Chapter 6
(Directed) Hypergraphs: q-Rung Orthopair Fuzzy Models and Beyond

A q-rung orthopair fuzzy set is a powerful tool for depicting fuzziness and uncertainty, as compared to the Pythagorean fuzzy model. In this chapter, we present concepts including q-rung orthopair fuzzy hypergraphs, (α, β)-level hypergraphs, and transversals and minimal transversals of q-rung orthopair fuzzy hypergraphs. We implement some interesting notions of q-rung orthopair fuzzy hypergraphs into decision-making. We describe additional concepts like q-rung orthopair fuzzy directed hypergraphs, dual directed hypergraphs, line graphs, and coloring of q-rung orthopair fuzzy directed hypergraphs. We also apply other interesting notions of q-rung orthopair fuzzy directed hypergraphs to real life problems. We introduce complex q-rung orthopair fuzzy graphs, complex Pythagorean fuzzy hypergraphs, and complex q-rung orthopair fuzzy hypergraphs. We study the transversals and minimal transversals of complex q-rung orthopair fuzzy hypergraphs. We present some algorithms to construct the minimal transversals and certain related concepts. Finally, we illustrate a collaboration network model through complex q-rung orthopair fuzzy hypergraphs to find the author having powerful collaboration skills using score and choice values. This chapter is basically due to [22–24, 35].

6.1 Introduction

Zadeh [37] proposed the notion of fuzzy sets in his monumental paper in 1965, to model uncertainty or vague ideas by nominating a degree of membership to each entity, ranging between 0 and 1. In 1983, intuitionistic fuzzy sets, primarily proposed by Atanassov [14], offered many significant advantages in representing human knowledge by denoting fuzzy membership not only with a single value but pairs of mutually orthogonal fuzzy sets called orthopairs, which allow the incorporation of uncertainty. Since intuitionistic fuzzy sets confine the selection of orthopairs to come only from a triangular region, as shown in Fig. 6.1, Pythagorean fuzzy sets, proposed

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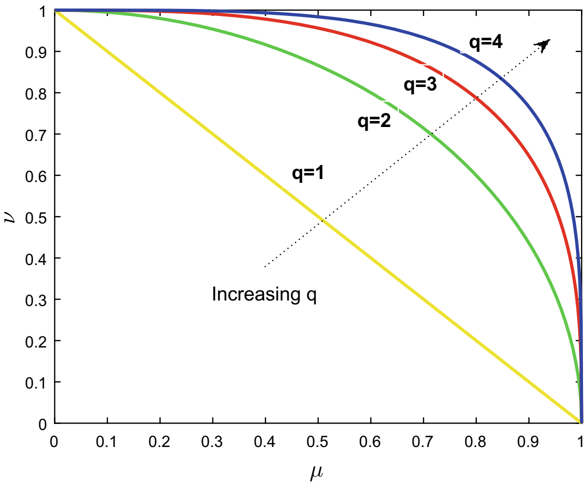


Fig. 6.1 Spaces of acceptable q-rung orthopairs

by Yager [32], as a new extension of intuitionistic fuzzy sets have emerged as an efficient tool for conducting uncertainty more properly in human analysis. Although both intuitionistic fuzzy sets and Pythagorean fuzzy sets make use of orthopairs to narrate assessment objects, they still have visible differences. The truth-membership function $T : X \to [0, 1]$ and falsity-membership function $F : X \to [0, 1]$ of intuitionistic fuzzy sets are required to satisfy the constraint condition $T(x) + F(x) \leq 1$. However, these two functions in Pythagorean fuzzy sets are needed to satisfy the condition $T(x)^2 + F(x)^2 \leq 1$, which shows that Pythagorean fuzzy sets have expanded space to assign orthopairs, as compared to intuitionistic fuzzy sets, displayed in Fig. 6.1.

A q-rung orthopair fuzzy set, originally proposed by Yager [35] in 2017, is a new generalization of orthopair fuzzy sets, which further relax the constraint of orthopair membership grades with $T(x)^q + F(x)^q \leq 1$ $(q \geq 1)$ [21]. As q increases, it is easy to see that the representation space of allowable orthopair membership grade increases. Figure 6.1 displays spaces of the most widely acceptable orthopairs for different q rungs. Ali [12] calculated the area of spaces with admissible orthopairs up to 10-rungs. Consider an example in the field of economics: in a market structure, a huge number of firms compete against each other with differentiated products with respect to branding or quality, which in nature are vague words. Since intuitionistic fuzzy sets have the capability to explore both aspects of ambiguous words, for example, it assigns an orthopair membership grade to "quality", i.e., support for quality and support for not-quality of an object with the condition that their sum is bounded by 1. This constraint clearly limits the selection of orthopairs.

The innovative concept of complex fuzzy sets was initiated by Ramot et al. [28] as an extension of fuzzy sets. Opposing to a fuzzy characteristic function, the range of complex fuzzy set's membership degrees is not restricted to [0, 1], but extends to the complex plane with the unit circle. Ramot et al. [29] discussed the union, intersection, and compliment of complex fuzzy sets with the help of illustrative examples. To generalize the concepts of intuitionistic fuzzy sets, complex intuitionistic fuzzy sets were introduced by Alkouri and Salleh [13]. As an extension of Pythagorean fuzzy sets and complex intuitionistic fuzzy sets, Ullah et al. [31] proposed complex Pythagorean fuzzy sets and discussed some applications. In complex Pythagorean fuzzy sets, membership $\mu = ue^{i\alpha}$ and nonmembership $\nu = ve^{i\beta}$ can take values in the unit circle subjected to the constraint $\mu^2 + \nu^2 \leq 1$. Complex Pythagorean fuzzy model, containing the phase term, is a more effective tool to capture the vague and uncertain data of periodic nature than the Pythagorean fuzzy model.

Definition 6.1 A *q-rung orthopair fuzzy set* Q in the universe X is an object having the representation

$$Q = (x, T_Q(x), F_Q(x)|x \in X),$$

where the function $T_Q : X \to [0, 1]$ defines the truth-membership and $F_Q : X \to [0, 1]$ defines the falsity-membership of the element $x \in X$ and for every $x \in X$, $0 \leq T_Q^q(x) + F_Q^q(x) \leq 1, q \geq 1$.

Furthermore, $\pi_Q(x) = \sqrt[q]{1 - T_Q^q(x) - F_Q^q(x)}$ is called a q-rung orthopair fuzzy index or indeterminacy degree of x to the set Q.

For convenience, Liu and Wang [21] called the pair $(T_Q^q(x), F_Q^q(x))$ as a q-rung orthopair fuzzy number, which is denoted as (T_Q^q, F_Q^q).

Definition 6.2 A *q-rung orthopair fuzzy relation* $\mathscr{R}$ in X is defined as $\mathscr{R} = \{x_1x_2, T_{\mathscr{R}}(x_1x_2), F_{\mathscr{R}}(x_1x_2)|x_1, x_2 \in X \times X\}$, where $T_{\mathscr{R}} : X \times X \to [0, 1]$ and $F_{\mathscr{R}} : X \times X \to [0, 1]$ represent the truth-membership and falsity-membership function of $\mathscr{R}$, respectively, such that $0 \leq T_{\mathscr{R}}^q(x_1x_2) + F_{\mathscr{R}}^q(x_1x_2) \leq 1$, for all $x_1x_2 \in X \times X$.

Example 6.1 Let $X = \{x_1, x_2, x_3\}$ be a non-empty set and $\mathscr{R}$ be a subset of $X \times X$ such that $\mathscr{R} = \{(x_1x_2, 0.9, 0.7),(x_1x_3, 0.7, 0.9), (x_2x_3, 0.6, 0.8)\}$. Note that, $0 \leq T_{\mathscr{R}}^5(x_1x_2) + F_{\mathscr{R}}^5(x_1x_2) \leq 1$, for all $x_1x_2 \in X \times X$. Hence, $\mathscr{R}$ is a 5-rung orthopair fuzzy relation on X.

For further terminologies and studies on Pythagorean fuzzy graphs and q-rung orthopair fuzzy graphs, readers are referred to [1–11, 15–20, 25–27, 30, 33, 34, 36].

6.2 q-Rung Orthopair Fuzzy Hypergraphs

Definition 6.3 A *q-rung orthopair fuzzy graph* on a non-empty set X is defined as an ordered pair $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V}$ is a q-rung orthopair fuzzy set on X and $\mathscr{E}$ is a q-rung orthopair fuzzy relation on X such that

$$T_{\mathscr{E}}(x_1x_2) \leq \min\{T_{\mathscr{V}}(x_1), T_{\mathscr{V}}(x_2)\}, F_{\mathscr{E}}(x_1x_2) \leq \max\{F_{\mathscr{V}}(x_1), F_{\mathscr{V}}(x_2)\},$$

and $0 \leq T_E^q(x_1x_2) + F_E^q(x_1x_2) \leq 1$, $q \geq 1$, for all $x_1, x_2 \in X$, where $T_{\mathscr{E}} : X \times X \to [0, 1]$ and $F_{\mathscr{E}} : X \times X \to [0, 1]$ represent the truth-membership and falsity-membership degrees of $\mathscr{E}$, respectively.

Remark 6.1

- When $q = 1$, 1-rung orthopair fuzzy graph is called an intuitionistic fuzzy graph.
- When $q = 2$, 2-rung orthopair fuzzy graph is called Pythagorean fuzzy graph.

Definition 6.4 The *support* of a q-rung orthopair fuzzy set $Q = (\text{x}, T_Q(\text{x}), F_Q(\text{x})| \text{x} \in X)$ is defined as $supp(Q) = \{\text{x}|T_Q(\text{x}) \neq 0, F_Q(\text{x}) \neq 1\}$.
The *height* of a q-rung orthopair fuzzy set $Q = (\text{x}, T_Q(\text{x}), F_Q(\text{x})|\text{x} \in X)$ is defined as $h(Q) = (\max_{\text{x}\in X} T_Q(\text{x}), \min_{\text{x}\in X} F_Q(\text{x}))$.
If $h(Q) = (1, 0)$, then q-rung orthopair fuzzy set Q is called *normal*.

Example 6.2 Let $Q = \{(q_1, 1, 0), (q_2, 0, 1), (q_3, 0.5, 0.6), (q_4, 0.6, 0.7), (q_5, 0.9, 0.3)\}$ be a 4-rung orthopair fuzzy set on X. Then, the support and height of Q are given as, $supp(Q) = \{q_1, q_3, q_4, q_5\}$, $h(Q) = (1, 0)$, respectively. Note that Q is normal.

Definition 6.5 Let X be a non-empty set. A *q-rung orthopair fuzzy hypergraph* $\mathscr{H}$ on X is defined in the form of an ordered pair $\mathscr{H} = (\mathscr{Q}, \zeta)$, where $\mathscr{Q} = \{\mathscr{Q}_1, \mathscr{Q}_2, \mathscr{Q}_3, \ldots \mathscr{Q}_n\}$ is a finite collection of nontrivial q-rung orthopair fuzzy subsets on X and ζ is a q-rung orthopair fuzzy relation on q-rung orthopair fuzzy sets $\mathscr{Q}_i$'s such that

1. $T_\zeta(E_k) = T_\zeta(x_1, x_2, x_3, \ldots, x_m) \leq \min\{\mathscr{Q}_i(x_1), \mathscr{Q}_i(x_2), \mathscr{Q}_i(x_3), \ldots, \mathscr{Q}_i(x_m)\}$,
   $F_\zeta(E_k) = F_\zeta(x_1, x_2, x_3, \ldots, x_m) \leq \max\{\mathscr{Q}_i(x_1), \mathscr{Q}_i(x_2), \mathscr{Q}_i(x_3), \ldots, \mathscr{Q}_i(x_m)\}$,
   for all $x_1, x_2, x_3, \ldots, x_m \in X$,
2. $\bigcup_i supp(\mathscr{Q}_i) = X$, for all $\mathscr{Q}_i \in \mathscr{Q}$.

Definition 6.6 The *height* of a q-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \zeta)$ is defined as $h(\mathscr{H}) = \{\max(\zeta_l), \min(\zeta_m)\}$, where $\zeta_l = \max T_{\zeta_j}(x_i)$ and $\zeta_m = \min F_{\zeta_j}(x_i)$. Here, $T_{\zeta_j}(x_i)$ and $F_{\zeta_j}(x_i)$ denote the truth-membership degree and falsity-membership degree of vertex x_i to the hyperedge ζ_j, respectively.

Definition 6.7 Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a q-rung orthopair fuzzy hypergraph. The *order* of $\mathscr{H}$, which is denoted by $O(\mathscr{H})$, and is defined as $O(\mathscr{H}) = \sum_{x\in X} \wedge \mathscr{Q}_i(x)$. The *size* of $\mathscr{H}$, which is denoted by $S(\mathscr{H})$, and is defined as $S(\mathscr{H}) = \sum_{x\in X} \vee \mathscr{Q}_i(x)$.

In a q-rung orthopair fuzzy hypergraph, adjacent vertices x_i and x_j are the vertices which are the part of the same q-rung orthopair fuzzy hyperedge. Two q-rung orthopair fuzzy hyperedges ζ_i and ζ_j are said to be adjacent hyperedges if they possess the non-empty intersection, i.e., $supp(\zeta_i) \cap supp(\zeta_i) \neq \emptyset$.

We now define the adjacent level between two q-rung orthopair fuzzy vertices and q-rung orthopair fuzzy hyperedges.

Definition 6.8 The *adjacent level* between two vertices x_i and x_j is denoted by $\gamma(x_i, x_j)$ and is defined as $\gamma(x_i, x_j) = (\max_k \min[T_k(x_i), T_k(x_j)], \min_k \max[F_k(x_i), F_k(x_j)])$.

The *adjacent level* between two hyperedges ζ_i and ζ_j is denoted by $\sigma(\zeta_i, \zeta_j)$ and is defined as $\sigma(\zeta_i, \zeta_j) = (\max_j \min[T_j(x), T_k(x)], \min_j \max[F_j(x), F_k(x)])$.

Definition 6.9 A *simple* q-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \zeta)$ is defined as a hypergraph, which has no repeated hyperedges contained in it, i.e., if $\zeta_i, \zeta_j \in \zeta$ and $\zeta_i \subseteq \zeta_j$, then $\zeta_i = \zeta_j$.

A q-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \zeta)$ is *support simple* if $\zeta_i, \zeta_j \in \zeta$, $supp(\zeta_i) = supp(\zeta_j)$, and $\zeta_i \subseteq \zeta_j$, then $\zeta_i = \zeta_j$.

A q-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \zeta)$ is *strongly support simple* if $\zeta_i, \zeta_j \in \zeta$ and $supp(\zeta_i) = supp(\zeta_j)$, then $\zeta_i = \zeta_j$.

Definition 6.10 A q-rung orthopair fuzzy set $Q : X \to [0, 1]$ is called an *elementary* set if T_Q and F_Q are single-valued on the support of Q.
A q-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \zeta)$ is *elementary* if all it's hyperedges are elementary.

Proposition 6.1 *A q-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \zeta)$ is the generalization of fuzzy hypergraph and intuitionistic fuzzy hypergraph.*

An upper bound on the cardinality of hyperedges of a q-rung orthopair fuzzy hypergraph of order n can be achieved by using the following result.

Theorem 6.1 *Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a simple q-rung orthopair fuzzy hypergraph of order n. Then, $|\zeta|$ acquires no upper bound.*

Proof Let $X = \{x_1, x_2\}$. Define $\zeta_N = \{\mathscr{Q}_j, j = 1, 2, 3, \ldots, N\}$, where

$$T_{\mathscr{Q}_j}(x_1) = \frac{1}{1+j}, F_{\mathscr{Q}_j}(x_1) = 1 - \frac{1}{1+j}$$

and

$$T_{\mathscr{Q}_j}(x_2) = \frac{1}{1+j}, F_{\mathscr{Q}_j}(x_2) = 1 - \frac{1}{1+j}.$$

Then, $\mathscr{H}_N = (\mathscr{Q}, \zeta_N)$ is a simple q-rung orthopair fuzzy hypergraph having N hyperedges.

Theorem 6.2 *Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be an elementary and simple q-rung orthopair fuzzy hypergraph on a non-empty set X having n elements. Then $|\zeta| \leq 2^n - 1$. The equality holds if and only if $\{supp(\zeta_j)|\zeta_j \in \zeta, \zeta \neq 0\} = P(X)\backslash\emptyset$.*

Proof Since $\mathscr{H}$ is elementary and simple then at most one $\zeta_i \in \zeta$ can have each nontrivial subset of X as its support, therefore, we have $|\zeta| \leq 2^n - 1$.

To prove that the relation satisfies the equality, consider a set of mappings $\zeta = \{(T_A, F_A)|A \subseteq X\}$ such that,

$$T_A(x) = \begin{cases} \frac{1}{|A|}, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}, \; F_A(x) = \begin{cases} \frac{1}{|A|}, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then each set containing single element has height (1, 1) and the height of the set having two elements is (0.5, 0.5) and so on. Hence, $\mathscr{H}$ is simple and elementary with $|\zeta| = 2^n - 1$.

Definition 6.11 The *cut level set* of a q-rung orthopair fuzzy set Q is defined to be a crisp set of the following form, $Q^{(\alpha,\beta)} = \{x \in X | T_Q(x) \geq \alpha, F_Q(x) \leq \beta\}$, where $\alpha, \beta \in [0, 1]$.

Definition 6.12 Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a q-rung orthopair fuzzy hypergraph. The (α, β)-*level hypergraph* of $\mathscr{H}$ is defined as $\mathscr{H}^{(\alpha,\beta)} = (\mathscr{Q}^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)})$, where

1. $\zeta^{(\alpha,\beta)} = \{\zeta_i^{(\alpha,\beta)} : \zeta_i \in \zeta\}$ and $\zeta_i^{(\alpha,\beta)} = \{x \in X | T_{\zeta_i}(x) \geq \alpha, F_{\zeta_i}(x) \leq \beta\}$,
2. $\mathscr{Q}^{(\alpha,\beta)} = \bigcup_{\zeta_i \in \zeta} \zeta_i^{(\alpha,\beta)}$.

Example 6.3 Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a 4-rung orthopair fuzzy hypergraph as shown in Fig. 6.2, where $\zeta = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$. Incidence matrix of $\mathscr{H}$ is given in Table 6.1.

By direct calculations, it can be seen that it is a 4-rung orthopair fuzzy hypergraph. All the above mentioned concepts can be well explained by considering this example. Here, $h(\mathscr{H}) = \{\max(\zeta_l), \min(\zeta_m)\} = (0.6, 0.2)$. Since, $\mathscr{H}$ does not contain repeated hyperedges, it is simple 4-rung orthopair fuzzy hypergraph. Also, $\mathscr{H}$ is support simple and strongly support simple, i.e., whenever $\zeta_i, \zeta_j \in \zeta$ and $supp(\zeta_i) = supp(\zeta_j)$, then $\zeta_i = \zeta_j$. Adjacency level between x_1, x_2 and between two hyperedges ζ_1, ζ_2 is given as follows:

Table 6.1 Incidence matrix of $\mathscr{H}$

I	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
x_1	(0.1, 0.2)	(0.1, 0.2)	(0.1, 0.2)	(0, 1)	(0, 1)
x_2	(0.2, 0.3)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
x_3	(0.3, 0.4)	(0, 1)	(0, 1)	(0, 1)	(0.3, 0.4)
x_4	(0, 1)	(0, 1)	(0.4, 0.5)	(0, 1)	(0, 1)
x_5	(0, 1)	(0.5, 0.6)	(0, 1)	(0, 1)	(0, 1)
x_6	(0, 1)	(0, 1)	(0, 1)	(0, 1)	(0.5, 0.4)
x_7	(0, 1)	(0, 1)	(0.4, 0.3)	(0.4, 0.3)	(0, 1)
x_8	(0, 1)	(0, 1)	(0, 1)	(0.6, 0.5)	(0, 1)
x_9	(0, 1)	(0, 1)	(0, 1)	(0.6, 0.7)	(0.6, 0.7)

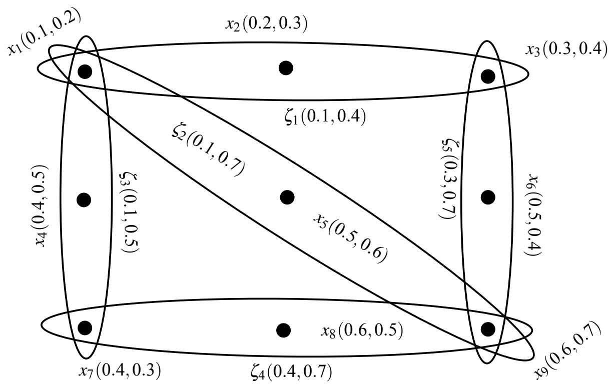
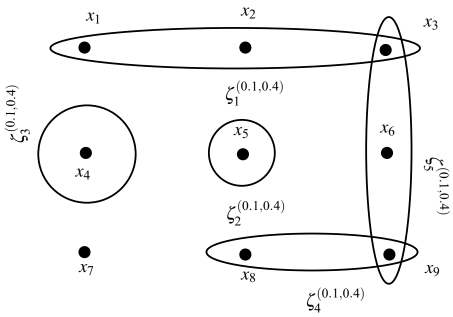


Fig. 6.2 4-rung orthopair fuzzy hypergraph

Fig. 6.3 (0.1, 0.4)-level hypergraph of $\mathscr{H}$



$$
\begin{aligned}
\gamma(x_1, x_2) &= (\max_k \min[T_k(x_1), T_k(x_2)], \min_k \max[F_k(x_1), F_k(x_2)]), k = 1, 2, 3, 4, 5. \\
&= (0.1, 0.3), \\
\sigma(\zeta_1, \zeta_2) &= (\max \min[T_1(x), T_2(x)], \min \max[F_1(x), F_2(x)]) \\
&= (0.2, 0.6).
\end{aligned}
$$

For $\alpha = 0.1,\ \beta = 0.4 \in [0, 1]$, $(0.1, 0.4)$-level hypergraph of $\mathscr{H}$ is $\mathscr{H}^{(0.1,0.4)} = (\mathscr{Q}^{(0.1,0.4)}, \zeta^{(0.1,0.4)})$, where

$$
\begin{aligned}
\zeta^{(0.1,0.4)} &= \{\zeta_1^{(0.1,0.4)}, \zeta_2^{(0.1,0.4)}, \zeta_3^{(0.1,0.4)}, \zeta_4^{(0.1,0.4)}, \zeta_5^{(0.1,0.4)}\} \\
&= \{\{x_1, x_2, x_3\}, \{x_5\}, \{x_4\}, \{x_8, x_9\}, \{x_3, x_6, x_9\}\}, \\
\mathscr{Q}^{(0.1,0.4)} &= \{x_1, x_2, x_3\} \cup \{x_5\} \cup \{x_4\} \cup \{x_8, x_9\} \cup \{x_3, x_6, x_9\} \\
&= \{x_1, x_2, x_3, x_4, x_5, x_6, x_8, x_9\}.
\end{aligned}
$$

Note that, $(0.1, 0.4)$-level hypergraph of $\mathscr{H}$ is a crisp hypergraph as shown in Fig. 6.3.

Remark 6.2 If $\alpha \geq \mu$ and $\beta \leq \nu$ and $\mathscr{Q}$ is a q-rung orthopair fuzzy set on X, then $\mathscr{Q}^{(\alpha,\beta)} \subseteq \mathscr{Q}^{(\mu,\nu)}$. Thus, we can have $\zeta^{(\alpha,\beta)} \subseteq \zeta^{(\mu,\nu)}$, for level hypergraphs of $\mathscr{H}$, i.e., if a q-rung orthopair fuzzy hypergraph has distinct hyperedges, its (α, β)-level hyperedges may be same and hence (α, β)-level hypergraphs of a simple q-rung orthopair fuzzy hypergraphs may have repeated edges.

Definition 6.13 Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a q-rung orthopair fuzzy hypergraph and $\mathscr{H}^{(\alpha,\beta)}$ be the (α, β)-level hypergraph of $\mathscr{H}$. The sequence of real numbers $\rho_1 = (T_{\rho_1}, F_{\rho_1})$, $\rho_2 = (T_{\rho_2}, F_{\rho_2})$, $\rho_3 = (T_{\rho_3}, F_{\rho_3})$, ..., $\rho_n = (T_{\rho_n}, F_{\rho_n})$, $0 < T_{\rho_1} < T_{\rho_2} < T_{\rho_3} < \cdots < T_{\rho_n}, F_{\rho_1} > F_{\rho_2} > F_{\rho_3} > \cdots > F_{\rho_n} > 0$, where $(T_{\rho_n}, F_{\rho_n}) = h(\mathscr{H})$ such that

(i) if $\rho_{i-1} = (T_{\rho_{i-1}}, F_{\rho_{i-1}}) < \rho = (T_\rho, F_\rho) \leq \rho_i = (T_{\rho_i}, F_{\rho_i})$, then $\zeta^\rho = \zeta^{\rho_i}$,
(ii) $\zeta^{\rho_i} \subseteq \zeta^{\rho_{i+1}}$,

is called the *fundamental sequence* of $\mathscr{H}$, denoted by $f_S(H)$. The set of ρ_i-level hypergraphs $\{\mathscr{H}^{\rho_1}, \mathscr{H}^{\rho_2}, \mathscr{H}^{\rho_3}, \ldots, \mathscr{H}^{\rho_n}\}$ is called the *core hypergraphs* of $\mathscr{H}$ or simply the *core set* of $\mathscr{H}$ and is denoted by $c(\mathscr{H})$.

Definition 6.14 A q-rung orthopair fuzzy hypergraph $\mathscr{H}_1 = (\mathscr{Q}_1, \zeta_1)$ is called *partial hypergraph* of $\mathscr{H}_2 = (\mathscr{Q}_2, \zeta_2)$ if $\zeta_1 \subseteq \zeta_2$ and is denoted as $\mathscr{H}_1 \subseteq \mathscr{H}_2$.

Definition 6.15 Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a q-rung orthopair fuzzy hypergraph having fundamental sequence $f_S(\mathscr{H}) = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ and let $\rho_{n+1} = 0$, if for all hyperedges $\zeta_k \in \zeta$, $k = 1, 2, 3, \ldots, n$, and for all $\rho \in (\rho_{i+1}, \rho_i]$, we have $\zeta_i^\rho = \zeta_i^{\rho_i}$ then $\mathscr{H}$ is called *sectionally elementary*.

Theorem 6.3 *Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be an elementary q-rung orthopair fuzzy hypergraph. Then the necessary and sufficient condition for $\mathscr{H} = (\mathscr{Q}, \zeta)$ to be strongly support simple is that $\mathscr{H}$ is support simple.*

Proof Suppose that $\mathscr{H}$ is support simple, elementary and $supp(\zeta_i) = supp(\zeta_j)$, for $\zeta_i, \zeta_j \in \zeta$. Let $h(\zeta_i) \leq h(\zeta_j)$. Since $\mathscr{H}$ is elementary, we have $\zeta_i \leq \zeta_j$ and since $\mathscr{H}$ is support simple, we have $\zeta_i = \zeta_j$. Hence, $\mathscr{H}$ is strongly support simple. On the same lines, the converse part may be proved.

Definition 6.16 A q-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \zeta)$ is said to be a $\mathscr{B} = (T_{\mathscr{B}}, F_{\mathscr{B}})$ *tempered* q-rung orthopair fuzzy hypergraph if for $H = (X, \xi)$, a crisp hypergraph, and a q-rung orthopair fuzzy set $\mathscr{B} = (T_{\mathscr{B}}, F_{\mathscr{B}}):X \to [0, 1]$ such that, $\zeta = \{D_A = (T_{D_A}, F_{D_A}) | A \subset X\}$, where

$$T_{D_A}(x) = \begin{cases} \min(T_{\mathscr{B}}(y)) : y \in A, & \text{if} \quad x \in A, \\ 0, & \text{otherwise}. \end{cases},$$

$$F_{D_A}(x) = \begin{cases} \max(F_{\mathscr{B}}(y)) : y \in A, & \text{if} \quad x \in A, \\ 0, & \text{otherwise}. \end{cases}$$

Table 6.2 Incidence matrix of $\mathscr{H}$

I	ζ_1	ζ_2	ζ_3	ζ_4
x_1	(0.6, 0.7)	(0, 1)	(0.6, 0.7)	(0, 1)
x_2	(0, 1)	(0.7, 0.6)	(0, 1)	(0.7, 0.6)
x_3	(0.8, 0.7)	(0.8, 0.7)	(0, 1)	(0, 1)
x_4	(0, 1)	(0.6, 0.5)	(0.6, 0.7)	(0, 1)
x_5	(0.7, 0.8)	(0, 1)	(0, 1)	(0.7, 0.8)

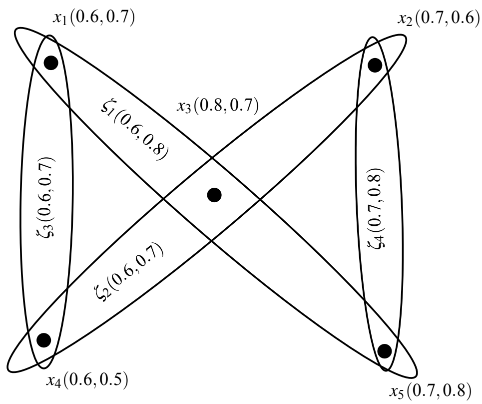


Fig. 6.4 $\mathscr{B}$-tempered 3-rung orthopair fuzzy hypergraph

Example 6.4 Consider a 3-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \zeta)$ as shown in Fig. 6.4. Incidence matrix of $\mathscr{H} = (\mathscr{Q}, \zeta)$ is given in Table 6.2.

Define a 3-rung orthopair fuzzy set $\mathscr{B} = \{(x_1, 0.6, 0.7), (x_2, 0.7, 0.6), (x_3, 0.8, 0.7), (x_4, 0.6, 0.5), (x_5, 0.7, 0.8)\}$. By direct calculations, we have

$$T_{D_{\{x_1,x_3,x_5\}}}(x_1) = \min\{0.6, 0.8, 0.7\} = 0.6, \ F_{D_{\{x_1,x_3,x_5\}}}(x_1) = \max\{0.7, 0.8, 0.7\} = 0.8,$$
$$T_{D_{\{x_2,x_3,x_4\}}}(x_2) = \min\{0.7, 0.8, 0.6\} = 0.6, \ F_{D_{\{x_2,x_3,x_4\}}}(x_2) = \max\{0.6, 0.5, 0.7\} = 0.7,$$
$$T_{D_{\{x_1,x_4\}}}(x_4) = \min\{0.6, 0.6\} = 0.6, \ F_{D_{\{x_1,x_4\}}}(x_4) = \max\{0.7, 0.7\} = 0.7,$$
$$T_{D_{\{x_2,x_5\}}}(x_5) = \min\{0.7, 0.7\} = 0.7, F_{D_{\{x_2,x_5\}}}(x_5) = \max\{0.6, 0.8\} = 0.8.$$

Similarly, all other values can be calculated by using the same method. Thus, we have $\zeta_1 = (T_{D_{\{x_1,x_3,x_5\}}}, F_{D_{\{x_1,x_3,x_5\}}}), \zeta_2 = (T_{D_{\{x_2,x_3,x_4\}}}, F_{D_{\{x_2,x_3,x_4\}}}), \zeta_3 = (T_{D_{\{x_1,x_4\}}}, F_{D_{\{x_1,x_4\}}}), \zeta_4 = (T_{D_{\{x_2,x_5\}}}, F_{D_{\{x_2,x_5\}}})$.

Hence, $\mathscr{H}$ is $\mathscr{B}$-tempered 3-rung orthopair fuzzy hypergraph.

6.3 Transversals of q-Rung Orthopair Fuzzy Hypergraphs

Definition 6.17 Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a q-rung orthopair fuzzy hypergraph on X. A q-rung orthopair fuzzy subset τ of X, which satisfies the condition $\tau^{h(\zeta_i)} \cap \zeta_i^{h(\zeta_i)} \neq \emptyset$, for all $\zeta_i \in \zeta$, is called a *q-rung orthopair fuzzy transversal* of $\mathscr{H}$.
τ is called *minimal transversal* of $\mathscr{H}$ if $\tau_1 \subset \tau$, τ_1 is not a q-rung orthopair fuzzy transversal. $t_r(\mathscr{H})$ denotes the collection of minimal transversals of $\mathscr{H}$.

We now discuss some results on q-rung orthopair fuzzy transversals.

