Chapter 5
Extended Bipolar Fuzzy (Directed) Hypergraphs to m-Polar Information

An m-polar fuzzy set is a useful tool to solve real-world problems that involve multi-agents, multi-attributes, multi-objects, multi-indexes, and multipolar information. In this chapter, we present the notions of regular m-polar fuzzy hypergraphs and totally regular m-polar fuzzy hypergraphs. We discuss applications of m-polar fuzzy hypergraphs in decision-making problems. Furthermore, we discuss the notion of m-polar fuzzy directed hypergraphs and depict certain operations on them. We also describe an application of m-polar fuzzy directed hypergraphs in business strategy. This chapter is based on [7–9, 12].

5.1 Introduction

Fuzzy set theory deals with real-life data incorporating vagueness. Zhang [20] extended the theory of fuzzy sets to bipolar fuzzy sets, which register the bipolar behavior of objects. Nowadays, analysts believe that the world is moving toward multipolarity. Therefore, it comes as no surprise that multipolarity in data and information plays a vital role in various fields of science and technology. In neurobiology, multipolar neurons in brain gather a great deal of information from other neurons. In information technology, multipolar technology can be exploited to operate large-scale systems. Based on this motivation, Chen et al. [12] introduced the concept of m-polar fuzzy set as a generalization of a bipolar fuzzy set and shown that 2-polar and bipolar fuzzy sets are cryptomorphic mathematical notions. The framework of this theory is that "multipolar information" (not like the bipolar information which gives two-valued logic) arises because information for a natural world is frequently from n factors ($n \geq 2$). For example, "Pakistan is a good country". The truth value of this statement may not be a real number in [0, 1]. Being good country may have several properties: good in agriculture, good in political awareness, good in regaining macroeconomic stability, etc. The each component may be a real number in [0, 1].

M. Akram and A. Luqman, *Fuzzy Hypergraphs and Related Extensions*,
Studies in Fuzziness and Soft Computing 390,
https://doi.org/10.1007/978-981-15-2403-5_5

If n is the number of such components under consideration, then the truth value of fuzzy statement is a n-tuple of real numbers in $[0, 1]$, that is, an element of $[0, 1]^n$.

Hypergraphs have many applications in various fields, including biological sciences, computer science, and natural sciences. To study the degree of dependence of an object to the other, Kaufamnn [14] applied the concept of fuzzy sets to hypergraphs. Mordeson and Nair [16] presented fuzzy graphs and fuzzy hypergraphs. Generalization and redefinition of fuzzy hypergraphs were discussed by Lee-Kwang and Lee [15]. The concept of interval-valued fuzzy sets was applied to hypergraphs by Chen [11]. Parvathi et al. [17] established the notion of intuitionistic fuzzy hypergraphs.

Definition 5.1 An *m-polar fuzzy set* C on a non-empty set X is a mapping $C : X \to [0, 1]^m$. The membership value of every element $x \in X$ is denoted by $C(x) = (P_1 \circ C(x), P_2 \circ C(x), \ldots, P_m \circ C(x))$, where $P_i \circ C : [0, 1]^m \to [0, 1]$ is defined as the $i-$th projection mapping.

Note that, $[0, 1]^m$ (mth-power of $[0, 1]$) is considered as a partially ordered set with the point-wise order $\leq$, where m is an arbitrary ordinal number (we make an appointment that $m = \{n|n < m\}$ when $m > 0$), $\leq$ is defined by $x \leq y \Leftrightarrow P_i(x) \leq P_i(y)$ for each $i \in m$ $(x, y \in [0, 1]^m)$, and $P_i : [0, 1]^m \to [0, 1]$ is the $i-$th projection mapping $(i \in m)$. $\mathbf{1} = (1, 1, \ldots, 1)$ is the greatest value and $\mathbf{0} = (0, 0, \ldots, 0)$ is the smallest value in $[0, 1]^m$. $m\mathscr{F}(X)$ is the power set of all m-polar fuzzy subsets on X.

1. When $m = 2$, $[0, 1]^2$ is the ordinary closed unit square in $\mathbb{R}^2$, the Euclidean plane. The righter (resp., the upper), the point in this square, the larger it is. Let $x = (0, 0) = \mathbf{0}$ (the smallest element of $[0, 1]^2$), $a = (0.35, 0.85)$, $b = (0.85, 0.35)$, and $y = (1, 1) = \mathbf{1}$ (the largest element of $[0, 1]^2$). Then $x \leq c \leq y, \quad \forall \quad c \in [0, 1]^2$, (especially, $x \leq a \leq y$ and $x \leq b \leq y$ hold). It is easy to note that $a \nleq b \nleq a$ because $P_0(a) = 0.35 < 0.85 = P_0(b)$ and $P_1(a) = 0.85 > 0.35 = P_1(b)$ hold. The "order relation $\leq$" on $[0, 1]^2$ can be described in at least two ways. It can be seen in Fig. 5.1.
2. When $m = 4$, the order relation can be seen in Fig. 5.2.

Example 5.1 Suppose that a democratic country wants to elect its leader. Let $C = \{\text{Irtiza, Moeed, Ramish, Ahad}\}$ be the set of four candidates and $X = \{a, b, c, \ldots, s, t\}$ be the set of voters. We assume that the voting is weighted. A voter in $\{a, b, c\}$ can send a value in $[0, 1]$ to each candidate but a voter in $X - \{a, b, c\}$ can

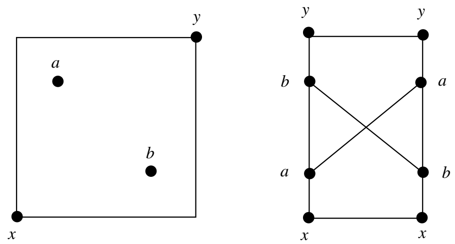


Fig. 5.1 Order relation when $m = 2$

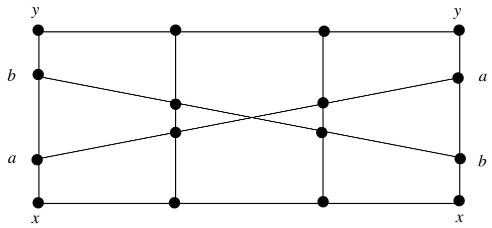


Fig. 5.2 Order relation when $m = 4$

only send a value in $[0.2, 0.7]$ to each candidate. Let $A(a) = (0.8, 0.6, 0.5, 0.1)$ (which shows that the preference degrees of a corresponding to Irtiza, Moeed, Ramish, and Ahad are 0.8, 0.6, 0.5, and 0.1, respectively.), $A(b) = (0.9, 0.7, 0.5, 0.8)$, $A(c) = (0.9, 0.9, 0.8, 0.4), \ldots, A(s) = (0.6, 0.7, 0.5, 0.3)$, and $A(t) = (0.5, 0.7, 0.2, 0.5)$. Thus, we obtain a 4-polar fuzzy set $A : X \to [0, 1]^4$ which can also be written as

$$A = \{(a, (0.8, 0.6, 0.5, 0.1)), (b, (0.9, 0.7, 0.5, 0.8)), (c, (0.9, 0.9, 0.8, 0.4)), \ldots, (s, (0.6, 0.7, 0.5, 0.3)), (t, (0.5, 0.7, 0.2, 0.5))\}.$$

Definition 5.2 Let C and D be two m-polar fuzzy sets on X. Then, the operations $C \cup D$, $C \cap D$, $C \subseteq D$, and $C = D$ are defined as

1. $P_i \circ (C \cup D)(x) = \sup\{P_i \circ C(x), P_i \circ D(x)\} = P_i \circ C(x) \vee P_i \circ D(x)$,
2. $P_i \circ (C \cap D)(x) = \inf\{P_i \circ C(x), P_i \circ D(x)\} = P_i \circ C(x) \wedge P_i \circ D(x)$,
3. $C \subseteq D$ if and only if $P_i \circ C(x) \leq P_i \circ D(x)$,
4. $C = D$ if and only if $P_i \circ C(x) = P_i \circ D(x)$,

for all $x \in X$, for each $1 \leq i \leq m$.

Definition 5.3 Let C be an m-polar fuzzy set on a non-empty crisp set X. An *m-polar fuzzy relation* on C is a mapping $(P_1 \circ D, P_2 \circ D, \ldots, P_m \circ D) = D : C \to C$ such that

$$D(xy) \leq \inf\{C(x), C(y)\}, \quad \text{for all } x, y \in X$$

that is, for each $1 \leq i \leq m$,

$$P_i \circ D(xy) \leq \inf\{P_i \circ C(x), P_i \circ C(y)\}, \quad \text{for all } x, y \in X$$

where $P_i \circ C(x)$ denotes the $i-$th degree of membership of the vertex x and $P_i \circ D(xy)$ denotes the $i-$th degree of membership of the edge xy. D is also an m-polar fuzzy relation in X defined by the mapping $D : X \times X \to [0, 1]^m$.

Definition 5.4 An *m-polar fuzzy graph* on a non-empty set X is a pair $G = (C, D)$, where $C : X \to [0, 1]^m$ is an m-polar fuzzy set on the set of vertices X and $D : X \times X \to [0, 1]^m$ is an m-polar fuzzy relation in X such that

$$D(xy) \le \inf\{C(x), C(y)\}, \qquad \text{for all } x, y \in X.$$

Note that, $D(xy) = \mathbf{0}$, for all $xy \in X \times X - E$, where $\mathbf{0} = (0, 0, \dots, 0)$ and $E \subseteq X \times X$ is the set of edges. C is called an *m-polar fuzzy vertex set* of G and D is an *m-polar fuzzy edge set* of G. An m-polar fuzzy relation D on X is symmetric if $P_i \circ D(xy) = P_i \circ D(yx)$, for all $x, y \in X$.

For further terminologies and studies on m-polar fuzzy hypergraphs, readers are referred to [1–6, 10, 13, 18, 19].

5.2 m-Polar Fuzzy Hypergraphs

Definition 5.5 An m-polar fuzzy hypergraph on a non-empty set X is a pair $H = (A, B)$, where $A = \{\zeta_1, \zeta_2, \dots, \zeta_r\}$ is a family of m-polar fuzzy subsets on X and B is an m-polar fuzzy relation on the m-polar fuzzy subsets ζ_i's such that

1. $B(E_i) = B(\{x_1, x_2, \dots, x_s\}) \le \inf\{\zeta_i(x_1), \zeta_i(x_2), \dots, \zeta_i(x_s)\}$, for all $x_1, x_2, \dots, x_s \in X$.
2. $\bigcup_k supp(\zeta_k) = X$, for all $\xi_k \in A$.

Example 5.2 Let $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$ be a family of 4-polar fuzzy subsets on $X = \{a, b, c, d, e, f, g\}$ given in Table 5.1. Let B be a 4-polar fuzzy relation on ζ_j's, $1 \le j \le 5$, given as, $B(\{a, c, e\}) = (0.2, 0.4, 0.1, 0.3)$, $B(\{b, d, f\}) = (0.2, 0.1, 0.1, 0.1)$, $B(\{a, b\}) = (0.3, 0.1, 0.1, 0.6)$, $B(\{e, f\}) = (0.2, 0.4, 0.3, 0.2)$, $B(\{b, e, g\}) = (0.2, 0.1, 0.2, 0.4)$. Thus, the 4-polar fuzzy hypergraph is shown in Fig. 5.3.

Example 5.3 Consider a 5-polar fuzzy hypergraph with vertex set $X = \{a, b, c, d, e, f, g\}$ whose degrees of membership are given in Table 5.2 and three hyperedges {a, b, c}, {b, d, e}, {b, f, g} such that $B(\{a, b, c\}) = (0.2, 0.1, 0.3, 0.1, 0.2)$,

Table 5.1 4-polar fuzzy subsets on $X = \{a, b, c, d, e, f, g\}$

$x \in X$	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
a	(0.3, 0.4, 0.5, 0.6)	(0, 0, 0, 0)	(0.3, 0.4, 0.5, 0.6)	(0, 0, 0, 0)	(0, 0, 0, 0)
b	(0, 0, 0, 0)	(0.4, 0.1, 0.1, 0.6)	(0.4, 0.1, 0.1, 0.6)	(0, 0, 0, 0)	(0.4, 0.1, 0.1, 0.6)
c	(0.3, 0.5, 0.1, 0.3)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
d	(0, 0, 0, 0)	(0.4, 0.2, 0.5, 0.1)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
e	(0.2, 0.4, 0.6, 0.8)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0.2, 0.4, 0.6, 0.8)	(0.2, 0.4, 0.6, 0.8)
f	(0, 0, 0, 0)	(0.2, 0.5, 0.3, 0.2)	(0, 0, 0, 0)	(0.2, 0.5, 0.3, 0.2)	(0, 0, 0, 0)
g	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0.3, 0.5, 0.1, 0.4)

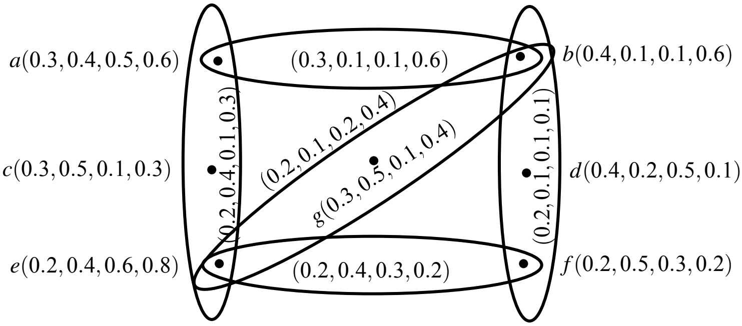


Fig. 5.3 4-polar fuzzy hypergraph

Table 5.2 5-polar fuzzy subsets on X

$x \in X$	ζ_1	ζ_2	ζ_3
a	(0.2, 0.1, 0.3, 0.1, 0.3)	(0, 0, 0, 0, 0)	(0, 0, 0, 0, 0)
b	(0.2, 0.3, 0.5, 0.6, 0.2)	(0.2, 0.3, 0.5, 0.6, 0.2)	(0.2, 0.3, 0.5, 0.6, 0.2)
c	(0.3, 0.2, 0.4, 0.5, 0.2)	(0, 0, 0, 0, 0)	(0, 0, 0, 0, 0)
d	(0, 0, 0, 0, 0)	(0.6, 0.2, 0.2, 0.3, 0.3)	(0, 0, 0, 0, 0)
e	(0, 0, 0, 0, 0)	(0.4, 0.5, 0.6, 0.7, 0.3)	(0, 0, 0, 0, 0)
f	(0, 0, 0, 0, 0)	(0, 0, 0, 0, 0)	(0.1, 0.2, 0.3, 0.4, 0.4)
g	(0, 0, 0, 0, 0)	(0, 0, 0, 0, 0)	(0.2, 0.4, 0.6, 0.8, 0.4)

$B(\{b, d, e\}) = (0.1, 0.2, 0.3, 0.4, 0.2)$, $\quad B(\{b, f, g\}) = (0.2, 0.2, 0.3, 0.3, 0.2)$. Hence, the 5-polar fuzzy hypergraph is shown in Fig. 5.4.

Example 5.4 Let $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$ be a family of 4-polar fuzzy subsets on $X = \{a, b, c, d, e, f, g\}$ as given in Table 5.3. Let B be a 4-polar fuzzy relation on $\zeta_i^{'}$s, $1 \leq i \leq 5$, which is given as follows.

$$B(\{a, c, e\}) = (0.2, 0.4, 0.1, 0.3), \qquad B(\{b, d, f\}) = (0.2, 0.1, 0.1, 0.1),$$
$$B(\{a, b\}) = (0.3, 0.1, 0.1, 0.6), \qquad B(\{e, f\}) = (0.2, 0.4, 0.3, 0.2),$$
$$B(\{b, e, g\}) = (0.2, 0.1, 0.2, 0.4).$$

By routine computations, it is easy to see that $H = (A, B)$ is a 4-polar fuzzy hypergraph as shown in Fig. 5.5.

Definition 5.6 An m-polar fuzzy hypergraph $H = (A, B)$ is called *m-polar fuzzy r-uniform hypergraph* if $|supp(B_i)| = r$ for each $\zeta_i \in B$, $1 \leq i \leq r$.

