called an *m-polar fuzzy equivalence relation* if it satisfies the conditions,

1. $N(z, z) = (\mathscr{P}_1 \circ N(z, z), \mathscr{P}_2 \circ N(z, z), \ldots, \mathscr{P}_m \circ N(z, z)) = (1, 1, \ldots, 1)$, for all $z \in X$,
2. $N(u, w) = (\mathscr{P}_1 \circ N(u, w), \mathscr{P}_2 \circ N(u, w), \ldots, \mathscr{P}_m \circ N(u, w)) = (\mathscr{P}_1 \circ N(w, u), \mathscr{P}_2 \circ N(w, u), \ldots, \mathscr{P}_m \circ N(w, u)) = N(w, u)$, for all $u, w \in X$,
3. for all $u, v, w \in X$, $N(u, w) = \sup_{v \in X}\{\min(N(u, v), N(v, w))\}$, i.e., $\mathscr{P}_j \circ N(u, w) = \sup_{v \in X}\{\min(\mathscr{P}_j \circ N(u, v), \mathscr{P}_j \circ N(v, w))\}, 1 \leq j \leq m$.

Definition 8.8 An *m-polar fuzzy quotient space* is denoted by a triplet $(X, \tilde{C}, N)$, where X is a finite domain, $\tilde{C}$ represents the attributes of X and N represents the m-polar fuzzy relationship between the objects of universe X, which is called the structure of the domain.

Definition 8.9 Let z_i and z_j be two objects in the universe X. The *similarity* between $z_i, z_j \in X$ having the attribute $\tilde{c}_k$ is defined as,

$$N(z_i, z_j) = \frac{|\tilde{c}_{ik} \cap \tilde{c}_{jk}|}{|\tilde{c}_{ik} \cup \tilde{c}_{jk}|},$$

where $\tilde{c}_{ik}$ represents that object z_i possesses the attribute $\tilde{c}_k$ and $\tilde{c}_{jk}$ represents that object z_j possesses the attribute $\tilde{c}_k$.

Proposition 8.1 *Let N be an m-polar fuzzy relation on a finite domain X and $N_\tau = \{(x, w)|\mathscr{P}_j \circ N(x, w) \geq t_j, 1 \leq j \leq m\}$, $\tau = (t_1, t_2, \ldots, t_j) \in [0, 1]$. Then, N_τ is an equivalence relation on X and is said to be cut-equivalence relation of N.*

Proposition 8.1 represents that N_τ is a crisp relation, which is equivalence on X and its *knowledge space* is given as $\xi_{N_\tau}(X) = X/N_\tau$.

The *value domain* of an equivalence relation N on X is defined as $D = \{N(w, y)|w, y \in X\}$ such that, $\mathscr{P}_j \circ X(w) \wedge \mathscr{P}_j \circ X(y) \wedge \mathscr{P}_j \circ N(x, y) > 0, 1 \leq j \leq m$.

Definition 8.10 Let N be an m-polar fuzzy equivalence relation on a finite set X and D be the value domain of N. The set given by $\xi_X(N) = \{X/N_\tau|\tau \in D\}$ is called *m-polar fuzzy hierarchical quotient space structure* of N.

Example 8.1 Let $X = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ be a finite set of elements and N_1 be a 4-polar fuzzy equivalence relation on X, the relation matrix $\tilde{M}_{N_1}$ corresponding to N_1 is given as follows:

$$\tilde{M}_{N_1} = \begin{bmatrix} (1,1,1,1) & (0.4,0.4,0.5,0.5) & (0.5,0.5,0.4,0.4) & (0.5,0.5,0.4,0.4) & (0.5,0.5,0.4,0.4) & (0.5,0.5,0.4,0.4) \\ (0.4,0.4,0.5,0.5) & (1,1,1,1) & (0.8,0.8,0.9,0.9) & (0.8,0.8,0.6,0.6) & (0.8,0.8,0.6,0.6) & (0.6,0.6,0.5,0.5) \\ (0.5,0.5,0.4,0.4) & (0.8,0.8,0.9,0.9) & (1,1,1,1) & (0.6,0.6,0.7,0.7) & (0.6,0.6,0.7,0.7) & (0.6,0.6,0.5,0.5) \\ (0.5,0.5,0.4,0.4) & (0.8,0.8,0.6,0.6) & (0.6,0.6,0.7,0.7) & (1,1,1,1) & (0.7,0.8,0.7,0.8) & (0.6,0.6,0.5,0.5) \\ (0.5,0.5,0.4,0.4) & (0.8,0.8,0.6,0.6) & (0.6,0.6,0.7,0.7) & (0.7,0.8,0.7,0.8) & (1,1,1,1) & (0.6,0.6,0.5,0.5) \\ (0.5,0.5,0.4,0.4) & (0.6,0.6,0.5,0.5) & (0.6,0.6,0.5,0.5) & (0.6,0.6,0.5,0.5) & (0.6,0.6,0.5,0.5) & (1,1,1,1) \end{bmatrix}.$$

Its corresponding m-polar fuzzy hierarchical quotient space structure is given as

$$\begin{aligned} X/N_{1\tau_1} &= X/N_{1(t_1,t_2,t_3,t_4)} = \{\{w_1, w_2, w_3, w_4, w_5, w_6\}\}, \\ X/N_{1\tau_2} &= X/N_{1(t_1',t_2',t_3',t_4')} = \{\{w_1\}, \{w_2, w_3, w_4, w_5, w_6\}\}, \\ X/N_{1\tau_3} &= X/N_{1(t_1'',t_2'',t_3'',t_4'')} = \{\{w_1\}, \{w_2, w_3, w_4, w_5\}, \{w_6\}\}, \\ X/N_{1\tau_4} &= X/N_{1(t_1''',t_2''',t_3''',t_4''')} = \{\{w_1\}, \{w_2, w_3\}, \{w_4, w_5\}, \{w_6\}\}, \\ X/N_{1\tau_5} &= X/N_{1(t_1'''',t_2'''',t_3'''',t_4'''')} = \{\{w_1\}, \{w_2, w_3\}, \{w_4\}, \{w_5\}, \{w_6\}\}, \\ X/N_{1\tau_6} &= X/N_{1(t_1''''',t_2''''',t_3''''',t_4''''')} = \{\{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}, \{w_5\}, \{w_6\}\}, \end{aligned}$$

where

$$\begin{aligned}
0 &< \tau_1 = (t_1, t_2, t_3, t_4) \leq 0.4,\\
0.4 &< \tau_2 = (t_1', t_2', t_3', t_4') \leq 0.5,\\
0.5 &< \tau_3 = (t_1'', t_2'', t_3'', t_4'') \leq 0.6,\\
0.6 &< \tau_4 = (t_1''', t_2''', t_3''', t_4''') \leq 0.7,\\
0.7 &< \tau_5 = (t_1'''', t_2'''', t_3'''', t_4'''') \leq 0.8,\\
0.8 &< \tau_6 = (t_1''''', t_2''''', t_3''''', t_4''''') \leq 1.
\end{aligned}$$

Hence, a 4-polar fuzzy hierarchical quotient space structure is given as $\xi_{X(N_1)} = \{X/N_{\tau_1}, X/N_{\tau_2}, X/N_{\tau_3}, X/N_{\tau_4}, X/N_{\tau_5}, X/N_{\tau_6}\}$ and is shown in Fig. 8.1.

It is worth to note that the same hierarchical quotient space structure can be formed by different 4-polar fuzzy equivalence relations. For instance, the relation matrix $\tilde{M}_{N_2}$ of 4-polar fuzzy equivalence relation generates the same hierarchical quotient space structure as given by $\tilde{M}_{N_1}$. The relation matrix $\tilde{M}_{N_2}$ is given as

$$\tilde{M}_{N_2} = \begin{bmatrix}
(1,1,1,1) & (0.2,0.2,0.5,0.5) & (0.6,0.6,0.4,0.4) & (0.6,0.6,0.4,0.4) & (0.6,0.6,0.4,0.4) & (0.6,0.6,0.4,0.4)\\
(0.2,0.2,0.5,0.5) & (1,1,1,1) & (0.2,0.2,0.5,0.5) & (0.2,0.2,0.5,0.5) & (0.2,0.2,0.5,0.5) & (0.2,0.2,0.5,0.5)\\
(0.6,0.6,0.4,0.4) & (0.2,0.2,0.5,0.5) & (1,1,1,1) & (0.7,0.7,0.7,0.7) & (0.7,0.7,0.7,0.7) & (0.7,0.7,0.7,0.7)\\
(0.6,0.6,0.4,0.4) & (0.2,0.2,0.5,0.5) & (0.7,0.7,0.7,0.7) & (1,1,1,1) & (0.8,0.8,0.7,0.8) & (0.6,0.6,0.5,0.5)\\
(0.6,0.6,0.4,0.4) & (0.2,0.2,0.5,0.5) & (0.7,0.7,0.7,0.7) & (0.8,0.8,0.7,0.8) & (1,1,1,1) & (0.6,0.6,0.5,0.5)\\
(0.6,0.6,0.4,0.4) & (0.2,0.2,0.5,0.5) & (0.7,0.7,0.7,0.7) & (0.6,0.6,0.5,0.5) & (0.6,0.6,0.5,0.5) & (1,1,1,1)
\end{bmatrix}.$$