Remark 6.3 Although τ can be regarded as a minimal transversal of $\mathscr{H}$, it is not necessary for $\tau^{(\alpha,\beta)}$ to be the minimal transversal of $\mathscr{H}^{(\alpha,\beta)}$, for all $\alpha, \beta \in [0, 1]$. Also, it is not necessary for the family of minimal q-rung orthopair fuzzy hypergraphs to form a hypergraph on X. For those q-rung orthopair fuzzy transversals that satisfy the above property, we have the following definition.

Definition 6.18 A q-rung orthopair fuzzy transversal τ with the property that $\tau^{(\alpha,\beta)}$ is a minimal transversal of $\mathscr{H}^{(\alpha,\beta)}$, for $\alpha, \beta \in [0, 1]$, is called *locally minimal q-rung orthopair fuzzy transversal* of $\mathscr{H}$. The collection of locally minimal q-rung orthopair fuzzy transversals of $\mathscr{H}$ is denoted by $t_r^*(\mathscr{H})$.

Lemma 6.1 *Let $f_S(\mathscr{H}) = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ be the fundamental sequence of a q-rung orthopair fuzzy hypergraph $\mathscr{H}$ and τ be the q-rung orthopair fuzzy transversal of $\mathscr{H}$. Then, $h(\tau) \geq h(\zeta_i)$, for each $\zeta_i \in \zeta$ and if τ is minimal, then $h(\tau) = \max\{h(\zeta_i)|\zeta_i \in \zeta\} = \rho_1$.*

Proof Since τ is a q-rung orthopair fuzzy transversal of $\mathscr{H}$ then $\tau^{h(\zeta_i)} \cap \zeta_i^{h(\zeta_i)} \neq \emptyset$. Consider an arbitrary element of $supp(\tau)$, then $\zeta_i(x) > h(\zeta_i)$ and we have $h(\tau) \geq h(\zeta_i)$. If τ is minimal transversal then $h(\zeta_i) = \{\max T_{\zeta_i}(x), \min F_{\zeta_i}(x)|x \in X$ and $\zeta_i \in \zeta\} = \rho_1$. Hence, $h(\tau) = \max\{h(\zeta_i)|\zeta_i \in \zeta\} = \rho_1$.

Theorem 6.4 *Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a q-rung orthopair fuzzy hypergraph then the statements,*

(i) *τ is a q-rung orthopair fuzzy transversal of $\mathscr{H}$,*
(ii) *For all $\zeta_i \in \zeta$ and for each $\rho = \{T_\rho, F_\rho\} \in [0, 1]$ satisfying $0 < (T_\rho, F_\rho) < h(\zeta_i)$, $\tau^\rho \cap \zeta^\rho \neq \emptyset$,*
(iii) *τ^ρ is a transversal of $\mathscr{H}^\rho$, for all $\rho \in [0, 1]$, $0 < \rho < \rho_1$,*

are equivalent.

Proof $(i) \Rightarrow (ii)$. Suppose τ is a q-rung orthopair fuzzy transversal of $\mathscr{H}$. For any $\rho \in [0, 1]$, which satisfies $0 < (T_\rho, F_\rho) < h(\zeta_i)$, $\tau^\rho \supseteq \tau^{h(\zeta_i)}$ and $\zeta_i^\rho \supseteq \zeta_i^{h(\zeta_i)}$. Hence, $\tau^\rho \cap \zeta^\rho \supseteq \tau^{h(\zeta_i)} \cap \zeta_i^{h(\zeta_i)} \neq \emptyset$, because τ is a transversal.
$(ii) \Rightarrow (iii)$. Let $\tau^\rho \cap \zeta_i^\rho \neq \emptyset$, for all $\zeta_i \in \zeta$ and $0 < T_\rho < T_{\rho_1}, 0 > F_\rho < F_{\rho_1}$, which implies that τ^ρ is a transversal of $\mathscr{H}^\rho$.
$(iii) \Rightarrow (i)$. This part can be proved trivially.

Theorem 6.5 *Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a q-rung orthopair fuzzy hypergraph. For each $x \in X$ such that $\tau(x) \in f_S(\mathscr{H})$ and for all $\tau \in t_r(\mathscr{H})$, the fundamental sequence of $t_r(\mathscr{H}) \subset f_S(\mathscr{H})$.*

Proof Let the fundamental sequence of $\mathscr{H}$ be $f_S(\mathscr{H}) = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ and $\tau \in t_r(\mathscr{H})$, for $\tau(x) \in (\rho_{i+1}, \rho_i]$. Consider a mapping ψ defined by

$$\psi(u) = \begin{cases} \rho_i, & \text{if} \quad x = u, \\ \tau(u), & \text{otherwise.} \end{cases}$$

Thus, from the definition of ψ, we have $\psi^{\rho_i} = \tau^{\rho_i}$ and the Definition 6.13 implies that $\mathscr{H}^{\rho} = \mathscr{H}^{\rho_i}$, for all $\rho \in (\rho_{i+1}, \rho_i]$. Since τ is a q-rung orthopair fuzzy transversal of $\mathscr{H}$ and $\psi^{\rho} = \tau^{\rho}$, for all $\rho \notin (\rho_{i+1}, \rho_i]$, ψ is a q-rung orthopair fuzzy transversal. Now $\psi \leq \tau$ and minimality of τ both implies that $\psi = \tau$. Hence, $\tau(x) = \psi(x) = \rho_1$. Thus, $\tau(x) \in f_S(\mathscr{H})$, therefore we have $f_S(t_r(\mathscr{H})) \subseteq f_S(\mathscr{H})$.

Theorem 6.6 *The collection of all minimal transversals $t_r(\mathscr{H})$ is sectionally elementary.*

Proof Let the fundamental sequence of $t_r(\mathscr{H})$ be $f_s(t_r(\mathscr{H})) = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$. Consider an element τ of $t_r(\mathscr{H})$ and some $\rho \in (\rho_{i+1}, \rho_i]$ such that $\tau^{\rho_i} \subset \tau^{\rho}$. In consideration of $[t_r(\mathscr{H})]^{\rho} = [t_r(\mathscr{H})]^{\rho_i}$, we have $\psi \in t_r(\mathscr{H})$ satisfying $\psi^{\rho} = \tau^{\rho_i}$. Then, the condition $\psi^{\rho} \supset \tau^{\rho_i}$ implies the existence of a q-rung orthopair fuzzy set $\mathscr{R}$ such that,

$$\mathscr{R}(x) = \begin{cases} \rho, & \text{if} \quad x \in \psi^{\rho_i} \backslash \tau^{\rho_i}, \\ \psi(x), & \text{otherwise,} \end{cases}$$

is the q-rung orthopair fuzzy transversal of $\mathscr{H}$. Now, $\rho < \psi$ yields a contradiction to the minimality of ψ.

Lemma 6.2 *Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a q-rung orthopair fuzzy hypergraph. Consider an element x of $supp(\tau)$, where $\tau \in t_r(\mathscr{H})$, then there exists a q-rung orthopair fuzzy hyperedge ζ of $\mathscr{H}$ such that,*

(i) $\tau(x) = h(\zeta) = \zeta(x) > 0$,
(ii) $\tau^{h(\zeta)} \cap \zeta^{h(\zeta)} = \{x\}$.

Proof (i) Let $\tau(x) > 0$ and Q denotes the set of all q-rung orthopair fuzzy hyperedges of $\mathscr{H}$ such that for each element ζ of Q, $\zeta(x) \geq \tau(x)$. Then this set is non-empty because $\tau^{\tau(x)}$ is a transversal of $\mathscr{H}^{\tau(x)}$ and $x \in \tau^{\tau(x)}$. Additionally, each element ζ of Q satisfies the inequality $h(\zeta) \geq \zeta(x) \geq \tau(x)$. Suppose on contrary, (i) is false then for each $\zeta \in Q$, $h(\zeta) > \tau(x)$ and we have an element $x^{\zeta} \neq x$, where $x^{\zeta} \in \zeta^{h(\zeta)} \cap \tau^{h(\zeta)}$. Here, we define a q-rung orthopair fuzzy set Q' as

$$Q'(v) = \begin{cases} \tau(v), & \text{if} \quad x \neq v, \\ \max\{h(\zeta) | h(\zeta) < \tau(x)\}, & \text{if} \quad x = v. \end{cases}$$

Note that, Q' is a q-rung orthopair fuzzy transversal of $\mathscr{H}$ and $Q' < \tau$, which is contradiction to the fact that τ is minimal. Hence, (i) holds for some ζ.

(ii) Suppose each element of Q satisfies (i) and also have an element $x^{\zeta} \neq x$, where $x^{\zeta} \in \zeta^{h(\zeta)} \cap \tau^{h(\zeta)}$. The same arguments as given above completes the proof.

Theorem 6.7 *Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be an ordered q-rung orthopair fuzzy hypergraph with $f_S(\mathscr{H}) = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ and $c(\mathscr{H}) = \{\mathscr{H}^{\rho_1}, \mathscr{H}^{\rho_2}, \mathscr{H}^{\rho_3}, \ldots, \mathscr{H}^{\rho_n}\}$. Then, $t_r^{\star}(\mathscr{H})$ is non-empty. Further, if τ_n is a minimal transversal of $\mathscr{H}^{\rho_n}$ then there exists $T \in t_r^{\star}(\mathscr{H})$ such that $supp(T) = \tau_n$.*

Proof Let τ_n be a minimal transversal of $\mathscr{H}^{\rho_n}$, $\mathscr{H}^{\rho_{n-1}}$ is a partial hypergraph of $\mathscr{H}^{\rho_n}$ because $\mathscr{H}$ is ordered and consequently τ_{n-1} is minimal transversal of $\mathscr{H}^{\rho_{n-1}}$ such that $\tau_{n-1} \subseteq \tau_n$. By continuing the same argument, we establish a nested sequence of minimal transversals $\tau_1 \subseteq \tau_2 \subseteq \tau_3 \subseteq \cdots \subseteq \tau_n$, where every τ_i is minimal transversal of $\mathscr{H}^{\rho_i}$. Let $\eta_j = \eta_j(\tau_j, \rho_j)$ is an elementary q-rung orthopair fuzzy set having height ρ_j and support τ_j. Then, $T = \max\{\eta_j | 1 \leq j \leq n\}$ is locally minimal transversal of $\mathscr{H}$ having support τ_n.

We now give an Algorithm 6.3.1 for finding $t_r(\mathscr{H})$.

Algorithm 6.3.1 Algorithm for finding $t_r(\mathscr{H})$
Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a q-rung orthopair fuzzy hypergraph having the set of core hypergraphs $c(\mathscr{H}) = \{\mathscr{H}^{\rho_1}, \mathscr{H}^{\rho_2}, \mathscr{H}^{\rho_3}, \ldots, \mathscr{H}^{\rho_n}\}$. An iterative procedure to find the minimal transversal τ of $\mathscr{H}$ is as follows,

1. Find a crisp minimal transversal τ_1 of $\mathscr{H}^{\rho_1}$.
2. Find a minimal transversal τ_2 of $\mathscr{H}^{\rho_2}$, which satisfies $\tau_1 \subseteq \tau_2$, i.e., formulate a new hypergraph $\mathscr{H}_2$ having hyperedges ζ^{ρ_2} which is augmented having a loop at each $x \in \tau_1$. In accordance with, we can say that $\zeta(H_2) = \zeta^{\rho_2} \cup \{\{x\} | x \in \tau_1\}$. Let τ_2 be an arbitrary minimal transversal of $\mathscr{H}_2$.
3. By continuing the same procedure repeatedly, we have a sequence of minimal transversals $\tau_1 \subseteq \tau_2 \subseteq \tau_3 \subseteq \cdots \subseteq \tau_j$ such that τ_j be the minimal transversal of $\mathscr{H}^{\rho_j}$ with the property $\tau_{j-1} \subseteq \tau_j$.
4. Consider an elementary q-rung orthopair fuzzy set μ_j having the support τ_j and $h(\mu_j) = \rho_j$, $1 \leq j \leq n$. Then, $\tau = \bigcup_{j=1}^{n} \{\mu_j | 1 \leq j \leq n\}$ is a minimal q-rung orthopair fuzzy transversal of $\mathscr{H}$.

Example 6.5 Consider a 5-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \zeta)$, as shown in Fig. 6.5, where $\zeta = \{\zeta_1, \zeta_2, \zeta_3\}$. Incidence matrix of $\mathscr{H} = (\mathscr{Q}, \zeta)$ is given in Table 6.3.

By routine calculations, we have $h(\zeta_1) = (0.8, 0.6)$, $h(\zeta_2) = (0.8, 0.5)$, and $h(\zeta_3) = (0.8, 0.5)$. Consider a 5-rung orthopair fuzzy subset τ_1 of X such that

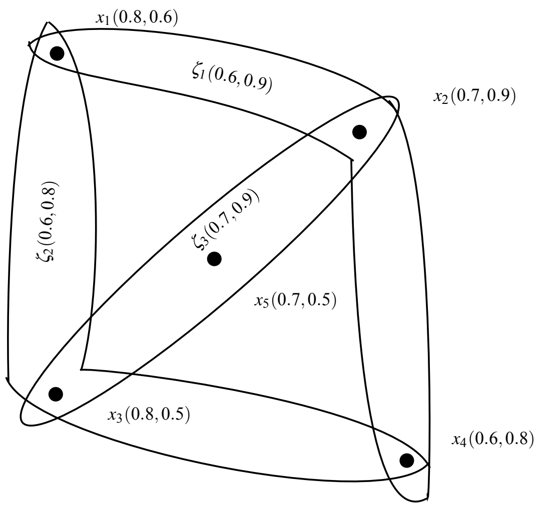


Fig. 6.5 5-rung orthopair fuzzy hypergraph

Table 6.3 Incidence matrix of $\mathscr{H}$

I	ζ_1	ζ_2	ζ_3
x_1	(0.8, 0.6)	(0.8, 0.6)	(0, 1)
x_2	(0.7, 0.9)	(0, 1)	(0.7, 0.9)
x_3	(0, 1)	(0.8, 0.5)	(0.8, 0.5)
x_4	(0.6, 0.8)	(0.6, 0.8)	(0, 1)
x_5	(0, 1)	(0, 1)	(0.7, 0.5)

$\tau_1 = \{(x_1, 0.8, 0.6), (x_2, 0.7, 0.9), (x_3, 0.8, 0.5)\}$. Note that, $\zeta_1^{h(\zeta_1)} = \{x_1\}$, $\zeta_2^{h(\zeta_2)} = \{x_3\}$ and $\zeta_3^{h(\zeta_3)} = \{x_3\}$. Also $\tau_1^{(0.8,0.6)} = \{x_1\}$, $\tau_2^{(0.8,0.5)} = \{x_3\}$ and $\tau_3^{(0.8,0.5)} = \{x_3\}$. It can be seen that $\tau_1^{h(\zeta_i)} \cap \zeta_i^{h(\zeta_i)} \neq \emptyset$, for all $\zeta_i \in \zeta$. Thus, τ_1 is a 5-rung orthopair fuzzy transversal of $\mathscr{H}$. Similarly, $\tau_2 = \{(x_1, 0.8, 0.6), (x_3, 0.8, 0.5)\}$, $\tau_3 = \{(x_1, 0.8, 0.6), (x_3, 0.8, 0.5), (x_4, 0.6, 0.8)\}$, $\tau_4 = \{(x_1, 0.8, 0.6), (x_3, 0.8, 0.5), (x_5, 0.7, 0.5), \}$ are other transversals of $\mathscr{H}$. The minimal transversal is τ_2, i.e., whenever $\tau \subseteq \tau_2$, τ is not a 5-rung orthopair fuzzy transversal.

Let $\alpha = 0.8, \beta = 0.5$, then $\zeta_1^{(0.8,0.5)} = \{\emptyset\}$, $\zeta_2^{(0.8,0.5)} = \{x_3\}$, $\zeta_3^{(0.8,0.5)} = \{x_3\}$ shows that $\tau_2^{(0.8,0.5)}$ is not a minimal transversal of $\mathscr{H}^{(0.8,0.5)}$.

Theorem 6.8 *Let $\mathscr{H} = (\mathscr{Q}, \zeta)$ be a q-rung orthopair fuzzy hypergraph and $x \in X$. Then, there exists an element τ of $t_r(\mathscr{H})$ such that $x \in supp(\tau)$ if and only if there is an hyperedge $\zeta_1 \in \zeta$ which satisfies,*

- $\zeta_1(x) = h(\zeta')$,
- *For every $\xi \in \zeta$ with $h(\xi) > h(\zeta_1)$, $\xi^{h(\zeta_i)} \not\subset \zeta_1^{h(\zeta_1)}$,*
- *$h(\zeta_1)$ level cut of ζ_1 is not a proper subset of any other hyperedge of $\mathscr{H}^{h(\zeta_1)}$.*

Proof Let us suppose that $\tau(x) > 0$ and τ is an element of $t_r(\mathscr{H})$, then first condition directly follows from Lemma 6.2.

To prove the second condition, suppose that for every ζ_1 which satisfies the first condition, there is $\xi \in \zeta$ such that $h(\xi) > h(\zeta_1)$ and $\xi^{h(\xi)} \subseteq \zeta_1^{h(\zeta_1)}$. Then there exists an element $v \neq x$, where $v \in \xi^{h(\xi)} \cap \tau^{h(\xi)} \subseteq \zeta_1^{h(\zeta_1)} \cap \tau^{h(\zeta_1)}$, which is a contradiction.

To prove that $h(\zeta_1)$ level cut of ζ_1 is not a proper subset of any other hyperedge of $\mathscr{H}^{h(\zeta_1)}$, suppose that for every ζ_1, which satisfies the above two conditions, there is $\xi \in \zeta$ with $\emptyset \subset \xi^{h(\xi)} \subset \zeta_1^{h(\zeta_1)}$, as $\xi^{h(\xi)} \neq \emptyset$ and from second condition, we have $h(\xi) = \zeta_1(x) = \tau(x)$. If $h(\xi) = \zeta_1(x)$, our supposition accommodates $\xi' \in \zeta$ such that $\emptyset \subset \xi'^{h(\zeta_1)} \subset \xi^{h(\zeta_1)} \subset \zeta_1^{h(\zeta_1)}$. This recursive procedure must end after a finite number of steps, so assume that $\xi(x) < h(\xi)$, which implies the existence of an element $v \neq x$, where $v \in \xi^{h(\zeta_1)} \cap \tau^{h(\zeta_1)} \subseteq \zeta_1^{h(\zeta_1)} \cap \tau^{h(\zeta_1)}$, which is again a contradiction.

The sufficient condition is proved by using the construction given in Algorithm 6.3.1. By using first condition, we have $h(\zeta_1) = \rho_1, \rho_1 \in f_S(\mathscr{H})$ and from other two conditions, we have $y_\xi \in \xi^{h(\xi)} \backslash \zeta_1^{h(\zeta_1)}$ such that $\xi \neq \zeta_1$ and $h(\xi) \geq h(\zeta_1)$. Then $Q \cap \zeta_1^{h(\zeta_1)}$, where Q is the collection of all such vertices. An initial sequence of transversals of is constructed in a way that $\tau_j \subseteq Q$, for $1 \leq j \leq n$ and $\tau_i \subseteq Q \cup \{x\}$. Continuing the construction given in Algorithm 6.3.1 will give a minimal q-rung orthopair fuzzy transversal with $\tau(x) = \zeta_1(x) = h(\zeta_1)$.

Definition 6.19 Let Q be a q-rung orthopair fuzzy set and $\alpha, \beta \in [0, 1]$. The *lower truncation* of Q at level α, β is a q-rung orthopair fuzzy set $Q_{\langle\alpha,\beta\rangle}$ given by

$$Q_{\langle\alpha,\beta\rangle}(x) = \begin{cases} Q(x), & \text{if} \quad x \in Q^{(\alpha,\beta)}, \\ (0, 1), & \text{otherwise.} \end{cases}$$

The *upper truncation* of Q at level α, β is a q-rung orthopair fuzzy set $Q^{\langle\alpha,\beta\rangle}$ given by

$$Q^{\langle\alpha,\beta\rangle}(x) = \begin{cases} (\alpha, \beta), & \text{if} \quad x \in Q^{(\alpha,\beta)}, \\ Q(x), & \text{otherwise.} \end{cases}$$

Definition 6.20 Let $\mathscr{E}$ be a collection of q-rung orthopair fuzzy sets of X and $\mathscr{E}^{\langle\alpha,\beta\rangle} = \{q^{\langle\alpha,\beta\rangle} | q \in \mathscr{E}\}$, $\mathscr{E}_{\langle\alpha,\beta\rangle} = \{q_{\langle\alpha,\beta\rangle} | q \in \mathscr{E}\}$. Then, the *upper* and *lower truncations* of a q-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \zeta)$ at α, β level are a pair of q-rung orthopair fuzzy hypergraphs, $\mathscr{H}^{\langle\alpha,\beta\rangle}$ and $\mathscr{H}_{\langle\alpha,\beta\rangle}$, defined by $\mathscr{H}^{\langle\alpha,\beta\rangle} = (X, \mathscr{E}^{\langle\alpha,\beta\rangle})$ and $\mathscr{H}_{\langle\alpha,\beta\rangle} = (X, \mathscr{E}_{\langle\alpha,\beta\rangle})$.

Definition 6.21 Let Q be a q-rung orthopair fuzzy set on X, then each $(\mu, \nu) \in (0, h(Q))$ for which $Q^{(\alpha,\beta)} \nsubseteq Q^{(\mu,\nu)}$, $(\mu, \nu) < (\alpha, \beta) \leq h(Q)$, is called the *transition level* of Q.

Definition 6.22 Let Q be a nontrivial q-rung orthopair fuzzy set of X. Then,

(i) the sequence $\mathscr{S}(Q) = \{t_1^Q, t_2^Q, t_3^Q, \dots, t_n^Q\}$ is called the *basic sequence* determined by Q, where

- $t_1^Q > t_2^Q > t_3^Q > \cdots > t_n^Q > 0$,
- $t_1^Q = h(Q)$,
- $\{t_2^Q, t_3^Q, \dots, t_n^Q\}$ is the set of transition levels of Q.

(ii) The set of cuts of Q, $\mathscr{C}(Q)$, is defined as $\mathscr{C}(Q) = \{Q^t | t \in \mathscr{S}(Q)\}$.

(iii) The join $\max\{\eta(Q^t, t)|t \in \mathscr{S}(Q)\}$ of basic elementary q-rung orthopair fuzzy sets $E(Q) = \{\eta(Q^t, t)|t \in \mathscr{S}(Q)\}$ is called the *basic elementary join* of Q.

Lemma 6.3 *Let $\mathscr{H}$ be a q-rung orthopair fuzzy hypergraph with $f_S(\mathscr{H}) = \{\rho_1, \rho_2, \rho_3, \dots, \rho_n\}$. Then,*

(i) If $t = (\mu, \nu)$ is a transition level of $\tau \in t_r(\mathscr{H})$, then there is an $\varepsilon > 0$ such that, $\forall\ (\alpha, \beta) \in (t, t+\varepsilon]$, $\tau^{(\mu,\nu)}$ is a minimal $\mathscr{H}^{(\mu,\nu)}$-transversal extension of $\tau^{(\alpha,\beta)}$, i.e., if $\tau^{(\alpha,\beta)} \subseteq \tau' \subseteq \tau^{(\mu,\nu)}$ then τ' is not a transversal of $\mathscr{H}^{(\mu,\nu)}$.

(ii) $t_r(\mathscr{H})$ is sectionally elementary.

(iii) $f_S(t_r(\mathscr{H}))$ is properly contained in $f_S(\mathscr{H})$.

(iv) $\tau^{(\alpha,\beta)}$ is a minimal transversal of $\mathscr{H}^{(\alpha,\beta)}$, for each $\tau \in t_r(\mathscr{H})$ and $\rho_2 < (\alpha, \beta) \leq \rho_1$.

Proof (i) Let $\tilde{t} = (\mu, \nu)$ be a transition level of $\tau \in t_r(\mathscr{H})$. Then by definition, we have $\tau^{(\alpha,\beta)} \not\subseteq \tau^{(\mu,\nu)}$, $(\mu, \nu) < (\alpha, \beta) \leq h(\mathscr{H})$, for all α, β. Since, τ possesses a finite support, this implies the existence of an $\varepsilon > 0$ such that $\tau^{(\alpha,\beta)}$ is constant on $(\tilde{t}, \tilde{t}+\varepsilon]$. Assume that there is a transversal T of $\mathscr{H}^{(\mu,\nu)}$ such that $\tau^{(\alpha',\beta')} \subseteq T \subseteq \tau^{(\mu,\nu)}$, for $\alpha', \beta' \in (\tilde{t}, \tilde{t}+\varepsilon]$. We claim that this supposition is false. To demonstrate the existence of this claim, we suppose that assumption is true and consider the collection of basic elementary q-rung orthopair fuzzy sets $E(\tau) = \{\eta(\tau^t, t)|t \in S(\tau)\}$ of τ. Note that a nested sequence of X is formed by $c(\tau) \cup T$, where $c(\tau)$ is used to denote the basic cuts of τ. Since $\mathscr{H} = (\mathscr{Q}, \zeta)$ is defined on a finite set X and $\mathscr{Q}$ is a finite collection of q-rung orthopair fuzzy sets of X, then each $\rho \in (0, h(\mathscr{H}))$ corresponds a number $\varepsilon_\rho > 0$ such that

- $\mathscr{H}^{(\alpha,\beta)}$ is constant on $(\rho, \rho+\varepsilon_\rho]$,
- $\mathscr{H}^{(\alpha,\beta)}$ is constant on $(\rho-\varepsilon_\rho, \rho]$.

It follows from these considerations that level cuts of $\tau^{\star(\alpha,\beta)}$ of the join $\tau^\star = \max\{\max\{E(\tau)\backslash\eta(\tau^{\tilde{t}}, \tilde{t}), \eta(\tau^{\tilde{t}}, \tilde{t}-\varepsilon_{\tilde{t}}), \eta(T, \tilde{t})\}\}$ persuade

$$\tilde{\tau}^{(\alpha,\beta)} = \begin{cases} T, & \text{if} \quad (\alpha, \beta) \in (\tilde{t}-\varepsilon_{\tilde{t}}, \tilde{t}), \\ \tau^{(\alpha,\beta)}, & \text{if} \quad (\alpha, \beta) \in (0, h(\mathscr{H}))\backslash(\tilde{t}, \tilde{t}-\varepsilon_{\tilde{t}})]. \end{cases}$$

This relation is derived because of supposition that $\varepsilon_{\tilde{t}}$ is so small that the open interval $(\tilde{t}-\varepsilon_{\tilde{t}}, \tilde{t})$ does not contain any other transition level of τ.

Since, it is assumed that T is a transversal of $\mathscr{H}^{\tilde{t}}$, T is a transversal of $\mathscr{H}^{(\alpha,\beta)}$, for all $(\alpha, \beta) \in (\tilde{t} - \varepsilon_{\tilde{t}}, \tilde{t})$ and $\mathscr{H}^{(\alpha,\beta)}$ is constant on $(\tilde{t} - \varepsilon_{\tilde{t}}, \tilde{t})$. Note that, $\tau^{(\alpha,\beta)}$ is a transversal of $\mathscr{H}^{(\alpha,\beta)}$, for all $(\alpha, \beta) \in (0, h(\mathscr{H})]$, therefore, it follows that $\tilde{\tau}$ is a q-rung orthopair fuzzy transversal of $\mathscr{H}$, as $\tilde{\tau} < \tau$, implies that $\tau \notin t_r(\mathscr{H})$, which leads to a contradiction. Hence, the supposition is false and claim is satisfied.

(ii) Let $\tau \in t_r(\mathscr{H})$, then $\tau^{(\alpha,\beta)}$ is a transversal of $\mathscr{H}^{(\alpha,\beta)}$ for $0 < (\alpha, \beta) < h(\mathscr{H})$. Suppose that a transition level t of τ corresponds an interval $(t, t + \varepsilon]$, $\varepsilon > 0$, on which $\tau^{(\alpha,\beta)}$ is constant. Then for $(\alpha', \beta') \in (t, t + \varepsilon]$, $\tau^{(\alpha',\beta')}$ is not a transversal of $\mathscr{H}^t$, which implies that $\tau^{(\alpha',\beta')} \notin (t_r(\mathscr{H}))^t$, where $t_r(\mathscr{H}))^t$ denotes the t-cut of $t_r(\mathscr{H})$. However, the definition of fundamental sequence of $t_r(\mathscr{H})$ implies that $t \in f_S(t_r(\mathscr{H}))$.

(iii) To prove (iii), we suppose that if $t = (\mu, \nu)$ is a transition level of some $\tau \in t_r(\mathscr{H})$, then t belongs to $f_S(\mathscr{H})$. On contrary, suppose that the transition level t of some $\tau \in t_r(\mathscr{H})$ does not belong to $f_S(\mathscr{H})$. Then for some $\rho_j \in f_S(\mathscr{H})$, we have $\rho_{j+1} < t < \rho_j$, where $\rho_{n+1} = 0$, as $\mathscr{H}^{(\alpha,\beta)} = \mathscr{H}^{\rho_j}$, for all $(\alpha, \beta) \in (\rho_{j+1}, \rho_j]$, follows that τ^t is a transversal of $\mathscr{H}^t = \mathscr{H}^{\rho_j}$. Furthermore, there exists an $\varepsilon > 0$, such that $\tau^{(\alpha,\beta)}$ is constant on $(t, t + \varepsilon]$. Without loss of generality, we assume that $t + \varepsilon \leq \rho_j$ and $(\alpha', \beta') \in (t, t + \varepsilon]$. Since t is a transition level of τ then $\tau^{(\alpha',\beta')} \subsetneq \tau^t$ and $\tau^{(\alpha',\beta')}$ is not a transversal of $\mathscr{H}^t$ (from i), which is not possible, as $\mathscr{H}^{(\alpha',\beta')} = \mathscr{H}^{\rho_j} = \mathscr{H}^t$, this proves our claim. Along with this result and the fact that $h(\tau) = \rho_1 \in f_S(\mathscr{H})$, it follows that $f_S(t_r(\mathscr{H})) \subseteq f_S(\mathscr{H})$, for all $\tau \in t_r(\mathscr{H})$.

(iv) First, we will show that τ^{ρ_1} is a minimal transversal of $\mathscr{H}^{\rho_1}$. Suppose on contrary that there is a minimal transversal T of $\mathscr{H}^{\rho_1}$ such that $T \subseteq \tau^{\rho_1}$. Let $\tilde{\tau} = \max\{\tau^{\rho_2}, \eta_1\}$, where η_1 is the basic elementary q-rung orthopair fuzzy set having support T and height ρ_1. τ^{ρ_2} is considered as the upper truncation of τ at level ρ_2. It is obvious that $\tilde{\tau}$ is a transversal of $\mathscr{H}$ with $\tilde{\tau} < \tau$, which is contradiction to the fact that τ is minimal. From (ii) and (iii) parts, it is followed that $\tau^{(\alpha,\beta)} \in t_r(\mathscr{H})^{(\alpha,\beta)}$, for $\rho_2 < (\alpha, \beta) < \rho_1$.