Example 5.5 Consider $H = (A, B)$ is a 3-polar fuzzy hypergraph as shown in Fig. 5.6, where $A = \{(v_1, 0.1, 0.3, 0.2), (v_2, 0.1, 0.1, 0.3), (v_3, 0.2, 0.1, 0.1),$

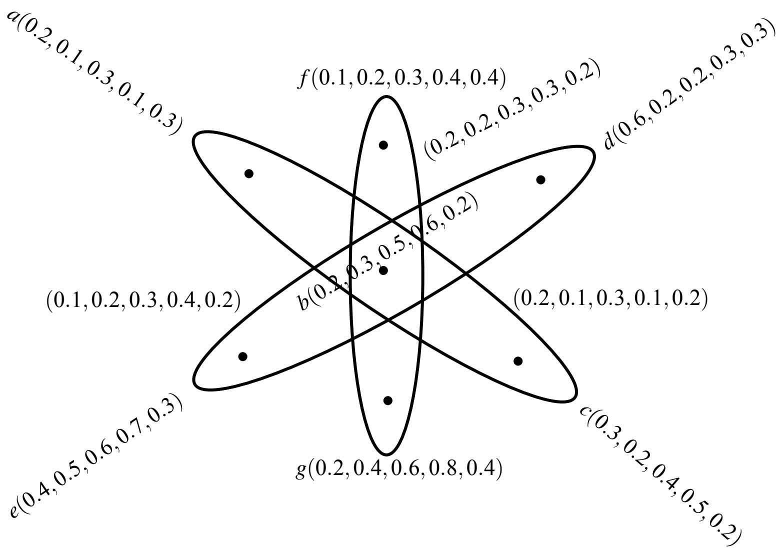


Fig. 5.4 5-polar fuzzy hypergraph

Table 5.3 4-polar fuzzy subsets on $X = \{a, b, c, d, e, f, g\}$

$x \in X$	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
a	(0.4, 0.5, 0.6, 0.7)	(0, 0, 0, 0)	(0.4, 0.5, 0.6, 0.7)	(0, 0, 0, 0)	(0, 0, 0, 0)
b	(0, 0, 0, 0)	(0.3, 0.2, 0.2, 0.7)	(0.3, 0.2, 0.2, 0.7)	(0, 0, 0, 0)	(0.3, 0.2, 0.2, 0.7)
c	(0.4, 0.6, 0.1, 0.4)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
d	(0, 0, 0, 0)	(0.5, 0.3, 0.6, 0.1)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)
e	(0.2, 0.4, 0.6, 0.8)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0.2, 0.4, 0.6, 0.8)	(0.2, 0.4, 0.6, 0.8)
g	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0, 0, 0, 0)	(0.4, 0.6, 0.5, 0.5)

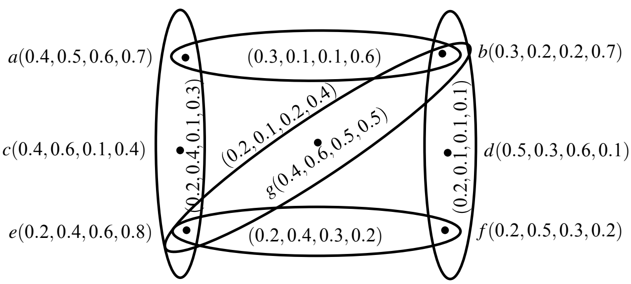


Fig. 5.5 4-polar fuzzy hypergraph

$(v_4, 0.1, 0.1, 0.2)\}$ is a 3-polar fuzzy set of vertices on $X = \{v_1, v_2, v_3, v_4\}$ and the B is defined as $B(\{v_1, v_2\}) = (0.1, 0.1, 0.2)$, $B(\{v_2, v_3\}) = (0.1, 0.1, 0.1)$, $B(\{v_3, v_4\}) = (0.1, 0.1, 0.2)$. Clearly, $|supp(\zeta_i)| = 2$, for each $i = 1, 2, 3$. Thus, $H = (A, B)$ is a 3-polar fuzzy 2-uniform hypergraph, as shown in Fig. 5.6.

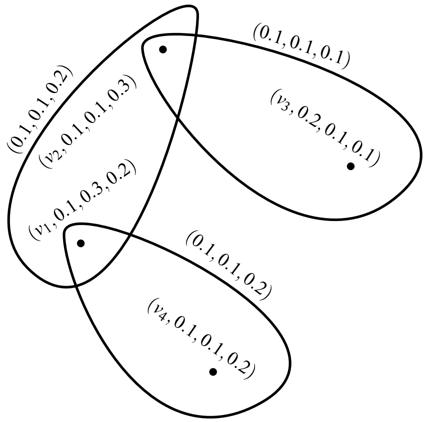


Fig. 5.6 3-polar fuzzy 2-uniform hypergraph

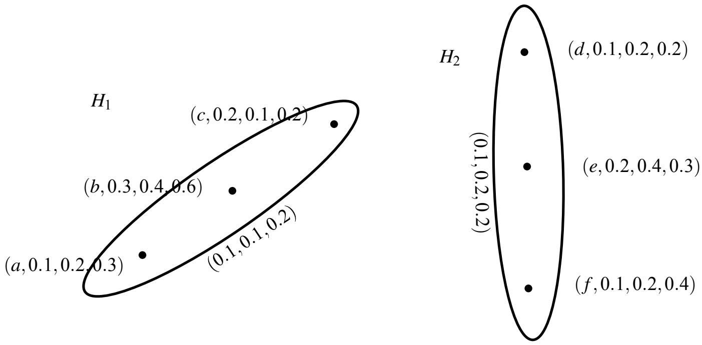


Fig. 5.7 3-polar fuzzy hypergraphs H_1 and H_2

Definition 5.7 Let $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ be two m-polar fuzzy hypergraphs on X_1 and X_2, respectively. The *Cartesian product* of H_1 and H_2 is an ordered pair $H = H_1 \Box H_2 = (A_1 \Box A_2, B_1 \Box B_2)$ such that

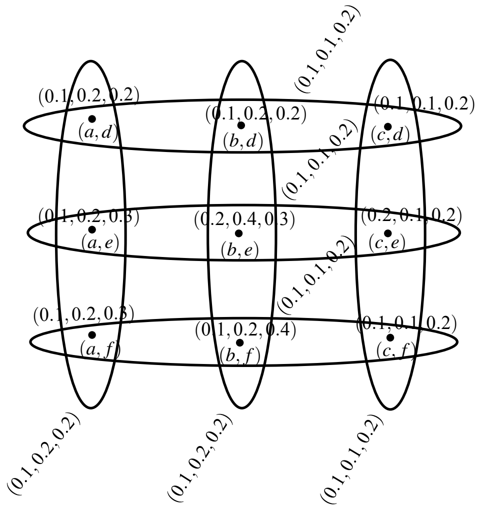
1. $P_i \circ (A_1 \Box A_2)(v_1, v_2) = \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_2)\}, \ \forall \ (v_1, v_2) \in X_1 \times X_2,$
2. $P_i \circ (B_1 \Box B_2)(\{v_1\} \times e_2) = \inf\{P_i \circ A_1(v_1), P_i \circ B_2(e_2)\}, \ \forall \ v_1 \in X_1, \ \forall \ e_2 \in E_2,$
3. $P_i \circ (B_1 \Box B_2)(e_1 \times \{v_2\} = \inf\{P_i \circ B_1(e_1), P_i \circ A_2(v_2)\}, \ \forall \ v_2 \in X_2, \ \forall \ e_1 \in E_1.$

Example 5.6 Let $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ be two 3-polar fuzzy hypergraphs on $X_1 = \{a, b, c\}$ and $X_2 = \{d, e, f\}$, respectively, as shown in Fig. 5.7.

The Cartesian product $H_1 \Box H_2$ is shown in Fig. 5.8.

Theorem 5.1 *If H_1 and H_2 are the m-polar fuzzy hypergraphs then $H_1 \Box H_2$ is as m-polar fuzzy hypergraph.*

Fig. 5.8 Cartesian product $H_1 \square H_2$



Proof Case (i): Let $v_1 \in X_1$, $e_2 = \{v_{21}, v_{22}, \ldots, v_{2q}\} \subseteq X_2$ then for each $1 \leq i \leq m$,

$$\begin{aligned}
&P_i \circ (B_1 \square B_2)(\{v_1\} \times e_2) \\
&\quad = \inf\{P_i \circ A_1(v_1), P_i \circ B_2(e_2)\} \\
&\quad \leq \inf\{P_i \circ A_1(v_1), \inf_{v_2 \in e_2} P_i \circ A_2(v_2)\} \\
&\quad = \inf\{P_i \circ A_1(v_1), \inf\{P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\}\} \\
&\quad = \inf\{\inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{21})\}, \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{22})\}, \\
&\quad \ldots, \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{2q})\}\} \\
&\quad = \inf\{P_i \circ (A_1 \square A_2)(v_1, v_{21}), P_i \circ (A_1 \square A_2)(v_1, v_{22}), \ldots, P_i \circ (A_1 \square A_2)(v_1, v_{2q})\} \\
&\quad = \inf_{v_1 \in e_1, v_2 \in e_2} P_i \circ (A_1 \square A_2)(v_1, v_2).
\end{aligned}$$

Case (ii): Let $v_2 \in X_2$, $e_1 = \{v_{11}, v_{12}, \ldots, v_{1p}\} \subseteq X_1$ then for each $1 \leq i \leq m$,

$$\begin{aligned}
&P_i \circ (B_1 \square B_2)(e_1 \times \{v_1\}) \\
&\quad = \inf\{P_i \circ B_1(e_1), P_i \circ A_2(v_2)\} \\
&\quad \leq \inf\{\inf_{v_1 \in e_1} P_i \circ A_1(v_1), P_i \circ A_2(v_2)\} \\
&\quad = \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_1(v_{12}), \ldots, P_i \circ A_1(v_{1p})\}, P_i \circ A_2(v_2)\} \\
&\quad = \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_2(v_2)\}, \inf\{P_i \circ A_1(v_{12}), P_i \circ A_2(v_2)\}, \\
&\quad \ldots, \inf\{P_i \circ A_1(v_{1p}), P_i \circ A_2(v_2)\}\} \\
&\quad = \inf\{P_i \circ (A_1 \square A_2)(v_{11}, v_2), P_i \circ (A_1 \square A_2)(v_{12}, v_2), \ldots, P_i \circ (A_1 \square A_2)(v_{1p}, v_2)\} \\
&\quad = \inf_{v_1 \in e_1, v_2 \in e_2} P_i \circ (A_1 \square A_2)(v_1, v_2).
\end{aligned}$$

Definition 5.8 Let $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ be two m-polar fuzzy hypergraphs on X_1 and X_2, respectively. Then, the *direct product* of H_1 and H_2 is an ordered pair $H = H_1 \times H_2 = (A_1 \times A_2, B_1 \times B_2)$ such that

1. $P_i \circ (A_1 \times A_2)(v_1, v_2) = \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_2)\}, \quad \forall \quad (v_1, v_2) \in X_1 \times X_2$,
2. $P_i \circ (B_1 \times B_2)(e_1 \times e_2) = \inf\{P_i \circ B_1(e_1), P_i \circ B_2(e_2)\}, \quad \forall\ e_1 \in E_1,\ e_2 \in E_2$.

Definition 5.9 Let $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ be two m-polar fuzzy hypergraphs on X_1 and X_2, respectively, then the *strong product* of H_1 and H_2 is an ordered pair $H = H_1 \boxtimes H_2 = (A_1 \boxtimes A, B_1 \boxtimes B_2)$ such that

1. $P_i \circ (A_1 \boxtimes A_2)(v_1, v_2) = \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_2)\}, \quad \forall\ (v_1, v_2) \in X_1 \times X_2$,
2. $P_i \circ (B_1 \boxtimes B_2)(\{v_1\} \times e_2) = \inf\{P_i \circ A_1(v_1), P_i \circ B_2(e_2)\}, \quad \forall\ v_1 \in X_1,\ \forall\, e_2 \in E_2$,
3. $P_i \circ (B_1 \boxtimes B_2)(e_1 \times \{v_2\}) = \inf\{P_i \circ B_1(e_1), P_i \circ A_2(v_2)\}, \quad \forall\ v_2 \in X_2,\ \forall\, e_1 \in E_1$,
4. $P_i \circ (B_1 \boxtimes B_2)(e_1 \times e_2) = \inf\{P_i \circ B_1(e_1), P_i \circ B_2(e_2)\}, \quad \forall\ e_1 \in E_1,\ e_2 \in E_2$.

Theorem 5.2 *If H_1 and H_2 are two m-polar fuzzy r-uniform hypergraphs, then $H_1 \boxtimes H_2$ is a m-polar fuzzy hypergraph.*

Proof Case (i): Let $v_1 \in X_1$, $e_2 = \{v_{21}, v_{22}, \ldots, v_{2q}\} \subseteq X_2$ then for each $1 \le i \le m$,

$$\begin{aligned}
&P_i \circ (B_1 \boxtimes B_2)(\{v_1\} \times e_2) \\
&\quad = \inf\{P_i \circ A_1(v_1), P_i \circ B_2(e_2)\} \\
&\quad \le \inf\{P_i \circ A_1(v_1), \inf_{v_2 \in e_2} P_i \circ A_2(v_2)\} \\
&\quad = \inf\{P_i \circ A_1(v_1), \inf\{P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\}\} \\
&\quad = \inf\{\inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{21})\}, \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{22})\}, \\
&\quad \ldots, \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{2q})\}\} \\
&\quad = \inf\{P_i \circ (A_1 \boxtimes A_2)(v_1, v_{21}), P_i \circ (A_1 \boxtimes A_2)(v_1, v_{22}), \\
&\quad \ldots, P_i \circ (A_1 \boxtimes A_2)(v_1, v_{2q})\} \\
&\quad = \inf_{v_1 \in e_1, v_2 \in e_2} P_i \circ (A_1 \boxtimes A_2)(v_1, v_2).
\end{aligned}$$

Case (ii): Let $v_2 \in X_2$, $e_1 = \{v_{11}, v_{12}, \ldots, v_{1p}\} \subseteq X_1$ then for each $1 \le i \le m$,

$$\begin{aligned}
&P_i \circ (B_1 \boxtimes B_2)(e_1 \times \{v_1\}) \\
&\quad = \inf\{P_i \circ B_1(e_1), P_i \circ A_2(v_2)\} \\
&\quad \le \inf\{\inf_{v_1 \in e_1} P_i \circ A_1(v_1), P_i \circ A_2(v_2)\} \\
&\quad = \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_1(v_{12}), \ldots, P_i \circ A_1(v_{1p})\}, P_i \circ A_2(v_2)\} \\
&\quad = \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_2(v_2)\}, \inf\{P_i \circ A_1(v_{12}), P_i \circ A_2(v_2)\}, \\
&\quad \ldots, \inf\{P_i \circ A_1(v_{1p}), P_i \circ A_2(v_2)\}\}
\end{aligned}$$

$$= \inf\{P_i \circ (A_1 \boxtimes A_2)(v_{11}, v_2), P_i \circ (A_1 \boxtimes A_2)(v_{12}, v_2),$$
$$\ldots, P_i \circ (A_1 \boxtimes A_2)(v_{1p}, v_2)\}$$
$$= \inf_{v_1 \in e_1, v_2 \in e_2} P_i \circ (A_1 \boxtimes A_2)(v_1, v_2).$$

Case (iii): Let $e_1 = \{v_{11}, v_{12}, \ldots, v_{1p}\} \subseteq X_1$ and $e_2 = \{v_{21}, v_{22}, \ldots, v_{2q}\} \subseteq X_2$ then for each $1 \leq i \leq m$,

$$P_i \circ (B_1 \boxtimes B_2)(e_1 \times e_2)$$
$$= \inf\{P_i \circ B_1(e_1), P_i \circ B_2(e_2)\}$$
$$\leq \inf\{\inf_{v_1 \in e_1} P_i \circ A_1(v_1), \inf_{v_2 \in e_2} P_i \circ A_2(v_2)\}$$
$$= \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_1(v_{12}), \ldots, P_i \circ A_1(v_{1p})\}$$
$$, \inf\{P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\}\}$$
$$= \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_2(v_{21})\}, \inf\{P_i \circ A_1(v_{12}), P_i \circ A_2(v_{22})\}$$
$$, \ldots, \inf\{P_i \circ A_1(v_{1p}), P_i \circ A_2(v_{2q})\}\}$$
$$= \inf\{P_i \circ (A_1 \boxtimes A_2)(v_{11}, v_{21}), P_i \circ (A_1 \boxtimes A_2)(v_{12}, v_{22}), \ldots, P_i \circ (A_1 \boxtimes A_2)(v_{1p}, v_{2q})\}$$
$$= \inf_{v_1 \in e_1, v_2 \in e_2} P_i \circ (A_1 \boxtimes A_2)(v_1, v_2).$$

Definition 5.10 Let $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ be two m-polar fuzzy hypergraphs on X_1 and X_2, respectively, then composition of H_1 and H_2 is an ordered pair $H = H_1 \diamond H_2 = (A_1 \diamond A_2, B_1 \diamond B_2)$ such that,

1. $P_i \circ (A_1 \diamond A_2)(v_1, v_2) = \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_2)\}, \ \forall \ (v_1, v_2) \in X_1 \times X_2$,
2. $P_i \circ (B_1 \diamond B_2)(\{v_1\} \times e_2) = \inf\{P_i \circ A_1(v_1), P_i \circ B_2(e_2)\}, \quad \forall \quad v_1 \in X_1, \ \forall \ e_2 \in E_2$,
3. $P_i \circ (B_1 \diamond B_2)(e_1 \times \{v_2\} = \inf\{P_i \circ B_1(e_1), P_i \circ A_2(v_2)\}, \quad \forall \quad v_2 \in X_2, \ \forall \ e_1 \in E_1$,
4. $P_i \circ (B_1 \diamond B_2)((v_{11}, v_{21})(v_{12}, v_{22}) \cdots (v_{1p}, v_{2q})) = \inf\{P_i \circ B_1(e_1), P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\}, \ \forall \ e_1 \in E_1, v_{21}, v_{22}, \ldots, v_{2q} \in X_2$.