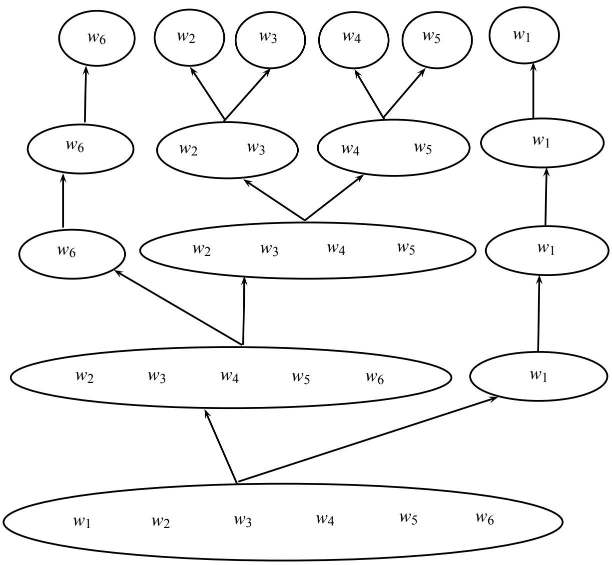


Fig. 8.1 A 4-polar fuzzy hierarchical quotient space structure

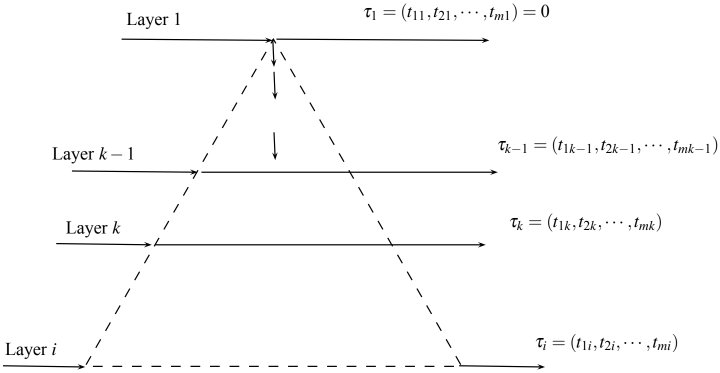


Fig. 8.2 Pyramid model of m-polar fuzzy hierarchical quotient space structure

Furthermore, assuming the number of blocks in every distinct layer of this hierarchical quotient space structure, a pyramid model can also be constructed as shown in Fig. 8.2.

8.2.2 *Information Entropy of m-Polar Fuzzy Hierarchical Quotient Space Structure*

Definition 8.11 Let N be an m-polar fuzzy equivalence relation on X. Let $\xi_X(N) = \{X(\tau_1), X(\tau_2), X(\tau_3), \ldots, X(\tau_j)\}$ be its corresponding hierarchical quotient space structure, where $\tau_i = (t_{1i}, t_{2i}, \ldots, t_{mi}), i = 1, 2, \ldots, j$ and $X(\tau_j) < X(\tau_{j-1}) < \cdots < X(\tau_1)$. Then, the *partition sequence* of $\xi_X(N)$ is given as $P(\xi_X(N)) = \{P_1, P_2, P_3, \ldots, P_j\}$, where $P_i = |X(\tau_i)|, i = 1, 2, \ldots, j$ and $|.|$ denotes the number of elements in a set.

Definition 8.12 Let N be an m-polar fuzzy equivalence relation on X. Let $\xi_X(N) = \{X(\tau_1), X(\tau_2), X(\tau_3), \ldots, X(\tau_j)\}$ be its corresponding hierarchical quotient space structure, where $\tau_i = (t_{1i}, t_{2i}, \ldots, t_{mi})$, $i = 1, 2, \ldots, j$ and $X(\tau_j) < X(\tau_{j-1}) < \cdots < X(\tau_1)$, $P(\xi_X(N)) = \{P_1, P_2, \ldots, P_j\}$ be the partition sequence of $\xi_X(N)$. Assume that $X(\tau_i) = \{X_{i1}, X_{i2}, \ldots, X_{iP_i}\}$. The *information entropy* $E_{X(\tau_i)}$ is defined as $E_{X(\tau_i)} = -\sum_{r=1}^{\mathscr{P}_i} \frac{|X_{ir}|}{|X|} ln(\frac{|X_{ir}|}{|X|})$.

Theorem 8.1 *Let N be an m-polar fuzzy equivalence relation on X. Let $\xi_X(N) = \{X(\tau_1), X(\tau_2), X(\tau_3), \ldots, X(\tau_j)\}$ be its corresponding hierarchical quotient space structure, where $\tau_i = (t_{1i}, t_{2i}, \ldots, t_{mi})$, $i = 1, 2, \ldots, j$, then the entropy sequence $E(\xi_X(N)) = \{E_{X(\tau_1)}, E_{X(\tau_2)}, \ldots, E_{X(\tau_j)}\}$ increases monotonically and strictly.*

Proof The terminology of hierarchical quotient space structure implies that $X(\tau_j) < X(\tau_{j-1}) < \cdots < X(\tau_1)$, i.e., $X(\tau_{j-1})$ is a quotient subspace of $X(\tau_j)$. Suppose that

$X(\tau_i) = \{X_{i1}, X_{i2}, \ldots, X_{iP_i}\}$ and $X(\tau_{i-1}) = \{X_{(i-1)1}, X_{(i-1)2}, \ldots, X_{(i-1)P_{(i-1)}}\}$, then every subblock of $X(\tau_{i-1})$ is an amalgam of subblocks of $X(\tau_i)$. Without loss of generality, it is assumed that only one subblock $X_{i-1,j}$ in $X(\tau_{i-1})$ is formed by the combination of two subblocks X_{ir}, X_{is} in $X(\tau_i)$ and all other remaining blocks are equal in both sequences. Thus,

$$\begin{aligned}
E_{X(\tau_{j-1})} &= -\sum_{r=1}^{P_{i-1}} \frac{|X_{i-1,r}|}{|X|}\ln(\frac{|X_{i-1,r}|}{|X|}) \\
&= -\sum_{r=1}^{P_{j-1}} \frac{|X_{i-1,r}|}{|X|}\ln(\frac{|X_{i-1,r}|}{|X|}) - \sum_{r=j+1}^{P_{i-1}} \frac{|X_{i-1,r}|}{|X|}\ln(\frac{|X_{i-1,r}|}{|X|}) - \frac{|X_{i-1,j}|}{|X|}\ln(\frac{|X_{i-1,j}|}{|X|}) \\
&= -\sum_{r=1}^{P_{j-1}} \frac{|X_{i,r}|}{|X|}\ln(\frac{|X_{i,r}|}{|X|}) - \sum_{r=j+1}^{P_{i}} \frac{|X_{i,r}|}{|X|}\ln(\frac{|X_{i,r}|}{|X|}) - \frac{|X_{i,r}|+|X_{i,s}|}{|X|}\ln(\frac{|X_{i,r}|+|X_{i,s}|}{|X|}).
\end{aligned}$$

Since,

$$\begin{aligned}
\frac{|X_{i,r}|+|X_{i,s}|}{|X|}\ln(\frac{|X_{i,r}|+|X_{i,s}|}{|X|}) &= \frac{|X_{i,r}|}{|X|}\ln(\frac{|X_{i,r}|+|X_{i,s}|}{|X|}) + \frac{|X_{i,s}|}{|X|}\ln(\frac{|X_{i,r}|+|X_{i,s}|}{|X|}) \\
&> \frac{|X_{i,r}|}{|X|}\ln(\frac{|X_{i,r}|}{|X|}) + \frac{|X_{i,s}|}{|X|}\ln(\frac{|X_{i,s}|}{|X|}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
E_{X(\tau_{j-1})} &< -\sum_{r=1}^{P_{j-1}} \frac{|X_{i,r}|}{|X|}\ln(\frac{|X_{i,r}|}{|X|}) - \sum_{r=j+1}^{P_{i}} \frac{|X_{i,r}|}{|X|}\ln(\frac{|X_{i,r}|}{|X|}) - \frac{|X_{i,r}|}{|X|}\ln(\frac{|X_{i,r}|}{|X|}) - \frac{|X_{i,s}|}{|X|}\ln(\frac{|X_{i,s}|}{|X|}), \\
&= E_{X(\tau_j)}, (2 \leq j \leq n).
\end{aligned}$$

Hence, $E_{X(\tau_1)} < E_{X(\tau_2)} < E_{X(\tau_2)} < \cdots < E_{X(\tau_j)}$.