Theorem 6.9 *At least one minimal q-rung orthopair fuzzy transversal is contained in every q-rung orthopair fuzzy transversal of a q-rung orthopair fuzzy hypergraph $\mathscr{H}$.*

Proof Let $f_S(\mathscr{H}) = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ be the fundamental sequence of $\mathscr{H}$ and suppose that ξ be a transversal of $\mathscr{H}$, which is not minimal. Let τ be a minimal transversal of $\mathscr{H}$, $\tau \leq \xi$, which is constructed in such a way, $\{q_i \in Q(X) | i = 0, 1, 2, \ldots, n\}$ satisfying $\tau = q_n \leq \cdots \leq q_1 \leq q_0 \leq \xi$, where $Q(X)$ is the collection of q-rung orthopair fuzzy sets on X. It can be noted that $h(\xi) \geq h(\mathscr{H}) = \rho_1$ and $\xi^{(\alpha,\beta)}$ is a transversal of $\mathscr{H}^{(\alpha,\beta)}$, for $0 < (\alpha, \beta) \leq \rho_1$. Therefore, the reduction process is started as $q_0 = \xi^{\langle \rho_1 \rangle}$, where $\xi^{\langle \rho_1 \rangle}$ represents the upper truncation level of ξ at ρ_1. Since the top level cut ξ^{ρ_1} of ρ_0 comprises a crisp minimal transversal T_1 of $\mathscr{H}^{\rho_1}$, we have $q_1 = \max\{\xi^{\langle \rho_2 \rangle}, \lambda^{T_1}\}$, where λ^{T_1} is elementary q-rung orthopair fuzzy set having height ρ_1 and support T_1. Note that, $q_1 \leq q_2 \leq \xi$. The same procedure will determine the all other remaining members. For instance, we have $q_2 = \max\{\xi^{\langle\rho_3\rangle}, \lambda^{T_1}, \lambda^{T_2}\}$, where λ^{T_2} is an elementary q-rung orthopair fuzzy set having height ρ_2 and support T_2, such that

$$T_2 = \begin{cases} T_1, & \text{if } T_1 \text{ is a transversal of } \mathscr{H}^{\rho_2}, \\ B_2, & \text{otherwise}, \end{cases}$$

where B_2 is the minimal transversal extension of T_1, i.e., if $T_1 \subseteq B \subseteq B_2$, then B_2 is not considered as a transversal of $\mathscr{H}^{\rho_2}$ and B_2 is contained in ρ-level of ξ because ξ^{ρ_2} contains a transversal of $\mathscr{H}^{\rho_2}$. Further, as $T_2 \subseteq \xi^{\rho_2}$, it is obvious that $q_2 \leq q_1$. When this process is finished, we certainly have $q_n = \tau$ a q-rung orthopair fuzzy transversal of $\mathscr{H}$ and is included in ξ. We now claim that τ is a minimal transversal of $\mathscr{H}$, i.e., $\tau \in t_r(\mathscr{H})$. On contrary, suppose that τ_1 is a transversal of $\mathscr{H}$ such that $\tau_1 < \tau$. Then, we have

(i) $\tau_1^{(\alpha,\beta)} \subseteq \tau^{(\alpha,\beta)}$ for all $\alpha, \beta \in (0, h(\mathscr{H})]$,
(ii) $\tau_1^{(\alpha',\beta')} \subseteq \tau^{(\alpha',\beta')}$ for some $\alpha', \beta' \in (0, h(\mathscr{H})]$.

However, no such α', β' exist. To prove this, let $\alpha, \beta \in (\rho_2, \rho_1]$, then as $\tau_1^{(\alpha,\beta)} \subseteq \tau^{(\alpha,\beta)}$, $\tau_1^{(\alpha,\beta)}$ is a transversal of $\mathscr{H}^{(\alpha,\beta)} = \mathscr{H}^{\rho_1}$ and $\tau^{(\alpha,\beta)} \in t_r(\mathscr{H}^{\rho_1})$, which implies that $\tau_1^{(\alpha,\beta)} = \tau^{(\alpha,\beta)}$ on $(\rho_2, \rho_1]$. Moreover, suppose that $\alpha, \beta \in (\rho_3, \rho_2]$ then by using $\tau_1^{(\alpha,\beta)} = \tau^{(\alpha,\beta)}$, we have $\tau_1^{(\alpha,\beta)} \supseteq \tau^{\rho_1}$ on $(\rho_3, \rho_2]$ and if $T_2 = T_1 = \tau^{\rho_1}$, then by previous arguments $\tau_1^{(\alpha,\beta)} = \tau^{(\alpha,\beta)}$ on $(\rho_3, \rho_2]$. Furthermore, if $T_1 \subseteq T_2$ and $T_1 \subseteq \tau_1^{(\alpha,\beta)} \subsetneq T_2$ then $\tau_1^{(\alpha,\beta)}$ is not a transversal of $\mathscr{H}^{(\alpha,\beta)} = \mathscr{H}^{\rho_2}$, which is contradiction to the fact that τ_1 is a transversal of $\mathscr{H}$. Hence, we have $\tau_1^{(\alpha,\beta)} = \tau^{(\alpha,\beta)}$ on $(\rho_3, \rho_2]$. In general, we have $\tau_1^{(\alpha,\beta)} = \tau^{(\alpha,\beta)}$ on $(0, h(\mathscr{H})]$, which completes the proof.

6.4 Applications to Decision-Making

Decision-making is considered as the abstract technique, which results in the selection of an opinion or a strategy among a couple of elective potential results. Every decision-making procedure delivers a final decision, which may or may not be appropriate for our problem. We have to make hundreds of decisions everyday, some are easy but others may be complicated, confused and miscellaneous. That is the reason which leads to the process of decision-making. Decision-making is the foremost way to choose the most desirable alternative. It is essential in real-life problems when there are many possible choices. Thus, decision makers evaluate numerous merits and demerits of every choice and try to select the most fitting alternative.

6.4.1 Selection of Most Desirable Appliance

Here, we consider a decision-making problem of selecting the most appropriate product from different brands or organizations. Suppose that a person wants to purchase a product, which is available in many brands. Let he/she considers the following nine organizations or brands $O = \{O_1, O_2, O_3, \ldots, O_9\}$, of which product can be chosen to purchase. We will discuss that how the (α, β)-level cuts can be applied to q-rung orthopair fuzzy hypergraph to make a good decision. A 6-rung orthopair fuzzy hypergraph model depicting the problem is shown in Fig. 6.6.

The truth-membership degrees and falsity-membership degrees of vertices (which represent the organizations) depicts that how much that organization fulfills the costumer's requirements and up to which percentage the product is not suitable. The hyperedges of our graph represent the characteristics of those organizations which are (as vertices) contained in that hyperedge. It can be shown from Table 6.4.

The attributes, which we have considered as hyperedges $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$ to describe the characteristics of different organizations are {Delivery and service, Durability, Affordability, Quality, Functionality, Marketability}. Note that, if ζ_2 is considered as durability, then the membership degrees $(0.9, 0.5)$ of O_3 describes that the product manufactured by organization O_3 is 90% durable and 50% lacks the requirements of the customer. Similarly, O_4 is 60% durable and 40% lacks the condition. In the same way, we can describe the characteristics of all products manufactured by different organizations. Now to select the most appropriate product, we will find out the (α, β)-level cuts of all hyperedges. We choose the values of α and β in such manner that they will be fixed according to customer's demand. Let $\alpha = 0.7$ and $\beta = 0.4$, it means that customer will consider that product, which will satisfy

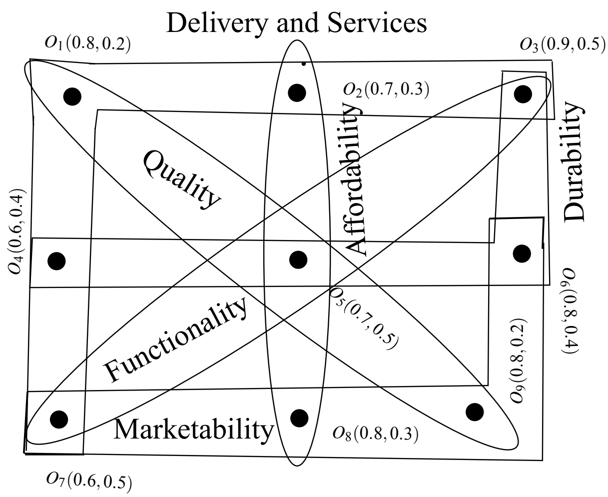


Fig. 6.6 6-rung orthopair fuzzy model for most appropriate appliance

Table 6.4 Incidence matrix

I	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
O_1	(0.8, 0.2)	(0, 1)	(0.8, 0.2)	(0, 1)	(0, 1)	(0, 1)
O_2	(0.7, 0.3)	(0, 1)	(0, 1)	(0.7, 0.3)	(0, 1)	(0, 1)
O_3	(0.9, 0.5)	(0.9, 0.5)	(0, 1)	(0, 1)	(0.9, 0.5)	(0, 1)
O_4	(0.6, 0.4)	(0.6, 0.4)	(0, 1)	(0, 1)	(0, 1)	(0, 1)
O_5	(0, 1)	(0.7, 0.5)	(0.7, 0.5)	(0.7, 0.5)	(0.7, 0.5)	(0, 1)
O_6	(0, 1)	(0.8, 0.4)	(0, 1)	(0, 1)	(0, 1)	(0.8, 0.4)
O_7	(0.6, 0.5)	(0, 1)	(0, 1)	(0, 1)	(0.6, 0.5)	(0.6, 0.5)
O_8	(0, 1)	(0, 1)	(0, 1)	(0.8, 0.3)	(0, 1)	(0.8, 0.3)
O_9	(0, 1)	(0, 1)	(0.8, 0.2)	(0, 1)	(0, 1)	(0.8, 0.2)

70% or more of the the characteristics mentioned above and will have deficiency less than or equal to 40%. The (α, β)-levels of all hyperedges are given as follows:

$$\zeta_1^{(0.7,0.4)} = \{O_1, O_2\}, \quad \zeta_2^{(0.7,0.4)} = \{O_6\}, \quad \zeta_3^{(0.7,0.4)} = \{O_1, O_5, O_9\},$$
$$\zeta_4^{(0.7,0.4)} = \{O_2, O_8\}, \quad \zeta_5^{(0.7,0.4)} = \{\emptyset\}, \quad \zeta_6^{(0.7,0.4)} = \{O_6, O_8, O_9\}.$$

Note that, $\zeta_1^{(0.7,0.4)}$ level set represents that O_1 and O_2 are the organizations that provide the best delivery services among all other organizations, $\zeta_2^{(0.7,0.4)}$ level set represents that O_6 is the organization, whose products are more durable as compared to all other organizations. Similarly, $\zeta_4^{(0.7,0.4)}$ indicates that the products proposed by O_2 and O_8 organizations, are more cheap and affordable in comparison to others. Thus, if a customer wants some specific speciality of product, for example he she wants to purchase a product with good marketablity, then the organizations O_6, O_8 and O_9 are more suitable. Similarly, if the satisfaction and dissatisfaction level of a customer are taken as $\alpha = 0.8$ and $\beta = 0.3$, respectively. Then, (0.8, 0.3)-level cuts are given as,

$$\zeta_1^{(0.8,0.3)} = \{O_1\}, \quad \zeta_2^{(0.8,0.3)} = \{\emptyset\}, \quad \zeta_3^{(0.8,0.3)} = \{O_1, O_9\},$$
$$\zeta_4^{(0.8,0.3)} = \{O_8\}, \quad \zeta_5^{(0.8,0.3)} = \{\emptyset\}, \quad \zeta_6^{(0.8,0.3)} = \{O_8, O_9\}.$$

Here, $\zeta_4^{(0.8,0.3)} = \{O_8\}$ indicates that the products proposed by organization O_8 satisfy the customer's requirement 80%, which is affordability and so on. For $\alpha = 0.7$ and $\beta = 0.3$, we have,

$$\zeta_1^{(0.7,0.3)} = \{O_1, O_2\}, \quad \zeta_2^{(0.7,0.3)} = \{\emptyset\}, \quad \zeta_3^{(0.7,0.3)} = \{O_1, O_9\},$$
$$\zeta_4^{(0.7,0.3)} = \{O_2, O_8\}, \quad \zeta_5^{(0.7,0.3)} = \{\emptyset\}, \quad \zeta_6^{(0.7,0.3)} = \{O_8, O_9\}.$$

Hence, by considering different (α, β)-levels corresponding to the satisfaction and dissatisfaction levels of customers, we can conclude that which organization fulfill the actual demands of a customer. The method adopted in this application is given in the following Algorithm 6.4.1.

Algorithm 6.4.1

Finding the most suitable organization

1. Input the degree of membership of all $q-$rung orthopair fuzzy vertices $O_1, O_2, O_3, \cdots, O_m$.
2. Calculate the membership degrees of $q-$rung orthopair fuzzy hyperedges using the formula,
$$T_\zeta(E_k) = T_\zeta(O_1, O_2, O_3, \ldots, O_m) \leq \min\{\mathscr{Q}_i(O_1), \mathscr{Q}_i(O_2), \mathscr{Q}_i(O_3), \ldots, \mathscr{Q}_i(O_m)\},$$
$$F_\zeta(E_k) = F_\zeta(O_1, O_2, O_3, \ldots, O_m) \leq \max\{\mathscr{Q}_i(O_1), \mathscr{Q}_i(O_2), \mathscr{Q}_i(O_3), \ldots, \mathscr{Q}_i(O_m)\},$$
for all $O_1, O_2, O_3, \cdots, O_m$ representing the organizations as vertices of hyperedge.
3. Calculate the $(\alpha, \beta)-$levels of $q-$rung orthopair fuzzy hyperedges by using,
$$\zeta_i^{(\alpha,\beta)} = \{O_j \in O | T_{\zeta_i}(O_j) \geq \alpha \ , \ F_{\zeta_i}(O_j) \leq \beta\},$$
for $i = 1, 2, 3, \cdots, k$, $j = 1, 2, 3 \cdots, m$ and $\alpha, \beta \in [0, 1]$.
4. Crisp sets describe the most suitable organization according to the customer's satisfaction levels.

6.4.2 *Adaptation of Most Alluring Residential Scheme*

The essential factor for any purchase of property is the budget and location for a purchaser, particularly. However, it is a complicated procedure to select a residential area for buying a house. In addition to scrutinizing the further details such as the pricing, loan options, payments, and developer's credentials a customer must examine closely some other facilities which should be possessed by every housing colony. Now, to adopt a favorable housing scheme, an obvious initial step is to compare the differen societies. After analyzing the characteristics of different societies, one will be able to make a wise decision. We will investigate the problem of adopting the most alluring residential scheme using 7-rung orthopair fuzzy hypergraph. Let the set of vertices of 7-rung orthopair fuzzy hypergraph is taken as the representative of those attributes characteristics, which one has been considered to make a comparison between different housing societies. The hyperedges of 7-rung orthopair fuzzy hypergraph represents some housing schemes, which will be compared. The portrayal of our problem is illustrated in Fig. 6.7.

The description of hyperedges $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7\}$ and vertices $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ of above hypergraph is given in Tables 6.5 and 6.6, respectively.

Note that, each hyperedge represents a distinct housing scheme and the vertices contained in hyperedges are those attributes, which will be provided by the societies represented through hyperedges. It means that Senate Avenue housing society provides 80% the basic facilities of life such as water, gas, and electricity and 20% deprives these facilities. Similarly, the same society 90% accommodates its residents being easy assessable and only 10% lacks the facility. In the same way, taking into account the truth-membership and falsity-membership degrees of all other attributes, we can identify the characteristics of all societies.

Now, to determine the overall comforts of each society, we will calculate the heights of all hyperedges and the society having the maximum truth-membership

Fig. 6.7 7-rung orthopair fuzzy hypergraph model

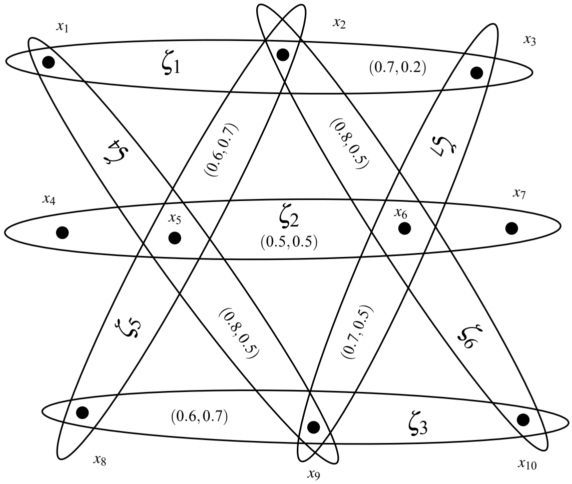


Table 6.5 Description of hyperedges

Set of hyperedges	Corresponding housing scheme	Provision of facilities (%)	Deprival of facilities (%)
ζ_1	Senate avenue	70	20
ζ_2	Soan gardens	50	50
ζ_3	CBR town	60	70
ζ_4	OPF housing scheme	80	50
ζ_5	Paradise city	60	70
ζ_6	RP corporation	80	50
ζ_7	Tele gardens housing scheme	70	50

Table 6.6 Description of attributes

Set of attributes	Depicting facility	Provision level of corresponding facility	Deprival of corresponding facility
x_1	Basic amenities of life	0.8	0.2
x_2	Easily assessable	0.9	0.1
x_3	Land ownership	0.7	0.2
x_4	The power back-up	0.6	0.3
x_5	Eco-friendly construction	0.9	0.4
x_6	Social infrastructure	0.8	0.5
x_7	Drainage system	0.5	0.6
x_8	Security	0.6	0.7
x_9	Regular sanitation	0.8	0.5
x_{10}	The parking area	0.9	0.3

Table 6.7 Heights of hyperedges

Heights of hyperedges	$(\max(\zeta_l), \min(\zeta_m))$
h (Senate Avenue)	(0.9, 0.1)
h (Soan Gardens)	(0.9, 0.3)
h (CBR Town)	(0.9, 0.3)
h (OPF Housing Scheme)	(0.9, 0.2)
h (Paradise City)	(0.9, 0.1)
h (RP Corporation)	(0.9, 0.1)
h (Tele Gardens Housing Scheme)	(0.8, 0.2)

and minimum falsity-membership will be considered as a most comfortable society to be live in. The calculated heights of all schemes are given in Table 6.7.

It can be noted from Table 6.7 that there are three societies which have the maximum membership and minimum nonmembership degrees, i.e., Senate Avenue, Paradise City, and RP Corporation are those housing societies which will provide 90% facilities to their habitants and only 10% amenities will be dispersed. Thus, it is more beneficial and substantial to select one of these three housing schemes.

The same problem can be speculated to a more extended idea that if some one wants to built a new housing scheme, which will carry out the facilities of all above societies. The concept of 7-rung orthopair fuzzy hypergraphs can be utilized to speculate such housing scheme. Consider a 7-rung orthopair fuzzy set of vertices given as follows,

$$\tau_1 = \{(x_1, 0.8, 0.2), (x_2, 0.9, 0.1), (x_5, 0.9, 0.3), (x_6, 0.8, 0.2), (x_{10}, 0.9, 0.3)\}.$$

By applying the definition of 7-rung orthopair fuzzy transversal, it can be seen that

$$\begin{aligned}
&\zeta_1^{(0.9,0.1)} \cap \tau_1^{(0.9,0.1)} = \{x_2\}, \qquad \zeta_2^{(0.9,0.3)} \cap \tau_1^{(0.9,0.3)} = \{x_5\},\\
&\zeta_3^{(0.9,0.3)} \cap \tau_1^{(0.9,0.3)} = \{x_{10}\}, \qquad \zeta_4^{(0.9,0.2)} \cap \tau_1^{(0.9,0.2)} = \{x_5\},\\
&\zeta_5^{(0.9,0.1)} \cap \tau_1^{(0.9,0.1)} = \{x_2\}, \qquad \zeta_6^{(0.9,0.1)} \cap \tau_1^{(0.9,0.1)} = \{x_2\},\\
&\zeta_7^{(0.8,0.2)} \cap \tau_1^{(0.8,0.2)} = \{x_6\},
\end{aligned}$$

that is, the q-rung orthopair fuzzy subset τ_1 satisfies the condition of transversal and the housing society that will be represented through this hyperedge will contain at least one attribute of each scheme mentioned above. Similarly, some other societies can be figured out by following the same method. Hence, some other 7-rung orthopair fuzzy subsets are given as

$$\begin{aligned}
\tau_2 &= \{(x_1, 0.8, 0.2), (x_2, 0.9, 0.1), (x_3, 0.7, 0.2), (x_5, 0.9, 0.3), (x_6, 0.8, 0.2), (x_{10}, 0.9, 0.3)\},\\
\tau_3 &= \{(x_2, 0.9, 0.1), (x_4, 0.6, 0.3), (x_5, 0.9, 0.3), (x_6, 0.8, 0.2), (x_{10}, 0.9, 0.3)\},\\
\tau_4 &= \{(x_2, 0.9, 0.1), (x_5, 0.9, 0.3), (x_6, 0.8, 0.2), (x_{10}, 0.9, 0.3)\},\\
\tau_5 &= \{(x_2, 0.9, 0.1), (x_5, 0.9, 0.3), (x_6, 0.8, 0.2), (x_7, 0.5, 0.5), (x_8, 0.6, 0.7), (x_{10}, 0.9, 0.3)\}.
\end{aligned}$$

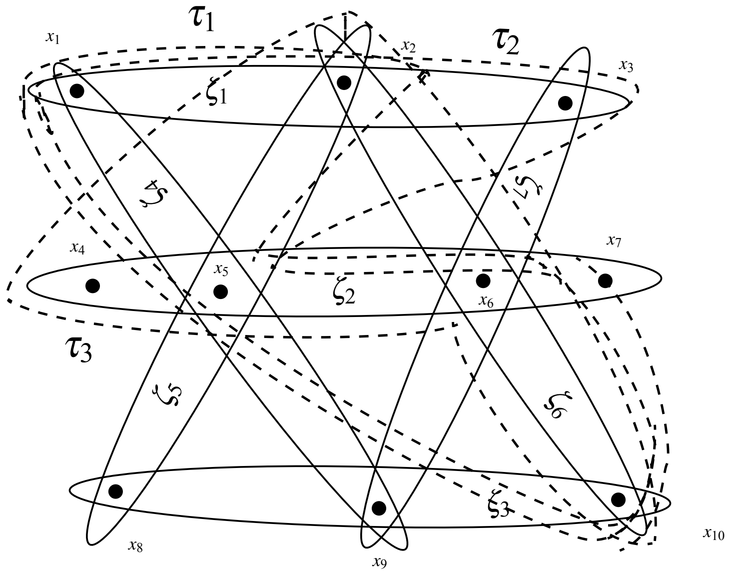


Fig. 6.8 7-rung orthopair fuzzy transversals

The graphical description of these schemes is displayed in Fig. 6.8 through dashed lines.

Thus, the schemes shown through dashed lines will contain the attributes of all other societies and may be more advantageous to their dwellers. The method adopted in our application is explained through the Algorithm 6.4.2.

Algorithm 6.4.2

Finding the more advantageous schemes
1. Input the degree of membership of all $q-$rung orthopair fuzzy vertices $x_1, x_2, x_3, \cdots, x_m$.
2. Calculate the membership degrees of $q-$rung orthopair fuzzy hyperedges using the formula, $T_\zeta(E_k) \leq \min\{\mathcal{Q}_i(x_1), \mathcal{Q}_i(x_2), \ldots, \mathcal{Q}_i(x_m)\}$, $F_\zeta(E_k) \leq \max\{\mathcal{Q}_i(x_1), \mathcal{Q}_i(x_2), \ldots, \mathcal{Q}_i(x_m)\}$, for all $x_1, x_2, x_3, \cdots, x_m$ representing the attributes of housing societies.
3. Calculate the heights of all $q-$rung orthopair fuzzy hyperedges using, $h(\zeta_j) = (\max T_{\zeta_j}(x_i), \min F_{\zeta_j}(x_i))$, $j = 1, 2, 3, \cdots, k$ and $i = 1, 2, 3, \cdots, m$.
4. Maximum truth-membership and minimum falsity-membership will denote the most alluring residential area.
5. Input the different $q-$rung orthopair fuzzy subsets.
6. Determine the $q-$rung orthopair fuzzy transversals using the formula,
7. Find the more advantageous schemes, which will contain the attributes of all other societies.

6.5 q-Rung Orthopair Fuzzy Directed Hypergraphs

In this section, we define q-rung orthopair fuzzy digraphs and q-rung orthopair fuzzy directed hypergraphs. A q-rung orthopair fuzzy directed hypergraph generalizes the concept of an intuitionistic fuzzy directed hypergraph and broaden the space of orthopairs. We also define and construct the dual and line graphs of q-rung orthopair fuzzy directed hypergraphs. All these concepts are explained and justified through concrete examples.

Definition 6.23 A *q-rung orthopair fuzzy digraph* on a non-empty set X is a pair $\overrightarrow{D} = (\mathscr{A}, \overrightarrow{\mathscr{B}})$, where $\mathscr{A}$ is a q-rung orthopair fuzzy set on X and $\overrightarrow{\mathscr{B}}$ is a q-rung orthopair fuzzy relation on X such that

$$T_{\overrightarrow{\mathscr{B}}}(x_1x_2) \leq \min\{T_{\mathscr{A}}(x_1), T_{\mathscr{A}}(x_2)\}, \ F_{\overrightarrow{\mathscr{B}}}(x_1x_2) \leq \max\{F_{\mathscr{A}}(x_1), F_{\mathscr{A}}(x_2)\},$$

and $0 \leq T^q_{\overrightarrow{\mathscr{B}}}(x_1x_2) + F^q_{\overrightarrow{\mathscr{B}}}(x_1x_2) \leq 1$, $q \geq 1$, for all $x_1, x_2 \in X$.

Remark 6.4
- When $q = 1$, 1-rung orthopair fuzzy digraph is called an intuitionistic fuzzy digraph.
- When $q = 2$, 2-rung orthopair fuzzy digraph is called Pythagorean fuzzy digraph.

Example 6.6 Let $X = \{x_1, x_2, x_3, x_4\}$ be the set of universe, $\mathscr{A} = \{(x_1, 0.7, 0.8), (x_2, 0.6, 0.9), (x_3, 0.5, 0.8), (x_4, 0.7, 0.8)\}$ be a 5-rung orthopair fuzzy set and $\overrightarrow{\mathscr{B}}$ be a 5-rung orthopair fuzzy relation on X such that, $0 \leq T^5_{\overrightarrow{\mathscr{B}}}(x_ix_j) + F^5_{\overrightarrow{\mathscr{B}}}(x_ix_j) \leq 1$, for all $x_i, x_j \in X$. The corresponding 5-rung orthopair fuzzy digraph $\overrightarrow{D} = (\mathscr{A}, \overrightarrow{\mathscr{B}})$ is shown in Fig. 6.9.

Definition 6.24 A *q-rung orthopair fuzzy directed hypergraph* $\mathscr{D}$ on X is defined as an ordered pair $\mathscr{D} = (Q, \xi)$, where Q is the collection of q-rung orthopair fuzzy subsets of X and ξ is a family of q-rung orthopair fuzzy directed hyperedges (or hyperarcs) such that,

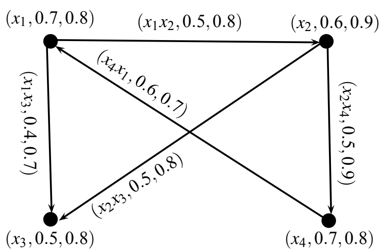


Fig. 6.9 5-rung orthopair fuzzy digraph $\overrightarrow{D}$

1.
$$T_\xi(E_k) = T_\xi(x_1, \ldots, x_m) \le \min\{T_{Q_i}(x_1), \ldots, T_{Q_i}(x_m)\},$$
$$F_\xi(E_k) = F_\xi(x_1, \ldots, x_m) \le \max\{F_{Q_i}(x_1), \ldots, F_{Q_i}(x_m)\},$$
for all $x_1, x_2, \ldots, x_m \in X$.

2. $\bigcup_i supp(Q_i) = X$, for all $Q_i \in Q$.

A q-rung orthopair fuzzy directed hyperedge $\xi_i \in \xi$ is defined as an ordered pair $(h(\xi_i), t(\xi_i))$, where $h(\xi_i)$ and $t(\xi_i) \in X - h(\xi_i)$, nontrivial subsets of X, are called the *head* of ξ_i and *tail* of ξ_i, respectively.
A *source vertex* v in ξ_i is defined as $h(\xi_i) \neq v$, for all $\xi_i \in \xi$ and a *destination vertex* v' in ξ_i is defined as $t(\xi_i) \neq v'$, for all $\xi_i \in \xi$.

Definition 6.25 A q-rung orthopair fuzzy directed hypergraph is called a *backward q-rung orthopair fuzzy directed hypergraph* if all of its hyperarcs are B-arcs, i.e., $\xi_i = (h(\xi_i), t(\xi_i))$ with $|h(\xi_i)| = 1$, for all $\xi_i \in \xi$.
A q-rung orthopair fuzzy directed hypergraph is called a *forward q-rung orthopair fuzzy directed hypergraph* if all of its hyperarcs are F-arcs, i.e., $\xi_i = (h(\xi_i), t(\xi_i))$ with $|t(\xi_i)| = 1$, for all $\xi_i \in \xi$.

Definition 6.26 The *height* of a q-rung orthopair fuzzy directed hypergraph $\mathscr{D} = (Q, \xi)$ is defined as $h^\star(\mathscr{D}) = \{\max(\xi_l), \min(\xi_m)\}$, where $\xi_l = \max T_{\xi_j}(x_i)$ and $\xi_m = \min F_{\xi_j}(x_i)$. Here, $T_{\xi_j}(x_i)$ and $F_{\xi_j}(x_i)$ denote the truth-membership and falsity-membership of vertex x_i to the directed hyperedge ξ_j, respectively.

Definition 6.27 Let $\mathscr{D} = (Q, \xi)$ be a q-rung orthopair fuzzy directed hypergraph. The *order* of $\mathscr{D}$, which is denoted by $O(\mathscr{D})$, and is defined as $O(\mathscr{D}) = \sum_{x \in X} \wedge \xi_i(x)$.
The *size* of $\mathscr{D}$, which is denoted by $S(\mathscr{D})$, and is defined as $S(\mathscr{D}) = \sum_{x \in X} \vee \xi_i(x)$.

Definition 6.28 A repeatedly occurring sequence $v_1, \xi_1, v_2, \xi_2, \ldots, v_{n-1}, \xi_{n-1}, v_n$ of definite vertices and directed hyperarcs such that,

- $0 < T_\xi(\xi_i) \le 1$ and $0 \le F_\xi(\xi_i) < 1$,
- $v_{i-1}, v_i \in \xi_i, i = 1, 2, 3, \ldots, n$,

is called a *q-rung orthopair fuzzy directed hyperpath* of length $n - 1$ from v_1 to v_n. If $v_1 = v_n$, then this q-rung orthopair fuzzy directed hyperpath is called a *q-rung orthopair fuzzy directed hypercycle*.

Definition 6.29 The *strength of q-rung orthopair fuzzy directed hyperpath* of length k, which connects the two vertices v_1 and v_2, is defined as $\lambda^k(v_1, v_2) = \{\min\{T_\xi(\xi_1), T_\xi(\xi_2), T_\xi(\xi_3), \ldots, T_\xi(\xi_k)\}, \max\{F_\xi(\xi_1), F_\xi(\xi_2), F_\xi(\xi_3), \ldots, F_\xi(\xi_k)\}\}, v_1 \in \xi_1, v_2 \in \xi_k$ and $\xi_1, \xi_2, \xi_3, \ldots, \xi_k$ are q-rung orthopair fuzzy directed hyperedges.

The *strength of connectedness* between v_1 and v_2 is given as, $\lambda^\infty(v_1, v_2) = \{\max_k T(\lambda^k(v_1, v_2)), \min_k F(\lambda^k(v_1, v_2))\}$.

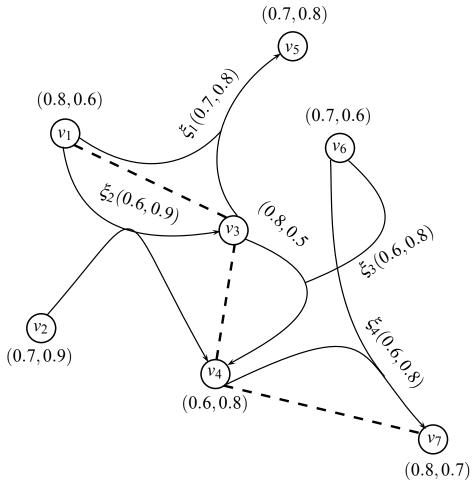


Fig. 6.10 A 5-rung orthopair fuzzy directed hypergraph

A *connected q-rung orthopair fuzzy directed hypergraph* is one in which we have at least one q-rung orthopair fuzzy directed hyperpath between each pair of vertices of $\mathscr{D}$.

We now illustrate the Definitions 6.24, 6.25, 6.26, 6.27, 6.28 and 6.29 through an example of 5-rung orthopair fuzzy directed hypergraph.

Example 6.7 Consider a 5-rung orthopair fuzzy directed hypergraph $\mathscr{D} = (Q, \xi)$, as shown in Fig. 6.10.