Theorem 5.3 *If H_1 and H_2 are two m-polar fuzzy hypergraphs, then $H_1 \diamond H_2$ is a m-polar fuzzy hypergraph.*

Proof Case(i): Let $v_1 \in X_1, e_2 \subseteq X_2$ then for each $1 \leq i \leq m$,

$$P_i \circ (B_1 \diamond B_2)(\{v_1\} \times e_2)$$
$$= \inf\{P_i \circ A_1(v_1), P_i \circ B_2(e_2)\}$$
$$\leq \inf\{P_i \circ A_1(v_1), \inf_{v_2 \in e_2} P_i \circ A_2(v_2)\}$$
$$= \inf\{P_i \circ A_1(v_1), \inf\{P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\}\}$$
$$= \inf\{\inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{21})\}, \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{22})\}$$
$$, \ldots, \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{2q})\}\}$$
$$= \inf\{P_i \circ (A_1 \diamond A_2)(v_1, v_{21}), P_i \circ (A_1 \diamond A_2)(v_1, v_{22})$$

$$
\begin{aligned}
&, \ldots, P_i \circ (A_1 \diamond A_2)(v_1, v_{2q})\} \\
&= \inf_{v_1 \in e_1, v_2 \in e_2} P_i \circ (A_1 \diamond A_2)(v_1, v_2).
\end{aligned}
$$

Case(ii): Let $v_2 \in X_2$, $e_1 \subseteq X_1$ then for each $1 \leq i \leq m$,

$$
\begin{aligned}
&P_i \circ (B_1 \diamond B_2)(e_1 \times \{v_1\}) \\
&= \inf\{P_i \circ B_1(e_1), P_i \circ A_2(v_2)\} \\
&\leq \inf\{\inf_{v_1 \in e_1} P_i \circ A_1(v_1), P_i \circ A_2(v_2)\} \\
&= \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_1(v_{12}), \ldots, P_i \circ A_1(v_{1p})\}, P_i \circ A_2(v_2)\} \\
&= \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_2(v_2)\}, \inf\{P_i \circ A_1(v_{12}), P_i \circ A_2(v_2)\} \\
&\quad, \ldots, \inf\{P_i \circ A_1(v_{1p}), P_i \circ A_2(v_2)\}\} \\
&= \inf\{P_i \circ (A_1 \diamond A_2)(v_{11}, v_2), P_i \circ (A_1 \diamond A_2)(v_{12}, v_2) \\
&\quad, \ldots, P_i \circ (A_1 \diamond A_2)(v_{1p}, v_2)\} \\
&= \inf_{v_1 \in e_1, v_2 \in e_2} P_i \circ (A_1 \diamond A_2)(v_1, v_2).
\end{aligned}
$$

Case(iii): Let $e_1 = \{v_{11}, v_{12}, \ldots, v_{1p}\} \subseteq X_1, v_{21}, v_{22}, \ldots, v_{2q} \in X_2$ then for each $1 \leq i \leq m$,

$$
\begin{aligned}
&P_i \circ (B_1 \diamond B_2)((v_{11}, v_{21})(v_{12}, v_{22}) \cdots (v_{1p}, v_{2q})) \\
&= \inf\{P_i \circ B_1(e_1), P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\} \\
&\leq \inf\{\inf_{v_1 \in e_1} P_i \circ A_1(v_1), P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\} \\
&= \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_1(v_{12}), \ldots, P_i \circ A_1(v_{1p})\} \\
&\quad, P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\} \\
&= \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_2(v_{21})\}, \inf\{P_i \circ A_1(v_{12}), P_i \circ A_2(v_{22})\} \\
&\quad, \ldots, \inf\{P_i \circ A_1(v_{1p}), P_i \circ A_2(v_{2q})\}\} \\
&= \inf\{P_i \circ (A_1 \diamond A_2)(v_{11}, v_{21}), P_i \circ (A_1 \diamond A_2)(v_{12}, v_{22}) \\
&\quad, \ldots, P_i \circ (A_1 \diamond A_2)(v_{1p}, v_{2q})\} \\
&= \inf_{v_1 \in e_1, v_2 \in e_2} P_i \circ (A_1 \diamond A_2)(v_1, v_2).
\end{aligned}
$$

Definition 5.11 Let $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ be two m-polar fuzzy hypergraphs on X_1 and X_2, respectively, then the *union* of H_1 and H_2 is an ordered pair $H = H_1 \cup H_2 = (A_1 \cup A_2, B_1 \cup B_2)$ such that

1. $P_i \circ (A_1 \cup A_2)(v) = \begin{cases} P_i \circ A_1(v), & \text{if } v \in X_1 - X_2, \\ P_i \circ A_2(v), & \text{if } v \in X_2 - X_1, \\ \sup\{P_i \circ A_1(v), P_i \circ A_2(v)\}, & \text{if } v \in X_1 \cap X_2. \end{cases}$
2. $P_i \circ (B_1 \cup B_2)(e) = \begin{cases} P_i \circ B_1(e), & \text{if } e \in E_1 - E_2, \\ P_i \circ B_2(e), & \text{if } e \in E_2 - E_1, \\ \sup\{P_i \circ B_1(e), P_i \circ B_2(e)\}, & \text{if } e \in E_1 \cap E_2. \end{cases}$

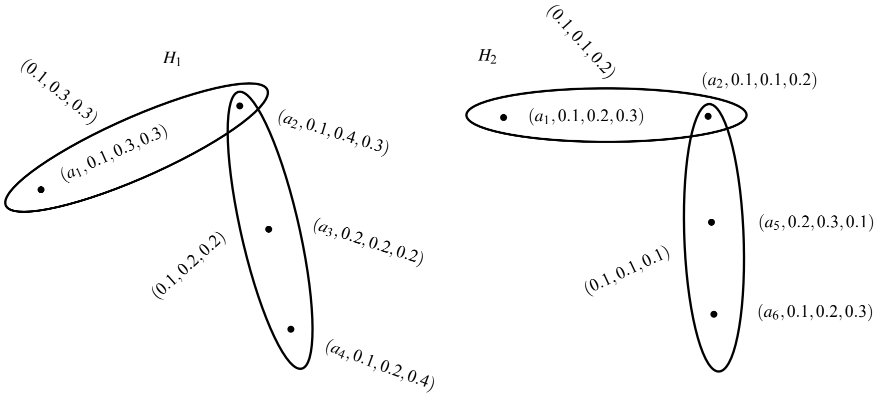
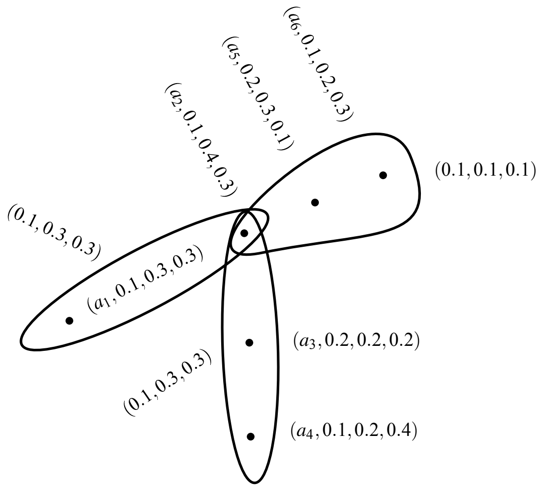


Fig. 5.9 3-polar fuzzy hypergraphs H_1 and H_2

Fig. 5.10 $H_1 \cup H_2$



where $E_1 = supp(B_1)$ and $E_2 = supp(B_2)$.

Example 5.7 Consider 3-polar fuzzy hypergraphs $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ as shown in Fig. 5.9.

The union of H_1 and H_2 is given in Fig. 5.10.

Theorem 5.4 *The union $H_1 \cup H_2 = (A_1 \cup A_2, B_1 \cup B_2)$ of two m-polar fuzzy hypergraphs $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ is an m-polar fuzzy hypergraph.*

Proof Let $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ be two m-polar fuzzy hypergraphs on X_1 and X_2, respectively, such that $E_1 = supp(B_1)$ and $E_2 = supp(B_2)$. It is to be shown that that $H_1 \cup H_2 = (A_1 \cup A_2, B_1 \cup B_2)$ is an m-polar fuzzy hypergraph.

Since, all conditions for $A_1 \cup A_2$ are satisfied automatically, therefore, it is enough to show that $B_1 \cup B_2$ is an m-polar fuzzy relation on $A_1 \cup A_2$.
Case(i): If $e \in E_1 - E_2$ then for each $1 \leq i \leq m$,

$$\begin{aligned}
P_i \circ (B_1 \cup B_2)(e) &= P_i \circ B_1(e_1) \\
&\leq \inf_{v_1 \in e_1} PioA_1(v_1) \\
&= \inf\{P_i \circ A_1(v_{11}), P_i \circ A_1(v_{12}), \ldots, P_i \circ A_1(v_{1p})\} \\
&= \inf\{P_i \circ (A_1 \cup A_2)(v_{11}), P_i \circ (A_1 \cup A_2)(v_{12}), \ldots, P_i \circ (A_1 \cup A_2)(v_{1p})\}.
\end{aligned}$$

Case(ii): If $e \in E_2 - E_1$ then for each $1 \leq i \leq m$,

$$\begin{aligned}
P_i \circ (B_1 \cup B_2)(e) &= P_i \circ B_2(e_2) \\
&\leq \inf_{v_2 \in e_2} PioA_2(v_2) \\
&= \inf\{P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\} \\
&= \inf\{P_i \circ (A_1 \cup A_2)(v_{21}), P_i \circ (A_1 \cup A_2)(v_{22}), \ldots, P_i \circ (A_1 \cup A_2)(v_{2q})\}.
\end{aligned}$$

Case(iii): If $e \in E_1 \cap E_2$ or $v_{j1}, v_{j2}, \ldots, v_{jp} \in X_1 \cap X_2$ then for each $1 \leq i \leq m$,

$$\begin{aligned}
P_i \circ (B_1 \cup B_2)(e) &= \sup\{P_i \circ B_1(e), P_i \circ B_2(e)\} \\
&\leq \sup\{\inf\{P_i \circ A_1(v_{j1}), P_i \circ A_1(v_{j2}), \ldots, P_i \circ A_1(v_{jp})\} \\
&\quad , \inf\{P_i \circ A_2(v_{j1}), P_i \circ A_2(v_{j2}), \ldots, P_i \circ A_2(v_{jp})\}\} \\
&= \inf\{\sup\{P_i \circ A_1(v_{j1}), P_i \circ A_2(v_{j1})\}, \sup\{P_i \circ A_1(v_{j2}), P_i \circ A_2(v_{j2})\} \\
&\quad , \ldots, \sup\{P_i \circ A_1(v_{jp}), P_i \circ A_2(v_{jp})\} \\
&= \inf\{P_i \circ (A_1 \cup A_2)(v_{11}), P_i \circ (A_1 \cup A_2)(v_{12}), \ldots, P_i \circ (A_1 \cup A_2)(v_{1p})\}.
\end{aligned}$$

Definition 5.12 Let $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ be two m-polar fuzzy hypergraphs on X_1 and X_2, respectively, then the *join* $H = H_1 + H_2$ of two m-polar fuzzy hypergraphs H_1 and H_2 is defined as follows:

1. $P_i \circ (A_1 + A_2)(v) = P_i \circ (A_1 \cup A_2)(v)$, if $v \in X_1 \cup X_2$,
2. $P_i \circ (B_1 + B_2)(e) = P_i \circ (B_1 \cup B_2)(e)$, if $e \in E_1 \cup E_2$,
3. $P_i \circ (B_1 + B_2)(e) = \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_2)\}$, if $e \in E'$,

where E' is the set of all the edges joining the vertices of X_1 and X_2 and $X_1 \cap X_2 = \emptyset$.

Example 5.8 Consider $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ be two 3-polar fuzzy hypergraphs as shown in Fig. 5.11 then their join is given in Fig. 5.12.

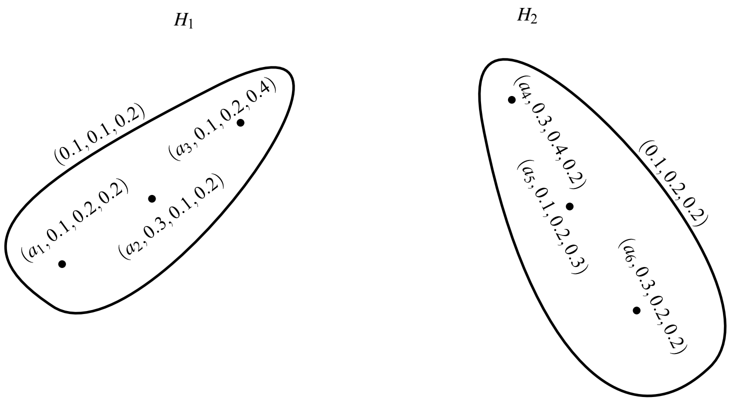
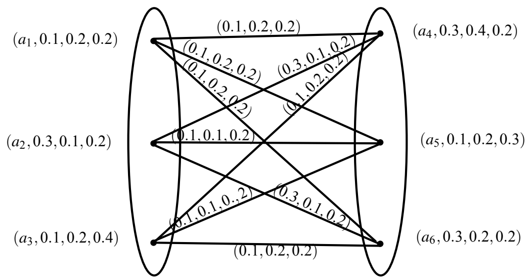


Fig. 5.11 3-polar fuzzy hypergraphs H_1 and H_2

Fig. 5.12 $H_1 + H_2$



Definition 5.13 Let $H_1 = (A_1, B_1)$ and $H_2 = (A_2, B_2)$ be two m-polar fuzzy hypergraphs on X_1 and X_2, respectively, then the *lexicographic product* of H_1 and H_2 is defined by the ordered pair $H = H_1 \bullet H_2 = (A_1 \bullet A_2, B_1 \bullet B_2)$ such that

1. $P_i \circ (A_1 \bullet A_2)(v_1, v_2) = \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_2)\}, \ \forall \ (v_1, v_2) \in X_1 \times X_2,$
2. $P_i \circ (B_1 \bullet B_2)(\{v_1\} \times e_2) = \inf\{P_i \circ A_1(v_1), P_i \circ B_2(e_2)\}, \ \forall \ v_1 \in X_1, \ \forall \ e_2 \in E_2,$
3. $P_i \circ (B_1 \bullet B_2)(e_1 \times e_2) = \inf\{P_i \circ B_1(e_1), P_i \circ B_2(v_2)\}, \ \forall \ e_1 \in E_1, \ \forall \ e_2 \in E_2.$

Theorem 5.5 *If H_1 and H_2 are m-polar fuzzy hypergraphs then $H_1 \bullet H_2$ is an m-polar fuzzy hypergraph.*

Proof Case(i): Let $v_1 \in X_1$, $e_2 = \{v_{21}, v_{22}, \ldots, v_{2q}\} \subseteq X_2$ then for each $1 \leq i \leq m$,

$$
\begin{aligned}
&P_i \circ (B_1 \bullet B_2)(\{v_1\} \times e_2)\\
&= \inf\{P_i \circ A_1(v_1), P_i \circ B_2(e_2)\}\\
&\leq \inf\{P_i \circ A_1(v_1), \inf_{v_2 \in e_2} P_i \circ A_2(v_2)\}\\
&= \inf\{P_i \circ A_1(v_1), \inf\{P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\}\}\\
&= \inf\{\inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{21})\}, \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{22})\}\\
&\quad, \ldots, \inf\{P_i \circ A_1(v_1), P_i \circ A_2(v_{2q})\}\}\\
&= \inf\{P_i \circ (A_1 \bullet A_2)(v_1, v_{21}), P_i \circ (A_1 \bullet A_2)(v_1, v_{22}), \ldots, P_i \circ (A_1 \bullet A_2)(v_1, v_{2q})\}\\
&= \inf_{v_1 \in e_1, v_2 \in e_2} P_i \circ (A_1 \bullet A_2)(v_1, v_2).
\end{aligned}
$$

Case(ii): Let $e_1 = \{v_{11}, v_{12}, \ldots, v_{1p}\} \subseteq X_1, e_2 = \{v_{21}, v_{22}, \ldots, v_{2q}\} \subseteq X_2$ then for each $1 \leq i \leq m$,

$$
\begin{aligned}
&P_i \circ (B_1 \bullet B_2)(e_1 \times e_2)\\
&= \inf\{P_i \circ B_1(e_1), P_i \circ B_2(e_2)\}\\
&\leq \inf\{\inf_{v_1 \in e_1} P_i \circ A_1(v_1), \inf_{v_2 \in e_2} P_i \circ A_2(v_2)\}\\
&= \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_1(v_{12}), \ldots, P_i \circ A_1(v_{1p})\}\\
&, \inf\{P_i \circ A_2(v_{21}), P_i \circ A_2(v_{22}), \ldots, P_i \circ A_2(v_{2q})\}\}\\
&= \inf\{\inf\{P_i \circ A_1(v_{11}), P_i \circ A_2(v_{21})\}, \inf\{P_i \circ A_1(v_{12}), P_i \circ A_2(v_{22})\}\\
&\quad, \ldots, \inf\{P_i \circ A_1(v_{1p}), P_i \circ A_2(v_{2q})\}\}\\
&= \inf\{P_i \circ (A_1 \bullet A_2)(v_{11}, v_{21}), P_i \circ (A_1 \bullet A_2)(v_{12}, v_{22})\\
&\quad, \ldots, P_i \circ (A_1 \bullet A_2)(v_{1p}, v_{2q})\}\\
&= \inf_{v_1 \in e_1, v_2 \in e_2} P_i \circ (A_1 \bullet A_2)(v_1, v_2).
\end{aligned}
$$

Definition 5.14 Let $H = (A, B)$ be an m-polar fuzzy hypergraph on a non-empty set X. The dual m-polar fuzzy hypergraph of H, denoted by $H^D = (A^*, B^*)$, is defined as

1. $A^* = B$ is the m-polar fuzzy set of vertices of H^D.
2. If $|X| = n$ then, B^* is an m-polar fuzzy set on the family of hyperedges $\{X_1, X_2, \ldots, X_n\}$ such that, $X_i = \{E_j \mid x_j \in E_j, E_j$ is a hyperedge of $H\}$, i.e., X_i is the m-polar fuzzy set of those hyperedges which share the common vertex x_i and $B^*(X_i) = \inf\{E_j \mid x_j \in E_j\}$.

Example 5.9 Consider the example of a 3-polar fuzzy hypergraph $H = (A, B)$ given in Fig. 5.13, where $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $E = \{E_1, E_2, E_3, E_4\}$. The dual 3-polar fuzzy hypergraph is shown in Fig. 5.14 with dashed lines with vertex set $E = \{E_1, E_2, E_3, E_4\}$ and set of hyperedges $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ such that $X_1 = X_3$.

Definition 5.15 The *open neighborhood* of a vertex x in an m-polar fuzzy hypergraph is the set of adjacent vertices of x excluding that vertex and it is denoted by $N(x)$.