Definition 8.13 Let $X = \{s_1, s_2, s_3, \ldots, s_n\}$ be a non-empty set of universe and let $P_d(X) = \{X_1, X_2, X_3, \ldots, X_d\}$ be a partition space of X, where $|P_d(X)| = d$ then $P_d(X)$ is called *d-order partition space* on X.

Definition 8.14 Let X be a finite non-empty universe and let $P_d(X) = \{X_1, X_2, X_3, \ldots, X_d\}$ be a d-order partition space on X. Let $|X_1| = l_1, |X_2| = l_2, \ldots, |X_d| = l_d$ and the sequence $\{l_1, l_2, \ldots, l_d\}$ is arranged in increasing order then we got a new sequence $\chi(d) = \{l'_1, l'_2, \ldots, l'_d\}$ which is also increasing and called a *subblock sequence* of $P_d(X)$.

Note that, two different d-order partition spaces on X may possess the similar subblock sequence $\chi(d)$.

Definition 8.15 Let X be a finite non-empty universe and let $P_d(X) = \{X_1, X_2, X_3, \ldots, X_d\}$ be a partition space of X. Suppose that $\chi_1(d) = \{l'_1, l'_2, \ldots, l'_d\}$ be a subblock sequence of $P_d(X)$, then the *ω-displacement* of $\chi_1(d)$ is defined as an increasing sequence $\chi_2(d) = \{l'_1, l'_2, \ldots, l'_r + 1, \ldots, l'_s - 1, \ldots, l'_d\}$, where $r < s$, $l'_r + 1 < l'_s - 1$.

An *ω-displacement* is obtained by subtracting 1 from some bigger term and adding 1 to some smaller element such that the sequence keeps its increasing property.

Theorem 8.2 *A single time ω-displacement $\chi_2(d)$ which is derived from $\chi_1(d)$ satisfies $E(\chi_1(d)) < E(\chi_2(d))$.*

Proof Let $\chi_1(d) = \{l'_1, l'_2, \ldots, l'_d\}$ and $\chi_2(d) = \{l'_1, l'_2, \ldots, l'_r + 1, \ldots, l'_s - 1, \ldots, l'_d\}$, $l'_1 + l'_2 + \cdots + l'_d = k$ then we have

$$E(\chi_2(t)) = -\sum_{j=1}^{d} \frac{l'_l}{k}\ln\frac{l'_l}{k} + \frac{l'_r}{k}\ln\frac{l'_r}{k} + \frac{l'_s}{k}\ln\frac{l'_s}{k} - \frac{l'_r+1}{k}\ln\frac{l'_r+1}{k} - \frac{l'_s-1}{k}\ln\frac{l'_s-1}{k}.$$

Let $g(z) = -\frac{z}{k}\ln\frac{z}{k} - \frac{l-z}{k}\ln\frac{l-z}{k}$, where $l = l'_r + l'_s$ and $g'(z) = \frac{1}{k}\ln\frac{l-z}{z}$. Suppose that $g'(z) = 0$, then we obtain a solution, i.e., $z = \frac{l}{2}$. Furthermore, $g''(z) = \frac{-l}{k(l-z)z} < 0$, $0 \leq z \leq \frac{l}{2}$ and $g(z)$ is increasing monotonically. Let $z_1 = l'_r$ and $z_2 = l'_r + 1$, $l'_r + 1 < l'_s - 1$, i.e., $z_1 < z_2 \leq \frac{l}{2} = \frac{l'_r + l'_s}{2}$. Since, $g(z)$ is monotone, then $g(z_2) - g(z_1) > 0$. Thus,

$$\frac{l'_r}{k}\ln\frac{l'_r}{k} + \frac{l'_s}{k}\ln\frac{l'_s}{k} - \frac{l'_r+1}{k}\ln\frac{l'_r+1}{k} - \frac{l'_s-1}{k}\ln\frac{l'_s-1}{k} > 0.$$

Hence,

$$\begin{aligned}
E(\chi_2(d)) &= -\sum_{j=1}^{d} \frac{l'_l}{k}\ln\frac{l'_l}{k} + \frac{l'_r}{k}\ln\frac{l'_r}{k} + \frac{l'_s}{k}\ln\frac{l'_s}{k} - \frac{l'_r+1}{k}\ln\frac{l'_r+1}{k} - \frac{l'_s-1}{k}\ln\frac{l'_s-1}{k} \\
&> -\left(\frac{l'_r+1}{k}\ln\frac{l'_r+1}{k} + \frac{l'_s-1}{k}\ln\frac{l'_s-1}{k}\right) \\
&> -\sum_{j=1}^{t} \frac{l'_l}{k}\ln\frac{l'_l}{k} \\
&= E(\chi_1(d)).
\end{aligned}$$

This completes the proof.

8.3 An m-Polar Fuzzy Hypergraph Model of Granular Computing

Definition 8.16 An *object space* is defined as a system (X, N), where X is a universe of objects or elements and $N = \{n_1, n_2, n_3, \ldots, n_k\}$, $k = |X|$ is a family of relations between the elements of X. For $r \leq k$, $n_r \in N$, $n_r \subseteq X \times X \times \cdots \times X$, if $(z_1, z_2, \ldots, z_r) \subseteq n_r$, then there exists an r-array relation n_r on $(z_1, z_2, \ldots, z_n)$.

A granule affiliates to a particular level. The whole view of granules at every level can be taken as a complete description of a particular problem at that level of granularity [11]. An m-polar fuzzy hypergraph formed by the set of relations N and membership degrees $X(w) = \mathscr{P}_j \circ X(w)$, $1 \leq j \leq m$ of objects in the space is considered as a specific level of granular computing model. All m-polar fuzzy hyperedges in that m-polar fuzzy hypergraph can be regarded as the complete granule in that particular level.

Definition 8.17 A *partition* of a set X established on the basis of relations between objects is defined as a collection of non-empty subsets which are pair-wise disjoint and whose union is whole of X. These subsets which form the partition of X are called *blocks*. Every partition of a finite set X contains the finite number of blocks. Corresponding to the m-polar fuzzy hypergraph, the constraints of partition $\psi = \{\mathscr{E}_i | 1 \leq i \leq n\}$.

(i) each $\mathscr{E}_i$ is non-empty,
(ii) for $i \neq j$, $\mathscr{E}_i \cap \mathscr{E}_j = \emptyset$,
(iii) $\cup\{supp(\mathscr{E}_i)|1 \leq i \leq n\} = X$.

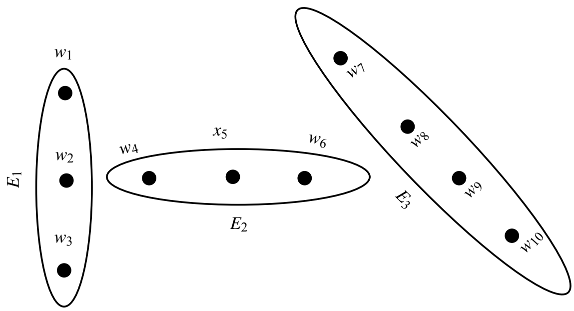
Definition 8.18 A *covering* of a set X is defined as a collection of non-empty subsets whose union is whole of X. The conditions for the covering $c = \{\mathscr{E}_i | 1 \leq i \leq n\}$ of X are stated as

(i) each $\mathscr{E}_i$ is non-empty,
(ii) $\cup\{supp(\mathscr{E}_i)|1 \leq i \leq n\} = X$.

The corresponding definitions in classical hypergraph theory are completely analogous to the above Definitions 8.17 and 8.18. In a crisp hypergraph, if the hyperedges E_i and E_j do not intersect each other, i.e., $E_i, E_j \in E$ and $E_i \cap E_j = \emptyset$ then these hyperedges form a *partition* of granules in this level. Furthermore, if $E_i, E_j \in E$ and $E_i \cap E_j \neq \emptyset$, i.e., the hyperedges E_i and E_j intersect each other, then these hyperedges form a *covering* in this level.

Example 8.2 Let $X = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9, w_{10}\}$. The partition and covering of X are given in Figs. 8.3 and 8.4, respectively.

Fig. 8.3 A partition of granules in a level



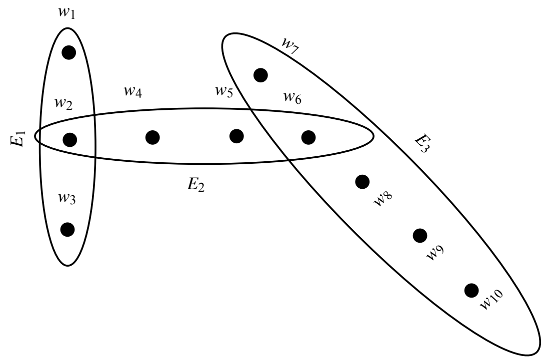


Fig. 8.4 A covering of granules in a level

A set-theoretic way to study the granular computing model uses the following operators in an m-polar fuzzy hypergraph model.