In this 5-rung orthopair fuzzy directed hypergraph, we have

$$\xi_1 = \{\{(v_1, 0.8, 0.6), (v_3, 0.8, 0.5)\}, \{(v_5, 0.7, 0.8)\}\} = \{t(\xi_1), h(\xi_1)\},$$
$$\xi_2 = \{\{(v_1, 0.8, 0.6), (v_2, 0.7, 0.9)\}, \{(v_3, 0.8, 0.5), (v_4, 0.6, 0.8)\}\} = \{t(\xi_2), h(\xi_2)\},$$
$$\xi_3 = \{\{(v_3, 0.8, 0.5), (v_6, 0.7, 0.6)\}, \{(v_4, 0.6, 0.8)\}\} = \{t(\xi_3), h(\xi_3)\},$$
$$\xi_4 = \{\{(v_4, 0.6, 0.8), (v_6, 0.7, 0.6)\}, \{(v_7, 0.8, 0.7)\}\} = \{t(\xi_4), h(\xi_4)\}.$$

A 5-rung orthopair fuzzy directed hyperpath from v_1 to v_7 of length 3 is shown through dashed lines and is given by an alternating sequence $v_1, \xi_2, v_3, \xi_3, v_4, \xi_4, v_7$ of distinct vertices and directed hyperarcs. The strength of this hyperpath is

$$\begin{aligned}\lambda^3(v_1, v_7) &= \{\min\{T_\xi(\xi_2), T_\xi(\xi_3), T_\xi(\xi_4)\}, \max\{F_\xi(\xi_2), F_\xi(\xi_3), F_\xi(\xi_4)\}\} \\ &= (0.6, 0.9), \\ \lambda^\infty(v_1, v_7) &= (0.6, 0.9).\end{aligned}$$

Note that, $\mathscr{D} = (Q, \xi)$ is not connected because we don't have a directed hyperpath between each pair of vertices, i.e., v_1 is not connected to v_6. A backward and forward 5-rung orthopair fuzzy directed hypergraph is shown in Fig. 6.11a, b, respectively.

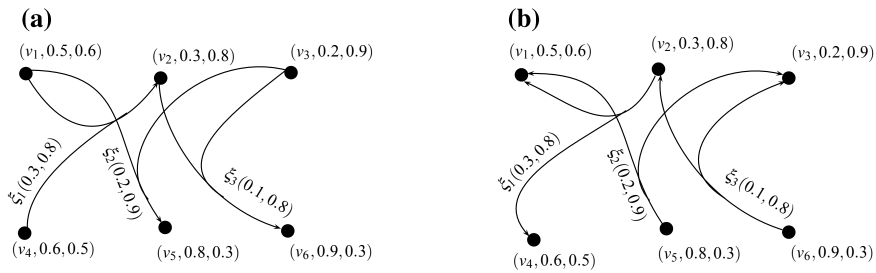


Fig. 6.11 Backward and forward 5-rung orthopair fuzzy directed hypergraphs

Definition 6.30 A q-rung orthopair fuzzy directed hypergraph $\mathscr{D} = (Q, \xi)$ is *linear* if every pair of q-rung orthopair fuzzy directed hyperedges $\xi_i, \xi_j \in \xi$ satisfies

- $supp(\xi_i) \subseteq supp(\xi_j) \Rightarrow i = j$,
- $|supp(\xi_i) \cap supp(\xi_j)| \leq 1$.

Example 6.8 Consider a 5-rung orthopair fuzzy directed hypergraph $\mathscr{D} = (Q, \xi)$, as shown in Fig. 6.10. In this 5-rung orthopair fuzzy directed hypergraph, we have $supp(\xi_1) = \{v_1, v_3, v_5\}$, $supp(\xi_2) = \{v_1, v_2, v_3, v_4\}$, $supp(\xi_3) = \{v_3, v_6, v_4\}$, $supp(\xi_4) = \{v_4, v_6, v_7\}$. Note that, $supp(\xi_i) \subseteq supp(\xi_j) \Rightarrow i = j$ and

$$\begin{aligned}
|supp(\xi_1) \cap supp(\xi_2)| &= |\{v_1, v_3\}| = 2,\\
|supp(\xi_1) \cap supp(\xi_3)| &= |\{v_3\}| = 1,\\
|supp(\xi_1) \cap supp(\xi_4)| &= |\{\emptyset\}| = 0,\\
|supp(\xi_2) \cap supp(\xi_3)| &= |\{v_4, v_3\}| = 2,\\
|supp(\xi_2) \cap supp(\xi_4)| &= |\{v_4\}| = 1,\\
|supp(\xi_3) \cap supp(\xi_4)| &= |\{v_4, v_6\}| = 2.
\end{aligned}$$

That is, $|supp(\xi_i) \cap supp(\xi_j)| \not\leq 1$, for all $\xi_i, \xi_j \in \xi$. Hence, $\mathscr{D} = (Q, \xi)$ is not linear.

Definition 6.31 Let $\mathscr{D} = (Q, \xi)$ be a q-rung orthopair fuzzy directed hypergraph. The *q-rung orthopair fuzzy line graph* of $\mathscr{D}$ is the graph $l(\mathscr{D}) = (X_l, \xi_l)$ such that,

1. $X_l = \xi$,
2. $\{\xi_i, \xi_j\} \in \xi_l \Leftrightarrow |supp(\xi_i) \cap supp(\xi_j)| \neq \emptyset$, for $i \neq j$.

The truth-membership and falsity-membership of vertices and edges of $l(\mathscr{D})$ are determined as follows:

- $X_l(\xi_i) = \xi(\xi_i)$,
- $T_{\xi_l}(\{\xi_i, \xi_j\}) = \min\{T_\xi(\xi_i), T_\xi(\xi_j) | \xi_i, \xi_j \in \xi\}$, $F_{\xi_l}(\{\xi_i, \xi_j\}) = \max\{F_\xi(\xi_i), F_\xi(\xi_j) | \xi_i, \xi_j \in \xi\}$.

Theorem 6.10 *Let $\mathscr{G} = (U, \varepsilon)$ be a simple q-rung orthopair fuzzy directed graph. Then $\mathscr{G}$ is the q-rung orthopair fuzzy line graph of a linear q-rung orthopair fuzzy directed hypergraph.*

Proof Let $\mathscr{G} = (U, \varepsilon)$ be a simple q-rung orthopair fuzzy directed graph. We suppose that $\mathscr{G} = (U, \varepsilon)$ is connected, with no loss of generality. A q-rung orthopair fuzzy directed hypergraph $\mathscr{D} = (Q, \xi)$ can be formulated from $\mathscr{G}$ as follows:

(i) The set of directed edges of $\mathscr{G}$ will be taken as vertices of $\mathscr{D}$, i.e., $\varepsilon = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n\}$ be the directed edges of $\mathscr{G}$ and hence the set of vertices of $\mathscr{D}$. Let $X = \{q_1, q_2, q_3, \ldots, q_k\}$ be the set of nontrivial q-rung orthopair fuzzy sets on U such that $q_i(\varepsilon_j) = (1, 0)$, $i = 1, 2, 3, \ldots, k$, $j = 1, 2, 3, \ldots, n$.
(ii) Let $U = \{u_1, u_2, u_3, \ldots, u_j\}$ then the directed hyperedges of $\mathscr{D}$ are $\xi = \{\xi_1, \xi_2, \xi_3, \ldots, \xi_n\}$, where ξ_i are those directed edges of $\mathscr{G}$, which contain the vertex u_i as their incidence vertex, i.e., $\xi_i = \{\varepsilon_j | u_i \in \varepsilon_j, j = 1, 2, 3, \ldots, n\}$. Moreover, $\xi(\xi_i) = U(u_i)$, $i = 1, 2, 3, \ldots, k$.

We now claim that $\mathscr{D} = (Q, \xi)$ is linear q-rung orthopair fuzzy directed hypergraph. Consider an arbitrary directed hyperedge $\xi_j = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_r\}$ and from the defining relation of q-rung orthopair fuzzy directed hypergraph, we have

$$T_\xi(\xi_j) = \min\{T_{q_j}(\varepsilon_1), T_{q_j}(\varepsilon_2), \ldots, T_{q_j}(\varepsilon_r)\} = T_U(u_i) \leq 1,$$
$$F_\xi(\xi_j) = \max\{F_{q_j}(\varepsilon_1), F_{q_j}(\varepsilon_2), \ldots, F_{q_j}(\varepsilon_r)\} = F_U(u_i) \geq 0,$$

$i = 1, 2, 3, \ldots, k$ and $\bigcup_k supp(q_k) = X$, for all q_k.
We now prove that $\mathscr{D} = (Q, \xi)$ is linear.

1. By our supposition, membership degree of each vertex ε_i of $\mathscr{D}$ is $(1, 0)$. Thus, we have $supp(\xi_i) \subseteq supp(\xi_j)$ implies $i = j$.
2. Suppose on contrary that $|supp(\xi_i) \cap supp(\xi_j)| = \{\varepsilon_l, \varepsilon_m\}$, i.e., these edges have two incidence vertices in common, which is contradiction to the fact that $\mathscr{G}$ is simple. Hence, $|supp(\xi_i) \cap supp(\xi_j)| \leq 1$, for $1 \leq i, j \leq r$.

Theorem 6.11 *A necessary and sufficient condition for $l(\mathscr{D})$ to be connected is that $\mathscr{D}$ is connected.*

Proof Let $\mathscr{D} = (Q, \xi)$ be a connected q-rung orthopair fuzzy directed hypergraph and $l(\mathscr{D}) = (X_l, \xi_l)$ be the line graph of $\mathscr{D}$. Suppose that ξ_i and ξ_j be two vertices of $l(\mathscr{D})$ and $v_i \in \xi_i$, $v_j \in \xi_j$, for $v_i \neq v_j$. Since $\mathscr{D}$ is connected then there exists an alternating sequence $v_i, \xi_i, v_{i+1}, \xi_{i+1}, \ldots, \xi_j, v_j$, which connects v_i and v_j. From the definition of strength of connectedness between v_i and v_j, we have

$$\begin{aligned}
\lambda^\infty(\xi_i, \xi_j) &= \max_k T(\lambda^k(\xi_i, \xi_j)), \min_k F(\lambda^k(\xi_i, \xi_j)) \\
&= \{\max_k(T_{\xi_l}(\xi_i, \xi_{i+1}) \wedge T_{\xi_l}(\xi_{i+1}, \xi_{i+2}) \wedge \cdots \wedge T_{\xi_l}(\xi_{j-1}, \xi_j)), \\
&\quad \min_k(F_{\xi_l}(\xi_i, \xi_{i+1}) \vee F_{\xi_l}(\xi_{i+1}, \xi_{i+2}) \vee \cdots \vee F_{\xi_l}(\xi_{j-1}, \xi_j))\}, k = 1, 2, \ldots \\
&= \{\max_k(T_{\xi_l}(\xi_i) \wedge T_{\xi_l}(\xi_{i+1}) \wedge T_{\xi_l}(\xi_{i+2}) \wedge \cdots \wedge T_{\xi_l}(\xi_{j-1}) \wedge T_{\xi_l}(\xi_j)), \\
&\quad \min_k(F_{\xi_l}(\xi_i) \vee F_{\xi_l}(\xi_{i+1}) \vee F_{\xi_l}(\xi_{i+2}) \vee \cdots \vee F_{\xi_l}(\xi_{j-1}) \vee F_{\xi_l}(\xi_j))\}, \\
&= \max T(\lambda^k(v_i, v_j)), \min F(\lambda^k(v_i, v_j)) \\
&= \lambda^\infty(v_i, v_j) > 0.
\end{aligned}$$

Hence, $l(\mathscr{D})$ is connected. By reversing the same procedure, we can easily prove that if $l(\mathscr{D})$ is connected then $\mathscr{D}$ is connected.

Let $\mathscr{D} = (Q, \xi)$ be a q-rung orthopair fuzzy directed hypergraph. The construction of a q-rung orthopair fuzzy directed line graph from a q-rung orthopair fuzzy directed hypergraph is illustrated in Algorithm 6.5.1.

Algorithm 6.5.1

Finding the $q-$rung orthopair fuzzy directed line graph

1. Input the number of directed hyperedges r of $q-$rung orthopair fuzzy directed hypergraph $\mathscr{D} = (Q, \xi)$.
2. Input the truth-membership and falsity membership of directed hyperedges $\xi_1, \xi_2, \xi_3, \cdots, \xi_r$.
3. Construct a $q-$rung orthopair fuzzy line graph $l(\mathscr{D}) = (X_l, \xi_l)$, whose vertices are taken as the directed hyperedges $\xi_1, \xi_2, \xi_3, \cdots, \xi_r$.
4. Calculate the degrees of membership of vertices $l(\mathscr{D}) = (X_l, \xi_l)$ as $X_l(\xi_j) = \xi(\xi_j)$.
5. Draw an edge between ξ_i and ξ_j in $l(\mathscr{D})$ if $|supp(\xi_i) \cap supp(\xi_j)| \geq 1$.
6. Calculate the degrees of membership of edges in $l(\mathscr{D})$ as,
   $\xi_l(\xi_i\xi_j) = (\min\{T_\xi(\xi_i), T_\xi(\xi_j)\}, \max\{F_\xi(\xi_i), F_\xi(\xi_j)\})$.

Definition 6.32 The 2-*section graph* of a q-rung orthopair fuzzy directed hypergraph $\mathscr{D} = (Q, \xi)$ is a q-rung orthopair fuzzy graph $[\mathscr{D}]_2 = (X', \mathscr{E})$ such that

(i) $X = X'$, i.e., the set of vertices of both graphs is same.
(ii) $\mathscr{E} = \{v_i v_j | v_i \neq v_j, v_i v_j \in \xi_k, k = 1,2,3, \ldots\}$, i.e., v_i and v_j are adjacent in $\mathscr{D}$.

We now justify the Definitions 6.31 and 6.32 through Example 6.9.

Example 6.9 Let $\mathscr{D} = (Q, \xi)$ be a 7-rung orthopair fuzzy directed hypergraph as shown in Fig. 6.12. By following the above Algorithm 6.5.1, it's line graph is constructed and shown by dashed lines.

The 2-section graph of 7-rung orthopair fuzzy directed hypergraph given in Fig. 6.12 is shown in Fig. 6.13.

Definition 6.33 Let $\mathscr{D} = (Q, \xi)$ be a q-rung orthopair fuzzy directed hypergraph. The *dual q–rung orthopair fuzzy directed hypergraph* $\mathscr{D}^d = (X^d, \xi^d)$ of $\mathscr{D} = (Q, \xi)$ is defined as,

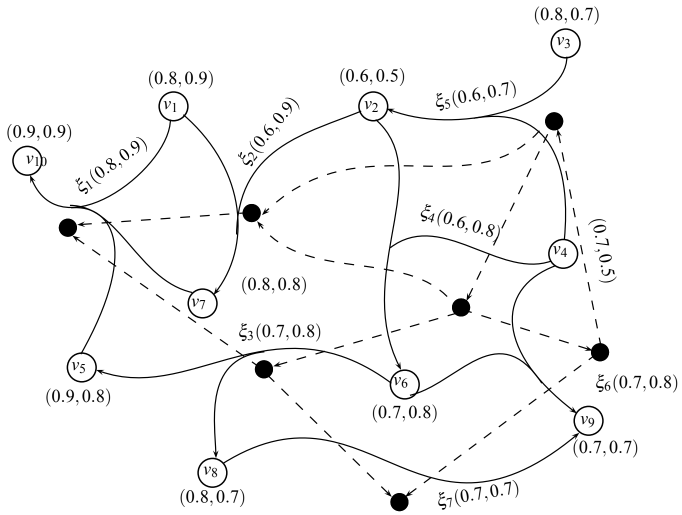


Fig. 6.12 A 7-rung orthopair fuzzy directed hypergraph and its line graph

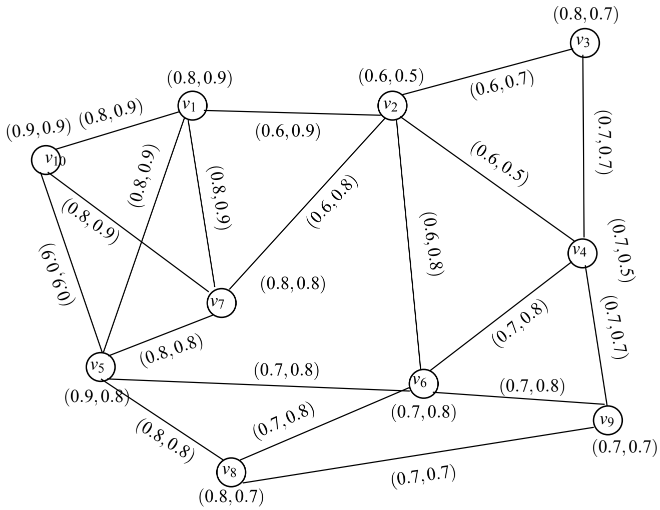


Fig. 6.13 The 2-section graph of 7-rung orthopair fuzzy directed hypergraph

(i) $X^d = \xi$ is the q-rung orthopair fuzzy set of vertices of $\mathscr{D}^d$.

(ii) If $|X| = n$, then ξ^d is q-rung orthopair fuzzy set on the set of directed hyperedges $\{X_1, X_2, X_3, \ldots, X_n\}$ such that $X_i = \{\xi_j | v_i \in \xi_j, \xi_j \in \xi\}$, i.e., X_i is the set of those directed hyperedges in which v_i is a common vertex.

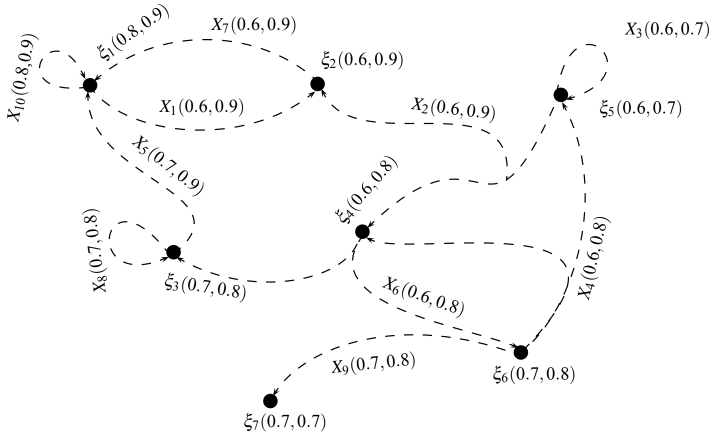


Fig. 6.14 Dual directed hypergraph of 7-rung orthopair fuzzy directed hypergraph

The membership degrees of X_i are defined as

$$T_{\xi^d}(X_i) = \min\{T_\xi(\xi_j)|v_i \in \xi_j\}, \ F_{\xi^d}(X_i) = \max\{F_\xi(\xi_j)|v_i \in \xi_j\}.$$

The method of forming the dual of q-rung orthopair fuzzy directed hypergraph is described in Algorithm 6.5.2. We also explain this concept through an example.

Algorithm 6.5.2

The dual of $q-$rung orthopair fuzzy directed hypergraph

1. Input $\{v_1, v_2, v_3, \cdots, v_n\}$ the set of vertices and $\{\xi_1, \xi_2, \xi_3, \cdots, \xi_m\}$ the set of directed hyperedges of $\mathscr{D}$.
2. Formulate a $q-$rung orthopair fuzzy set of vertices of $\mathscr{D}^d$ as $X^d = \xi$.
3. Define a mapping $\psi : X \to \xi$, which maps the set of vertices to the directed hyperedges of $\mathscr{D}$, i.e., if vertex v_i is contained in $\xi_l, \xi_{l+1}, \xi_{l+2}, \cdots, \xi_m$ then v_i is mapped onto $\xi_l, \xi_{l+1}, \xi_{l+2}, \cdots, \xi_m$.
4. Construct the directed hyperedges $\{X_1, X_2, X_3, \cdots, X_n\}$ of $\mathscr{D}^d$ such that $X_i = \{\xi_j|\psi(v_i) = \xi_j\}$.
5. Draw the $q-$rung orthopair fuzzy directed hyperedge, the vertex ξ_j of $\mathscr{D}^d$ is associated to $h(X_i)$ if and only if $v_i \in t(\xi_j)$ in $\mathscr{D}$ and viceversa.
6. Formulate the truth-membership and falsity-membership of directed hyperedges of $\mathscr{D}^d$ as,
$$T_{\xi^d}(X_i) = \min\{T_\xi(\xi_j)|v_i \in \xi_j\}, \ F_{\xi^d}(X_i) = \max\{F_\xi(\xi_j)|v_i \in \xi_j\}.$$

Example 6.10 Let $\mathscr{D} = (Q, \xi)$ be a 7-rung orthopair fuzzy directed hypergraph as shown in Fig. 6.12. The dual 7-rung orthopair fuzzy directed hypergraph $\mathscr{D}^d = (X^d, \xi^d)$ of $\mathscr{D} = (Q, \xi)$ is shown in Fig. 6.14, which is constructed by following the Algorithm 6.5.2.

Theorem 6.12 *The 2-section of dual of q-rung orthopair fuzzy directed hypergraph* $[\mathscr{D}^d]_2$ *is same as the line graph of* $\mathscr{D}$*, i.e.,* $[\mathscr{D}^d]_2 = l(\mathscr{D})$.

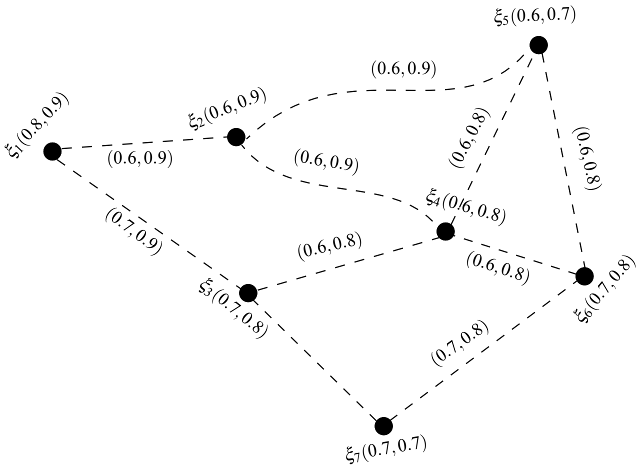


Fig. 6.15 $l(\mathscr{D})$

Proof Let $\mathscr{D} = (Q, \xi)$ be a q-rung orthopair fuzzy directed hypergraph having $\{v_1, v_2, v_3, \ldots, v_n\}$ the set of vertices and $\{\xi_1, \xi_2, \xi_3, \ldots, \xi_m\}$ the set of directed hyperedges. Suppose that $l(\mathscr{D}) = (X_l, \xi_l)$, $\mathscr{D}^d = (X^d, \xi^d)$ and $[\mathscr{D}^d]_2 = (X^d, \mathscr{E})$ be the line graph, dual directed hypergraph, and 2-section of dual of $\mathscr{D}$, respectively. The 2-section $[\mathscr{D}^d]_2$ has the same vertex set as that of $l(\mathscr{D})$. Assume that the set of directed hyperedges of $\mathscr{D}^d$ be $\{X_1, X_2, X_3, \ldots, X_n\}$. Obviously $\{\xi_i\xi_j | \xi_i, \xi_j \in X_i\}$ are the edges of $[\mathscr{D}^d]_2$ and also the set of edges of $l(\mathscr{D})$. We now show that $\xi_l(\xi_i\xi_j) = \mathscr{E}(\xi_i\xi_j)$.

$$\begin{aligned}\xi_l(\xi_i\xi_j) &= (\max\{T_\xi(\xi_i), T_\xi(\xi_i)\}, \min\{F_\xi(\xi_i), F_\xi(\xi_i)\}),\\ &= (\max\{T_{\xi^d}(\xi_i), T_{\xi^d}(\xi_i)\}, \min\{F_{\xi^d}(\xi_i), F_{\xi^d}(\xi_i)\}),\\ &= \mathscr{E}(\xi_i\xi_j),\end{aligned}$$

which completes the proof.

We now justify the result of Theorem 6.12 through a concrete example.

Example 6.11 Let $\mathscr{D} = (Q, \xi)$ be a 7-rung orthopair fuzzy directed hypergraph as shown in Fig. 6.12. Its line graph is constructed and shown by dashed lines in Fig. 6.15.

The dual of $\mathscr{D}$ is shown in Fig. 6.14. We now determine the 2-section of $\mathscr{D}^d$, which is given in Fig. 6.16.

Thus, Figs. 6.15 and 6.16 show that $[\mathscr{D}^d]_2 = l(\mathscr{D})$.

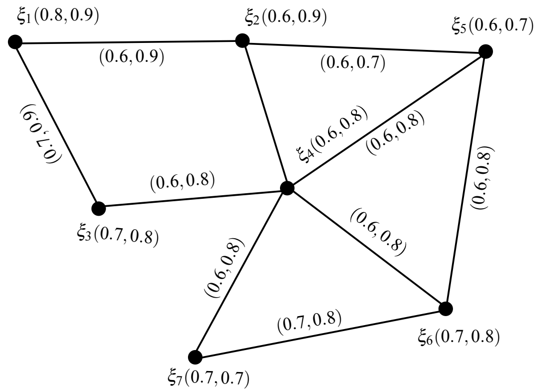


Fig. 6.16 $[\mathscr{D}^d]_2$

6.6 Coloring of q-Rung Orthopair Fuzzy Directed Hypergraphs

In this section, we define the (α, β)-level hypergraph of $\mathscr{D}$, which is a useful concept in the coloring of q-rung orthopair fuzzy directed hypergraphs. A sequence of real numbers, called the fundamental sequence of $\mathscr{D}$, is also defined using the (α, β)-level sets. The concept of the fundamental sequence is used to prove various results related to the coloring of q-rung orthopair fuzzy directed hypergraphs. Moreover, we define $\mathscr{L}$-coloring, chromatic number, and p-coloring of $\mathscr{D}$. We also prove some useful results, which simplify the complicated procedure of coloring and finding the chromatic number of q-rung orthopair fuzzy directed hypergraphs.

Definition 6.34 Let $\mathscr{D} = (X, \xi)$ be a q-rung orthopair fuzzy directed hypergraph. The (α, β)-*level hypergraph* of $\mathscr{D}$ is defined as $\mathscr{D}^{(\alpha,\beta)} = (X^{(\alpha,\beta)}, \xi^{(\alpha,\beta)})$, where

1. $\xi^{(\alpha,\beta)} = \{\xi_i^{(\alpha,\beta)} : \xi_i \in \xi\}$ and $\xi_i^{(\alpha,\beta)} = \{x \in X | T_{\xi_i}(x) \geq \alpha, F_{\xi_i}(x) \leq \beta\}$,
2. $X^{(\alpha,\beta)} = \bigcup\limits_{\xi_i \in \xi} \xi_i^{(\alpha,\beta)}$.

Definition 6.35 Let $\mathscr{D} = (X, \xi)$ be a q-rung orthopair fuzzy directed hypergraph and $\mathscr{D}^{(\alpha,\beta)}$ be the (α, β)-level hypergraph of $\mathscr{D}$. The sequence of real numbers $\rho_1 = (T_{\rho_1}, F_{\rho_1})$, $\rho_2 = (T_{\rho_2}, F_{\rho_2})$, $\rho_3 = (T_{\rho_3}, F_{\rho_3})$,...,$\rho_n = (T_{\rho_n}, F_{\rho_n})$, $0 < T_{\rho_1} < T_{\rho_2} < T_{\rho_3} < \cdots < T_{\rho_n}, F_{\rho_1} > F_{\rho_2} > F_{\rho_3} > \cdots > F_{\rho_n} > 0$, where $(T_{\rho_n}, F_{\rho_n}) = h(\mathscr{H})$ such that,

(i) if $\rho_{i-1} = (T_{\rho_{i-1}}, F_{\rho_{i-1}}) < \rho = (T_\rho, F_\rho) \leq \rho_i = (T_{\rho_i}, F_{\rho_i})$ then $\xi^\rho = \xi^{\rho_i}$,
(ii) $\xi^{\rho_i} \subseteq \xi^{\rho_{i+1}}$,

is called the *fundamental sequence* of $\mathscr{D}$, denoted by $f_S(\mathscr{D})$. The set of ρ_i-level hypergraphs $\{\mathscr{D}^{\rho_1}, \mathscr{D}^{\rho_2}, \mathscr{D}^{\rho_3}, \ldots, \mathscr{D}^{\rho_n}\}$ is called the *core hypergraphs* of $\mathscr{D}$ or simply the *core set* of $\mathscr{D}$ and is denoted by $c(\mathscr{D})$.

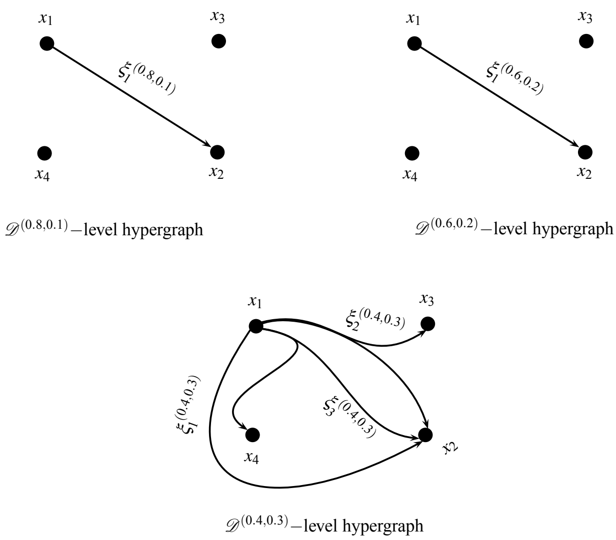


Fig. 6.17 Fundamental sequence of $\mathscr{D}$

Definition 6.36 A q-rung orthopair fuzzy directed hypergraph $\mathscr{D} = (X, \xi)$ is *ordered* if $c(\mathscr{D}) = \{\mathscr{D}^{\rho_1}, \mathscr{D}^{\rho_2}, \mathscr{D}^{\rho_3}, \ldots, \mathscr{D}^{\rho_n}\}$ is ordered, i.e., $\mathscr{D}^{\rho_1} < \mathscr{D}^{\rho_2} < \mathscr{D}^{\rho_3} < \cdots < \mathscr{D}^{\rho_n}$ and is *simply ordered* if $c(\mathscr{D})$ is simply ordered.

Example 6.12 Consider a 2-rung orthopair fuzzy directed hypergraph $\mathscr{D} = (X, \xi)$, where $X = \{x_1, x_2, x_3, x_4\}$ and $\xi = \{\xi_1, \xi_2, \xi_3\}$ such that $\xi_1 = \{(x_1, 0.8, 0.1), (x_2, 0.8, 0.1)\}, \xi_2 = \{(x_1, 0.6, 0.2), (x_2, 0.6, 0.2), (x_3, 0.4, 0.3)\}, \xi_3 = \{(x_1, 0.4, 0.3), (x_2, 0.4, 0.3), (x_4, 0.4, 0.3)\}$. By determining the (α, β)-level hypergraphs of $\mathscr{D}$, we have $\mathscr{D}^{(0.8,0.1)} = \mathscr{D}^{(0.6,0.2)}$ and $f_S(D) = \{(0.6, 0.2), (0.8, 0.1)\}$. Further, $\mathscr{D}^{(0.4,0.3)} = \mathscr{D}^{(0.6,0.2)}$. The corresponding sequence of level hypergraphs is shown in Fig. 6.17.

We now define the primitive k-coloring (or simply a p-coloring), $\mathscr{L}$-coloring, and chromatic number of q-rung orthopair fuzzy directed hypergraphs and illustrate these concepts by considering a concrete example.

Definition 6.37 Let $\mathscr{D} = (X, \xi)$ be a q-rung orthopair fuzzy directed hypergraph. A *primitive k-coloring C (or simply a p-coloring)* is defined as a partition of X in k subgroups, called colors, such that the elements from at least two colors of C are contained in the support of every q-rung orthopair fuzzy directed hyperedge of $\mathscr{D}$.

Definition 6.38 Let $\mathscr{D} = (X, \xi)$ be a q-rung orthopair fuzzy directed hypergraph and $c(\mathscr{D}) = \{\mathscr{D}^{\rho_1}, \mathscr{D}^{\rho_2}, \mathscr{D}^{\rho_3}, \ldots, \mathscr{D}^{\rho_n}\}$ be the set of core hypergraphs of $\mathscr{D}$. An *$\mathscr{L}$-coloring* is defined as a partition of X, with k components, into k subgroups $\{s_1, s_2, s_3, \ldots, s_k\}$ such that C persuades a coloring for each core hypergraph $\mathscr{D}^{\rho_i} = (X^{\rho_i}, \xi^{\rho_i})$.

Remark 6.5 Note that, an $\mathscr{L}$-coloring of $\mathscr{D}$ is a p-coloring, but in general, the converse does not hold. The preceding theorem states the condition under which an $\mathscr{L}$-coloring and p-coloring of $\mathscr{D}$ coincides.