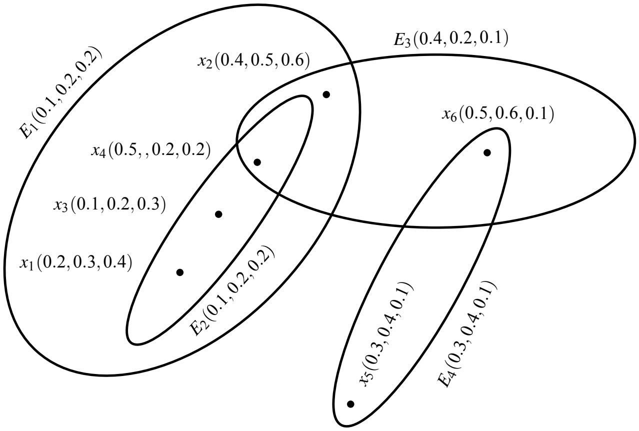


Fig. 5.13 3-polar fuzzy hypergraph

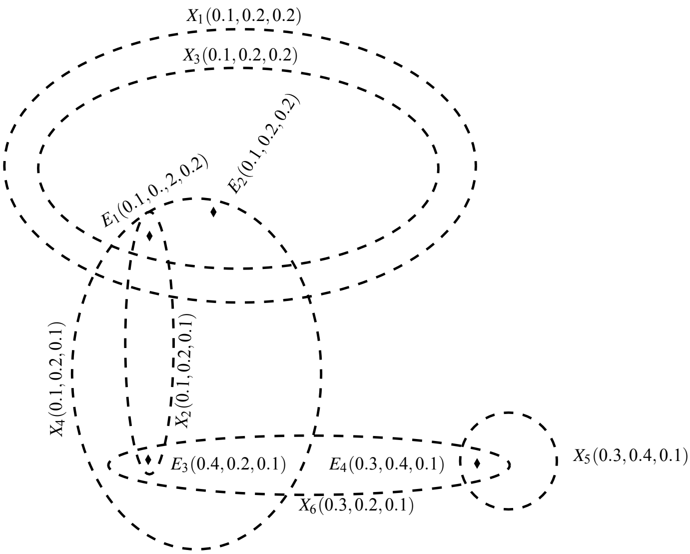


Fig. 5.14 Dual 3-polar fuzzy hypergraph

Example 5.10 Consider the 3-polar fuzzy hypergraph $H = (A, B)$, where $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ is a family of 3-polar fuzzy subsets on $X = \{a, b, c, d, e\}$ and B is a 3polar fuzzy relation on the 3-polar fuzzy subsets ζ_i's such that $\zeta_1 = \{(a, 0.3, 0.4, 0.5), (b, 0.2, 0.4, 0.6)\}, \zeta_2 = \{(c, 0.2, 0.1, 0.4), (d, 0.5, 0.1, 0.1), (e, 0.2, 0.3, 0.1)\}$, $\zeta_3 = \{(b, 0.1, 0.2, 0.4), (c, 0.4, 0.5, 0.6)\}$, $\zeta_4 = \{(a, 0.1, 0.3, 0.2), (d, 0.3, 0.4, 0.4)\}$. In this example, the open neighborhood of the vertex a is $\{b, d\}$ as shown in Fig. 5.15.

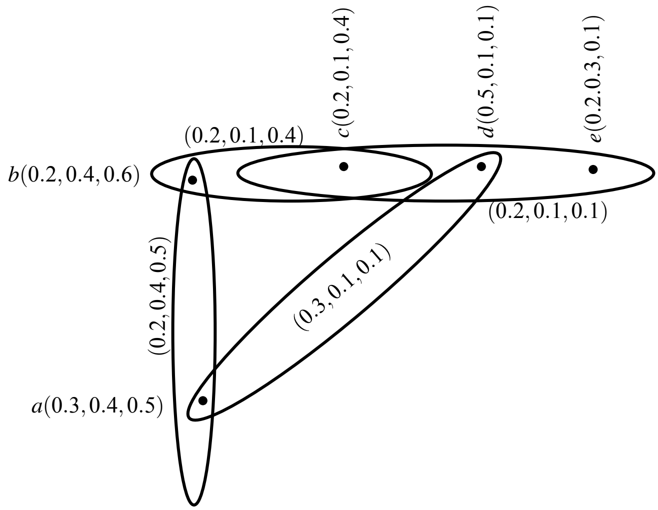


Fig. 5.15 3-polar fuzzy hypergraph

Definition 5.16 The *closed neighborhood* of a vertex x in an m-polar fuzzy hypergraph is the set of adjacent vertices of x including x and it is denoted by $N[x]$.

Example 5.11 Consider a 3-polar fuzzy hypergraph $H = (A, B)$ as shown in Fig. 5.15. In this example, closed neighborhood of the vertex a is $\{a, b, d\}$.

Definition 5.17 The *open neighborhood degree* of a vertex x in H is denoted by $deg(x)$ and defined as an m-tuple $deg(x) = (deg^{(1)}(x), deg^{(2)}(x), deg^{(3)}(x), \ldots, deg^{(m)}(x))$, such that

$$deg^{(1)}(x) = \Sigma_{x \in N(x)} P_1 \circ \zeta_j(x),$$

$$deg^{(2)}(x) = \Sigma_{x \in N(x)} P_2 \circ \zeta_j(x),$$

$$deg^{(3)}(x) = \Sigma_{x \in N(x)} P_3 \circ \zeta_j(x),$$

$$\vdots$$

$$deg^{(m)}(x) = \Sigma_{x \in N(x)} P_m \circ \zeta_j(x).$$

Definition 5.18 Let $H = (A, B)$ be an m-polar fuzzy hypergraph on a non-empty set X. If all vertices in A have the same open neighborhood degree n, then H is called *n-regular m*-polar fuzzy hypergraph.

Definition 5.19 The *closed neighborhood degree* of a vertex x in H is denoted by $deg[x]$ and defined as an m-tuple such that $deg[x] = (deg^{(1)}[x], deg^{(2)}[x],$

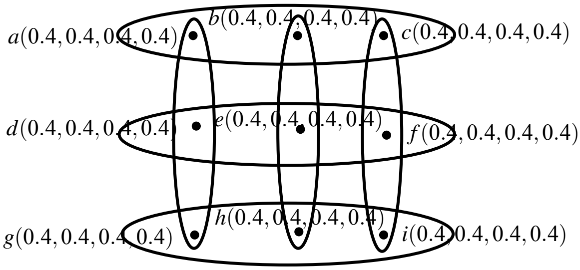


Fig. 5.16 Regular and totally regular 4-polar fuzzy hypergraph

$deg^{(3)}[x], \ldots, deg^{(m)}[x])$, where

$$deg^{(1)}[x] = deg^{(1)}(x) + \wedge_j P_1 \circ \zeta_j(x),$$

$$deg^{(2)}[x] = deg^{(2)}(x) + \wedge_j P_2 \circ \zeta_j(x),$$

$$deg^{(3)}[x] = deg^{(3)}(x) + \wedge_j P_3 \circ \zeta_j(x),$$

$$\vdots$$

$$deg^{(m)}[x] = d_G^{(m)}(x) + \wedge_j P_m \circ \zeta_j(x).$$

Example 5.12 Consider the example of a 3-polar fuzzy hypergraph $H = (A, B)$, where $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ is a family of 3-polar fuzzy subsets on $X = \{a, b, c, d, e\}$ and B is a 3-polar fuzzy relation on the 3-polar fuzzy subsets ζ_j, where $\zeta_1 = \{(a, 0.3, 0.4, 0.5), (b, 0.2, 0.4, 0.6)\}$, $\zeta_2 = \{(c, 0.2, 0.1, 0.4), (d, 0.5, 0.1, 0.1), (e, 0.2, 0.3, 0.1)\}, \zeta_3 = \{(b, 0.1, 0.2, 0.4), (c, 0.4, 0.5, 0.6)\}, \zeta_4 = \{(a, 0.1, 0.3, 0.2), (d, 0.3, 0.4, 0.4)\}$. Then, $deg(a) = (0.5, 0.8, 1)$ and $deg[a] = (0.6, 1.1, 1.2)$.

Definition 5.20 Let $H = (A, B)$ be an m-polar fuzzy hypergraph on X. If all vertices in A have the same closed neighborhood degree m, then H is called *m-totally regular* m-polar fuzzy hypergraph.

Example 5.13 Consider the 3-polar fuzzy hypergraph $H = (A, B)$, where $A = \{\zeta_1, \zeta_2, \zeta_3\}$ is a family of 3-polar fuzzy subsets on $X = \{a, b, c, d, e\}$ and B is a 3-polar fuzzy relation on the 3-polar fuzzy subsets ζ_j such

$\zeta_1 = \{(a, 0.5, 0.4, 0.1), (b, 0.3, 0.4, 0.1), (c, 0.4, 0.4, 0.3)\}$,
$\zeta_2 = \{(a, 0.3, 0.1, 0.1), (d, 0.2, 0.3, 0.2), (e, 0.4, 0.6, 0.1)\}$,
$\zeta_3 = \{(b, 0.3, 0.4, 0.3), (d, 0.4, 0.3, 0.4), (e, 0.4, 0.3, 0.1)\}$.

By routine calculations, it easy to see that the H is neither regular nor totally regular 3-polar fuzzy graph.

Example 5.14 The 4-polar fuzzy hypergraph shown in Fig. 5.16 is both regular and totally regular.

Remark 5.1 (a) For an m-polar fuzzy hypergraph $H = (A, B)$ to be both regular and totally regular, the number of vertices in each hyperedge E_j must be same. Suppose that $|E_j| = k$ for every j, then H is said to be k-uniform.

(b) Each vertex lies in exactly same number of hyperedges.

Definition 5.21 Let $H = (A, B)$ be a regular m-polar fuzzy hypergraph. The *order* of a regular m-polar fuzzy hypergraph H is an m-tuple of the form,

$$O(H) = (\Sigma_{x\in X} \wedge P_1 \circ \zeta_j(x), \Sigma_{x\in X} \wedge P_2 \circ \zeta_j(x), \ldots, \Sigma_{x\in X} \wedge P_m \circ \zeta_j(x)).$$

The *size* of a regular m-polar fuzzy hypergraph is $S(H) = \sum_{E_j\subseteq X} B(E_j)$.

Example 5.15 Consider the 4-polar fuzzy hypergraph $H = (A, B)$ on $X = \{a, b, c, d, e, f, g, h, i\}$ and $A = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6\}$, where

$\zeta_1 = \{(a, 0.4, 0.4, 0.4, 0.4), (b, 0.4, 0.4, 0.4, 0.4), (c, 0.4, 0.4, 0.4, 0.4)\}$,
$\zeta_2 = \{(d, 0.4, 0.4, 0.4, 0.4), (e, 0.4, 0.4, 0.4, 0.4), (f, 0.4, 0.4, 0.4, 0.4)\}$,
$\zeta_3 = \{(g, 0.4, 0.4, 0.4, 0.4), (h, 0.4, 0.4, 0.4, 0.4), (i, 0.4, 0.4, 0.4, 0.4)\}$,
$\zeta_4 = \{(a, 0.4, 0.4, 0.4, 0.4), (d, 0.4, 0.4, 0.4, 0.4), (g, 0.4, 0.4, 0.4, 0.4)\}$,
$\zeta_5 = \{(b, 0.4, 0.4, 0.4, 0.4), (e, 0.4, 0.4, 0.4, 0.4), (h, 0.4, 0.4, 0.4, 0.4)\}$,
$\zeta_6 = \{(c, 0.4, 0.4, 0.4, 0.4), (f, 0.4, 0.4, 0.4, 0.4), (i, 0.4, 0.4, 0.4, 0.4)\}$.

Clearly, $O(H) = (3.6, 3.6, 3.6, 3.6)$ and $S(H) = (7.2, 7.2, 7.2, 7.2)$.

Theorem 5.6 *Let $H = (A, B)$ be an m-polar fuzzy hypergraph on X. Then, $A : X \longrightarrow [0, 1]^m$ is a constant function if and only if the following statements are equivalent,*

(a) H is a regular m-polar fuzzy hypergraph,

(b) H is a totally regular m-polar fuzzy hypergraph.

Proof Suppose that $A : X \longrightarrow [0, 1]^m$, where $A = \{\zeta_1, \zeta_2, ..., \zeta_r\}$ is a constant function. That is, $P_i \circ \zeta_j(x) = c_i$, for all $x \in \zeta_j$, $1 \le i \le m$, $1 \le j \le r$.

$(a) \Rightarrow (b)$ Suppose that H is n-regular m-polar fuzzy hypergraph. Then $deg^{(i)}(x) = n_i$, for all $x \in X$, $1 \le i \le m$. By using Definition 5.19, $deg^{(i)}[x] = n_i + k_i$, for all $x \in X$, $1 \le i \le m$. Hence, H is a totally regular m-polar fuzzy hypergraph.

$(b) \Rightarrow (a)$ Suppose that H is a k-totally regular m-polar fuzzy hypergraph. Then, $deg^{(i)}[x] = k_i$, for all $x \in X$, $1 \le i \le m$.

$\Rightarrow deg^{(i)}(x) + \wedge_j P_i \circ \zeta_j(x) = k_i$ for all $x \in \zeta_j$,

$\Rightarrow deg^{(i)}(x) + c_i = k_i$, for all $x \in \zeta_j$,

$\Rightarrow deg^{(i)}(x) = k_i - c_i$, for all $x \in \zeta_j$. Thus, H is a regular m-polar fuzzy hypergraph. Hence, (a) and (b) are equivalent.

Conversely, suppose that (a) and (b) are equivalent, i.e., H is regular if and only if H is a totally regular. On contrary suppose that A is not constant, that is, $P_i \circ \zeta_j(x) \ne P_i \circ \zeta_j(y)$ for some x and y in A. Let $H = (A, B)$ be a n-regular m-polar fuzzy hypergraph then, $deg^{(i)}(x) = n_i$ for all $x \in \zeta_j(x)$. Consider,

$$deg^{(i)}[x] = deg^{(i)}(x) + \wedge_j P_i \circ \zeta_j(x) = n_i + \wedge_j P_i \circ \zeta_j(x),$$
$$deg^{(i)}[y] = deg^{(i)}(y) + \wedge_j P_i \circ \zeta_j(y) = n_i + \wedge_j P_i \circ \zeta_j(y).$$

Since, $P_i \circ \zeta_j(x)$ and $P_i \circ \zeta_j(y)$ are not equal for some x and y in X, hence $deg[x]$ and $deg[y]$ are not equals, thus H is not a totally regular m-poalr fuzzy hypergraph, which is a contradiction to our assumption. Next, let H be a totally regular m-polar fuzzy hypergraph, then $deg[x] = deg[y]$, that is,

$$deg^{(i)}(x) + \wedge_j P_i \circ \zeta_j(x) = deg^{(i)}(y) + \wedge_j P_i \circ \zeta_j(y),$$
$$deg^{(i)}(x) - deg^{(i)}(y) = \wedge_j P_i \circ \zeta_j(y) - \wedge_j P_i \circ \zeta_j(x).$$

It follows that $deg(x)$ and $deg(y)$ are not equal, so H is not a regular m-polar fuzzy hypergraph, which is again a contradiction to our assumption. Hence, A must be constant and it completes the proof.

Theorem 5.7 *If an m-polar fuzzy hypergraph is both regular and totally regular then $A : X \longrightarrow [0, 1]^m$ is constant function.*

Proof Let H be a regular and totally regular m-polar fuzzy hypergraph then,

$$deg^{(i)}(x) = n_i \text{ for all } x \in X, 1 \le i \le m.$$
$$deg^{(i)}[x] = k_i \text{ for all } x \in \zeta_j(x),$$
$$\Leftrightarrow deg^{(i)}(x) + \wedge_j P_i \circ \zeta_j(x) = k_i, \text{ for all } x \in \zeta_j(x),$$
$$\Leftrightarrow n_1 + \wedge_j P_i \circ \zeta_j(x) = k_i, \text{ for all } x \in \zeta_j(x),$$
$$\Leftrightarrow \wedge_j P_i \circ \zeta_j(x) = k_i - n_i, \text{ for all } x \in \zeta_j(x),$$
$$\Leftrightarrow P_i \circ \zeta_j(x) = k_i - n_i, \text{ for all } x \in X, 1 \le i \le m.$$

Hence, $A : X \longrightarrow [0, 1]^m$ is a constant function.

Remark 5.2 The converse of Theorem 5.7 may not be true, in general as it can be seen in the following example.

Consider a 3-polar fuzzy hypergraph $H = (A, B)$ on $X = \{a, b, c, d, e\}$,
$\zeta_1 = \{(a, 0.2, 0, 2, 0.2), (b, 0.2, 0.2, 0.2), (c, 0.2, 0.2, 0.2)\}$,
$\zeta_2 = \{(a, 0.2, 0, 2, 0.2), (d, 0.2, 0.2, 0.2)\}$,
$\zeta_3 = \{(b, 0.2, 0.2, 0.2), (e, 0.2, 0.2, 0.2)\}$,
$\zeta_4 = \{(c, 0.2, 0.2, 0.2), (e, 0.2, 0.2, 0.2)\}$. Then, $A : X \longrightarrow [0, 1]^m$, where $A = \{\zeta_1, \zeta_2, ..., \zeta_r\}$ is a constant function. But $deg(a) = (0.6, 0.6, 0.6) \neq (0.4, 0.4, 0.4) = deg(e)$. Also $(deg[a] = (0.8, 0.8, 0.8) \neq (0.6, 0.6, 0.6) = deg[e])$. So H is neither regular nor totally regular m-polar fuzzy hypergraph.

Definition 5.22 An m-polar fuzzy hypergraph $H = (A, B)$ is called *complete* if for every $x \in X$, $N(x) = \{xy | \ y \in X - x\}$, that is, $N(x)$ contains all the remaining vertices of X except x.

Example 5.16 Consider a 3-polar fuzzy hypergraph $H = (A, B)$ on $X = \{a, b, c, d\}$ as shown in Fig. 5.17 then $N(a) = \{b, c, d\}$, $N(b) = \{a, c, d\}$, and $N(c) = \{a, b, d\}$.

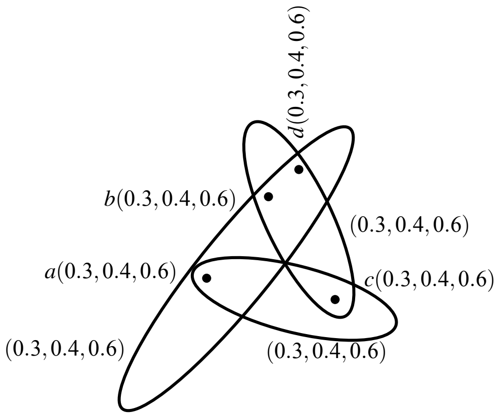


Fig. 5.17 Complete 3-polar fuzzy hypergraph

Remark 5.3 For a complete m-polar fuzzy hypergraph, the cardinality of $N(x)$ is same for every vertex.