Definition 8.19 Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two granules in our model and the m-polar fuzzy hyperedges $\mathscr{E}_1$, $\mathscr{E}_2$ represent their external properties. The *union* of two granules $\mathscr{G}_1 \cup \mathscr{G}_2$ is defined as a larger m-polar fuzzy hyperedge that contains the vertices of both $\mathscr{E}_1$ and $\mathscr{E}_2$. If $w_i \in \mathscr{G}_1 \cup \mathscr{G}_2$, then the membership degree $(\mathscr{G}_1 \cup \mathscr{G}_2)(w_i)$ of w_i in larger granule $\mathscr{G}_1 \cup \mathscr{G}_2$ is defined as follows:

$$\mathscr{P}_j \circ (\mathscr{G}_1 \cup \mathscr{G}_2)(w_i) = \begin{cases} \max\{\mathscr{P}_j \circ (\mathscr{E}_1)(w_i), \mathscr{P}_j \circ (\mathscr{E}_2)(w_i)\}, & \text{if} \quad w_i \in \mathscr{E}_1 \text{ and } w_i \in \mathscr{E}_2, \\ \mathscr{P}_j \circ (\mathscr{E}_1)(w_i), & \text{if} \quad w_i \in \mathscr{E}_1 \text{ and } w_i \notin \mathscr{E}_2, \\ \mathscr{P}_j \circ (\mathscr{E}_2)(w_i), & \text{if} \quad w_i \in \mathscr{E}_2 \text{ and } w_i \notin \mathscr{E}_1, \end{cases}$$

$1 \leq j \leq m$.

Definition 8.20 Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two granules in our model and the m-polar fuzzy hyperedges $\mathscr{E}_1, \mathscr{E}_2$ represent their external properties. The *intersection* of two granules $\mathscr{G}_1 \cap \mathscr{G}_2$ is defined as a larger m-polar fuzzy hyperedge that contains the vertices of both $\mathscr{E}_1$ and $\mathscr{E}_2$. If $w_i \in \mathscr{G}_1 \cap \mathscr{G}_2$, then the membership degree $(\mathscr{G}_1 \cap \mathscr{G}_2)(w_i)$ of w_i in smaller granule $\mathscr{G}_1 \cap \mathscr{G}_2$ is defined as follows,

$$\mathscr{P}_j \circ (\mathscr{G}_1 \cap \mathscr{G}_2)(w_i) = \begin{cases} \min\{\mathscr{P}_j \circ (\mathscr{E}_1)(w_i), \mathscr{P}_j \circ (\mathscr{E}_2)(w_i)\}, & \text{if} \quad w_i \in \mathscr{E}_1 \text{ and } w_i \in \mathscr{E}_2, \\ \mathscr{P}_j \circ (\mathscr{E}_1)(w_i), & \text{if} \quad w_i \in \mathscr{E}_1 \text{ and } w_i \notin \mathscr{E}_2, \\ \mathscr{P}_j \circ (\mathscr{E}_2)(w_i), & \text{if} \quad w_i \in \mathscr{E}_2 \text{ and } w_i \notin \mathscr{E}_1, \end{cases}$$

$1 \leq j \leq m$.

Definition 8.21 Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two granules in our model and the m-polar fuzzy hyperedges $\mathscr{E}_1$, $\mathscr{E}_2$ represent their external properties. The *difference* between two granules $\mathscr{G}_1 - \mathscr{G}_2$ is defined as a smaller m-polar fuzzy hyperedge that contains those vertices belonging to $\mathscr{E}_1$ but not to $\mathscr{E}_2$.

Note that, if a vertex $w_i \in \mathscr{E}_1$ and $w_i \notin \mathscr{E}_2$, then $\mathscr{P}_j \circ (\mathscr{E}_1)(w_i) > 0$ and $\mathscr{P}_j \circ (\mathscr{E}_2)(w_i) = 0, 1 \leq j \leq m$.

Definition 8.22 A granule $\mathscr{G}_1$ is said to be the *sub-granule* of $\mathscr{G}_2$, if each vertex w_i of $\mathscr{E}_1$ also belongs to $\mathscr{E}_2$, i.e., $\mathscr{E}_1 \subseteq \mathscr{E}_2$. In such case, $\mathscr{G}_2$ is called the *super-granule* of $\mathscr{G}_1$.

Note that, if $\mathscr{E}(w_i) = \{0, 1\}$, then the all above described operators are reduced to classical hypergraphs theory of granular computing.

8.4 Formation of Hierarchical Structures

We can interpret a problem in distinct levels of granularities. These granular structures at different levels produce a set of m-polar fuzzy hypergraphs. The upper set of these hypergraphs constructs a hierarchical structure in distinct levels. The relationships between granules are expressed by lower level, which represents the problem as a concrete example of granularity. The relationships between granule sets are expressed by higher level, which represents the problem as an abstract example of granularity. Thus, the single-level structures can be constructed and then can be subdivided into hierarchical structures using the relational mappings between different levels.

Definition 8.23 Let $H^1 = (A^1, B^1)$ and $H^2 = (A^2, B^2)$ be two m-polar fuzzy hypergraphs. In an hierarchy structure, their level cuts are H^1_τ and H^2_τ, respectively, where $\tau = (t_1, t_2, \ldots, t_m)$. Let $\tau \in [0, 1]$ and $\mathscr{P}_j \circ \mathscr{E}^1_i \geq t_j$,$1 \leq j \leq m$, where $\mathscr{E}^1_i \in B^1$, then a mapping $\phi : H^1_\tau \to H^2_\tau$ from H^1_τ to H^2_τ maps the $\mathscr{E}^1_{\tau_i}$ in H^1_τ to a vertex w^2_i in H^2_τ. Furthermore, the mapping $\phi^{-1} : H^2_\tau \to H^1_\tau$ maps a vertex w^2_i in H^2_τ to τ-cut of m-polar fuzzy hyperedge $\mathscr{E}^1_\tau i$ in $H^1_{\tau_i}$. It can be denoted as $\phi(\mathscr{E}^1_{\tau_i}) = w^2_i$ or $\phi^{-1}(w^2_i) = \mathscr{E}^1_{\tau_i}$, for $1 \leq i \leq n$.

In an m-polar fuzzy hypergraph model, the mappings are used to describe the relations among different levels of granularities. At each distinct level, the problem is interpreted w.r.t the m-PF granularity of that level. The mapping associates the different descriptions of the same problem at distinct levels of granularities. There are two fundamental types to construct the method of hierarchical structures, the *top-down construction procedure* and *the bottom-up construction procedure* [24].

A formal discussion is provided to interpret an m-polar fuzzy hypergraph model ingranular computing, which is more compatible to human thinking. Zhang and Zhang [30] highlighted that one of the most important and acceptable characteristic of human intelligence is that the same problem can be viewed and analyzed in different granularities. Their claim is that the problem can not only be solved using various world of granularities but also can be switched easily and quickly. Hence, the procedure of solving a problem can be considered as the calculations in different hierarchies within that model.

A multilevel granularity of the problem is represented by an m-polar fuzzy hypergraph model, which allows the problem solvers to decompose it into various minor problems and transform it in other granularities. The transformation of problem in other granularities is performed by using two operators, i.e., zooming-in and zooming-out operators. The transformation from weaker level to finer level of granularity is done by zoom-in operator and the zoom-out operator deals with the shifting of problem from coarser to finer granularity.

Definition 8.24 Let $H^1 = (A^1, B^1)$ and $H^2 = (A^2, B^2)$ be two m-polar fuzzy hypergraphs, which are considered as two levels of hierarchical structures and H^2 owns the coarser granularity than H^1. Suppose $H^1_\tau = (X^1, E^1_\tau)$ and $H^2_\tau = (X^2, E^2_\tau)$ are the corresponding τ-level hypergraphs of H^1 and H^2, respectively. Let $e^1_i \in E^1_\tau$, $z^1_j \in X^1$, $e^2_j \in E^2_\tau$, $z^2_l, z^2_m \in X^2$ and $z^2_l, z^2_m \in e^2_j$. If $\phi(e^1_i) = z^2_l$, then $n(z^1_j, z^2_m)$ is the relationship between z^1_j and z^2_m and is obtained by the characteristics of granules.

Definition 8.25 Let the hyperedge $\phi^{-1}(z_l)$ be a vertex in a new level and the relation between hyperedges in this level is same as that of relationship between vertices in previous level. This is called the *zoom-in operator* and transforms a weaker level to a stronger level. The function $r(z^1_j, z^2_m)$ defines the relation between vertices of original level as well as new level.