Theorem 6.13 *Let $\mathscr{D} = (X, \xi)$ be an ordered q-rung orthopair fuzzy directed hypergraph and C is a p-coloring of $\mathscr{D}$ then $\mathscr{L}$-coloring of $\mathscr{D}$ is also C.*

Definition 6.39 Let $\mathscr{D} = (X, \xi)$ be a q-rung orthopair fuzzy directed hypergraph and let $k \geq 2$ be an integer then the *k-coloring of vertex set* is defined as a function $\kappa : X \to \{1, 2, 3, \dots, k\}$ such that for all $\rho \in f_S(\mathscr{D})$ and for each hyperedge ξ^ρ, which is not a loop, κ is not a constant on ξ^ρ.
The minimum integer k, for which there exists a k-coloring of $\mathscr{D}$ is called *chromatic number* of $\mathscr{D}$, denoted by $\chi(\mathscr{D})$.

Example 6.13 Let $\mathscr{D} = (X, \xi)$ be a 1-rung orthopair fuzzy directed hypergraph, where $X = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$ and $\xi = \{\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7\}$ such that

$$\begin{aligned}
\xi_1 &= \{(t_1, 0.6, 0.3), (t_2, 0.6, 0.3), (t_4, 0.5, 0.2)\},\\
\xi_2 &= \{(t_1, 0.6, 0.3), (t_3, 0.6, 0.3), (t_5, 0.3, 0.1), (t_7, 0.5, 0.2)\},\\
\xi_3 &= \{(t_1, 0.6, 0.3), (t_3, 0.6, 0.3), (t_6, 0.2, 0.1), (t_7, 0.5, 0.2)\},\\
\xi_4 &= \{(t_2, 0.6, 0.3), (t_3, 0.6, 0.3), (t_4, 0.5, 0.2)\},\\
\xi_5 &= \{(t_2, 0.6, 0.3), (t_4, 0.5, 0.2), (t_5, 0.3, 0.1), (t_7, 0.5, 0.2)\},\\
\xi_6 &= \{(t_2, 0.6, 0.3), (t_4, 0.5, 0.2), (t_6, 0.2, 0.1)\},\\
\xi_7 &= \{(t_4, 0.5, 0.2), (t_5, 0.3, 0.1), (t_6, 0.2, 0.1)\},
\end{aligned}$$

Let $\rho_1 = (0.6, 0.3)$, $\rho_2 = (0.5, 0.2)$, $\rho_3 = (0.30.1)$ and $\rho_4 = (0.2, 0.1)$. The corresponding ρ_i-level hyperedges are given as follows:

$$\begin{aligned}
\xi^{\rho_1} &= \{\{t_1, t_2\}, \{t_1, t_3\}, \{t_2, t_3\}\},\\
\xi^{\rho_2} &= \{\{t_1, t_2, t_4\}, \{t_1, t_3, t_7\}, \{t_2, t_3, t_4\}, \{t_2, t_7, t_4\}\},\\
\xi^{\rho_3} &= \{\{t_1, t_2, t_4\}, \{t_1, t_3, t_5, t_7\}, \{t_1, t_3, t_7\}, \{t_2, t_3, t_4\}, \{t_2, t_4, t_5\}, \{t_2, t_4\}, \{t_4, t_5\}\},\\
\xi^{\rho_3} &= \{\{t_1, t_2, t_4\}, \{t_1, t_3, t_5, t_7\}, \{t_1, t_3, t_6, t_7\}, \{t_2, t_3, t_4\}, \{t_2, t_4, t_5, t_7\}, \{t_2, t_4, t_6\}, \{t_4, t_5, t_6\}\}.
\end{aligned}$$

Suppose $\{C_1, C_2\}$ is a coloring of $\mathscr{D}^{\rho_1}$. Then, $\{t_1, t_2\} \cap \{C_1, C_2\} \neq \emptyset$, $\{t_1, t_3\} \cap \{C_1, C_2\} \neq \emptyset$ and $\{t_2, t_3\} \cap \{C_1, C_2\} \neq \emptyset$. Thus, $C_1 \cap C_2 \neq \emptyset$, which is a contradiction. Hence, $\chi(\mathscr{D}^{\rho_1}) = 3$. $\{\{t_1, t_2, t_3\}, \{t_4, t_5, t_6, t_7\}\}$ is the coloring of $\mathscr{D}^{\rho_2}$. Hence, $\chi(\mathscr{D}^{\rho_2}) = 2$. Similarly, $\chi(\mathscr{D}^{\rho_3}) = 3$ and $\chi(\mathscr{D}^{\rho_4}) = 3$.

Definition 6.40 Let $\mathscr{D} = (X, \xi)$ be a q-rung orthopair fuzzy directed hypergraph and $Q = \{q_1, q_2, q_3, \dots, q_k\}$ be the collection of non trivial q-rung orthopair fuzzy sets on X then Q is a *q-rung orthopair fuzzy k-coloring* if Q satisfies the following:

- $\min\{q_i, q_j\} = (0, 1)$, if $i \neq j$,
- for every $(\alpha, \beta) \in (0, 1]$, $\bigcup_i q_i^{(\alpha,\beta)} = X$,

- for every $(\alpha, \beta) \in (0, 1]$, each hyperedge $\xi_j^{(\alpha,\beta)}$ possesses non-empty intersection with at least two color classes $q_i^{(\alpha,\beta)}$.

Observation 6.14 *Let $\mathscr{D} = (X, \xi)$ be a q-rung orthopair fuzzy directed hypergraph having the fundamental sequence $f_S(\mathscr{D}) = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$. Then, the coloring of core hypergraph $\mathscr{D}^{\rho_i}$ can be enlarged to the coloring of $\mathscr{D}^{\rho_{i+1}}$ if and only if a single color class of κ does not contain any hyperedge of $\mathscr{D}^{\rho_{i+1}}$. Particularly, if $\mathscr{D}$ is simply ordered then any coloring κ of $\mathscr{D}^{\rho_i}$ maybe elongated to the coloring of $\mathscr{D}$.*

Theorem 6.14 *Let $\mathscr{D} = (X, \xi)$ be a q-rung orthopair fuzzy directed hypergraph having the fundamental sequence $f_S(\mathscr{D}) = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$. Let $\widetilde{\mathscr{D}}^{\rho_n}$ be the core coloring of $\mathscr{D}^{\rho_n}$ then every coloring of $\mathscr{D}^{\rho_n}$ is a coloring of $\mathscr{D}$ if and only if for every $\rho \in f_S(\mathscr{D})$ there exists $A \in \widetilde{\mathscr{D}}^{\rho_n}$ such that $A \subseteq \xi_i^{\rho}$, for each $\xi_i \in \xi$ for which ξ_i^{ρ} is a non loop edge.*

Proof Suppose the existance of some $\rho \in f_S(\mathscr{D})$ and $\xi_i \in \xi$ such that $|\xi_i^{\rho}| \geq 2$ and $A \nsubseteq \xi_i^{\rho}$, for every $A \in \widetilde{\mathscr{D}}^{\rho_n}$. Let a color class is defined for the vertex set of ξ_i^{ρ}. Construct a sub-hypergraph $\mathscr{D}'$ of $\mathscr{D}$, which is constructed by removing ξ_i^{ρ} from the vertices of $\widetilde{\mathscr{D}}^{\rho_n}$. Thus, $\{A \backslash \xi_i^{\rho} | A \in \widetilde{\mathscr{D}}^{\rho_n}\}$ is the set of hyperedges of $\mathscr{D}'$. Since every $\xi_j^{\rho_n} \in \mathscr{D}^{\rho_n}$, which is not a loop and also including $\xi_i^{\rho_n}$, contains some $A \in \widetilde{\mathscr{D}}^{\rho_n}$ and this non loop edge $\xi_j^{\rho_n}$ has non empty intersection with the vertices of $\mathscr{D}'$. Let $\{q_2, q_3, \ldots, q_k\}$ be the coloring of $\mathscr{D}'$ then the coloring of $\mathscr{D}^{\rho_n}$ is $\{\xi^{\rho}, q_2, q_3, \ldots, q_k\}$, where ξ^{ρ} is contained in single color class. Hence, there exists a coloring of $\mathscr{D}^{\rho_n}$ which is not a coloring of $\mathscr{D}$.
Conversely, assume that there exists some $\rho \in f_S(\mathscr{D})$ and $\xi_i \in \xi$ such that $|\xi_i^{\rho}| \geq 2$ and $A \subseteq \xi_i^{\rho}$, for every $A \in \widetilde{\mathscr{D}}^{\rho_n}$. Suppose that ρ and ξ_i are taken as arbitrary but fixed and κ be the coloring of $\mathscr{D}^{\rho_n}$. Since κ is not a constant on A, it is also non constant on ξ_i^{ρ}, hence κ is a coloring of $\mathscr{D}$.

The coloring problem of $\mathscr{D}$ can be reduced to the correlated crisp coloring. It can be done by replacing $\mathscr{D}$ with a more simpler framework $\mathscr{D}^{\Lambda}$, it will be noted that $\mathscr{D}^{\Lambda}$ is ordered, simpler to color and every p-coloring of $\mathscr{D}^{\Lambda}$ will generate the $\mathscr{L}$-coloring of $\mathscr{D}$.

Definition 6.41 A *spike reduction* of $\xi_i \in P(X)$, which is denoted by $\widetilde{\xi}_i$, is defined as

$$\widetilde{\xi}_i^{(\alpha,\beta)} = \begin{cases} \xi_i^{(\alpha,\beta)}, & \text{if } |\xi_i^{(\alpha,\beta)}| \geq 2, \\ \emptyset, & \text{if } |\xi_i^{(\alpha,\beta)}| \leq 1, \end{cases}$$

for $0 < \alpha, \beta \leq 1$. Particularly, if ξ_i is a loop then $\widetilde{\xi}_i = \emptyset$.

Definition 6.42 Given $\mathscr{D} = (Q, \xi)$ then $\widetilde{\mathscr{D}} = (\widetilde{X}, \widetilde{\xi})$, where $\widetilde{\xi} = \{\widetilde{\xi}_i | \xi_i \in \xi\}$.

Construction 6.2 Let $\mathscr{D} = (X, \xi)$ be a q-rung orthopair fuzzy directed hypergraph having the fundamental sequence $f_S(\mathscr{D}) = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ and $c(\mathscr{D}) = \{\mathscr{D}^{\rho_1}, \mathscr{D}^{\rho_2}, \mathscr{D}^{\rho_3}, \ldots, \mathscr{D}^{\rho_n}\}$. Then, the conversion of $\mathscr{D}$ into $\mathscr{D}^s$ is given in the following construction.

1. Obtain a partial hypergraph $\overline{\mathscr{D}}^{\rho_1}$ of $\mathscr{D}^{\rho_1}$ by abolishing all those directed hyperedges of $\mathscr{D}^{\rho_1}$ that properly accommodate any other hyperedge of $\mathscr{D}^{\rho_1}$.
2. Subsequently, obtain a partial hypergraph $\overline{\mathscr{D}}^{\rho_2}$ of $\mathscr{D}^{\rho_2}$ by abolishing all those directed hyperedges of $\mathscr{D}^{\rho_2}$ that properly accommodate any other hyperedge of $\mathscr{D}^{\rho_2}$ or (properly or improperly) contain a hyperedge of partial hypergraph $\overline{\mathscr{D}}^{\rho_1}$. (It may be possible that $\overline{\mathscr{D}}^{\rho_2}$ possesses no hyperedges, in such case existance of $\overline{\mathscr{D}}^{\rho_2}$ is ignored.)
3. By following the same procedure, obtain a partial hypergraph $\overline{\mathscr{D}}^{\rho_3}$ of $\mathscr{D}^{\rho_3}$ by abolishing all those directed hyperedges of $\mathscr{D}^{\rho_3}$ that properly accommodate any other hyperedge of $\mathscr{D}^{\rho_3}$ or (properly or improperly) contain a hyperedge of partial hypergraph either $\overline{\mathscr{D}}^{\rho_1}$ or $\overline{\mathscr{D}}^{\rho_2}$.
4. Following this iterative procedure, we obtain a subsequence of $f_S(\mathscr{D})$, $\rho_m^s \cdots < \rho_1^s = \rho_1$ and the set of partial hypergraphs corresponding to this subsequence is $c(\overline{\mathscr{D}}) = \{\overline{\mathscr{D}}^{\rho_1^s}, \overline{\mathscr{D}}^{\rho_2^s}, \overline{\mathscr{D}}^{\rho_3^s}, \ldots, \overline{\mathscr{D}}^{\rho_m^s}\}$ from the $c(\mathscr{D})$. It is obvious from above procedure that each $\overline{\mathscr{D}}^{\rho_i^s}$, $1 \leq i \leq m$, contain non-empty set of hyperedges because all those hypergraphs having empty set of hyperedges have been eliminated from the consideration.
5. Construct the elementary q-rung orthopair fuzzy directed hypergraph $\mathscr{D}^s = (X^s, \xi^s)$ satisfying the following conditions

   - $f_S(\mathscr{D}^s) = \{\rho_1^s, \rho_2^s, \rho_3^s, \ldots, \rho_m^s\}$,
   - if $\xi_j \in \xi^s$ then $h(\xi_j) \in \{\rho_1^s, \rho_2^s, \rho_3^s, \ldots, \rho_m^s\}$,
   - the family of hyperedges in ξ^s having heights ρ_k^s is the collection of elementary q-rung orthopair fuzzy sets $\{\eta(Q, \rho_k^s) | Q \in \overline{\mathscr{D}}^{\rho_k^s}\}$, for all k, $1 \leq k \leq m$.

Definition 6.43 Let $\mathscr{D}^{\Lambda}$ be a q-rung orthopair fuzzy directed hypergraph obtained from $\widetilde{\mathscr{D}}$ by the procedure described above, i.e., $\mathscr{D}^{\Lambda} = (\widetilde{\mathscr{D}})^s$.

Definition 6.44 Let $\mathscr{D} = (X, \xi)$ be a q-rung orthopair fuzzy directed hypergraph having the fundamental sequence $f_S(\mathscr{D}) = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ and $c(\mathscr{D}) = \{\mathscr{D}^{\rho_1}, \mathscr{D}^{\rho_2}, \mathscr{D}^{\rho_3}, \ldots, \mathscr{D}^{\rho_n}\}$ with $\mathscr{D}^{\rho_i} = (X_i, \mathscr{E}_i)$ and the elements of $f_S(\mathscr{D})$ are ordered then $\mathscr{D}$ is called *sequentially simple* if whenever $\mathscr{E} \in \mathscr{E}_i \backslash \mathscr{E}_{i-1}$ then $\mathscr{E} \nsubseteq X_{i-1}$, $i = 1, 2, 3, \ldots, n$.

Theorem 6.15 *Let $\mathscr{D} = (X, \xi)$ be a sequentially simple q-rung orthopair fuzzy directed hypergraph having core set $c(\mathscr{D}) = \{\mathscr{D}^{\rho_i} = (X_i, \mathscr{E}_i) | i = 1, 2, 3, \ldots, n\}$ and the elements of $f_S(\mathscr{D})$ are ordered. Suppose that $\mathscr{E} \in \mathscr{E}_{j+k} \backslash \mathscr{E}_j$, $j < n$ and $k \in \{1, 2, 3, \ldots, n-j\}$ then $\mathscr{E} \nsubseteq X_j$.*

Proof The general proof of this theorem is illustrated by considering an example. Assume that $\mathscr{E} \in \mathscr{E}_{j+3} \backslash \mathscr{E}_j$, then

(i) either $\mathscr{E} \in \mathscr{E}_{j+2}$ or $\mathscr{E} \notin \mathscr{E}_{j+2}$. In the succeeding condition $\mathscr{E} \in \mathscr{E}_{j+3} \backslash \mathscr{E}_{j+2}$, which indicates that $\mathscr{E} \nsubseteq X_{j+2}$, thus $\mathscr{E} \nsubseteq X_j$ because $X_j \subseteq X_{j+2}$. Now suppose that $\mathscr{E} \in \mathscr{E}_{j+2}$. Then

(ii) either $\mathscr{E} \in \mathscr{E}_{j+1}$ or $\mathscr{E} \notin \mathscr{E}_{j+1}$. In the succeeding condition $\mathscr{E} \in \mathscr{E}_{j+2} \backslash \mathscr{E}_{j+1}$, which indicates that $\mathscr{E} \nsubseteq X_{j+1}$, thus $\mathscr{E} \nsubseteq X_j$ because $X_j \subseteq X_{j+1}$. Now suppose that $\mathscr{E} \in \mathscr{E}_{j+1}$. Then

(iii) since $\mathscr{E} \notin \mathscr{E}_j$, this implies that $\mathscr{E} \in \mathscr{E}_{j+1} \backslash \mathscr{E}_j$. Thus, $\mathscr{E} \nsubseteq X_j$. Hence it is clear that $\mathscr{E} \nsubseteq X_j$.

Theorem 6.16 *Let $\mathscr{D} = (X, \xi)$ be a sequentially simple q-rung orthopair fuzzy directed hypergraph the $\widetilde{\mathscr{D}}$, $\mathscr{D}^s$ and $\mathscr{D}^\Lambda$ are also sequentially simple q-rung orthopair fuzzy directed hypergraphs.*

Proof Since $\mathscr{D} = (X, \xi)$ is a sequentially simple q-rung orthopair fuzzy directed hypergraph. Since $\widetilde{\mathscr{D}}$ is obtained by removing all those hyperedges of $\mathscr{D}$, which are spikes(loops) and also by eliminating all terminal spikes from the directed hyperedges of $\mathscr{D}$. Certainly $\widetilde{\mathscr{D}}$ is a sequentially simple q-rung orthopair fuzzy directed hypergraph. Also the skeleton of $\mathscr{D}$, denoted by $\mathscr{D}^s$, is a sequentially simple q-rung orthopair fuzzy directed hypergraph. Therefore, $\mathscr{D}^\Lambda = (\widetilde{\mathscr{D}})^s$ is also sequentially simple q-rung orthopair fuzzy directed hypergraph.

6.7 Applications

6.7.1 The Most Proficient Arrangement for Hazardous Chemicals

Hazardous waste is a type of waste that is considered to have potential and substantial threats to the environment and human health. There are many human activities, including medical practice, industrial manufacturing procedures, and batteries that generate the hazardous waste in various categories, including solids, gases, liquids, and sludges. The improper arrangement of these hazardous wastes results in many serious tragedies. Serious health issues, including cancer, birth defects, and nerve damage may occur due to improper handling for those who ingest the contaminated air, water or food. Remediation and cleanup cost of these hazardous substances may amount to millions and billions of dollars. To ensure the well being of the population, protection of the surrounding environment, and to avoid any type of threat or hazard proper management of hazardous chemicals is extremely important. A q-rung orthopair fuzzy directed hypergraph can be used to well demonstrate the management system of hazardous elements. The 5-rung orthopair fuzzy directed hypergraph model of some compatible and incompatible elements is shown in Fig. 6.18.

The set of oval vertices $G = \{G_1, G_2, G_3, G_4, G_5\}$ of this directed hypergraph represents the types of those elements, which are adjacent to them. The description of these vertices is given in Table 6.8.

For the cost efficient and secure management of hazardous elements, it is imperative to fill the containers up to 75% and also the container's material should be

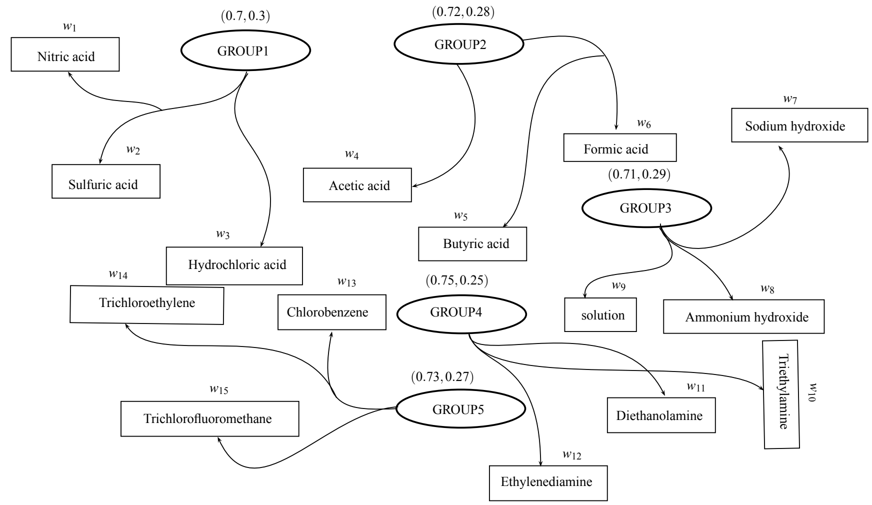


Fig. 6.18 5-rung orthopair fuzzy directed hypergraph model

Table 6.8 Description of oval vertices

	Category	Membership values	Proficiency (%)	Ineptness (%)
GROUP1	Inorganic acids	(0.7, 0.3)	70	30
GROUP2	Organic acids	(0.72, 0.28)	72	28
GROUP3	Caustics	(0.71, 0.29)	71	29
GROUP4	Amines and alkanolamines	(0.75, 0.25)	75	25
GROUP5	Halogenated compounds	(0.73, 0.27)	73	27

compatible to the elements stored in it. Only those chemical substances are connected through the same directed hyperedges, which are compatible to each other and are not dangerous when stored together. For a proficient management of such elements, one should know the characteristics of hazardous elements such as corrosivity, reactivity or toxicity of these elements. A 5-rung orthopair fuzzy set Q describes the corrosivity of these chemical substances.

$$Q = \{(w_1, 0.81, 0.23), (w_2, 0.81, 0.23), (w_3, 0.81, 0.23), (w_4, 0.90, 0.17), (w_5, 0.90, 0.17), (w_6, 0.90, 0.17), (w_7, 0.87, 0.13), (w_8, 0.87, 0.13), (w_9, 0.87, 0.13), (w_{10}, 0.75, 0.30), (w_{11}, 0.70, 0.20), (w_{12}, 0.85, 0.20), (w_{13}, 0.70, 0.10), (w_{14}, 0.70, 0.10), (w_{15}, 0.90, 0.20)\}.$$

Table 6.9 describes the importance of defining this 5-rung orthopair fuzzy set.

Table 6.9 Corrosivity and fortifying level of square vertices

Square vertices	Corrosivity (%)	Vitriolicity (%)	Square vertices	Corrosivity (%)	Vitriolicity (%)
Nitric acid	81	23	Triethylamine	75	30
Sulfuric acid	81	23	Diethanolamine	70	20
Hydrochloric acid	81	23	Ethylenediamine	85	20
Acetic acid	90	17	Chlorobenzene	70	10
Butyric acid	90	17	Trichloroethylene	70	10
Formic acid	90	17	Trichlorofluoromethane	90	20
Sodium hydroxide	87	13	Solutions	87	13
Ammonium hydroxide	87	13			

Table 6.10 Compatibility and incompatibility levels of containers to chemicals

C	Inorganic	Organic	Caustics	Alkanolamines	Compounds
C_1	(0.81, 0.23)	(0.001, 0.980)	(0.10, 0.75)	(0.001, 0.980)	(0.010, 0.908)
C_2	(0.10, 0.83)	(0.90, 0.17)	(0.75, 0.10)	(0.010, 0.908)	(0.75, 0.10)
C_3	(0.001, 0.980)	(0.81, 0.23)	(0.10, 0.83)	(0.10, 0.83)	(0.91, 0.23)
C_4	(0.10, 0.83)	(0.81, 0.23)	(0.90, 0.17)	(0.81, 0.23)	(0.81, 0.23)
C_5	(0.001, 0.980)	(0.71, 0.23)	(0.930, 0.200)	(0.001, 0.980)	(0.870, 0.210)

The containers which are holding these chemicals should be in good condition, non-leaking and compatible and these wastes should not be kept in a container that is made of an incompatible material. For example, acids must not be stored in metal material, hydrofluoric acid should not be stored in glass and lightweight polyethylene containers should not be used to store or transfer solvents. Thus, one should make sure that containers possess a high-level of compatibility with chemicals. We now consider a set of containers/cabinets $C = \{C_1, C_2, C_3, C_4, C_5\}$ and define five 5-rung orthopair fuzzy sets on C according to their compatibility with these elements. For example, the membership degrees $C_1(G_1) = (0.001, 0.980)$ implies that C_1 container is made up of such material which is incompatible to store inorganic acids and suitable to store organic acids as $C_1(G_2) = (0.81, 0.23)$. Similarly, by taking the same assumptions, we define other 5-rung orthopair fuzzy sets as given in Table 6.10.

It can be noted from Table 6.10 that inorganic acids should be stored in C_1 container as this is highly compatible to inorganic acids, so this storage will be most secure and risk less. Note that, the material of C_2 is compatible with organic acids, caustics, and halogenated compounds but we will use this container to store organic acids because the truth-membership degree is greatest in this case. In the same way, we find that C_3 is good for halogenated compounds, C_4 is used to store amines and

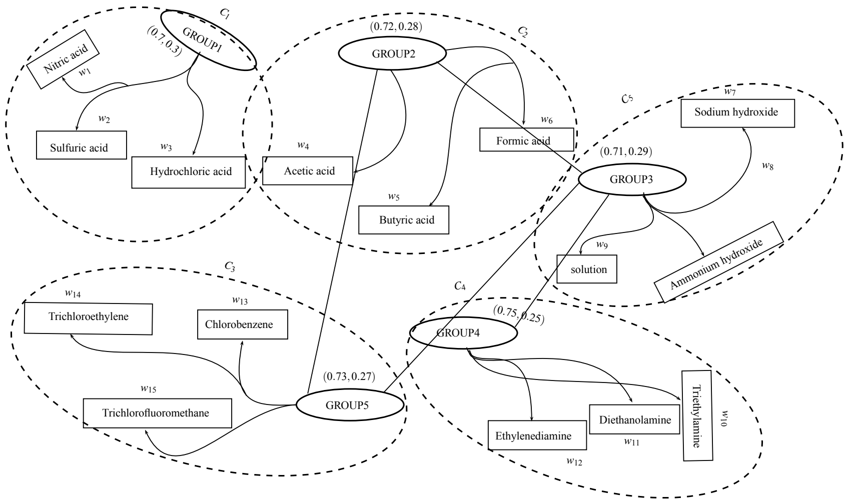


Fig. 6.19 Graphical representations of storages of chemical substances

alkanolamines and C_5 is suitable for storing caustics. The graphical representations of these storages are shown in Fig. 6.19.

Thus, by taking the above model under consideration, hazardous chemicals can be systemized in a more appropriate and acceptable manner to reduce the precarious risks to human health and environment.

6.7.2 Assessment of Collaborative Enterprise to Achieve a Particular Objective

Collaboration is the demonstration of working as a team of members to achieve some piece of work, including research projects. Many organizations are realizing the significance of collaboration as a key factor in innovations. The collaborative work provides more opportunities for studying team-work skills and improves personal and professional relationships. Here, we consider a few projects in chemical industry, which are assigned to different groups of trainees. A 7-rung orthopair fuzzy directed hypergraph model is used to well demonstrate this collaborative activity of different teams/groups.

6.7.2.1 The Project Possessesing the Powerful Collaboration

Consider the peculiar projects in the field of chemical industry, including *Zero Energy Homes*, *Heat Exchanger Network Retrofit*, *Genetic Algorithms for Process Optimization*, *Progressive Crude Distillation*, *Water Management* (for pollution prevention)

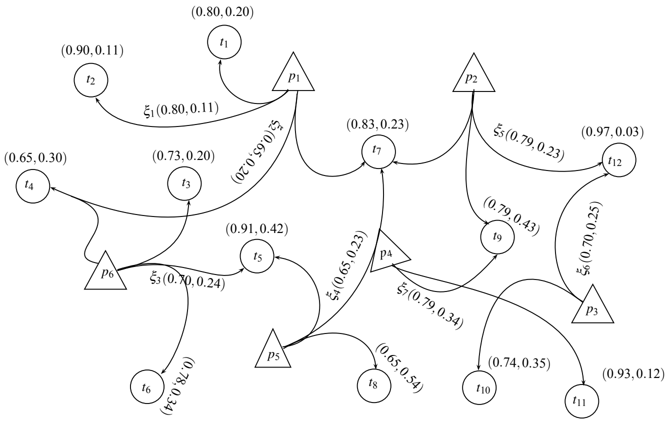


Fig. 6.20 7-rung orthopair fuzzy directed hypergraph model

Table 6.11 Collaboration capabilities of groups to projects

Assigned projects	Collaboration team	Collaborative competency (%)	Collaborative incompetency (%)
Zero energy homes	$\{t_1, t_2, t_4, t_7\}$	65	11
Heat exchanger network retrofit	$\{t_7, t_9, t_{12}\}$	79	23
Genetic algorithms for process optimization	$\{t_{10}, t_{12}\}$	70	25
Progressive crude distillation	$\{t_9, t_{11}\}$	79	34
Water management	$\{t_5, t_7, t_8\}$	65	23
Design of LNG facilities	$\{t_3, t_4, t_5, t_6\}$	70	23

and *Design of LNG Facilities*. The assignment of these projects to different groups is well explained through a 7-rung orthopair fuzzy directed hypergraph model as shown in Fig. 6.20.

Note that, the set of triangular vertices $\{p_1, p_2, p_3, p_4, p_5, p_6\}$ represents the projects that are considered to be worked on and the set of circular vertices $\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}, t_{12}\}$ represents the trainees, to whom these projects are assigned. Each directed hyperedge connects the corresponding project to it's allocated trainees. The projects assigned to different groups are illustrated through Table 6.11.

Note that, collaborative competency levels of different teams narrate that how much mutual understanding is there between the members of corresponding teams towards their projects. For example, the trainees of "*Zero Energy Homes*" project have 65% collaborative competency, i.e., they give respect to each other's ideas, contribution, and acknowledge the opinions of other trainees and their collective strength to achieve the goal is 65%. Incompetency degree shows that they have 11% conflicts of ideas and opinions. Similarly, the collaborative competency of all other

Table 6.12 Heights of all directed hyperedges

$h(\xi_1)$	(0.90, 0.11)	$h(\xi_2)$	(0.83, 0.23)
$h(\xi_3)$	(0.91, 0.20)	$h(\xi_4)$	(0.91, 0.23)
$h(\xi_5)$	(0.97, 0.03)	$h(\xi_6)$	(0.97, 0.03)
$h(\xi_7)$	(0.93, 0.12)		

teams can be studied through the table. Now, to evaluate the strength of determination and competent behavior of all teams towards their collaborative project, we calculate the heights of all directed hyperedges, which are given in Table 6.12.

The directed hyperedge having a maximum height, i.e., maximum truth-membership and minimum falsity-membership will correspond to the most efficient team working in collaboration. Note that, ξ_5 and ξ_6 have maximum heights showing that $\{t_7, t_9, t_{12}\}$ and $\{t_{10}, t_{12}\}$ share the most powerful collaborative characteristics. The method adopted in this part can be explained by a simple algorithm given in Table 6.13.

6.7.2.2 The Enduring Connection Between Projects:

Now, the line graph of the above 7-rung orthopair fuzzy directed hypergraph model can be used to determine the common trainees of distinct projects. The corresponding line graph is shown in Fig. 6.21.

The dashed lines between the projects demonstrate that they share some common trainees. The truth-membership and falsity-membership of these edges are given here.

$$\begin{aligned}
(T_{p_1p_2}, F_{p_1p_2}) &= (0.80, 0.11),\\
(T_{p_1p_5}, F_{p_1p_5}) &= (0.80, 0.11),\\
(T_{p_1p_6}, F_{p_1p_6}) &= (0.80, 0.11),\\
(T_{p_2p_5}, F_{p_2p_5}) &= (0.79, 0.23),\\
(T_{p_2p_3}, F_{p_2p_3}) &= (0.79, 0.23),\\
(T_{p_3p_4}, F_{p_3p_4}) &= (0.79, 0.25).
\end{aligned}$$

The maximum truth-membership and minimum falsity-membership reveal the robust connection among the distinct projects. For instance, projects p_1 and p_5 are 80% connected to each other, i.e., the trainees of these projects can share their ideas, creative thinkings and motives among themselves to enhance the output of their projects. The method adopted in this section can be explained by a simple algorithm given in Table 6.14.