Theorem 5.8 *Every complete m-polar fuzzy hypergraph is a totally regular m-polar fuzzy hypergraph.*

Proof Since given m-polar fuzzy hypergraph H is complete, each vertex lies in exactly same number of hyperedges and each vertex have same closed neighborhood degree m. That is, $deg[x_1] = deg[x_2]$ for all $x_1, x_2 \in X$. Hence, H is m-totally regular.

5.3 Applications of m-Polar Fuzzy Hypergraphs

Analysis of human nature and their culture has been tangled with assessment of social networks from many years. Such networks are refined by designating one or more relations on the set of individuals and the relations can be taken from efficacious relationships, facets of some management and from a large range of others means. For super-dyadic relationships between the nodes, network models represented by simple graph are not sufficient. Natural presence of hyperedges can be found in co-citation, e-mail networks, co-authorship, web log networks, and social networks, etc. Representation of these models as hypergraphs maintain the dyadic relationships.

5.3.1 Super-Dyadic Managements in Marketing Channels

In marketing channels, dyadic correspondence organization has been a basic implementation. Marketing researchers and managers are realized that their common engagement in marketing channels is a central key for successful marketing and to yield benefits for company. m-polar fuzzy hypergraphs consist of marketing managers as vertices and hyperedges show their dyadic communication involving their parallel thoughts, objectives, plans, and proposals. The more powerful close relation in the researchers is more beneficial for the marketing strategies and the production of an organization. A 3-polar fuzzy network model showing the dyadic communications among the marketing managers of an organization is given in Fig. 5.18. The membership degrees of each person symbolize the percentage of its dyadic behavior toward the other persons of the same dyad group. Adjacent level between any pair of vertices illustrates that how much their dyadic relationship is proficient. The adjacent levels are given in Table 5.4. It can be seen that the most capable dyadic pair is (Kashif, Kaamil). 3-polar fuzzy hyperedges are taken as the different digital marketing strategies adopted by the different dyadic groups of the same organization. The vital goal of this model is to figure out the most potent dyad of digital marketing techniques. The six different groups are made by the marketing managers and the digital marketing strategies adopted by these six groups are represented by hyperedges, i.e., the 3-polar fuzzy hyperedges $\{T_1, T_2, T_3, T_4, T_5, T_6\}$ show the following strategies {Product pricing, Product planning, Environment analysis and marketing research, Brand name, Build the relationships, Promotions}, respectively. The exclusive effects

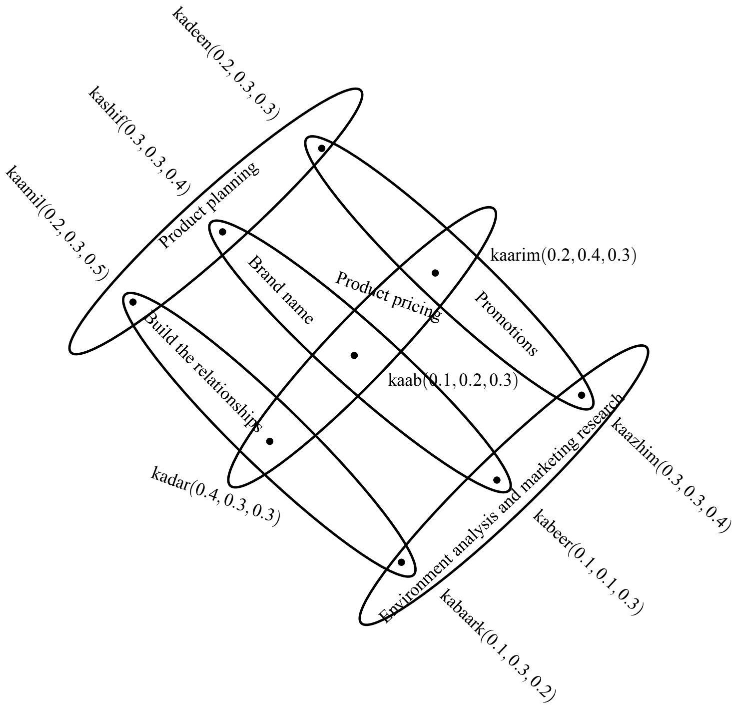


Fig. 5.18 Super-dyadic managements in marketing channels

Table 5.4 Adjacent levels of 3-polar fuzzy hypergraph

Dyad pairs	Adjacent level	Dyad pairs	Adjacent level
γ(Kadeen, Kashif)	(0.2, 0.3, 0.3)	γ(Kaarim, Kaazhim)	(0.2, 0.3, 0.3)
γ(Kadeen, Kaamil)	(0.2, 0.3, 0.3)	γ(Kaarim, Kaab)	(0.1, 0.2, 0.3)
γ(Kadeen, Kaarim)	(0.2, 0.3, 0.3)	γ(Kaarim, Kadar)	(0.2, 0.3, 0.3)
γ(Kadeen, Kaazhim)	(0.2, 0.3, 0.3)	γ(Kaab, Kadar)	(0.1, 0.2, 0.3)
γ(Kashif, Kaamil)	(0.2, 0.3, 0.4)	γ(Kaab, Kabeer)	(0.1, 0.1, 0.3)
γ(Kashif, Kaab)	(0.1, 0.2, 0.3)	γ(Kadar, Kabaark)	(0.1, 0.3, 0.2)
γ(Kashif, Kabeer)	(0.1, 0.1, 0.3)	γ(Kaazhim, Kabeer)	(0.1, 0.1, 0.3)
γ(Kaamil, Kadar))	(0.2, 0.2, 0.3)	γ(Kaazhim, Kabaark)	(0.1, 0.3, 0.2)
γ(Kaamil, Kabaark)	(0.1, 0.3, 0.2)	γ(Kabeer, Kabaark)	(0.1, 0.1, 0.2)

Table 5.5 Effects of marketing strategies

Marketing strategy	Profitable growth	Instruction manual for company success	Create longevity of the business
Product pricing	0.1	0.2	0.3
Product planning	0.2	0.3	0.3
Environment analysis and marketing research	0.1	0.2	0.2
Brand name	0.1	0.3	0.3
Build the relationships	0.1	0.3	0.2
Promotions	0.2	0.3	0.3

of membership degrees of each marketing strategy toward the achievements of an organization are given in Table 5.5. Effective dyads of market strategies enhance the performance of an organization and discover the better techniques to be adopted. The adjacency of all dyadic communication managements is given in Table 5.6. The most dominant and capable marketing strategies adopted mutually are Product planning and Promotions. Thus to increase the efficiency of an organization, dyadic managements should make the powerful planning for products and use the promotions skill to attract customers to purchase their products. The membership degrees of this dyad is (0.2, 0.3, 0.3) which shows that the amalgamated effect of this dyad will increase the profitable growth of an organization up to 20%, instruction manual for company success up to 30%, create longevity of the business up to 30% . Thus, to promote the performance of an organization, super dyad marketing communications are more energetic. The method of finding out the most effective dyads is explained in Algorithm 5.3.1.

Table 5.6 Adjacency of all dyadic communication managements

Dyadic strategies	Effects
σ (Product pricing, Product planning)	(0.1, 0.2, 0.3)
σ (Product pricing, Environment analysis and marketing research)	(0.1, 0.2, 0.2)
σ (Product pricing, Brand name)	(0.1, 0.2, 0.3)
σ (Product pricing, Build the relationships)	(0.1, 0.2, 0.2)
σ (Product pricing, Promotions)	(0.1, 0.2, 0.3)
σ (Product planning, Environment analysis and marketing research)	(0.1, 0.2, 0.2)
σ (Product planning, Brand name)	(0.1, 0.3, 0.3)
σ (Product planning, Build the relationships)	(0.1, 0.3, 0.2)
σ (Product planning, Promotions)	(0.2, 0.3, 0.3)
σ (Environment analysis and marketing research, Brand name)	(0.1, 0.2, 0.2)
σ (Environment analysis and marketing research, Build the relationships)	(0.1, 0.2, 0.2)
σ (Environment analysis and marketing research, Promotions)	(0.1, 0.2, 0.2)
σ (Brand name, Build the relationships)	(0.1, 0.3, 0.2)
σ (Brand name, Promotions)	(0.1, 0.3, 0.3)
σ (Build the relationships, Promotions)	(0.1, 0.3, 0.2)

Algorithm 5.3.1 Finding the most effective dyads

Input the membership values $A(x_i)$ of all nodes (marketing managers) $x_1, x_2, ..., x_n$.
Input the membership values $B(T_i)$ of all hyperedges $T_1, T_2, ..., T_r$.
Find the adjacent level between nodes x_i and x_j as,
do i from $1 \to n-1$
  do j from $i+1 \to n$
    do k from $1 \to r$
      if $x_i, x_j \in E_k$ **then**
        $\gamma(x_i, x_j) = \sup_k \inf\{A(x_i), A(x_j)\}$.
      end if
    end do
  end do
end do
Find the best capable dyadic pair as $\sup_{i,j} \gamma(x_i, x_j)$.
do i from $1 \to r-1$
  do j from $i+1 \to r$
    do k from $1 \to r$
      if $x_k \in T_i \cap T_j$ **then**
        $\sigma(T_i, T_j) = \sup_k \inf\{B(T_i), B(T_j)\}$.
      end if
    end do
  end do
end do
Find the best effective super dyad management as $\sup_{i,j} \sigma(T_i, T_j)$.

Description of Algorithm 5.3.1: Lines 1 and 2 passes the input of m-polar fuzzy set A on n vertices $x_1, x_2, \ldots, x_n$ and m-polar fuzzy relation B on r edges T_1, T_2, ..., T_r. Lines 3–12 calculate the adjacent level between each pair of nodes. Line 14 calculates the best capable dyadic pair. The loop initializes by taking the value $i = 1$ of do loop which is always true, i.e., the loop runs for the first iteration. For any ith iteration of do loop on line 3, the do loop on line 4 runs $n - i$ times and, the do loop on line 5 runs r times. If there exists a hyperedge E_k containing x_i and x_j then, line 7 is executed otherwise the if conditional terminates. For every ith iteration of the loop on line 3, this process continues n times and then increments i for the next iteration maintaining the loop throughout the algorithm. For $i = n - 1$, the loop calculates the adjacent level for every pair of distinct vertices and terminates successfully at line 12. Similarly, the loops on lines 13, 14 , and 15 maintain and terminate successfully.

5.3.2 m-Polar Fuzzy Hypergraphs in Work Allotment Problem

In customer care centers, availability of employees plays a vital to solve people's problems. Such a department should ensure that the system has been managed carefully to overcome practical difficulties. A lot of customers visit such centers to find a solution of their problems. In this part, focus is given to alteration of duties for the employees taking leave. The problem is that employees are taking leave without proper intimation and alteration. We now show the importance of m-polar fuzzy hypergraphs for the allocation of duties to avoid any difficulties.

Consider the example of a customer care center consisting of 30 employees. Assuming that six workers are necessary to be available at their duties. We present the employees as vertices and degree of membership of each employee represents the workload, percentage of available time and number of workers who are also aware of the employee's work type. The range of values for present time and the workers knowing the type of work is given in Tables 5.7 and 5.8, respectively. The degree of membership of each edge represents the common work load, percentage of available time and number of workers who are also aware of the employee's work type. This phenomenon can be represented by a 3-polar fuzzy graph as shown in Fig. 5.19. Using Algorithm 5.3.2, the strength of allocation and alteration of duties among employees is given in Table 5.9. Column 3 in Table 5.9 shows the percentage of alteration of duties. For example, in case of leave, duties of a_1 can be given to a_3 and similarly for other employees. The method for the calculation of alteration of duties is given in Algorithm 5.3.2.

Table 5.7 Range of membership values of table time

Time	Membership value
5 h	0.40
6 h	0.50
8 h	0.70
10 h	0.90

Table 5.8 Workers knowing the work type

Workers	Membership value
3	0.40
4	0.60
5	0.80
6	0.90

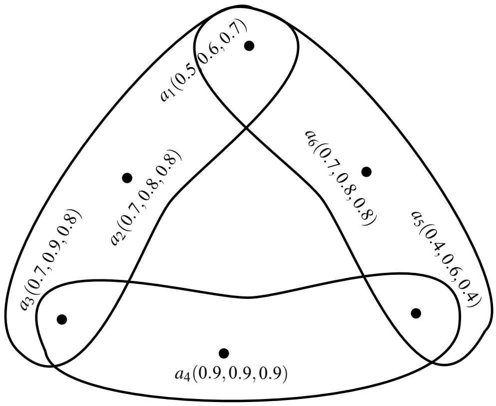


Fig. 5.19 3-polar fuzzy hypergraph

Table 5.9 Alteration of duties

Workers	$A(a_i, a_j)$	$S(a_i, a_j)$
a_1, a_2	(0.7, 0.8, 0.8)	0.77
a_1, a_3	(0.7, 0.9, 0.8)	0.80
a_2, a_3	(0.5, 0.7, 0.7)	0.63
a_3, a_4	(0.7, 0.6, 0.8)	0.70
a_3, a_5	(0.7, 0.9, 0.8)	0.80
a_4, a_5	(0.9, 0.9, 0.9)	0.90
a_5, a_6	(0.7, 0.8, 0.8)	0.77
a_5, a_1	(0.5, 0.6, 0.7)	0.60
a_1, a_6	(0.6, 0.8, 0.5)	0.63

Algorithm 5.3.2 Calculation of alteration of duties

1. Input the n number of employees $a_1, a_2, \ldots, a_n$.
2. Input the number of edges $E_1, E_2, \ldots, E_r$.
3. Input the incident matrix B_{ij} where, $1 \leq i \leq n$, $1 \leq j \leq r$.
4. Input the membership values of edges $\xi_1, \xi_2, \ldots, \xi_r$
5. **do** i from $1 \to n$
6.   **do** j from $1 \to n$
7.   **do** k from $1 \to r$
8.   **if** $a_i, a_j \in E_k$ **then**
9.   **do** t from $1 \to m$
10. $P_t \circ A(a_i, a_j) = |P_t \circ B_{ik} - P_t \circ B_{jk}| + P_t \circ \xi_k$
11.   **end do**
12.   **end if**
13.   **end do**
14.   **end do**
15. **end do**
16. **do** i from $1 \to n$
17.   **do** j from $1 \to n$
18.   **if** $A(a_i, a_j) > \mathbf{0}$ **then**
19. $S(a_i, a_j) = \dfrac{P_1 \circ A(a_i, a_j) + P_2 \circ A(a_i, a_j) + \ldots + P_m \circ A(a_i, a_j)}{m}$
20.   **end if**
21.   **end do**
22. **end do**

Description of Algorithm 5.3.2: Lines 1, 2, 3 and 4 passes the input of membership values of vertices, hyperedges and an m-polar fuzzy adjacency matrix B_{ij}. The nested loops on lines 5 to 15 calculate the rth, $1 \leq r \leq m$, strength of allocation and alteration of duties between each pair of employees. The nested loops on lines 16 to 22 calculate the strength of allocation and alteration of duties between each pair of employees. The net time complexity of the algorithm is $O(n^2 rm)$.

5.3.3 Availability of Books in Library

A library in college is a collection of sources of information and similar resources, made accessible to student community for reference and examination preparation. A student preparing for some examination will use the knowledge sources such as

1. Prescribed textbooks (A)
2. Reference books in syllabus (B)
3. Other books from library (C)
4. Knowledgeable study materials (D)
5. E-gadgets and internet (E)

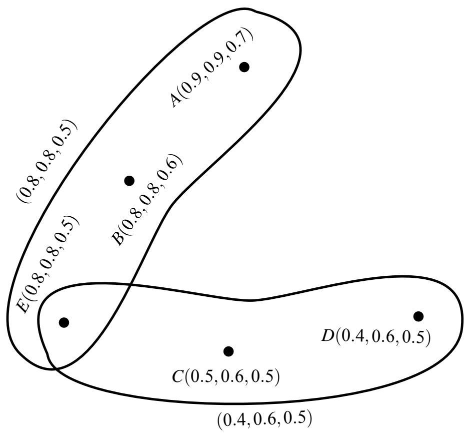


Fig. 5.20 3-polar fuzzy hypergraph

Table 5.10 Library sources

Sources s_i	$T(s_i)$	$S(a_i, a_j)$
A	(1.7, 1.7, 1.4)	1.60
B	(1.6, 1.6, 1.1)	1.43
E	(1.6, 1.6, 1.0)	1.40
C	(0.9, 1.2, 1.0)	1.03
D	(0.8, 1.2, 1.0)	1.0

The important thing is to consider the maximum availability of the sources which students mostly use. This phenomenon can be discussed using m-polar fuzzy hypergraphs. We now calculate the importance of each source in student community.

Consider the example of five library resources $\{A, B, C, D, E\}$ in a college. We represent these sources as vertices in a 3-polar fuzzy hypergraph. The degree of membership of each vertex represents the percentage of students using a particular source for exam preparation, percentage of faculty of members using the sources and number of sources available. The degree of membership of each edge represents the common percentage. The 3-polar fuzzy hypergraph is shown in Fig. 5.20. Using Algorithm 5.3.3, the strength of each library source is given in Table 5.10.

Column 3 in Table 5.10 shows that sources A and B are mostly used by students and faculty. Therefore, these should be available in maximum number. There is also a need to confirm the availability of source E to students and faculty. The method for the calculation of percentage importance of the sources is given in Algorithm 5.3.3 whose net time complexity is O(nrm).