Let the vertex $\phi(e_i)$ be a hyperedge in a new level and the relation between vertices in this level is same as that of relationship between hyperedges in corresponding level. This is called the *zoom-out operator* and transforms a finer level to a coarser level.

By using these zoom-in and zoom-out operators, a problem can be viewed at multi-levels of granularities. These operations allow us to solve the problem more appropriately and granularity can be switched easily at any level of problem-solving.

In an m-polar fuzzy hypergraph model of granular computing, the membership degrees of elements reflect the actual situation more efficiently and a wide variety of complicated problems in uncertain and vague environments can be presented by means of m-polar fuzzy hypergraphs. The previous analysis conclude that this model of granular computing generalizes the classical hypergraph model and fuzzy hypergraph model.

Definition 8.26 Let H^1 and H^2 be two crisp hypergraphs. Suppose that H^1 owns the finer m-polar fuzzy granularity than H^2. A mapping from H^1 to H^2 $\psi : H^1 \to H^2$ maps a hyperedge of H^1 to the vertex of H^2 and the mapping $\psi^{-1} : H^2 \to H^1$ maps a vertex of H^2 to the hyperedge of H^1.

The procedure of bottom-up construction for level hypergraph model is illustrated in Algorithm 8.4.1.

Algorithm 8.4.1

The procedure of bottom-up construction for level hypergraph

1. Determine an $m-$polar fuzzy equivalence relation matrix according to the actual circumstances.
2. For fixed $\tau \in [0, 1]$, obtain the corresponding hierarchical quotient space structure.
3. Obtain the hyperedges through the hierarchical quotient space structure.
4. Granules in i-level are mapped to $(i+1)-$level.
5. Calculate the $m-$polar fuzzy relationships between the vertices of $(i+1)-$level and determine the $m-$polar fuzzy equivalence relation matrix.
6. Determine the corresponding hierarchical quotient space structure according to τ, which is fixed in Step 2.
7. Get the hyperedges in $(i+1)-$level and $(i+1)-$level of the model is constructed.
8. Step 1 - Step 5 are repeated until the whole universe is formulated to a single granule.

Definition 8.27 Let N be an m-polar fuzzy equivalence relation on X. A *coarse gained universe* X/N_τ can be obtained by using m-polar fuzzy equivalence relation, where $[w_i]_{N_\tau} = \{w_j \in X | w_i N w_j\}$. This equivalence class $[w_i]_{N_\tau}$ is considered as an hyperedge in the level hypergraph.

Definition 8.28 Let $H_1 = (X_1, E_1)$ and $H_2 = (X_2, E_2)$ be level hypergraphs of m-polar fuzzy hypergraphs and H_2 has weaker granularity than H_1. Suppose that e_i^1, $e_j^2 \in E_1$ and $w_i^2, w_j^2 \in X^2$, $i, j = 1, 2, \ldots, n$. The *zoom-in operator* $\omega : H_2 \to H_1$ is defined as $\omega(w_i^2) = e_i^1$, $e_i^1 \in E_1$. The relations between the vertices of H^2 define the relationships among the hyperedges in new level. The zoom-in operator of two levels is shown in Fig. 8.5.

Remark 1 For all $X_2', X_2'' \subseteq X_2$, we have $\omega(X_2') = \bigcup_{w_i^2 \in X_2'} \omega(w_i^2)$ and $\omega(X_2'') = \bigcup_{w_j^2 \in X_2''} \omega(w_j^2)$.

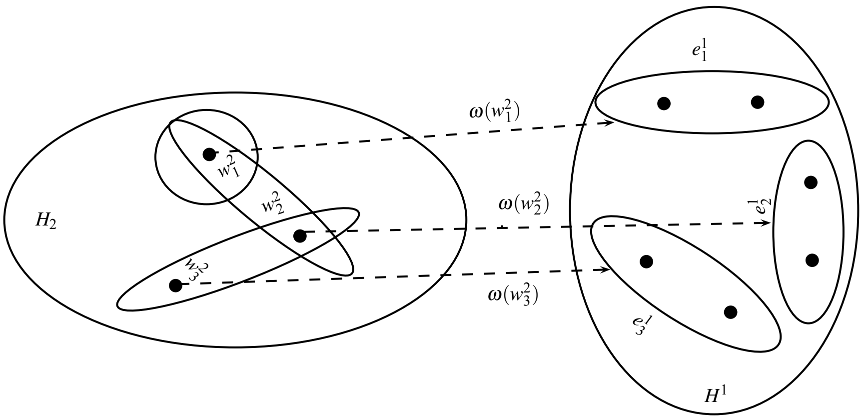


Fig. 8.5 Zoom-in operator

Theorem 8.3 *Let $H_1 = (X_1, E_1)$ and $H_2 = (X_2, E_2)$ be two levels and $\omega : H_2 \to H_1$ be the zoom-in operator. Then for all $X_2', X_2'' \subseteq X_2$, the zoom-in operator satisfies*

(i) *ω maps the empty set to an empty set, i.e., $\omega(\emptyset) = \emptyset$,*
(ii) *$\omega(X_2) = E_1$,*
(iii) *$\omega([X_2']^c) = [\omega(X_2')]^c$,*
(iv) *$\omega(X_2' \cap X_2'')= \omega(X_2') \cap \omega(X_2'')$,*
(v) *$\omega(X_2' \cup X_2'')= \omega(X_2') \cup \omega(X_2'')$,*
(vi) *$X_2' \subseteq X_2''$ if and only if $\omega(X_2') \subseteq \omega(X_2'')$.*

Proof (i) It is trivially satisfied that $\omega(\emptyset) = \emptyset$.

(ii) As we know that for all $w_i^2 \in X_2$, we have $\omega(X_2') = \bigcup_{w_i^2 \in X_2'} \omega(w_i^2)$. Since $\omega(w_i^2) = e_i^1$, we have $\omega(X_2') = \bigcup_{w_i^2 \in X_2'} \omega(w_i^2) = \bigcup_{e_i^1 \in E_1} e_i^1 = E_1$.

(iii) Let $[X_2']^c = X_2'$ and $[X_2'']^c = X_2''$, then it is obvious that $X_2' \cap X_2' = \emptyset$ and $X_2' \cup X_2' = X_2$. It follows from (ii) that $\omega(X_2) = E_1$ and we denote by W_1' that edge set of H_1 on which the vertex set X_2' of H_2 is mapped under ω, i.e., $\omega(X_2') = W_1'$. Then $\omega([X_2']^c) = \omega(X_2') = \bigcup_{w_i^2 \in X_2'} \omega(w_i^2) = \bigcup_{e_i^1 \in W_1'} e_i^1 = X_1'$ and $[\omega(X_2')]^c = [\bigcup_{w_j^2 \in X_2'} \omega(w_j^2)]^c = [\bigcup_{e_j^1 \in E_1'} e_j^1]^c = (E_1')^c$. Since, the relationship between hyperedges in new level is same as that of relations among vertices in original level so we have $(E_1')^c = X_1'$. Hence, we conclude that $\omega([X_2']^c) = [\omega(X_2')]^c$.

(iv) Assume that $X_2' \cap X_2'' = \tilde{X}_2$ then for all $w_i^2 \in \tilde{X}_2$ implies that $w_i^2 \in X_2'$ and $w_i^2 \in X_2''$. Further, we have $\omega(X_2' \cap X_2'') = \omega(\tilde{X}_2) = \bigcup_{w_i^2 \in \tilde{X}_2} \omega(w_i^2)= \bigcup_{e_i^1 \in \tilde{E}_1} \omega(e_i^1) = \tilde{E}_1$.
$\omega(X_2') \cap \omega(X_2'')=\{\bigcup_{w_i^2 \in X_2'} \omega(w_i^2)\} \cap \{\bigcup_{w_j^2 \in X_2''} \omega(w_j^2)\} = \bigcup_{e_i^1 \in E_1'} e_i^1 \cap \bigcup_{e_j^1 \in E_1''} e_j^1 = E_1' \cap E_1''$. Since, the relationship between hyperedges in new level is same as that of relations among vertices in original level so we have $E_1' \cap E_1'' = \tilde{E}_1$. Hence, we conclude that $\omega(X_2' \cap X_2'')= \omega(X_2') \cap \omega(X_2'')$.