Table 6.13 Algorithm

Algorithm for powerful collaboration

```
m =input('enter the number of trainees');
T =input('enter the degrees of membership of vertices(trainees) as m × 2');
r =input('enter the number of directed hyperedges');
Xi =input('enter the degrees of membership of directed hyperedges r × 2');
Y =input('enter the set valued function that tells us how many vertices are
                contained in a hyperedge as r × m');
J=[zeros(r,1) ones(r,1)];
for i = 1 : r
      for k = 1 : m
            if Y(i, k) == 1;
                  J(i, 1) = max(J(i, 1), T(k, 1));
                  T(i, 2) = min(J(i, 2), T(k, 2));
            end
      end
end
H = max(J(:, 1));j=0;v=zeros(r,2); b=1;
for l = 1 : r
      if J(l,1)==H
            j=j+1;v(l,1)=l;b=min(b,J(l,2));
      end
end
if j>1
      for l = 1 : r
            if J(l,2)==b
                  k=k+1;v(l,2)=l;
                  fprintf('you can choice (any of these) hyperedge(s) %d',l)
            end
      end
else
      for l = 1 : r
            if J(l,1)==H
                  fprintf('you can choice edge %d',l)
            end
      end
end
```

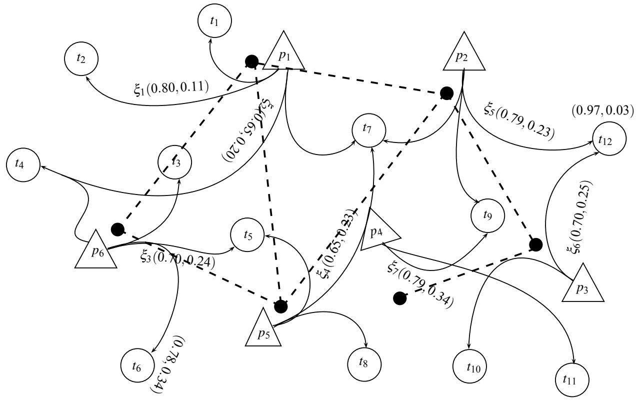


Fig. 6.21 Line graph of 7-rung orthopair fuzzy directed hypergraph

Table 6.14 Algorithm for the enduring connection between projects

```
m =input('enter the number of vertices');
T =input('enter the degrees of membership of vertices as m × 2');
r =input('enter the number of directed hyperedges');
Xi =input('enter the degrees of membership of directed hyperedges r × 2');
s =input('enter the number of edges in line graph');
P =input('enter the degrees of membership of edges s × 2');
H = max(P(:, 1));j=0;v=zeros(s,2); b=1;
for n=1:s
    if P(n,1)==H
        j=j+1;v(n,1)=l;b=min(b,P(n,2));
    end
end
if j>1
    for n=1:s
        if P(n,2)==b
            k=k+1;v(n,2)=n;
            fprintf('you can choice (any of these) hyperedge(s) %d',n)
        end
    end
else
    for n=1:s
        if P(n,1)==H
            fprintf('you can choice edge %d',n)
        end
    end
end
```

6.8 Comparative Analysis

Orthopair fuzzy sets are defined as those fuzzy sets in which the membership degrees of an element is taken as the pair of values in the unit interval [0, 1], given as $(T(x), F(x), T(x))$ indicates support for membership (truth-membership), and $F(x)$ indicates support against membership (falsity-membership) to the fuzzy set. Intuitionistic fuzzy sets and Pythagorean fuzzy sets are examples of orthopair fuzzy sets. Atanassov's [14] intuitionistic fuzzy set has been studied widely by various researchers, but the range of applicability of intuitionistic fuzzy set is limited because of its constraint that the sum of truth-membership and falsity-membership must be equal to or less than one. Under this condition, intuitionistic fuzzy sets cannot express some decision evaluation information effectively, because a decision maker may provide information for a particular attribute such that the sum of the degrees of truth-membership and the degrees of falsity-membership becomes greater than one. In order to solve such types of problems, Pythagorean fuzzy sets were defined by Yager [32], whose prominent characteristic is that the square sum of the truth-membership degree and the falsity-membership degree is less than or equal to one. Thus, a Pythagorean fuzzy set can solve a number of practical problems that cannot be handled using intuitionistic fuzzy set and is a generalization of intuitionistic fuzzy set. Due to the more complicated information in society and the development of theories, q-rung orthopair fuzzy sets were proposed by Yager [35]. A q-rung orthopair fuzzy set is characterized in such a way that the sum of the q^{th} power of the truth-membership degree and the q^{th} power of the degrees of falsity-membership is restricted to less than or equal to one. Note that, intuitionistic fuzzy sets and Pythagorean fuzzy sets are particular cases of q-rung orthopair fuzzy sets. The flexibility and effectiveness of a q-rung orthopair fuzzy model can be proven as follows: Suppose that (x, y) is an intuitionistic fuzzy grade, where $x \in [0, 1]$, $y \in [0, 1]$, and $0 \leq x + y \leq 1$, since $x^q \leq x$, $y^q \leq y$, $q \geq 1$, so we have $0 \leq x^q + y^q \leq 1$. Thus, every intuitionistic fuzzy grade is also a Pythagorean fuzzy grade, as well as a q-rung orthopair fuzzy grade. However, there are q-rung orthopair fuzzy grades that are not intuitionistic fuzzy nor Pythagorean fuzzy grades. For example, (0.9, 0.8), here $(0.9)^5 + (0.8)^5 \leq 1$, but $0.9 + 0.8 = 1.7 > 1$ and $(0.9)^2 + (0.8)^2 = 1.45 > 1$. This implies that the class of q-rung orthopair fuzzy sets extend the classes of intuitionistic fuzzy sets and Pythagorean fuzzy sets. It is worth noting that as the parameter q increases, the space of acceptable orthopairs also increases, and thus, the bounding constraint is satisfied by more orthopairs. Thus, a wider range of uncertain information can be expressed by using q-rung orthopair fuzzy sets. We can adjust the value of the parameter q to determine the expressed information range; thus, q-rung orthopair fuzzy sets are more effective and more practical for the uncertain environment. Based on these advantages of q-rung orthopair fuzzy sets, we proposed q-rung orthopair fuzzy hypergraphs and q-rung orthopair fuzzy directed hypergraphs to combine the benefits of both theories. A wider range of uncertain information can be expressed using the methods proposed in this paper, and they are closer to real decision-making. Our proposed models are more general as compared to the intuitionistic fuzzy and Pythagorean fuzzy models, as when $q = 1$, the model reduces to the intuitionistic fuzzy model, and when $q = 2$, it reduces to the Pythagorean fuzzy model. Hence, our approach is more flexible and generalized, and different values of q can be chosen by decision makers according to the different attitudes.

6.9 Complex Pythagorean Fuzzy Hypergraphs

A complex Pythagorean fuzzy set is an extension of a Pythagorean fuzzy set that is used to handle the vagueness with the degrees whose ranges are enlarged from real to complex subset with unit disc. For example, a clothing brand considers five locations to open new outlet regarding some particular criteria. If an expert assign membership 0.8 and nonmembership 0.6 to a location with respect to a criterion then intuitionistic fuzzy set fails to deal with this problem because $0.8 + 0.6 \geq 1$, but this problem can be effectively handled by Pythagorean fuzzy set as $0.8^2 + 0.6^2 \leq 1$. On the other hand, if we consider the maximum number of people visiting the outlet at a particular time then Pythagorean fuzzy set also fails because to handle time we have to introduce the periodic term. Now expert assign membership $0.8e^{\iota(1.4\pi)}$ and nonmembership $0.6e^{\iota(1.1\pi)}$ which satisfy the conditions of complex Pythagorean fuzzy set as $0.8^2 + 0.6^2 \leq 1$. Therefore, complex Pythagorean fuzzy set is proficient in dealing with data involving time period (periodic nature) due to complex membership and nonmembership grades along with the constraints.

Definition 6.45 A *complex Pythagorean fuzzy set* P on the universal set X is defined as, $P = \{(u, T_P(u)e^{i\phi_P(u)}, F_P(u)e^{i\psi_P(u)})|u \in X\}$, where $i = \sqrt{-1}$, $T_P(u), F_P(u) \in [0, 1]$, $\phi_P(u), \psi_P(u) \in [0, 2\pi]$, and for every $u \in X$, $0 \leq T_P^2(u) + F_P^2(u) \leq 1$. Here, $T_P(u), F_P(u)$ and $\phi_P(u), \psi_P(u)$ are called the amplitude terms and phase terms for truth membership and falsity membership grades, respectively.

Definition 6.46 A *complex Pythagorean fuzzy graph* on X is an ordered pair $G^* = (C, D)$, where C is a complex Pythagorean fuzzy set on X and D is complex Pythagorean fuzzy relation on X such that,

$$\begin{aligned} &T_D(ab) \leq \min\{T_C(a), T_C(b)\},\\ &F_D(ab) \leq \max\{F_C(a), F_C(b)\}, \text{ (for amplitude terms)}\\ &\phi_D(ab) \leq \min\{\phi_C(a), \phi_C(b)\},\\ &\psi_D(ab) \leq \max\{\psi_C(a), \psi_C(b)\}, \text{ (for phase terms)} \end{aligned}$$

$0 \leq T_D^2(ab) + F_D^2(ab) \leq 1$, for all $a, b \in X$.

Definition 6.47 A *complex Pythagorean fuzzy hypergraph* on X is defined as an ordered pair $H^* = (\mathscr{C}^*, \mathscr{D}^*)$, where $\mathscr{C}^* = \{\beta_1, \beta_2, \ldots, \beta_k\}$ is a finite family of complex Pythagorean fuzzy sets on X and $\mathscr{D}^*$ is a complex Pythagorean fuzzy relation on complex Pythagorean fuzzy sets β_j's such that

(i)

$$
\begin{aligned}
&T_{\mathscr{D}^*}(\{s_1, s_2, \ldots, s_l\}) \le \min\{T_{\beta_j}(s_1), T_{\beta_j}(s_2), \ldots, T_{\beta_j}(s_l)\},\\
&F_{\mathscr{D}^*}(\{s_1, s_2, \ldots, s_l\}) \le \max\{F_{\beta_j}(s_1), F_{\beta_j}(s_2), \ldots, F_{\beta_j}(s_l)\}, \text{ (for amplitude terms)}\\
&\phi_{\mathscr{D}^*}(\{s_1, s_2, \ldots, s_l\}) \le \min\{\phi_{\beta_j}(s_1), \phi_{\beta_j}(s_2), \ldots, \phi_{\beta_j}(s_l)\},\\
&\psi_{\mathscr{D}^*}(\{s_1, s_2, \ldots, s_l\}) \le \max\{\psi_{\beta_j}(s_1), \psi_{\beta_j}(s_2), \ldots, \psi_{\beta_j}(s_l)\}, \text{ (for phase terms)}
\end{aligned}
$$

$0 \le T^2_{\mathscr{D}^*} + F^2_{\mathscr{D}^*} \le 1$, for all $s_1, s_2, \ldots, s_l \in X$.

(ii) $\bigcup_j supp(\beta_j) = X$, for all $\beta_j \in \mathscr{C}^*$.

Note that, $E_k = \{s_1, s_2, \ldots, s_l\}$ is the crisp hyperedge of $H^* = (\mathscr{C}^*, \mathscr{D}^*)$.

Example 6.15 Consider a complex Pythagorean fuzzy hypergraph $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ on $X = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. The complex Pythagorean fuzzy relation is defined as, $\mathscr{D}^*(s_1, s_2, s_3) = ((0.6e^{i(0.2)2\pi}, 0.5e^{i(0.9)2\pi}))$, $\mathscr{D}^*(s_4, s_5, s_6) = (0.6e^{i(0.4)2\pi}, 0.4e^{i(0.6)2\pi})$, $\mathscr{D}^*(s_3, s_6) = (0.6e^{i(0.6)2\pi}, 0.5e^{i(0.6)2\pi})$, $\mathscr{D}^*(s_2, s_5) = (0.6e^{i(0.4)2\pi}, 0.5e^{i(0.6)2\pi})$, and $\mathscr{D}^*(s_1, s_4) = (0.6e^{i(0.2)2\pi}, 0.9e^{i(0.9)2\pi})$. The corresponding complex Pythagorean fuzzy hypergraph is shown in Fig. 6.22.

Definition 6.48 A complex Pythagorean fuzzy hypergraph $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ is *simple* if whenever $\mathscr{D}^*_j, \mathscr{D}^*_k \in \mathscr{D}^*$ and $\mathscr{D}^*_j \subseteq \mathscr{D}^*_k$, then $\mathscr{D}^*_j = \mathscr{D}^*_k$.
A complex Pythagorean fuzzy hypergraph $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ is *support simple* if whenever $\mathscr{D}^*_j, \mathscr{D}^*_k \in \mathscr{D}^*$, $\mathscr{D}^*_j \subseteq \mathscr{D}^*_k$, and $supp(\mathscr{D}^*_j) = supp(\mathscr{D}^*_k)$, then $\mathscr{D}^*_j = \mathscr{D}^*_k$.

Definition 6.49 Let $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ be a complex Pythagorean fuzzy hypergraph. Suppose that $\alpha_1, \beta_1 \in [0, 1]$ and $\theta, \varphi \in [0, 2\pi]$ such that $0 \le \alpha_1^2 + \beta_1^2 \le 1$. The

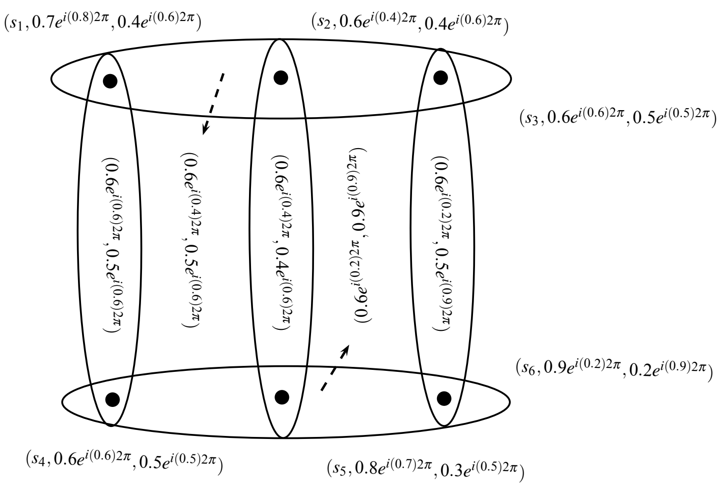


Fig. 6.22 Complex Pythagorean fuzzy hypergraph

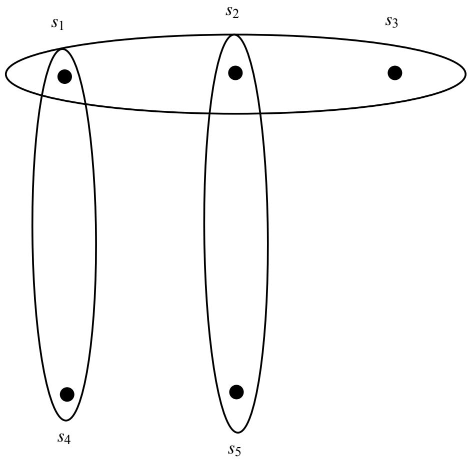


Fig. 6.23 $(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})$-level hypergraph of H^*

$(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})$-*level hypergraph* of H^* is defined as an ordered pair $H^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})} = (\mathscr{C}^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})}, \mathscr{D}^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})})$, where

(i) $\mathscr{D}^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})} = \{D_j^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})} : D_j^* \in \mathscr{D}^*\}$ and $D_j^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})} = \{y \in X : T_{D_j^*}(y) \geq \alpha_1, \phi_{D_j^*}(y) \geq \theta,$ and $F_{D_j^*}(y) \leq \beta_1, \psi_{D_j^*}(y) \leq \varphi\}$,

(ii) $\mathscr{C}^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})} = \bigcup_{D_j^* \in \mathscr{D}^*} D_j^{*(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})}$.

Note that, $(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})$-level hypergraph of H^* is a crisp hypergraph.

Example 6.16 Consider a complex Pythagorean fuzzy hypergraph $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ as shown in Fig. 6.22. Let $\alpha_1 = 0.5$, $\beta_1 = 0.6$, $\theta = 0.3\pi$, and $\varphi = 0.7\pi$. Then, $(\alpha_1 e^{i\theta}, \beta_1 e^{i\varphi})$-level hypergraph of H^* is shown in Fig. 6.23.

Definition 6.50 Let $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ be a complex Pythagorean fuzzy hypergraph. The *complex Pythagorean fuzzy line graph* of H^* is defined as an ordered pair $l(H^*) = (\mathscr{C}_l^*, \mathscr{D}_l^*)$, where $\mathscr{C}_l^* = \mathscr{D}^*$ and there exists an edge between two vertices in $l(H^*)$ if $|supp(D_j) \cap supp(D_k)| \geq 1$, for all $D_j, D_k \in \mathscr{D}^*$. The membership degrees of $l(H^*)$ are given as

(i) $\mathscr{C}_l^*(E_k) = \mathscr{D}^*(E_k)$,

(ii) $\mathscr{D}_l^*(E_j E_k) = (\min\{T_{\mathscr{D}^*}(E_j), T_{\mathscr{D}^*}(E_k)\}e^{i \min\{\phi_{\mathscr{D}^*}(E_j), \phi_{\mathscr{D}^*}(E_k)\}}, \max\{F_{\mathscr{D}^*}(E_j), F_{\mathscr{D}^*}(E_k)\}e^{i \max\{\psi_{\mathscr{D}^*}(E_j), \psi_{\mathscr{D}^*}(E_k)\}})$.

Definition 6.51 A complex Pythagorean fuzzy hypergraph $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ is said to be *linear* if for every $D_j, D_k \in \mathscr{D}^*$,

(i) $supp(D_j) \subseteq supp(D_k) \Rightarrow j = k$,

(ii) $|supp(D_j) \cap supp(D_k)| \leq 1$.

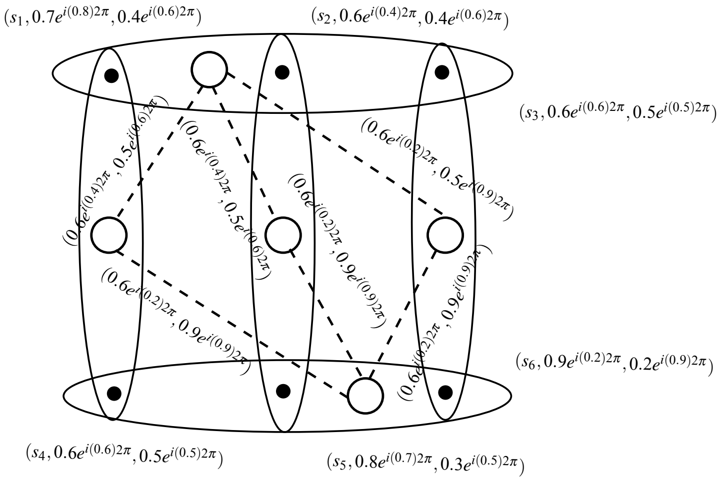


Fig. 6.24 Line graph of complex Pythagorean fuzzy hypergraph $H^* = (\mathscr{C}^*, \mathscr{D}^*)$

Example 6.17 Consider a complex Pythagorean fuzzy hypergraph $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ as shown in Fig. 6.22. By direct calculations, we have

$$supp(\mathscr{D}_1) = \{s_1, s_2, s_3\},\ supp(\mathscr{D}_2) = \{s_4, s_5, s_6\},\ supp(\mathscr{D}_3) = \{s_1, s_4\},$$
$$supp(\mathscr{D}_4) = \{s_2, s_5\},\ supp(\mathscr{D}_5) = \{s_3, s_6\}.$$

Note that, $supp(D_j) \subseteq supp(D_k) \Rightarrow j = k$ and $|supp(D_j) \cap supp(D_k)| \leq 1$. Hence, complex Pythagorean fuzzy hypergraph $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ is linear. The corresponding complex Pythagorean fuzzy hypergraph $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ and its line graph is shown in Fig. 6.24.

Theorem 6.17 *A simple strong complex Pythagorean fuzzy hypergraph is the complex Pythagorean fuzzy line graph of a linear complex Pythagorean fuzzy hypergraph.*

Definition 6.52 The 2-*section* $H_2^* = (\mathscr{C}_2^*, \mathscr{D}_2^*)$ of a complex Pythagorean fuzzy hypergraph $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ is a complex Pythagorean fuzzy graph having same set of vertices as that of H^*, $\mathscr{D}_2^*$ is a complex Pythagorean fuzzy set on $\{e = u_j u_k | u_j, u_k \in E_l,\ l = 1, 2, 3, \ldots\}$, and $\mathscr{D}_2^*(u_j u_k) = (\min\{\min T_{\beta_l}(u_j), \min T_{\beta_l}(u_k)\} e^{i \min\{\min \phi_{\beta_l}(u_j), \min \phi_{\beta_l}(u_k)\}}, \max\{\max F_{\beta_l}(u_j), \max F_{\beta_l}(u_k)\} e^{i \max\{\max \psi_{\beta_l}(u_j), \max \psi_{\beta_l}(u_k)\}})$ such that $0 \leq T^2_{\mathscr{D}_2^*}(u_j u_k) + F^2_{\mathscr{D}_2^*}(u_j u_k) \leq 1$.

Example 6.18 An example of a complex Pythagorean fuzzy hypergraph is given in Fig. 6.25. The 2-section of H^* is presented with dashed lines.

Definition 6.53 Let $H^* = (\mathscr{C}^*, \mathscr{D}^*)$ be a complex Pythagorean fuzzy hypergraph. A *complex Pythagorean fuzzy transversal* τ is a complex Pythagorean fuzzy set of X

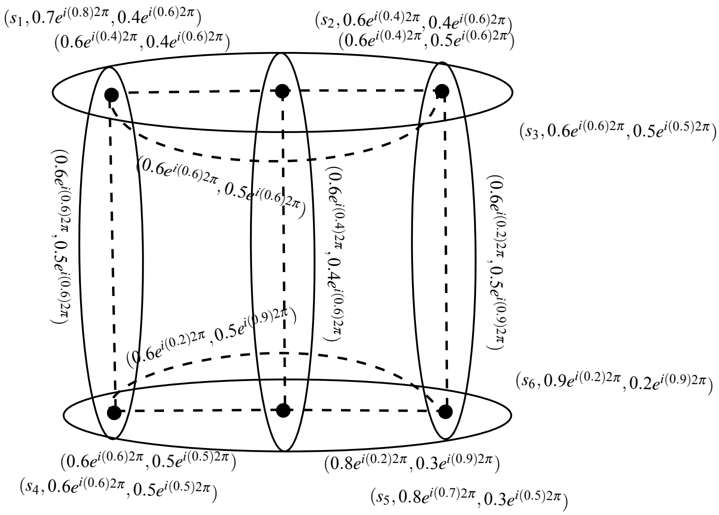


Fig. 6.25 2-section of complex Pythagorean fuzzy hypergraph H^*

satisfying the condition $\rho^{h(\rho)} \cap \tau^{h(\rho)} \neq \emptyset$, for all $\rho \in \mathscr{D}^*$, where $h(\rho)$ is the height of ρ.

A *minimal complex Pythagorean fuzzy transversal* t is the complex Pythagorean fuzzy transversal of H^* having the property that if $\tau \subset t$, then τ is not a complex Pythagorean fuzzy transversal of H^*.

6.10 Complex q-Rung Orthopair Fuzzy Hypergraphs

A complex q-rung orthopair fuzzy model provides more flexibility due to its most prominent feature that is the sum of the qth powers of the truth-membership, falsity-membership must be less than or equal to one, and the sum of qth powers of the corresponding phase angles should lie between 0 and 2π. A complex q-rung orthopair fuzzy hypergraph model proves to be more generalized framework to deal with vagueness in complex hypernetworks when the relationships are more generalized rather than the pairwise interactions. The generalization of our proposed model can be observed from the reduction of complex q-rung orthopair fuzzy model to complex intuitionistic fuzzy and complex Pythagorean fuzzy models for $q = 1$ and $q = 2$, respectively.

Definition 6.54 A *complex q-rung orthopair fuzzy set S* in the universal set X is given as

$$S = \{(u, T_S(u)e^{i\phi_S(u)}, F_S(u)e^{i\psi_S(u)})|u \in X\},$$

where $i = \sqrt{-1}$, $T_S(u)$, $F_S(u) \in [0, 1]$ are named as amplitude terms, $\phi_S(u)$, $\psi_S(u) \in [0, 2\pi]$ are named as phase terms, and for every $u \in X$, $0 \leq T_S^q(u) + F_S^q(u) \leq 1$, $q \geq 1$.

Remark 6.6
- When $q = 1$, complex 1-rung orthopair fuzzy set is called a complex intuitionistic fuzzy set.
- When $q = 2$, complex 1-rung orthopair fuzzy set is called a complex Pythagorean fuzzy set.

Definition 6.55 Let $S_1 = \{(u, T_{S_1}(u)e^{i\phi_{S_1}(u)}, F_{S_1}(u)e^{i\psi_{S_1}(u)})|u \in X\}$ and $S_2 = \{(u, T_{S_2}(u)e^{i\phi_{S_2}}(u), F_{S_2}(u)e^{i\psi_{S_2}}(u))|u \in X\}$ be two complex q-rung orthopair fuzzy sets in X, then

(i) $S_1 \subseteq S_2 \Leftrightarrow T_{S_1} \leq T_{S_2}(u)$, $F_{S_1}(u) \geq F_{S_2}(u)$, and $\phi_{S_1}(u) \leq \phi_{S_2}(u)$, $\psi_{S_1}(u) \geq \psi_{S_2}(u)$ for amplitudes and phase terms, respectively, for all $u \in X$.
(ii) $S_1 = S_2 \Leftrightarrow T_{S_1} = T_{S_2}(u)$, $F_{S_1}(u) = F_{S_2}(u)$, and $\phi_{S_1}(u) = \phi_{S_2}(u)$, $\psi_{S_1}(u) = \psi_{S_2}(u)$ for amplitudes and phase terms, respectively, for all $u \in X$.

Definition 6.56 Let $S_1 = \{(u, T_{S_1}(u)e^{i\phi_{S_1}(u)}, F_{S_1}(u)e^{i\psi_{S_1}(u)})|u \in X\}$ and $S_2 = \{(u, T_{S_2}(u)e^{i\phi_{S_2}}(u), F_{S_2}(u)e^{i\psi_{S_2}}(u))|u \in X\}$ be two complex q-rung orthopair fuzzy sets in X, then

(i) $S_1 \cup S_2 = \{(u, \max\{T_{S_1}(u), T_{S_2}(u)\}e^{i\max\{\phi_{S_1}(u),\phi_{S_2}(u)\}}, \min\{F_{S_1}(u), F_{S_2}(u)\} e^{i\min\{\psi_{S_1}(u),\psi_{S_2}(u)\}})|u \in X\}$.
(ii) $S_1 \cap S_2 = \{(u, \min\{T_{S_1}(u), T_{S_2}(u)\}e^{i\min\{\phi_{S_1}(u),\phi_{S_2}(u)\}}, \max\{F_{S_1}(u), F_{S_2}(u)\} e^{i\max\{\psi_{S_1}(u),\psi_{S_2}(u)\}})|u \in X\}$.

Definition 6.57 A *complex q-rung orthopair fuzzy relation* is a complex q-rung orthopair fuzzy set on $X \times X$ given as

$$R = \{(rs, T_R(rs)e^{i\phi_R(rs)}, F_R(rs)e^{i\psi_R(rs)})|rs \in X \times X\},$$

where $i = \sqrt{-1}$, $T_R : X \times X \to [0, 1]$, $F_R : X \times X \to [0, 1]$ characterize the amplitudes of truth and falsity degrees of R, and $\phi_R(rs)$, $\psi_R(rs) \in [0, 2\pi]$ are called the phase terms such that for all $rs \in X \times X$, $0 \leq T_R^q(rs) + F_R^q(rs) \leq 1, q \geq 1$.

Example 6.19 Let $X = \{b_1, b_2, b_3\}$ be the universal set and $\{b_1b_2, b_2b_3, b_1b_3\}$ be the subset of $X \times X$. Then, the complex 5-rung orthopair fuzzy relation R is given as

$$R = \{(b_1b_2, 0.9e^{i(0.7)\pi}, 0.7e^{i(0.9)\pi}), (b_2b_3, 0.6e^{i(0.7)\pi}, 0.8e^{i(0.9)\pi}), (b_1b_3, 0.7e^{i(0.8)\pi}, 0.5e^{i(0.6)\pi})\}.$$

Note that, $0 \leq T_R^5(xy) + F_R^5(xy) \leq 1$, for all $xy \in X \times X$. Hence, R is a complex 5-rung orthopair fuzzy relation on X.

Definition 6.58 A *complex q-rung orthopair fuzzy graph* on X is an ordered pair $\mathscr{G} = (\mathscr{A}, \mathscr{B})$, where $\mathscr{A}$ is a complex q-rung orthopair fuzzy set on X and $\mathscr{B}$ is complex q-rung orthopair fuzzy relation on X such that

$$T_{\mathscr{B}}(ab) \leq \min\{T_{\mathscr{A}}(a), T_{\mathscr{A}}(b)\},$$
$$F_{\mathscr{B}}(ab) \leq \max\{F_{\mathscr{A}}(a), F_{\mathscr{A}}(b)\}, \text{ (for amplitude terms)}$$
$$\phi_{\mathscr{B}}(ab) \leq \min\{\phi_{\mathscr{A}}(a), \phi_{\mathscr{A}}(b)\},$$
$$\psi_{\mathscr{B}}(ab) \leq \max\{\psi_{\mathscr{A}}(a), \psi_{\mathscr{A}}(b)\}, \text{ (for phase terms)}$$

$0 \leq T^q_{\mathscr{B}}(ab) + F^q_{\mathscr{B}}(ab) \leq 1, q \geq 1$, for all $a, b \in X$.

Remark 6.7 Note that,

- When $q = 1$, complex 1-rung orthopair fuzzy graph is called a complex intuitionistic fuzzy graph.
- When $q = 2$, complex 2-rung orthopair fuzzy graph is called a complex Pythagorean fuzzy graph.

Example 6.20 Let $\mathscr{G} = (\mathscr{A}, \mathscr{B})$ be a complex 6-rung orthopair fuzzy graph on $X = \{s_1, s_2, s_3, s_4\}$, where $\mathscr{A} = \{(s_1, 0.7e^{i(0.9)\pi}, 0.9e^{i(0.7)\pi}), (s_2, 0.5e^{i(0.6)\pi}, 0.6e^{i(0.5)\pi}), (s_3, 0.7e^{i(0.4)\pi}, 0.4e^{i(0.7)\pi}), (s_4, 0.8e^{i(0.5)\pi}, 0.5e^{i(0.8)\pi})\}$ and $\mathscr{B} = \{(s_1s_4, 0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi}), (s_2s_4, 0.5e^{i(0.5)\pi}, 0.6e^{i(0.8)\pi}), (s_3s_4, 0.7e^{i(0.4)\pi}, 0.5e^{i(0.8)\pi})\}$ are complex 6-rung orthopair fuzzy set and complex 6-rung orthopair fuzzy relation on X, respectively. The corresponding complex 6-rung orthopair fuzzy graph $\mathscr{G}$ is shown in Fig. 6.26.

We now define the more extended concept of complex q-rung orthopair fuzzy hypergraphs.

Definition 6.59 The *support* of a complex q-rung orthopair fuzzy set $S = \{(u, T_S(u)e^{i\phi_S(u)}, F_S(u)e^{i\psi_S(u)})|u \in X\}$ is defined as $supp(S) = \{u|T_S(u) \neq 0, F_S(u) \neq 1, 0 < \phi_S(u), \psi_S(u) < 2\pi\}$.

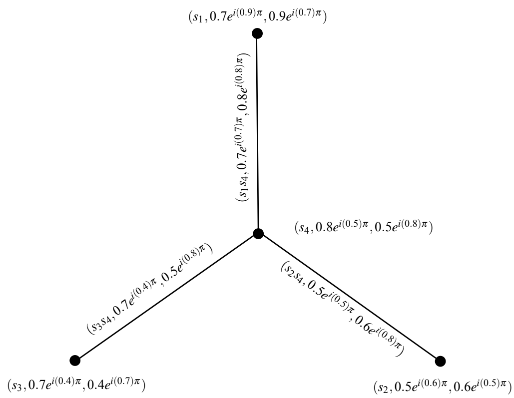


Fig. 6.26 Complex 6-rung orthopair fuzzy graph

The *height* of a complex q-rung orthopair fuzzy set $S = \{(u, T_S(u)e^{i\phi_S(u)}, F_S(u)e^{i\psi_S(u)}) | u \in X\}$ is defined as

$$h(S) = \{\max_{u\in X} T_S(u)e^{i \max_{u\in X} \phi_S(u)}, \min_{u\in X} F_S(u)e^{i \min_{u\in X} \psi_S(u)}\}.$$

If $h(S) = (1e^{i2\pi}, 0e^{i0})$, then S is called *normal.*

Definition 6.60 Let X be a nontrivial set of universe. A *complex q-rung orthopair fuzzy hypergraph* is defined as an ordered pair $\mathscr{H} = (\mathscr{Q}, \eta)$, where $\mathscr{Q} = \{Q_1, Q_2, \ldots, Q_k\}$ is a finite family of complex q-rung orthopair fuzzy sets on X and η is a complex q-rung orthopair fuzzy relation on complex q-rung orthopair fuzzy sets Q_j's such that

(i)

$$\begin{aligned}
&T_\eta(\{a_1, a_2, \ldots, a_l\}) \le \min\{T_{Q_j}(a_1), T_{Q_j}(a_2), \ldots, T_{Q_j}(a_l)\},\\
&F_\eta(\{a_1, a_2, \ldots, a_l\}) \le \max\{F_{Q_j}(a_1), F_{Q_j}(a_2), \ldots, F_{Q_j}(a_l)\}, \text{ (for amplitude terms)}\\
&\phi_\eta(\{a_1, a_2, \ldots, a_l\}) \le \min\{\phi_{Q_j}(a_1), \phi_{Q_j}(a_2), \ldots, \phi_{Q_j}(a_l)\},\\
&\psi_\eta(\{a_1, a_2, \ldots, a_l\}) \le \max\{\psi_{Q_j}(a_1), \psi_{Q_j}(a_2), \ldots, \psi_{Q_j}(a_l)\}, \text{ (for phase terms)}
\end{aligned}$$

$0 \le T_\eta^q + F_\eta^q \le 1$, $q \ge 1$, for all $a_1, a_2, \ldots, a_l \in X$.