Algorithm 5.3.3 *Calculation of percentage importance of the sources*

1. Input the n number of sources $s_1, s_2, \ldots, s_n$.
2. Input the number of edges $E_1, E_2, \ldots, E_r$.
3. Input the incident matrix B_{ij}, where $1 \le i \le n$, $1 \le j \le r$.
4. Input the membership values of edges $\xi_1, \xi_2, \ldots, \xi_r$
5. **do** i from $1 \to n$
6. $\quad A(s_i) = \mathbf{1}$
7. $\quad C(s_i) = \mathbf{1}$
8. $\quad$ **do** k from $1 \to r$
9. $\quad\quad$ **if** $s_i \in E_k$ **then**
10. $\quad\quad\quad A(s_i) = \sup\{A(s_i), \xi_k\}$
11. $\quad\quad\quad C(s_i) = \inf\{C(s_i), B_{ik}\}$
12. $\quad\quad$ **end if**
13. $\quad$ **end do**
14. $\quad T(s_i) = C(s_i) + A(s_i)$
15. **end do**
16. **do** i from $1 \to n$
17. $\quad$ **if** $T(s_i) > \mathbf{0}$ **then**
18. $\quad\quad S(s_i) = \dfrac{P_1 \circ T(s_i) + P_2 \circ T(s_i) + \ldots + P_m \circ T(s_i)}{m}$
19. $\quad$ **end if**
20. **end do**

Description of Algorithm 5.3.3: Lines 1, 2, 3, and 4 passes the input of membership values of vertices, hyperedges and an m-polar fuzzy adjacency matrix B_{ij}. The nested loops on lines 5 to 15 calculate the degree of usage and availability of library sources. The nested loops on lines 16–20 calculate the strength of each library source.

5.3.4 Selection of a Pair of Good Team for Competition

Competition grants the inspiration to achieve a goal; to demonstrate determination, creativity, and perseverance to overcome challenges; and to understand that hard work and commitment leads to a greater chance of success. It is inarguably accepted that a bit of healthy competition in any field is known to enhance motivation and generate increased effort from those competing. The sporting field is no exception to this rule. While there will always be varying levels of sporting talent and interest across any group of people, the benefits that competitive sport provides are still accessible to all. There is a role for both competitive and noncompetitive sporting pursuits. To get success in any competition, a strong team can be held largely accountable for the success.

The purpose of this application is to select a pair of good player team for competition with other country. For example, we have three teams of players (three

Table 5.11 3-polar subsets of teams

Players	Self confidence	Strong sense of motivation	Adaptability
Adnan	0.5	0.6	0.5
Usman	0.6	0.4	0.8
Awais	0.5	0.8	0.9
Hamza	0.7	0.7	0.6
Waseem	0.3	0.7	0.4
Usama	0.4	0.2	0.3
Iqbal	0.5	0.5	0.5
Noman	0.3	0.6	0.6
Arshad	0.4	0.3	0.7
Saeed	0.4	0.2	0.9
Nawab	0.7	0.5	0.6
Haris	0.6	0.6	0.5

3-polar fuzzy hypergraphs) and we have to select only one pair of team for competition with other country. Then to select it, we use union operation of m-polar fuzzy hypergraphs. Hypergraph is used because there is a link in one team more than two players and m-polar represents different qualities of players and teams. Consider three teams, team 1 consists of players $\{Adnan, Usman, Hamza, Awais\}$. Team 2 consists of players $\{Waseem, Usama, Iqbal, Noman\}$. Team 3 consists of players $\{Arshad, Saeed, Nawab, Haris\}$. The 3-polar fuzzy set of players represent the three different qualities of each player, i.e., self confidence, strong sense of motivation, adaptability. 3-polar fuzzy hyperedges represent the three characteristics of a good team. First membership degree of 3-polar fuzzy hyperedges represents the focus of team on goals, second represents the communication with each other, third represents how much team is organized. We want to select a pair of good team which qualify these three properties with maximum membership degrees values (Tables 5.11 and 5.12).

Let $A = \{(Adnan, 0.5, 0.6, 0.5), (Usman, 0.6, 0.4, 0.8), (Awais, 0.5, 0.8, 0.9), (Hamza, 0.7, 0.7, 0.6),$

$(Waseem, 0.3, 0.7, 0.4), (Usama, 0.4, 0.2, 0.3), (Iqbal, 0.5, 0.5, 0.5), (Noman, 0.3, 0.6, 0.6),$

$(Arshad, 0.4, 0.3, 0.7), (Saeed, 0.4, 0.2, 0.9), (Nawab, 0.7, 0.5, 0.6), (Haris, 0.6, 0.6, 0.5)\}$ be a 3-polar fuzzy set of players and $B = \{(Team\ 1, 0.5, 0.4, 0.5), (Team\ 2, 0.3, 0.2, 0.3), (Team\ 3, 0.4, 0.2, 0.6)\}$ is a set of 3-polar fuzzy hyperedges.

We select that pair of team whose union is strong, i.e., we select that union whose edges have maximum membership degrees. It represents the focus of teams on goals, second represents the communication with each other of both teams, and

Table 5.12 3-polar fuzzy qualities of teams

Teams	Focus on goals	Communication skills	Organization
1	0.5	0.4	0.5
2	0.3	0.2	0.3
3	0.4	0.2	0.6

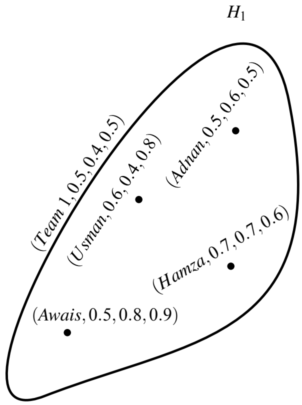


Fig. 5.21 3-polar fuzzy hypergraph H_1

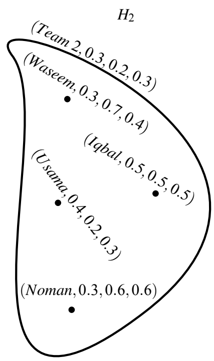


Fig. 5.22 3-polar fuzzy hypergraph H_2

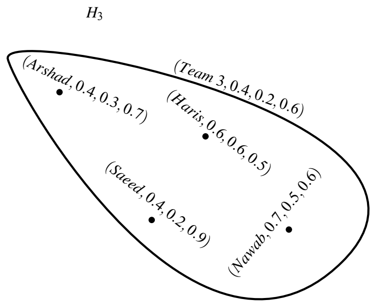


Fig. 5.23 3-polar fuzzy hypergraph H_3

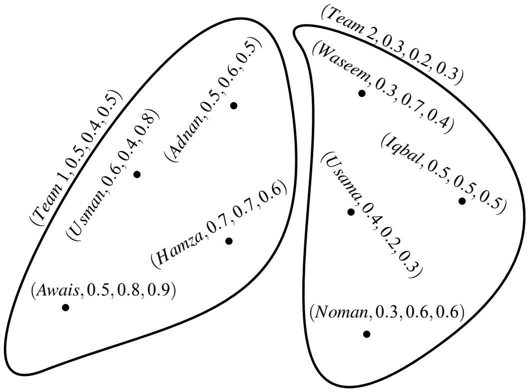


Fig. 5.24 $H_1 \cup H_2$

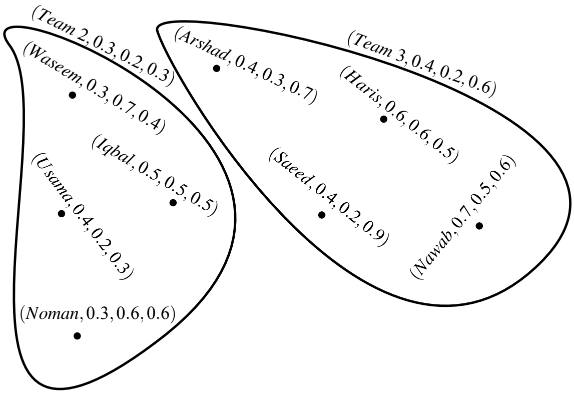


Fig. 5.25 $H_2 \cup H_3$

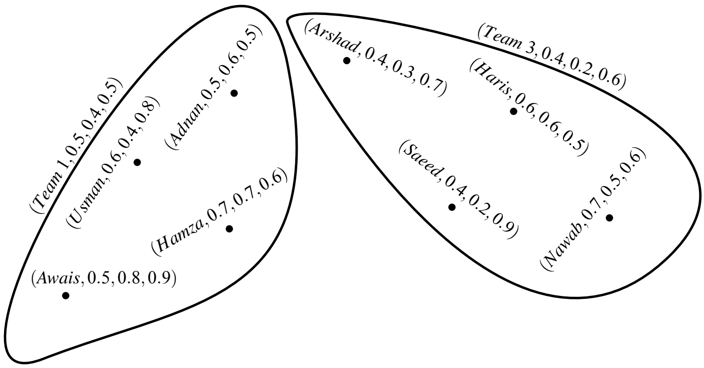


Fig. 5.26 $H_1 \cup H_3$

third represents how much team is organized. So, we select the pair of team 1 and team 3 (Figs. 5.21, 5.22, 5.23, 5.24, 5.25 and 5.26).

We present our proposed method in Algorithm 5.3.4.

Algorithm 5.3.4 *Selection of team for competition*

Step 1: **Input**

The set of players.

Assign the membership values to each player.

Select the players of each team.

Step 2: Compute the membership values of each team(edges) by using the relation $B(E_i) = B(\{x_1, x_2, \ldots, x_r\}) \le \inf\{\zeta_i(x_1), \zeta_i(x_2), \ldots, \zeta_i(x_s)\}$, for all $x_1, x_2, \ldots, x_s \in X$.

Step 3: Compute union of teams.

Compute their union by using the relation

(i) $P_i \circ (A_1 \cup A_2)(v) = \begin{cases} P_i \circ A_1(v) \; if \; v \in X_1 - X_2, \\ P_i \circ A_2(v) \; if \; v \in X_2 - X_1, \\ \sup\{P_i \circ A_1(v), P_i \circ A_2(v)\} \; if \; v \in X_1 \cap X_2. \end{cases}$

(ii) $P_i \circ (B_1 \cup B_2)(e) = \begin{cases} P_i \circ B_1(e) \; if \; e \in E_1 - E_2, \\ P_i \circ B_2(e) \; if \; e \in E_2 - E_1, \\ \sup\{P_i \circ B_1(e), P_i \circ B_2(e)\} \; if \; e \in E_1 \cap E_2. \end{cases}$

Step 4: **Output**

Select that pair of team for competition for which edges of union have maximum membership degree.

5.4 *m*-Polar Fuzzy Directed Hypergraphs

Definition 5.23 A directed hypergraph is a hypergraph with directed hyperedges. A directed hyperedge or hyperarc is an ordered pair $E = (X, Y)$ of (possibly empty) disjoint subsets of vertices. X is the tail of E, while Y is its head. A sequence of crisp hypergraphs $H_i = (V_i, E_i)$, $1 \le i \le n$, is said to ordered if $H_1 \subset H_2 \subset \ldots, H_n$. The sequence $\{H_i \mid 1 \le i \le n\}$ is said to be simply ordered if it is ordered, and if whenever $E \subset E_{i+1} \backslash E_i$, then $E \not\subseteq V_i$.

We now define an m-polar fuzzy directed hypergraph.

Definition 5.24 An m-polar fuzzy directed hypergraph with underlying set X is an ordered pair $H = (\sigma, \varepsilon)$, where σ is non-empty set of vertices and ε is a family of m-polar fuzzy (m-polar fuzzy) directed hyperarcs (or hyperedges). An m-polar fuzzy directed hyperarc (or hyperedge) $e_i \in \varepsilon$ is an ordered pair $(t(e_i), h(e_i))$, such that, $t(e_i) \ne \emptyset$, is called its tail and $h(e_i) \ne t(e_i)$ is its head, such that $P_k o \varepsilon_i(\{v_1, v_2, \ldots, v_s\}) \le \inf\{P_k o\sigma_i(v_1), P_k o\sigma_i(v_2), \ldots, P_k o\sigma_i(v_s)\}$, for all $v_1, v_2, \ldots, v_s \in V,\; 1 \le k \le m$.

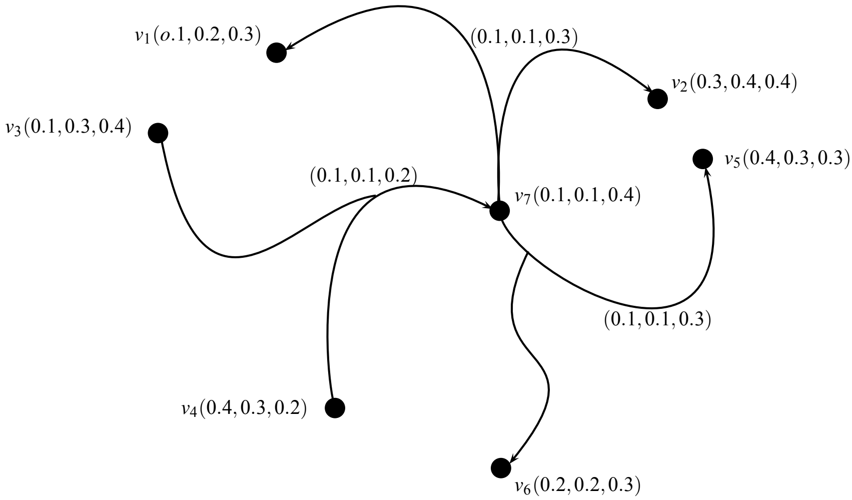


Fig. 5.27 3-polar fuzzy directed hypergraph

Definition 5.25 Let $H = (\sigma, \varepsilon)$ be an m-polar fuzzy directed hypergraph. The order of H, denoted by $O(H)$, is defined as $O(H) = \sum_{x \in V} \wedge \sigma_i(x)$. The size of H, denoted by $S(H)$, is defined by $S(H) = \sum_{e_k \subset V} \varepsilon(e_k)$.

In an m-polar fuzzy directed hypergraph, the vertices v_i and v_j are adjacent vertices if they both belong to the same m-polar fuzzy directed hyperedge. Two m-polar fuzzy directed hyperedges e_i and e_j are called adjacent if they have non-empty intersection. That is, $supp(e_i) \cap supp(e_j) \neq \emptyset, i \neq j$.

Definition 5.26 An m-polar fuzzy directed hypergraph $H = (\sigma, \varepsilon)$ is *simple* if it contains no repeated directed hyperedges, i.e., if $e_j, e_k \in \varepsilon$ and $e_j \subseteq e_k$ then $e_j = e_k$. An m-polar fuzzy directed hypergraph $H = (\sigma, \varepsilon)$ is called *support simple* if $e_j, e_k \in \varepsilon$ and $supp(e_j) = supp(e_k)$ and $e_j \subseteq e_k$, then $e_j = e_k$. An m-polar fuzzy directed hypergraph, $H = (\sigma, \varepsilon)$ is called *strongly support simple* if $e_j, e_k \in \varepsilon$ and $supp(e_j) = supp(e_k)$, then $e_j = e_k$.

Example 5.17 Consider a 3-polar fuzzy directed hypergraph $H = (\sigma, \varepsilon)$, such that $\sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ is the family of 3-polar fuzzy subsets on $X = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, as shown in Fig. 5.27, such that

$\sigma_1 = \{(v_1, 0.1, 0.2, 0.3), (v_2, 0.3., 0.4, 0.4), (v_3, 0.1, 0.3, 0.4)\}$,
$\sigma_2 = \{(v_5, 0.4, 0.3, 0.3), (v_6, 0.2, 0.2, 0.3), (v_7, 0.1, 0.1, 0.4)\}$,
$\sigma_3 = \{(v_3, 0.1, 0.3, 0.4), (v_4, 0.4, 0.3, 0.2), (v_7, 0.1, 0.1, 0.4)\}$.

3-polar fuzzy relation ε is defined as, $\varepsilon(v_1, v_2, v_7) = (0.1, 0.1, 0.3)$, $\varepsilon(v_5, v_6, v_7) = (0.1, 0.1, 0.3)$, $\varepsilon(v_3, v_4, v_7) = (0.1, 0.1, 0.2)$.

Clearly, H is simple, strongly support simple, and support simple, that is, it contains no repeated directed hyperedges and if whenever $e_j, e_k \in \varepsilon$ and $supp(e_j) = supp(e_k)$, then $e_j = e_k$. Further, $O(H) = (1.6, 1.8, 2.3)$ and $S(H) = (0.3, 0.3, 0.8)$.

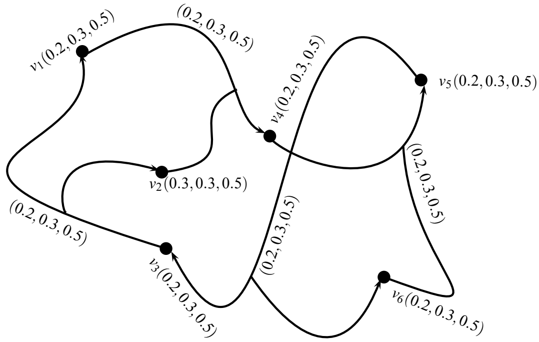


Fig. 5.28 Regular 3-polar fuzzy hypergraph

Definition 5.27 Let $\varepsilon = (\varepsilon^-, \varepsilon^+)$ be a directed m-polar fuzzy hyperedge in an m-polar fuzzy directed hypergraph. Then, the vertex set ε^- is called the m-polar fuzzy in-set and the vertex set ε^+ is called the m-polar fuzzy out-set of the directed hyperedge ε. It is not necessary that the sets ε^-, ε^+ will be disjoint. The hyperedge ε is called the join of the vertices of ε^- and ε^+.

Definition 5.28 The in-degree $D_H^-(v)$ of a vertex v in an m-polar fuzzy directed hypergraph is defined as the sum of membership degrees of all those directed hyperedges such that v is contained in their out-set, that is,

$$D_H^-(v) = \sum_{v \in h(e_i)} \varepsilon(e_i),\ 1 \le k \le m.$$

The out-degree $D_H^+(v)$ of a vertex v in an m-polar fuzzy directed hypergraph is defined as the sum of membership degrees of all those directed hyperedges such that v is contained in their in-set, that is,

$$D_H^+(v) = \sum_{v \in t(e_i)} \varepsilon(e_i),\ 1 \le k \le m.$$

Definition 5.29 An m-polar fuzzy directed hypergraph $H = (\sigma, \varepsilon)$ is said to be k-regular if in-degrees and out-degrees of all vertices in H are same.