(v) Assume that $X_2' \cup X_2'' = \bar{X}_2$. Then we have $\omega(X_2' \cup X_2'') = \omega(\bar{X}_2) = \bigcup_{w_i^2 \in \bar{X}_2} \omega(w_i^2)= \bigcup_{e_i^1 \in \bar{E}_1} \omega(e_i^1) = \bar{E}_1$.
$\omega(X_2') \cup \omega(X_2'')=\{\bigcup_{w_i^2 \in X_2'} \omega(w_i^2)\} \cup \{\bigcup_{w_j^2 \in X_2''} \omega(w_j^2)\} = \bigcup_{e_i^1 \in E_1'} e_i^1 \cup \bigcup_{e_j^1 \in E_1''} e_j^1 = E_1' \cup E_1''$. Since, the relationship between hyperedges in new level is same as that of relations among vertices in original level so we have $E_1' \cup E_1'' = \bar{E}_1$. Hence, we conclude that $\omega(X_2' \cup X_2'')= \omega(X_2') \cup \omega(X_2'')$.

(vi) First we show that $X_2' \subseteq X_2''$ implies that $\omega(X_2') \subseteq \omega(X_2'')$. Since, $X_2' \subseteq X_2''$, which implies that $X_2' \cap X_2'' = X_2'$ and $\omega(X_2') = \bigcup_{w_i^2 \in X_2'} \omega(w_i^2) = \bigcup_{e_i^1 \in E_1'} e_i^1 = E_1'$. Also $\omega(X_2'') = \bigcup_{w_j^2 \in X_2''} \omega(w_j^2) = \bigcup_{e_j^1 \in E_1''} e_j^1 = E_1''$. Since, the relationship between

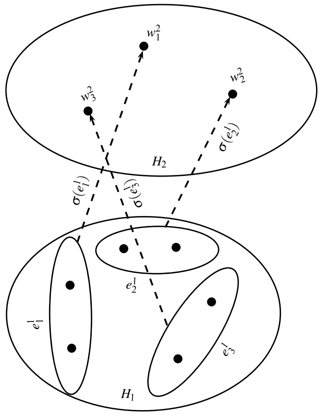


Fig. 8.6 Zoom-out operator

hyperedges in new level is same as that of relations among vertices in original level so we have $E_1' \subseteq E_1''$, i.e., $\omega(X_2') \subseteq \omega(X_2'')$. Hence, $X_2' \subseteq X_2''$ implies that $\omega(X_2') \subseteq \omega(X_2'')$.
We now prove that $\omega(X_2') \subseteq \omega(X_2'')$ implies that $X_2' \subseteq X_2''$. Suppose on contrary that whenever $\omega(X_2') \subseteq \omega(X_2'')$ then there is at least one vertex $w_i^2 \in X_2'$ but $w_i^2 \notin X_2''$, i.e., $X_2' \nsubseteq X_2''$. Since, $\omega(w_i^2) = e_i^1$ and the relationship between hyperedges in new level is same as that of relations among vertices in original level so we have $e_i^1 \in E_1'$ but $e_i^1 \notin E_1''$, i.e., $E_1' \nsubseteq E_1''$, which is contradiction to the supposition. Thus, we have $\omega(X_2') \subseteq \omega(X_2'')$ implies that $X_2' \subseteq X_2''$. Hence, $X_2' \subseteq X_2''$ if and only if $\omega(X_2') \subseteq \omega(X_2'')$.

Definition 8.29 Let $H_1 = (X_1, E_1)$ and $H_2 = (X_2, E_2)$ be level hypergraphs of m-polar fuzzy hypergraphs and H_2 has weaker granularity than H_1. Suppose that e_i^1, $e_j^2 \in E_1$ and $w_i^2, w_j^2 \in X_2$, $i, j = 1, 2, \ldots, n$. The *zoom-out operator* $\sigma : H_1 \to H_2$ is defined as $\sigma(e_i^1) = w_i^2, w_i^2 \in X_2$. The zoom-out operator of two levels is shown in Fig. 8.6.

Theorem 8.4 *Let $\sigma : H_1 \to H_2$ be the zoom-out operator from $H_1 = (X_1, E_1)$ to $H_2 = (X_2, E_2)$ and let $E_1' \subseteq E_1$. Then, the zoom-out operator σ satisfies the following properties:*

(i) $\sigma(\emptyset) = \emptyset$,
(ii) *σ maps the set of hyperedges of H_1 onto the set of vertices of H_2, i.e., $\sigma(E_1) = X_2$,*
(iii) $\sigma([E_1']^c) = [\sigma(E_1')]^c$.

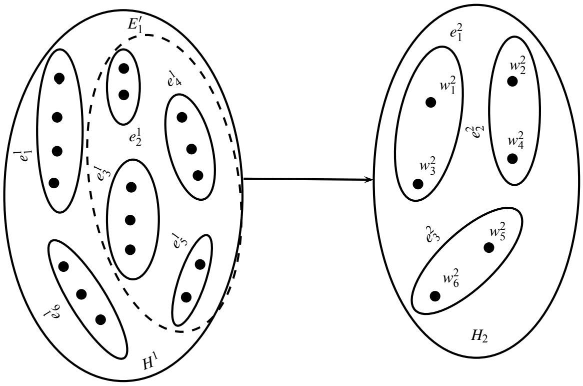


Fig. 8.7 Internal and external zoom-out operators

Proof (i) This part is trivially satisfied.

(ii) According to the definition of σ, we have $\sigma(e_i^1) = w_i^2$. Since, the hyperedges define a partition of hypergraph so we have $E_1 = \{e_1^1, e_2^1, e_3^1, \ldots, e_n^1\} = \bigcup_{e_i^1 \in E_1} e_i^1$.

Then

$\sigma(E_1) = \sigma(\bigcup_{e_i^1 \in E_1} e_i^1) = \bigcup_{e_i^1 \in E_1} \sigma(e_i^1) = \bigcup_{w_i^2 \in X_2} w_i^2 = X_2.$

(iii) Assume that $[E_1']^c = V_1'$ then it is obvious that $E_1' \cap V_1' = \emptyset$ and $E_1' \cup V_1' = E_1$. Suppose on contrary that there exists at least one vertex $w_i^2 \in \sigma([E_1']^c)$ but $w_i^2 \notin [\sigma(E_1')]^c$. $w_i^2 \in \sigma([E_1']^c)$ implies that $w_i^2 \in \sigma(V_1') \Rightarrow w_i^2 \in \bigcup_{e_i^1 \in V_1'} \sigma(e_i^1) \Rightarrow w_i^2 \in \bigcup_{e_i^1 \in E_1 \setminus E_1'} \sigma(e_i^1)$. Since, $w_i^2 \notin [\sigma(E_1')]^c \Rightarrow w_i^2 \in \sigma(E_1') \Rightarrow w_i^2 \in \bigcup_{e_i^1 \in E_1'} \sigma(e_i^1)$, which is contradiction to our assumption. Hence, $\sigma([E_1']^c) = [\sigma(E_1')]^c$.

Definition 8.30 Let $H_1 = (X_1, E_1)$ and $H_2 = (X_2, E_2)$ be two levels of m-polar fuzzy hypergraphs and H_1 possesses the stronger granularity than H_2. Let $E_1' \subseteq E_1$ then $\hat{\sigma}(E_1') = \{e_i^2 | e_i^2 \in E_2, \kappa(e_i^2) \subseteq E_1'\}$ is called *internal zoom-out operator*.

The operator $\check{\sigma}(E_1') = \{e_i^2 | e_i^2 \in E_2, \kappa(e_i^2) \cap E_1' \neq \emptyset\}$ is called *external zoom-out operator*.

Example 8.3 Let $H_1 = (X_1, E_1)$ and $H_2 = (X_2, E_2)$ be two levels of m-polar fuzzy hypergraphs and H_1 possesses the stronger granularity than H_2, where $E_1 = \{e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1\}$ and $E_2 = \{e_1^2, e_2^2, e_3^2\}$. Furthermore, $e_1^2 = \{w_1^2, w_3^2\}$, $e_2^2 = \{w_2^2, w_4^2\}$, $e_3^2 = \{w_5^2, w_6^2\}$ as shown in Fig. 8.7.

Let $E_1' = \{e_2^1, e_3^1, e_4^1, e_5^1\}$ be the subset of hyperedges of H_1 then we can not zoom-out to H_2 directly, thus by using the internal and external zoom-out operators we have the following relations.

$\hat{\sigma}(\{e_2^1, e_3^1, e_4^1, e_5^1\}) = \{e_2^2\}$,

$\check{\sigma}(\{e_2^1, e_3^1, e_4^1, e_5^1\}) = \{e_1^2, e_2^2, e_3^2\}$.