(ii) $\bigcup_j supp(Q_j) = X$, for all $Q_j \in \mathscr{Q}$.

Note that, $E_k = \{a_1, a_2, \ldots, a_l\}$ is the crisp hyperedge of $\mathscr{H} = (\mathscr{Q}, \eta)$.

Remark 6.8 Note that,

- When $q = 1$, complex 1-rung orthopair fuzzy hypergraph is a complex intuitionistic fuzzy hypergraph.
- When $q = 2$, complex 2-rung orthopair fuzzy hypergraph is a complex Pythagorean fuzzy hypergraph.

Definition 6.61 Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be a complex q-rung orthopair fuzzy hypergraph. The *height* of $\mathscr{H}$, given as $h(\mathscr{H})$, is defined as $h(\mathscr{H}) = (\max \eta_l e^{i \max \phi}, \min \eta_m e^{i \min \psi})$, where $\eta_l = \max T_{\rho_j}(x_k)$, $\phi = \max \phi_{\rho_j}(x_k)$, $\eta_m = \min F_{\rho_j}(x_k)$, $\psi = \min \psi_{\rho_j}(x_k)$. Here, $T_{\rho_j}(x_k)$ and $F_{\rho_j}(x_k)$ denote the truth and falsity degrees of vertex x_k to hyperedge ρ_j, respectively.

Definition 6.62 Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be a complex q-rung orthopair fuzzy hypergraph. Suppose that $\mu, \nu \in [0, 1]$ and $\theta, \varphi \in [0, 2\pi]$ such that $0 \le \mu^q + \nu^q \le 1$. The $(\mu e^{i\theta}, \nu e^{i\varphi})$*-level hypergraph* of $\mathscr{H}$ is defined as an ordered pair $\mathscr{H}^{(\mu e^{i\theta}, \nu e^{i\varphi})} = (\mathscr{Q}^{(\mu e^{i\theta}, \nu e^{i\varphi})}, \eta^{(\mu e^{i\theta}, \nu e^{i\varphi})})$, where

(i) $\eta^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \{\rho_j^{(\mu e^{i\theta}, \nu e^{i\varphi})} : \rho_j \in \eta\}$ and $\rho_j^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \{u \in X : T_{\rho_j}(u) \ge \mu, \phi_{\rho_j}(u) \ge \theta, \text{ and } F_{\rho_j}(u) \le \nu, \psi_{\rho_j}(u) \le \varphi\}$,

(ii) $\mathscr{Q}^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \bigcup_{\rho_j \in \eta} \rho_j^{(\mu e^{i\theta}, \nu e^{i\varphi})}$.

Table 6.15 Incidence matrix of complex 6-rung orthopair fuzzy hypergraph $\mathscr{H}$

$u \in X$	η_1	η_2	η_3	η_4
u_1	$(0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi})$	(0, 0)	(0, 0)	$(0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi})$
u_2	$(0.7e^{i(0.7)\pi}, 0.6e^{i(0.6)\pi})$	(0, 0)	(0, 0)	(0, 0)
u_3	$(0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$	$(0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$	(0, 0)	(0, 0)
u_4	(0, 0)	$(0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$	$(0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$	(0, 0)
u_5	(0, 0)	$(0.6e^{i(0.6)\pi}, 0.8e^{i(0.8)\pi})$	(0, 0)	(0, 0)
u_6	(0, 0)	(0, 0)	$(0.9e^{i(0.9)\pi}, 0.8e^{i(0.8)\pi})$	$(0.9e^{i(0.9)\pi}, 0.8e^{i(0.8)\pi})$

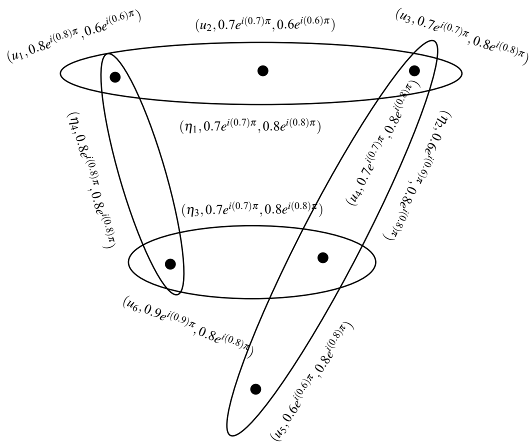


Fig. 6.27 Complex 6-rung orthopair fuzzy hypergraph

Note that, $(\mu e^{i\theta}, \nu e^{i\varphi})$-level hypergraph of $\mathscr{H}$ is a crisp hypergraph.

Example 6.21 Consider a complex 6-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \eta)$ on $X = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. The complex 6-rung orthopair fuzzy relation η is given as, $\eta(u_1, u_2, u_3) = (0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$, $\eta(u_3, u_4, u_5) = (0.6e^{i(0.6)\pi}, 0.8e^{i(0.8)\pi})$, $\eta(u_1, u_6) = (0.8e^{i(0.8)\pi}, 0.8e^{i(0.8)\pi})$ and $\eta(u_4, u_6) = (0.7e^{i(0.7)\pi}, 0.8e^{i(0.8)\pi})$. The incidence matrix of $\mathscr{H}$ is given in Table 6.15.

The corresponding complex 6-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \eta)$ is shown in Fig. 6.27.

Let $\mu = 0.7$, $\nu = 0.6$, $\theta = 0.7\pi$, and $\varphi = 0.6\pi$, then $(0.7e^{i(0.7)\pi}, 0.6e^{i(0.6)\pi})$-level hypergraph of $\mathscr{H}$ is shown in Fig. 6.28.

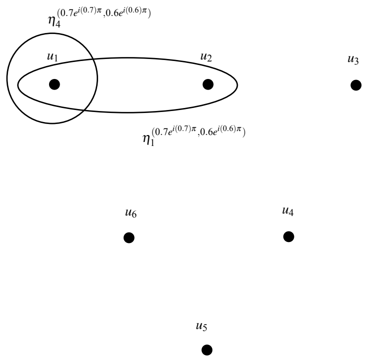


Fig. 6.28 $(0.7e^{i(0.7)\pi}, 0.6e^{i(0.6)\pi})$-level hypergraph of $\mathscr{H}$

Note that,

$$\eta_1^{(0.7e^{i(0.7)\pi},0.6e^{i(0.6)\pi})} = \{u_1, u_2\}, \quad \eta_2^{(0.7e^{i(0.7)\pi},0.6e^{i(0.6)\pi})} = \{\emptyset\},$$
$$\eta_3^{(0.7e^{i(0.7)\pi},0.6e^{i(0.6)\pi})} = \{\emptyset\}, \quad \eta_4^{(0.7e^{i(0.7)\pi},0.6e^{i(0.6)\pi})} = \{u_1\}.$$

6.11 Transversals of Complex q-Rung Orthopair Fuzzy Hypergraphs

Definition 6.63 Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be a complex q-rung orthopair fuzzy hypergraph and for $0 < \mu \leq T(h(\mathscr{H})), \nu \geq F(h(\mathscr{H})) > 0, 0 < \theta \leq \phi(h(\mathscr{H}))$, and $\varphi \geq \psi(h(\mathscr{H})) > 0$ let $\mathscr{H}^{(\mu e^{i\theta}, \nu e^{i\varphi})} = (\mathscr{Q}^{(\mu e^{i\theta}, \nu e^{i\varphi})}, \eta^{(\mu e^{i\theta}, \nu e^{i\varphi})})$ be the level hypergraph of $\mathscr{H}$. The sequence of complex numbers $\{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \ldots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$ such that $0 < \mu_1 < \mu_2 < \cdots < \mu_n = T(h(\mathscr{H})), \nu_1 > \nu_2 > \cdots > \nu_n = F(h(\mathscr{H})) > 0$, $0 < \theta_1 < \theta_2 < \cdots < \theta_n = \phi(h(\mathscr{H}))$, and $\varphi_1 > \varphi_2 > \cdots > \varphi_n = \psi(h(\mathscr{H})) > 0$ satisfying the conditions

(i) if $\mu_{k+1} < \alpha \leq \mu_k$, $\nu_{k+1} > \beta \geq \nu_k$, $\theta_{k+1} < \phi \leq \theta_k$, $\varphi_{k+1} > \psi \geq \varphi_k$, then $\eta^{(\alpha e^{i\phi}, \beta e^{i\psi})} = \eta^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$, and
(ii) $\eta^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})} \subset \eta^{(\mu_{k+1} e^{i\theta_{k+1}}, \nu_{k+1} e^{i\varphi_{k+1}})}$,

is called the *fundamental sequence* of $\mathscr{H} = (\mathscr{Q}, \eta)$, denoted by $\mathscr{F}_s(\mathscr{H})$. The set of $(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})$-level hypergraphs $\{\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}, \mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}, \ldots, \mathscr{H}^{(\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})}\}$ is called the set of core hypergraphs or the *core set* of $\mathscr{H}$, denoted by $cor(\mathscr{H})$.

Definition 6.64 Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be a complex q-rung orthopair fuzzy hypergraph. A *complex q-rung orthopair fuzzy transversal* τ is a complex q-rung orthopair fuzzy set of X satisfying the condition $\rho^{h(\rho)} \cap \tau^{h(\rho)} \neq \emptyset$, for all $\rho \in \eta$, where $h(\rho)$ is the height of ρ.

Table 6.16 Incidence matrix of complex 5-rung orthopair fuzzy hypergraph $\mathscr{H}$

$a \in X$	η_1	η_2	η_3
a_1	$(0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi})$	$(0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi})$	$(0, 0)$
a_2	$(0.7e^{i(0.7)\pi}, 0.9e^{i(0.9)\pi})$	$(0, 0)$	$(0.7e^{i(0.7)\pi}, 0.9e^{i(0.9)\pi})$
a_3	$(0, 0)$	$(0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})$	$(0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})$
a_4	$(0.6e^{i(0.6)\pi}, 0.8e^{i(0.8)\pi})$	$(0.6e^{i(0.6)\pi}, 0.8e^{i(0.8)\pi})$	$(0, 0)$
a_5	$(0, 0)$	$(0, 0)$	$(0.7e^{i(0.7)\pi}, 0.5e^{i(0.5)\pi})$

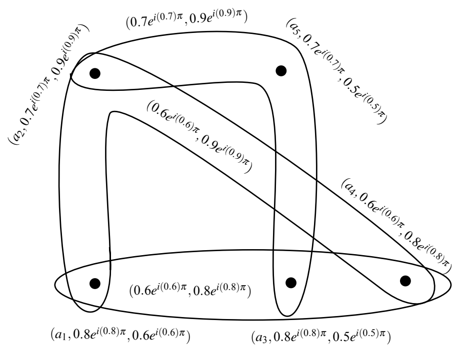


Fig. 6.29 Complex 5-rung orthopair fuzzy hypergraph

A *minimal complex q-rung orthopair fuzzy transversal* t is the complex q-rung orthopair fuzzy transversal of $\mathscr{H}$ having the property that if $\tau \subset t$, then τ is not a complex q-rung orthopair fuzzy transversal of $\mathscr{H}$.

Let us denote the family of minimal complex q-rung orthopair fuzzy transversals of $\mathscr{H}$ by $t_r(\mathscr{H})$.

Example 6.22 Consider a complex 5-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \eta)$ on $X = \{a_1, a_2, a_3, a_4, a_5\}$. The complex 5-rung orthopair fuzzy relation η is given as, $\eta(\{a_1 a_3, a_4\}) = (0.6e^{i(0.6)\pi}, 0.9e^{i(0.9)\pi})$, $\eta(\{a_2, a_3, a_5\}) = (0.7e^{i(0.7)\pi}, 0.9e^{i(0.9)\pi})$, and $\eta(\{a_1, a_2, a_4\}) = (0.6e^{i(0.6)\pi}, 0.9e^{i(0.9)\pi})$. The incidence matrix of $\mathscr{H}$ is given in Table 6.16.

The corresponding complex 5-rung orthopair fuzzy hypergraph is shown in Fig. 6.29.

By routine calculations, we have $h(\eta_1) = (0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi})$, $h(\eta_2) = (0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})$, and $h(\eta_3) = (0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})$. Consider a complex 5-rung orthopair fuzzy set τ_1 of X such that

$$\tau_1 = \{(a_1, 0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi}), (a_2, 0.7e^{i(0.7)\pi}, 0.9e^{i(0.9)\pi}), (a_3, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})\}.$$

Note that,

$$\eta_1^{(0.8e^{i(0.8)\pi},0.6e^{i(0.6)\pi})} = \{a_1\},\quad \eta_2^{(0.8e^{i(0.8)\pi},0.5e^{i(0.5)\pi})} = \{a_3\},\quad \eta_3^{(0.8e^{i(0.8)\pi},0.5e^{i(0.5)\pi})} = \{a_3\},$$
$$\tau_1^{(0.8e^{i(0.8)\pi},0.6e^{i(0.6)\pi})} = \{a_1, a_3\},\quad \tau_1^{(0.8e^{i(0.8)\pi},0.5e^{i(0.5)\pi})} = \{a_3\},\quad \tau_1^{(0.8e^{i(0.8)\pi},0.5e^{i(0.5)\pi})} = \{a_3\}.$$

Thus, we have $\eta_j^{h(\eta_j)} \cap \tau_1^{h(\eta_j)} \neq \emptyset$, for all $\eta_j \in \eta$. Hence, τ_1 is a complex 5-rung orthopair fuzzy transversal of $\mathscr{H}$. Similarly,

$$\tau_2 = \{(a_1, 0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi}), (a_3, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi})\},$$
$$\tau_3 = \{(a_1, 0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi}), (a_3, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi}), (a_4, 0.6e^{i(0.6)\pi}, 0.8e^{i(0.8)\pi})\},$$
$$\tau_4 = \{(a_1, 0.8e^{i(0.8)\pi}, 0.6e^{i(0.6)\pi}), (a_3, 0.8e^{i(0.8)\pi}, 0.5e^{i(0.5)\pi}), (a_5, 0.7e^{i(0.7)\pi}, 0.5e^{i(0.5)\pi})\},$$

are complex 5-rung orthopair fuzzy transversals of $\mathscr{H}$.

Definition 6.65 A complex q-rung orthopair fuzzy hypergraph $\mathscr{H}_1 = (\mathscr{Q}_1, \eta_1)$ is a *partial* complex q-rung orthopair fuzzy hypergraph of $\mathscr{H}_2 = (\mathscr{Q}_2, \eta_2)$ if $\eta_1 \subseteq \eta_2$, denoted by $\mathscr{H}_1 \subseteq \mathscr{H}_2$.
A complex q-rung orthopair fuzzy hypergraph $\mathscr{H}_1 = (\mathscr{Q}_1, \eta_1)$ is *ordered* if the core set $cor(\mathscr{H}) = \{\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}, \mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}, \ldots, \mathscr{H}^{(\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})}\}$ is ordered, i.e., $\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})} \subseteq \mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})} \subseteq \cdots \subseteq \mathscr{H}^{(\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})}$. $\mathscr{H}$ is *simply ordered* if $\mathscr{H}$ is ordered and $\eta' \subset \eta^{(\mu_{l+1} e^{i\theta_{l+1}}, \nu_{l+1} e^{i\varphi_{l+1}})} \backslash \eta^{(\mu_l e^{i\theta_l}, \nu_l e^{i\varphi_l})} \Rightarrow \eta' \nsubseteq \mathscr{Q}^{(\mu_l e^{i\theta_l}, \nu_l e^{i\varphi_l})}$.

Definition 6.66 A complex q-rung orthopair fuzzy set S on X is *elementary* if S is single-valued on $supp(S)$. A complex q-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \eta)$ is *elementary* if every $Q_j \in \mathscr{Q}$ and η are elementary.

Proposition 6.2 *If τ is a complex q-rung orthopair fuzzy transversal of $\mathscr{H} = (\mathscr{Q}, \eta)$, then $h(\tau) \geq h(\rho)$, for all $\rho \in \eta$. Furthermore, if τ is minimal complex q-rung orthopair fuzzy transversal of $\mathscr{H} = (\mathscr{Q}, \eta)$, then $h(\tau) = \max\{h(\rho) | \rho \in \eta\} = h(\mathscr{H})$.*

Lemma 6.4 *Let $\mathscr{H}_1 = (\mathscr{Q}_1, \eta_1)$ be a partial complex q-rung orthopair fuzzy hypergraph of $\mathscr{H}_2 = (\mathscr{Q}_2, \eta_2)$. If τ_2 is minimal complex q-rung orthopair fuzzy transversal of $\mathscr{H}_2$, then there is a minimal complex q-rung orthopair fuzzy transversal of $\mathscr{H}_1$ such that $\tau_1 \subseteq \tau_2$.*

Proof Let S_1 be a complex q-rung orthopair fuzzy set on X, which is defined as $S_1 = \tau_2 \cap (\cup_{Q_{1j} \in \mathscr{Q}_1} Q_{1j})$. Then, S_1 is a complex q-rung orthopair fuzzy transversal of $\mathscr{H}_1 = (\mathscr{Q}_1, \eta_1)$. Thus, there exists a minimal complex q-rung orthopair fuzzy transversal of $\mathscr{H}_1$ such that $\tau_1 \subseteq S_1 \subseteq \tau_2$.

Lemma 6.5 *Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be a complex q-rung orthopair fuzzy hypergraph then $f_s(t_r(\mathscr{H})) \subseteq f_s(\mathscr{H})$.*

Proof Let $f_s(\mathscr{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \ldots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$ and $\tau \in t_r(\mathscr{H})$. Suppose that for $u \in supp(\tau)$, $(T_\tau(u), F_\tau(u)) \in (\mu_{j+1}, \mu_j] \times (\nu_{j+1}, \nu_j]$, $\phi_\tau(u) \in (\theta_{j+1}, \theta_j]$, and $\psi_\tau(u) \in (\varphi_{j+1}, \varphi_j]$. Define a function λ by

$$T_\lambda(v)e^{i\phi} = \begin{cases} \mu_j e^{i\theta_j}, & \text{if} \quad u = v, \\ T_\tau(u)e^{i\phi_\tau(u)}, & \text{otherwise.} \end{cases}, \quad F_\lambda(v)e^{i\psi} = \begin{cases} \mu_j e^{i\varphi_j}, & \text{if} \quad u = v, \\ F_\tau(u)e^{i\psi_\tau(u)}, & \text{otherwise.} \end{cases}$$

From definition of λ, we have $\lambda^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})} = \tau^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$. Definition 6.63 implies that for every $t \in (\mu_{j+1}e^{i\theta_{j+1}}, \mu_j e^{\theta_j}] \times (\nu_{j+1}e^{i\varphi_{j+1}}, \nu_j e^{i\varphi_j}]$, $\mathscr{H}^t = \mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$. Thus, $\lambda^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$ is a complex q-rung orthopair fuzzy transversal of $\mathscr{H}^t$. Since, τ is minimal complex q-rung orthopair fuzzy transversal and $\lambda^t = \tau^t$, for all $t \notin (\mu_{j+1}e^{i\theta_{j+1}}, \mu_j e^{\theta_j}] \times (\nu_{j+1}e^{i\varphi_{j+1}}, \nu_j e^{i\varphi_j}]$. This implies that λ is also a complex q-rung orthopair fuzzy transversal and $\lambda \leq \tau$ but the minimality of τ implies that $\lambda = \tau$. Hence, $\tau(u) = \lambda(u) = (\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})$, which implies that for every complex q-rung orthopair fuzzy transversal $\tau \in t_r(\mathscr{H})$ and for each $u \in X$, $\tau(u) \in f_s(\mathscr{H})$ and so we have $f_s(t_r(\mathscr{H})) \subseteq f_s(\mathscr{H})$.

We now illustrate a recursive procedure to find $t_r(\mathscr{H})$ in Algorithm 6.11.1.

Algorithm 6.11.1 To find the family of minimal complex q-rung orthopair fuzzy transversals $t_r(\mathscr{H})$

Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be a complex q-rung orthopair fuzzy hypergraph having the fundamental sequence $f_s(\mathscr{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \ldots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$ and core set $cor(\mathscr{H}) = \{\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}, \mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}, \ldots, \mathscr{H}^{(\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})}\}$. The minimal transversal of $\mathscr{H} = (\mathscr{Q}, \eta)$ is determined as follows:

1. Determine a crisp minimal transversal t_1 of $\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$.
2. Determine a crisp minimal transversal t_2 of $\mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ satisfying the condition $t_1 \subseteq t_2$, i.e., obtain an hypergraph H_2 having the hyperedges $\eta^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ and a loop at every vertex $u \in t_1$. Thus, we have $\eta(H_2) = \eta(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}) \cup \{\{u \in t_1\}\}$.
3. Let t_2 be the minimal transversal of H_2.
4. Obtain a sequence of minimal transversals $t_1 \subseteq t_2 \subseteq \cdots \subseteq t_j$ such that t_j is the minimal transversal of $\mathscr{H}^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$ satisfying the condition $t_{j-1} \subseteq t_j$.
5. Define an elementary complex q-rung orthopair fuzzy set S_j having the support t_j and $h(S_j) = (\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})$, $1 \leq j \leq n$.
6. Determine a minimal complex q-rung orthopair fuzzy transversal of $\mathscr{H}$ as $\tau = \bigcup_{j=1}^{n} \{S_j | 1 \leq j \leq n\}$.

Example 6.23 Consider a complex 5-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \eta)$ on $X = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ as shown in Fig. 6.30. Let $(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}) = (0.9e^{i(0.9)2\pi}, 0.7e^{i(0.7)2\pi})$, $(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}) = (0.8e^{i(0.8)2\pi}, 0.5e^{i(0.5)2\pi})$, $(\mu_3 e^{i\theta_3}, \nu_3 e^{i\varphi_3}) = (0.6e^{i(0.6)2\pi}, 0.4e^{i(0.4)2\pi})$, and $(\mu_4 e^{i\theta_4}, \nu_4 e^{i\varphi_4}) = (0.3e^{i(0.3)2\pi}, 0.2e^{i(0.2)2\pi})$. Clearly, the sequence $\{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), (\mu_3 e^{i\theta_3}, \nu_3 e^{i\varphi_3}), (\mu_4 e^{i\theta_4}, \nu_4 e^{i\varphi_4})\}$ satisfies all the conditions of Definition 6.63. Hence, it is the fundamental sequence of $\mathscr{H}$.

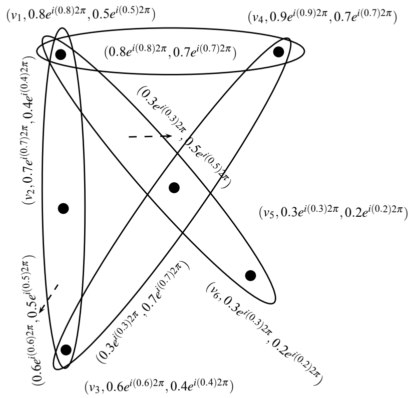


Fig. 6.30 Complex 5-rung orthopair fuzzy hypergraph

Note that, $t_1 = t_2 = \{v_4\}$ is the minimal transversal of $\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$ and $\mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$, $t_3 = \{v_1\}$ is the minimal transversal of $\mathscr{H}^{(\mu_3 e^{i\theta_3}, \nu_3 e^{i\varphi_3})}$, and $t_4 = \{v_1, v_4\}$ is the minimal transversal of $\mathscr{H}^{(\mu_4 e^{i\theta_4}, \nu_4 e^{i\varphi_4})}$. Consider

$$S_1 = \{(v_4, 0.9e^{i(0.9)2\pi}, 0.7e^{i(0.7)2\pi})\} = S_2,$$
$$S_3 = \{(v_1, 0.8e^{i(0.8)2\pi}, 0.5e^{i(0.5)2\pi})\},$$
$$S_4 = \{(v_1, 0.8e^{i(0.8)2\pi}, 0.5e^{i(0.5)2\pi}), (v_4, 0.9e^{i(0.9)2\pi}, 0.7e^{i(0.7)2\pi})\}.$$

Hence, $\bigcup_{j=1}^{4} = \{(v_1, 0.8e^{i(0.8)2\pi}, 0.5e^{i(0.5)2\pi}), (v_4, 0.9e^{i(0.9)2\pi}, 0.7e^{i(0.7)2\pi})\}$ is a complex 5-rung orthopair fuzzy transversal of $\mathscr{H}$.

Lemma 6.6 *Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be a complex q-rung orthopair fuzzy hypergraph with $f_s(\mathscr{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \ldots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$. If τ is a complex q-rung orthopair fuzzy transversal of $\mathscr{H}$, then $h(\tau) \geq h(Q_j)$, for every $Q_j \in \mathscr{Q}$. If $\tau \in t_r(\mathscr{H})$ then $h(\tau) = \max\{h(Q_j) | Q_j \in \mathscr{Q}\} = (\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})$.*

Proof Since τ is a complex q-rung orthopair fuzzy transversal of $\mathscr{H}$, implies that $\tau^{h(Q_j)} \cap Q_j^{h(Q_j)} \neq \emptyset$. Let $a \in supp(\tau)$, then $T_\tau(a) \geq T(h(Q_j))$, $F_\tau(a) \leq F(h(Q_j))$, $\phi_\tau(a) \geq \phi(h(Q_j))$, and $\psi_\tau(a) \leq \psi(h(Q_j))$. This shows that $h(\tau) \geq h(Q_j)$. If $\tau \in t_r(\mathscr{H})$, i.e., τ is minimal complex q-rung orthopair fuzzy transversal then $h(Q_j) = (\max T_{Q_j}(a) e^{i \max \phi_{Q_j}(a)}, \min F_{Q_j}(a) e^{i \min \psi_{Q_j}(a)}) = (\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})$. Thus, we have $h(\tau) = \max\{h(Q_j) | Q_j \in \mathscr{Q}\} = (\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})$.

Lemma 6.7 *Let β be a complex q-rung orthopair fuzzy transversal of a complex q-rung orthopair fuzzy hypergraph $\mathscr{H}$. Then, there exists $\gamma \in t_r(\mathscr{H})$ such that $\gamma \leq \beta$.*

Proof Let $f_s(\mathscr{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \ldots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$. Suppose that $\lambda^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$ is a transversal of $\mathscr{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$ and $\tau^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})} \in t_r(\mathscr{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})})$, for $1 \leq k \leq n$ such that $\tau^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})} \subseteq \lambda^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$. Let β_k be an elementary complex q-rung orthopair fuzzy set having support λ_k and γ_k be an elementary complex q-rung orthopair fuzzy set having support τ_k, for $1 \leq k \leq n$. Then, Algorithm 6.11.1 implies that $\beta = \bigcup_{k=1}^{n} \beta_k$ is a complex q-rung orthopair fuzzy transversal of $\mathscr{H}$ and $\gamma = \bigcup_{k=1}^{n} \gamma_k$ is minimal complex q-rung orthopair fuzzy transversal of $\mathscr{H}$ such that $\gamma \leq \beta$.

Theorem 6.18 *Let $\mathscr{H}_1 = (\mathscr{Q}_1, \eta_1)$ and $\mathscr{H}_2 = (\mathscr{Q}_2, \eta_2)$ be complex q-rung orthopair fuzzy hypergraphs. Then, $\mathscr{Q}_2 = t_r(\mathscr{H}_1) \Leftrightarrow \mathscr{H}_2$ is simple, $\mathscr{Q}_2 \subseteq \mathscr{Q}_1$, $h(\eta_k) = h(\mathscr{H}_1)$, for every $\rho_k \in \eta_2$, and for every complex q-rung orthopair fuzzy set $\xi \in \mathscr{P}(X)$, exactly one of the conditions must satisfy,*

(i) $\rho \leq \xi$, for some $\rho \in \mathscr{Q}_2$ or
(ii) there is $Q_j \in \mathscr{Q}_1$ and $(\mu e^{i\theta}, \nu e^{i\varphi})$, where $(\mu, \nu) \in [0, T_{h(Q_j)}] \times [0, F_{h(Q_j)}]$, $\theta \in [0, \phi_{h(Q_j)}]$, $\varphi \in [0, \psi_{h(Q_j)}]$ such that $Q_j^{(\mu e^{i\theta}, \nu e^{i\varphi})} \cap \xi^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \emptyset$, i.e., ξ is not a complex q-rung orthopair fuzzy transversal of $\mathscr{H}_1$.

Proof Let $\mathscr{Q}_2 = t_r(\mathscr{H}_1)$. Since, the family of all minimal complex q-rung orthopair fuzzy transversals form a simple complex q-rung orthopair fuzzy hypergraph on $X_1 \subseteq X_2$. Lemma 6.6 implies that every edge of $t_r(\mathscr{H}_1)$ has height $(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}) = h(\mathscr{H}_1)$. Let ξ be an arbitrary complex q-rung orthopair fuzzy set.

Case (i) If ξ is a complex q-rung orthopair fuzzy transversal of $\mathscr{H}_1$), then Lemma 6.7 implies the existence of a minimal complex q-rung orthopair fuzzy transversal ρ such that $\rho \leq \xi$. Thus, the condition (i) holds and (ii) violates.

Case (ii) If ξ is not a complex q-rung orthopair fuzzy transversal of $\mathscr{H}_1$), then there is an edge $Q_j \in \mathscr{Q}_1$ such that $Q_j^{(\mu e^{i\theta}, \nu e^{i\varphi})} \cap \xi^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \emptyset$. If condition (i) holds, $\rho \leq \xi$ implies that $Q_j^{(\mu e^{i\theta}, \nu e^{i\varphi})} \cap \rho^{(\mu e^{i\theta}, \nu e^{i\varphi})} = \emptyset$, which is the contradiction against the fact that ρ is complex q-rung orthopair fuzzy transversal. Hence, condition (i) does not hold and (ii) is satisfied.

Conversely, suppose that $\mathscr{Q}_2$ satisfies all properties as mentioned above and $\rho \in \mathscr{Q}_2$. Let $\rho = \xi$, then we obtain $\rho \leq \rho$ and conditions (ii) is not satisfied, so ρ is complex q-rung orthopair fuzzy transversal of $\mathscr{H}_1$. If t is minimal complex q-rung orthopair fuzzy transversal of $\mathscr{H}_1$ and $t \leq \rho$, t does not satisfy (ii), this implies the existence of $\rho_2 \in \mathscr{Q}_2$ such that $\rho_2 \leq t$, hence $\mathscr{Q}_2 \subseteq t_r(\mathscr{H}_1)$. Since, t is minimal complex q-rung orthopair fuzzy which implies that $\rho = t$, ρ and t were chosen arbitrarily therefore, we have $\mathscr{Q}_2 = t_r(\mathscr{H}_1)$.

The construction of fundamental subsequence and subcore of complex q-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \eta)$ is discussed in Algorithm 6.11.2.

Algorithm 6.11.2 Construction of fundamental subsequence and subcore
Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be a complex q-rung orthopair fuzzy hypergraph and $\mathscr{H}_1 = (\mathscr{Q}_1, \eta_1)$ be a partial complex q-rung orthopair fuzzy hypergraph of $\mathscr{H}$. The fundamental subsequence $f_{ss}(\mathscr{H})$ is constructed as follows:

Let $f_s(\mathscr{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \ldots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$ and $cor(\mathscr{H}) = \{\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}, \mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}, \ldots, \mathscr{H}^{(\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})}\}$.