Example 5.18 Consider a 3-polar fuzzy directed hypergraph $H = (\sigma, \varepsilon)$ as shown in Fig. 5.28, where $\sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is the family of 3-polar fuzzy subsets on $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and

$\sigma_1 = \{(v_1, 0.2, 0.3, 0.5),(v_2, 0.2, 0.3, 0.5),(v_4, 0.2, 0.3, 0.5)\}$,
$\sigma_2 = \{(v_4, 0.2, 0.3, 0.5),(v_5, 0.2, 0.3, 0.5),(v_6, 0.2, 0.3, 0.5)\}$,
$\sigma_3 = \{(v_3, 0.2, 0.3, 0.5),(v_5, 0.2, 0.3, 0.5),(v_6, 0.2, 0.3, 0.5)\}$,
$\sigma_4 = \{(v_1, 0.2, 0.3, 0.5),(v_2, 0.2, 0.3, 0.5),(v_3, 0.2, 0.3, 0.5)\}$. By routine calculations, we see that the 3-polar fuzzy directed hypergraph is regular.

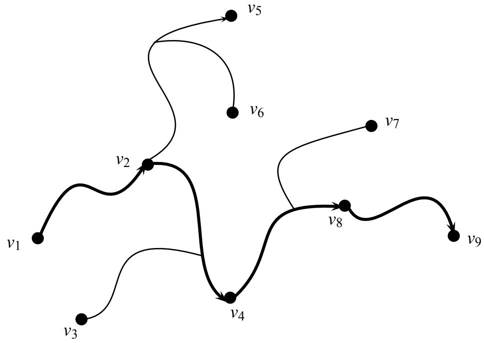


Fig. 5.29 Directed hyperpath (denoted by a thick line)

Note that, $D_H^-(v_1) = (0.2, 0.3, 0.5) = D_H^+(v_1)$ and $D_H^-(v_2) = (0.2, 0.3, 0.5) = D_H^+(v_2)$. Similarly, $D_H^-(v_3) = D_H^+(v_3)$, $D_H^-(v_4) = D_H^+(v_4)$, $D_H^-(v_5) = D_H^+(v_5)$. Hence, H is regular 3-polar fuzzy directed hypergraph.

Definition 5.30 An m-polar fuzzy directed hyperpath of length k in an m-polar fuzzy directed hypergraph is defined as a sequence $v_1, e_1, v_2, e_2, \ldots, e_k, v_{k+1}$ of distinct vertices and directed hyperedges such that

1. $\varepsilon(e_i) > 0, i = 1, 2, ..., k,$
2. $v_i, v_{i+1} \in e_i$.

The consecutive pairs (v_i, v_{i+1}) are called the directed arcs of the directed hyperpath. The path is shown by a thick line in Fig. 5.29.

Definition 5.31 The incidence matrix of an m-polar fuzzy directed hypergraph $H = (\sigma, \varepsilon)$ is characterized by an $n \times m$ matrix $[a_{ij}]$ as follows:

$$a_{ij} = \begin{cases} P_k o \varepsilon_j(v_i), & \text{if } v_i \in \varepsilon_j, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 5.32 An m-polar fuzzy directed hypergraph is called elementary if $P_k o \varepsilon_{ij} : V \longrightarrow [0, 1]^m$ are constant functions, $P_k o \varepsilon_{ij}$ is taken as the membership degree of vertex i to hyperedge j.

Proposition 5.1 *In an m-polar fuzzy directed hypergraph, when m-polar fuzzy vertices have constant membership degrees, then m-polar fuzzy directed hyperedges are elementary.*

Example 5.19 Consider a 3-polar fuzzy directed hypergraph $H = (\sigma, \varepsilon)$, where $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$ be the family of 3-polar fuzzy subsets on $V = \{v_1, v_2, v_3, v_4, v_5\}$. The corresponding incidence matrix is given in Table 5.13.

The corresponding elementary 3-polar fuzzy directed hypergraph is shown in Fig. 5.30.

Table 5.13 Elementary 3-polar fuzzy directed hypergraph

I	ε_1	ε_2	ε_3
v_1	(0.1, 0.2, 0.3)	0	(0.1, 0.2, 0.3)
v_2	(0.1, 0.2, 0.3)	0	(0.1, 0.2, 0.3)
v_3	(0.1, 0.2, 0.3)	(0.1, 0.2, 0.3)	0
v_4	0	(0.1, 0.2, 0.3)	(0.1, 0.2, 0.3)
v_5	0	(0.1, 0.2, 0.3)	0

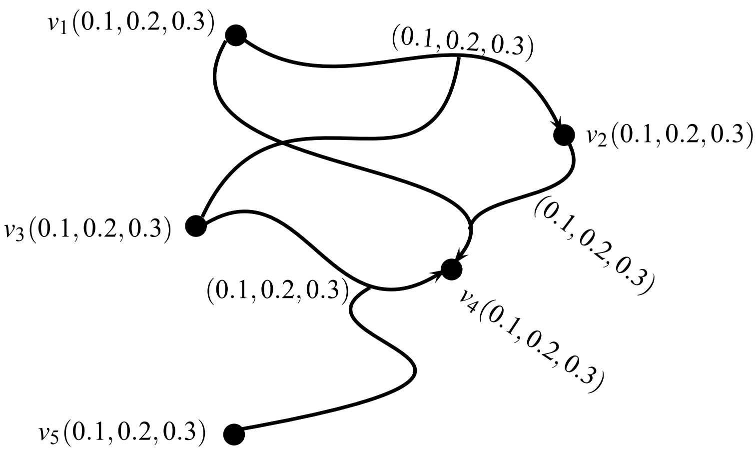


Fig. 5.30 Elementary 3-polar fuzzy directed hypergraph

Definition 5.33 Let $H = (\sigma, \varepsilon)$ be an m-polar fuzzy directed hypergraph. Suppose $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_m) \in [0, 1]^m$. The $\boldsymbol{\mu}$-level is defined as $\varepsilon_{\boldsymbol{\mu}} = \{v \in \sigma \mid P_k o\sigma(v) \geq \mu_k\}$. The crisp directed hypergraph $H_{\boldsymbol{\mu}} = (\sigma_{\boldsymbol{\mu}}, \varepsilon_{\boldsymbol{\mu}})$, such that

- $\varepsilon_{\boldsymbol{\mu}} = \{v \in \sigma \mid P_k o\sigma(v) \geq \mu_k\},\ 1 \leq k \leq m.$
- $\sigma_{\boldsymbol{\mu}} = \bigcup \varepsilon_{\boldsymbol{\mu}},$

is called the $\boldsymbol{\mu}$-level directed hypergraph of H.

Definition 5.34 Let $H = (\sigma, \varepsilon)$ be an m-polar fuzzy directed hypergraph and $H_{\boldsymbol{\mu}_i} = (\sigma_{\boldsymbol{\mu}_i}, \boldsymbol{\varepsilon}_{\boldsymbol{\mu}_i})$ be the $\boldsymbol{\mu_i}$-level directed hypergraphs of H. The sequence $\{\boldsymbol{\mu_1}, \boldsymbol{\mu_2}, \boldsymbol{\mu_3}, ..., \boldsymbol{\mu_n}\}$ of m-tuples, where $\boldsymbol{\mu_1} > \boldsymbol{\mu_2} > ...\boldsymbol{\mu_n} > 0$ and $\boldsymbol{\mu_n} = h(H)$(height of m-polar fuzzy directed hypergraph), such that the following properties,

1. if $\boldsymbol{\mu_{i+1}} < \boldsymbol{\alpha} \leq \boldsymbol{\mu_i}$, then $\varepsilon_{\boldsymbol{\alpha}} = \varepsilon_{\boldsymbol{\mu}_i}$,
2. $\varepsilon_{\boldsymbol{\mu}_i} \sqsubset \varepsilon_{\boldsymbol{\mu}_{i+1}}$,

are satisfied, is called a *fundamental sequence* of H. The sequence is denoted by $FS(H)$. The $\boldsymbol{\mu_i}$-level hypergraphs $\{H_{\boldsymbol{\mu}_1}, H_{\boldsymbol{\mu}_2}, ..., H_{\boldsymbol{\mu}_n}\}$ are called the core hypergraphs of H. This is also called core set of H and is denoted by $c(H)$.

Table 5.14 3-polar fuzzy directed hypergraph

I	ε_1	ε_2	ε_3
v_1	(0.8, 0.6, 0.1)	0	0
v_2	(0.8, 0.6, 0.5)	(0.6, 0.4, 0.3)	(0.5, 0.3, 0.2)
v_3	(0.8, 0.6, 0.5)	(0.6, 0.4, 0.3)	(0.5, 0.3, 0.2)
v_4	0	(0.6, 0.4, 0.1)	0
v_5	0	0	(0.5, 0.3, 0.2)
v_6	0	0	(0.5, 0.3, 0.2)

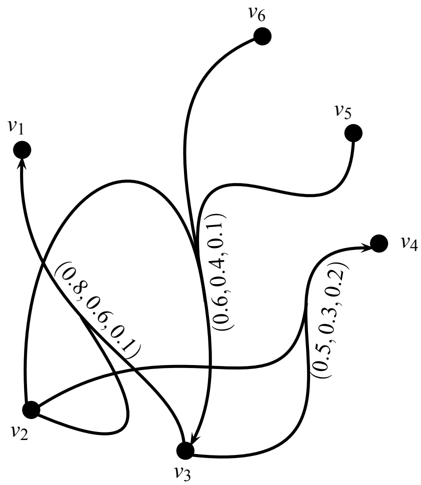


Fig. 5.31 3-polar fuzzy directed hypergraph

Definition 5.35 Let $H = (\sigma, \varepsilon)$ be an m-polar fuzzy directed hypergraph and $FS(H) = \{\boldsymbol{\mu_1}, \boldsymbol{\mu_2}, \boldsymbol{\mu_3}, \dots, \boldsymbol{\mu_n}\}$. If for each $e \in \varepsilon$ and each $\boldsymbol{\mu_i} \in FS(H)$, $e_{\boldsymbol{\mu}} = \varepsilon_{\boldsymbol{\mu_i}}$, for all $\boldsymbol{\mu} \in (\boldsymbol{\mu_{i+1}}, \boldsymbol{\mu_i}]$, then H is called *sectionally elementary*.

Definition 5.36 Let $H = (\sigma, \varepsilon)$ be an m-polar fuzzy directed hypergraph and $c(H) = \{H_{\boldsymbol{\mu_1}}, H_{\boldsymbol{\mu_2}}, \dots, H_{\boldsymbol{\mu_n}}\}$. H is said to be *ordered* if $c(H)$ is ordered. That is, $H_{\boldsymbol{\mu_1}} \subset H_{\boldsymbol{\mu_2}} \subset \dots \subset H_{\boldsymbol{\mu_n}}$. The m-polar fuzzy directed hypergraph is called simply ordered if the sequence $\{H_{\boldsymbol{\mu_1}}, H_{\boldsymbol{\mu_2}}, \dots, H_{\boldsymbol{\mu_n}}\}$ is simply ordered.

Example 5.20 Consider a 3-polar fuzzy directed hypergraph $H = (\sigma, \varepsilon)$ as shown in Fig. 5.31 and given by incidence matrix in Table 5.14.

By computing the μ_i-level 3-polar fuzzy directed hypergraphs of H, we have $\varepsilon_{(0.8,0.6,0.5)} = \{v_2, v_3\}$, $\varepsilon_{(0.6,0.4,0.3)} = \{v_2, v_3\}$ and $\varepsilon_{(0.5,0.3,0.2)} = \{v_2, v_3, v_5, v_6\}$. Note that, $H_{(0.8,0.6,0.5)} = H_{(0.6,0.4,0.3)}$ and $H_{(0.8,0.6,0.5)} \subseteq H_{(0.5,0.3,0.2)}$. The fundamental sequence is $FS(H) = \{(0.8, 0.6, 0.5), (0.5, 0.3, 0.2)\}$. Furthermore, $H_{(0.8,0.6,0.5)} \neq H_{(0.6,0.4,0.3)}$. H is not sectionally elementary since $\varepsilon_{2(\mu)} \neq \varepsilon_{2(0.8,0.6,0.5)}$ for $\mu = (0.6, 0.4, 0.3)$. The 3-polar fuzzy directed hypergraph is ordered, and the set of core

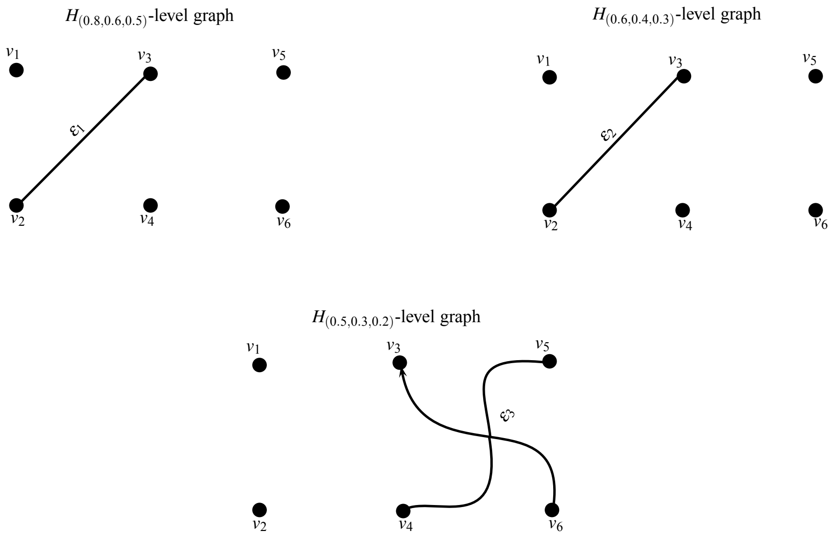


Fig. 5.32 H induced fundamental sequence

Table 5.15 Index matrix of an m-polar fuzzy hypergraph

I	t_1	t_2	. . .	t_n
t_1	$\varepsilon(t_1t_1)$	$\varepsilon(t_1t_2)$	. . .	$\varepsilon(t_1t_n)$
t_2	$\varepsilon(t_2t_1)$	$\varepsilon(t_2t_2)$	. . .	$\varepsilon(t_2t_n)$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
t_n	$\varepsilon(t_nt_1)$	$\varepsilon(t_nt_2)$	. . .	$\varepsilon(t_nt_n)$

hypergraphs is $c(H) = \{H_1 = H_{(0.8,0.6,0.5)}, H_2 = H_{(0.5,0.3,0.2)}\}$. The induced fundamental sequence of H is given in Fig. 5.32 (Table 5.15).

Proposition 5.2 *Let $H = (\sigma, \varepsilon)$ be an m-polar fuzzy directed hypergraph, the following conditions hold*

(a) If $H = (\sigma, \varepsilon)$ is an elementary m-polar fuzzy directed hypergraph, then H is ordered.
(b) If H is an ordered m-polar fuzzy directed hypergraph with $c(H) = \{H_{\mu_1}, H_{\mu_2}, \ldots, H_{\mu_n}\}$ and if H_{μ_n} is simple, then H is elementary.

Definition 5.37 Let $H = (\sigma, \varepsilon)$ be an m-polar fuzzy directed hypergraph. The index matrix of H is defined by

Now we present certain operations on m-polar fuzzy directed hypergraphs.

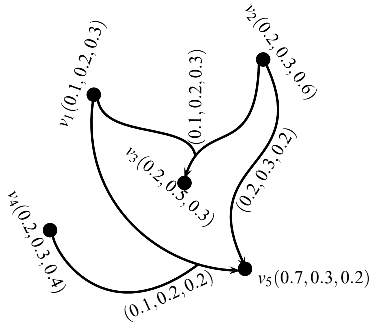


Fig. 5.33 3-polar fuzzy directed hypergraph H_1

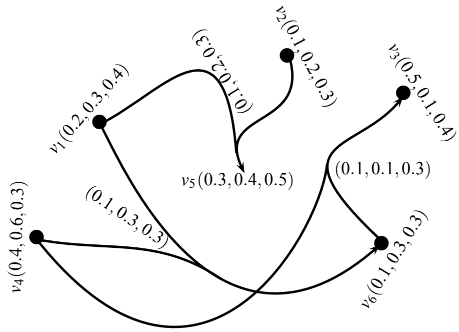


Fig. 5.34 3-polar fuzzy directed hypergraph H_2

Definition 5.38 Let $H_1 = (\sigma_1, \varepsilon_1)$ and $H_2 = (\sigma_2, \varepsilon_2)$ be two m-polar fuzzy directed hypergraphs. The *addition* of two m-polar fuzzy directed hypergraphs over a fixed set X is denoted by $H_1 \boxplus H_2 = (\sigma_1 \cup \sigma_2, \varepsilon_1 \cup \varepsilon_2)$ and defined as

$$P_k o(\sigma_1 \cup \sigma_2)(v_r) = \begin{cases} P_k o\sigma_1(v_r), & \text{if } v_r \in \sigma_1 \setminus \sigma_2, \\ P_k o\sigma_2(v_r), & \text{if } v_r \in \sigma_2 \setminus \sigma_1, \\ \sup\{P_k o\sigma_1(v_r),\, P_k o\sigma_2(v_r)\}, & \text{if } v_r \in \sigma_1 \cap \sigma_2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

$$P_k o(\varepsilon_1 \cup \varepsilon_2)(e_{rs}) = \begin{cases} P_k o\varepsilon_1(e_{ij}), & \text{if } v_r = v_i \in \sigma_1 \text{ and } v_s = v_j \in \sigma_1 \setminus \sigma_2, \\ P_k o\varepsilon_2(e_p q), & \text{if } v_r = v_p \in \sigma_2 \text{ and } v_s = v_q \in \sigma_2 \setminus \sigma_1, \\ \sup\{P_k o\varepsilon_1(e_{ij}),\, P_k o\varepsilon_2(e_{pq})\}, & \text{if } v_r = v_i = v_p \in \sigma_1 \cap \sigma_2, v_s = v_j = v_q \in \sigma_1 \cap \sigma_2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

Example 5.21 Let $H_1 = (\sigma_1, \varepsilon_1)$ and $H_2 = (\sigma_2, \varepsilon_2)$ be two 3-polar fuzzy directed hypergraphs, where $\sigma_1 = \{v_1, v_2, ..., v_5\}$, $\varepsilon_1 = \{(\{v_1, v_2\}, v_3), (\{v_1, v_4\}, v_5), \{\{v_2\}, v_5\}$ and $\sigma_2 = \{v_1, v_2, ..., v_6\}$, $\varepsilon_2 = \{(\{v_1, v_2\}, v_5), (\{v_4, v_6\}, v_3), \{\{v_1, v_4\}, v_6\}$ as shown in Figs. 5.33 and 5.34, respectively.