8.5 A Granular Computing Model of Web Searching Engines

The most fertile way to direct a search on the Internet is through a search engine. A web search engine is defined as a system software which is designed to search for queries on World Wide Web. A user may utilize a number of search engines to gather information and similarly various searchers may make an effective use of same engine to fulfill their queries. In this section, we construct a granular computing model of web searching engines based on 4-polar fuzzy hypergraph. In a web searching hypernetwork, the vertices denote the various search engines. According to the relation set N, the vertices having some relationship are united together as an hyperedge, in which the search engines serve only one user. After assigning the membership degrees to that unit, a 4-polar fuzzy hyperedge is constructed, which is also considered as a granule. A 4-polar fuzzy hyperedge indicates a user who wants to gather some information and the vertices in that hyperedge represent those search engines which provide relevant data to the user. Let us consider there are ten search engines and the corresponding 4-polar fuzzy hypergraph $H = (A, B)$ is shown in Fig. 8.8. Note that, $A = \{e_1, e_2, e_3, \ldots, e_{10}\}$ and $B = \{U_1, U_2, U_3, U_4, U_5\}$.

The incidence matrix of 4-polar fuzzy hypergraph is given in Table 8.1.

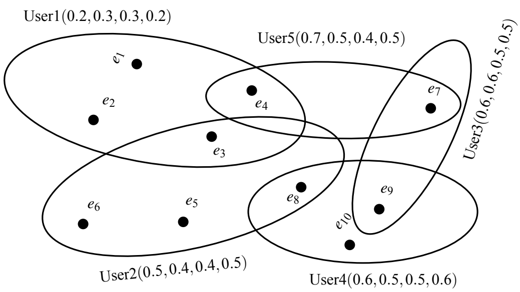


Fig. 8.8 A 4-polar fuzzy hypergraph representation of web searching

Table 8.1 Incidence matrix

X	$U1$	$U2$	$U3$	$U4$	$U5$
e_1	(0.2, 0.3, 0.3, 0.2)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
e_2	(0.2, 0.3, 0.3, 0.2)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
e_3	(0.2, 0.3, 0.3, 0.2)	(0.5, 0.4, 0.4, 0.5)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
e_4	(0.2, 0.3, 0.3, 0.2)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0.7, 0.5, 0.4, 0.5)
e_5	(0, 0, 0, 0)	(0.5, 0.4, 0.4, 0.5)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
e_6	(0, 0, 0, 0)	(0.5, 0.4, 0.4, 0.5)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
e_7	(0, 0, 0, 0)	(0, 0, 0, 0)	(0.6, 0.6, 0.5, 0.5)	(0, 0, 0, 0)	(0.7, 0.5, 0.4, 0.5)
e_8	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0.6, 0.5, 0.5, 0.6)	(0, 0, 0, 0)
e_9	(0, 0, 0, 0)	(0, 0, 0, 0)	(0.6, 0.6, 0.5, 0.5)	(0.6, 0.5, 0.5, 0.6)	(0, 0, 0, 0)
e_{10}	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0.6, 0.5, 0.5, 0.6)	(0, 0, 0, 0)

Table 8.2 The information table

X	Core technology	Scalability	Content processing	Query functionality
e_1	0.7	0.6	0.5	0.7
e_2	0.6	0.5	0.5	0.6
e_3	0.7	0.8	0.8	0.7
e_4	0.8	0.6	0.6	0.8
e_5	0.7	0.5	0.5	0.7
e_6	0.7	0.6	0.5	0.7
e_7	0.6	0.5	0.5	0.6
e_8	0.7	0.8	0.8	0.7
e_9	0.8	0.6	0.6	0.8
e_{10}	0.7	0.8	0.8	0.7

An m-polar fuzzy hypergraph model of granular computing illustrates a vague set having some membership degrees. In this model, there are five users need the search engines to gather information. Note that, the membership degrees of these engines are different to the users because whenever a user selects a search engine, he/she considers various factors or attributes. Hence, an m-polar fuzzy hypergraph in granular computing is more meaning full and effective.

Let us suppose that each search engine possesses four attributes which are *Core Technology, Scalability, Content Processing, Query Functionality*. The information table for various search engines having these attributes is given in Table 8.2.

The membership degrees of search engines reveal the percentage of attributes possessed by them, e.g., e_1 own 70% of *core technology*, 60% *scalability*, 50% provide *content processing* and *query functionality* of this engine is 70%. The 4-polar fuzzy equivalence relation matrix describes the similarities between these search engines and is given as follows:

$$\tilde{P}_N = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{0.7} & \mathbf{0.7} & \mathbf{0.7} & \mathbf{0.7} & \mathbf{0.7} & \mathbf{0.7} & \mathbf{0.7} & \mathbf{0} \\ \mathbf{0.6} & \mathbf{0.7} & \mathbf{1} & \mathbf{0.8} & \mathbf{0.8} & \mathbf{0.8} & \mathbf{0.8} & \mathbf{0.8} & \mathbf{0.8} & \mathbf{0} \\ \mathbf{0.6} & \mathbf{0.7} & \mathbf{0.8} & \mathbf{1} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.6} \\ \mathbf{0.6} & \mathbf{0.7} & \mathbf{0.8} & \mathbf{0.6} & \mathbf{1} & \mathbf{0.5} & \mathbf{0.5} & \mathbf{0.5} & \mathbf{0.5} & \mathbf{0} \\ \mathbf{0.6} & \mathbf{0.7} & \mathbf{0.8} & \mathbf{0.6} & \mathbf{0.5} & \mathbf{1} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0.6} & \mathbf{0} \\ \mathbf{0.6} & \mathbf{0.7} & \mathbf{0.8} & \mathbf{0.6} & \mathbf{0.5} & \mathbf{0.6} & \mathbf{1} & \mathbf{0.7} & \mathbf{0.7} & \mathbf{0.7} \\ \mathbf{0.6} & \mathbf{0.7} & \mathbf{0.8} & \mathbf{0.6} & \mathbf{0.5} & \mathbf{0.6} & \mathbf{0.7} & \mathbf{1} & \mathbf{0.8} & \mathbf{0.8} \\ \mathbf{0.6} & \mathbf{0.7} & \mathbf{0.8} & \mathbf{0.6} & \mathbf{0.5} & \mathbf{0.6} & \mathbf{0.7} & \mathbf{0.8} & \mathbf{1} & \mathbf{0.8} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0.6} & \mathbf{0} & \mathbf{0} & \mathbf{0.7} & \mathbf{0.8} & \mathbf{0.8} & \mathbf{1} \end{bmatrix},$$

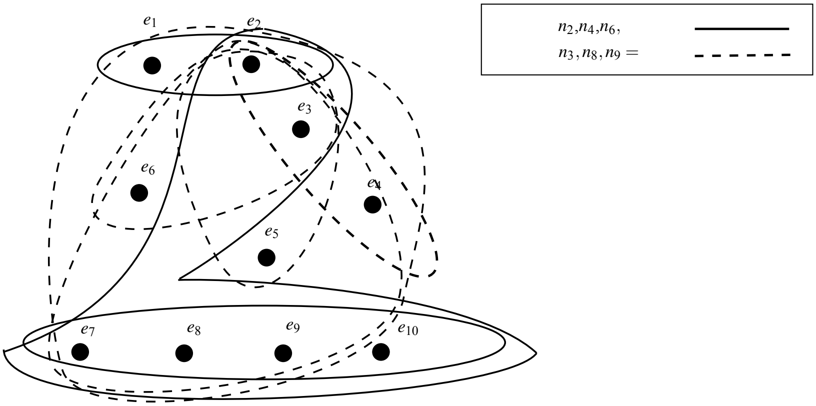


Fig. 8.9 A single-level model of 4-polar fuzzy hypergraph

where $\mathbf{1} = (1, 1, 1, 1)$, $\mathbf{0} = (0, 0, 0, 0)$, $\mathbf{0.5} = (0.5, 0.5, 0.5, 0.5)$, $\mathbf{0.6} = (0.6, 0.6, 0.6, 0.6)$, $\mathbf{0.7} = (0.7, 0.7, 0.7, 0.7)$ and $\mathbf{0.8} = (0.8{,}0.8{,}0.8{,}0.8)$. Let $\tau = (t_1, t_2, t_3, t_4) = (0.7, 0.7, 0.7, 0.7)$, then its corresponding hierarchical quotient space structure is given as follows:

$$X/N_\tau = X/N_{(0.7,0.7,0.7,0.7)} = \{\{e_1, e_2\}, \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}, \{e_2, e_3, e_5\}, \{e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}, \{e_2, e_3, e_4\}, \{e_2, e_3, e_6\}, \{e_2, e_3, e_7, e_8, e_9, e_{10}\}, \{e_7, e_8, e_9, e_{10}\}\}.$$

Note that, $n_1 = n_5 = n_7 = n_{10} = \{\emptyset\}$, $n_2 = \{(e_1, e_2)\}$, $n_3 = \{(e_2, e_3, e_4), (e_2, e_3, e_5), (e_2, e_3, e_6)\}$, $n_4 = \{(e_7, e_8, e_9, e_{10})\}$, $n_6 = \{(e_2, e_3, e_7, e_8, e_9, e_{10})\}$, $n_8 = \{(e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9)\}$, $n_9 = \{(e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9)\}$. Hence, a single level of 4-polar fuzzy hypergraph model is constructed and is shown in Fig. 8.9.