1. Construct $\widetilde{\mathscr{H}}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$, a partial hypergraph of $\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$, by removing all hyperedges of $\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$, which contain properly any other hyperedge of $\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$.
2. In the same way, a partial hypergraph $\widetilde{\mathscr{H}}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ of $\mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ is constructed by removing all hyperedges of $\mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$, which contain properly any other hyperedge of $\mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ or any other hyperedge of $\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}$. $\widetilde{\mathscr{H}}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ is nontrivial iff there exists a complex q-rung orthopair fuzzy transversal $\tau \in t_r(\mathscr{H})$ and a vertex $u \in \mathscr{Q}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}$ such that $(T_\tau(u) e^{i\phi_\tau(u)}, F_\tau(u) e^{i\psi_\tau(u)}) = (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})$.
3. Continuing the same procedure, construct $\widetilde{\mathscr{H}}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$, a partial hypergraph of $\mathscr{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$, by removing all hyperedges of $\mathscr{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$, which contain properly any other hyperedge of $\mathscr{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$ or contain any other hyperedge of $\mathscr{H}^{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1})}, \mathscr{H}^{(\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2})}, \ldots, \mathscr{H}^{(\mu_{k-1} e^{i\theta_{k-1}}, \nu_{k-1} e^{i\varphi_{k-1}})}$. $\widetilde{\mathscr{H}}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$ is nontrivial if and only if there exists a complex q-rung orthopair fuzzy transversal $\tau \in t_r(\mathscr{H})$ and an element $u \in \mathscr{Q}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$ such that $(T_\tau(u) e^{i\phi_\tau(u)}, F_\tau(u) e^{i\psi_\tau(u)}) = (\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})$.
4. Let $\{(\tilde{\mu}_1 e^{i\tilde{\theta}_1}, \tilde{\nu}_1 e^{i\tilde{\varphi}_1}), (\tilde{\mu}_2 e^{i\tilde{\theta}_2}, \tilde{\nu}_2 e^{i\tilde{\varphi}_2}), \ldots, (\tilde{\mu}_l e^{i\tilde{\theta}_l}, \tilde{\nu}_l e^{i\tilde{\varphi}_l})\}$ be the set of complex numbers such that the corresponding partial hypergraphs $\widetilde{\mathscr{H}}^{(\tilde{\mu}_1 e^{i\tilde{\theta}_1}, \tilde{\nu}_1 e^{i\tilde{\varphi}_1})}, \widetilde{\mathscr{H}}^{(\tilde{\mu}_2 e^{i\tilde{\theta}_2}, \tilde{\nu}_2 e^{i\tilde{\varphi}_2})}, \ldots, \widetilde{\mathscr{H}}^{(\tilde{\mu}_l e^{i\tilde{\theta}_l}, \tilde{\nu}_l e^{i\tilde{\varphi}_l})}$ are non-empty.
5. Then, $f_{ss}(\mathscr{H}) = \{(\tilde{\mu}_1 e^{i\tilde{\theta}_1}, \tilde{\nu}_1 e^{i\tilde{\varphi}_1}), (\tilde{\mu}_2 e^{i\tilde{\theta}_2}, \tilde{\nu}_2 e^{i\tilde{\varphi}_2}), \ldots, (\tilde{\mu}_l e^{i\tilde{\theta}_l}, \tilde{\nu}_l e^{i\tilde{\varphi}_l})\}$ and $\widetilde{cor}(\mathscr{H}) = \{\widetilde{\mathscr{H}}^{(\tilde{\mu}_1 e^{i\tilde{\theta}_1}, \tilde{\nu}_1 e^{i\tilde{\varphi}_1})}, \widetilde{\mathscr{H}}^{(\tilde{\mu}_2 e^{i\tilde{\theta}_2}, \tilde{\nu}_2 e^{i\tilde{\varphi}_2})}, \ldots, \widetilde{\mathscr{H}}^{(\tilde{\mu}_l e^{i\tilde{\theta}_l}, \tilde{\nu}_l e^{i\tilde{\varphi}_l})}\}$ are subsequence and subcore set of $\mathscr{H}$, respectively.

Definition 6.67 Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be a complex q-rung orthopair fuzzy hypergraph having fundamental subsequence $f_{ss}(\mathscr{H})$ and subcore $\widetilde{cor}(\mathscr{H})$ of $\mathscr{H}$. The *complex q-rung orthopair fuzzy transversal core* of $\mathscr{H}$ is defined as an elementary complex q-rung orthopair fuzzy hypergraph $\widehat{\mathscr{H}} = (\widehat{\mathscr{Q}}, \widehat{\eta})$ such that,

(i) $f_{ss}(\mathscr{H}) = f_{ss}(\widehat{\mathscr{H}})$, i.e., $f_{ss}(\mathscr{H})$ is also a fundamental subsequence of $\widehat{\mathscr{H}}$,
(ii) height of every $\widehat{Q}_j \in \widehat{\mathscr{Q}}$ is $(\tilde{\mu}_j e^{i\tilde{\theta}_j}, \tilde{\nu}_j e^{i\tilde{\varphi}_j}) \in f_{ss}(\mathscr{H})$ iff $supp(\widehat{Q}_j)$ is an hyperedge of $\widetilde{\mathscr{H}}^{(\tilde{\mu}_j e^{i\tilde{\theta}_j}, \tilde{\nu}_j e^{i\tilde{\varphi}_j})}$.

Theorem 6.19 *For every complex q-rung orthopair fuzzy hypergraph, we have $t_r(\mathscr{H}) = t_r(\widehat{\mathscr{H}})$.*

Proof Let $t \in t_r(\mathscr{H})$ and $\widehat{Q}_j \in \widehat{\mathscr{Q}}$. Definition 6.67 implies that $h(\widehat{Q}_j) = (\tilde{\mu}_j e^{i\tilde{\theta}_j}, \tilde{\nu}_j e^{i\tilde{\varphi}_j})$ and $\widehat{Q}_j^{(\tilde{\mu}_j e^{i\tilde{\theta}_j}, \tilde{\nu}_j e^{i\tilde{\varphi}_j})}$ is an hyperedge of $\widetilde{\mathscr{H}}^{(\tilde{\mu}_j e^{i\tilde{\theta}_j}, \tilde{\nu}_j e^{i\tilde{\varphi}_j})}$. Since $\widetilde{\mathscr{H}}^{(\tilde{\mu}_j e^{i\tilde{\theta}_j}, \tilde{\nu}_j e^{i\tilde{\varphi}_j})} \subseteq \mathscr{H}^{(\tilde{\mu}_j e^{i\tilde{\theta}_j}, \tilde{\nu}_j e^{i\tilde{\varphi}_j})}$ and $\tau^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$ is a transversal of $\mathscr{H}^{(\tilde{\mu}_j e^{i\tilde{\theta}_j}, \tilde{\nu}_j e^{i\tilde{\varphi}_j})}$ therefore $\widehat{Q}_j^{(\tilde{\mu}_j e^{i\tilde{\theta}_j}, \tilde{\nu}_j e^{i\tilde{\varphi}_j})} \cap \tau^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})} \neq \emptyset$. Thus, τ is acomplex q-rung orthopair fuzzy transversal of $\widehat{\mathscr{H}}$.

Let $\widehat{\tau} \in t_r(\widehat{\mathscr{H}})$ and $Q_j \in \mathscr{Q}$. Definition 6.63 implies that $Q_j^{h(Q_j)} \in \mathscr{H}^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$, for $h(Q_j) \leq (\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j}) \in f_s(\mathscr{H})$. Definition of subcore $\widetilde{cor}(\mathscr{H})$ implies the

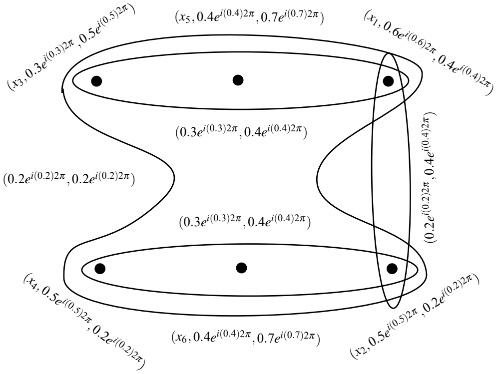


Fig. 6.31 Complex 6-rung orthopair fuzzy hypergraph

existence of an hyperedge $\widehat{Q}_j^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$ of $\widetilde{\mathscr{H}}^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$ such that $\widehat{Q}_j^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})} \subseteq Q_j^{h(Q_j)}$ and $(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k}) \geq (\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j}) \geq h(Q_j)$. For $\widehat{\tau} \in t_r(\widehat{\mathscr{H}})$, we have $u \in \widehat{Q}_j^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})} \cap \widehat{\tau}^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})} \subseteq \widehat{Q}_j^{h(Q_j)} \cap \widehat{\tau}^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})}$. Hence, $\widehat{\tau}$ is a complex q-rung orthopair fuzzy transversal of $\mathscr{H}$.

Let $\tau \in t_r(\mathscr{H}) \Rightarrow \tau$ is a complex q-rung orthopair fuzzy transversal of $\widehat{\mathscr{H}}$. This implies that there is $\widehat{\tau}$ such that $\widehat{\tau} \subseteq \tau$. But $\widehat{\tau}$ is a complex q-rung orthopair fuzzy transversal of $\mathscr{H}$ and $\tau \in t_r(\mathscr{H})$ implies that $\widehat{\tau} = \tau$. Thus, $t_r(\mathscr{H}) \subseteq t_r(\widehat{\mathscr{H}})$. Also $t_r(\widehat{\mathscr{H}}) \subseteq t_r(\mathscr{H})$ implies that $t_r(\mathscr{H}) = t_r(\widehat{\mathscr{H}})$.

Although τ can be taken as a minimal transversal of $\mathscr{H}$, it is not necessary for $\tau^{(\mu e^{i\theta}, \nu e^{i\varphi})}$ to be the minimal transversal of $\mathscr{H}^{(\mu e^{i\theta}, \nu e^{i\varphi})}$, for all $\mu, \nu \in [0, 1]$, and $\theta, \varphi \in [0, 2\pi]$. Furthermore, it is not necessary for the family of minimal complex q-rung orthopair fuzzy transversals to form a hypergraph on X. For those complex q-rung orthopair fuzzy transversals that satisfy the above property, we have

Definition 6.68 A complex q-rung orthopair fuzzy transversal τ having the property that $\tau^{(\mu e^{i\theta}, \nu e^{i\varphi})} \in t_r(\mathscr{H}^{(\mu e^{i\theta}, \nu e^{i\varphi})})$, for all $\mu, \nu \in [0, 1]$, and $\theta, \varphi \in [0, 2\pi]$ is called the *locally minimal complex q-rung orthopair fuzzy transversal* of $\mathscr{H}$. The collection of all locally minimal complex q-rung orthopair fuzzy transversals of $\mathscr{H}$ is represented by $t_r^*(\mathscr{H})$.

Note that, $t_r^*(\mathscr{H}) \subseteq t_r(\mathscr{H})$, but the converse is not generally true.

Example 6.24 Consider a complex 6-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \eta)$ as shown in Fig. 6.31. The complex 6-rung orthopair fuzzy set

$$\{(x_1, 0.6e^{i(0.6)2\pi}, 0.4e^{i(0.4)2\pi}), (x_5, 0.4e^{i(0.4)2\pi}, 0.7e^{i(0.7)2\pi}), (x_6, 0.4e^{i(0.4)2\pi}, 0.7e^{i(0.7)2\pi})\}$$

is a locally minimal complex 6-rung orthopair fuzzy transversal of $\mathscr{H}$.

Theorem 6.20 *Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be an ordered complex q-rung orthopair fuzzy hypergraph with $f_s(\mathscr{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \ldots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$.*

If λ_k is a minimal transversal of $\mathscr{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$, then there exists $\alpha \in t_r(\mathscr{H})$ such that $\alpha^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})} = \lambda_k$ and $\alpha^{(\mu_l e^{i\theta_l}, \nu_l e^{i\varphi_l})}$ is a minimal transversal of $\mathscr{H}^{(\mu_l e^{i\theta_l}, \nu_l e^{i\varphi_l})}$, for all $l < k$. In particular, if $\lambda_j \in t_r(\mathscr{H}^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})})$, then there exists a locally minimal complex q-rung orthopair fuzzy transversal $\alpha^{(\mu_j e^{i\theta_j}, \nu_j e^{i\varphi_j})} = \lambda_j$ and $t_r^*(\mathscr{H}) \neq \emptyset$.

Proof Let $\lambda_k \in t_r(\mathscr{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})})$. Since, $\mathscr{H} = (\mathscr{Q}, \eta)$ is an ordered complex q-rung orthopair fuzzy hypergraph, therefore, $\mathscr{H}^{(\mu_{k-1} e^{i\theta_{k-1}}, \nu_{k-1} e^{i\varphi_{k-1}})} \subseteq \mathscr{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})}$. Also, there exists $\lambda_{k-1} \in t_r(\mathscr{H}^{(\mu_{k-1} e^{i\theta_{k-1}}, \nu_{k-1} e^{i\varphi_{k-1}})})$ such that $\lambda_{k-1} \subseteq \lambda_k$. Following this iterative procedure, we have a nested sequence $\lambda_1 \subseteq \lambda_2 \subseteq \cdots \subseteq \lambda_{k-1} \subseteq \lambda_k$ of minimal transversals, where every $\lambda_l \in t_r(\mathscr{H}^{(\mu_l e^{i\theta_l}, \nu_l e^{i\varphi_l})})$. Let α_l be an elementary complex q-rung orthopair fuzzy set having height $(\mu_l e^{i\theta_l}, \nu_l e^{i\varphi_l})$ and support α_l. Let us define $\alpha(x)$ such that $\alpha(x) = \{(\max T_{\alpha_l}(x) e^{i \max \phi_{\alpha_l}(x)}, \min F_{\alpha_l}(x) e^{i \min \psi_{\alpha_l}(x)}) | 1 \leq l \leq n\}$, that generates the required minimal complex q-rung orthopair fuzzy transversal of $\mathscr{H}$. If $k = n$, α is locally minimal complex q-rung orthopair fuzzy transversal of $\mathscr{H}$. Hence, $t_r^*(\mathscr{H}) \neq \emptyset$.

Theorem 6.21 *Let $\mathscr{H} = (\mathscr{Q}, \eta)$ be a simply ordered complex q-rung orthopair fuzzy hypergraph with $f_s(\mathscr{H}) = \{(\mu_1 e^{i\theta_1}, \nu_1 e^{i\varphi_1}), (\mu_2 e^{i\theta_2}, \nu_2 e^{i\varphi_2}), \ldots, (\mu_n e^{i\theta_n}, \nu_n e^{i\varphi_n})\}$. If $\lambda_k \in t_r(\mathscr{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})})$, then there exists $\alpha \in t_r^*(\mathscr{H})$ such that $\alpha^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})} = \lambda_k$.*

Proof Let $\lambda_k \in t_r(\mathscr{H}^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})})$ and $\mathscr{H} = (\mathscr{Q}, \eta)$ is a simply ordered complex q-rung orthopair fuzzy hypergraph. Theorem 6.20 implies that a nested sequence $\lambda_1 \subseteq \lambda_2 \subseteq \cdots \subseteq \lambda_{k-1} \subseteq \lambda_k$ of minimal transversals can be constructed. Let α_l be an elementary complex q-rung orthopair fuzzy set having height $(\mu_l e^{i\theta_l}, \nu_l e^{i\varphi_l})$ and support α_l such that $\alpha(x) = \{(\max T_{\alpha_l}(x) e^{i \max \phi_{\alpha_l}(x)}, \min F_{\alpha_l}(x) e^{i \min \psi_{\alpha_l}(x)}) | 1 \leq l \leq n\}$ generates the locally minimal complex q-rung orthopair fuzzy transversal of $\mathscr{H}$ with $\alpha^{(\mu_k e^{i\theta_k}, \nu_k e^{i\varphi_k})} = \lambda_k$.

6.12 Application of Complex q-Rung Orthopair Fuzzy Hypergraphs

Definition 6.69 Let $\mathscr{Q} = (T e^{i\phi}, F e^{i\psi})$ be a complex q-rung orthopair fuzzy number. Then, *score function* of $\mathscr{Q}$ is defined as

$$s(\mathscr{Q}) = (T^q - F^q) + \frac{1}{2^q \pi^q}(\phi^q - \psi^q).$$

The *accuracy* of $\mathscr{Q}$ is defined as

$$a(\mathscr{Q}) = (T^q + F^q) + \frac{1}{2^q \pi^q}(\phi^q + \psi^q).$$

For two complex q-rung orthopair fuzzy numbers $\mathscr{Q}_1$ and $\mathscr{Q}_2$

1. if $s(\mathscr{Q}_1) > s(\mathscr{Q}_2)$, then $\mathscr{Q}_1 \succ \mathscr{Q}_2$,
2. if $s(\mathscr{Q}_1) = s(\mathscr{Q}_2)$, then
   - if $a(\mathscr{Q}_1) > a(\mathscr{Q}_2)$, then $\mathscr{Q}_1 \succ \mathscr{Q}_2$,
   - if $a(\mathscr{Q}_1) = a(\mathscr{Q}_2)$, then $\mathscr{Q}_1 \sim \mathscr{Q}_2$.

A complex 6-rung orthopair fuzzy hypergraph model of research collaboration network A collaboration network is a group of independent organizations or people that interact to complete a particular goal for achieving better collective results by the means of joint execution of task. The entities of a collaborative network may be geographically distributed and heterogenous in terms of their culture, goals, and operating environment but they collaborate to achieve compatible or common goals. For decades, science academies have been interested in research collaboration. The most common reasons of research collaboration are funding, more experts working on the same project imply the more chances for effectiveness, productivity, and innovativeness. Nowadays, most of the public research is based on collaboration of different types of expertise from different disciples and different economic sectors. In this section, we study a research collaboration network model through complex 6-rung orthopair fuzzy hypergraph. Consider a science academy wants to select an author among a group of researchers which has best collaborative skills. For this purpose, following are the characteristics that can be considered,

- Cooperative spirit
- Mutual respect
- Critical thinking
- Innovations
- Creativity
- Embrace diversity.

Consider a complex 6-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \eta)$ on $X = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}\}$. The set of universe X represents the group of authors as the vertices of $\mathscr{H}$ and these authors are grouped through hyperedges if they have worked together on some projects. The truth-membership of each author represents the collaboration strength and falsity-membership describes the opposite behavior of corresponding author. Suppose that a team of experts assigns that the collaboration power of A_1 is 60% and non-collaborative behavior is 50% after carefully observing the different attributes. The corresponding phase terms illustrate the specific period of time in which the collaborative behavior of an author varies. We model this data as $(A_1, 0.6e^{i(0.5)2\pi}, 0.5e^{i(0.5)2\pi})$. The complex 6-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \eta)$ model of collaboration network is shown in Fig. 6.32.

The membership degrees of hyperedges represent the collective degrees of collaboration and non-collaboration of the corresponding authors combined through an hyperedge. The adjacency matrix of this network is given in Tables 6.17, 6.18, and 6.19.

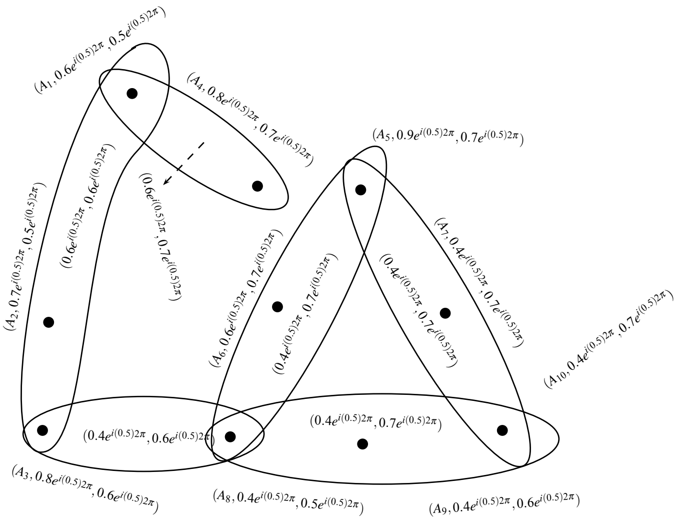


Fig. 6.32 Complex 6-rung orthopair fuzzy hypergraph model of collaboration network

Table 6.17 Adjacency matrix of collaboration network

η	A_1	A_2	A_3	A_4
A_1	(0, 0)	$(0.6e^{i(0.5)2\pi}, 0.6e^{i(0.5)2\pi})$	$(0.6e^{i(0.5)2\pi}, 0.6e^{i(0.5)2\pi})$	$(0.6e^{i(0.5)2\pi}, 0.6e^{i(0.5)2\pi})$
A_2	$(0.6e^{i(0.5)2\pi}, 0.6e^{i(0.5)2\pi})$	(0, 0)	$(0.6e^{i(0.5)2\pi}, 0.6e^{i(0.5)2\pi})$	(0, 0)
A_3	$(0.6e^{i(0.5)2\pi}, 0.6e^{i(0.5)2\pi})$	$(0.6e^{i(0.5)2\pi}, 0.6e^{i(0.5)2\pi})$	(0, 0)	(0, 0)
A_4	$(0.6e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	(0, 0)	(0, 0)	(0, 0)
A_5	(0, 0)	(0, 0)	(0, 0)	(0, 0)
A_6	(0, 0)	(0, 0)	(0, 0)	(0, 0)
A_7	(0, 0)	(0, 0)	(0, 0)	(0, 0)
A_8	(0, 0)	(0, 0)	$(0.4e^{i(0.5)2\pi}, 0.6e^{i(0.5)2\pi})$	(0, 0)
A_9	(0, 0)	(0, 0)	(0, 0)	(0, 0)
A_{10}	(0, 0)	(0, 0)	(0, 0)	(0, 0)

Table 6.18 Adjacency matrix of collaboration network

η	A_5	A_6	A_7	A_8
A_1	(0, 0)	(0, 0)	(0, 0)	(0, 0)
A_2	(0, 0)	(0, 0)	(0, 0)	(0, 0)
A_3	(0, 0)	(0, 0)	(0, 0)	$(0.4e^{i(0.5)2\pi}, 0.6e^{i(0.5)2\pi})$
A_4	(0, 0)	(0, 0)	(0, 0)	(0, 0)
A_5	(0, 0)	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	$(0.6e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$
A_6	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	(0, 0)	(0, 0)	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$
A_7	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	(0, 0)	(0, 0)	(0, 0)
A_8	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	(0, 0)	(0, 0)
A_9	(0, 0)	(0, 0)	(0, 0)	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$
A_{10}	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	(0, 0)	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$

Table 6.19 Adjacency matrix of collaboration network

η	A_9	A_{10}
A_1	(0, 0)	(0, 0)
A_2	(0, 0)	(0, 0)
A_3	(0, 0)	(0, 0)
A_4	(0, 0)	(0, 0)
A_5	(0, 0)	$(0.6e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$
A_6	(0, 0)	(0, 0)
A_7	(0, 0)	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$
A_8	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$
A_9	(0, 0)	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$
A_{10}	$(0.4e^{i(0.5)2\pi}, 0.7e^{i(0.5)2\pi})$	(0, 0)

The score values and choice values of a complex 6-rung orthopair fuzzy hypergraph $\mathscr{H} = (\mathscr{Q}, \eta)$ are calculated as follows:

$$s_{jk} = (T_{jk}^q + F_{jk}^q) + \frac{1}{2^q \pi^q}(\phi_{jk}^q + \psi_{jk}^q), \ c_j = \sum_k s_{jk} + (T_j^q + F_j^q) + \frac{1}{2^q \pi^q}(\phi_j^q + \psi_j^q),$$

respectively. These values are given in Table 6.20.

The choice values of Table 6.20 show that A_5 is the author having maximum strength of collaboration and good collective skills among all the authors. Similarly,

Table 6.20 Score and choice values

s_{jk}	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	c_j
A_1	0	0.1245	0.1245	0.1245	0	0	0	0	0	0	0.88690
A_2	0.1245	0	0.1245	0	0	0	0	0	0	0	0.41377
A_3	0.1245	0.1245	0	0	0	0	0	0.0820	0	0	0.67105
A_4	0.1955	0	0	0	0	0	0	0	0	0	0.60654
A_5	0	0	0	0	0	0.1529	0.1955	0.1529	0	0.1955	1.37714
A_6	0	0	0	0	0.1529	0	0	0.1529	0	0	0.53480
A_7	0	0	0	0	0.1955	0	0	0	0	0.1529	0.50139
A_8	0	0	0.0820	0	0.1529	0.1529	0	0	0.1529	0.1529	0.74457
A_9	0	0	0	0	0	0	0	0.1529	0	0.1529	0.38780
A_{10}	0	0	0	0	0.1529	0	0.1529	0.1529	0.1529	0	0.76459

the choice values of all authors represent the strength of their respective collaboration skills in a specific period of time. The method adopted in our model to select the author having best collaboration skills is given in Algorithm 6.12.1.

Algorithm 6.12.1 Selection of author having maximum collaboration skills

1. Input the set of vertices (authors) $A_1, A_2, \ldots, A_j$.
2. Input the complex q-rung orthopair fuzzy set Q of vertices such that $Q(A_k) = (T_k e^{i\phi_k}, F_k e^{i\psi_k})$, $1 \leq k \leq j, 0 \leq T_k^q + F_k^q \leq 1, q \geq 1$.
3. Input the adjacency matrix $\eta = [(T_{kl}e^{i\phi_{kl}}, F_{kl}e^{i\psi_{kl}})]_{j\times j}$ of vertices.
4. **do** k from $1 \to j$
5. $c_k = 0$
6. **do** l from $1 \to j$
7. $s_{jk} = (T_{kl}^q + F_{kl}^q) + \frac{1}{2^q\pi^q}(\phi_{kl}^q + \psi_{kl}^q)$
8. $c_k = c_k + s_{jk}$
9. **end do**
10. $c_k = c_k + (T_k^q + F_k^q) + \frac{1}{2^q\pi^q}(\phi_k^q + \psi_k^q)$
11. **do**
12. Select a vertex of $\mathscr{H} = (\mathscr{Q}, \eta)$ having maximum choice value as the author possessing strong collaboration powers.

6.13 Comparative Analysis

The proposed complex q-rung orthopair fuzzy model is more flexible and compatible to the system when the given data ranges over complex subset with a unit disk instead of the real subset with [0, 1]. We illustrate the flexibility of our proposed model by taking an example. Consider an educational institute that wants to establish its minimum branches in a particular city in order to facilitate the maximum number of

Table 6.21 Comparative analysis of three models

Methods	Score values	Ranking
Complex intuitionistic fuzzy model	0.4 1.0 0.6	$p_2 > p_3 > p_1$
Complex Pythagorean fuzzy model	0.4 0.9 0.42	$p_2 > p_3 > p_1$
Complex 3-rung orthopair fuzzy model	0.104 0.67 0.234	$p_2 > p_3 > p_1$

students according to some parameters such as transportation, suitable place, connectivity with the main branch, and expenditures. Suppose a team of three decision makers selects the different places. Let $X = \{p_1, p_2, p_3\}$ be the set of places where the team is interested to establish the new branches. After carefully observing the different attributes, the first decision makers assign the membership and nonmembership degrees to support the place p_1 as 60% and 40%, respectively. The phase terms represent the period of time for which the place p_1 can attract maximum number of students. This information is modeled using a complex intuitionistic fuzzy set as $(p_1, 0.6e^{i(0.6)2\pi}, 0.4e^{i(0.4)2\pi})$. Note that, $0 \leq 0.6 + 0.4 \leq 1$. Similarly, he models the other places as, $(p_2, 0.7e^{i(0.7)2\pi}, 0.2e^{i(0.2)2\pi})$, $(p_3, 0.5e^{i(0.5)2\pi}, 0.2e^{i(0.2)2\pi})$. We denote this complex intuitionistic fuzzy model as

$$I = \{(p_1, 0.6e^{i(0.6)2\pi}, 0.4e^{i(0.4)2\pi}), (p_2, 0.7e^{i(0.7)2\pi}, 0.2e^{i(0.2)2\pi}), (p_3, 0.5e^{i(0.5)2\pi}, 0.2e^{i(0.2)2\pi})\}.$$

Since, all complex intuitionistic fuzzy grades are complex Pythagorean fuzzy as well as complex q-rung orthopair fuzzy grades. We find the score functions of the above values using the formulas $s(p_j) = (T - F) + \frac{1}{2\pi}(\phi - \psi)$, $s(p_j) = (T^2 - F^2) + \frac{1}{2^2\pi^2}(\phi^2 - \psi^2)$, and $s(p_j) = (T^3 - F^3) + \frac{1}{2^3\pi^3}(\phi^3 - \psi^3)$. The results corresponding to these three approaches are given in Table 6.21.

Suppose that the second decision-maker assigns the membership values to these places as, $(p_1, 0.6e^{i(0.6)2\pi}, 0.4e^{i(0.4)2\pi})$, $(p_2, 0.7e^{i(0.7)2\pi}, 0.2e^{i(0.2)2\pi})$, $(p_3, 0.7e^{i(0.7)2\pi}, 0.5e^{i(0.5)2\pi})$. This information can not be modeled using complex intuitionistic fuzzy set as $0.7 + 0.5 = 1.2 > 1$. We model this information using a complex Pythagorean fuzzy set and the corresponding model is given as

$$P = \{(p_1, 0.6e^{i(0.6)2\pi}, 0.4e^{i(0.4)2\pi}), (p_2, 0.7e^{i(0.7)2\pi}, 0.2e^{i(0.2)2\pi}), (p_3, 0.7e^{i(0.7)2\pi}, 0.5e^{i(0.5)2\pi})\}.$$

Since, all complex Pythagorean fuzzy grades are also complex q-rung orthopair fuzzy grades. We find the score functions of the above values using the formulas $s(p_j) = (T^2 - F^2) + \frac{1}{2^2\pi^2}(\phi^2 - \psi^2)$ and $s(p_j) = (T^3 - F^3) + \frac{1}{2^3\pi^3}(\phi^3 - \psi^3)$. The results corresponding to these two approaches are given in Table 6.22.

Table 6.22 Comparative analysis of two models

Methods	Score values	Ranking
Complex Pythagorean fuzzy model	0.4 0.9 0.48	$p_2 > p_3 > p_1$
Complex 3-rung orthopair fuzzy model	0.104 0.67 0.436	$p_2 > p_3 > p_1$

We now suppose that the third decision maker assigns the membership values to these places as

$$(p_1, 0.6e^{i(0.6)2\pi}, 0.4e^{i(0.4)2\pi}), (p_2, 0.8e^{i(0.8)2\pi}, 0.7e^{i(0.7)2\pi}), (p_3, 0.7e^{i(0.7)2\pi}, 0.5e^{i(0.5)2\pi}).$$

This information cannot be modeled using complex intuitionistic fuzzy set and complex Pythagorean fuzzy set as $0.7 + 0.8 = 1.5 > 1$, $0.7^2 + 0.8^2 = 1.13 > 1$. We model this information using a complex 3-rung orthopair fuzzy set and the corresponding model is given as

$$Q = \{(p_1, 0.6e^{i(0.6)2\pi}, 0.4e^{i(0.4)2\pi}), (p_2, 0.8e^{i(0.8)2\pi}, 0.7e^{i(0.7)2\pi}), (p_3, 0.7e^{i(0.7)2\pi}, 0.5e^{i(0.5)2\pi})\}.$$

We find the score functions of the above values using the formula $s(p_j) = (T^3 - F^3) + \frac{1}{2^3\pi^3}(\phi^3 - \psi^3)$. The score values of complex 3-rung orthopair fuzzy information are given as

$$s(p_1) = 0.304,\ s(p_2) = 0.438,\ s(p_3) = 0.436.$$

Note that, p_2 is the best optimal choice to establish a new branch according to the given parameters. We see that every complex intuitionistic fuzzy grade is a complex Pythagorean fuzzy grade, as well as a complex q-rung orthopair fuzzy grade, however there are complex q-rung orthopair fuzzy grades that are not complex intuitionistic fuzzy nor complex Pythagorean fuzzy grades. This implies the generalization of complex q-rung orthopair fuzzy values. Thus, the proposed complex q-rung orthopair fuzzy model provides more flexibility due to its most prominent feature that is the adjustment of the range of demonstration of given information by changing the value of parameter q, $q \geq 1$. The generalization of our proposed model can also be observed from the reduction of complex q-rung orthopair fuzzy model to complex intuitionistic fuzzy and complex Pythagorean fuzzy models for $q = 1$ and $q = 2$, respectively.

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