The index matrix of H_1 is given in Table 5.16, where $\sigma_1 = \{v_1, v_2, ..., v_5\}$.

The index matrix of H_2 is given in Table 5.17, where $\sigma_2 = \{v_1, v_2, ..., v_6\}$.

Table 5.16 Index matrix of H_1

I	v_1	v_2	v_3	v_4	v_5
v_1	0	0	0	0	0
v_2	0	0	0	0	0
v_3	(0.1, 0.2, 0.3)	(0.1, 0.2, 0.3)	0	0	0
v_4	0	0	0	0	0
v_5	(0.1, 0.2, 0.2)	(0.2, 0.3, 0.2)	0	(0.1, 0.2, 0.2)	0

Table 5.17 Index matrix of H_2

I	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	0	0	0
v_2	0	0	0	0	0	0
v_3	(0.1, 0.2, 0.3)	(0.1, 0.2, 0.3)	0	(0.1, 0.1, 0.3)	0	(0.1, 0.1, 0.3)
v_4	0	0	0	0	0	0
v_5	(0.1, 0.2, 0.2)	0	0	(0.1, 0.2, 0.2)	0	0
v_6	(0.1, 0.3, 0.3)	0	0	(0.1, 0.3, 0.3)	0	0

Table 5.18 Index matrix of $H_1 \boxplus H_2$

$H_1 \boxplus H_2$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	0	0	0
v_2	0	0	0	0	0	0
v_3	(0.1, 0.2, 0.3)	(0.1, 0.2, 0.3)	0	(0.1, 0.1, 0.3)	0	(0.1, 0.1, 0.3)
v_4	0	0	0	0	0	0
v_5	(0.1, 0.2, 0.2)	0	0	(0.1, 0.2, 0.2)	0	0
v_6	(0.1, 0.3, 0.3)	0	0	(0.1, 0.3, 0.3)	0	0

The index matrix of $H_1 \boxplus H_2$ is given in Table 5.18, where $\sigma_1 \cup \sigma_2 = \{v_1, v_2, ..., v_6\}$. The corresponding hypergraph is shown in Fig. 5.35.

Definition 5.39 Let $H_1 = (\sigma_1, \varepsilon_1)$ and $H_2 = (\sigma_2, \varepsilon_2)$ be two m-polar fuzzy directed hypergraphs. The vertex-wise multiplication of two m-polar fuzzy directed hypergraphs over a fixed set V is denoted by $H_1 \otimes H_2 = (\sigma_1 \otimes \sigma_2, \varepsilon_1 \otimes \varepsilon_2)$ and defined as

$$P_k o(\sigma_1 \otimes \sigma_2) = \inf\{P_k o\sigma_1(v_r), P_k o\sigma_2(v_r)\} \text{ if } v_r \in \sigma_1 \cap \sigma_2, \tag{5.3}$$

$$\begin{aligned} &P_k o(\varepsilon_1 \otimes \varepsilon_2)(e_{rs}) \\ &\quad = \inf\{P_k o\varepsilon_1(e_{ij}), P_k o\sigma_2(e_{pq})\} \text{ if } v_r = v_i = v_p \in \sigma_1 \cap \sigma_2, v_s = v_j = v_q \in \sigma_1 \cap \sigma_2. \end{aligned} \tag{5.4}$$

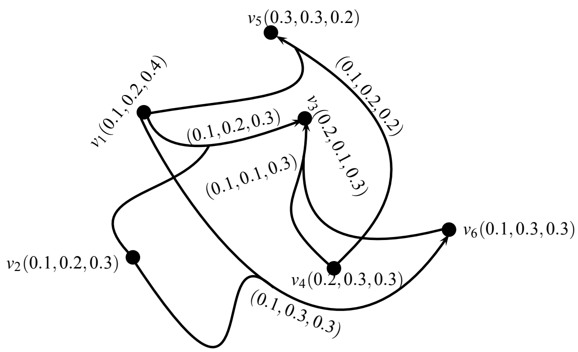


Fig. 5.35 $H_1 \boxplus H_2$

Table 5.19 Index matrix of $H_1 \otimes H_2$

$H_1 \otimes H_2$	v_1	v_2	v_3	v_4	v_5
v_1	0	0	0	0	0
v_2	0	0	0	0	0
v_3	0	0	0	0	0
v_4	0	0	0	0	0
v_5	(0.1, 0.2, 0.2)	(0.1, 0.2, 0.3)	0	0	0

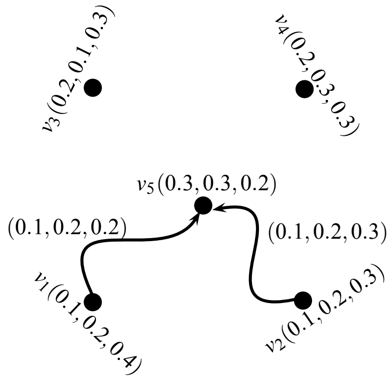


Fig. 5.36 $H_1 \otimes H_2$

Example 5.22 Let $H_1 = (\sigma_1, \varepsilon_1)$ and $H_2 = (\sigma_2, \varepsilon_2)$ be two 3-polar fuzzy directed hypergraphs as shown in Figs. 5.33 and 5.34, respectively. The index matrix of $H_1 \otimes H_2$ is shown in Table 5.19, where $\sigma_1 \cap \sigma_2 = \{v_1, v_2, ..., v_5\}$.

The graph of $H_1 \otimes H_2$ is shown in the Fig. 5.36.

Definition 5.40 Let $H_1 = (\sigma_1, \varepsilon_1)$ and $H_2 = (\sigma_2, \varepsilon_2)$ be two m-polar fuzzy directed hypergraphs. The structural subtraction of two m-polar fuzzy directed hypergraphs over a fixed set V is denoted by $H_1 \boxminus H_2 = (\sigma_2 - \sigma_1, \varepsilon_2 - \varepsilon_1)$ and defined as

Table 5.20 Index matrix of $H_1 \boxminus H_2$

$H_1 \boxminus H_2$	v_6
v_6	0

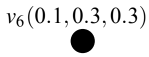


Fig. 5.37 $H_1 \boxminus H_2$

$$P_k o(\sigma_2 - \sigma_1)(v_r) = \begin{cases} P_k o\sigma_1(v_r), & \text{if } v_r \in \sigma_1, \\ P_k o\sigma_2(v_r), & \text{if } v_r \in \sigma_2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

$$P_k o(\varepsilon_2 - \varepsilon_1)(e_{rs}) = P_k o\varepsilon_1(e_{ij}) \text{ if } v_r = v_i \in \sigma_2 - \sigma_1 \text{ and } v_s = v_j \in \sigma_2 - \sigma_1. \tag{5.6}$$

The graph $H_1 \boxminus H_2$ is empty when $\sigma_2 - \sigma_1 = \emptyset$.

Example 5.23 Let $H_1 = (\sigma_1, \varepsilon_1)$ and $H_2 = (\sigma_2, \varepsilon_2)$ be two 3-polar fuzzy directed hypergraphs as shown in Figs. 5.33 and 5.34, respectively. The index matrix of $H_1 \boxminus H_2$ is shown in Table 5.20, where $\sigma_2 - \sigma_1 = \{v_6\}$.

The graph $H_1 \boxminus H_2$ is shown in the following Fig. 5.37

Definition 5.41 Let $H_1 = (\sigma_1, \varepsilon_1)$ and $H_2 = (\sigma_2, \varepsilon_2)$ be two m-polar fuzzy directed hypergraphs. The multiplication of two m-polar fuzzy directed hypergraphs H_1 and H_2, denoted by $H_1 \odot H_2 = (\sigma_1 \odot \sigma_2, \varepsilon_1 \odot \varepsilon_2)$ is defined as

$$P_k o(\sigma_1 \odot \sigma_2)(v_r) = \begin{cases} P_k o\sigma_1(v_r), & \text{if } v_r \in \sigma_1, \\ P_k o\sigma_2(v_r), & \text{if } v_r \in \sigma_2, \\ \inf\{P_k o\sigma_1(v_r), P_k o\sigma_2(v_r)\}, & \text{if } v_r \in \sigma_1 \cap \sigma_2. \end{cases} \tag{5.7}$$

$$\begin{aligned} &P_k o(\varepsilon_1 \odot \varepsilon_2)(e_{rs}) \\ &= \begin{cases} P_k o\varepsilon_1(e_{ij}), & \text{if } v_r = v_i \in \sigma_1 \text{ and } v_s = v_j \in \sigma_1 \setminus \sigma_2, \\ P_k o\varepsilon_2(e_{pq}), & \text{if } v_r = v_p \in \sigma_2 \text{ and } v_s = v_q \in \sigma_2 \setminus \sigma_1, \\ \sup_{i,q}\{\inf_{j,p}\{P_k o\varepsilon_1(e_{ij}), P_k o\varepsilon_2(e_{pq})\}\}, & \text{if } v_r = v_i \in \sigma_1 \cap \sigma_2 \text{ and } v_s = v_q \in \sigma_1 \cap \sigma_2. \end{cases} \end{aligned} \tag{5.8}$$

Table 5.21 Index matrix of $H_1 \odot H_2$

$H_1 \odot H_2$	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	0	0	0	0	0
v_2	0	0	0	0	0	0
v_3	(0.1, 0.2, 0.3)	(0.1, 0.2, 0.3)	0	(0.1, 0.2, 0.3)	0	(0.1, 0.1, 0.3)
v_4	0	0	0	0	0	0
v_5	(0.1, 0.2, 0.2)	(0.1, 0.2, 0.2)	0	(0.1, 0.2, 0.2)	0	0
v_6	(0.1, 0.3, 0.3)	0	0	(0.1, 0.3, 0.3)	0	0

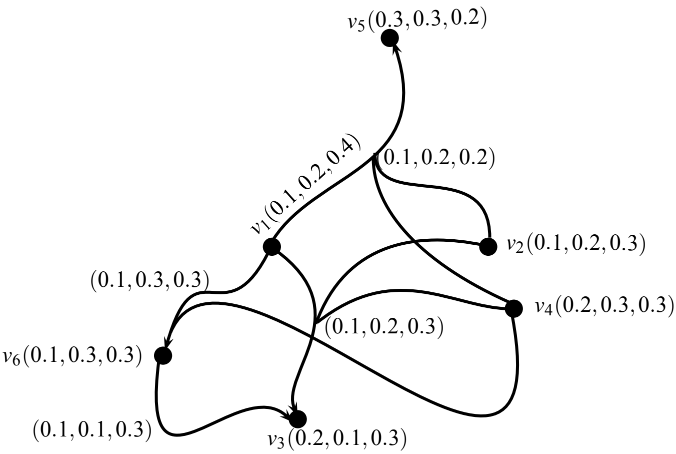


Fig. 5.38 $H_1 \odot H_2$

Example 5.24 The index matrix of graph $H_1 \odot H_2$ is shown in Table 5.21, where $\sigma_2 \cup (\sigma_1 - \sigma_2) = \{v_1, v_2, v_3, ..., v_6\}$ is given in Table 5.20.

The corresponding hypergraph is shown in Fig. 5.38.

5.5 Application of m-Polar Fuzzy Directed Hypergraphs

Decision-making is regarded as the intellectual process resulting in the selection of a belief or a course of action among several alternative possibilities. Every decision-making process produces a final choice, which may or may not prompt action. Decision-making is the process of identifying and choosing alternatives based on the values, preferences, and beliefs of the decision-maker. Problems in almost every credible discipline, including decision-making can be handled using graphical models.

5.5.1 Business Strategy Company

A business strategy is a registered plan on how an organization is setting out to fulfill their ambitions. A business strategy has a variety of successful key of principles that sketch how a company will go about achieving their dreams in business. It deals with competitors, look at their needs and expectations of customers and will examine the long-term growth and sustainability of their organization.

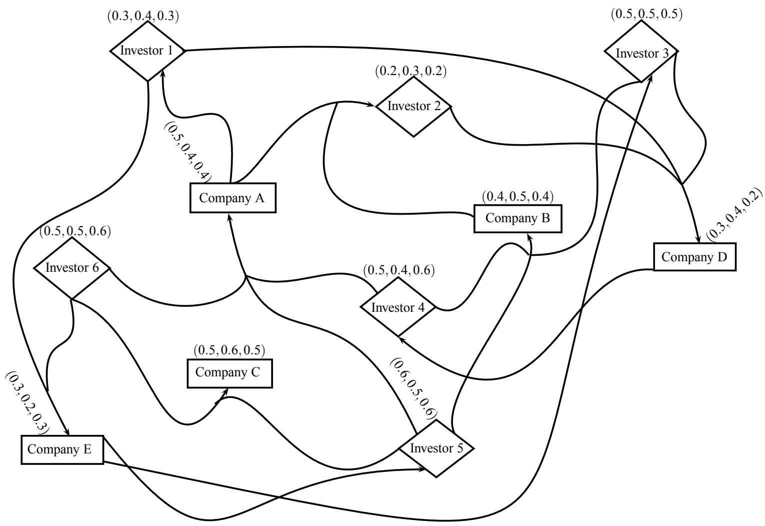


Fig. 5.39 3-polar fuzzy directed hypergraph model

Table 5.22 Collective interest of investors toward companies

Business strategy company	Positive effects of investors
Company A	(0.5, 0.4, 0.6)
Company B	(0.5, 0.4, 0.5)
Company C	(0.5, 0.5, 0.6)
Company D	(0.2, 0.3, 0.2)
Company E	(0.3, 0.4, 0.3)

In this fast running world where every investor is searching out a best business strategy company so that they invest their money on the company to promote the business and to compete their competitors. Then to select a good marketing business company which will achieve its goals, meet the expectations and sustain a competitive advantage in the marketplace, we develop a 3-polar fuzzy directed hypergraphical model that how an investor can choice the greatest salubrious company to promote the business by following a step by step procedure. A 3-polar fuzzy directed hypergraph demonstrating a group of investors as members of different business strategy companies is shown in Fig. 5.39.

If an investor wants to adopt the most suitable and powerful business company to which he works and get the progress in business, the following procedure can help the investors. Firstly, one should think about the cooperative contribution of investors toward the company, which can be found out by means of membership values of 3-polar fuzzy directed hypergraphs. The membership values given in Table 5.22 shows the collective interest of investors toward the company.

Table 5.23 Benefits of company on the investors

Business strategy company	Effects of company on investors
Company A	(0.5, 0.4, 0.4)
Company B	(0.4, 0.5, 0.4)
Company C	(0.5, 0.6, 0.5)
Company D	(0.3, 0.4, 0.2)
Company E	(0.3, 0.2, 0.3)

Table 5.24 In-degrees and out-degrees of companies

Business strategy company	In-degrees	out-degrees
Company A	(0.5, 0.4, 0.4)	(0.5, 0.7, 0.5)
Company B	(0.4, 0.4, 0.4)	(0.2, 0.3, 0.2)
Company C	(0.5, 0.5, 0.5)	(0, 0, 0)
Company D	(0.2, 0.3, 0.2)	(0.3, 0.4, 0.3)
Company E	(0.3, 0.2, 0.3)	(0.6, 0.4, 0.6)

The first membership value showing how much investors invest money on company, second showing the sharp-minded quality of investors to run the business and third showing how can strongly they make production by working with company. It can be noticed that the company C has strong collective interest in investors which is maximum among all other companies. Secondly, one should do his research on the powerful impacts of all under consideration companies on their investors. The membership degrees of all company nodes show their effects on their investors as given in Table 5.23.

The membership values showing three different positive effects of company on investor, first one shows how much a company is financially strong already, second showing its business growth in the market, and third one showing the strong competitive position of company. Note that, company C has the most benefits for investors. Thirdly, an investor can observe the influence of a company by calculating its in-degrees and out-degrees. In-degrees show the percentage of investors joining the company and out-degrees show the percentage of investors leaving that company. The in-degrees and out-degrees of all business strategy companies are given in Table 5.24.

Hence, a best business strategy company has maximum in-degrees and minimum out-degrees. However, in case when two companies have same minimum out-degrees, then we compare their in-degrees. Similarly, when in-degrees same, we compare out-degrees. From all the above discussion, we conclude that company C is the most appropriate company to fulfill the requirements of the investors because it is more financially strong, best in competitive position and business growth of this company is more suitable to run the business and compete with the competitors. The method of searching out the constructive and profitable business strategy company is explained in the following Algorithm 5.5.1.

Algorithm 5.5.1 To find out the constructive and profitable business strategy company

1. Input the membership values of all nodes(investors) $v_1, v_2, ..., v_n$.
2. Determine the augmentation of investors toward companies by calculating the membership values of all directed hyperedges as

$$P_k o\varepsilon_r \leq \inf\{P_k ov_1, P_k ov_2, ..., P_k ov_n\}, 1 \leq k \leq m.$$

3. Obtain the most suitable company as

$$\sup P_k o\varepsilon_r.$$

4. Find the company having strong and more benefits for investors as,

$$\sup P_k ov_r,$$

where all v_r here are vertices represent the different business strategy company.
5. Find the profitable influence of companies v_r on the investors by calculating the in-degrees $D^-(v_r)$ as

$$\sum_{v_r \in h(\varepsilon_r)} P_k o\varepsilon_r.$$

6. Find the profitless impact of companies v_k on the investors by calculating the out-degrees $D^+(v_r)$ as,

$$\sum_{v_r \in t(\varepsilon_r)} P_k o\varepsilon_r.$$

7. Obtain the most advantageous business strategy company as

$$(\sup D^-(v_r), \inf D^+(v_r)).$$

The algorithm runs linearly and its net time complexity is $\bigcirc(n)$, where n is the number of membership values of all nodes(investors).

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