Thus, we can obtain eight hyperedges $E_1 = \{e_1, e_2\}$, $E_2 = \{e_2, e_3, e_4\}$, $E_3 = \{e_2, e_3, e_5\}$, $E_4 = \{e_2, e_3, e_6\}$, $E_5 = \{e_7, e_8, e_9, e_{10}\}$, $E_6 = \{e_2, e_3, e_7, e_8, e_9, e_{10}\}$, $E_7 = \{e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$, $E_8 = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$. The procedure of constructing this single-level model is explained in the following flow chart Fig. 8.10.

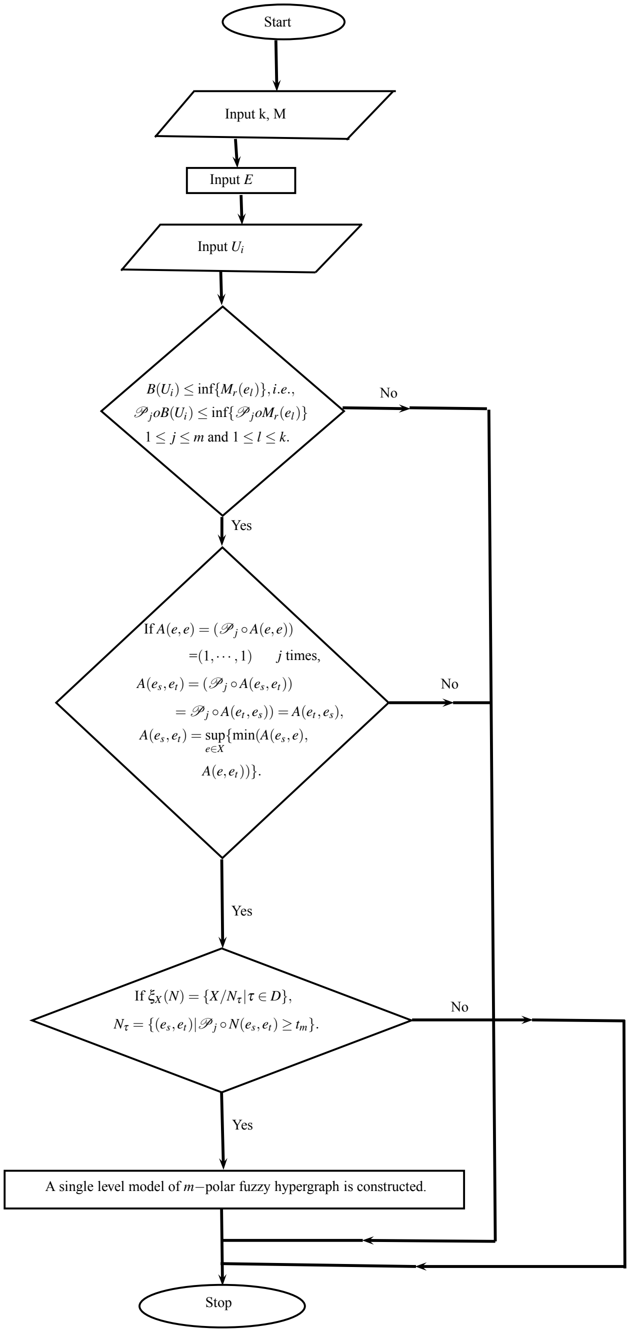


Fig. 8.10 Flow chart of single-level model of m-polar fuzzy hypergraph

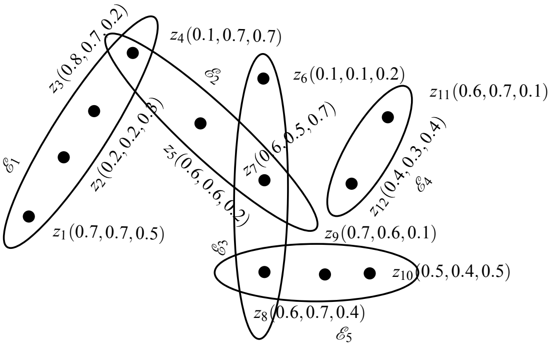
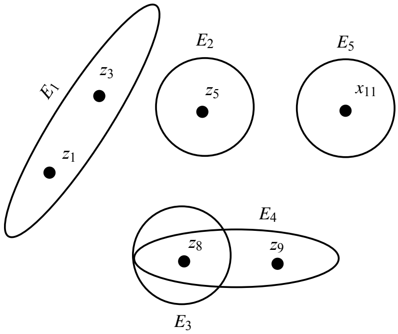


Fig. 8.11 A 3-polar fuzzy hypergraph

Fig. 8.12 (0.5, 0.5, 0.6)-level hypergraph of H



Example 8.4 Let $H = (A, B)$ be a 3-polar fuzzy hypergraph as shown in Fig. 8.11. Let $X = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}, z_{11}, z_{12}\}$ and $B = \{\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \mathscr{E}_4, \mathscr{E}_5\}$.

For $t_1 = 0.5$, $t_2 = 0.5$ and $t_3 = 0.6$, the (0.5, 0.5, 0.6)-level hypergraph of H is given in Fig. 8.12.

By considering the fixed t_1, t_2, t_3 and following the Algorithm 8.4.1, the bottom-up construction of this model is given in Fig. 8.13.

The possible method for *the bottom-up construction* is described in Algorithm 8.5.1.

Algorithm 8.5.1

Algorithm for the method of the bottom-up construction

```
clc
𝒫_j ∘ z_i=input('𝒫_j ∘ z_i='); T=input('τ='); q=1;
while q==1
[r, m]=size(𝒫_j ∘ z_i);N=zeros(r, r);N=input('N='); [r_1, r]=size(N); D=ones(r_1, m)+1;
    for l=1:r_1
        if N(l,:)==zeros(1, r)
            D(l,:)=zeros(1, m);
        else
            for k=1:r
                if N(l, k)==1
                    for j=1:m
                        D(l, j)=min(D(l, j),𝒫_j ∘ z_i(k, j));
                    end
                else
                    s=0;
                end
            end
        end
    end
    D
    𝒫_j ∘ ℰ_i=input('𝒫_j ∘ ℰ_i=');
    if size(𝒫_j ∘ ℰ_i)==[r_1, m]
        if 𝒫_j ∘ ℰ_i <=D
            if size(T)==[1, m]
                S=zeros(r_1, r);s=zeros(r_1, 1);
                for l=1:r_1
                    for k=1:r
                        if N(l, k)==1
                            if 𝒫_j ∘ z_i(k,:)>=T(1,:)
                                S(l, k)=1;
                                s(l, 1)=s(l, 1)+1;
                            else
                                S(l, k)=0;
                            end
                        end
                    end
                end
                S
                if s==ones(r_1, 1)
                    q=2;
                else
                    𝒫_j ∘ z_i = 𝒫_j ∘ ℰ_i;
                end
            else
                fprintf('error')
            end
        else
            fprintf('error')
        end
    else
        fprintf('error')
    end
end
```

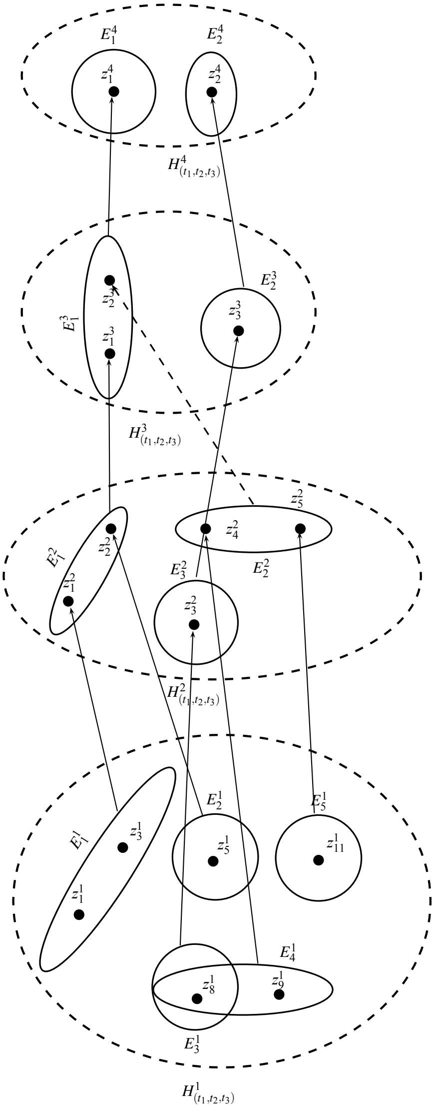


Fig. 8.13 Bottom-up construction procedure

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